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On the existence of unique equilibria in location models*

H.M. Webers[†]

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Abstract

In this paper we study a variant of the two-stage location-then-price game where consumers are distributed piecewise uniformly, each piece being referred to as an interval. Clearly, only the *exact* interval in which the indifferent consumer is located may be uncertain for the firms. Therefore, we encompass the firms with beliefs about the interval in which the indifferent consumer is located. Given their beliefs, the firms' expected demands are differentiable everywhere and the firms' expected profit functions are quasi-concave. We define the game where firms first choose beliefs and then maximize the corresponding expected profit in two stages to be a psychological game. We show that there exists a unique psychological equilibrium for this game, which consists of a subgame perfect Nash equilibrium for the two-stage game given certain beliefs and the beliefs are such that the equilibrium outcome is consistent with these beliefs. We give a coordination argument in order to easily find this equilibrium.

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1 Introduction

In this paper, we present a generalization of the standard Hotelling (1929) model of spatial competition where firms first choose locations and then, given these locations, compete in prices. In the model consumers are distributed according to a piecewise uniform density along a line segment, whereas in the standard model consumers are distributed with uniform density. The different pieces are referred to as intervals. This specification of the consumers' density could make our results widely applicable, because any density function can be approximated with a piecewise uniform density, by passing to finer partitions. As a consequence the firms' profit functions are piecewise quasi-concave and they are not differentiable everywhere due to the fact that demand is kinked. For any given locations and prices, however, both firms know that there is a unique indifferent consumer. This means that only the *exact* interval in which the indifferent consumer is located, may be uncertain for both firms. Therefore we encompass the firms with beliefs about the interval in which the indifferent consumer is located, called the state. Recently Geanakoplos, Pearce, and Stacchetti (1989) and Kolpin (1992) introduced the psychological game to provide a framework for the formal analysis of strategic settings in which expectations play a role[1]. The principal characteristic of a psychological game is that the firms' expected profits depend on what everybody believes. Given their beliefs, the firms' expected demands are differentiable everywhere and the firms' expected profit functions are quasi-concave. We define the game where firms first choose beliefs and then maximize the corresponding expected profit in two stages to be a psychological game.

We show that there exists a unique psychological equilibrium for this game. Such an equilibrium consists of a subgame perfect Nash equilibrium for the two-stage game given certain beliefs and the beliefs are such that the equilibrium outcome is consistent with these beliefs, i.e., the state corresponds to the location of the indifferent consumer. In equilibrium, the indifferent consumer is exactly the median consumer while both firms have identical profits. Furthermore we present a natural way to find the psychological equilibrium, which implicitly requires firms to coordinate on the interval that contains the indifferent consumer.

This paper is related to the work of Goeree and Ramer (1994), who generalize the results for the traditional two-stage location-then-price game,

[1] In another context these, subjective, expectations are often referred to as emotions.

by allowing for log-concave densities, and to the work of Tabuchi and Thisse (1995), who look at the specific example of a triangular density in more detail. They find that in general equilibrium profits differ over the two firms. In case of a triangular density we show, however, that the firm with the lower profit has a profit that is lower than the psychological equilibrium profit, which casts doubt on the credibility of the asymmetric outcome. As Geanakoplos, Pearce, and Stacchetti (1989) argue, the usual solution concepts are valid only as long as the correct payoff function is employed.

There are several reasons to favour the psychological game approach. First, the psychological equilibrium outcome is robust in the sense that the indifferent consumer is exactly the median consumer. The firms' locations and prices are determined completely by the density function then. Second, equilibrium outcomes are such that both firms have the same profit, so, even without restricting one firm to locate to the left of the other firm, neither firm has an incentive to deviate. In that sense the coordination problem arising in Goeree and Ramer (1994) does not appear. Third, the psychological game approach is also applicable in case of density functions that are not log-concave.

This paper is based on Webers (1994) and is organized as follows. In Section 2 the model is presented and the definition of psychological game is given. In Section 3 the psychological equilibrium for this game is introduced and the equilibrium conditions are derived. In Section 4 we prove the existence of a, generically, unique psychological equilibrium with consistent beliefs. In Section 5 we use a coordination argument to find this equilibrium. In Section 6 we discuss the case of a triangular density and in Section 7 we briefly look at the two-dimensional case. The proofs are gathered in the Appendix.

2 The model

There is a continuum of consumers distributed along the line segment $[0, 1]$ with cumulative density function $\mathcal{F}_1 : [0,1] \mapsto [0,1]$. There are two firms, indexed $i \in I = \{1,2\}$. Firm $i \in I$ locates at x_i along the real line and sells the commodity at price $p_i \in \mathbb{R}_+$. Real income of the consumers is given by w. Each consumer buys one unit of the commodity from the firm that offers the highest indirect utility, for firm $i \in I$ being given by

$$V_i(x, x_i, p_i) = w - p_i - t(x, x_i), \tag{1}$$

where x is the consumer's location in the unit interval. The number $t(x, x_i)$ is the transportation cost for shipping the product of firm i to this consumer's location. We assume this transportation cost to be quadratic in distance with unit cost equal to one, i.e., $t(x, x_i) = (x - x_i)^2$. The market area of the product of firm $i \in I$ at given locations x_1 and x_2 and prices p_1 and p_2 is therefore given by

$$M_i(x_1, x_2, p_1, p_2) = \{x \in [0,1] \mid V_i(x, x_i, p_i) \geq V_j(x, x_j, p_j), j \neq i\},$$

i.e., the set of consumers that prefer the commodity of firm i over the commodity of firm j, $j \neq i \in I$. The demand $X_i(x_1, x_2, p_1, p_2)$ for the commodity of firm $i \in I$ then is equal to

$$X_i(x_1, x_2, p_1, p_2) = \int_{M_i(x_1, x_2, p_1, p_2)} dx. \tag{2}$$

By definition, the sum of the commodity demands equals one. Given x_1, x_2, p_1 and p_2 the location of the consumer indifferent between buying from firm 1 and buying from firm 2 is given by

$$x(x_1, x_2, p_1, p_2) = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2(x_2 - x_1)}, \tag{3}$$

being the midpoint between the firms' locations corrected for price differences. Under the assumption that the price differences are not too large, both firms will sell their products. From equations (2) and (3) it follows that for $i \in I$

$$X_i(x_1, x_2, p_1, p_2) = \mathcal{F}_i(x(x_1, x_2, p_1, p_2)), \tag{4}$$

where for all $x \in [0,1]$, $\mathcal{F}_2(x) = 1 - \mathcal{F}_1(x)$. Given the locations x_1 and x_2 and prices p_1 and p_2 the profit of firm $i \in I$ is equal to $p_i \mathcal{F}_i(x(x_1, x_2, p_1, p_2))$, where costs are assumed to be normalized to zero.

The function $\mathcal{F}_1$ is assumed to be continuous, but is allowed to be non-differentiable in a finite number of, say $n - 1$, points $t_1, \ldots, t_{n-1}$, with $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$. For $k \in \mathcal{K} = \{0, \ldots, n-1\}$, we assume there is a differentiable function $\mathcal{F}_{ik} : [0,1] \mapsto \mathbb{R}_+$ such that $\mathcal{F}_i$ coincides with $\mathcal{F}_{ik}$ for $t_k \leq x \leq t_{k+1}$, $i \in I$. We assume that each firm has beliefs about the state, being the interval in which the indifferent consumer is located. The beliefs b_{ik} firm $i \in I$ has about the location of the indifferent consumer to lie in the interval $[t_k, t_{k+1}]$, $k \in \mathcal{K}$, is represented by the vector of beliefs $b_i = (b_{i0}, \ldots, b_{i(n-1)})$ in the set $B_i = \{y \in [0,1]^n \mid \sum_{k=0}^{n-1} y_k = 1\}$. At beliefs

$b_i \in B_i$ and with the indifferent consumer located at $x \in [0,1]$, the expected demand $F_i(b_i, x)$ for firm $i \in I$ is given by

$$F_i(b_i, x) = \sum_{k=0}^{n-1} b_{ik} \mathcal{F}_{ik}(x). \tag{5}$$

Clearly, for given $b_i \in B_i$, the expected demand function $F_i(b_i, .)$, $i \in I$, is differentiable. For simplicity, given $b_i \in B_i$ the function F_i is assumed to be three times continuously differentiable. Note that in case $\mathcal{F}_1$ is (three times) continuously differentiable and hence $n = 1$, expected demand is equal to demand.

For given beliefs $b_i \in B_i$, the expected profit for firm $i \in I$ is equal to

$$\Pi_i(b_i, x_1, x_2, p_1, p_2) = p_i F_i(b_i, x(x_1, x_2, p_1, p_2)). \tag{6}$$

Now we are able to introduce the psychological game.

Definition 2.1 *The game where firms first choose beliefs and then maximize the corresponding expected profit in a two-stage location-then-price game is a (strategic form) psychological game G.*

At given beliefs $b = (b_1, b_2) \in B = B_1 \times B_2$, the two-stage game with payoffs $\Pi_1(b_1, .)$ and $\Pi_2(b_2, .)$ is a conventional two-stage location-then-price game, which we refer to as $G(b)$.

3 The equilibrium concept

First we define the solution concept for the two-stage location-then-price game G with beliefs. At given beliefs $b \in B$ and at given locations $x_1 < x_2$, suppose $p_i^*(b, x_1, x_2)$ is the unique corresponding Nash equilibrium price for firm $i \in I$. Given these prices the firms choose locations as to maximize their expected profits. Because equilibrium prices depend on the other firm's beliefs, also the firm's location choices for the game $G(b)$ will depend on the other firm's beliefs. Suppose the corresponding Nash equilibrium locations are unique also, to be denoted by $x_1^*(b)$ for firm 1 and $x_2^*(b)$ for firm 2. The subgame perfect Nash equilibrium strategy at beliefs $b \in B$ for the game $G(b)$ is denoted by $s^*(b) = < x_1^*(b), x_2^*(b), \{(p_1^*(b, x_1, x_2), p_2^*(b, x_1, x_2)) \mid x_1 < x_2\} >$. Consequently, at given beliefs $b \in B$, the indifferent consumer is located at

$$x^*(b) = \frac{x_1^*(b) + x_2^*(b)}{2} + \frac{p_2^*(b, x_1^*(b), x_2^*(b)) - p_1^*(b, x_1^*(b), x_2^*(b))}{2(x_2^*(b) - x_1^*(b))}. \tag{7}$$

We say that beliefs $b \in B$ are *consistent* if for all $i \in I$ and for all $k \in \mathcal{K}$

$$\begin{array}{ll} b_{ik} = 1 & \text{if } t_k < x^*(b) < t_{k+1} \\ b_{ik} = 0 & \text{if } x^*(b) < t_k \text{ or } x^*(b) > t_{k+1} \\ b_{ik} \in [0,1] & \text{otherwise,} \end{array} \tag{8}$$

so, expected demand is equal to realized demand.

Definition 3.1 *A psychological equilibrium for a psychological game G with unique subgame perfect Nash equilibria at any beliefs is a pair of strategies s^* and beliefs $b^* \in B$ such that b^* is consistent and s^* is a subgame perfect Nash equilibrium strategy for the game $G(b^*)$, i.e., $s^* = s^*(b^*)$.*

First we determine the solution to the location-then-price game for fixed beliefs $b \in B$. For $i \in I$, let f_i denote the first order derivative of F_i with respect to x, let f_i' denote the second order derivative of F_i with respect to x and let f_i'' denote the third order derivative of F_i with respect to x. For simplicity we let $f_i(b_i, x) > 0$ for all $b_i \in B_i$ and $x \in [0,1]$. For firm $i \in I$, the corresponding first order condition for the price stage at locations x_1 and x_2 reads

$$p_i = -\frac{F_i(b_i,x)}{f_i(b_i,x)} \left(\frac{\partial x}{\partial p_i}\right)^{-1}, \tag{9}$$

while the second order condition for a maximum is given by

$$\left(2f_i(b_i,x) - \frac{F_i(b_i,x)}{f_i(b_i,x)} f_i'(b_i,x)\right) \frac{\partial x}{\partial p_i} < 0, \tag{10}$$

where x is given by equation (3). In general, the set of equations (9) and (10) has multiple solutions, or no solutions at all. For the case where $\mathcal{F}_1$ is three times continuosly differentiable and consequently $\mathcal{F}_1 = F_1$, i.e., beliefs do not matter, Goeree and Ramer (1994) prove the existence of a unique price equilibrium assuming log-concavity of $\mathcal{F}_1$. They apply a theorem of Caplin and Nalebuff (1991) in order to show the quasi-concavity of the profit functions. For the case where $\mathcal{F}_1$ is continuous but not differentiable everywhere, in the next section, for any beliefs $b \in B$, we prove the existence of a unique price equilibrium for the situation of piecewise linearity of $\mathcal{F}_1$. In this case profit is piecewise quasi-concave. With a piecewise linear $\mathcal{F}_1$ we are able to approximate any cumulative density function by passing to finer partitions, i.e., by increasing n.

However, for the time being we need only the existence of a unique solution $p_i^*(b, x_1, x_2)$ for $i \in I$ to the price stage at any beliefs $b \in B$. For ease of notation we write $x^*(b, x_1, x_2) = x(x_1, x_2, p_1^*(b, x_1, x_2), p_2^*(b, x_1, x_2))$. The profit of firm $i \in I$ at the equilibrium prices is denoted by $\Pi_i^*(b, x_1, x_2) = \Pi_i(b, x_1, x_2, p_1^*(b, x_1, x_2), p_2^*(b, x_1, x_2))$. Given these prices, firm $i \in I$ strategically chooses at $b \in B$ location x_i as to maximize its profit $\Pi_i^*(b, x_1, x_2)$. For firm $i \neq j \in I$, the corresponding first order condition for the location stage yields

$$F_i(b_i, x^*(b)) f_i(b_i, x^*(b)) = \left(2(f_i(b_i, x^*(b)))^2 - F_i(b_i, x^*(b)) f_i'(b_i, x^*(b))\right) \times (x_j^*(b) - x_i^*(b)) \frac{\partial x^*(b, x_1, x_2)}{\partial x_i}\Big|_{(x_1^*(b), x_2^*(b))}, \tag{11}$$

while the second order condition for a maximum is given by

$$\frac{\partial}{\partial x_i}\Big|_{(x_1^*(b), x_2^*(b))} \left\{ \left(2 - \frac{F_i(b_i, x^*(b, x_1, x_2))}{f_i(b_i, x^*(b, x_1, x_2))} f_i'(b_i, x^*(b, x_1, x_2))\right) \frac{\partial x^*(b, x_1, x_2)}{\partial x_i} \times (x_j - x_i) F_i(b_i, x^*(b, x_1, x_2)) - \left(\frac{(F_i(b_i, x^*(b, x_1, x_2)))^2}{f_i(b_i, x^*(b, x_1, x_2))}\right) \right\} < 0. \tag{12}$$

It is easily checked that for all $i \in I$ it holds

$$\frac{\partial x^*(b, x_1, x_2)}{\partial x_i} = \frac{1}{2} \left(3 - \sum_{j \in I} \frac{F_j(b_j, x^*(b, x_1, x_2)) f_j'(b_j, x^*(b, x_1, x_2))}{(f_j(b_j, x^*(b, x_1, x_2)))^2} \right)^{-1}. \tag{13}$$

Consequently, assuming that $x_1^*(b)$ and $x_2^*(b)$ are the unique solutions to (11), $s^*(b) = < x_1^*(b), x_2^*(b), \{(p_1^*(b, x_1, x_2), p_2^*(b, x_1, x_2)) \mid x_1 < x_2\} >$ is the unique subgame perfect Nash equilibrium strategy for the game $G(b)$.

4 Existence of psychological equilibria

We consider a piecewise uniform density function having a piecewise linear cumulative density. We assume that for some $n \in \mathbb{N}$, on each interval $[\frac{k}{n}, \frac{k+1}{n}]$, $k \in \mathcal{K}$, consumers are located with density $d_{k+1} > 0$ such that

$$\sum_{k=0}^{n-1} d_{k+1} = n.$$

This means that $t_k = \frac{k}{n}$ for $k \in \mathcal{K}$. The density will be denoted by the tuple $< d_1, \ldots, d_n >$. Note that we have the standard uniform case when $n = 1$ or $d_{k+1} = 1$ for each $k \in \mathcal{K}$. We denote $\mathcal{D}_k = \sum_{m=0}^{k} d_m$ for all $k \in \mathcal{K}$, where we define $d_0 = 0$. For given beliefs $b_i \in B_i$, the expected demand for firm $i \in I$ is linear in x and can be written as

$$F_i(b_i, x) = \gamma_i(b_i) + \delta_i(b_i)x, \tag{14}$$

where

$$\gamma_1(b_1) = \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1} b_{1k}(\mathcal{D}_k - kd_{k+1}), \quad \gamma_2(b_2) = 1 - \tfrac{1}{n}\textstyle\sum_{k=1}^{n-1} b_{2k}(\mathcal{D}_k - kd_{k+1}),$$

$$\delta_1(b_1) = \textstyle\sum_{k=0}^{n-1} b_{1k}d_{k+1}, \quad \delta_2(b_2) = -\textstyle\sum_{k=0}^{n-1} b_{2k}d_{k+1}.$$

Clearly, $f_i(b_i, x) = \delta_i(b_i)$ and $f_i'(b_i, x) = f_i''(b_i, x) = 0$ for all $i \in I$ and $b_i \in B_i$. Equation (9) can be rewritten then as

$$p_i(b, x_1, x_2) = -(x + \frac{\gamma_i(b_i)}{\delta_i(b_i)}) \left(\frac{\partial x}{\partial p_i}\right)^{-1} \tag{15}$$

and the second order conditions for a maximum are fulfilled because $\delta_1(b_1) > 0$ and $\delta_2(b_2) < 0$. Note that $\gamma_2(b_2) = 1 - \gamma_1(b_1)$ and $\delta_2(b_2) = -\delta_1(b_1)$ in case firms have identical beliefs, i.e., $b_1 = b_2$.

Proposition 4.1 *For any density $< d_1, \ldots, d_n >$ with $n \in \mathbb{N}$, any $b \in B$ and any $x_1 < x_2$, there exists a unique solution to the price stage for the game $G(b)$ given by*

$$p_i^*(b, x_1, x_2) = \frac{x_2 - x_1}{3} \left(x_1 + x_2 + \frac{4\gamma_i(b_i)}{\delta_i(b_i)} - \frac{2\gamma_j(b_j)}{\delta_j(b_j)} \right)$$

for $i \neq j \in I$.

Proof
Substitution of the prices from equation (15) into equation (3) yields

$$x^*(b, x_1, x_2) = \frac{x_1 + x_2}{6} - \frac{\gamma_1(b_1)}{3\delta_1(b_1)} - \frac{\gamma_2(b_2)}{3\delta_2(b_2)}.$$

The corresponding prices are given then by

$$p_i^*(b, x_1, x_2) = \frac{x_2 - x_1}{3}\left(x_1 + x_2 + \frac{4\gamma_i(b_i)}{\delta_i(b_i)} - \frac{2\gamma_j(b_j)}{\delta_j(b_j)}\right)$$

for $i \neq j \in I$

□

Requiring the prices to be positive yields a condition on x_1 and x_2, namely

$$\frac{2\gamma_2(b_2)}{\delta_2(b_2)} - \frac{4\gamma_1(b_1)}{\delta_1(b_1)} < x_1 + x_2 < \frac{2\gamma_1(b_1)}{\delta_1(b_1)} - \frac{4\gamma_2(b_2)}{\delta_2(b_2)}. \tag{16}$$

Furthermore it is easily found from equation (13) that for all $i \in I$ it holds that

$$\frac{\partial x^*(b, x_1, x_2)}{\partial x_i} = \frac{1}{6}. \tag{17}$$

Given the prices $p_i^*(b, x_1, x_2)$, $i \in I$, the firms choose locations as to maximize their expected profits. The profit of firm $i \neq j \in I$ can be written then as

$$\Pi_i^*(b, x_1, x_2) = \frac{\delta_i(b_i)(x_2 - x_1)}{18}\left(x_1 + x_2 + \frac{4\gamma_i(b_i)}{\delta_i(b_i)} - \frac{2\gamma_j(b_j)}{\delta_j(b_j)}\right)^2. \tag{18}$$

Proposition 4.2 *For any density $< d_1, \ldots, d_n >$ with $n \in \mathbb{N}$ and any beliefs $b \in B$, there exists a unique solution to the location stage for the game $G(b)$ given by*

$$x_i^*(b) = \frac{1}{4}(\frac{\gamma_j(b_j)}{\delta_j(b_j)} - \frac{5\gamma_i(b_i)}{\delta_i(b_i)})$$

for $i \neq j \in I$.

Proof See Appendix.

It is easily checked that equation (16) is satisfied at the solution $x_1^*(b)$ and $x_2^*(b)$ in case of identical beliefs. In equilibrium the indifferent consumer is located at

$$x^*(b) = -\frac{1}{2}(\frac{\gamma_1(b_1)}{\delta_1(b_1)} + \frac{\gamma_2(b_2)}{\delta_2(b_2)}). \tag{19}$$

From this we derive the following theorem.

Theorem 4.3 *For any density $< d_1, \ldots, d_n >$ with $n \in \mathbb{N}$, generically, there exists a unique psychological equilibrium (s^*, b^*) with $s^* = s(b^*)$. In equilibrium, for all $i \in I$, $b^*_{ik^*} = 1$ for the (generically) unique value $k^* \in \mathcal{K}$ for which $\frac{\mathcal{D}_{k^*}}{n} \leq \frac{1}{2} \leq \frac{\mathcal{D}_{k^*+1}}{n}$ and $b^*_{ik} = 0$ for $k \neq k^* \in \mathcal{K}$.*

Proof See Appendix.

In the non-generic case, $\frac{\mathcal{D}_{k^*}}{n} = \frac{1}{2}$ for some $k^* \in \mathcal{K}$, i.e., the median consumer is exactly located at a corner, and both $k^* - 1$ and k^* induce a psychological equilibrium. Profits for both firms, however, are maximized for the value of $k \in \{k^* - 1, k^*\}$ for which the corresponding value d_k is minimal, because then equilibrium prices are higher while demand is $\frac{1}{2}$ in both cases. When $d_{k^*-1} = d_{k^*}$, profits are the same for both values of k. Consequently, for all $i \in I$ and for all symmetric beliefs $b \in B$ with $b_{i(k^*-1)} = 1 - b_{ik^*}$ and $b_{ik^*} \in [0, 1]$, profits are maximized then and the indifferent consumer is exactly the median consumer.

For convenience we write $\gamma_1^* = 1 - \gamma_2^* = \frac{1}{n}(\mathcal{D}_{k^*} - k^* d_{k^*+1})$ and $\delta_1^* = -\delta_2^* = d_{k^*+1}$, which are the corresponding values in case b_1^* and b_2^* are such that $b^*_{1k^*} = b^*_{2k^*} = 1$. Equilibrium beliefs $b^* \in B$ are such that both firms have the same (expected) demands. We denote $x^* = x^*(b^*)$. In equilibrium the indifferent consumer is located at

$$x^* = \frac{1 - 2\gamma_1^*}{2\delta_1^*} \tag{20}$$

which is exactly the median consumer because $F_1(x^*) = 1 - F_2(x^*) = \frac{1}{2}$. In case of symmetric densities equation (20) reduces to $x^* = \frac{1}{2}$. For n odd, we have $k^* = \frac{n-1}{2}$ then, which gives $x^* = \frac{n - 2(\mathcal{D}_{k^*} - k^* d_{k^*+1})}{2n d_{k^*+1}} = \frac{2k^*+1}{2n} = \frac{1}{2}$. For n even, k^* is either $\frac{n}{2} - 1$ or $\frac{n}{2}$ and similarly it follows $x^* = \frac{1}{2}$.

Corollary 4.4 *For any density $< d_1, \ldots, d_n >$ with $n \in \mathbf{N}$, generically, the unique subgame perfect Nash equilibrium outcome for the game $G(b^*)$ is given by locations $x_1^* = \frac{-1-4\gamma_1^*}{4\delta_1^*}$ and $x_2^* = \frac{5-4\gamma_1^*}{4\delta_1^*}$ and by prices $p_1^* = p_2^* = \frac{3}{2(\delta_1^*)^2}$.*

Proof

For the equilibrium beliefs $b^* \in B$ specified in Theorem 4.3 it is easy to calculate that $x_i^* = x_i^*(b^*)$ and $p_i^* = p_i^*(b^*, x_1^*(b^*), x_2^*(b^*))$ for $i \in I$ from Propositions 4.1 and 4.2 by substituting γ_1^*, γ_2^*, δ_1^*, and δ_2^*.

□

Thus the two firms always charge the same price in equilibrium and furthermore both firms have different locations in equilibrium. This means that the principle of minimum differentiation no longer holds. Note that these results are similar to the results Lederer and Hurter (1986) find for the situation of discriminatory pricing. In case of a symmetric density the result in Corollary 4.4 reduces to locations $x_1^* = \frac{1}{2} - \frac{3}{4d_{k^*+1}^*}$ and $x_2^* = \frac{1}{2} + \frac{3}{4d_{k^*+1}^*}$. Intuitively this 'symmetric' result is what we could expect. It is exactly the result Goeree and Ramer (1994) find for symmetric densities in case $\mathcal{F}_1$ is differentiable. Because the prices and demands are the same for both firms, equilibrium profits are the same for both firms,

$$\Pi_1^* = \Pi_2^* = \frac{3}{4(d_{k^*+1})^2}.$$

Finally we look at the degree of differentiation in equilibrium.

Lemma 4.5 *For any density $< d_1, \ldots, d_n >$ with $n \in \mathbf{N}$, generically, the degree of differentiation in equilibrium is equal to $x_2^* - x_1^* = \frac{3}{2d_{k^*+1}}$.*

Proof

In equilibrium x_1^* and x_2^* are according to Corollary 4.4. But then we can write $x_2^* - x_1^* = \frac{5-4\gamma_1^*}{4\delta_1^*} - \frac{-1-4\gamma_1^*}{4\delta_1^*} = \frac{3}{2\delta_1^*} = \frac{3}{2d_{k^*+1}}$.

□

In particular if the consumers are distributed uniformly then the degree of differentiation is equal to $\frac{3}{2}$ with locations $x_1^* = -\frac{1}{4}$ and $x_2^* = \frac{5}{4}$. Furthermore, if $d_{k^*+1} \leq 1$ then $x_2^* - x_1^* \geq \frac{3}{2}$ and if $d_{k^*+1} \geq 1$ then $x_2^* - x_1^* \leq \frac{3}{2}$. Consequently, for the limiting case $d_{k^*+1} = n$, i.e., demand is concentrated entirely in an infinitesimal interval, the degree of differentiation tends to zero if n goes to infinity. Only for this case, Hotelling's principle of minimum differentiation is restored. In general we find that, when demand is more concentrated at the centre both firms will locate closer to the centre, and, when demand is concentrated at the endpoints, both firms will locate further away from the centre.

5 A coordination argument

As argued before, both firms know that for any tuple of locations and prices, generically, there is a unique interval in which the indifferent consumer is located. The only uncertainty the firms face is that a priori they do not know what interval will result. Let the beliefs of firm $i \in I$ about the location of the indifferent consumer to lie in any of the intervals $[\frac{k}{n}, \frac{k+1}{n}]$, $k \in \mathcal{K}$, be represented by the vector $b_i = (b_{i0}, \ldots, b_{i(n-1)})$ in the set $B_i^c = \{y \in \{0,1\}^n \mid \sum_{k=0}^{n-1} y_k = 1\}$. This means that the vector $b_i \in B_i^c$, $i \in I$, is a unit vector. We will write $b_i(k) \in B_i^c$ for the kth unit vector, $k \in \mathcal{K}$. Because of consistency, it is natural to assume that the firms coordinate on the same beliefs. Let these identical beliefs be denoted by $b(k) \in B^c = \{(y,y) \in B_1^c \times B_2^c\}$. The corresponding conventional two-stage location-then-price game with beliefs $b(k) \in B^c$ is referred to as $G(b(k))$. Because, generically, equilibrium beliefs b^* are in B^c, the unique equilibrium stated in Theorem 4.3 will be found again, generically, but the approach in this section to find this equilibrium is based on coordination. At beliefs $b(k) \in B^c$, expected demand for firm $i \in I$ is linear in x and can be written as

$$F_i^k(x) = \gamma_i^k + \delta_i^k x \tag{21}$$

where

$$\gamma_1^k = 1 - \gamma_2^k = \tfrac{1}{n}(\mathcal{D}_k - k d_{k+1}),$$

$$\delta_1^k = -\delta_2^k = d_{k+1}.$$

Applying Proposition 4.1 then yields the solution

$$p_i^*(b(k), x_1, x_2) = \frac{x_2 - x_1}{3} \left(x_1 + x_2 + \frac{2n + 2(\mathcal{D}_k - k d_{k+1})}{n d_{k+1}} \right) \tag{22}$$

to the price stage for the game $G(b(k))$ by substituting γ_i^k and δ_i^k for $\gamma_i(b_i)$ and $\delta_i(b_i)$, respectively, for all $i \in I$. The solution to the location stage is found from Proposition 4.2 and is given by

$$\begin{aligned} x_1^*(b(k)) &= \tfrac{k}{n} - \tfrac{n+4\mathcal{D}_k}{4nd_{k+1}}, \\ x_2^*(b(k)) &= \tfrac{k}{n} + \tfrac{5n-4\mathcal{D}_k}{4nd_{k+1}}. \end{aligned} \tag{23}$$

Note that the firms are located symmetrically around $\frac{k}{n}$ if $\mathcal{D}_k = \frac{n}{2}$. At these locations and prices the indifferent consumer is expected to be located at

$$x^*(b(k)) = \frac{k}{n} + \frac{n - 2\mathcal{D}_k}{2nd_{k+1}}. \tag{24}$$

We say that this outcome is consistent if $\frac{k}{n} \leq x^*(b(k)) \leq \frac{k+1}{n}$. Rewriting equation (24) then yields the following result.

Lemma 5.1 *For any density $< d_1, \ldots, d_n >$ with $n \in \mathbb{N}$, the subgame perfect Nash equilibrium outcome for the game $G(b(k))$ is consistent if $k \in \mathcal{K}$ satisfies the condition*

$$0 \leq n - 2\mathcal{D}_k \leq 2d_{k+1}.$$

As we know already from Theorem 4.3, generically, there exists a unique psychological equilibrium.

Proposition 5.2 *For any density $< d_1, \ldots, d_n >$ with $n \in \mathbb{N}$, there is at least one value $k^* \in \mathcal{K}$ for which $b(k^*)$ induces a psychological equilibrium. If $k^* \in \mathcal{K}$ is not unique, either $b(k^*-1)$ or $b(k^*+1)$ also induces a psychological equilibrium.*

Proof See Appendix.

From Lemma 5.1 we know that k^* must satisfy the condition

$$\frac{\mathcal{D}_{k^*}}{n} \leq \frac{1}{2} \leq \frac{\mathcal{D}_{k^*+1}}{n}.$$

If $\mathcal{D}_k = \frac{n}{2}$ for some $k \in \mathcal{K}$ then we end up at the corner solution $\frac{k}{n}$ where the equilibria are paired, i.e., the indifferent consumer is the same. Otherwise we end up at an interior solution, as we saw already in Theorem 4.3.

Corollary 5.3 *For any density* $< d_1, \ldots, d_n >$ *with* $n \in \mathbf{N}$*, generically, the unique subgame perfect Nash equilibrium outcome for the game* $G(b(k^*))$ *where* $b(k^*)$*, generically, induces the psychological equilibrium is given by locations* $x_1^{c*} = \frac{k^*}{n} - \frac{n+4\mathcal{D}_{k^*}}{4nd_{k^*+1}}$ *and* $x_2^{c*} = \frac{k^*}{n} + \frac{5n-4\mathcal{D}_{k^*}}{4nd_{k^*+1}}$ *and prices* $p_1^{c*} = p_2^{c*} = \frac{3}{2(d_{k^*+1})^2}$.

Proof
This follows easily from Corollary 4.4.

It is easy to check that, given k^*, $x_1^{c*} < \frac{k^*}{n} \leq x^*(b^{c*}) \leq \frac{k^*+1}{n} < x_2^{c*}$, where the last inequality results from the fact that $5n - 4\mathcal{D}_{k^*+1}$ is positive. This means that the indifferent consumer is located to the right of firm 1 and to the left of firm 2 and, furthermore, the firms are located outside the interval $[\frac{k^*}{n}, \frac{k^*+1}{n}]$. This formalizes Smithies' (1941) notions of 'competitive region' for the region $[x_1^{c*}, x_2^{c*}]$ and of 'hinterlands' for the regions $(-\infty, x_1^{c*})$ and (x_2^{c*}, ∞). It is obvious by now that the size of the competitive region crucially depends on the density. Nevertheless the density is irrelevant for the size of the market areas.

6 The triangular density

In this section we apply the psychological game approach to an example introduced by Tabuchi and Thisse (1995). Consider the cumulative density function $\mathcal{F}_1 : [0,1] \mapsto [0,1]$ given by

$$\mathcal{F}_1(x) = \begin{cases} \mathcal{F}_{10}(x) & \text{for } 0 \leq x \leq \frac{1}{2} \\ \mathcal{F}_{11}(x) & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases}$$

where $\mathcal{F}_{1k} : [0,1] \mapsto \mathbf{R}$, $k \in \{0,1\}$, are given by $\mathcal{F}_{10}(x) = 2x^2$ and $\mathcal{F}_{11}(x) = 4x - 2x^2 - 1$. Note that at $x = \frac{1}{2}$, $\mathcal{F}_1$ is continuously differentiable only once. In case of identical beliefs $\hat{b} = (b_s, b_s) \in B$, expected demand for firm $i \in I$ is quadratic in x and can be written as

$$F_i(b_s, x) = \hat{\gamma}_i(b_s) + \hat{\delta}_i(b_s)x + \hat{\epsilon}_i(b_s)x^2, \tag{25}$$

where

$$\begin{aligned} \hat{\gamma}_1(b_s) = 1 - \hat{\gamma}_2(b_s) &= -b_{s1}, \\ \hat{\delta}_1(b_s) = -\hat{\delta}_2(b_s) &= 4b_{s1}, \\ \hat{\epsilon}_1(b_s) = -\hat{\epsilon}_2(b_s) &= 2(b_{s0} - b_{s1}) = 2 - 4b_{s1}. \end{aligned}$$

Because F_1 is logconcave there exists a unique solution to the price stage for all $\hat{b} \in B$, given implicitly by equation (9). The solution to the location stage can be found from equation (11).

Lemma 6.1 *For beliefs $b_s = (1,0)$, the subgame perfect Nash equilibrium outcome for the game $G(\hat{b})$ is given by $< -\frac{2}{3}(6)^{-\frac{1}{2}}, \frac{5}{3}(6)^{-\frac{1}{2}}, \frac{7}{18}, \frac{14}{18} >$ and the indifferent consumer is located at $(6)^{-\frac{1}{2}}$. For beliefs $b_s = (0,1)$, the subgame perfect Nash equilibrium outcome for the game $G(\hat{b})$ is given by $< 1 - \frac{5}{3}(6)^{-\frac{1}{2}}, 1 + \frac{2}{3}(6)^{-\frac{1}{2}}, \frac{14}{18}, \frac{7}{18} >$ and the indifferent consumer is located at $1 - (6)^{-\frac{1}{2}}$. For beliefs $b_s = (\frac{1}{2}, \frac{1}{2})$, the subgame perfect Nash equilibrium outcome for the game $G(\hat{b})$ is given by $< \frac{1}{8}, \frac{7}{8}, \frac{3}{8}, \frac{3}{8} >$ and the indifferent consumer is located at $\frac{1}{2}$.*

Proof See Appendix.

Because $(6)^{-\frac{1}{2}} < \frac{1}{2} < 1 - (6)^{-\frac{1}{2}}$, it is easy to see that all these three equilibrium outcomes are consistent. In fact these are the only three consistent equilibrium outcomes. The first two, asymmetric, solutions are the ones found by Tabuchi and Thisse (1995) and Goeree and Ramer (1994), which in fact require firms to be treated asymmetrically. The third, symmetric, solution is the one which follows by approximating the cumulative density function with a piecewise linear cumulative density.

Consider therefore the following piecewise uniform density with $\frac{n+1}{2} \in \mathbb{N}$, i.e., n odd. On each interval $[\frac{k}{n}, \frac{k+1}{n}]$, $k \in \mathcal{K}$, consumers are located with density $d_{k+1} > 0$ such that

$$d_{k+1} = \begin{cases} \frac{4n(k+1)}{(n+1)^2} & \text{for} \quad k \in \{0, \ldots, \frac{n-1}{2}\} \\ d_{n-k} & \text{for} \quad k \in \{\frac{n-1}{2} + 1, \ldots, n-1\}. \end{cases} \tag{26}$$

Then we get the following result.

Lemma 6.2 *For the density $< d_1, \ldots, d_n >$ with $\frac{n+1}{2} \in \mathbb{N}$ specified in equation (26), the unique psychological equilibrium outcome is found for $k^* = \frac{n-1}{2}$ and is equal to $< \frac{n-3}{8n}, \frac{7n+3}{8n}, \frac{3(n+1)^2}{8n^2}, \frac{3(n+1)^2}{8n^2} >$. For $n \to \infty$ we get the unique symmetric equilibrium outcome $< \frac{1}{8}, \frac{7}{8}, \frac{3}{8}, \frac{3}{8} >$ as a limiting case.*

Proof
For $n \in \mathbb{N}$ odd it is clear from Theorem 4.3 that $k^* = \frac{n-1}{2}$. But then we can calculate $D_{k^*} = \frac{n(n-1)}{2(n+1)}$ and $d_{k^*+1} = \frac{2n}{n+1}$ from which it follows that $\gamma_1^* = -\frac{n-1}{2(n+1)}$ and $\delta_1^* = \frac{2n}{n+1}$. Corollary 4.4 then gives the required result. The limiting result is found immediately.

It is clear by now that approximating a cumulative density function with a piecewise linear cumulative density lets the asymmetric equilibria disappear. In equilibrium the indifferent consumer is exactly the median consumer. Consequently, firms' demands, prices and profits are the same.

7 The two-dimensional case

In this section we point out briefly the applicability of our results to the two-dimensional case, i.e., the case where consumers are located on the square $S = [0,1] \times [0,1]$. For ease of exposition we let consumers be distributed uniformly over S. Firm $i \in I$ is located at $x_i \in S$ and sells the commodity at price p_i. For convenience we let $x_1 \neq x_2$. Because transportation costs are assumed to be quadratic in distance, the set of indifferent consumers is a line segment perpendicular to the line passing through the firms' locations. For any fixed location pair, the demand for the commodity of firm $i \in I$ can be approximated by a one-dimensional density $< d_1, \ldots, d_n >$ with $n \in \mathbb{N}$, and consequently Theorem 4.3 can be applied. Note, however, that the density depends on the line on which they are located. Essentially this means that firms also have to decide upon the line along which to locate. In Figure 1 the bold line depicts the possible equilibrium locations in case firms locate symmetrically on a fixed line through the centre $(\frac{1}{2}, \frac{1}{2})$. The two dashed lines depict the degree of differentiation at the corresponding equilibrium configuration.

We see that the degree of differentiation lies between its minimum $\frac{3}{4}(2)^{\frac{1}{2}}$ for locations $x_1 = (\frac{1}{8}, \frac{7}{8})$ and $x_2 = (\frac{7}{8}, \frac{1}{8})$, and its maximum $\frac{3}{2}$ for locations $x_1 = (-\frac{1}{4}, \frac{1}{2})$ and $x_2 = (\frac{5}{4}, \frac{1}{2})$. The first result is exactly the result stated in Lemma 6.2, but then on the interval $[0, (2)^{\frac{1}{2}}]$ instead of $[0,1]$. The second result is exactly the result for the uniform case as can be seen from Lemma 4.5. In case the degree of differentiation equals $\frac{3}{4}(2)^{\frac{1}{2}}$ per firm profit is $\frac{3}{8}$ and in case the degree of differentiation equals $\frac{3}{2}$ per firm profit is $\frac{3}{4}$. So profits for both firms are maximized in case the degree of differentiation is maximal.

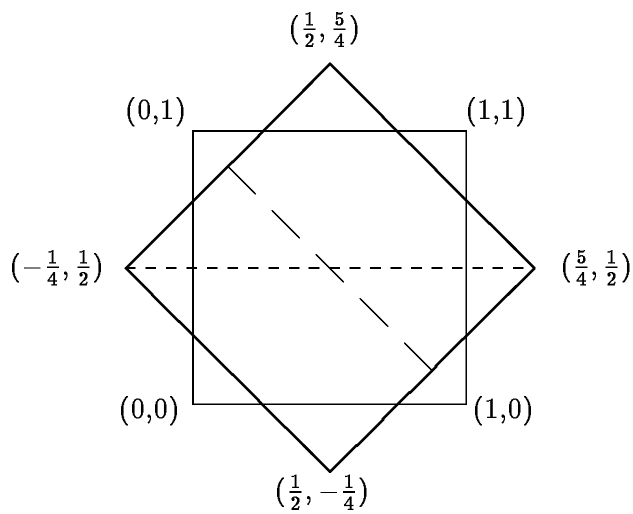

Figure 1: Possible equilibrium locations.

Appendix

Proof of Proposition 4.2

Recall that for each density $< d_1, \ldots, d_n >$ with $n \in \mathbb{N}$ and beliefs $b \in B$, expected demand for firm $i \in I$ is linear in x. Equation (11) then reduces to

$$\frac{\gamma_i(b_i)}{\delta_i(b_i)} + x^*(b) = \frac{x_j^*(b) - x_i^*(b)}{3} \tag{27}$$

for $i \neq j \in I$. Summation over $i \in I$ yields

$$x^*(b) = -\frac{1}{2}\left(\frac{\gamma_1(b_1)}{\delta_1(b_1)} + \frac{\gamma_2(b_2)}{\delta_2(b_2)}\right). \tag{28}$$

Substitution of (28) into (27) then yields the required result

$$x_i^*(b) = \frac{1}{4}\left(\frac{\gamma_j(b_j)}{\delta_j(b_j)} - \frac{5\gamma_i(b_i)}{\delta_i(b_i)}\right).$$

The second order conditions for a maximum are fulfilled (at least in case of identical beliefs), because $F_i(b_i, .)$ being affine, equation (12) reduces to $\frac{\gamma_2(b_2)}{\delta_2(b_2)} < \frac{\gamma_1(b_1)}{\delta_1(b_1)}$ for all $i \in I$, where we use the fact that $\frac{\partial x^*(b, x_i, x_j)}{\partial x_i} = \frac{1}{6}$ for all $i \neq j \in I$.

Proof of Theorem 4.3

From Propositions 4.1 and 4.2 we know that (up to symmetry) for any beliefs $b \in B$ there is a unique subgame perfect Nash equilibrium $s^*(b) =< x_1^*(b), x_2^*(b), \{(p_1^*(b, x_1, x_2), p_2^*(b, x_1, x_2)) \mid x_1 < x_2\} >$ for the game $G(b)$. Given beliefs $b^* \in B$, there generically exists a unique corresponding state, i.e., there is a unique value $k^* \in \mathcal{K}$ such that $\frac{k^*}{n} \leq x^*(b^*) \leq \frac{k^*+1}{n}$. According to (8), consistency requires that, for all $i \in I$, $b_{ik^*}^* = 1$ and $b_{ik}^* = 0$ for $k \neq k^* \in \mathcal{K}$. This means that equilibrium beliefs are identical. From equation (19) we find that $x^*(b^*) = \frac{1-2\gamma_1^*}{2\delta_1^*}$. Substitution in the constraint $\frac{k^*}{n} \leq x^*(b^*) \leq \frac{k^*+1}{n}$ yields

$$\frac{\mathcal{D}_{k^*}}{n} \leq \frac{1}{2} \leq \frac{\mathcal{D}_{k^*+1}}{n}. \tag{29}$$

Finally we have to show that there is a unique value $k^* \in \mathcal{K}$ that satisfies equation (29). Suppose without loss of generality that equation (29) is also satisfied for some $k \geq k^* + 1$. Then $\frac{1}{2} \leq \frac{\mathcal{D}_{k^*+1}}{n} \leq \frac{\mathcal{D}_k}{n}$, which contradicts $\frac{\mathcal{D}_k}{n} \leq \frac{1}{2}$, unless $k = k^* + 1$ and $\frac{\mathcal{D}_{k^*+1}}{n} = \frac{1}{2}$.

Proof of Proposition 5.2

The first part we prove by contradiction using an induction argument. Let $\mathcal{K}_k$ be defined as $\{0, \ldots, k\}$ for any $k \in \mathcal{K}$. Suppose that there is no equilibrium for $k \in \mathcal{K}_0$. Then from Lemma 5.1 we know that the condition $0 \leq \frac{n-2\mathcal{D}_k}{2nd_{k+1}} \leq \frac{1}{n}$ does not hold for $k = 0$. Since $\mathcal{D}_0 = 0$ the condition can be rewritten as $0 \leq \frac{1}{2d_1} \leq \frac{1}{n}$. Clearly $0 \leq \frac{1}{2d_1}$ always holds, therefore it must be that $d_1 < \frac{n}{2}$. Next suppose that there is no equilibrium for $k \in \mathcal{K}_1$. If there is no equilibrium for $k = 1$ then the condition $0 \leq \frac{n-2\mathcal{D}_1}{2nd_2} \leq \frac{1}{n}$ does not hold. Because there is no equilibrium for $k = 0$ we furthermore know that $d_1 < \frac{n}{2}$. But then the condition simplifies to $\mathcal{D}_2 < \frac{n}{2}$. By induction we see that there is no equilibrium for all $k \in \mathcal{K}_{n-1}$ if and only if $\mathcal{D}_n < \frac{n}{2}$, which contradicts $\mathcal{D}_n = 1$. Therefore there exists an equilibrium for some $k^* \in \mathcal{K}_{n-1} = \mathcal{K}$. But then $\frac{\mathcal{D}_{k^*}}{n} \leq \frac{1}{2} \leq \frac{\mathcal{D}_{k^*+1}}{n}$. Now suppose without loss of generality that there also exists an equilibrium for some $k > k^*$ where $k \in \mathcal{K}$. This means that $\frac{\mathcal{D}_k}{n} \leq \frac{1}{2} \leq \frac{\mathcal{D}_{k+1}}{n}$. But then $\frac{\mathcal{D}_{k+1}}{n} \geq \frac{1}{2} \geq \frac{\mathcal{D}_k}{n} \geq \frac{\mathcal{D}_{k^*+1}}{n} \geq \frac{\mathcal{D}_{k^*}}{n}$ which means that $\frac{\mathcal{D}_{k^*+1}}{n}$ must be equal to $\frac{1}{2}$ and k be equal to $k^* + 1$. It is easy to see that there are at most two equilibria. Suppose to the contrary that there exists a third equilibrium for say $m > k \in \mathcal{K}$, then we get $\frac{\mathcal{D}_m}{n} \geq \frac{\mathcal{D}_{k+1}}{n} = \frac{\mathcal{D}_k + d_{k+1}}{n} \geq \frac{1}{2} + \frac{d_{k+1}}{n} > \frac{1}{2}$ while $\frac{\mathcal{D}_m}{n} \leq \frac{1}{2}$ is required.

Proof of Lemma 6.1

For beliefs $b_s = (1, 0)$ we have $\hat{\gamma_1} = 1 - \hat{\gamma_2}$, $\hat{\delta}_1 = \hat{\delta}_2 = 0$, and $\hat{\epsilon}_1 = -\hat{\epsilon}_2 = 2$. Consequently, $F_1(b_s, x) = 2x^2$ and $F_2(b_s, x) = 1 - 2x^2$. From the first order conditions for the location stage, equation (11), we then find the solution $x^*(\hat{b}) = (6)^{-\frac{1}{2}}$. It is easily checked that $x_1^*(\hat{b}) = -\frac{2}{3}(6)^{-\frac{1}{2}}$ and $x_2^*(\hat{b}) = \frac{5}{3}(6)^{-\frac{1}{2}}$. The corresponding prices are $\frac{7}{18}$ and $\frac{14}{18}$, respectively. The proof for the situation $b_s = (0, 1)$ is similar and is left to the reader. For beliefs $b_s = (\frac{1}{2}, \frac{1}{2})$ we have $\hat{\gamma_1} = 1 - \hat{\gamma_2} = -\frac{1}{2}$, $\hat{\delta_1} = -\hat{\delta_2} = 2$, and $\hat{\epsilon_1} = -\hat{\epsilon_2} = 0$. Consequently, $F_1(b_s, x) = 2x - \frac{1}{2}$ and $F_2(b_s, x) = \frac{3}{2} - 2x$. From equation (11) we then find the solution $x^*(\hat{b}) = \frac{1}{2}$. Furthermore $x_1^*(\hat{b}) = \frac{1}{8}$ and $x_2^*(\hat{b}) = \frac{7}{8}$. The corresponding price is $\frac{3}{8}$ for both firms.

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