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Chapter 17

THE OWEN SET AND THE CORE OF SEMI-INFINITE LINEAR PRODUCTION SITUATIONS

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Abstract We study linear production situations with an infinite number of production techniques. Such a situation gives rise to a semi-infinite linear program. Related to this program, we introduce primal and dual games and study relations between these games, the cores of these games and the so-called Owen set.

1 INTRODUCTION

Linear production (LP) situations are situations where several producers own resource bundles. They can use these resources to produce various products via linear production techniques that are available to all the producers. The goal of each producer is to maximize his profit, which equals the revenue of his products at the given market prices. These situations and corresponding cooperative games are introduced in Owen ([4]). He showed that these games have a nonempty core by constructing a core-element via a related dual linear program. Samet and Zemel ([5]) study relations between the set of all core-elements we can find in this way and the core, and the emphasis in their study is placed on replication of players. Gellekom, Potters, Reijnierse, Tijs and Engel

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([2]) named the set of all the core-elements that can be found in the same way as Owen did, the "Owen set" and they give a characterization of this set.

More general are situations involving the linear transformation of products (LTP) introduced by Timmer, Borm and Suijs ([7]) because it is shown in this paper that any LP situation can be written as an LTP situation. In an LTP situation different producers may control different transformation techniques and each of these techniques can have more than one output good. LTP situations give rise to LTP games, which also have a nonempty core.

A part of Fragnelli, Patrone, Sideri and Tijs ([1]) is devoted to the study of semi-infinite LP situations. These are LP situations where there is a countably infinite number of products that can be produced. Semi-infinite LTP situations, in which there is a countably infinite number of transformation techniques, are analyzed in Timmer, Llorca and Tijs ([8]).

In this work we study the Owen set and the core of semi-infinite LP and LTP situations and relations between these two concepts. For this reason, we introduce primal and dual games corresponding to the primal and dual programs of both semi-infinite LP and LTP situations. Using these primal and dual games we show that if these games have the same value then the Owen set is included in the core and otherwise, they are disjoint. Our main result is that for both semi-infinite LP and LTP situations the core of the corresponding game is nonempty if there exists a finite upper bound for the maximal profit obtained by the coalition of all producers. Because LTP situations are more general than LP situations, the use of more sophisticated tools is required to show that the results for LP situations also hold for LTP situations. This is why we analyze LP situations before turning our attention to LTP situations.

This work is organized as follows. The Sections 2 and 4 present the most relevant results of respectively finite LP and LTP situations and their corresponding games. Semi-infinite LP and LTP situations are introduced in the Sections 3 and 5, respectively. Relations between the Owen set, the core and the primal and dual games are investigated and we show that the core is nonempty if there exists a finite upper bound for the maximal profit obtained by the coalition of all producers. Section 6 concludes.

2 FINITE LINEAR PRODUCTION SITUATIONS

Finite linear production (LP) situations describe situations with a set of producers, a bundle of resources for each producer and a set of linear production techniques that all the producers may apply. The resources are used in the various linear production techniques to produce some products that can be sold on the market at given market prices. We assume that there are no costs involved. The goal of each producer is to maximize his profits. Producers are also allowed to cooperate and pool their resources. Such a coalition of producers also maximizes its profit given the joint resources. Cooperation pays off because the maximal profit of the group is at least as much as the sum of the individual profits.

More formally, denote by N, R and Q respectively the finite sets of producers, resources and products. The *technology matrix* $A \in \mathbb{R}_+^{R\times Q}$, that is $A \in \mathbb{R}^{R\times Q}$ and $A_{ij} \geq 0$ for all $i \in R$ and $j \in Q$, describes all the available linear production techniques in the following way. Each production technique produces one product and you need A_{ij} units of resource $i \in R$ to produce one unit of product $j \in Q$. The resources owned by the producers are described by the *resource matrix* $B \in \mathbb{R}_+^{R\times N}$ where producer $k \in N$ owns B_{ik} units of resource $i \in R$. Prices are denoted by the price vector $c \in \mathbb{R}_+^Q \setminus \{0\}$. We assume that there is a positive quantity available of each resource, that is, for all resources $i \in R$ there is a producer k such that $B_{ik} > 0$. Furthermore, if there is a product j with a positive market price, then we do not allow for "output without input" and therefore there exists at least one resource $i \in R$ with $A_{ij} > 0$. Finally, all producers are price-takers and all products can be sold on the market.

To maximize his profit, producer k needs an optimal production plan $x \in \mathbb{R}_+^Q$ that tells him how much he should produce of each good. Not all production plans are feasible since the producer has to take into account his limited amount of resources. The amount of resources needed in production plan x, Ax, should not exceed the amount of resources of producer k, $Be_{\{k\}}$, where $e_{\{k\}}$ denotes the k^{th} unit vector in $\mathbb{R}^N$ with $e_{\{k\},t} = 1$ if $t = k$ and $e_{\{k\},t} = 0$ otherwise. Furthermore, the production plan has to be nonnegative since we are only interested in producing nonnegative quantities of the products, and its profit equals $x^T c$. The following linear program maximizes the profit of producer k.

$$\max\left\{x^T c \,\middle|\, Ax \leq Be_{\{k\}},\ x \geq 0\right\}$$

Next to producing on their own, producers are allowed to cooperate. If a coalition S of producers cooperates then they put all their resources together and so, this coalition has the resource bundle Be_S at its disposal, where $e_S \in \mathbb{R}^N$ with $e_{S,t} = 1$ if $t \in S$ and $e_{S,t} = 0$ if $t \notin S$. Given this amount of resources, the coalition wants to maximize its profit,

$$P_S: \quad \max\left\{x^T c \,|\, Ax \leq Be_S,\ x \geq 0\right\},$$

where P_S denotes the *primal* linear program for coalition S. The corresponding *dual* problem for this coalition, D_S, is the following program.

$$D_S: \quad \min\left\{y^T Be_S \,\middle|\, A^T y \geq c,\ y \geq 0\right\}$$

The vector y can be seen as a vector of *shadow prices* for the resources since the condition $A^T y \geq c$ can be interpreted as follows. If a company wants to buy the resources Be_S of coalition S and is willing to pay y_i per unit of good $i \in M$ then for any product $j \in Q$, the value of the resources needed to produce one unit of this product according to the prices in y should be at least as large as the market price c_j. Otherwise, coalition S will not agree with this sale. Therefore, the program D_S minimizes the value of the resources owned by coalition S according to the shadow prices and subject to the restrictions above.

For ease of notation, let F_{pS} and F_{dS} denote the set of feasible solutions of respectively the primal and dual program for coalition S,

$$\begin{aligned} F_{pS} &= \{x \in \mathbb{R}^Q \,|\, Ax \le Be_S,\ x \ge 0\}, \\ F_{dS} &= \{y \in \mathbb{R}^R \,|\, A^T y \ge c,\ y \ge 0\}, \end{aligned}$$

denote by w_{pS} and w_{dS} the optimal values of the programs,

$$\begin{aligned} w_{pS} &= \max\{x^T c \,|\, x \in F_{pS}\}, \\ w_{dS} &= \min\{y^T Be_S \,|\, y \in F_{dS}\}, \end{aligned}$$

and let O_{pS} and O_{dS} be the sets of optimal solutions,

$$\begin{aligned} O_{pS} &= \left\{x \in F_{pS} \,\middle|\, x^T c = w_{pS}\right\}, \\ O_{dS} &= \left\{y \in F_{dS} \,\middle|\, y^T Be_S = w_{dS}\right\}. \end{aligned}$$

The assumptions we made ensure that F_{pS}, F_{dS}, O_{pS} and O_{dS} are nonempty sets and w_{pS} and w_{dS} exist and are finite. It follows from duality theory ([9, p. 281]) that $w_{pS} = w_{dS}$ for all coalitions S.

We see that an LP situation can be described by the tuple (N, A, B, c). Corresponding to such a situation we define two games, (N, v_p) and (N, v_d). The first one, (N, v_p), is the well known LP game where $v_p(S) = w_{pS}$ for all coalitions S. The second game, (N, v_d), is the game that gives each coalition S the value of its dual program, $v_d(S) = w_{dS}$.

If two producers cooperate then they can produce at least the amount that they can produce on their own, so, their joint profit will be at least as large as the sum of their individual profits. Similar reasoning shows that the highest profit will be obtained if all the producers work together. But how should this joint profit be divided among the producers? We could divide the profit according to a so-called *core-allocation*. The *core* of a game (N, v), $C(v)$, allocates the profit in such a way that no coalition of producers has an incentive to start producing on their own. More precisely,

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \ge v(S) \text{ for all } S \subset N \right\}.$$

We define the core of an LP situation, Core(A, B, c), to be the core of the corresponding LP game, Core$(A, B, c) = C(v_p)$. Owen ([4]) shows that LP games are *totally balanced*, that is, the games themselves have a nonempty core and so do all of their subgames. He obtains this result by showing that we can easily obtain a core-element of an LP game as follows. Instead of solving the programs P_S for all coalitions $S \subset N$ in order to calculate $v_p(S)$ and the core, we only solve D_N, the dual program of the grand coalition. Let y be an optimal solution of D_N. If each producer k gets the value of his resources according to the shadow prices, $y^T Be_{\{k\}}$, then this distribution of values is a core-allocation. The set of all core-allocations that we can obtain in this way, is called the Owen set corresponding to the LP situation (N, A, B, c):

$$\text{Owen}(A, B, c) = \left\{ (y^T Be_{\{k\}})_{k \in N} \mid y \in O_{dN} \right\}.$$

This set has been studied extensively by Gellekom et al. ([2]) and they also provide a characterization of the Owen set. Because the set O_{dN} is nonempty, so is Owen(A, B, c). Furthermore, each vector in this set is an element of $C(v_p)$ and therefore Owen$(A, B, c) \subset$ Core(A, B, c).

We end this section with an example of an LP situation and corresponding LP game.

Example 2.1 Consider the following LP situation. There are two producers, $N = \{1, 2\}$, two resources, two products and

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}, \; c = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Producer 1 owns nothing of the second resource (see the first column of the resource matrix B) and producer 2 owns nothing of the first resource. Since both products require a positive amount of input of each of the two resources, a single producer cannot produce anything. Consequently, $v_p(\{1\}) = v_p(\{2\}) = 0$. If both producers cooperate then they own a positive amount of each resource and they have many production plans at their disposal, namely all plans $x \in F_{pN}$:

$$F_{pN} = \{x \in \mathbb{R}^2 | \, 2x_1 + 2x_2 \leq 6, \; x_1 + 3x_2 \leq 7, \; x \geq 0\}$$

The profit of such a production plan x is $c^T x = 3x_1 + 4x_2$ and so, the profit maximization problem P_N of the grand coalition equals $\max\{3x_1 + 4x_2 | \, x \in F_{pN}\}$. The maximal profit $w_{pN} = 11$ is attained in the plan $x = (1, 2)^T$, so $O_{pN} = \{(1, 2)^T\}$, and Core$(A, B, c) = \{(a, 11 - a)^T | \, 0 \leq a \leq 11\}$. For the dual game (N, v_d) it holds that $v_d(\{i\}) = v_p(\{i\}) = 0$ for all $i \in N$. The set of all feasible shadow prices for the grand coalition is the set

$$F_{dN} = \{y \in \mathbb{R}^2 | \, 2y_1 + y_2 \geq 3, \; 2y_1 + 3y_2 \geq 4, \; y \geq 0\}.$$

We want to minimize the value of the resources of coalition N according to the shadow prices y, $y^T Be_N = 6y_1 + 7y_2$, over all feasible shadow prices: $\min\{6y_1 + 7y_2 | \, y \in F_{dN}\}$. The minimum $w_{dN} = 11$ is attained in $y = (5/4, 1/2)^T$ and so $O_{dN} = \{(5/4, 1/2)^T\}$. The Owen set, $\text{Owen}(A, B, c) = \{(15/2, 7/2)^T\}$, consists of one point and we see that $\text{Owen}(A, B, c) \subset \text{Core}(A, B, c)$, $\text{Owen}(A, B, c) \neq \text{Core}(A, B, c)$.

3 SEMI-INFINITE LP SITUATIONS

If we extend the set Q such that it contains a countable infinite number of products then we arrive at semi-infinite LP situations. Without loss of generality we may assume that $Q = \mathbf{N} = \{1, 2, \dots\}$, the set of natural numbers. An example of a production process with a countable infinite number of products is the "process" of baking pancakes at home. Pancakes are made of milk, flour, eggs, salt, butter and perhaps a little sugar. If you have a recipe for baking pancakes and you change the amounts of the ingredients slightly (e.g. you add a little flour or you use a little bit less milk) then you get another recipe for pancakes. This set of processes will be countable infinite if you require that all quantities should be integer multiples of one gram, for example.

A semi-infinite LP situation (N, A, B, c) thus has $A \in \mathbf{R}_+^{R \times Q}$, $B \in \mathbf{R}_+^{R \times N}$ and $c \in \mathbf{R}_+^Q$ with $Q = \mathbf{N}$. As opposed to LP situations, we impose no further restrictions on these variables since we want to keep our analysis as general as possible. The restrictions we imposed in the previous section turn up by themselves in the proof of Theorem 3.5. Because we have a countable infinite number of products, the linear programs, which determine the "maximal" profits of the coalitions, and their dual programs are now *semi-infinite linear programs*. The primal program for a coalition S of producers that determines its maximal profit, now equals

$$P_S: \ \sup\{x^T c | \, Ax \leq Be_S, \ x \geq 0\},$$

where we replaced the maximum by the *supremum* since the optimal value may not be reached by any production plan x. This program contains an infinite number of variables x_j, $j \in Q$. Similarly, in the dual program

$$D_S: \ \inf\{y^T Be_S | \, A^T y \geq c, \ y \geq 0\}$$

we replaced the minimum by the *infimum* because we have an infinite number of restrictions. The set of feasible dual solutions, F_{dS}, may now be empty and the same holds for the sets of optimal solutions O_{pS} and O_{dS}. The optimal values are

$$w_{pS} = \sup\{x^T c | \, x \in F_{pS}\}$$
$$w_{dS} = \inf\{y^T Be_S | \, y \in F_{dS}\}.$$

Once again we define two games, the LP game (N, v_p) and the dual game (N, v_d) where $v_p(S) = w_{pS}$ and $v_d(S) = w_{dS}$. Notice that in this setting the values w_{pS} and w_{dS} may take any nonnegative number *including* $+\infty$. Several nice properties of these games are mentioned in the next theorem.

Theorem 3.1 *Let* (N, A, B, c) *be a semi-infinite LP situation. Then*

(a) $F_{dS} = F_{dN}$ *for all* $S \subset N$;

(b) v_p *and* v_d *are monotonic games; and*

(c) $v_p(S) \leq v_d(S)$ *for all* $S \subset N$.

Proof. First, by definition it holds that $F_{dS} = \{y \in \mathbb{R}^R | A^T y \geq c,\ y \geq 0\} = F_{dN}$ for all $S \subset N$.

Second, let $S \subset T \subset N$ be coalitions of agents. Then a game (N, v) is monotonic if $v(S) \leq v(T)$. Here, $Be_S \leq Be_T$ implies that $F_{pS} \subset F_{pT}$ and so $v_p(S) = \sup\{x^T c | \, x \in F_{pS}\} \leq v_p(T)$. From the first part of this proof it follows that $F_{dS} = F_{dT}$ and together with $Be_S \leq Be_T$ this gives $v_d(S) = \inf\{y^T Be_S | y \in F_{dS}\} \leq v_d(T)$.

Third, let $S \subset N$ be a coalition of agents. If $F_{dS} = \emptyset$ then $v_p(S) \leq \infty = v_d(S)$. Otherwise, take feasible solutions $x \in F_{pS}$ and $y \in F_{dS}$. Then $x^T c = c^T x \leq y^T Ax \leq y^T Be_S$ and therefore $v_p(S) = \sup\{x^T c | \, x \in F_{pS}\} \leq \inf\{y^T Be_S | \, y \in F_{dS}\} = v_d(S)$. □

We use these properties to prove the next results about the relations between the Owen set and the cores of the LP and dual games.

Theorem 3.2 *Let* (N, A, B, c) *be a semi-infinite LP situation. Then*

(a) $\text{Owen}(A, B, c) \subset C(v_d)$*; and*

(b) if $v_p(N) = v_d(N)$ *then* $C(v_d) \subset C(v_p)$.

Proof. To show the first item, if $\text{Owen}(A, B, c) = \emptyset$ then we are finished. Otherwise, take an element $z \in \text{Owen}(A, B, c)$. Then there exists an optimal dual solution $y' \in O_{dN}$ such that $z_i = (y')^T Be_{\{i\}}$ for all $i \in N$. So, $\sum_{i \in N} z_i = \sum_{i \in N} (y')^T Be_{\{i\}} = (y')^T Be_N = v_d(N)$ because $y' \in O_{dN}$. It also holds that $\sum_{i \in S} z_i = (y')^T Be_S \geq \inf\{y^T Be_S | \, y \in F_{dS}\} = v_d(S)$ where the inequality follows from $y' \in F_{dS}$. We conclude that $z \in C(v_d)$.

For the second item, we are finished if $C(v_d) = \emptyset$. Otherwise take an element $z \in C(v_d)$. By definition it holds that $\sum_{i \in N} z_i = v_d(N) = v_p(N)$. It also holds that $\sum_{i \in S} z_i \geq v_d(S) \geq v_p(S)$ where the first inequality follows from $z \in C(v_d)$ and the second one from statement 3 in Theorem 3.1. Hence, $z \in C(v_p)$. □

A corollary of this theorem is that if $v_p(N) = v_d(N)$, that is, there is no duality gap, then $\text{Owen}(A, B, c) \subset \text{Core}(A, B, c)$. In the first part of the proof we noticed that $\text{Owen}(A, B, c) = \emptyset$ may hold. The following example provides a semi-infinite LP situation where this is true and where the cores of the two games are nonempty.

Example 3.1 Consider the semi-infinite LP situation (N, A, B, c) where N is a set of one agent,

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 & \dots \\ 1 & 4 & \dots & n^2 & \dots \end{bmatrix}, \; Be_N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{and}$$

$$c^T = \begin{bmatrix} 2 & 4 & \dots & 2n & \dots \end{bmatrix}.$$

Then

$$\begin{aligned} v_d(N) &= \inf\{y^T Be_N | \, A^T y \geq c, \; y \geq 0\} \\ &= \inf\{y_2 | \, y_1 + n^2 y_2 \geq 2n, \; n = 1, 2, \dots, \; y \geq 0\} \\ &= 0, \end{aligned}$$

$F_{dN} \neq \emptyset$ but $O_{dN} = \emptyset$. Consequently, $\text{Owen}(A, B, c) = \emptyset$. However, $v_p(N) = 0$ implies that $\text{Core}(A, B, c) = C(v_p) = \{0\} \neq \emptyset$. Similarly we can show that $C(v_d) = \{0\}$.

Two other relations between the Owen set and the core, depending on the values $v_p(N)$ and $v_d(N)$, are presented in the next theorem.

Theorem 3.3 *Let (N, A, B, c) be a semi-infinite LP situation. Then*

(a) if $v_p(N) < v_d(N) < \infty$ then $\text{Owen}(A, B, c) \cap \text{Core}(A, B, c) = \emptyset$; and

(b) if $v_p(N) < v_d(N) = \infty$ then $\text{Owen}(A, B, c) = \emptyset$ and the core $\text{Core}(A, B, c)$ is nonempty.

Proof. Concerning the first item, if $\text{Owen}(A, B, c) = \emptyset$ then the proof is finished. Otherwise, let $z \in \text{Owen}(A, B, c)$ and take $y \in O_{dN}$ such that $z_i = y^T Be_{\{i\}}$ for all $i \in N$. Then $\sum_{i \in N} z_i = \sum_{i \in N} y^T Be_{\{i\}} = y^T Be_N = v_d(N) > v_p(N)$. Hence, $z \notin C(v_p) = \text{Core}(A, B, c)$.

Secondly, since Be_N contains finite quantities, $v_d(N) = \infty$ can occur only if $F_{dN} = \emptyset$. In this case, $O_{dN} = \emptyset$ and therefore $\text{Owen}(A, B, c) = \emptyset$. The latter part of this statement, the nonemptiness of $\text{Core}(A, B, c)$, will be shown in Theorem 3.5. □

All the above relations between the Owen set and the core of a semi-infinite LP situation can be summarized as follows.

Theorem 3.4 *Let (N, A, B, c) be a semi-infinite LP situation.*

(a) If $v_p(N) = v_d(N)$ then $\text{Owen}(A, B, c) \subset \text{Core}(A, B, c)$.

(b) If $v_p(N) < v_d(N)$ then $\text{Owen}(A, B, c) \cap \text{Core}(A, B, c) = \emptyset$.

Proof. The proof follows immediately from the Theorems 3.2 and 3.3. □

As we stated in the second part of the proof of Theorem 3.3 there is one thing left to show, namely that the core of a semi-infinite LP situation is nonempty whenever the "profit" of the grand coalition is finite, $v_p(N) < \infty$.

Theorem 3.5 *Let (N, A, B, c) be a semi-infinite LP situation where the corresponding LP game (N, v_p) has $v_p(N) < \infty$. Then Core$(A, B, c) \neq \emptyset$.*

Proof. This proof is an exhaustive list of all possible semi-infinite LP situations that we may come across. In each of these situations we will show that if $v_p(N)$ is finite then Core(A, B, c) is a nonempty set.

First, suppose that $Be_N = 0$, where 0 denotes the vector with each element equal to zero. Thus, all the agents have no resources available. But then no producer can produce a positive quantity of any product, so $F_{pS} = \{0\}$ for all coalitions S and consequently $v_p(S) = 0$. In particular, $v_p(N) = 0 < \infty$ and Core$(A, B, c) = C(v_p) = \{(0, \ldots, 0)\} \neq \emptyset$.

What happens if $Be_N \neq 0$ but every product needs a resource that is not available? Let $h(t)$ describe for all resource vectors $t \in \mathbf{R}_+^R$ those resources that are available in a positive quantity, so, $h(t) = \{i \in R | \, t_i > 0\}$. Denote by e'_j the jth unit vector in $\mathbf{R}^Q$ with $e'_{j,l} = 1$ if $l = j$ and $e'_{j,l} = 0$ otherwise. Then Ae'_j is a vector in $\mathbf{R}_+^R$ that describes how much we need of each resource to produce one unit of product $j \in Q$. Thus, $h(Be_N) \not\supset h(Ae'_j)$ for all $j \in Q$ means that each product $j \in Q$ needs some unavailable resources. Consequently, no producer can produce a positive quantity of some product, $F_{pS} = \{(0, 0, \ldots)\}$ and $v_p(S) = 0$ for all coalitions S of producers. In particular, $v_p(N) = 0 < \infty$ and Core$(A, B, c) = \{(0, \ldots, 0)\} \neq \emptyset$.

Assume now that $Be_N \neq 0$ and that some products can be produced, that is, $h(Be_N) \supset h(Ae'_j)$ for some $j \in Q$. All coalitions of producers want to maximize their profit and therefore they will restrict their production to the products that can be produced. So, without changing the values of the coalitions we remove all products $j \in Q$ that cannot be produced, that is, for which $h(Be_N) \not\supset h(Ae'_j)$, as well as all unavailable resources $i \in R$, which have $(Be_N)_i = 0$. For simplicity of notation, let (N, A, B, c) also denote this reduced semi-infinite LP situation.

This brings us to the next situation where $Be_N > 0$ and consequently, $h(Be_N) = M \supset h(Ae'_j)$ for all $j \in Q$. What happens if $c = 0$, prices are zero? If all products have a price equal to zero then anything a producer sells on the market will give him a revenue of zero. So, $v_p(S) = 0$ for all coalitions S of producers and in particular it holds that $v_p(N) = 0 < \infty$ and $\text{Core}(A, B, c) \neq \emptyset$.

If $Be_N > 0$ and there is a product $j \in Q$ for which $c_j > 0$ then we can remove all products j for which $c_j = 0$ without changing any of the values $v_p(S)$. This holds because each coalition of producers will restrict its production to the products with a positive price.

This leads to $Be_N > 0$ and $c > 0$. If there exists a product $j \in Q$ that uses no resources, $Ae'_j = 0$, then the producers can produce an infinite amount of this good, because it needs no input, and sell it at price $c_j > 0$ to obtain an infinite profit. Hence, $v_p(N) = \infty$ and we may say that the producers are in heaven since they can take as much of the profit as they want.

Finally, we end up with $Be_N > 0$, $c > 0$ and $Ae'_j \neq 0$ for all $j \in Q$. In this case we use a theorem of Tijs ([6]) that says that we either have $O_{dN} = \emptyset$ and $v_p(N) = v_d(N) = \infty$ (heaven once again), or $v_p(N) = v_d(N) < \infty$ and $O_{dN} \neq \emptyset$. In the latter case $\text{Owen}(A, B, c) \neq \emptyset$, which implies that $\text{Core}(A, B, c) \neq \emptyset$. □

We may conclude from this theorem that if $v_p(N) < \infty$ then there exists a core-allocation, a division of the value $v_p(N)$ upon which no coalition S can improve. If we are in the heavenly situation $v_p(N) = \infty$, then we do not need shadow prices or core-allocations since any producer can get what he wants from $v_p(N)$, even if it is an infinitely large amount.

4 FINITE LTP SITUATIONS

Another kind of linear production is described by situations involving the linear transformation of products (LTP), where the "T" stands for the *transformation* of a set of input goods into a set of output goods. Timmer, Borm and Suijs ([7]) show that an LP situation is a special kind of LTP situation.

In LTP situations each transformation technique may have more than one output good. Recall that each production process in an LP situation has only one output good, namely its product. Furthermore, different producers may have different transformation techniques at their disposal, while in an LP situation all producers use the same set of production techniques. LTP situations are introduced in Timmer, Borm and Suijs ([7]) and defined as follows.

Let M be the finite set of goods and N the finite set of producers. Producer $i \in N$ owns the bundle of goods $\omega(i) \in \mathbb{R}^M_+$ and we assume, as we do for LP situations, that all producers together own something of each good, that is, $\sum_{i \in N} \omega(i) > 0$. We do make this assumption although in this model there need not be a clear distinction between input and output goods. A good may be an output good of one transformation technique while it is an input good of another technique. A *transformation technique* is described by a vector $a \in \mathbf{R}^M$, for example

$$a = \begin{bmatrix} 5 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

if M contains four goods. Positive elements in such a vector a indicate that the corresponding good is an output of the transformation technique, negative elements indicate input goods and zero means that the corresponding good is not used in this technique. In this example, the first and third good are outputs of the transformation process, the fourth good is an input and the second good is not used. More precisely, 3 units of the fourth good can be transformed into 5 units of the first good and 1 unit of the third good. We assume that each transformation technique uses at least one good to produce another good, so, it contains at least one positive and one negative element.

Denote by D_i the finite set of transformation techniques of producer $i \in N$. Then $k \in D_i$ means that producer i can use technique a^k. The set of all transformation techniques is $D = \cup_{i \in N} D_i$. We assume that all producers are price-takers and that all goods can be sold at the exogenous market prices $p \in \mathbf{R}_+^M \setminus \{0\}$. All transformation techniques are *linear*, so, $2a^k$ means that twice the amount of input is used to produce twice the amount of output with technique k. The factor 2 is called the activity level of technique k. Denote by $y = (y_k)_{k \in D}$ the vector of activity levels. Because we cannot reverse any transformation process, all activity levels are nonnegative. The *transformation matrix* $A \in \mathbf{R}^{M \times D}$ is the matrix with transformation technique a^k at column k. Related to this is the matrix $G \in \mathbf{R}_+^{M \times D}$ that describes which and how many of the goods are needed as inputs in the various transformation techniques. For all $j \in M$ and $k \in D$ we have $G_{jk} = g_j^k = \max\{0, -a_j^k\}$. From this it follows that $(a^k + g^k)_j = \max\{a_j^k, 0\}$, so the vector $a^k + g^k$ describes which and how many of the goods are outputs in technique k. Thus, when technique k has activity level $y_k \geq 0$ then the vector $g^k y_k$ describes the amount of input goods we need and $(a^k + g^k)y_k$ describes the output of this transformation technique.

Consider first a single producer $i \in N$. He should choose his activity vector y such that the amount of goods he needs does not exceed the amount of goods he owns, $Gy \leq \omega(i)$. Furthermore, this producer can only use his own transformation techniques. Therefore $y_k = 0$ if $k \notin D_i$. The amount of output of the transformation techniques will be $(A + G)y$. We see that the producer started with $\omega(i)$ from which he uses Gy as inputs and he obtains $(A + G)y$ as outputs, so he can sell the goods that remain after the transformation, $\omega(i) - Gy + (A + G)y = \omega(i) + Ay$, on the market. His goal is to maximize his profit $p^T(\omega(i) + Ay)$ such that the activity vector y is feasible:

$$\max\{p^T(\omega(i) + Ay) |\, Gy \leq \omega(i),\ y \geq 0,\ y_k = 0 \text{ if } k \notin D_i\}.$$

Producers are also allowed to work together. When they cooperate then they will pool their techniques and their bundles of goods. A coalition $S \subset N$ of producers has the bundle $\omega(S) = \sum_{i \in S} \omega(i)$ at its disposal and it can use all the transformation techniques in $D(S) = \cup_{i \in S} D_i$. The profit maximization problem of such a coalition is similar to that of a single producer and equals

$$\max\{p^T(\omega(S) + Ay) |\, Gy \leq \omega(S),\ y \geq 0,\ y_k = 0 \text{ if } k \notin D(S)\}.$$

When we want to determine the dual problem of this profit-maximization problem then the last constraint, $y_k = 0$ if $k \notin D(S)$, gives some trouble. We will replace this constraint by another one with the same interpretation. For this, define for all $k \in D$ and $S \subset N$, $S \neq \emptyset$

$$\beta(S)_k = \begin{cases} \infty & , k \in D(S) \\ 0 & , k \notin D(S). \end{cases}$$

This vector $\beta(S)$ gives an upper bound for the activity vector that can be chosen by coalition S and it implies that

$$\begin{cases} y_k = 0 \text{ if } k \notin D(S) \\ y \geq 0 \end{cases} \Leftrightarrow \begin{cases} y \leq \beta(S) \\ y \geq 0. \end{cases}$$

The (primal) maximization problem P_S for coalition S can thus be rewritten to

$$P_S: \ \max\{p^T(\omega(S) + Ay) |\, Gy \leq \omega(S),\ y \leq \beta(S),\ y \geq 0\}.$$

Because of the vector $\beta(S)$ it is now very easy to determine the dual program D_S of P_S (cf. [9]):

$$D_S: \ \min\left\{ (z_M + p)^T \omega(S) + z_D^T \beta(S) \;\middle|\; \begin{array}{c} G^T z_M + z_D \geq A^T p, \\ z_M \geq 0,\ z_D \geq 0 \end{array} \right\}.$$

The vector $A^T p \in \mathbf{R}^D$ denotes the profits for all transformation techniques per activity level. The matrix G is denoted in units of goods per activity level. Therefore, the vector $z_M \in \mathbf{R}^M$ is denoted in units of dollars per good an the vector $z_D \in \mathbf{R}^D$ in dollars per activity level. A nice interpretation for the vector z_M follows from the *complementary slackness conditions*: if $\hat{y}$, $\hat{z}_M$ and $\hat{z}_D$ are optimal solutions of the primary and dual programs of coalition S then

$$0 = \hat{z}_M^T[\omega(S) - G\hat{y}], \tag{4.1}$$

$$0 = \hat{z}_D^T[\beta(S) - \hat{y}] \tag{4.2}$$

and

$$0 = \hat{y}^T[G^T\hat{z}_M + \hat{z}_D - A^T p].$$

Equation (4.1) is equal to $\sum_{j \in M} \hat{z}_{M,j}(\omega(S) - G\hat{y})_j = 0$. This sum of nonnegative elements is zero if and only if each element equals zero. So, for all goods $j \in M$ it holds that $\hat{z}_{M,j}(\omega(S) - G\hat{y})_j = 0$. If $\hat{z}_{M,j} > 0$ then $\omega(S)_j = (G\hat{y})_j$: the available amount of good j is precisely enough to cover the amount of good j that is needed. From the objective function of the dual program D_S it follows that an extra unit of good j will raise the profit by $\hat{z}_M$ because duality theory says that the optimal values of P_S and D_S are equal. If, on the other hand, the amount of good j available is too large, $\omega(S)_j > (G\hat{y})_j$, then $\hat{z}_{M,j} = 0$: an extra unit of good j will not raise the profit. We can therefore think of $\hat{z}_M$ as the vector of prices that the coalition S of producers is willing to pay for an extra unit of the goods. We will call the vector $\hat{z}_M + p$ the vector of *shadow prices* for the goods of this coalition. The following theorem shows a nice result that follows from (4.2).

Theorem 4.1 *The equality $\hat{z}_D^T\beta(S) = 0$ holds for all optimal solutions $(\hat{z}_M, \hat{z}_D)$ of D_S.*

Proof. Because the set of feasible solutions of D_S is the nonempty intersection of a finite number of halfspaces that is bounded from below by the zero-vector, the program D_S can be solved and a minimum exists. Let $(\hat{z}_M, \hat{z}_D)$ be an optimal solution. By the complementary slackness conditions equation (4.2) holds and is equal to $\sum_{k \in D} \hat{z}_{D,k}(\beta(S) - \hat{y})_k = 0$. Again, this is a sum of nonnegative elements, so it should hold that $\hat{z}_{D,k}(\beta(S) - \hat{y})_k = 0$ for all transformation techniques $k \in D$. If $\hat{z}_{D,k} > 0$ then $\beta(S)_k = \hat{y}_k$. The definition of $\beta(S)$ implies that in this case $\beta(S)_k = 0$, so $k \notin D(S)$. If $\beta(S)_k > \hat{y}_k$, which is equivalent to $k \in D(S)$, then $\hat{z}_{D,k} = 0$. We conclude that $\hat{z}_{D,k}\beta(S)_k = 0$ for all transformation techniques $k \in D$. □

For ease of notation let F_{pS} and F_{dS} denote the sets of feasible solutions of respectively the primal and the dual program for coalition S,

$$F_{pS} = \{y \in \mathbf{R}^D | \, Gy \leq \omega(S),\ y \leq \beta(S),\ y \geq 0\},$$
$$F_{dS} = \{(z_M, z_D) \in \mathbf{R}^M \times \mathbf{R}^D | \, G^T z_M + z_D \geq A^T p,\ z_M \geq 0, z_D \geq 0\},$$

denote by u_{pS} and u_{dS} the optimal values of the programs,

$$u_{pS} = \max\{p^T(\omega(S) + Ay) | \, y \in F_{pS}\},$$
$$u_{dS} = \min\{(z_M + p)^T\omega(S) + z_D^T\beta(S) | \, (z_M, z_D) \in F_{dS}\},$$

and let O_{pS} and O_{dS} be the sets of optimal solutions,

$$O_{pS} = \{y \in F_{pS} | \, p^T(\omega(S) + Ay) = u_{pS}\},$$
$$O_{dS} = \{(z_M, z_D) \in F_{dS} | \, (z_M + p)^T \omega(S) + z_D^T \beta(S) = u_{dS}\}.$$

The sets F_{pS}, F_{dS}, O_{pS} and O_{dS} are nonempty and the values u_{pS} and u_{dS} exist and are finite. By duality theory it holds that $u_{pS} = u_{dS}$ for any coalition S of producers.

An LTP situation will be described by the tuple (N, A, D, ω, p) where $\omega = (\omega(i))_{i \in N}$. Corresponding to an LTP situation we define two cooperative games. The first one, (N, v_p), is the LTP game as defined in Timmer, Borm and Suijs ([7]) where $v_p(S) = u_{pS}$, the maximal profit that coalition S can obtain. The second one is the dual game (N, v_d) that gives each coalition S the value of its dual program, $v_d(S) = u_{dS}$.

The core of an LTP situation, Core(A, ω, p), is defined as the core of an LTP game, Core$(A, \omega, p) = C(v_p)$. Furthermore, we know that for all $(z_M, z_D) \in O_{dN}$

$$v_p(N) = v_d(N) = (z_M + p)^T \omega(N) + z_D^T \beta(N) = (z_M + p)^T \omega(N),$$

where the last equality follows from theorem 4.1. Timmer, Borm and Suijs ([7]) show that $((z_M + p)^T \omega(i))_{i \in N} \in C(v_p)$. Thus it follows from $O_{dN} \neq \emptyset$ that LTP games are totally balanced: each LTP game has a nonempty core and because each subgame is another LTP game, these subgames also have a nonempty core. Although G. Owen did not show that you can find a core-element of an LTP game via the dual program D_N, we let Owen(A, ω, p) denote the set of all core-elements that we can find in this way:

$$\text{Owen}(A, \omega, p) = \{((z_M + p)^T \omega(i))_{i \in N} | \, (z_M, z_D) \in O_{dN}\}.$$

From $O_{dN} \neq \emptyset$ it follows that Owen$(A, \omega, p) \neq \emptyset$ and Owen$(A, \omega, p) \subset$ Core(A, ω, p).

The following example of an LTP situation with its two corresponding games illustrates the concepts introduced in this section.

Example 4.1 Consider the following LTP situation. There are two producers, $N = \{1, 2\}$. They work with two goods in their transformation techniques and

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}, \; \omega(1) = \omega(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The first column of A contains the technique of producer 1 and the second column contains the technique of the other producer, so, $D_i = \{i\}$, $i \in N$.

When each of the producers works on her own then she will transform her single unit of the first good into respectively 2 and 3 units of the second good. This producer already owns a unit of the second good and therefore $v_p(\{1\}) = 3$ and $v_p(\{2\}) = 4$.

When the producers cooperate then they own $\omega(N) = (2,2)^T$ and their set of feasible activity vectors is

$$F_{pN} = \{y \in \mathbf{R}^2 | \, y_1 + y_2 \leq 2, \; y \geq 0\}.$$

Producer 2 has a more efficient transformation technique than producer 1 because it generates a larger profit from the same amount of input, namely 2 dollars per activity level against 1 dollar per activity level for producer 1.

$$P_N: \; \max\{4 + y_1 + 2y_2 | \, y \in F_{pN}\}$$

The maximal profit $u_{pN} = 8$ is attained in $y = (0,2)^T$, so $O_{pN} = \{(0,2)^T\}$. The core equals $\text{Core}(A, \omega, p) = \{(b, 8-b) | \, 3 \leq b \leq 4\}$.

For the dual game (N, v_d) it holds that $v_d(\{1\}) = 3$ and $v_d(\{2\}) = 3$. The set of feasible solutions of D_N is

$$F_{dN} = \{(z_M, z_D) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+ | \, z_{M,1} + z_{D,1} \geq 1, \; z_{M,1} + z_{D,2} \geq 2\}.$$

When we solve the program

$$D_N: \; \min\{4 + 2z_{M,1} + 2z_{M,2} + \infty(z_{D,1} + z_{D,2}) | \, (z_M, z_D) \in F_{dN}\},$$

then we get $O_{dN} = \{((2,0),(0,0))\}$ and $u_{dN} = 8 = u_{pN}$. Thus the Owen set consists of only one point, $\text{Owen}(A, \omega, p) = \{(4,4)\}$ and is contained in $\text{Core}(A, \omega, p)$

5 SEMI-INFINITE LTP SITUATIONS

In this section we will study semi-infinite LTP situations where the set D contains a countable infinite number of transformation techniques. Without loss of generality we assume that $D = \{1, 2, 3, \ldots\}$. A *semi-infinite LTP situation* (N, A, D, ω, p) thus has $A \in \mathbf{R}^{M \times D}$, $\omega(i) \in \mathbf{R}^M_+$ for all $i \in N$ and $p \in \mathbf{R}^M_+$. As opposed to the previous section, we do not put any further restrictions on A, ω and p.

Because of the infinite number of transformation techniques, the linear programs that determine the maximal profits of the coalitions and their dual programs are semi-infinite linear programs. Therefore, we will replace the maximum by the supremum in the definitions of P_S and u_{pS} and the minimum will be replaced by the infimum in the definitions of D_S and u_{dS}. As opposed to finite LTP situations, the set of feasible dual solutions F_{dS} may now be empty and the same holds for the sets of optimal solutions O_{pS} and O_{dS}. The two games (N, v_p) and (N, v_d) are defined as before, so, $v_p(S) = u_{pS}$ for the LTP game and $v_d(S) = u_{dS}$ for the dual game. In this semi-infinite situation the values u_{pS} and u_{dS} can take any nonnegative value as well as $+\infty$.

The Owen set, as defined in the previous section, is based on the dual program for the grand coalition:

$$D_N : \inf\left\{ (z_M + p)^T \omega(N) + z_D^T \beta(N) \;\middle|\; \begin{array}{c} G^T z_M + z_D \geq A^T p, \\ z_M \geq 0,\ z_D \geq 0 \end{array} \right\}.$$

In our definition of the Owen set we use that for finite LTP situations it holds that $z_D^T \beta(N) = 0$ for any optimal solution (z_M, z_D) of D_N. But this property need not hold for semi-infinite LTP situations. When $u_{dN} = \infty$ then an optimal solution (z_M, z_D) (if it exists) has $z_D^T \beta(N) = \infty$ but when $u_{dN} < \infty$ then $z_D^T \beta(N) = 0$. For this reason we will define the Owen set only if $u_{dN} < \infty$:

$$\text{Owen}(A, \omega, p) = \{((z_M + p)^T \omega(i))_{i \in N} | \, (z_M, z_D) \in O_{dN}\}.$$

The next theorem states some nice properties of the LTP and dual games.

Theorem 5.1 *Let (N, A, D, ω, p) be a semi-infinite LTP situation. Then*

(a) $F_{dS} = F_{dN}$ for all $S \subset N$;

(b) v_p and v_d are monotonic games; and

(c) $v_p(S) \leq v_d(S)$ for all $S \subset N$.

Proof. For the first item, by definition $F_{dS} = \{(z_M, z_D) \in \mathbb{R}_+^M \times \mathbb{R}_+^D | \, G^T z_M + z_D \geq A^T p\} = F_{dN}$ for all $S \subset N$.

To show the second item, let $S \subset T \subset N$, then $\omega(S) \leq \omega(T)$ and $\beta(S) \leq \beta(T)$. So, $F_{pS} = \{y \in \mathbb{R}^D | \, Gy \leq \omega(S),\ y \leq \beta(S),\ y \geq 0\} \subset F_{pT}$ and therefore $v_p(S) = \sup\{p^T(\omega(S) + Ay) | \, y \in F_{pS}\} \leq v_p(T)$. From the first part of this proof it follows that $F_{dS} = F_{dT}$ and together with $\omega(S) \leq \omega(T)$ and $\beta(S) \leq \beta(T)$ this implies that $v_d(S) = \inf\{(z_M + p)^T \omega(S) + z_D^T \beta(S) | \, (z_M, z_D) \in F_{dS}\} \leq v_d(T)$.

Finally, for the third item, let S be a coalition of producers. If $F_{dS} = \emptyset$ then $v_p(S) \leq \infty = v_d(S)$. Otherwise, take feasible solutions $y \in F_{pS}$ and $(z_M, z_D) \in F_{dS}$. Then $p^T(\omega(S) + Ay) = p^T \omega(S) + y^T A^T p \leq p^T \omega(S) + y^T(G^T z_M + z_D) = p^T \omega(S) + z_M^T G y + z_D^T y \leq p^T \omega(S) + z_M^T \omega(S) + z_D^T \beta(S) = (z_M + p)^T \omega(S) + z_D^T \beta(S)$ and from this it follows that $v_p(S) = \sup\{p^T(\omega(S) + Ay) | \, y \in F_{pS}\} \leq \inf\{(z_M + p)^T \omega(S) + z_D^T \beta(S) | \, (z_M, z_D) \in F_{dS}\} = v_d(S)$. $\square$

Some relations between the Owen set and the cores of the LTP and dual games are stated below.

Theorem 5.2 *Let (N, A, D, ω, p) be a semi-infinite LTP situation. Then the following two relations hold:*

(a) If $v_d(N) < \infty$ then Owen$(A, \omega, p) \subset C(v_d)$.

(b) If $v_p(N) = v_d(N)$ then $C(v_d) \subset C(v_p)$.

Proof. For item (a), if Owen$(A, \omega, p) = \emptyset$ then the result holds. Otherwise, let $x \in$ Owen(A, ω, p). Then there exists a solution $(z'_M, z'_D) \in O_{dN}$ such that $x_i = (z'_M + p)^T \omega(i)$ for all $i \in N$. By definition, $\sum_{i \in N} x_i = \sum_{i \in N} (z'_M + p)^T \omega(i) = (z'_M + p)^T \omega(N) = (z'_M + p)^T \omega(N) + (z'_D)^T \beta(N) = v_d(N)$, where $(z'_D)^T \beta(N) = 0$ because $v_d(N) < \infty$. Second, $(z'_D)^T \beta(N) = 0$ implies $(z'_D)^T \beta(S) = 0$ because $\beta(N) \geq \beta(S)$. Also, $(z'_M, z'_D) \in O_{dN} \subset F_{dN} = F_{dS}$, where the last equality follows from Theorem 5.1(a). Thus, $\sum_{i \in S} x_i = (z'_M + p)^T \omega(S) = (z'_M + p)^T \omega(S) + (z'_D)^T \beta(S) \geq \inf\{(z_M + p)^T \omega(S) + z_D^T \beta(S) \mid (z_M, z_D) \in F_{dS}\} = v_d(S)$. Hence, $x \in C(v_d)$.

For item (b), if $C(v_d) = \emptyset$ then we are done. Otherwise, take an element $x \in C(v_d)$. By definition it holds that $\sum_{i \in N} x_i = v_d(N) = v_p(N)$. Furthermore, $\sum_{i \in S} x_i \geq v_d(S) \geq v_p(S)$ by Theorem 5.1(c). We conclude that $x \in C(v_p)$. □

A consequence of this theorem is that if $v_p(N) = v_d(N) < \infty$ then Owen$(A, \omega, p) \subset$ Core(A, ω, p). We can now have Owen$(A, \omega, p) = \emptyset$ even if $v_p(N) = v_d(N)$, as the following example shows.

Example 5.1 Consider the semi-infinite LTP situation (N, A, D, ω, p), where N is a set of one player,

$$A = \begin{bmatrix} -1 & -4 & & -k^2 & \\ -1 & -1 & \cdots & -1 & \cdots \\ 1 & 9/2 & & (k+1)^2/2 & \end{bmatrix}, \; p = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \text{ and}$$

$$\omega(N) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Then

$$\begin{aligned} v_d(N) &= \inf\left\{ (z_M + p)^T \omega(N) + z_D^T \beta(N) \;\middle|\; \begin{array}{l} G^T z_M + z_D \geq A^T p, \\ z_M \geq 0, \; z_D \geq 0 \end{array} \right\} \\ &= \inf\left\{ z_{M,1} + 2 z_{M,3} + 5 + \infty \sum_{k \in D} z_{D,k} \;\middle|\; \begin{array}{c} k^2 z_{M,1} + z_{M,2} + z_{D,k} \\ \geq 2k, \; k = 1, 2, \ldots, \\ z_M \geq 0, \; z_D \geq 0 \end{array} \right\} \\ &= 5, \end{aligned}$$

where $F_{dN} \neq \emptyset$, but $O_{dN} = \emptyset$ and this implies that Owen$(A, \omega, p) = \emptyset$. There is no duality gap in this example because $v_p(N) = 5 = v_d(N)$.

In case of a duality gap, $v_p(N) < v_d(N)$, another relation between the Owen set and the core exists.

Theorem 5.3 *Let (N, A, D, ω, p) be a semi-infinite LTP situation where $v_p(N) < v_d(N) < \infty$. Then $Owen(A, \omega, p) \cap Core(A, \omega, p) = \emptyset$.*

Proof. The proof of this theorem goes analogously to the proof of the first part in Theorem 3.3. □

Finally, we obtain the same result as for semi-infinite LP, namely, that if $v_p(N)$ is finite in a semi-infinite LTP situation then the core is nonempty. For this, we need two intermediate theorems. The first one is a theorem by Karlin and Studden ([3]), which we translated to semi-infinite LTP situations.

Theorem 5.4 *Suppose that $v_p(N)$ is finite and that $\omega_j(N) > 0$ for all $j \in M$. Then there is no duality gap, $v_p(N) = v_d(N)$, and the dual program D_N has an optimal solution.*

The second intermediate theorem shows that we have no duality gap, $v_p(N) = v_d(N)$, and $C(v_p) \neq \emptyset$ if certain conditions hold.

Theorem 5.5 *If $\omega(N) \in \mathbb{R}_+^M \setminus \{0\}$, $p \in \mathbb{R}_+^M \setminus \{0\}$, $\omega_j(N) = 0 \Rightarrow g_j^k = 0$ for all $k \in D$, $p^T a^k > 0$ for all $k \in D$, $a^k \notin \mathbb{R}_+^M$ for all $k \in D$ and $v_p(N) < \infty$, then $v_p(N) = v_d(N)$, $O_{dN} \neq \emptyset$ and $C(v_p) \neq \emptyset$.*

Proof. If $\omega_j(N) > 0$ for all $j \in M$ then together with $v_p(N) < \infty$ and Theorem 5.4 it follows that there is no duality gap and there exists an optimal dual solution $\hat{z}$. Define $x \in \mathbb{R}^N$ by $x_i = (\hat{z} + p)^T \omega(i)$ for all $i \in N$. We leave it to the reader to show that $x \in C(v_p)$.

If $\omega_j(N) = 0$ for some $j \in M$ then define $M_0 = \{j \in M \mid \omega_j(N) = 0\}$ and $M_+ = \{j \in M \mid \omega_j(N) > 0\}$. Then $M_0 \neq \emptyset$ and $M_+ \neq \emptyset$. Now the primal problem can be rewritten to

$$v_p(N) = p^T \omega(N) + \sup \left\{ \sum_{k \in D} p^T a^k y_k \;\middle|\; \begin{array}{l} \sum_{k \in D} g_j^k y_k \leq \omega_j(N), \; j \in M_+, \\ y \geq 0 \end{array} \right\}$$

and similarly, we obtain for the dual problem

$$v_d(N) = p^T \omega(N) + \inf \left\{ \sum_{j \in M_+} z_j \omega_j(N) \;\middle|\; \begin{array}{l} \sum_{j \in M_+} g_j^k z_j \geq p^T a^k, \; k \in D, \\ z_j \geq 0, \; j \in M_+ \end{array} \right\}$$

where we observe that the assumptions imply that for all $k \in D$ there exists a $j \in M_+$ such that $g_j^k > 0$. Thus, the latter problem is feasible. Let e^j denote the jth unit vector in $\mathbb{R}^{M_+}$, with $e_l^j = 1$ if $l = j$ and $e_l^j = 0$ otherwise. Define the cone K_1 by

$$K_1 = \text{cone}\left(\left(\left\{g_j^k\right\}_{j\in M_+}\right)_{k\in D}, \left(e^j\right)_{j\in M_+}\right) = \mathbf{R}_+^{M_+},$$

where the last equality follows from $g_j^k \geq 0$ for all $j \in M_+, k \in D$. But then

$$\{\omega_j(N)\}_{j\in M_+} \in \text{int}\,(K_1) = \mathbf{R}_{++}^{M_+},$$

where int (K_1) denotes the interior of the cone K_1, because $\omega_j(N) > 0$ for all $j \in M_+$. Together with $v_p(N) < \infty$ and Theorem 5.4 it follows once again that $v_p(N) = v_d(N)$ and there exists an optimal dual solution $\hat{z}$. To obtain an element of the core $C(v_p)$, we define $\underline{z}_j = \hat{z}_j$ for all $j \in M_+$ and $\underline{z}_j = 0$ otherwise. Also, define $x \in \mathbf{R}^N$ by $x_i = (\underline{z} + p)^T \omega(i)$ for all $i \in N$. First,

$$\begin{aligned}\sum_{i\in N} x_i &= \sum_{i\in N} (\underline{z} + p)^T \omega(i) = (\underline{z} + p)^T \omega(N) \\ &= \sum_{j\in M_+} \hat{z}_j\omega_j(N) + p^T\omega(N) = v_d(N) = v_p(N).\end{aligned}$$

Second, let $S \subset N$, $S \neq \emptyset$, be a coalition of players. Notice that $\omega_j(N) = 0$ for some $j \in M_0$ implies that $\omega_j(S) = 0$ for all $S \subset N$ because $\omega(S) = \sum_{i\in S}\omega(i)$. Then,

$$\begin{aligned}&(\underline{z} + p)^T \omega(S) = p^T\omega(S) + \sum_{j\in M_+} \hat{z}_j\omega_j(S) \\ &\geq p^T\omega(S) + \inf\left\{\sum_{j\in M_+} z_j\omega_j(S) \,\middle|\, \begin{array}{l}\sum_{j\in M_+} g_j^k z_j \geq p^T a^k,\ k \in D, \\ z_j \geq 0,\ j \in M_+\end{array}\right\} \\ &= p^T\omega(S) + \sup\left\{\sum_{k\in D} p^T a^k y_k \,\middle|\, \begin{array}{l}\sum_{k\in D} g_j^k y_k \leq \omega_j(S),\ j \in M_+, \\ y \geq 0\end{array}\right\} \\ &\geq p^T\omega(S) + \sup\left\{p^T Ay \,|\, Gy \leq \omega(S);\ y_k = 0 \text{ if } k \notin D(S);\ y \geq 0\right\} \\ &= v_p(S).\end{aligned}$$

We conclude that $\sum_{i\in S} x_i = (\underline{z} + p)^T \omega(S) \geq v_p(S)$ and hence, $x \in C(v_p)$. □

With the help of these two theorems we prove our main result about semi-infinite LTP situations, which states that if there exists a finite upper bound for the maximal profit that all producers together can obtain then the core of the LTP game is nonempty.

Theorem 5.6 *Let (N, A, D, ω, p) be a semi-infinite LTP situation and let (N, v_p) be the corresponding LTP game with $v_p(N) < \infty$. Then $C(v_p) \neq \emptyset$.*

Proof. In this proof, we consider one-by-one all the possible semi-infinite LTP situations that we may come across. In each of these situations we show that either $v_p(N) = \infty$ or $C(v_p) \neq \emptyset$.

First, suppose that $\omega(N) = 0$. This implies that $\omega(S) = 0$ for all coalitions S. No coalition of producers can transform any goods or sell any on the market. Hence, $v_p(S) = 0$ for all S and $C(v_p) = \{(0, \ldots, 0)\}$.

Second, consider the situation where $\omega(N) \neq 0$ but every transformation technique k needs a good j for which $\omega_j(N) = 0$. Let $h(t)$ describe for all bundles of goods $t \in \mathbf{R}^M_+$ those goods that are available in a positive quantity, so, $h(t) = \{j \in M \mid t_j > 0\}$. Then $h(\omega(N)) \not\supset h(g^k)$ for all $k \in D$ means that each technique k needs some unavailable goods. Consequently, no coalition S can transform any goods. The only thing it can do is sell its goods at the market and obtain $v_p(S) = p^T\omega(S)$. From $\omega(S) = \sum_{i \in S} \omega(i)$ we derive that the core consists of a single element, $C(v_p) = \{(p^T\omega(1), \ldots, p^T\omega(n))\}$, where $N = \{1, 2, \ldots, n\}$.

Assume now that $\omega(N) \neq 0$ and that some transformation techniques can be used because they only need goods that are available, $h(\omega(N)) \supset h(g^k)$ for some $k \in D$. All the coalitions of producers want to maximize their profit and therefore they will restrict their transformation to those techniques that can be used because the other techniques will not generate any profit. Therefore, without changing the values of the coalitions we remove all the transformation techniques k for which $h(\omega(N)) \not\supset h(g^k)$. If this removal implies that $D(S) = \emptyset$ for some coalition S then define $v_p(U) = p^T\omega(U)$ for all $U \subset S$. For convenience, let (N, A, D, ω, p) also denote this reduced semi-infinite LTP situation.

This leads us to the next situation where $\omega(N) \neq 0$, $h(\omega(N)) \supset h(g^k)$ for all $k \in D$, and also $p = 0$. If all the goods have a price of zero then $v_p(S) = 0$ for all coalitions S and consequently, $C(v_p) = \{(0, \ldots, 0)\}$.

If $\omega(N) \neq 0$, $h(\omega(N)) \supset h(g^k)$ for all $k \in D$, $p \neq 0$ and $p^Ta^k \leq 0$ for all $k \in D$ then no transformation technique gives a positive profit. For all optimal solutions $y \in O_{pS}$ it holds that $p^Ta^ky_k = 0$ for all techniques k. Hence, $v_p(S) = p^T\omega(S)$ for all coalitions S and $C(v_p) = \{(p^T\omega(1), \ldots, p^T\omega(n))\}$.

Now assume that $\omega(N) \neq 0$, $h(\omega(N)) \supset h(g^k)$ for all $k \in D$, $p \neq 0$ and $p^Ta^k > 0$ for some $k \in D$. In the previous situation we have seen that if $p^Ta^k \leq 0$ then in the optimum $p^Ta^ky_k = 0$. This technique k will not have any influence on the profit and so, removal of these techniques will not change the values of the coalitions. Also in this case, we define $v_p(U) = p^T\omega(U)$ for all $U \subset S$ if the removal implies that $D(S) = \emptyset$.

In the next situation, we consider $\omega(N) \neq 0$, $h(\omega(N)) \supset h(g^k)$ for all $k \in D, p \neq 0, p^T a^k > 0$ for all $k \in D$ and $a^k \in \mathbf{R}^M_+$ for some $k \in D$. Notice that for this technique k we have $a^k \in \mathbf{R}^M_+ \setminus \{0\}$, because $a^k = 0$ implies $p^T a^k = 0$, which is in contradiction to $p^T a^k > 0$. If $a^k \in \mathbf{R}^M_+$ then $g^k = 0$, which means that technique k needs no input goods to generate the positive profit $p^T a^k$. Consequently, the coalition N of all players will set the activity level y_k to infinity and so, $v_p(N) = \infty$. The total profit is infinitely large. We may say that we are in heaven because all the producers can take as much of the profit as they want.

Finally, we consider $\omega(N) \neq 0$, $h(\omega(N)) \supset h(g^k)$ for all $k \in D, p \neq 0$, $p^T a^k > 0$ for all $k \in D$ and $a^k \notin \mathbf{R}^M_+$ for all $k \in D$. Notice that $p^T a^k > 0$ implies that $a^k \notin \mathbf{R}^M_-$ for all $k \in D$. Together with $a^k \notin \mathbf{R}^M_+$ we get that each vector a^k contains at least one positive and one negative element. Each transformation technique needs at least one input good to produce at least one output good. Now, two situations may occur. Either we have $v_p(N) = v_d(N) = \infty$, heaven once again, or $v_p(N) < \infty$. In the latter case, Theorem 5.5 shows that the core is a nonempty set. □

6 CONCLUSIONS

We studied the Owen set, the core and relations between these two sets of two types of semi-infinite situations. These are situations involving linear production (LP) and those involving the linear transformation of products (LTP). We showed that if the underlying primal and dual problems of the grand coalition of players have the same value, that is, there is no duality gap, then the Owen set is a (possibly empty) subset of the core. Otherwise, the Owen set and the core have nothing in common. In the case of LTP situations we had to exclude situations where the underlying dual problem takes the value infinite. Finally, we showed that if there exists a finite upper bound of the maximal profit then the core is a nonempty set.

After completing this study, some questions remain. Throughout the paper we use cones consisting of real numbers like $\mathbf{R}^N$ and $\mathbf{R}^M_+$. What would happen if we replace these cones by more general cones? How do the results change if we consider an infinite number of producers (implying an infinite number of production techniques)? And finally, what happens if we assume that the set of production techniques is no longer countable? We intend to study these questions in the near future.

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