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Engwerda, J.C.; Weeren, A.J.T.M.

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The open-loop Nash equilibrium in LQ-games revisited

by

Jacob C. Engwerda

and

Arie J.T.M. Weeren*

both

Tilburg University

Department of Econometrics

P.O. Box 90153

5000 LE Tilburg

The Netherlands

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Abstract
In this paper we reconsider the conditions under which the finite-planning-horizon linear-quadratic differential game has an open-loop Nash equilibrium solution. Both necessary and sufficient conditions are presented for the existence of a unique solution in terms of an invertibility condition on a matrix. Moreover, we show that the often encountered solvability conditions stated in terms of Riccati equations are in general not correct. In an example we show that existence of a solution of the associated Riccati-type differential equations may fail to exist whereas an open-loop Nash equilibrium solution exists.
The scalar case is studied in more detail, and we show that solvability of the associated Riccati equations is in that case both necessary and sufficient.
Furthermore we consider convergence properties of the open-loop Nash equilibrium solution if the planning horizon is extended to infinity. To study this aspect we consider the existence of real solutions of the associated algebraic Riccati equation in detail and show how all solutions can be easily calculated from the eigenstructure of a matrix.

Keywords: Linear quadratic games, open-loop Nash equilibrium solution, solvability conditions, Riccati equations

I. Introduction

A well known problem studied in the literature on dynamic games is the existence of a unique open-loop Nash equilibrium solution in the two-player linear quadratic differential game defined by (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)):

$$\dot{x} = Ax + B_1u_1 + B_2u_2,\ x(0) = x_0 \tag{1}$$

with cost functionals:

$$J_1(u_1,u_2) := \frac{1}{2}x(t_f)^T K_{1f}x(t_f) + \frac{1}{2}\int_0^{t_f}\{x(t)^TQ_1x(t) + u_1(t)^TR_{11}u_1(t) + u_2(t)^TR_{12}u_2(t)\}dt,$$

and

$$J_2(u_1,u_2) := \frac{1}{2}x(t_f)^T K_{2f}x(t_f) + \frac{1}{2}\int_0^{t_f}\{x(t)^TQ_2x(t) + u_1(t)^TR_{21}u_1(t) + u_2(t)^TR_{22}u_2(t)\}dt,$$

in which all matrices are symmetric and, moreover, both Q_i and K_{if} are semi-positive definite and R_{ii} are positive definite, $i = 1, 2$.
It is often stated (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Abou-Kandil and Bertrand (1986) and Abou-Kandil et al. (1993)) that the open-loop Nash equilibrium solution is given by

$$u_1^*(t) = -R_{11}^{-1}B_1^TK_1(t)\Phi(t,0)x_0 \tag{2}$$

$$u_2^*(t) = -R_{22}^{-1}B_2^TK_2(t)\Phi(t,0)x_0 \tag{3}$$

provided that the set of coupled asymmetric Riccati-type differential equations

$$\dot{K}_1 = -A^TK_1 - K_1A - Q_1 + K_1S_1K_1 + K_1S_2K_2;\ K_1(t_f) = K_{1f} \tag{4}$$

$$\dot{K}_2 = -A^TK_2 - K_2A - Q_2 + K_2S_2K_2 + K_2S_1K_1;\ K_2(t_f) = K_{2f} \tag{5}$$

has a solution $K_1(t)$, $K_2(t)$. Here $\Phi(t,0)$ satisfies the transition equation

$$\dot{\Phi}(t,0) = (A - S_1K_1 - S_2K_2)\Phi(t,0);\ \Phi(t,t) = I$$

and $S_i = B_iR_{ii}^{-1}B_i^T, i = 1, 2$.
We will show by means of an example that, stated this way, this assertion is in general not correct. As correctly stated by Başar and Olsder (1982 theorem 6.5 A-2, or 1995 theorem 6.12) existence of a solution to the above mentioned Riccati differential equations is just a sufficient condition to conclude that there exists an open-loop Nash equilibrium for the game. Unfortunately, Başar and Olsder make an additional assumption in their proof on the costate variable (that it can be written as the product of a differentiable matrix and the state variable), under which, as we will show the existence of a solution of the Riccati equations is both a necessary and sufficient condition for existence of an open-loop Nash equilibrium. Therefore we present a correct proof of this theorem.

We will analyze problem (1) from its roots: the corresponding Hamiltonian equations. In section 2 we show how both necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium can be derived from these Hamiltonian equations, in terms of the invertibility of a certain matrix M. In section 3 we give a correct proof of the theorem stated in Başar and Olsder. Moreover, we present a sufficient condition which guarantees the existence of the set of Riccati differential equations.
One area where games of this type are widely used is in policy coordination models (see e.g. van Aarle et al. (1995), Dockner et al. (1985), Fershtman et al. (1987) Hughes Hallett et al. (1990), Petit (1989)). In many economic policy coordination problems an interesting problem is to analyse the effect of an expanding planning horizon on the resulting equilibria. Therefore we consider this effect if one expands the planning horizon t_f in (1) to infinity in a separate section. One nice property is that the equilibrium solution becomes much easier to calculate and implement than for a finite planning horizon. Before we present the results on this subject in section 5, we first consider the algebraic Riccati equations associated with (4,5) and their solutions. In section 4 we show how all solutions of these equations can be determined from the eigenstructure of the matrix M, and that the eigenvalues of the associated closed-loop system, obtained by applying the state feedback control $u_i^*(t) = -R_{ii}^{-1}B_i^TK_ix(t)$ in (1), are completely determined by the eigenvalues of matrix M. A number of the results presented in sections 4 an 5 are also reported by Abou-Kandil et al. (1993). The conditions under which they derive the results are however not always completely specified and their proofs are of a more analytic nature. Therefore we choose to give here a selfcontained expositure including their results.
Finally, in section 6 we study the scalar case which is of particular interest for many economic applications. We show that in the scalar case the above mentioned invertibility condition is always satisfied and that as a consequence the equilibrium solution is given by (2,3) and that this solution converges to a stationary stabilizing feedback policy if the planning horizon expands.
The paper ends with some concluding remarks.

II. Existence conditions for an open-loop Nash equilibrium

In this section we consider in detail the existence of a unique open-loop Nash equilibrium of the differential game (1). Due to the stated assumptions both cost functionals $J_i, i = 1, 2$, are strictly convex functions of u_i for all admissible control functions $u_j, j \neq i$ and for all x_0. This implies that the conditions following from the minimum principle are both necessary and sufficient (see e.g. Başar and Olsder (1982, section 6.5)).
Minimization of the Hamiltonian

$$H_i = (x^TQ_ix + u_1^TR_{i1}u_1 + u_2^TR_{i2}u_2) + \psi_i^T(Ax + B_1u_1 + B_2u_2),\ i = 1, 2$$

with respect to u_i yields the optimality conditions (see e.g. Başar and Olsder (1982) or

Abou-Kandil and Bertrand (1986)):

$$u_1^*(t) = -R_{11}^{-1} B_1^T \psi_1(t) \tag{6}$$

$$u_2^*(t) = -R_{22}^{-1} B_2^T \psi_2(t), \tag{7}$$

where the n-dimensional vectors $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1 x(t) - A^T \psi_1(t), \text{ with } \psi_1(t_f) = K_{1f} x(t_f)$$

$$\dot{\psi}_2(t) = -Q_2 x(t) - A^T \psi_2(t), \text{ with } \psi_2(t_f) = K_{2f} x(t_f)$$

and

$$\dot{x}(t) = Ax(t) - S_1 \psi_1(t) - S_2 \psi_2(t);\ x(0) = x_0.$$

In other words, the problem has a unique open-loop Nash equilibrium if and only if the differential equation

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = - \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} \begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

with boundary conditions $x(0) = x_0$, $\psi_1(t_f) - K_{1f} x(t_f) = 0$ and $\psi_2(t_f) - K_{2f} x(t_f) = 0$, has a unique solution. Denoting the state variable $(x^T(t)\ \psi_1^T(t)\ \psi_2^T(t))^T$ by $y(t)$, we can rewrite this two-point boundary value problem in the standard form

$$\dot{y}(t) = -My(t), \text{ with } Py(0) + Qy(t_f) = (x_0^T\ 0\ 0)^T, \tag{8}$$

where $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$, $P = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix}$

From (8) we have immediately that problem (1) has a unique open-loop Nash equilibrium if and only if

$$(P + Qe^{-Mt_f}) y(0) = (x_0^T\ 0\ 0)^T,$$

or equivalently,

$$(Pe^{Mt_f} + Q) e^{-Mt_f} y(0) = (x_0^T\ 0\ 0)^T, \tag{9}$$

is uniquely solvable for every x_0. Elementary matrix analysis then shows that

Theorem 1:

The two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state if and only if the following matrix $H(t_f)$ is invertible:

$$H(t_f) := W_{11}(t_f) + W_{12}(t_f) K_{1f} + W_{13}(t_f) K_{2f},$$

with $W(t_f) = (W_{ij}(t_f))\ \{i, j = 1, 2, 3;\ W_{ij} \in R^{n \times n}\} := \exp(Mt_f)$.

Moreover, the open-loop Nash equilibrium solution as well as the associated state trajectory can be calculated from the linear two-point boundary value problem (8). $\square$

III. Sufficient conditions for existence of an open-loop Nash equilibrium

In this section we consider the usual approach to the problem in terms of the Riccati equations (4,5) in more detail. First we show that whenever the set of Riccati equations (4,5) has a solution there exists an open-loop Nash equilibrium.

Theorem 2:
Problem (1) has a solution if the set of Riccati equations (4,5) has a solution.

Proof:
Let $K_i(t)$ satisfy the set of Riccati equations (4,5). Assume that the feedback control $u_i(t) = -R_{ii}^{-1}B_i^T K_i(t)x(t)$ is used to control system (1).
Now, define $\psi_i(t) := K_i(t)x(t)$. Then, obviously $\dot{\psi}_i(t) = \dot{K}_i(t)x(t) + K_i(t)\dot{x}(t)$.
Substitution of $\dot{K}_i$ from (4,5) and $\dot{x}$ from (1) yields

$$\dot{\psi}_i = (-A^T K_i - Q_i)x = -A^T\psi_i - Q_i x.$$

From this we conclude that the two-point boundary value problem (8) has a solution, which proves the claim. □

Now, under the assumption that the open-loop problem has a solution, it follows immediately from theorem 1 and (9) that

$$y_0 = e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0.$$

Since $y(t) = e^{-Mt}y_0$, it follows that the entries of $y(t)$ can be rewritten as

$$x(t) = (I\ 0\ 0)e^{M(t_f-t)}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0 \tag{10}$$

$$\psi_1(t) = (0\ I\ 0)e^{M(t_f-t)}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0 \tag{11}$$

$$\psi_2(t) = (0\ 0\ I)e^{M(t_f-t)}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0 \tag{12}$$

Using these formulas the following theorem can easily be proved.

Theorem 3:
If

$$H(t_f - t)(:= W_{11}(t_f - t) + W_{12}(t_f - t)K_{1f} + W_{13}(t_f - t)K_{2f})$$

is invertible for all $t \in [0, t_f]$, then

$$\psi_1(t) = K_1(t)x(t) \text{ and } \psi_2(t) = K_2(t)x(t)$$

for some continuously differentiable matrix functions $K_1(t)$ and $K_2(t)$, respectively.

Proof:
From (10) we have that $x(t) = H(t_f - t)H^{-1}(t_f)x_0$.
Since by assumption the matrix $H(t_f - t)$ is invertible it follows that $H^{-1}(t_f)x_0 = H^{-1}(t_f - t)x(t)$.
Substitution of this expression into the equations for $\psi_i, i = 1, 2$, in (11,12) yields:

$$\psi_1(t) = G_1(t_f - t)H^{-1}(t_f - t)x(t) \text{ and} \tag{13}$$
$$\psi_2(t) = G_2(t_f - t)H^{-1}(t_f - t)x(t) \tag{14}$$

for some continuously differentiable matrix functions $G_i, i = 1, 2$. Since also $H^{-1}(.)$ is a continuously differentiable matrix function the advertised result is obvious now. □

We like to stress here that the condition as stated in theorem 3 is just a sufficient condition to derive the fact that the adjoint state variables $\psi_i, i = 1, 2$ can be written as the product of a differentiable matrix and the state variable. Given the fact that such a representation is possible, the next corollary shows that then the open-loop Nash equilibrium can be obtained by solving the set of Riccati differential equations. This implies in particular (see the result of theorem 2) that whenever this representation is possible a unique open-loop Nash equilibrium exists **if and only if** the set of Riccati differential equations (4,5) has a solution.

Corollary 4:
If $H(t_f - t)$ is invertible $\forall t \in [0, t_f]$, then the unique open-loop Nash equilibrium solution is given by (2-5).

Proof:
From (6,7) we have that $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1x(t) - A^T\psi_1(t), \text{ with } \psi_1(t_f) = K_{1f}x(t_f) \text{and}$$

$$\dot{\psi}_2(t) = -Q_2x(t) - A^T\psi_2(t), \text{ with } \psi_2(t_f) = K_{2f}x(t_f)$$

and

$$\dot{x}(t) = Ax(t) - S_1\psi_1(t) - S_2\psi_2(t); \; x(0) = x_0.$$

According to theorem 3, under the above mentioned invertibility condition, $\psi_1(t)$ and $\psi_2(t)$ can be factorized as $K_1(t)x(t)$ and $K_2(t)x(t)$ for some continuous differentiable matrix functions $K_1(t)$ and $K_2(t)$, respectively. So, in particular we have that $\dot{\psi}_i = \dot{K}_i x + K_i \dot{x}$, $i = 1, 2$. Substitution of $\dot{\psi}_i$ and ψ_i, $i = 1, 2$ into the above formulas yields

$(\dot{K}_1 + A^T K_1 + K_1 A + Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2) e^{Mt} x_0 = 0$ with $(K_1(t_f) - K_{1f}) e^{M t_f} x_0 = 0$, and
$(\dot{K}_2 + A^T K_2 + K_2 A + Q_2 - K_2 S_2 K_2 - K_2 S_1 K_1) e^{Mt} x_0 = 0$ with $(K_2(t_f) - K_{2f}) e^{M t_f} x_0 = 0$,

for arbitrarily chosen x_0.
From this the stated result is obvious. $\square$

Note that this result in particular implies that under the above mentioned invertibility condition the existence of a solution to the set of Riccati equations is guaranteed. So verification of the solvability condition becomes superfluous.
The next example shows that there exist situations where the set of Riccati differential equations (4,5) does not have a solution, whereas there exists an open-loop Nash equilibrium for the game.

<u>Example 5:</u>
Let $A = \begin{pmatrix} -1 & 0 \\ 0 & -0.9 \end{pmatrix}$, $B_1 = B_2 = Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R_{11} = \begin{pmatrix} 500 & -200 \\ -200 & 100 \end{pmatrix}^{-1}$, $R_{22} = \begin{pmatrix} 1000 & 200 \\ 200 & 50 \end{pmatrix}^{-1}$, and $Q_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
Now, choose $t_f = 0.1$. Then, numerical calculation shows

$$H(0.1) = \begin{pmatrix} 35.0323 & 9.3217 \\ -1.8604 & 0.4729 \end{pmatrix} + \begin{pmatrix} 366.4330 & -142.9873 \\ -36.5968 & 16.4049 \end{pmatrix} K_{1f} + \begin{pmatrix} 850.3143 & 172.4050 \\ -22.0161 & -3.5423 \end{pmatrix} K_{2f} =: V(I\ K_{1f}\ K_{2f})^T.$$

Now, choose $K_{1f} = \begin{pmatrix} -\frac{1}{V(2,3)} & \frac{1-V(2,1)}{V(2,4)} \\ \frac{1-V(2,1)}{V(2,4)} & 2 \end{pmatrix}$, and $K_{2f} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{V(2,2)+V(2,3)K_{1f}(1,2)+V(2,4)K_{1f}(2,2)}{V(2,6)} \end{pmatrix}$,
then $H(0.1) = \begin{pmatrix} 20.11 & 1096.54 \\ 0 & 0 \end{pmatrix}$ is not invertible.
So, according to theorem 1 the problem has no open-loop Nash equilibrium, and therefore (see theorem 2) the corresponding set of Riccati differential equations has no solution.
Next consider $H(0.11)$. Numerical calculation shows that with the system parameters as chosen above, $H(0.11)$ is invertible. So, according to theorem 1 again, the game does have an open-loop Nash equilibrium for $t_f = 0.11$. However, since the set of Riccati differential equations can be rewritten as one autonomous vector differential equation, whose solutions are known to be shift invariant, it is clear that the corresponding set of Riccati differential equations can not have a solution for $t_f = 0.11$, since it has no solution for $t_f = 0.1$. $\square$

IV. The solutions for the algebraic Riccati equation

To study the asymptotic behaviour of the open-loop Nash equilibrium solution of game (1), in this section we first consider the set of solutions satisfying the set of so-called algebraic Riccati equations corresponding with (4,5)

$$\left.\begin{array}{l} 0 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \\ 0 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1; \end{array}\right\} \quad (ARE)$$

MacFarlane (1963) and Potter (1966) independently discovered that there exists a relationship between the stabilizing solution of the algebraic Riccati equation and the eigenvectors of a related Hamiltonian matrix in linear quadratic regulator problems. We will follow their approach here and formulate similar results for our problem (1). In fact Abou-Kandil et al. (1993) already pointed out the existence of a similar relationship. One of their results is that if the planning horizon t_f in (1) tends to infinity, under some technical conditions on the matrix M, the solution of the above mentioned set of Riccati differential equations converges to a solution of the set of (ARE) which can be calculated from the eigenspaces of matrix M.
In this section we elaborate on the relationship between solutions of (ARE) and matrix M in detail. We present both necessary and sufficient conditions in terms of the matrix M under which (ARE) has (a) real solution(s). In particular we will see that all solutions of (ARE) can be calculated from the invariant subspaces of M and that the eigenvalues of the associated closed-loop system, obtained by applying the control $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i(t) x(t)$, are completely determined by the eigenvalues of matrix M. As a corollary from these results we obtain both necessary and sufficient conditions for the existence of a stabilizing control of this type, a result which will be used in the next section.
In our analysis the set of all M-invariant subspaces play a crucial role. Therefore we introduce a separate notation for this set:

$$\mathcal{M}^{inv} := \{\mathcal{T} \mid M\mathcal{T} \subset \mathcal{T}\}.$$

It is well-known (see e.g. Lancaster and Tismenetsky (1985)) that this set contains only a finite number of (distinct) elements if and only if all eigenvalues of M have a geometric multiplicity one.
The set of possible solutions for the algebraic Riccati equation can, as will be shown in the next theorem, directly be calculated from the following collection of M invariant subspaces:

$$\mathcal{K}^{pos} := \left\{\mathcal{K} \in \mathcal{M}^{inv} \middle| \mathcal{K} \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = R^{3n}\right\}.$$

Note that elements in the set $\mathcal{K}^{pos}$ can be calculated using the set of matrices

$$K^{pos} := \left\{ K \in R^{3n \times n} | \mathrm{ImK} \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = R^{3n} \right\}.$$

The exact result on how all solutions of (ARE) can be calculated reads as follows:

<u>Theorem 6:</u>
(ARE) has a real solution (K_1, K_2) if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some $\mathcal{K} =: Im \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$,
such that

$$\mathcal{K} \in \mathcal{K}^{pos}.$$

Moreover, if the control functions $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t) x_0$ are used to control the system (1) the spectrum of the matrix $-A + S_1K_1 + S_2K_2$, coincides with $\sigma(M|_{\mathcal{K}})$.

<u>Proof:</u>
" $\Rightarrow$ " Assume (K_1, K_2) solve (ARE). Then simple calculations show that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} -A + S_1K_1 + S_2K_2 \\ Q_1 + A^T K_1 \\ Q_2 + A^T K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} (-A + S_1K_1 + S_2K_2).$$

Now, introducing $X := I, Y := K_1$, and $Z := K_2$, we see that $M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} J$, for some matrix J and matrix X invertible, which completes this part of the proof.
" $\Leftarrow$ " Let $\mathcal{K} \in \mathcal{K}^{pos}$. Then there exist K_1 and K_2 such that $\mathcal{K} = Im \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$, and a matrix J such that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J.$$

Spelling out the left hand side of this equation gives

$$\begin{pmatrix} -A + S_1K_1 + S_2K_2 \\ Q_1 + A^T K_1 \\ Q_2 + A^T K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J,$$

which immediately yields that $J = -A + S_1K_1 + S_2K_2$. Substitution of this equality into the right hand side of the equality shows then that $Q_1 + A^TK_1 = K_1(-A + S_1K_1 + S_2K_2)$ and $Q_2 + A^TK_2 = K_2(-A + S_1K_1 + S_2K_2)$, or stated differently, K_1, K_2 satisfy (ARE). This proves the second part of the theorem.
The last statement of the theorem concerning the spectrum of the matrix $-A+S_1K_1+S_2K_2$ follows directly from the above arguments by noting that if we choose as a basis for $\mathbb{R}^{3n}$ $\begin{pmatrix} I & 0 & 0 \\ K_1 & I & 0 \\ K_2 & 0 & I \end{pmatrix}$, matrix M has the block-triangular structure $\begin{pmatrix} -A + S_1K_1 + S_2K_2 & S_1 & S_2 \\ 0 & A^T - K_1S_1 & -K_1S_2 \\ 0 & -K_2S_1 & A^T - K_2 \end{pmatrix}$ which completes the proof. □

From the above theorem a number of interesting properties concerning the solvability of (ARE) follow. First of all we observe that every element of $\mathcal{K}^{pos}$ defines exactly one solution of (ARE). Furthermore, this set contains only a finite number of elements if and only if the geometric multiplicities of all eigenvalues of M is one. So, in that case we immediately conclude that (ARE) will have at most a finite number of solutions and that (ARE) will have no real solution if and only if $\mathcal{K}^{pos}$ is empty.
Another conclusion which immediately follows from the above theorem is that

Corollary 7:
(ARE) will have a set of solutions (K_1, K_2) stabilizing the closed-loop system matrix $A - S_1K_1 - S_2K_2$ if and only if there exists an M invariant subspace $\mathcal{K}$ in $\mathcal{K}^{pos}$ such that $Re\ \lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$. □

To illustrate some of the above mentioned properties, reconsider example 5.

Example 5 (continued):
Numerical calculations show that the eigenvalues of M are $\{-42.1181, -0.8866, -0.3441 \pm 4.6285i, -0.3168, 42.1096\}$, and the corresponding eigenspaces

$$\begin{aligned}
\mathcal{T}_1 &= \text{Span}\{ \ (-0.9968\ 0.0549\ 0.0471\ 0.0229\ 0.0242\ -0.0013)^T \ \}; \\
\mathcal{T}_2 &= \text{Span}\{ \ (-0.0178\ 0.0108\ -0.2191\ -0.5272\ -0.1570\ 0.8056)^T \ \}; \\
\mathcal{T}_3 &= \text{Span}\{ \ (-0.0439\ 0.1636\ -0.085\ -0.1382\ 0.0519\ -0.1906)^T, \\
&\qquad\qquad (0.2512\ -0.9146\ -0.0284\ -0.0425\ 0.0168\ -0.0582)^T \}; \\
\mathcal{T}_4 &= \text{Span}\{ \ (0.1145\ -0.4047\ -0.2570\ -0.4975\ 0.1676\ -0.6939)^T \ \} \text{ and} \\
\mathcal{T}_5 &= \text{Span}\{ \ (-0.9970\ 0.0545\ -0.0450\ -0.0219\ -0.0231\ 0.0013)^T \ \}.
\end{aligned}$$

According to theorem 6, the corresponding set of algebraic Riccati equations has at most $\begin{pmatrix} 4 \\ 2 \end{pmatrix} + 1 = 7$ real solutions. Furthermore, there is no solution which stabilizes the closed-loop system matrix.

As an example consider $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} := (\mathcal{T}_1\ \mathcal{T}_2)$. This yields the solution

$K_1 = YX^{-1} = \begin{pmatrix} 0.0471 & -0.2191 \\ 0.0229 & -0.5272 \end{pmatrix} \begin{pmatrix} -0.9968 & -0.0178 \\ 0.0549 & 0.0108 \end{pmatrix}^{-1}$ and

$K_2 = ZX^{-1} = \begin{pmatrix} 0.0242 & -0.1570 \\ -0.0013 & 0.8056 \end{pmatrix} \begin{pmatrix} -0.9968 & -0.0178 \\ 0.0549 & 0.0108 \end{pmatrix}^{-1}$.

The eigenvalues of the closed-loop system (1) using the control $u_i(t) = -R_{ii}^{-1} B_i^T K_i x(t)$ are $\{42.1181, 0.8866\}$. It is easily verified that the rank of the first two rows of every other candidate solution is also two, so we conclude that (ARE) has seven solutions, none of which is stabilizing. □

V. Convergence results

As argued in the introduction, it is interesting to see how the open-loop equilibrium solution changes when the planning horizon t_f tends to infinity. To study convergence properties for problem (1), it seems reasonable to require that problem (1) has a properly defined solution for every finite planning horizon. Therefore in this section we will make the following well-posedness assumption (see theorem 1)

$$H(t_f) \text{ is invertible for all } t_f < \infty. \tag{15}$$

Furthermore, we will see that general convergence results can only be derived if the eigenstructure of matrix M satisfies an additional property, which we define first.

Definition 8:
M is called dichotomically separable if there exist subspaces V_1 and V_2 such that $MV_i \subset V_i, i = 1,2$, $V_1 \oplus V_2 = \mathbb{R}^{3n}$, where $\dim V_1 = n$, $\dim V_2 = 2n$, and moreover $Re\ \lambda > Re\ \mu$ for all $\lambda \in \sigma(M|V_1), \mu \in \sigma(M|V_2)$. □

Using corollary 4 we have now immediately from (15) that to study the convergence of the open-loop Nash equilibrium solution we can restrict ourselves to the study of the set of Riccati differential equations (4-5) at time 0. We will denote the corresponding solutions of (4-5) by $K_i(0, t_f)$, respectively. So the question is under which conditions the solutions of this set of equations will converge if t_f increases. Note that $K_i(0, t_f)$ can be viewed as the solution $k(t)$ of an autonomous vector differential equation $\dot{k} = f(k)$, with $k(0) = k_0$ for some fixed k_0, and where f is a smooth function. Elementary analysis shows then that $K_i(0, t_f)$ converges to a limit $\bar{k}$ only if this limit $\bar{k}$ satisfies $f(\bar{k}) = 0$. Therefore, we immediately deduce from theorem 6 the following necessary condition for convergence.

Lemma 9:
$K_i(0, t_f)$ can only converge to a limit $\bar{K}_i(0)$ if the set $\mathcal{K}^{pos}$ is nonempty. □

Note that dichotomic separability of M implies that $\mathcal{K}^{pos}$ is nonempty. On the other hand it is not difficult to construct an example where $\mathcal{K}^{pos}$ is nonempty, whereas M is not dichotomically separable.
To study the convergence of $K_i(0,t_f)$ we reconsider (13) and (14) in theorem 3. From these formulas we have that

$$K_1(0,t_f) = (0\ I\ 0)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}\left((I\ 0\ 0)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}\right)^{-1}, \text{ and} \tag{16}$$

$$K_2(0,t_f) = (0\ 0\ I)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}\left((I\ 0\ 0)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}\right)^{-1}. \tag{17}$$

We are now able to give an elementary proof of the following result (see also Abou-Kandil et al (1993, section 4))

<u>Theorem 10:</u>
Assume that the well-posedness assumption (15) holds.
Then, if M is dichotomically separable and Span $\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$,

$$K_1(0,t_f) \to Y_0X_0^{-1}, \text{ and } K_2(0,t_f) \to Z_0X_0^{-1},$$

where X_0, Y_0, Z_0 are defined by (using the notation of definition 8) $V_1 =: \mathrm{Span}(X_0^T\ Y_0^T\ Z_0^T)^T$.

<u>Proof:</u>
Choose $\begin{pmatrix} I & 0 & 0 \\ K_{1f} & I & 0 \\ K_{2f} & 0 & I\end{pmatrix}$ as a basis for $\mathbb{R}^{3n}$. Then, because

$$\mathrm{Span}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix} \oplus V_2 = \mathbb{R}^{3n},$$

there exists an invertible matrix $V_{22} \in \mathbb{R}^{2n\times 2n}$ such that $V_2 = \mathrm{Span}\begin{pmatrix} 0 \\ V_{22}\end{pmatrix}$.
Moreover, because M is dichotomically separable, there exist matrices J_1, J_2 such that

$$M = V\begin{pmatrix} J_1 & 0 \\ 0 & J_2\end{pmatrix}V^{-1},$$

where

$$V = \begin{pmatrix} X_0 & 0 \\ \begin{pmatrix} Y_0 \\ Z_0\end{pmatrix} & V_{22}\end{pmatrix},$$

and $\sigma(J_i) = \sigma(M|V_i)$, $i = 1, 2$.

Using this, we can rewrite $K_1(0, t_f)$ and $K_2(0, t_f)$ in (16,17) as $\tilde{G}_i(t_f)\tilde{H}^{-1}(t_f)$, $i = 1, 2$, where

$$\begin{aligned}
\tilde{G}_1(t_f) &= (0\ I\ 0)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f}\end{pmatrix}V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}, \\
\tilde{G}_2(t_f) &= (0\ 0\ I)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f}\end{pmatrix}V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}, \\
\tilde{H}(t_f) &= (I\ 0\ 0)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f}\end{pmatrix}V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}.
\end{aligned}$$

Here λ_n is the element of $\sigma(M|V_1)$ which has the smallest real part.

Next, consider $\tilde{G}_1(t_f) - Y_0X_0^{-1}\tilde{H}(t_f)$.
Simple calculations show that this matrix can be rewritten as

$$e^{-\lambda_n t_f}(-Y_0X_0^{-1}\ I\ 0)V\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f}\end{pmatrix}V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix}, \tag{18}$$

Since $(-Y_0X_0^{-1}\ I\ 0)\left(X_0^T\ Y_0^T\ Z_0^T\right)^T = 0$, (18) equals

$$e^{-\lambda_n t_f}\begin{pmatrix} I & 0 \end{pmatrix}V_{22}e^{J_2 t_f}V_{22}^{-1}\begin{pmatrix} K_{1f} - Y_0X_0^{-1} \\ K_{2f} - Z_0X_0^{-1}\end{pmatrix}.$$

As $e^{-\lambda_n t_f}e^{J_2 t_f}$ converges to zero for $t_f \to \infty$, it is obvious now that $\tilde{G}_1(t_f) - Y_0X_0^{-1}\tilde{H}(t_f)$ converges to zero for $t_f \to \infty$. Similarly it can be shown that also $\tilde{G}_2(t_f) - Z_0X_0^{-1}\tilde{H}(t_f)$ converges to zero for $t_f \to \infty$. To conclude from this that $K_1(0, t_f) \to Y_0X_0^{-1}$, and $K_2(0, t_f) \to Z_0X_0^{-1}$, it suffices to show that $\tilde{H}^{-1}(t_f)$ remains bounded for $t_f \to \infty$. This follows, however, directly by spelling out $\tilde{H}(t_f)$ as

$$\tilde{H}(t_f) = e^{-\lambda_n t_f}X_0e^{J_1 t_f}X_0^{-1}.$$

$\square$

Combination of the results from theorem 10 and corollary 7 yields then

<u>Corollary 11:</u>
If the planning horizon t_f in the differential game (1) tends to infinity, the unique open-loop

Nash equilibrium solution converges to a stationary feedback strategy $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i x(t)$, $i = 1, 2$, which stabilizes the associated closed-loop system, if the following conditions are satisfied:

1. all conditions mentioned in theorem 10
2. $Re\ \lambda > 0, \forall \lambda \in \sigma(M|V_1)$.

Moreover, these constant feedback matrices can be calculated from the eigenspaces of matrix M (see theorem 10). $\square$

VI. The scalar case

We start this section by showing that the invertibility condition mentioned in corollary 3 is always satisfied if the dimensions of both the state and the input variables in system (1) equal one. This implies that for this kind of systems the usually stated assertion that the open-loop Nash strategy is given by (2-5) is correct and, moreover, that the associated Riccati equations yield the appropriate solution. To prove this result we first calculate the exponential of matrix M. To stress the fact that in this section we are dealing with the scalar case, we will put the system parameters in lower case, so e.g. a instead of A.

Lemma 12:
Consider matrix M in (8). The exponential of matrix M, e^{Ms}, is given by

$$V \begin{pmatrix} e^{-\mu s} & 0 & 0 \\ 0 & e^{as} & 0 \\ 0 & 0 & e^{\mu s} \end{pmatrix} V^{-1}, \tag{19}$$

where

$$V = \begin{pmatrix} a+\mu & 0 & a-\mu \\ -q_1 & -s_2 & -q_1 \\ -q_2 & s_1 & -q_2 \end{pmatrix}$$

and its inverse

$$V^{-1} = \frac{1}{detV} \begin{pmatrix} (s_1q_1 + s_2q_2) & s_1(a-\mu) & s_2(a-\mu) \\ 0 & -2q_2\mu & 2q_1\mu \\ -(s_1q_1+s_2q_2) & -s_1(a+\mu) & -s_2(a+\mu) \end{pmatrix},$$

with the determinant of V, $detV = 2\mu(s_1q_1 + s_2q_2)$, and $\mu = \sqrt{a^2 + s_1q_1 + s_2q_2}$.

Proof:
Straightforward multiplication shows that we can factorize M as $M = V\text{diag}(a, \mu, -\mu)V^{-1}$. So (see e.g. Lancaster et al (1985, theorem 9.4.3)), the exponential of matrix M, e^{Ms}, is as stated above. $\square$

Next consider the matrix $H(s)$ from theorem 3 for an arbitrarily chosen $s \in [0, t_f]$. Obviously, $H(s) = (1\ 0\ 0)e^{Ms}\begin{pmatrix} 1 \\ k_{1f} \\ k_{2f} \end{pmatrix}$. Using the expressions in lemma 12 for V and V^{-1} we find

$$H(s) = \frac{1}{detV}[(s_1q_1 + s_2q_2)\{(\mu - a)e^{\mu s} + (a+\mu)e^{-\mu s}\} + (\mu^2 - a^2)(e^{\mu s} - e^{-\mu s})(s_1k_{1f} + s_2k_{2f})].$$

Clearly, $H(s)$ is positive for every $s \geq 0$. This implies in particular that $H(s)$ differs from zero for every $s \in [0, t_f]$, whatever $t_f > 0$ is. So from corollary 4 we now immediately have the following conclusion.

<u>Theorem 13:</u>
Problem (1) has a unique open-loop Nash equilibrium solution:

$$u_1^*(t) = -\frac{1}{r_{11}}b_1k_1(t)x(t)$$

$$u_2^*(t) = -\frac{1}{r_{22}}b_2k_2(t)x(t)$$

where $k_1(t)$ and $k_2(t)$ are the solutions of the coupled asymmetric Riccati-type differential equations

$$\begin{aligned} \dot{k}_1 &= -ak_1 - k_1a - q_1 + k_1^2s_1 + k_1s_2k_2;\ k_1(t_f) = k_{1f} \\ \dot{k}_2 &= -ak_2 - k_2a - q_2 + k_2^2s_2 + k_2s_1k_1;\ k_2(t_f) = k_{2f}. \end{aligned}$$

Here $s_i = \frac{1}{r_{ii}}b_i^2, i = 1, 2.$ □

We conclude this section by considering the convergence properties of the open-loop equilibrium solution mentioned above. It turns out that in the scalar case we can prove that this solution always converges.

<u>Theorem 14:</u>
Assume that $s_1q_1 + s_2q_2 > 0$.
Then, the open-loop Nash equilibrium solution from theorem 13 converges to the (stationary feedback) strategies:

$$u_1^*(t) = -\frac{1}{r_{11}}b_1k_1x(t)$$

$$u_2^*(t) = -\frac{1}{r_{22}}b_2k_2x(t)$$

where $k_1 = \frac{(a+\mu)q_1}{s_1q_1+s_2q_2}$ and $k_2 = \frac{(a+\mu)q_2}{s_1q_1+s_2q_2}$.
Moreover, these strategies stabilize the closed-loop system (1).

Proof:
Since $s_1q_1 + s_2q_2 > 0$, it is clear from (19) that M is dichotomic separable. Furthermore we showed above that the well-posedness assumption is always satisfied in the scalar case. Note that $\mu > 0$, so according to corollary 11 the open-loop Nash strategies converge to a stationary feedback strategy whenever k_{if}, $i = 1, 2$, are such that $s_1q_1 + s_2q_2 + s_1(a - \mu)k_{1f} + s_2(a - \mu)k_{2f} \neq 0$.
Now consider the case that $s_1q_1 + s_2q_2 + s_1(a - \mu)k_{1f} + s_2(a - \mu)k_{2f} = 0$. To study this case, reconsider (16) and (17) for $t_f \to \infty$. Elementary spelling out of these formulas, using (19), shows that also in this case both $k_1(0, t_f)$ and $k_2(0, t_f)$ converge to the limits as advertised above, which concludes the proof. $\square$

VII. Concluding remarks

In this paper we reconsidered the existence and asymptotic behaviour of a unique open-loop Nash equilibrium solution in the two-player linear quadratic game. We analyzed the problem starting from its basics: the Hamiltonian equations. We derived necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium solution in terms of a full rank condition on a modified fundamental matrix. An open problem remains to find general conditions on the system matrices which guarantee that the rank condition is satisfied. Furthermore we showed by means of an example that in general a solution to the set of associated differential Riccati equations may fail to exist whereas an open-loop Nash equilibrium solution exists. A sufficient condition is given under which the open-loop equilibrium solution can be obtained via the solutions of these Riccati differential equations. Again, an open problem remains to interpret this solvability condition in terms of the system matrices.
To study convergence of the open-loop equilibrium solution if the planning horizon is extended to infinity, we argued that for well-posedness reasons we can restrict ourselves to study the asymptotic behavior of the Riccati differential equations. To that end we first considered the existence of real solutions for the corresponding algebraic Riccati equations. We showed how every real solution to (ARE) can be calculated from the invariant subspaces of the matrix $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$. Furthermore, we showed how the eigenvalues of the system if the corresponding feedback control strategies are used in (1) correspond to the eigenvalues of this matrix.
In particular this approach makes it possible to conclude whether (ARE) has a real solution, and if so, how many solutions there are (there are always only a finite number of solutions if the geometric multiplicity of all eigenvalues of M is one) and which of them gives rise to control strategies that stabilize the closed-loop system. We noted that in general (ARE) will have more than one stabilizing solution. We like to note that it is not difficult to show by means of an example that this property is independent of the fact whether matrix M is dichotomically separable or not.

These results raise a number of interesting open questions, namely, is it possible to say a priori something on the relationship between the eigenstructure of matrix M (in particular the structure which guarantees the existence of stabilizing solutions to (ARE), and more in particular the structure which generically implies convergence of the solutions of the Riccati differential equations) and geometric properties of the system parameters in (1). A first attempt to answer the question under which conditions on the system matrices there may exist a stabilizing solution was addressed in Engwerda and Weeren (1994), where for a number of particular situations it was shown that matrix M always has at least n eigenvalues (counted with their algebraic multiplicities) with a positive real part. On the other hand, by means of an example it was shown there that this property does not always hold. The results on the existence of real solutions to (ARE) were used to show that if the dimension of the direct sum of the invariant subspaces corresponding with the n largest eigenvalues (counted again with algebraic multiplicities) equals n, then generically the solution to the Riccati differential equations converges to a solution which can be directly calculated from this direct sum.

Since there are a number of applications which just involve scalar systems we concluded the paper by a detailed analysis of that case. We showed that for those systems, the unique open-loop Nash equilibrium solution can always be found by solving the associated set of Riccati differential equations, and that this solution converges to a stationary state feedback strategy, which stabilizes the associated closed-loop system if the planning horizon tends to infinity.

Finally we note that the obtained results can be straightforwardly generalized to the N player game.

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