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Kooperation bei Geldanlagen

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Abstract. The rate of return earned on a deposit can depend on its term, the amount of money invested in it, or both. Most banks, for example, offer a higher interest rate for longer term deposits. This implies that if one individual has capital available for investment now, but needs it in the next period, whereas the opposite holds for another individual, then they can both benefit from cooperation since it allows them to invest in a longer term deposit. A similar situation arises when the rate of return on a deposit depends on the amount of capital invested in it. Although the benefits of such cooperative behavior may seem obvious to all individuals, the actual participation of an individual depends on what part of the revenues he eventually receives. The allocation of the jointly earned benefits to the investors thus plays an important part in the stability of the cooperation. This paper provides a game theoretical analysis of this allocation problem. Several classes of corresponding deposit games are introduced. For each class, necessary conditions for a nonempty core are provided, and allocation rules that yield core-allocations are examined.

Zusammenfassung. Die Verzinsung einer Geldanlage kann von der Fristigkeit der Anlage, von der Höhe der Anlagesumme oder von beiden Parametern abhängig sein. Die meisten Banken zum Beispiel bieten eine höhere Verzinsung für längerfristige Anlagen an. Daraus folgt, dass, wenn ein Wirtschaftssubjekt jetzt über Kapital

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verfügt, das es erst in der nächsten Periode benötigt, während für ein anderes Wirtschaftssubjekt das Gegentei gilt, beide einen Vorteil aus einer Kooperation ziehen, da sie ihnen erlaubt, in eine längerfristige Anlage zu investieren. Eine ähnliche Situation liegt vor, wenn die Verzinsung der Anlage von der Anlagesumme abhängt. Obwohl die Vorteile solch kooperativen Verhaltens allen Wirtschaftssubjekten offensichtlich erscheinen, hängt die tatsächliche Partizipation eines Wirtschaftssubjekts von dem Anteil ab, den es letztendlich vom Gesamtergebnis erhalten wird. Die Aufteilung der gemeinsam erreichten Vorteile auf die Investoren spielt also eine wichtige Rolle für die Stabilität der Kooperation. Diese Arbeit stellt eine spieltheoretische Analyse dieses Allokationsproblems vor. Einige Klassen von entsprechenden Anlagespielen werden eingeführt. Für jede Klasse werden die notwendigen Bedingungen für das Vorhandensein eines nicht leeren Kerns aufgezeigt und Allokationsregeln für die Aufteilung des Kerns untersucht.

Key words: Cooperative game theory – Capital deposits

Schlüsselwörter: Kooperative Spieltheorie – Geldanlagen

1 Introduction

During their lives, people save part of their income so as to better deal with any (unforeseen) expenses in the future. These savings can be deposited at a bank to obtain some additional earnings. Depending on the type of deposit, the rate of return earned on it can depend on the amount of money invested, the term of the deposit, or both.

Deposit banks, for example, usually pay a higher interest rate when the term of the deposit increases, i.e. there is a so-called term structure of interest rates. This term structure implies that one would prefer long term deposits to short term deposits. However, long term deposits are less liquid so that future consumption needs may prevent an individual from investing in a long term deposit. The ideal deposit would thus be the one that earns the interest of a long term deposit and possesses the liquidity of a short term deposit. Though no deposit bank offers such deposits, they are not completely out of this world. Consider, for instance, two individuals, one having \$1000 to invest for this year only and one having \$1000 to invest for the subsequent year only. Furthermore, suppose that one-year deposits earn 4% interest per year and that two-year deposits earn 6% per year. Then each person individually can invest \$1000 in a one-year deposit only, earning \$40 interest. Now, if they pool their savings, they have \$1000 available for the next two years. Investing in a two-year deposit then earns them \$120, which exceeds the earnings of two one-year deposits. Moreover, each individual maintains his desired level of liquidity (see also Diamond and Dybvig, 1983).

The above example illustrates that under a given term structure of interest rates, individuals can obtain higher returns on investments by cooperating. Starting with the seminal work of Vasicek (1977) and Cox et al. (1985), the finance literature has devoted considerable attention to understanding the determinants of the term structure of interest rates. The ultimate goal is to be able to predict how changes in the underlying variables affect the yield curve. Whereas Vasicek (1977) builds on the no-arbitrage argument, Cox et al. (1985) departs from an intertemporal general equilibrium model with utility maximizing agents. This leads to an equilibrium structure for the spot rates, and consequently also for the term structure of interest rates. In such general equilibrium models agents act individually, i.e. cooperation between individuals is not taken into account. In this paper we take the term structure as given and study how cooperation between individuals can lead to higher returns on deposits. Of course, one could go one step further and try and incorporate possible consequences of our cooperative approach in analyzing term structures. This idea, however, is not exploited in the current exposition which focuses on specific issues that might induce cooperation between investors. Moreover, the current paper not only focuses on the issue of term structures but on more general types of deposit structures. We consider a finite time horizon consisting of a number of periods during which individuals have certain amounts of money available for depositing. There is a number of deposits available, each of which generates revenues that may depend on the term of the deposit or the amount of money invested in it.

In this type of situations, the following issues are prominent. First, what is the optimal strategy, i.e. how should the money optimally be divided over the different available deposits, and, second, what division of the revenues is considered to be acceptable to all cooperating individuals?

Determining the optimal strategy is a combinatorial optimization problem: what combination of deposits earns the highest benefits given the amount of money that the individuals have available for such deposits. This optimization problem, however, is not the main issue of this paper. Instead, we mainly focus on the allocation problem. Although individuals may recognize the benefits of cooperation when depositing their savings, it does not necessarily imply that they are also willing to participate in such a cooperation. The participation of each individual depends, amongst other things, on what share in the revenues he eventually receives. In this regard, the allocation of the revenues plays an important role in establishing an enduring and stable cooperation.

To tackle this allocation problem we turn to cooperative game theory. We model the situation as a cooperative game, called a *deposit game*. In a deposit game, the value of cooperation for a coalition equals the maximal revenue that this group can obtain by pooling their individual savings. In particular, our attention goes out to stability conditions of the grand coalition in which all individuals cooperate. We therefore examine balancedness of deposit games. In particular, we look for (simple) allocation rules to obtain core-allocations. We focus on three special subclasses of deposit games. Each subclass is characterized by properties of the revenue function, that is, how the revenue generated by a deposit depends on the term and the amount of capital of this deposit.

For the first subclass, called *term dependent deposit games*, the rate of return of a deposit depends on its term, but not on the amount of capital invested in it. We show that term dependent deposit games are (totally) balanced and the other way around, that is each nonnegative totally balanced cooperative game can be written as a term dependent deposit game. Furthermore, we show how to obtain particular core-allocations by constructing Owen-vectors (cf. Owen, 1975).

For the second subclass of *capital dependent deposit games*, the yearly rate of return of a deposit depends on the amount of capital invested in it, but not on the length of its term. The revenue of such a deposit is therefore additive over time. The capital dependent deposit games are an extension of games that were first introduced in Lemaire (1983), and further analyzed in Izquierdo and Rafels (1996). As opposed to our model, the latter only considers a time span of one period. We show that capital dependent deposit games are (totally) balanced if the revenue per unit of capital is increasing in the amount of capital invested. Furthermore, we show that in that case the proportional rule results in a core-allocation.

For the third and final class of *fixed term deposit games*, the revenue of a deposit is positive only if the term covers the whole time horizon that is under consideration. Hence, the name fixed term deposit game. We show that the class of fixed term deposit games contains the class of term dependent deposit games. Moreover, we show that fixed term deposit games are balanced if the rate of return is increasing in the amount of capital, and furthermore, that some specific class of proportional-like rules yields core-allocations.

Our results show that proportional-like allocation rules perform remarkably well when considering stability of cooperation. This is particularly interesting since it is common practice for investment funds to allocate revenues in a proportional way: each participant of the investment fund obtains the same rate of return, irrespective of the amount of capital he contributed to the fund.

The paper is organized as follows. Section 2 introduces deposit games and shows that they need not be balanced. Sections 3 through 5 analyze term dependent, capital dependent, and fixed term deposit games, respectively. Attention is focused on balancedness issues and the construction of allocation rules leading to core elements. A brief survey on cooperative game theory found in Appendix A. Proofs are stated in Appendix B.

2 Deposit games

Consider a group of individuals, each having amounts of money available for depositing at a bank during τ time periods. Let N denote the set of individuals and let $\omega^i \in \mathbb{R}^\tau_+$ describe individual i's endowment of money over time, i.e., ω^i_t is the amount of money available to individual i in period t.

A deposit is described by a fixed amount of capital c and a consecutive number of periods $t_1, t_1+1, \ldots, t_2$ with $1 \leq t_1 \leq t_2 \leq \tau$, in which the amount c is deposited in the bank. $T = \{t_1, t_1+1, \ldots, t_2\}$ is called the term of this deposit. Let

$$\mathcal{T} = \left\{T \subset \{1, 2, \ldots, \tau\} \,|\, \exists_{t_1, t_2 \in \{1,2,\ldots,\tau\}} : T = \{t_1, t_1+1, \ldots, t_2\}\right\}$$

denote the set of possible terms of a deposit, and let

$$D = \left\{d \in \mathbb{R}^\tau_+ \,|\, \exists_{c \geq 0}\, \exists_{T \in \mathcal{T}} : d = c e_T\right\}$$

denote the set of all possible deposits in τ periods where $(e_T)_t = 1$ if $t \in T$ and $(e_T)_t = 0$ if $t \notin T$. Given a deposit $d = c e_T \in D$, d_t is the amount of capital deposited in period t and it equals c if $t \in T$, and zero otherwise. Each deposit that is made in a bank yields a certain revenue. In this regard one can think of interest that is paid by the bank in each period for the duration of the deposit. Let $R : D \to \mathbb{R}_+$ denote the revenue function that assigns to each deposit $d \in D$ a revenue $R(d)$. Furthermore, assume that the zero deposit pays zero revenue, i.e. $R(0) = 0$.

Depending on the structure of the revenue function, and on the individuals' endowments, they may be able to obtain higher returns on deposits by pooling their money. We therefore define a deposit game, which is a cooperative game where the value of a coalition is given by the maximal revenue this coalition can obtain by depositing their available money in the bank. Let $S \subset N$, then $\omega(S) = \sum_{i \in S} \omega^i$ describes the total amount of money available for depositing in each period to coalition S. A collection $d_1, d_2, \dots d_m$ of deposits is feasible for coalition S if they have the money to make these deposits, that is, $\sum_{k=1}^{m} d_k \leq \omega(S)$. The total revenue then equals $\sum_{k=1}^{m} R(d_k)$. Hence, the value of coalition S is given by

$$v(S) = \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \dots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \omega(S) \right\},$$

provided that the supremum exists[1]. The class of all deposit games with N the set of individuals is denoted by DG^N. From the definition of the game it follows that deposit games are superadditive. For if two disjoint coalitions merge, they can at least make the deposits they can make separately, earning at least the revenues they can obtain separately.

Once individuals cooperate, they also have to divide the benefits that emerge from cooperation. The question that arises in this regard is what distributions are 'fair'. In most cases, a core-allocation is considered to be fair. A core-allocation divides the benefits $v(N)$ of the grand coalition N in such a way that no coalition S has an incentive to part company with the grand coalition N and decide on her own what deposits to make. The next example shows that the core of a deposit game can be empty.

Example 2.1 Consider the following three-person deposit game with a one period time span. So, let $\tau = 1$, $N = \{1, 2, 3\}$ and let $\omega^i = 500$ for $i = 1, 2, 3$. Next, suppose that the agents can only deposit their money in a one year bond of $\$1,000$ paying 4% interest. Then the revenue of one such a bond equals $\$40$. The revenue function $R : \mathbb{R}_+ \to \mathbb{R}$ thus equals $R(d) = 40\delta$ with δ the number of bonds that one can buy with d dollars.

Since individual i cannot buy any bonds, we have that $v(\{i\}) = 0$. Two individuals on the other hand, possess $\omega^i + \omega^j = 1,000$ so that they can invest their money in exactly one bond. Hence, $v(\{i, j\}) = 40$ for $i, j \in N$ with $i \neq j$. The grand coalition N possesses $\omega^1 + \omega^2 + \omega^3 = 1,500$. Since this enables them to invest in exactly one bond, it holds that $v(N) = 40$.

The core of this game is empty. For x to be a core-allocation it must hold that $x_1 + x_2 \geq 40$, $x_1 + x_3 \geq 40$, and $x_2 + x_3 \geq 40$. Adding the three inequalities

[1] Given the nature of the problem we are considering, this assumption is justified. For if the supremum would not exist, a coalition could obtain unlimited revenues with a limited amount of money. Realistically, this is not considered to be possible.

yields that $2(x_1 + x_2 + x_3) \geq 120$. Since $x_1 + x_2 + x_3 = 40 = v(N)$ we obtain the contradiction $80 \geq 120$. Hence, the core of this game is empty.

In order to obtain balancedness (i.e. a nonempty core) for deposit games we need to impose some additional restrictions on the revenue function $R : D \to \mathbb{R}_+$. In the remainder of this paper we focus on three subclasses of deposit games, each of which is characterized by properties of the revenue function.

For the first class under consideration, the rate of return of a deposit depends on its term, but not on the amount of capital invested in it. In case the revenue consists of interest payments, this means that the interest rate is independent of the amount of capital deposited. We refer to this class of games as term dependent deposit games.

The second class under consideration is the counterpart of the first one, which means that the yearly rate of return of a deposit depends on the amount of capital invested in it, but not on the length of its term. We refer to this class of games as capital dependent deposit games.

Finally, for the third class the revenue function is such that only deposits with a fixed term of τ periods yield a strictly positive revenue. Therefore, we refer to this class as fixed term deposit games. Note that in this case the rate of return can depend on the amount of capital deposited.

3 Term dependent deposit games

For term dependent deposit games, the rate of return of a deposit depends on its term, but not on the amount of capital deposited in it. Mathematically, this means that the revenue $R(d)$ of a deposit d is linear in the amount of capital c deposited, i.e.

$$R(\alpha d) = \alpha R(d), \tag{1}$$

for all $\alpha \geq 0$ and all $d \in D$.

The class of all term dependent deposit games with agent set N is denoted by TDG^N. Note that $TDG^N \subset DG^N$.

We denote by BA^N and $TOBA^N$ the class of all balanced games and all totally balanced games, respectively. Totally balanced means that not only the game (N, v) has a nonempty core, but also every subgame $(S, v|_S)$, $S \subset N$. In particular, $TOBA_+^N$ denotes the class of all nonnegative totally balanced games. The next theorem shows that term dependent deposit games are totally balanced.

Theorem 3.1 *Each term dependent deposit game is totally balanced.*

Theorem 3.1 states that $TDG^N \subset TOBA_+^N$. The reverse of this statement also holds, that is, every nonnegative totally balanced game can be written as a term dependent deposit game.

Theorem 3.2 *A nonnegative cooperative game is totally balanced if and only if it is a term dependent deposit game.*

Since term dependent deposit games are totally balanced and nonnegative, they can be formulated in terms of linear production games (see Owen, 1975). This enables us to construct a core-allocation by means of an Owen-vector. For this purpose, define $p \in \mathbb{R}^{\#\mathcal{T}}$, $A \in \mathbb{R}^{\tau \times \#\mathcal{T}}$, and $b_i \in \mathbb{R}^{\tau}$, $i \in N$ by $p = (R(e_T))_{T\in\mathcal{T}}$, $A = [(e_T)_{T\in\mathcal{T}}]$, and $b_i = \omega^i$, $i \in N$, respectively. Then

$$\begin{aligned} v(S) &= \max\left\{ p^\top x \,\middle|\, x \geq 0,\ Ax \leq \sum_{i\in S} b_i \right\} \\ &= \max\left\{ \sum_{T\in\mathcal{T}} R(e_T)x_T \,\middle|\, \forall_{T\in\mathcal{T}} : x_T \geq 0,\ \sum_{T\in\mathcal{T}} e_T x_T \leq \sum_{i\in S} \omega^i \right\} \end{aligned}$$

for each $S \subset N$. In terms of linear production games, the endowments of the agents serve as the resources and the goods they can produce are deposits. Since $R(\alpha d) = \alpha R(d)$ we have that $R(x_T e_T) = x_T R(e_T)$ for all $x_T \geq 0$. So, x_T represents the quantity that is produced of deposit e_T. One unit of a deposit e_T yields a revenue of $R(e_T)$. Thus the price at which one unit of the deposit e_T can be sold is set at $p_T = R(e_T)$.

In case $S = N$ the dual of this linear program equals

$$\min\left\{ \sum_{t=1}^{\tau} \omega_t(N) y_t \,\middle|\, \forall_{t\in\{1,2,\ldots,\tau\}} : y_t \geq 0,\ \forall_{T\in\mathcal{T}} : \sum_{t=1}^{\tau} y_t \geq R(e_T) \right\}.$$

Now, if $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_\tau)$ is an optimal solution of this minimization problem, then the Owen-vector $z \in \mathbb{R}^N$ defined by $z_i = \sum_{t=1}^{\tau} \omega_t^i \hat{y}_t$ for all $i \in N$ is a core-allocation for the corresponding term dependent deposit game.

Let us illustrate this procedure with the following example.

Example 3.3 Consider the following two-period situation with two individuals. Let $\omega^1 = (1500, 0)$ and $\omega^2 = (0, 1000)$. Since $\tau = 2$ we have that $\mathcal{T} = \{\{1\}, \{2\}, \{1,2\}\}$. Now suppose that a one year deposit in period 1 or 2 yields a revenue of 4%, and a two year deposit yields a revenue of 12%, i.e. $R(e_{\{1\}}) = R(e_{\{2\}}) = 0.04$, and $R(e_{\{1,2\}}) = 0.12$.

The corresponding term dependent deposit game (N, v) is given by $v(\{1\}) = 60$, $v(\{2\}) = 40$, and $v(\{1,2\}) = 120 + 20 = 140$. By taking $p = (0.04, 0.04, 0.12)$, $b_1 = (1500, 0)$, $b_2 = (0, 1000)$, and

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

we also have that $v(S) = \max\{p^\top x | x \geq 0, Ax \leq \sum_{i\in S} b_i\}$, for all $S \subset N$. An Owen-vector is now constructed as follows. Duality theory states that

$$\begin{aligned} &\max\left\{ p^\top x \,\middle|\, x \geq 0, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 1500 \\ 1000 \end{bmatrix} \right\} \\ &\quad = \min\left\{ y^\top \begin{bmatrix} 1500 \\ 1000 \end{bmatrix} \,\middle|\, y \geq 0, y^\top \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \geq \begin{bmatrix} 0.04 \\ 0.04 \\ 0.12 \end{bmatrix} \right\}. \end{aligned}$$

The optimal solution of the latter minimization problem is $\hat{y} = (0.04, 0.08)$. The corresponding Owen-vector $z = (\hat{y}^\top b_1, \hat{y}^\top b_2)$ equals $(60, 80)$ and belongs to the core of the term dependent deposit game (N, v).

4 Capital dependent deposit games

For the second subclass we consider deposits for which the yearly rate of return is independent of the length of its term, so that the revenue function R is additive over time, that is,

$$R(ce_T) = \sum_{t \in T} R(ce_{\{t\}}) \tag{2}$$

for all $T \in \mathcal{T}$ and all $c \geq 0$. A deposit game with a revenue function that satisfies expression (2) is called a capital dependent deposit game and the class of capital dependent deposit games is denoted by CDG^N. Note that if $\tau = 1$ we obtain the model of financial games as introduced in Lemaire (1983) and further analyzed in Izquierdo and Rafels (1996).

Capital dependent deposit games need not be balanced. In fact, the deposit game of Example 2.1 is a capital dependent deposit game with an empty core. The following theorem states that capital dependent deposit games have a nonempty core if the rate of return per unit of capital is nondecreasing in the amount of capital deposited.

Theorem 4.1 *If $\frac{R(ce_{\{t\}})}{c}$ is nondecreasing in c on $(0, \infty)$ for all $t \in \{1, 2, \ldots, \tau\}$, then the corresponding capital dependent deposit game (N, v) is totally balanced.*

Theorem 4.1 is proved by showing that the proportional rule defined by

$$\pi_i(v) = \sum_{t \in \{1,2,\ldots,\tau\}: \omega_t(N) > 0} \frac{\omega_t^i}{\omega_t(N)} R(\omega_t(N))$$

for all $i \in N$, belongs to the core of the capital dependent deposit game. Note that this proportional rule can easily be extended to a population monotonic allocation scheme as defined in Sprumont (1991). For this purpose, define the allocation scheme $\{\pi^S(v)\}_{S \subset N}$ by:

$$\pi_i^S(v) = \sum_{t \in \{1,2,\ldots,\tau\}: \omega_t(S) > 0} \frac{\omega_t^i}{\omega_t(S)} R(\omega_t(S))$$

for all $i \in S$ and all $S \subset N$.

Contrary to term dependent deposit games, not every nonnegative totally balanced game is a capital dependent deposit game, as the following example shows.

Example 4.2 Remember that, for each $C \subset N$, the unanimity game (N, u_C) is defined as follows

$$u_C(S) = \begin{cases} 1, \text{ if } C \subset S, \\ 0, \text{ otherwise.} \end{cases} \tag{3}$$

Consider the simple game (N, v) with $N = \{1, 2, 3, 4, 5, 6\}$ and $v(S) = u_{\{1,2,3,4\}}(S) + u_{\{1,2,5,6\}}(S) - u_{\{1,2,3,4,5,6\}}(S)$ for all $S \subset N$. So, $v(S) = 1$ if and only if $\{1, 2, 3, 4\} \subset S$ or $\{1, 2, 5, 6\} \subset S$. We will not give the formal proof that this game cannot be written as a capital dependent deposit game. Instead, we suffice by giving the intuition.

Suppose that we can write the game (N, v) as a capital dependent deposit game. Then there are certain time periods in which coalition $\{1, 2, 3, 4\}$ can make a capital deposit to obtain their revenue $v(\{1, 2, 3, 4\}) = 1$. The same holds for coalition $\{1, 2, 5, 6\}$. Now, it can be shown that coalitions $\{1, 2, 3, 4\}$ and $\{1, 2, 5, 6\}$ have to make their deposits in different time periods. Hence, coalition $\{1, 2, 3, 4, 5, 6\}$ can make the deposits that both coalitions $\{1, 2, 3, 4\}$ and $\{1, 2, 5, 6\}$ can make, yielding a revenue of at least $v(\{1, 2, 3, 4\} + v(\{1, 2, 5, 6\}) = 2$, which contradicts $v(N) = 1$.

Proposition 4.3 *The nonnegative cone of unanimity games* $\{u_S | S \subset N\}$ *is contained in the class* CDG^N *of capital dependent deposit games.*

The reverse of Proposition 4.3, however, is not true. The next example provides a game that is not a positive combination of unanimity games but that can be written as a capital dependent deposit game.

Example 4.4 Let $(N, v) \in G^N$ be a three-person game with $v(S) = u_{\{1,2\}}(S) + u_{\{1,3\}}(S) - u_{\{1,2,3\}}(S)$ for all $S \subset N$. Define a capital dependent deposit game $(N, w) \in CDG^N$ with $\tau = 1$ and $R(d) = 1$ if $d \geq 3$ and $R(d) = 0$ otherwise. Furthermore, let $\omega_1^1 = 2$, $\omega_1^2 = 1$, and $\omega_1^3 = 1$. Since $\sum_{i \in S} \omega_1^i \geq 3$ if and only if $S \in \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$, we have that $w(S) = 1$ if and only if $S \in \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Thus, $w(S) = v(S)$ for all $S \subset N$.

5 Fixed term deposit games

For the third and final subclass we consider the situation in which a deposit only yields a strictly positive revenue if the term covers all τ periods. Mathematically, this means that $R(ce_T) = 0$ if $T \neq \{1, 2, \ldots, \tau\}$. The class of all fixed term deposit games with agent set N is denoted by FDG^N. In fact, we have already seen this type of deposit games in the proof of Theorem 3.2. To show that each nonnegative totally balanced game is a term dependent deposit game, we constructed a deposit game in which deposits only earn a strictly positive revenue if they cover the whole time span of τ periods. Thus, the following result immediately follows from the proof of Theorem 3.2.

Theorem 5.1 *Each nonnegative totally balanced game is a fixed term deposit game.*

According to Theorem 3.1 and Theorem 3.2 the class of term dependent deposit games is equal to the class of nonnegative totally balanced games. Theorem 5.1 then implies

Theorem 5.2 *Every term dependent deposit game is a fixed term deposit game.*

Although fixed term deposit games exhaust the class of nonnegative totally balanced games, they are not totally balanced in general. Example 2.1 is an example of a fixed term deposit game with an empty core. We can, however, derive a sufficient condition for totally balancedness similar to the one for capital dependent deposit games.

Theorem 5.3 *Let* $T = \{1, 2, \ldots, \tau\}$. *If* $\frac{R(ce_T)}{c}$ *is nondecreasing in* c *on* $(0, \infty)$ *then the fixed term deposit game* (N, v) *is totally balanced.*

In order to show that fixed term deposit games are balanced, an allocation rule that belongs to the core is constructed. For defining this rule, let (N, v) be a fixed term deposit game. Next, consider the game (N, w) with $w(S) = \min_{t \in T} \omega_t(S)$ for all $S \subset N$. Here, $w(S)$ represents the amount of money coalition S can invest with term $\{1, 2, \ldots, \tau\}$. It is shown in Kalai and Zemel (1982) that a non-negative cooperative game is totally balanced if and only if it is a minimum of a finite collection of additive games. Therefore, (N, w) is totally balanced so that there exists a core-allocation $x \in \mathbb{R}^N$. A core-allocation for the game (N, v) is then found by allocating $v(N)$ proportionally with respect to x to the investors. This means that, as long as $\sum_{j \in N} x_j \neq 0$, investor $i \in N$ receives

$$\rho_i(x) = \frac{x_i}{\sum_{j \in N} x_j} v(N).$$

Otherwise, $v(N) = x(N) = 0$, and all investors receive $\rho_i(x) = 0$. Since this allocation rule depends on a core-allocation x of the game (N, w), we can obtain several proportional-like allocation rules by specifying the allocation x. For instance, by taking x to be the nucleolus $n(w)$ (see Schmeidler, 1969) of the game (N, w), we obtain the allocation $\rho(n(w)) \in C(v)$. Alternatively, since (N, w) can be interpreted as a flow game, a minimum cut solution $mc(w)$ (see Kalai and Zemel, 1982) also results in an allocation $\rho(mc(w)) \in C(v)$.

A Cooperative game theory

A cooperative TU-game is a pair (N, v) with N the set of agents and $v : 2^N \to \mathbb{R}$ the characteristic function describing the worth $v(S)$ of coalition $S \subset N$. The worth of the empty coalition is defined zero, that is, $v(\emptyset) = 0$. In particular, if $v(S) \geq 0$ for all $S \subset N$ the game is called nonnegative. The class of all cooperative games is denoted by G^N and the class of all nonnegative cooperative games is denoted by G^N_+.

A cooperative game $(N, v) \in G^N$ is called additive if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subset N$. The game $(N, u_C) \in G^N$ denotes the unanimity game with respect to the nonempty coalition C, that is, $u_C(S) = 1$ if $C \subset S$ and $u_C(S) = 0$ if $C \not\subset S$. The interpretation is that a coalition S obtains the value 1 if and only if it contains all the agents in C. Note that any cooperative game $(N, v) \in G^N$ can be written in a unique way as a linear combination of the unanimity games $\{u_C | C \subset N\}$.

A game $(N, v) \in G^N$ is called superadditive if for any $S, C \subset N$ with $S \cap C = \emptyset$ it holds true that $v(S) + v(C) \leq v(S \cup C)$. The interpretation of superadditivity is that two disjoint coalitions can do (weakly) better by merging into one large coalition.

One of the major topics of cooperative game theory is how to divide the benefits from cooperation. In this regard it is assumed that all agents are willing to cooperate so that the grand coalition N is formed. The core of a game $(N, v) \in G^N$ is defined by the set

$$C(v) = \left\{x \in \mathbb{R}^N \,|x(N) = v(N),\ \forall_{S \subset N} : x(S) \geq v(S)\right\}, \tag{4}$$

where $x(S) = \sum_{i \in S} x_i$. So, core-allocations induce a stable cooperation in the sense that no coalition has an incentive to part company with the grand coalition.

The core of a game can be empty. A game with a nonempty core is called balanced. A game (N, v) for which each subgame $(S, v_{|S})$ is balanced is called totally balanced. Here, the subgame $(S, v_{|S})$ is defined by $v_{|S}(C) = v(C)$ for all $C \subset S$. We denote by BA^N and $TOBA^N$ the class of all balanced games and all totally balanced games, respectively. In particular, $TOBA_+^N$ denotes the class of all nonnegative totally balanced games. The following theorem, which is due to Bondareva (1963) and Shapley (1967) provides a necessary and sufficient condition for balancedness.

Theorem A.1 *Let $(N, v) \in G^N$. Then $C(v) \neq \emptyset$ if and only if $\sum_{S \subset N} (S)v(S) \leq v(N)$ for all $: 2^N \to \mathbb{R}_+$ satisfying $\sum_{S \subset N : i \in S} (S) = 1$ for each $i \in N$.*

For a characterization of nonnegative totally balanced games, let $(N, v) \in G_+^N$. Then the game (N, v) is the minimum of a finite collection $(N, a_1), (N, a_2), \ldots, (N, a_q)$ of additive games, if $v(S) = \min\{a_k(S) | k = 1, 2, \ldots, q\}$ for all $S \subset N$. The following result is due to Kalai and Zemel (1982).

Theorem A.2 *A cooperative game $(N, v) \in G_+^N$ is totally balanced if and only if it is the minimum of a finite collection of additive games.*

Another characterization of all nonnegative totally balanced games is given in Owen (1975) by means of linear production games. In a linear production game each agent is endowed with a resource bundle, which can be employed to produce a bundle of consumption goods. The production technique is linear and available to all agents. Furthermore, produced consumption goods are sold at exogenously given prices. Mathematically, a linear production situation is described by a tuple $(N, A, p, (b_i)_{i \in N})$, where N denotes the set of agents, $A \in \mathbb{R}^{r \times m}$ the production technique, $p \in \mathbb{R}^m$ the price vector of the consumption goods, and $b_i \in \mathbb{R}^r$ the resources of agent i. The objective of each coalition is to maximize the revenues given their joint resources. The value of coalition $S \subset N$ in the corresponding linear production game is thus given by

$$v(S) = \max\left\{p^\top x \,\middle|\, x \geq 0, Ax \leq \sum_{i \in S} b_i\right\}.$$

The following result is due to Owen (1975).

Theorem A.3 *Linear production games are totally balanced.*

In particular, Owen (1975) shows how to determine a core-allocation for a linear production game. For this purpose, consider the optimization problem

$$v(N) = \max\left\{ p^\top x \,\middle|\, x \geq 0,\ Ax \leq \sum_{i \in N} b_i \right\}$$

for the grand coalition. Let $\hat{y}$ denote an optimal solution of the dual problem

$$\min\left\{ y^\top \left(\sum_{i \in N} b_i \right) \,\middle|\, y \geq 0,\ y^\top A \geq c^\top \right\}.$$

Then the allocation $(\hat{y}^\top b_i)_{i \in N}$ is called an Owen vector and is a core-allocation for the corresponding linear production game. For an interpretation of this allocation, recall that the vector $\hat{y}$ represents the shadow prices for the resources. The allocation $(\hat{y}^\top b_i)_{i \in N}$ then gives each player the value of his resources. Theorem A.3 states that linear production games are totally balanced. The reverse, however, also holds true: each nonnegative totally balanced game is a linear production game.

B Proofs

Proof of Theorem 3.1. Let $(N, v) \in TDG^N$. We first show that (N, v) is balanced. For this we use the necessary and sufficient condition for balancedness by Bondareva (1963) and Shapley (1967).

Take $\lambda : 2^N \to \mathbb{R}_+$ such that $\sum_{S \subset N : i \in S} \lambda(S) = 1$ for all $i \in N$. Then

$$\begin{aligned}
&\sum_{S \subset N} \lambda(S) v(S) \\
&= \sum_{S \subset N} \lambda(S) \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \omega(S) \right\} \\
&= \sum_{S \subset N} \sup \left\{ \sum_{k=1}^{m} \lambda(S) R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \omega(S) \right\} \\
&= \sum_{S \subset N} \sup \left\{ \sum_{k=1}^{m} R(\lambda(S) d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \omega(S) \right\} \\
&= \sum_{S \subset N} \sup \left\{ \sum_{k=1}^{m} R(\lambda(S) d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} \lambda(S) d_k \leq \lambda(S) \omega(S) \right\} \\
&= \sum_{S \subset N} \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \lambda(S) \omega(S) \right\} \\
&= \sup \left\{ \sum_{S \subset N} \sum_{k=1}^{m^S} R(d_k^S) \,\middle|\, \forall_{S \subset N} \exists_{m^S \in \boldsymbol{N}} \exists_{d_1^S, d_2^S, \ldots, d_{m^S}^S \in D} : \sum_{k=1}^{m^S} d_k^S \leq \lambda(S) \omega(S) \right\} \\
&\leq \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \sum_{S \subset N} \lambda(S) \omega(S) \right\}
\end{aligned}$$

$$= \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \sum_{i \in N} \omega^i \sum_{S \subset N : i \in S} \lambda(S) \right\}$$

$$= \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \sum_{i \in N} \omega^i \right\} = v(N),$$

where the third equality follows from (1). Since any subgame $(S, v_{|S})$ is again a term dependent deposit game, it follows that $(N, v_{|S})$ is balanced. Hence, term dependent deposit games are totally balanced. □

Proof of Theorem 3.2. The 'if'-part follows from Theorem 3.1. For the 'only if'-part we use Theorem A.2. So, take $(N, v) \in TOBA_+^N$ and let $(N, a_1), (N, a_2), \ldots, (N, a_q)$ be the finite collection of additive games such that $v(S) = \min\{a_k(S) | k \in \{1, 2, \ldots, q\}\}$ for all $S \subset N$. We construct a term dependent deposit game $(N, w) \in TDG^N$ such that $w(S) = v(S)$ for all $S \subset N$.

Take $\tau = q$ and define $R(d) = \min\{d_t | t \in \{1, 2, \ldots, \tau\}\}$. So, we have q time periods and a deposit only yields a positive reward if the term equals exactly q time periods. Next, define $\omega_t^i = a_t(\{i\})$ for all $i \in N$ and all $t \in \{1, 2, \ldots, q\}$. Then

$$w(S) = \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} : \sum_{k=1}^{m} d_k \leq \omega(S) \right\}$$

$$= \sup \left\{ \sum_{k=1}^{m} \min_{t \in \{1,2,\ldots,q\}} (d_k)_t \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} \forall_{t \in \{1,2,\ldots,q\}} : \sum_{k=1}^{m} (d_k)_t \leq \omega_t(S) \right\}$$

$$\leq \sup \left\{ \min_{t \in \{1,2,\ldots,q\}} \sum_{k=1}^{m} (d_k)_t \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_m \in D} \forall_{t \in \{1,2,\ldots,q\}} : \sum_{k=1}^{m} (d_k)_t \leq \omega_t(S) \right\}$$

$$= \sup \left\{ \min_{t \in \{1,2,\ldots,q\}} d_t \,\middle|\, \exists_{d \in D} \forall_{t \in \{1,2,\ldots,q\}} : d_t \leq \omega_t(S) \right\}$$

$$= \min_{t \in \{1,2,\ldots,q\}} \omega_t(S)$$

$$= \min_{t \in \{1,2,\ldots,q\}} a_t(S)$$

$$= v(S).$$

Furthermore, let $d = \left(\min_{t \in \{1,2,\ldots,q\}} \omega_t(S)\right) e_{\{1,2,\ldots,\tau\}}$. Then d is a feasible deposit for coalition S, hence

$$w(S) \geq R(d) = \min_{t \in \{1,2,\ldots,q\}} \omega_t(S) = \min_{t \in \{1,2,\ldots,q\}} a_t(S) = v(S).$$

Consequently, $w(S) = v(S)$. □

Proof of Theorem 4.1. Since each subgame is also a capital dependent deposit game, we only need to show that a capital dependent deposit game is balanced. To prove nonemptiness of the core we explicitly construct a core-allocation. First we derive some preliminary results.

Let $t \in \{1, 2, \ldots, \tau\}$ and let $c_1, c_2 > 0$. Since $\frac{R(ce_t)}{c}$ is increasing in c it follows that

$$\begin{aligned} R(c_1 e_t) + R(c_2 e_t) &= c_1 \frac{R(c_1 e_t)}{c_1} + c_2 \frac{R(c_2 e_t)}{c_2} \\ &\leq c_1 \frac{R((c_1 + c_2)e_t)}{c_1 + c_2} + c_2 \frac{R((c_1 + c_2)e_t)}{c_1 + c_2} \\ &= R((c_1 + c_2)e_t). \end{aligned} \quad (5)$$

This implies that in each period one should make only one deposit and make it as high as possible.

Take $(N, v) \in CDG^N$ and recall that $R(ce_T) = \sum_{t \in T} R(ce_t)$ for all $T \in \mathcal{T}$ and all $c \geq 0$. Hence, for $S \subset N$ it holds that

$$\begin{aligned} v(S) &= \sup \left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_k \in D} : \sum_{k=1}^{m} d_k \leq \omega(S) \right\} \\ &= \sup \left\{ \sum_{k=1}^{m} \sum_{t=1}^{\tau} R((d_k)_t) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_k \in D} \forall_{t \in \{1,2,\ldots,\tau\}} : \right. \\ &\qquad \left. \sum_{k=1}^{m} (d_k)_t \leq \omega_t(S) \right\} \\ &= \sup \left\{ \sum_{t=1}^{\tau} \sum_{k=1}^{m} R((d_k)_t) \,\middle|\, \exists_{m \in \boldsymbol{N}} \exists_{d_1, d_2, \ldots, d_k \in D} \forall_{t \in \{1,2,\ldots,\tau\}} : \right. \\ &\qquad \left. \sum_{k=1}^{m} (d_k)_t \leq \omega_t(S) \right\} \\ &= \sup \left\{ \sum_{t=1}^{\tau} R(d_t) \,\middle|\, \exists_{d \in D} \forall_{t \in \{1,2,\ldots,\tau\}} : d_t \leq \omega_t(S) \right\} \\ &= \sum_{t=1}^{\tau} R(\omega_t(S)), \end{aligned} \quad (6)$$

where the last two equalities follow from (5), that is invest in one deposit only. In particular we have that $v(N) = \sum_{t=1}^{\tau} R(\sum_{i \in N} \omega_t^i)$.

Now, we construct a core-allocation for the game (N, v). For each period t we divide the reward $R(\omega_t(S))$ proportional to the contribution ω_t^i of the agents. So, agent $i \in N$ receives in aggregate

$$\pi_i(v) = \sum_{t \in \{1,2,\ldots,\tau\} : \omega_t(N) > 0} \frac{\omega_t^i}{\omega_t(N)} R(\omega_t(N)). \quad (7)$$

In order to show that $\pi(v) = (\pi_i(v))_{i \in N} \in C(v)$, take $S \subset N$. Then

$$\sum_{i \in S} \pi_i(v) = \sum_{i \in S} \sum_{t \in \{1,2,\ldots,\tau\} : \omega_t(N) > 0} \frac{\omega_t^i}{\omega_t(N)} R(\omega_t(N))$$

$$
\begin{aligned}
&= \sum_{i \in S} \sum_{t \in \{1,2,\ldots,\tau\}: \omega_t(N) > 0} \frac{R(\omega_t(N))}{\omega_t(N)} \omega_t^i \\
&\geq \sum_{i \in S} \sum_{t \in \{1,2,\ldots,\tau\}: \omega_t(S) > 0} \frac{R(\omega_t(S))}{\omega_t(S)} \omega_t^i \\
&= \sum_{t \in \{1,2,\ldots,\tau\}: \omega_t(S) > 0} \frac{R(\omega_t(S))}{\omega_t(S)} (\omega_t(S)) \\
&= \sum_{t=1}^{\tau} R(\omega_t(S)) \\
&= v(S),
\end{aligned}
$$

where the inequality follows from the fact that $\frac{R(ce_t)}{c}$ is increasing in c for all $t \in \{1, 2, \ldots, \tau\}$. Since the inequality is an equality for $S = N$, it follows that $\pi(v) \in C(v)$. □

Proof of Proposition 4.3. Let $(N, v) \in G^N$ be a nonnegative linear combination of unanimity games, that is, there exists $c_S \geq 0$, $S \subset N$ such that that $v = \sum_{S \subset N} c_S u_S$. We construct a game $(N, w) \in CDG^N$ such that $w = v$.

Take the number of time periods equal to $\tau = 2^N - 1$ and make a one-to-one correspondence between the time periods and all the non-empty coalitions. More precisely, let $S_t \subset N$ be the coalition corresponding to time period t. Next, define for $t = 1, 2, \ldots, 2^N - 1$

$$
R(ce_t) = \begin{cases} c_{S_t} & \text{, if } c \geq c_{S_t} \\ 0 & \text{, if } c < c_{S_t}. \end{cases} \tag{8}
$$

Furthermore, take $\omega_t^i = (\#S_t)^{-1} c_{S_t}$ if $i \in S_t$ and $\omega_t^i = 0$ if $i \notin S_t$. Here, $\#S_t$ denotes the number of agents in coalition S_t. Then

$$
\begin{aligned}
w(C) &= \sum_{t=1}^{2^N-1} R\left(\sum_{i \in C} \omega_t^i\right) \\
&= \sum_{t=1}^{2^N-1} R\left(\sum_{i \in S_t \cap C} c_{S_t} (\#S_t)^{-1}\right) \\
&= \sum_{t=1}^{2^N-1} c_{S_t} u_{S_t}(C) \\
&= \sum_{S \subset N} c_S u_S(C) \\
&= v(C),
\end{aligned}
$$

where the first equality follows from (8) and the third equality follows from (8) and $\sum_{i \in S_t \cap C} c_{S_t} (\#S_t)^{-1} \geq c_{S_t}$ if and only if $S_t \subset C$. □

Proof of Theorem 5.3. Since each subgame is also a fixed term deposit game, we only need to show that a fixed term deposit game is balanced.

Let $T = \{1, 2, \ldots, \tau\}$ and take $c_1, c_2 > 0$. Since $\frac{R(ce_T)}{c}$ is nondecreasing in c expression (5) holds, that is

$$R(c_1 e_T) + R(c_2 e_T) \leq R((c_1 + c_2)e_T). \tag{9}$$

Hence, one should make only one deposit with term T and make it as high as possible.

Next, take $(N, v) \in FDG^N$ and let $T = \{1, 2, \ldots, \tau\}$. Then

$$\begin{aligned}
v(S) &= \sup\left\{ \sum_{k=1}^{m} R(d_k) \,\middle|\, \exists_{m\in \boldsymbol{N}} \exists_{d_1,d_2,\ldots,d_k \in D} : \sum_{k=1}^{m} d_k \leq \omega(S) \right\} \\
&= \sup\left\{ \sum_{k=1}^{m} R(c_k e_T) \,\middle|\, \exists_{m\in \boldsymbol{N}} \exists_{c_1,c_2,\ldots,c_m \in \mathbb{R}_+} : \sum_{k=1}^{m} c_k e_T \leq \omega(S) \right\} \\
&= \sup\left\{ R(ce_T) \,\middle|\, \exists_{c\in\mathbb{R}_+} : ce_T \leq \omega(S) \right\} \\
&= \sup\left\{ R(ce_T) \,\middle|\, \exists_{c\in\mathbb{R}_+} : c \leq \min_{t\in T} \omega_t(S) \right\} \\
&= R\left(\min_{t\in T} \omega_t(S) \right)
\end{aligned} \tag{10}$$

where the third and fourth equality follows from (9). In particular we have that $v(N) = R(\min_{t\in T} \omega_t(N))$.

If $v(N) = 0$ then $v(S) = 0$ for all $S \subset N$ and $0 \in C(v)$. Let us assume that $v(N) > 0$. So, $\min_{t\in T} \omega_t(N) > 0$. Define a cooperative game (N, w) with $w(S) = \min_{t\in T} \omega_t(S)$ for all $S \subset N$. Note that (N, w) is the minimum of a finite collection of additive games $(N, a_1), (N, a_2), \ldots, (N, a_\tau)$, where $a_t(S) = \omega_t(S) = \sum_{i\in S} \omega_t^i$ for all $S \subset N$ and all $t \in T$. Hence, (N, w) is totally balanced by Theorem A.2.

Let $x \in \mathbb{R}^N$ be a core-allocation of the game (N, w). Thus, for each $S \subset N$ it holds true that $\sum_{i\in S} x_i \geq \min_{t\in T} \omega_t(S)$. Define for each $i \in N$

$$\rho_i(x) = \frac{x_i}{\min_{t\in T} \omega_t(N)} \, R\left(\min_{t\in T} \omega_t(N) \right).$$

We show that ρ is a core-allocation of (N, v). Therefore, take $S \subset N$. If $\min_{t\in T} \omega_t(S) = 0$ then $\sum_{i\in S} \rho_i(x) \geq 0 = v(S)$. If $\min_{t\in T} \omega_t(S) > 0$ then

$$\begin{aligned}
\sum_{i\in S} \rho_i(x) &= \frac{\sum_{i\in S} x_i}{\min_{t\in T} \omega_t(N)} \, R\left(\min_{t\in T} \omega_t(N) \right) \\
&\geq \frac{\min_{t\in T} \omega_t(S)}{\min_{t\in T} \omega_t(N)} \, R\left(\min_{t\in T} \omega_t(N) \right) \\
&= \frac{R(\min_{t\in T} \omega_t(N))}{\min_{t\in T} \omega_t(N)} \, \min_{t\in T} \omega_t(S) \\
&\geq \frac{R(\min_{t\in T} \omega_t(S))}{\min_{t\in T} \omega_t(S)} \, \min_{t\in T} \omega_t(S) \\
&= R(\min_{t\in T} \omega_t(S)) \\
&= v(S),
\end{aligned}$$

where the first inequality follows from $x \in C(w)$ and the second inequality follows from the fact that $\frac{R(ce_T)}{c}$ is nondecreasing in c. Since both inequalities are equalities for $S = N$, we have that $\rho \in C(v)$. □

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