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# The location model with reservation prices\*

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## Abstract

In this paper, we analyze a variant of the standard Hotelling model of spatial competition where firms first choose locations along the line and then, given these locations, compete in prices. Consumers have a finite reservation price and they incur a quadratic transportation cost. We show that there exists a unique subgame perfect Nash equilibrium for the location-then-price game if the reservation price is high enough. In that case the degree of differentiation is nondecreasing in the reservation price, because differentiation relaxes price competition. If the reservation price is lower, there is a continuum of subgame perfect Nash equilibria due to the fact that firms can act as local monopolists and the other firm's location choice becomes of less importance. However, all equilibria yield the same profit.

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## 1 Introduction

In this paper, we analyze a variant of the standard Hotelling (1929) model of spatial competition where firms first choose locations along the line and then, given these locations, compete in prices. Consumers have a finite reservation price, specifying how much they are willing to pay for the commodity offered by the firms. This idea of introducing a finite reservation price goes back to Lerner and Singer (1937) and Smithies (1941). Each consumer buys at most one unit of the mutually exclusive commodities. If a consumer buys the product, he buys from the firm that offers the highest indirect utility. The indirect utility function of a consumer incorporates both transportation costs and the reservation price. Firms are aware of the reservation price and take into account the impact of their location decisions on their profits. To make the analysis tractable we assume that firms coordinate on their location choices, which implies that firms coordinate on the degree of differentiation. If the reservation price is relatively high, the maximum differentiation result of d'Aspremont, Gabszewicz and Thisse (1979) is obtained. If, however, the reservation price is relatively low, the degree of differentiation is lower. On the other hand, there cannot be minimal differentiation, because then price competition would drive profits down too much.

We show that there exists a unique subgame perfect Nash equilibrium for the location-then-price game if the reservation price is high enough. In that case the degree of differentiation is nondecreasing in the reservation price, because differentiation relaxes price competition. If the reservation price is lower, there is a continuum of subgame perfect Nash equilibria due to the fact that firms can act as local monopolists and the other firm's location choice becomes of less importance. However, all equilibria yield the same profit.

In equilibrium the degree of differentiation between the firms is non decreasing in the reservation price. Economides (1984) already indicates maximal 'profitable' differentiation in the original Hotelling model with linear transportation costs, whereas we consider the model with quadratic transportation costs. Due to the fact that in case of linear transportation costs, the existence of a price equilibrium is not guaranteed for all location pairs, the analysis then has to be restricted to the local monopoly situation (see also Friedman (1983)). In case of quadratic transportation costs, however, we are able to analyze both the local monopoly situation and the competitive situation.

In general the firms can do better than locate at the end points of the

line as in Boyer and Moreaux (1993). As long as there are consumers at the edges of the markets that do not buy the product, firms have an incentive to move towards the edges.

The paper is organized as follows. In Section 2 the location-then-price game with reservation prices is formulated. Section 3 and Section 4 discuss the price stage and the location stage, respectively. In Section 5 the main result is stated, being the existence of a unique subgame perfect Nash equilibrium for the location-then-price game if the reservation price is high enough, and otherwise there exists a continuum of subgame perfect Nash equilibria. In Section 6 we look at the situation where the firms are located at the endpoints of the line, in more detail. The proofs are gathered in the Appendix.

## 2 The model

There is a continuum of consumers distributed uniformly with density one along the line segment  $[0,1]$ . Disposable income for the consumers to buy one unit of a certain commodity is given by some fixed number  $\bar{p} \in \mathbb{R}_+$ . The number  $\bar{p}$  will be referred to as the reservation price and specifies how much the consumers are willing to pay for the product. There are two firms on the market, denoted firm 1 and firm 2. Firm  $i \in I = \{1,2\}$  locates at  $x_i$  along the real line and sells the commodity at price  $p_i \in \mathcal{P} = [0, \bar{p}]$ . It is clear that a firm will not charge a price higher than  $\bar{p}$ , because then demand for the commodity of this firm is always zero. We assume that firm 1 locates to the left of firm 2.

**Assumption 2.1** *Firm 1 locates to the left of firm 2, i.e.,  $x_1 < x_2$ .*

To make the analysis tractable we furthermore assume that both firms locate symmetrically, which includes the benchmark case  $x_1 = 1 - x_2 = 0$ , originally considered by Boyer and Moreaux (1993). Essentially this assumption means that, in the first stage, firms coordinate on the degree of differentiation.

**Assumption 2.2** *Firm 1 and firm 2 locate symmetrically, i.e.,  $x_1 = 1 - x_2$ .*

If a consumer buys one unit of the commodity from firm  $i \in I$  at price  $p_i \in \mathcal{P}$  the indirect utility is given by

$$V_i(x, x_i, p_i) = \bar{p} - p_i - t(x, x_i), \quad (1)$$

where  $x$  is the consumer's location in the unit interval. The number  $t(x, x_i)$  is the transportation cost for shipping the product of firm  $i$  to this consumer's location. We assume this transportation cost to be quadratic in distance with unit cost equal to one, i.e.,  $t(x, x_i) = (x - x_i)^2$ . Each consumer buys from the firm that offers the highest non-negative indirect utility. A consumer does not buy if he is offered a negative indirect utility by both firms. The market area of the product of firm  $i \in I$  at given locations  $x_1$  and  $x_2$  and prices  $p_1 \in \mathcal{P}$  and  $p_2 \in \mathcal{P}$  is therefore given by

$$M_i(x_1, x_2, p_1, p_2) = \{x \in [0, 1] \mid V_i(x, x_i, p_i) \geq \max\{0, V_j(x, x_j, p_j)\}, j \neq i\},$$

i.e., the set of consumers that prefer to buy the commodity from firm  $i$ .

At locations  $x_1$  and  $x_2 = 1 - x_1$  and at prices  $p_1$  and  $p_2$ , the demand  $X_i(x_1, p_1, p_2)$  for the commodity of firm  $i \in I$  is equal to

$$X_i(x_1, p_1, p_2) = \int_{M_i(x_1, x_2, p_1, p_2)} dx. \quad (2)$$

We may distinguish three different types of indifferent consumers, namely two or less consumers being indifferent between buying from firm 1 and not buying at all, two or less consumers being indifferent between buying from firm 2 and not buying at all, and, finally, a consumer being indifferent between buying from firm 1 and buying from firm 2.

For  $i \in I$ , let

$$\begin{aligned} x_i^- &= x_i - (\bar{p} - p_i)^{\frac{1}{2}} \\ \text{and } x_i^+ &= x_i + (\bar{p} - p_i)^{\frac{1}{2}}, \end{aligned} \quad (3)$$

for given locations  $x_1$  and  $x_2$  and prices  $p_1$  and  $p_2$ .

If  $x_1^+ \geq x_2^-$ , the location  $x$  of the consumer being indifferent between buying from firm 1 and buying from firm 2 is given by

$$x = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2(x_2 - x_1)}, \quad (4)$$

being the midpoint between the firms' locations corrected for price differences. Otherwise  $x_1^+$  and  $x_2^-$  denote the locations of the consumers indifferent between not buying at all and buying from firm 1 and firm 2, respectively. Furthermore, if  $x_1^- \geq 0$ ,  $x_1^-$  and, if  $x_2^+ \leq 1$ ,  $x_2^+$  denote the locations of the consumers indifferent between not buying at all and buying from firm 1 and firm 2, respectively. Notice that when there are two consumers being indifferent between buying from a firm and not buying at all, they are located symmetrically around the firm's location.

Given the locations  $x_1$  and  $x_2 = 1 - x_1$ , at prices  $p_1 \in \mathcal{P}$  and  $p_2 \in \mathcal{P}$ , the demand for the commodity of firm 1 can be expressed as

$$X_1(x_1, p_1, p_2) = \begin{cases} x & \text{if } x_1^+ \geq x \text{ and } x_1^- \leq 0 \\ x - x_1^- & \text{if } x_1^+ \geq x \text{ and } x_1^- \geq 0 \\ x_1^+ & \text{if } x_1^+ \leq x \text{ and } x_1^- \leq 0 \\ x_1^+ - x_1^- & \text{if } x_1^+ \leq x \text{ and } x_1^- \geq 0, \end{cases} \quad (5)$$

and the demand for the commodity of firm 2 can be expressed as

$$X_2(x_1, p_1, p_2) = \begin{cases} 1 - x & \text{if } x_2^- \leq x \text{ and } x_2^+ \geq 1 \\ x_2^+ - x & \text{if } x_2^- \leq x \text{ and } x_2^+ \leq 1 \\ 1 - x_2^- & \text{if } x_2^- \geq x \text{ and } x_2^+ \geq 1 \\ x_2^+ - x_2^- & \text{if } x_2^- \geq x \text{ and } x_2^+ \leq 1. \end{cases} \quad (6)$$

At locations  $x_1$  and  $x_2 = 1 - x_1$  and at prices  $p_1$  and  $p_2$ , the profit of firm  $i \in I$  is equal to

$$\Pi_i(x_1, p_1, p_2) = p_i X_i(x_1, p_1, p_2). \quad (7)$$

Given the locations  $x_1$  and  $x_2 = 1 - x_1$  we look for a Nash equilibrium for the price stage where the two firms simultaneously choose prices as to maximize their profit. The price stage is solved by prices  $p_1^*(x_1) \in \mathcal{P}$  and  $p_2^*(x_1) \in \mathcal{P}$  such that

$$\begin{aligned} \Pi_1(x_1, p_1^*(x_1), p_2^*(x_1)) &\geq \Pi_1(x_1, p_1, p_2^*(x_1)) \\ \Pi_2(x_1, p_1^*(x_1), p_2^*(x_1)) &\geq \Pi_2(x_1, p_1^*(x_1), p_2) \end{aligned}$$

for all  $p_1 \in \mathcal{P}$  and  $p_2 \in \mathcal{P}$ , respectively. For ease of notation, we denote

$$\Pi_i^*(x_1) = \Pi_i(x_1, p_1^*(x_1), p_2^*(x_1)), \quad i \in I.$$

An equilibrium of the location stage is then given by some pair  $(x_1^*, 1 - x_1^*) \in \mathbb{R}^2$  satisfying  $x_1^* < \frac{1}{2}$  and for any  $x_1 < \frac{1}{2}$

$$\Pi_i^*(x_1^*) \geq \Pi_i^*(x_1) \text{ for all } i \in I.$$

A subgame perfect Nash equilibrium for the location-then-price game is defined by  $(x_1^*, 1 - x_1^*)$  and  $(p_1^*(x_1), p_2^*(x_1))$  for all location pairs  $(x_1, 1 - x_1)$  with  $x_1 < \frac{1}{2}$ . The corresponding equilibrium path is  $(x_1^*, 1 - x_1^*)$  and  $(p_1^*(x_1^*), p_2^*(x_1^*))$ .

### 3 The price stage

In this section we consider the price stage. With symmetric locations we may restrict ourselves to the situation  $x_1 \in [-\frac{1}{4}, \frac{1}{2}]$ , because for  $x_1 = 1 - x_2 = -\frac{1}{4}$  the degree of differentiation is relatively maximal (see for example Tirole (1988)). The solution to the price stage is most easily found by looking at three different cases, because in these three cases the structure of competition differs. First we consider the case  $-\frac{1}{4} \leq x_1 = 1 - x_2 \leq 0$ . In this case the demands for the commodity of firm 1 and firm 2 as expressed in equations (5) and (6), respectively, reduce to

$$X_1(x_1, p_1, p_2) = \begin{cases} x & \text{if } x_1^+ \geq x \\ x_1^+ & \text{if } x_1^+ \leq x \end{cases} \quad (8)$$

and

$$X_2(x_1, p_1, p_2) = \begin{cases} 1 - x & \text{if } x_2^- \leq x \\ 1 - x_2^- & \text{if } x_2^- \geq x, \end{cases} \quad (9)$$

because  $x_1^- \leq 0$  and  $x_2^+ \geq 1$  then. It is clear that this case, which includes the result of Boyer and Moreaux (1993) for  $x_1 = 1 - x_2 = 0$ , is the easiest because of the relatively simple demand function.

**Proposition 3.1** *Let  $-\frac{1}{4} \leq x_1 = 1 - x_2 \leq 0$ . Then, there exists a unique symmetric Nash equilibrium  $(p_1^*(x_1), p_2^*(x_1)) \in \mathcal{P} \times \mathcal{P}$  for the price stage given by  $p_1^*(x_1) = p_2^*(x_1) =$*

$$\begin{cases} \frac{2\bar{p}}{3} - \frac{2}{9}(x_1)^2 + \frac{2}{9}x_1((x_1)^2 + 3\bar{p})^{\frac{1}{2}} & \text{if } \rho_1 \leq \bar{p} \leq \rho_2 \\ \bar{p} - (\frac{1}{2} - x_1)^2 & \text{if } \rho_2 \leq \bar{p} \leq \rho_3 \\ 1 - 2x_1 & \text{if } \rho_3 \leq \bar{p}, \end{cases}$$

where  $\rho_1 = (x_1)^2$ ,  $\rho_2 = (x_1)^2 - 2x_1 + \frac{3}{4}$ ,  $\rho_3 = 1 - 2x_1 + (\frac{1}{2} - x_1)^2$ . If  $0 \leq \bar{p} \leq \rho_1$ , for both firms profits are zero and a possible Nash equilibrium  $(p_1^*(x_1), p_2^*(x_1))$  for the price game is given by  $p_1^*(x_1) = p_2^*(x_1) = 0$  then.

**Proof** See Appendix.

It is easy to check that for  $\bar{p} \geq \rho_1$ , equilibrium prices are nondecreasing in the reservation price  $\bar{p}$  and also that the corresponding equilibrium demands are nondecreasing in  $\bar{p}$ . For  $\bar{p} = \rho_1$  the consumer located at zero is just indifferent between buying from firm 1 and not buying at all. Equilibrium demand is equal to 0 for  $0 \leq \bar{p} \leq \rho_1$ ,  $\frac{1}{2}$  for  $\bar{p} \geq \rho_2$ , and it increases continuously from 0 to  $\frac{1}{2}$  for  $\rho_1 \leq \bar{p} \leq \rho_2$ .

Next we consider the case  $0 \leq x_1 = 1 - x_2 \leq \frac{1}{4}$ . The demands for the commodity of firm 1 and firm 2 are given by equations (5) and (6) then, respectively.

**Proposition 3.2** *Let  $0 \leq x_1 = 1 - x_2 \leq \frac{1}{4}$ . Then, there exists a unique symmetric Nash equilibrium  $(p_1^*(x_1), p_2^*(x_1)) \in \mathcal{P} \times \mathcal{P}$  for the price stage given by  $p_1^*(x_1) = p_2^*(x_1) =$*

$$\begin{cases} \frac{2\bar{p}}{3} & \text{if } 0 \leq \bar{p} \leq \rho_1 \\ \bar{p} - (x_1)^2 & \text{if } \rho_1 \leq \bar{p} \leq \rho_2 \\ \frac{2\bar{p}}{3} - \frac{2}{9}(x_1)^2 + \frac{2}{9}x_1((x_1)^2 + 3\bar{p})^{\frac{1}{2}} & \text{if } \rho_2 \leq \bar{p} \leq \rho_3 \\ \bar{p} - (\frac{1}{2} - x_1)^2 & \text{if } \rho_3 \leq \bar{p} \leq \rho_4 \\ 1 - 2x_1 & \text{if } \rho_4 \leq \bar{p}, \end{cases}$$

where  $\rho_1 = 3(x_1)^2$ ,  $\rho_2 = 5(x_1)^2$ ,  $\rho_3 = (x_1)^2 - 2x_1 + \frac{3}{4}$ ,  $\rho_4 = 1 - 2x_1 + (\frac{1}{2} - x_1)^2$ .

**Proof** See Appendix.

Again equilibrium prices are nondecreasing in the reservation price  $\bar{p}$  and also the corresponding equilibrium demands are nondecreasing in  $\bar{p}$ . Equilibrium demand increases continuously from 0 to  $2x_1$  for  $0 \leq \bar{p} \leq \rho_1$ , it is equal to  $2x_1$  for  $\rho_1 \leq \bar{p} \leq \rho_2$ , it increases continuously from  $2x_1$  to  $\frac{1}{2}$  for  $\rho_2 \leq \bar{p} \leq \rho_3$ , and it is equal to  $\frac{1}{2}$  for  $\bar{p} \geq \rho_3$ . In equilibrium the situation in which both  $x_1^+ \geq x_2^-$  and  $x_1^- > 0$  or  $x_2^+ < 1$  cannot occur. If equilibrium prices are relatively so low that  $x_1^+ \geq x_2^-$ , then it holds that  $x_1^- \leq 0$  and  $x_2^+ \geq 1$ .

Finally we consider the case  $\frac{1}{4} \leq x_1 = 1 - x_2 \leq \frac{1}{2}$ . The demands for the commodity of firm 1 and firm 2 are given again by equations (5) and (6), respectively.

**Proposition 3.3** *Let  $\frac{1}{4} \leq x_1 = 1 - x_2 \leq \frac{1}{2}$ . Then, there exists a unique symmetric Nash equilibrium  $(p_1^*(x_1), p_2^*(x_1)) \in \mathcal{P} \times \mathcal{P}$  for the price stage given by  $p_1^*(x_1) = p_2^*(x_1) =$*

$$\begin{cases} \frac{2\bar{p}}{3} & \text{if } 0 \leq \bar{p} \leq \rho_1 \\ \bar{p} - (\frac{1}{2} - x_1)^2 & \text{if } \rho_1 \leq \bar{p} \leq \rho_2 \\ \bar{p} - (z + 2x_1 - 1)^2 & \text{if } \rho_2 \leq \bar{p} \leq \rho_3 \\ \bar{p} - (x_1)^2 & \text{if } \rho_3 \leq \bar{p} \leq \rho_4 \\ 1 - 2x_1 & \text{if } \rho_4 \leq \bar{p}, \end{cases}$$



where  $z = 2\left(\frac{\bar{p}+2(1-2x_1)^2}{3}\right)^{\frac{1}{2}}\cos\left(\frac{\phi}{3}\right)$  with  $\phi \in [0, \pi]$  satisfying the condition  $\cos(\phi) = -\frac{(1-2x_1)^3}{2}\left(\frac{\bar{p}+2(1-2x_1)^2}{3}\right)^{-\frac{3}{2}}$ ,  $\rho_1 = 3\left(\frac{1}{2} - x_1\right)^2$ ,  $\rho_2$  is the unique value of  $\bar{p}$  for which  $z = \frac{3}{2} - 3x_1$ ,  $\rho_3$  is the unique value of  $\bar{p}$  for which  $z = 1 - x_1$ ,  $\rho_4 = 1 - 2x_1 + (x_1)^2$ .

**Proof** See Appendix.

One can check that in Proposition 3.3 it holds that  $\rho_2 = \rho_3 = \frac{11}{48}$  for  $x_1 = 1 - x_2 = \frac{1}{4}$ . Contrary to the previous two cases, equilibrium prices may be decreasing now in the reservation price for  $\rho_2 \leq \bar{p} \leq \rho_3$ . The reason is that price competition may be increasing in the reservation price, because the firms are located 'too' close. The equilibrium demands, however, are nondecreasing in the reservation prices. Equilibrium demand increases continuously from 0 to  $1 - 2x_1$  for  $0 \leq \bar{p} \leq \rho_1$ , it is equal to  $1 - 2x_1$  for  $\rho_1 \leq \bar{p} \leq \rho_2$ , it increases continuously from  $1 - 2x_1$  to  $\frac{1}{2}$  for  $\rho_2 \leq \bar{p} \leq \rho_3$ , and it is equal to  $\frac{1}{2}$  for  $\bar{p} \geq \rho_3$ .

As we know from Economides (1984) and Boyer and Moreaux (1993) there also may exist asymmetric equilibria in case the local monopoly situation and the competitive situation are touching, i.e.,  $x_1^+ = x = x_2^-$ .

**Lemma 3.4** *Let  $-\frac{1}{4} \leq x_1 = 1 - x_2 \leq \frac{1}{2}$ . Then, there are no asymmetric Nash equilibria for the price stage, unless the local monopoly situation and the competitive situation are touching. In the situation of touching, there exists a continuum of Nash equilibria. The symmetric solution in this situation is  $p_1^*(x_1) = p_2^*(x_1) = \bar{p} - \left(\frac{1}{2} - x_1\right)^2$ .*

**Proof**

In the situation of touching the reaction functions, when intersecting, are in fact overlapping. For given  $\bar{p}$  and  $x_1$ , the profit maximizing price for firm  $i \neq j \in I$  is given then by

$$p_i^*(p_j) = \bar{p} - (1 - 2x_1 - (\bar{p} - p_j)^{\frac{1}{2}})^2.$$

Because  $p_i^*(p_j^*(p_i)) = p_i$  the two reaction functions are overlapping, which proves the existence of a continuum of Nash equilibria. It is easy to see then that  $p_1^*(x_1) = p_2^*(x_1) = \bar{p} - \left(\frac{1}{2} - x_1\right)^2$  is the unique symmetric solution. For the strict local monopoly situation and competitive situation it is obvious that there do not exist asymmetric equilibria.  $\square$

The question arises, which 'touching' equilibrium is going to be picked up by the competitors. Symmetry and Harsanyi's (1975) tracing procedure suggest equal Nash equilibrium prices for both firms. Following Economides (1984) we pick up the symmetric Nash equilibrium as 'the' Nash equilibrium.

## 4 The location stage

In this section we look at the location stage. Given the (symmetric) equilibrium prices in the price stage we determine the optimal locations in each of the three cases for all possible values of the reservation price. In Section 5 then, we combine the three cases in order to determine the subgame perfect Nash equilibria for the location-then-price game.

**Proposition 4.1** *Suppose the location choice of firm 1 is restricted to  $-\frac{1}{4} \leq x_1 = 1 - x_2 \leq 0$ . Then, the optimal locations for the location stage are given by*

$$x_1^* = 1 - x_2^* = \begin{cases} 0 & \text{if } 0 \leq \bar{p} \leq \frac{5}{4} \\ \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} & \text{if } \frac{5}{4} \leq \bar{p} \leq \frac{33}{16} \\ -\frac{1}{4} & \text{if } \frac{33}{16} \leq \bar{p}. \end{cases}$$

**Proof** See Appendix.

In case  $-\frac{1}{4} \leq x_1 = 1 - x_2 \leq 0$ , the degree of horizontal differentiation is nondecreasing in the reservation price. When the reservation price is relatively low, the horizontal differentiation is relatively minimal, because of the demand effect. When the reservation price is relatively high, the horizontal differentiation is relatively maximal, because of the price effect. When the reservation price is intermediate, horizontal differentiation is increasing in the reservation price in order to soften price competition.

At the locations specified in Proposition 4.1, demand equals  $(\frac{\bar{p}}{3})^{\frac{1}{2}}$  for  $0 \leq \bar{p} \leq \frac{3}{4}$  and  $\frac{1}{2}$  otherwise. The corresponding prices are  $\frac{2\bar{p}}{3}$  for  $0 \leq \bar{p} \leq \frac{3}{4}$ ,  $\bar{p} - \frac{1}{4}$  for  $\frac{3}{4} \leq \bar{p} \leq \frac{5}{4}$ ,  $2(\bar{p} + 1)^{\frac{1}{2}} - 2$  for  $\frac{5}{4} \leq \bar{p} \leq \frac{33}{16}$ , and  $\frac{3}{2}$  for  $\bar{p} \geq \frac{33}{16}$ .

**Proposition 4.2** *Suppose the location choice of firm 1 is restricted to  $0 \leq x_1 = 1 - x_2 \leq \frac{1}{4}$ . Then, the optimal locations for the location stage are given by*

$$x_1^* = 1 - x_2^* = \begin{cases} \frac{1}{4} & \text{if } \frac{3}{16} \leq \bar{p} \leq \frac{9}{16} \\ \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} & \text{if } \frac{9}{16} \leq \bar{p} \leq \frac{5}{4} \\ 0 & \text{if } \frac{5}{4} \leq \bar{p}, \end{cases}$$

and by  $x_1^* = 1 - x_2^* \in [(\frac{\bar{p}}{3})^{\frac{1}{2}}, \frac{1}{4}]$  if  $0 \leq \bar{p} \leq \frac{3}{16}$ .

**Proof** See Appendix.

In case  $0 \leq x_1 = 1 - x_2 \leq \frac{1}{4}$ , the degree of horizontal differentiation is nondecreasing in the reservation price. When the reservation price is relatively low, the horizontal differentiation is relatively minimal, because of the demand effect. When the reservation price is relatively high, the horizontal differentiation is relatively maximal, because of the price effect. When the reservation price is intermediate, horizontal differentiation is increasing in the reservation price in order to soften price competition.

At the locations specified in Proposition 4.2, demand equals  $2(\frac{\bar{p}}{3})^{\frac{1}{2}}$  for  $0 \leq \bar{p} \leq \frac{3}{16}$  and  $\frac{1}{2}$  otherwise. The corresponding prices are  $\frac{2\bar{p}}{3}$  for  $0 \leq \bar{p} \leq \frac{3}{16}$ ,  $\bar{p} - \frac{1}{16}$  for  $\frac{3}{16} \leq \bar{p} \leq \frac{9}{16}$ ,  $2(\bar{p} + 1)^{\frac{1}{2}} - 2$  for  $\frac{9}{16} \leq \bar{p} \leq \frac{5}{4}$ , and 1 for  $\bar{p} \geq \frac{5}{4}$ .

**Proposition 4.3** *Let  $\frac{1}{4} \leq x_1 = 1 - x_2 \leq \frac{1}{2}$ . Then, the optimal locations for the location stage are given by  $x_1^* = 1 - x_2^* = \frac{1}{4}$  if  $\bar{p} \geq \frac{3}{16}$ , and by  $x_1^* = 1 - x_2^* \in [\frac{1}{4}, \frac{1}{2} - (\frac{\bar{p}}{3})^{\frac{1}{2}}]$  if  $0 \leq \bar{p} \leq \frac{3}{16}$ .*

**Proof** See Appendix.

In case  $\frac{1}{4} \leq x_1 = 1 - x_2 \leq \frac{1}{2}$ , the degree of horizontal differentiation again is nondecreasing in the reservation price. When the reservation price is relatively high, horizontal differentiation is relatively maximal in order to soften price competition.

At the locations specified in Proposition 4.3, demand equals  $2(\frac{\bar{p}}{3})^{\frac{1}{2}}$  for  $0 \leq \bar{p} \leq \frac{3}{16}$  and  $\frac{1}{2}$  otherwise. The corresponding prices are  $\frac{2\bar{p}}{3}$  for  $0 \leq \bar{p} \leq \frac{3}{16}$ ,  $\bar{p} - \frac{1}{16}$  for  $\frac{3}{16} \leq \bar{p} \leq \frac{9}{16}$ , and  $\frac{1}{2}$  for  $\bar{p} \geq \frac{9}{16}$ .

## 5 Subgame perfect Nash equilibria

In the sequel, the game in which firms first choose locations and then compete in prices is referred to as  $G$ . In the previous two sections we have derived all the ingredients to prove the following two theorems.

**Theorem 5.1** *For  $0 \leq \bar{p} \leq \frac{3}{16}$ , there is a continuum of subgame perfect Nash equilibria for the game  $G$ . In equilibrium the firms' locations are given by  $x_1^* = 1 - x_2^* \in [(\frac{\bar{p}}{3})^{\frac{1}{2}}, \frac{1}{2} - (\frac{\bar{p}}{3})^{\frac{1}{2}}]$ .*

**Proof** See Appendix.

When the reservation price is low enough, we are in a local monopoly situation. The two firms have some freedom in their location choices. For equilibrium it is only required that the firms do not locate too close to each other on the one hand and do not locate too close to the endpoints on the other hand. Indeed, when the firms locate too close to each other, competition drives profits down. When the firms locate too close to the endpoints, demand decreases. In equilibrium, for all location choices profits are the same and are equal to the local monopoly profit  $4(\frac{\bar{p}}{3})^{\frac{3}{2}}$ .

**Theorem 5.2** For  $\bar{p} \geq \frac{3}{16}$ , there is a unique subgame perfect Nash equilibrium for the game  $G$ . In equilibrium firms' locations are given by

$$x_1^* = 1 - x_2^* = \begin{cases} \frac{1}{4} & \text{for } \frac{3}{16} \leq \bar{p} \leq \frac{9}{16} \\ \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} & \text{for } \frac{9}{16} \leq \bar{p} \leq \frac{33}{16} \\ -\frac{1}{4} & \text{for } \frac{33}{16} \leq \bar{p}. \end{cases}$$

**Proof** See Appendix.

When the reservation price is relatively high, we are in a situation of competition. The degree of differentiation is non-decreasing in the reservation price. Firms will not locate too close because then competition drives profits down too much. In contrast to the previous situation, firms will locate close to the endpoints and possibly outside the interval when the reservation price is relatively very high. The reason is that this relaxes price competition.

The following corollary states the equilibrium outcome or path. The proof follows immediately from Theorems 5.1 and 5.2 and the propositions in Section 3.

**Corollary 5.3** The subgame perfect Nash equilibrium outcome for the game  $G$  is given by locations  $x_1^* = 1 - x_2^* = x^*$  and prices  $p_1^* = p_2^* = p^*$  satisfying

$$\begin{array}{llll} x^* \in [(\frac{\bar{p}}{3})^{\frac{1}{2}}, \frac{1}{2} - \bar{p}^{\frac{1}{2}}] & \text{and } p^* = \frac{2\bar{p}}{3} & \text{for } 0 \leq \bar{p} \leq \frac{3}{16} \\ x^* = \frac{1}{4} & \text{and } p^* = \bar{p} - \frac{1}{16} & \text{for } \frac{3}{16} \leq \bar{p} \leq \frac{9}{16} \\ x^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} & \text{and } p^* = 2(\bar{p} + 1)^{\frac{1}{2}} - 2 & \text{for } \frac{9}{16} \leq \bar{p} \leq \frac{33}{16} \\ x^* = -\frac{1}{4} & \text{and } p^* = \frac{3}{2} & \text{for } \frac{33}{16} \leq \bar{p}. \end{array}$$

Equilibrium profit  $\Pi^*$  for both firms as a function of  $\bar{p}$  equals  $4(\frac{\bar{p}}{3})^{\frac{3}{2}}$  for  $0 \leq \bar{p} \leq \frac{3}{16}$ ,  $\frac{1}{2}(\bar{p} - \frac{1}{16})$  for  $\frac{3}{16} \leq \bar{p} \leq \frac{9}{16}$ ,  $(\bar{p} + 1)^{\frac{1}{2}} - 1$  for  $\frac{9}{16} \leq \bar{p} \leq \frac{33}{16}$ , and  $\frac{3}{4}$  for  $\bar{p} \geq \frac{33}{16}$ . Equilibrium locations and equilibrium profits are drawn in Figures 1 and 2, respectively.

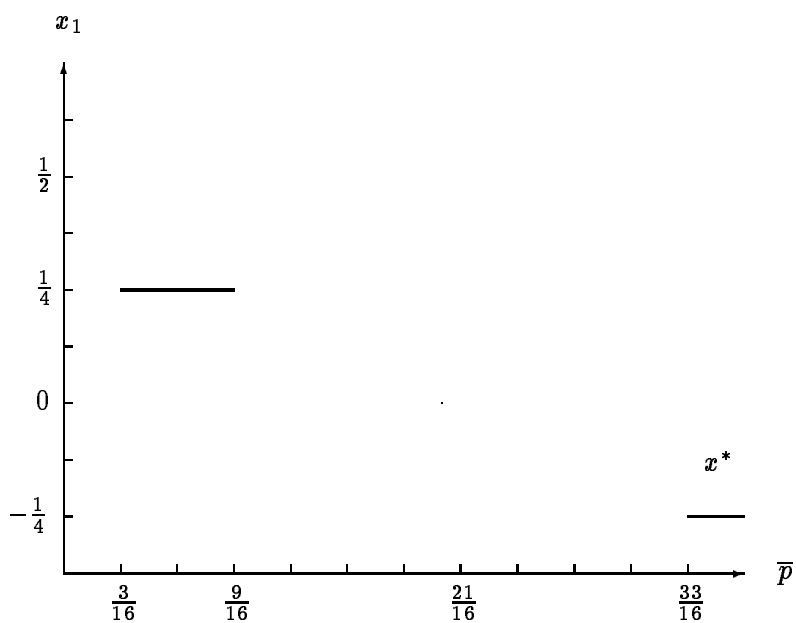


Figure 1: Equilibrium locations as a function of the reservation price  $\bar{p}$ .

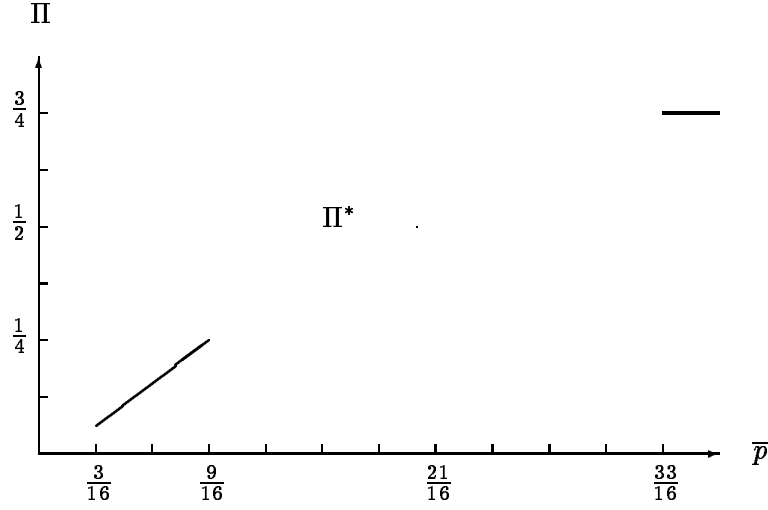


Figure 2: Equilibrium profits as a function of the reservation price  $\bar{p}$ .

## 6 A special case

In this section we look at the situation where both firms choose as location the two endpoints of the unit interval, i.e.,  $x_1 = 1 - x_2 = 0$ , in more detail. First we describe the price reaction functions.

**Proposition 6.1** *Suppose the firms choose  $x_1 = 1 - x_2 = 0$ . For given reservation price  $\bar{p} \in \mathbb{R}_+$  the price reaction function of firm  $i \in I$  as a function of price  $p_j \in \mathcal{P}$  of firm  $j \neq i$  is given by  $p_i(p_j) =$*

$$\begin{cases} \frac{p_j+1}{2} & \text{if } 0 \leq p_j \leq \tilde{p}_j \\ p_j - 1 + 2(\min\{3, \bar{p}\} - p_j)^{\frac{1}{2}} & \text{if } \tilde{p}_j \leq p_j \leq \hat{p}_j \\ \max\{\frac{2\bar{p}}{3}, p_j - 1\} & \text{if } \hat{p}_j \leq p_j \leq \bar{p}, \end{cases}$$

where

$$\tilde{p}_j = \begin{cases} 0 & \text{if } 0 \leq \bar{p} \leq \frac{9}{16} \\ -5 + 4(\bar{p} + 1)^{\frac{1}{2}} & \text{if } \frac{9}{16} \leq \bar{p} \leq 3 \\ 3 & \text{if } 3 \leq \bar{p} \end{cases}$$

and

$$\hat{p}_j = \begin{cases} \frac{2\bar{p}}{3} - 1 + 2\left(\frac{\bar{p}}{3}\right)^{\frac{1}{2}} & \text{if } 0 \leq \bar{p} \leq 3 \\ 3 & \text{if } 3 \leq \bar{p}. \end{cases}$$

**Proof** See Appendix.

It is easily checked that the price reaction function of firm  $i \neq j \in I$  is continuous in  $p_j$ . Moreover, in case  $0 \leq \bar{p} \leq 3$ ,  $p_i(p_j)$  is linearly increasing for relatively low values of  $p_j$ , constant for relatively high values of  $p_j$ , and decreasing and concave for intermediate values of  $p_j$ . More precisely, in this case the prices are strategic complements for  $0 \leq p_j \leq \tilde{p}_j$ , strategic substitutes for  $\tilde{p}_j \leq p_j \leq \hat{p}_j$ , and strategically independent for  $\hat{p}_j \leq p_j \leq \bar{p}$ . Note that in case  $\bar{p} \geq 3$  prices are strategic complements, because then  $p_i(p_j) = p_j - 1$  for  $\hat{p}_j \leq p_j \leq \bar{p}$  and furthermore  $\tilde{p}_j = \hat{p}_j$ .

From the price reaction functions we get the unique symmetric Nash equilibrium

$$p_1^* = p_2^* = \begin{cases} \frac{2\bar{p}}{3} & \text{if } 0 \leq \bar{p} \leq \frac{3}{4} \\ \bar{p} - \frac{1}{4} & \text{if } \frac{3}{4} \leq \bar{p} \leq \frac{5}{4} \\ 1 & \text{if } \frac{5}{4} \leq \bar{p}. \end{cases}$$

which also follows directly from Proposition 3.1.

Equilibrium profits  $\Pi^*$  for both firms as a function of  $\bar{p}$  in case  $x_1 = 1 - x_2 = 0$  are equal to

$$\Pi^* = \begin{cases} 2\left(\frac{\bar{p}}{3}\right)^{\frac{1}{2}} & \text{if } 0 \leq \bar{p} \leq \frac{3}{4} \\ \frac{1}{2}\left(\bar{p} - \frac{1}{4}\right) & \text{if } \frac{3}{4} \leq \bar{p} \leq \frac{5}{4} \\ \frac{1}{2} & \text{if } \frac{5}{4} \leq \bar{p}. \end{cases}$$

These profits are drawn in Figure 3. From Figures 2 and 3 we see that for any given reservation price  $\bar{p}$  equilibrium profits in case firms choose for the endpoints are at most equal to the equilibrium profits in case firms choose their locations strategically. Equal profits only occur for  $\bar{p} = 0$  and  $\bar{p} = \frac{5}{4}$ . For these values of the reservation price,  $x_1 = 1 - x_2 = 0$  is indeed the optimal location choice.

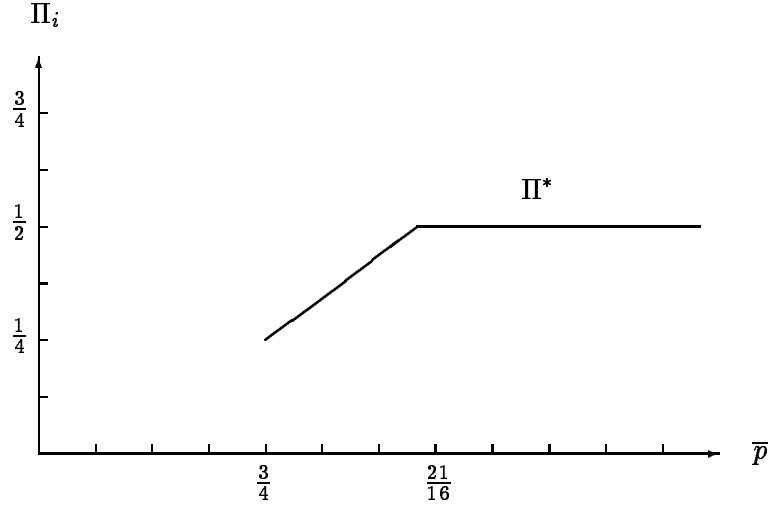


Figure 3: Equilibrium profits in case  $x_1 = 1 - x_2 = 0$ .

## Appendix

**Proof of Proposition 3.1** Let  $-\frac{1}{4} \leq x_1 = 1 - x_2 \leq 0$ . Firm  $i \in I$  wants to maximize its profit  $p_i X_i(x_1, p_1, p_2)$ , where  $X_i(x_1, p_1, p_2)$  is given by equations (8) and (9). A price equilibrium with  $x_1^+ \geq x$  and  $x_2^- \geq x$  or with  $x_1^+ \leq x$  and  $x_2^- \leq x$  cannot exist because it must hold that either  $x_1^+ \leq x \leq x_2^-$  or  $x_2^- \leq x \leq x_1^+$ . But then a price equilibrium is a pair of prices  $(p_1, p_2)$  such that  $p_1 x$  and  $p_2(1 - x)$  are maximal in case  $x_2^- \leq x_1^+$  and such that  $p_1 x_1^+$  and  $p_2(1 - x_2^-)$  are maximal in case  $x_1^+ \leq x_2^-$ . For  $\bar{p} \geq 1 - 2x_1 + (\frac{1}{2} - x_1)^2$  this yields the competitive outcome  $p_1^*(x_1) = p_2^*(x_1) = 1 - 2x_1$ . For  $(x_1)^2 - 2x_1 + \frac{3}{4} \leq \bar{p} \leq 1 - 2x_1 + (\frac{1}{2} - x_1)^2$  we have the boundary solution  $p_1^*(x_1) = p_2^*(x_1) = \bar{p} - (\frac{1}{2} - x_1)^2$  with per firm demand equal to  $\frac{1}{2}$ . For  $(x_1)^2 \leq \bar{p} \leq (x_1)^2 - 2x_1 + \frac{3}{4}$  we have the one-sided local monopoly outcome  $p_1^*(x_1) = p_2^*(x_1) = \frac{2\bar{p}}{3} - \frac{2}{9}(x_1)^2 + \frac{2}{9}x_1((x_1)^2 + 3\bar{p})^{\frac{1}{2}}$ . Finally, for  $\bar{p} \leq (x_1)^2$  the reservation prices are relatively too low to have the firms in the market. For any price in the interval  $[0, \bar{p}]$  profits are zero. Hence,  $p_1^*(x_1) = p_2^*(x_1) = 0$  yields a Nash equilibrium.



**Proof of Proposition 3.2** Let  $0 \leq x_1 = 1 - x_2 \leq \frac{1}{4}$ . Firm  $i$  wants to maximize its profit  $p_i X_i(x_1, p_1, p_2)$ , where  $X_i(x_1, p_1, p_2)$  is given by equations (5) and (6). Similar as in the proof of Proposition 3.1 a price equilibrium is a pair of prices such that  $p_1 x$  and  $p_2(1 - x)$  are maximal in case  $x_2^- \leq x_1^+$ ,  $x_1^- \leq 0$ ,  $x_2^+ \geq 1$ . In case  $x_1^+ \leq x_2^-$ ,  $x_1^- \leq 0$ ,  $x_2^+ \geq 1$  prices are such that  $p_1 x_1^+$  and  $p_2(1 - x_2^-)$  are maximal. Finally, in case  $x_1^+ \leq x_2^-$ ,  $x_1^- \geq 0$ ,  $x_2^+ \leq 1$  prices are such that  $p_1(x_1^+ - x_1^-)$  and  $p_2(x_2^+ - x_2^-)$  are maximal. The case  $x_2^- \leq x_1^+$ ,  $x_1^- \geq 0$ ,  $x_2^+ \leq 1$  cannot occur because then  $x_1 \geq \frac{1}{2} \max\{x, 1 - x\}$  which contradicts  $0 \leq x_1 \leq \frac{1}{4}$ . For  $\bar{p} \geq 1 - 2x_1 + (\frac{1}{2} - x_1)^2$  this yields the competitive outcome  $p_1^*(x_1) = p_2^*(x_1) = 1 - 2x_1$ . For  $(x_1)^2 - 2x_1 + \frac{3}{4} \leq \bar{p} \leq 1 - 2x_1 + (\frac{1}{2} - x_1)^2$  we have the boundary solution  $p_1^*(x_1) = p_2^*(x_1) = \bar{p} - (\frac{1}{2} - x_1)^2$  with per firm demand equal to  $\frac{1}{2}$ . For  $5(x_1)^2 \leq \bar{p} \leq (x_1)^2 - 2x_1 + \frac{3}{4}$  we have the one-sided local monopoly outcome  $p_1^*(x_1) = p_2^*(x_1) = \frac{2\bar{p}}{3} - \frac{2}{9}(x_1)^2 + \frac{2}{9}x_1((x_1)^2 + 3\bar{p})^{\frac{1}{2}}$ . For  $3(x_1)^2 \leq \bar{p} \leq 5(x_1)^2$  we have the boundary solution  $p_1^*(x_1) = p_2^*(x_1) = \bar{p} - (x_1)^2$  with per firm demand equal to  $2x_1$ . Finally, for  $\bar{p} \leq 3(x_1)^2$  we have the two-sided local monopoly outcome  $p_1^*(x_1) = p_2^*(x_1) = \frac{2\bar{p}}{3}$ .

**Proof of Proposition 3.3** Let  $\frac{1}{4} \leq x_1 = 1 - x_2 \leq \frac{1}{2}$ . Firm  $i \in I$  wants to maximize its profit  $p_i X_i(x_1, p_1, p_2)$ , where  $X_i(x_1, p_1, p_2)$  is given by equations (5) and (6). We look for a pair of prices such that  $p_1 x$  and  $p_2(1 - x)$  are maximal in case  $x_2^- \leq x_1^+$ ,  $x_1^- \leq 0$ ,  $x_2^+ \geq 1$ . In case  $x_2^- \leq x_1^+$ ,  $x_1^- \geq 0$ ,  $x_2^+ \leq 1$  prices are such that  $p_1(x - x_1^-)$  and  $p_2(x_2^+ - x)$  are maximal. Finally, in case  $x_1^+ \leq x_2^-$ ,  $x_1^- \geq 0$ ,  $x_2^+ \leq 1$  prices are such that  $p_1(x_1^+ - x_1^-)$  and  $p_2(x_2^+ - x_2^-)$  are maximal. The case  $x_1^+ \leq x_2^-$ ,  $x_1^- \leq 0$ ,  $x_2^+ \geq 1$  cannot occur because then  $x_1 \leq \frac{1}{2} \min\{x, 1 - x\}$  which contradicts  $\frac{1}{4} \leq x_1 \leq \frac{1}{2}$ . For  $\bar{p} \geq 1 - 2x_1 + (x_1)^2$  this yields the competitive outcome  $p_1^*(x_1) = p_2^*(x_1) = 1 - 2x_1$ . For  $0 \leq \bar{p} \leq 3(\frac{1}{2} - x_1)^2$  we have the two-sided local monopoly outcome  $p_1^*(x_1) = p_2^*(x_1) = \frac{2\bar{p}}{3}$ . The most difficult part is to find the price equilibrium in case  $x_2^- \leq x_1^+$ ,  $x_1^- \geq 0$ ,  $x_2^+ \leq 1$ , where prices are such that  $p_1(x - x_1^-)$  and  $p_2(x_2^+ - x)$  are maximal. The first order conditions for profit maximization are given by

$$\frac{x_2 - x_1}{2} + \frac{p_j - 2p_i}{2(x_2 - x_1)} + (\bar{p} - p_i)^{\frac{1}{2}} - p_i(2(\bar{p} - p_i))^{-\frac{1}{2}} = 0, \text{ for } j \neq i \in I. \quad (10)$$

Due to symmetry the first order conditions are solved by  $p_1^* = p_2^* = p^*$  where  $p^*$  solves the cubic equation

$$(\bar{p} - p^*)^{\frac{3}{2}} + 3(x_2 - x_1)(\bar{p} - p^*) + ((x_2 - x_1)^2 - \bar{p})(\bar{p} - p^*)^{\frac{1}{2}} - (x_2 - x_1)\bar{p} = 0. \quad (11)$$

By substituting  $y = (\bar{p} - p^*)^{\frac{1}{2}} + (x_2 - x_1)$  into equation (11) we get the reduced form of the cubic equation,

$$y^3 - (\bar{p} + 2(x_2 - x_1)^2)y + (x_2 - x_1)^3 = 0. \quad (12)$$

Because  $\Delta = \frac{((x_2 - x_1)^3)^2}{4} - \frac{(\bar{p} + 2(x_2 - x_1)^2)^3}{27} < 0$  the reduced form equation (12) has three real roots. The three solutions  $y_0, y_1,$  and  $y_2$  are given by

$$y_k = 2\left(\frac{\bar{p} + 2(x_2 - x_1)^2}{3}\right)^{\frac{1}{2}} \cos\left(\frac{\phi + 2k\pi}{3}\right), \quad k \in \{0, 1, 2\}, \quad (13)$$

where  $\phi$  follows from  $\cos(\phi) = -\frac{(x_2 - x_1)^3}{2} \left(\frac{\bar{p} + 2(x_2 - x_1)^2}{3}\right)^{-\frac{3}{2}}$ . For more details about the derivation see for example Turnbull (1952) or Uspensky (1948). From equation (11) it is easy to see that there is a unique value for  $p^*$  such that  $\bar{p} - p^* \geq 0$ . Without loss of generality we take  $\phi \in [0, \pi]$ . Then it is easy to see that  $y_1 < y_2 \leq y_0$  where the equality sign holds for  $\phi = \pi$ . This implies that the unique equilibrium price is found for  $k = 0$ . But then  $(\bar{p} - p^*)^{\frac{1}{2}} = y_0 + 2x_1 - 1$  which yields the result stated in the proposition. Finally it can be checked that  $\bar{p} - (x_1)^2 \leq \bar{p} - (z + 2x_1 - 1)^2 \leq \bar{p} - (\frac{1}{2} - x_1)^2$  requires that the reservation price  $\bar{p}$  is between some bounds, where the lower bound is greater than or equal to  $3(\frac{1}{2} - x_1)^2$  and the upper bound is smaller than or equal to  $1 - 2x_1 + (x_1)^2$ . The boundary solutions  $\bar{p} - (\frac{1}{2} - x_1)^2$  and  $\bar{p} - (x_1)^2$  can be determined easily then.

**Proof of Proposition 4.1** Let  $-\frac{1}{4} \leq x_1 = 1 - x_2 \leq 0$ . The Nash equilibrium for the price stage is given then by Proposition 3.1. If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = \frac{2\bar{p}}{3} - \frac{2}{9}(x_1)^2 + \frac{2}{9}x_1((x_1)^2 + 3\bar{p})^{\frac{1}{2}}$ , equilibrium profit  $\Pi_i^*(x_1)$  for firm  $i \in I$  is increasing in  $x_1$  because

$$\frac{\partial \Pi_i^*(x_1)}{\partial x_1} = p^*(x_1) + \frac{\partial p^*(x_1)}{\partial x_1} (x_1 + (\bar{p} - p^*(x_1))^{\frac{1}{2}}) - \frac{1}{2}(\bar{p} - p^*(x_1))^{-\frac{1}{2}} = p^*(x_1).$$

This means that the optimal value for  $x_1$  is the maximal value for  $x_1$  such that  $x_1 \geq -(\bar{p})^{\frac{1}{2}}$ ,  $x_1 \leq 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$  and  $-\frac{1}{4} \leq x_1 \leq 0$ . Note that this requires that  $\bar{p} \leq \frac{21}{16}$ . We find  $x_1^* = 0$  for  $0 \leq \bar{p} \leq \frac{3}{4}$  and  $x_1^* = 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$  for  $\frac{3}{4} \leq \bar{p} \leq \frac{21}{16}$ . If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = \bar{p} - (\frac{1}{2} - x_1)^2$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = \frac{1}{2}(\bar{p} - (\frac{1}{2} - x_1)^2)$  is increasing in  $x_1$ . This means that the optimal value for  $x_1$  is the maximal value for  $x_1$  such that  $x_1 \geq 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$ ,  $x_1 \leq \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$ , and  $-\frac{1}{4} \leq x_1 \leq 0$ .

This requires that  $\frac{3}{4} \leq \bar{p} \leq \frac{33}{16}$ . We find  $x_1^* = 0$  for  $\frac{3}{4} \leq \bar{p} \leq \frac{5}{4}$  with per firm profit equal to  $\frac{1}{2}(\bar{p} - \frac{1}{4})$  and  $x_1^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$  for  $\frac{5}{4} \leq \bar{p} \leq \frac{33}{16}$  with per firm profit equal to  $(\bar{p} + 1)^{\frac{1}{2}} - 1$ . If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = 1 - 2x_1$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = \frac{1}{2}(1 - 2x_1)$  is decreasing in  $x_1$ . This means that the optimal value for  $x_1$  is the minimal value for  $x_1$  such that  $x_1 \geq \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$  and  $-\frac{1}{4} \leq x_1 \leq 0$ . This requires that  $\bar{p} \geq \frac{5}{4}$ . We find  $x_1^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$  for  $\frac{5}{4} \leq \bar{p} \leq \frac{33}{16}$  with per firm profit equal to  $(\bar{p} + 1)^{\frac{1}{2}} - 1$  and  $x_1^* = -\frac{1}{4}$  for  $\bar{p} \geq \frac{33}{16}$  with per firm profit equal to  $\frac{3}{4}$ . Finally, profits are zero if  $x_1 \leq -(\bar{p})^{\frac{1}{2}}$  and  $-\frac{1}{4} \leq x_1 \leq 0$ , which requires that  $\bar{p} \leq \frac{1}{16}$ . Combining these different cases yields maximum profits for the locations stated in the proposition. Note that for  $\frac{3}{4} \leq \bar{p} \leq \frac{5}{4}$ , per firm profit is higher at  $x_1 = 0$  than at  $x_1 = 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$  because both the price and demand are higher in the first case. For  $\frac{5}{4} \leq \bar{p} \leq \frac{21}{16}$  per firm profit is higher at  $x_1 = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$  than at  $x_1 = 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$  for the same reason.

**Proof of Proposition 4.2** Let  $0 \leq x_1 = 1 - x_2 \leq \frac{1}{4}$ . The Nash equilibrium for the price stage is given then by Proposition 3.2. If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = \frac{2\bar{p}}{3}$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = 4(\frac{\bar{p}}{3})^{\frac{3}{2}}$  does not depend on  $x_1$ , so any  $(\frac{\bar{p}}{3})^{\frac{1}{2}} \leq x_1 \leq \frac{1}{4}$  yields maximum profits for  $0 \leq \bar{p} \leq \frac{3}{16}$ . If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = \bar{p} - (x_1)^2$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = 2x_1(\bar{p} - (x_1)^2)$  is increasing in  $x_1$ . This means that the optimal value for  $x_1$  is the maximal value for  $x_1$  such that  $(\frac{\bar{p}}{5})^{\frac{1}{2}} \leq x_1 \leq (\frac{\bar{p}}{3})^{\frac{1}{2}}$  and  $0 \leq x_1 \leq \frac{1}{4}$ . This requires that  $\bar{p} \leq \frac{5}{16}$ . We find  $x_1^* = (\frac{\bar{p}}{3})^{\frac{1}{2}}$  for  $0 \leq \bar{p} \leq \frac{3}{16}$  with per firm profit equal to  $4(\frac{\bar{p}}{3})^{\frac{3}{2}}$  and  $x_1^* = \frac{1}{4}$  for  $\frac{3}{16} \leq \bar{p} \leq \frac{5}{16}$  with per firm profit equal to  $\frac{1}{2}(\bar{p} - \frac{1}{16})$ . If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = \frac{2\bar{p}}{3} - \frac{2}{9}(x_1)^2 + \frac{2}{9}x_1((x_1)^2 + 3\bar{p})^{\frac{1}{2}}$ , equilibrium profit  $\Pi_i^*(x_1)$  for firm  $i \in I$  is increasing in  $x_1$  as we have seen in the proof of Proposition 4.1. This means that the optimal value for  $x_1$  is the maximal value for  $x_1$  such that  $x_1 \leq (\frac{\bar{p}}{5})^{\frac{1}{2}}$ ,  $x_1 \leq 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$ , and  $0 \leq x_1 \leq \frac{1}{4}$ . Note that this requires that  $\bar{p} \leq \frac{3}{4}$ . We find  $x_1^* = (\frac{\bar{p}}{5})^{\frac{1}{2}}$  for  $0 \leq \bar{p} \leq \frac{5}{16}$  and  $x_1^* = 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$  for  $\frac{5}{16} \leq \bar{p} \leq \frac{3}{4}$ . If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = \bar{p} - (\frac{1}{2} - x_1)^2$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = \frac{1}{2}(\bar{p} - (\frac{1}{2} - x_1)^2)$  is increasing in  $x_1$ . This means that the optimal value for  $x_1$  is the maximal value for  $x_1$  such that  $x_1 \geq 1 - (\bar{p} + \frac{1}{4})^{\frac{1}{2}}$ ,

$x_1 \leq \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$ , and  $0 \leq x_1 \leq \frac{1}{4}$ . This requires that  $\frac{5}{16} \leq \bar{p} \leq \frac{5}{4}$ . We find  $x_1^* = \frac{1}{4}$  for  $\frac{5}{16} \leq \bar{p} \leq \frac{9}{16}$  with per firm profit equal to  $\frac{1}{2}(\bar{p} - \frac{1}{16})$  and  $x_1^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$  for  $\frac{9}{16} \leq \bar{p} \leq \frac{5}{4}$  with per firm profit equal to  $(\bar{p} + 1)^{\frac{1}{2}} - 1$ . If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = p^*(x_1) = 1 - 2x_1$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = \frac{1}{2}(1 - 2x_1)$  is decreasing in  $x_1$ . This means that the optimal value for  $x_1$  is the minimal value for  $x_1$  such that  $x_1 \geq \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$  and  $0 \leq x_1 \leq \frac{1}{4}$ . This requires that  $\bar{p} \geq \frac{9}{16}$ . We find  $x_1^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}}$  for  $\frac{9}{16} \leq \bar{p} \leq \frac{5}{4}$  with per firm profit equal to  $(\bar{p} + 1)^{\frac{1}{2}} - 1$  and  $x_1^* = 0$  for  $\bar{p} \geq \frac{5}{4}$  with per firm profit equal to  $\frac{1}{2}$ . Combining these different cases yields, after some calculations, maximum profits for the locations stated in the proposition.

**Proof of Proposition 4.3** Let  $\frac{1}{4} \leq x_1 = 1 - x_2 \leq \frac{1}{2}$ . The Nash equilibrium for the price stage is given then by Proposition 3.3. If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = \frac{2\bar{p}}{3}$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = 4(\frac{\bar{p}}{3})^{\frac{3}{2}}$  does not depend on  $x_1$ , so any  $\frac{1}{4} \leq x_1 \leq \frac{1}{2} - (\frac{\bar{p}}{3})^{\frac{1}{2}}$  yields maximum profits for  $0 \leq \bar{p} \leq \frac{3}{16}$ . If the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = \bar{p} - (x_1)^2$  or by  $p_1^*(x_1) = p_2^*(x_1) = \bar{p} - (\frac{1}{2} - x_1)^2$ , for firm  $i \in I$  equilibrium profit is decreasing in  $x_1$ . The optimal value for  $x_1$  equals  $\frac{1}{4}$  then, which yields  $\rho_2 = \rho_3$ . But then  $x_1^* = \frac{1}{4}$  for  $\frac{3}{16} \leq \bar{p} \leq \frac{9}{16}$ . Finally, if the equilibrium prices are given by  $p_1^*(x_1) = p_2^*(x_1) = 1 - 2x_1$ , for firm  $i \in I$  equilibrium profit  $\Pi_i^*(x_1) = \frac{1}{2}(1 - 2x_1)$  is decreasing in  $x_1$ . Consequently  $x_1^* = \frac{1}{4}$  for  $\bar{p} \geq \frac{9}{16}$ .

**Proof of Theorem 5.1** Recall that for every  $x_1$  there exists a unique price equilibrium given by Propositions 3.1, 3.2, 3.3. We prove that for  $0 \leq \bar{p} \leq \frac{3}{16}$ , profits are maximal for all  $x_1^* \in [(\frac{\bar{p}}{3})^{\frac{1}{2}}, \frac{1}{2} - (\frac{\bar{p}}{3})^{\frac{1}{2}}]$ , which gives the required result. Let  $0 \leq \bar{p} \leq \frac{3}{16}$ . In case  $-\frac{1}{4} \leq x_1 \leq 0$  maximum profits are achieved for  $x_1^* = 0$  and are equal to  $2(\frac{\bar{p}}{3})^{\frac{3}{2}}$ . In case  $0 \leq x_1 \leq \frac{1}{2}$  maximum profits are achieved for all  $x_1^* \in [(\frac{\bar{p}}{3})^{\frac{1}{2}}, \frac{1}{2} - (\frac{\bar{p}}{3})^{\frac{1}{2}}]$  and are equal to  $4(\frac{\bar{p}}{3})^{\frac{3}{2}}$ . It is obvious that profits are higher for  $x_1^* \in [(\frac{\bar{p}}{3})^{\frac{1}{2}}, \frac{1}{2} - (\frac{\bar{p}}{3})^{\frac{1}{2}}]$  than for  $x_1^* = 0$ .

**Proof of Theorem 5.2** Recall that for all  $x_1$  there exists a unique price equilibrium given by Propositions 3.1, 3.2, 3.3. Let  $\bar{p} \geq \frac{3}{16}$  and define

$$x_1^*(\bar{p}) = \begin{cases} \frac{1}{4} & \text{if } \frac{3}{16} \leq \bar{p} \leq \frac{9}{16} \\ \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} & \text{if } \frac{9}{16} \leq \bar{p} \leq \frac{33}{16} \\ -\frac{1}{4} & \text{if } \frac{33}{16} \leq \bar{p}. \end{cases} \quad (14)$$

We prove that for  $\bar{p} \geq \frac{3}{16}$ , profits are maximal for  $x_1^* = x_1^*(\bar{p})$ . For  $\frac{3}{16} \leq \bar{p} \leq \frac{9}{16}$ , maximum profits in case  $-\frac{1}{4} \leq x_1 \leq 0$  are  $2(\frac{\bar{p}}{3})^{\frac{3}{2}}$ . In case  $0 \leq x_1 \leq \frac{1}{2}$  maximum profits are  $\frac{1}{2}(\bar{p} - \frac{1}{16})$ , being achieved for  $x_1^* = \frac{1}{4}$ . It is easily checked that  $\frac{1}{2}(\bar{p} - \frac{1}{16}) \geq 2(\frac{\bar{p}}{3})^{\frac{3}{2}}$  for  $\frac{3}{16} \leq \bar{p} \leq \frac{9}{16}$ . For  $\frac{9}{16} \leq \bar{p} \leq \frac{3}{4}$ , maximum profits in case  $-\frac{1}{4} \leq x_1 \leq 0$  are  $2(\frac{\bar{p}}{3})^{\frac{3}{2}}$ . In case  $0 \leq x_1 \leq \frac{1}{4}$ , profits are  $(\bar{p} + 1)^{\frac{1}{2}}$ , and in case  $\frac{1}{4} \leq x_1 \leq \frac{1}{2}$ , profits are  $\frac{1}{4}$ . For  $\frac{9}{16} \leq \bar{p} \leq \frac{3}{4}$  we have that  $2(\frac{\bar{p}}{3})^{\frac{3}{2}} \leq \frac{1}{4} \leq (\bar{p} + 1)^{\frac{1}{2}} - 1$ . This means that profits are maximal for  $x_1^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} \in [0, \frac{1}{4}]$ . For  $\frac{3}{4} \leq \bar{p} \leq \frac{5}{4}$ , maximum profits in case  $-\frac{1}{4} \leq x_1 \leq 0$  are  $\frac{1}{2}(\bar{p} - \frac{1}{4})$ . In case  $0 \leq x_1 \leq \frac{1}{4}$ , profits are  $(\bar{p} + 1)^{\frac{1}{2}}$ , and in case  $\frac{1}{4} \leq x_1 \leq \frac{1}{2}$ , profits are  $\frac{1}{4}$ . For  $\frac{3}{4} \leq \bar{p} \leq \frac{5}{4}$  we have that  $\frac{1}{4} \leq \frac{1}{2}(\bar{p} - \frac{1}{4}) \leq (\bar{p} + 1)^{\frac{1}{2}} - 1$ . This means that profits are maximal for  $x_1^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} \in [0, \frac{1}{4}]$ . For  $\bar{p} \geq \frac{5}{4}$ , maximum profits in case  $0 \leq x_1 \leq \frac{1}{4}$  are equal to  $\frac{1}{2}$  and are higher than profits  $\frac{1}{4}$  in case  $\frac{1}{4} \leq x_1 \leq \frac{1}{2}$ . For  $\bar{p} \geq \frac{5}{4}$  we thus only need to compare profits in case  $-\frac{1}{4} \leq x_1 \leq 0$  and in case  $0 \leq x_1 \leq \frac{1}{4}$ . For  $\frac{5}{4} \leq \bar{p} \leq \frac{33}{16}$ , maximum profits in case  $-\frac{1}{4} \leq x_1 \leq 0$  are  $(\bar{p} + 1)^{\frac{1}{2}} - 1$  and maximum profits in case  $0 \leq x_1 \leq \frac{1}{4}$  are  $\frac{1}{2}$ . Because  $(\bar{p} + 1)^{\frac{1}{2}} - 1 \geq \frac{1}{2}$  for  $\frac{5}{4} \leq \bar{p} \leq \frac{33}{16}$ , profits are maximal for  $x_1^* = \frac{3}{2} - (\bar{p} + 1)^{\frac{1}{2}} \in [-\frac{1}{4}, 0]$ . For  $\bar{p} \geq \frac{33}{16}$  it is easy to see that profits are maximal for  $x_1^* = -\frac{1}{4}$ .

**Proof of Proposition 6.1** We prove the proposition for  $\bar{p} \geq 3$ . For  $0 \leq \bar{p} \leq 3$  we refer to Boyer and Moreaux (1993). The proof goes for firm 1, but for firm 2 the proof is similar. There are two relevant maximization problems for firm 1. Either it maximizes its competitive profit  $p_1 x$  subject to the constraints  $x \leq x_1^+$  and  $x \leq 1$ , or it maximizes its local monopoly profit  $p_1 x_1^+$  subject to the constraints  $x_1^+ \leq x$  and  $x_1^+ \leq 1$ . For  $0 \leq p_2 \leq 3$  the competitive solution equals  $p_1 = \frac{p_2 + 1}{2}$  with profit  $\frac{(p_2 + 1)^2}{8}$ . For  $3 \leq p_2 \leq \bar{p}$  the competitive solution equals  $p_1 = p_2 - 1$  with profits  $p_2 - 1$ . The relevant local monopoly solution equals  $p_1 = p_2 - 1 + 2(\bar{p} - p_2)^{\frac{1}{2}}$  with profits  $(p_2 - 1 + 2(\bar{p} - p_2)^{\frac{1}{2}})(1 - (\bar{p} - p_2)^{\frac{1}{2}}) \leq p_2 - 1 \leq \frac{(p_2 + 1)^2}{8}$ . Therefore, the price reaction function of firm 1 is given by

$$p_1(p_2) = \begin{cases} \frac{p_2 + 1}{2} & \text{if } 0 \leq p_2 \leq 3 \\ p_2 - 1 & \text{if } 3 \leq p_2. \end{cases}$$

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