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Henk Norde[a,*], Fioravante Patrone[b], Stef Tijs[a]

[a]*Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands*
[b]*Department of Mathematics, Via Dodecaneso 35, University of Genoa, 16146 Genoa, Italy*

Abstract

Approximate solutions for optimization problems become of interest if the 'true' optimum cannot be found: this may happen for the simple reason that an optimum does not exist or because of the 'bounded rationality' (or bounded accuracy) of the optimizer. This paper characterizes several approximate solutions by means of consistency and additional requirements. In particular we consider invariance properties. We prove that, where the domain contains optimization problems without maximum, there is no non-trivial consistent solution satisfying non-emptiness, translation and multiplication invariance. Moreover, we show that the class of 'satisficing' solutions is obtained, if the invariance axioms are replaced with Chernoff's Choice Axiom. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we will try to give an answer to an apparently silly question: is the concept of 'approximate solution' in optimization meaningful? We will show that, moving from exact to approximate optimization, some serious problems may arise. One reason to focus on approximate optimization can be derived from the increasing interest for issues related with 'bounded rationality' in game theory. More and more the emphasis is shifted from maximization to approximate maximization. On this point, we only refer to Fudenberg and Tirole (1991), Myerson (1991) and Radner (1980). The

*Corresponding author. Tel.: +31-13-466-9111; fax: +31-13-466-3072.
E-mail address: h.norde@kub.nl (H. Norde).

latter paper is interesting, both to understand the kind of results that can be achieved along this path and for the remarks about the problems that arise when approximate maximization enters the scene. The interest in approximate optimization arising from game theory is only one particular case of a general issue. We mention that our results, obtained using Chernoff's Choice Axiom (see Chernoff, 1954), can be related to the idea of 'satisficing' (see Simon, 1955; March and Simon, 1958).

In many situations it happens that, given an optimization problem, one does not look for the maximum. This can happen for the obvious reason that a maximum does not exist or for the difficulty of finding it. In both cases, one should have some 'rule' that says when the search for a maximum could stop. Clearly, many different kinds of rules can be devised, from some 'rule of thumb' to a sophisticated analysis that compares the computational costs for improving the degree of approximation and the benefits that result.

The approach that will be used in this paper is axiomatic, i.e. we will state some desirable properties of an 'approximate solution concept' and we will analyze their consequences and mutual compatibility. To be more specific, we will investigate a special issue related with these rules: how should they be if one wants to behave in a consistent way across different optimization problems and, at the same time, one has to take into account some invariance properties. The invariance requirement is due to the fact that, in many cases, the function to be maximized is only a representative of a class of equivalent functions (let us recall at least utility theory, and the fact that in hard sciences the origin or the scale of measurement quite often can be chosen freely). The remarkable result that we get is an 'impossibility theorem', which asserts that there are no consistent rules for choosing truly approximate solutions if one wants to take into account translation and multiplication invariance (as one should do, for example, when dealing with expected utilities). We also investigate the cases in which one takes into account separately these invariance requirements. For example, taking into account only translation invariance, leads to the class of ε-optimal solutions (see Tijs, 1981).

Special emphasis is given to rules that satisfy Chernoff's Choice Axiom. The main reason to take this point of view is that we try to consider classes of optimization problems which contain both bounded and unbounded problems. This interest is an outgrowth of previous research done by the authors in the context of semi-infinite bimatrix games (see Jurg and Tijs, 1993; Lucchetti et al., 1986; Norde and Potters, 1997). Under appropriate assumptions we get the class of 'satisficing' solutions to which belong the (ε, k)-solutions investigated in the papers quoted above.

Sections 2 and 3 are devoted to the setting of the problem. The axioms and motivations are introduced and some examples are provided. Characterizations involving the translation and multiplication invariance axioms are given in Section 4 and characterizations using Chernoff's Choice Axiom in Section 5. Conclusions are made in Section 6. Moreover, we devote some room to emphasize, by means of examples, that careless specification of the domain of the rule can give quite strange results. We show how it could be possible to strengthen the consistency requirement in order to overcome these difficulties.

Notation. Throughout this paper we denote the set $\mathbb{R} \cup \{ + \infty \}$ by $\mathbb{R}^*$ and the set $\mathbb{R} \cup \{ - \infty, + \infty \}$ by $\overline{\mathbb{R}}$.

2. Optimization problems

An *optimization problem* is a pair (A, u) where A is a non-empty set of alternatives and u is a real-valued function with domain A. Let $\mathscr{P}$ be a non-empty collection of optimization problems. A *solution* β on $\mathscr{P}$ is a map which assigns to every optimization problem $(A, u) \in \mathscr{P}$ a subset of A.

Example 2.1. For the following examples no special restriction is imposed on $\mathscr{P}$.

(a) The solution β_{tot} is defined by:

$$\beta_{\text{tot}}(A, u) := A$$

(b) The solution $\beta_{\max}$ is defined by:

$$\beta_{\max}(A, u) := \{a \in A : u(a) \geq u(a') \text{ for every } a' \in A\}$$

(c) For $\varepsilon > 0$ the solution β_ε is defined by:

$$\beta_\varepsilon(A, u) := \{a \in A : u(a) \geq u(a') - \varepsilon \text{ for every } a' \in A\}$$

(d) For $k \in \mathbb{R}$ the solution β^k is defined by:

$$\beta^k(A, u) := \{a \in A : u(a) \geq k\}$$

(e) For $\varepsilon > 0$, $k \in \mathbb{R}$ the solution $\beta_{\varepsilon,k}$ is defined by:

$$\beta_{\varepsilon,k}(A, u) := \begin{cases} \beta_{\max}(A, u) & \text{if } \beta_{\max}(A, u) \neq \emptyset \\ \beta_\varepsilon(A, u) & \text{if } \beta_{\max}(A, u) = \emptyset \text{ and } \beta_\varepsilon(A, u) \neq \emptyset \\ \beta^k(A, u) & \text{otherwise} \end{cases}$$

Notice that $\beta_{\max}(A, u)$, $\beta_\varepsilon(A, u)$ and $\beta^k(A, u)$ can be empty; on the contrary, $\beta_{\text{tot}}(A, u) \neq \emptyset$ and $\beta_{\varepsilon,k}(A, u) \neq \emptyset$ for every $(A, u) \in \mathscr{P}$.

Consider the optimization problems (A, u) and (B, v), defined by $A := (-\infty, -\varepsilon/2]$, $u(x) := x$ for every $x \in A$ (where $\varepsilon > 0$) and $B := (-\infty, 0)$, $v(x) := x$ for every $x \in B$. Then, for every $k \in \mathbb{R}$, $\beta_{\varepsilon,k}(A, u) = \{-\varepsilon/2\}$ and $\beta_{\varepsilon,k}(B, v) = [-\varepsilon, 0)$. So an optimizer, using solution $\beta_{\varepsilon,k}$, is satisfied with alternative $b = -\varepsilon$ in optimization problem (B, v), but not with alternative $a = -\varepsilon$ in optimization problem (A, u), which has however a lower supremum. Solutions which, contrary to $\beta_{\varepsilon,k}$, exhibit an increasing level of satisfaction as the supremum of the problem raises, are called monotonic. Formally, a solution β on $\mathscr{P}$ is *monotonic* if for every pair of problems (A, u), $(B, v) \in \mathscr{P}$ with $\sup_{x \in A} u(x) \leq \sup_{y \in B} v(y)$ the following statement is true:

if $b \in \beta(B, v)$ and $a \in A$ is such that $u(a) \geq v(b)$ then $a \in \beta(A, u)$

A weaker axiom, stating that in sup-equivalent problems (i.e. problems with the same supremum) the same level of satisfaction should be used, is the axiom of approximation

consistency: a solution β on $\mathscr{P}$ is *approximation consistent* if for every pair of sup-equivalent problems (A, u), $(B, v) \in \mathscr{P}$ the following statement is true:

if $b \in \beta(B, v)$ and $a \in A$ is such that $u(a) \geq v(b)$ then $a \in \beta(A, u)$

So, if a solution β is approximation consistent, selection by β of an alternative $b \in B$ in some problem $(B, v) \in \mathscr{P}$, induces selection by β of all 'non-worse' alternatives in sup-equivalent problems.

To illustrate the idea of approximation consistency consider the following example. Let $\mathscr{P}$ be a collection of optimization problems dealing with profit maximization. Suppose that for every $(A, u) \in \mathscr{P}$ the set of actions A is finite, and that for every action $a \in A$ the monetary profit $u(a)$ is evaluated in Dutch guilders. In practice, optimizers in $\mathscr{P}$ will be satisfied with the highest profit which (i) is feasible, and (ii) represents a round figure. Condition (ii) is caused by the desire of the optimizer for not too complex payments. Let us assume that the optimizers in $\mathscr{P}$ do not like payments involving more than five notes or coins. So, every optimizer in $\mathscr{P}$, whose maximal profit is, for example, fl. 263, will only be satisfied when he gets this amount precisely, since $263 = 250 + 10 + 1 + 1 + 1$. On the other hand, every optimizer in $\mathscr{P}$, whose maximal profit is fl. 2630, is already satisfied when he receives 2600 $(= 1000 + 1000 + 250 + 250 + 100)$ guilders. Note that this solution can be extended easily to optimization problems which are expressed in other monetary units, provided that the same monetary system is used.

In fact, the solution in the previous paragraph exhibits different levels of accuracy of the optimizers, where this level of accuracy *only* depends on the supremum of the optimization problem under consideration. This, however, is precisely the job that many numerical methods for optimization problems do. These methods, which manipulate the functions u directly without reference to scales, 'stop' when a precision up to a fixed number of decimals is reached. Since real numbers are stored by computers using floating point representation, this level of accuracy also depends only on the supremum of the optimization problem under consideration. Therefore, such methods can be seen as examples of approximation consistent solutions. Clearly, the solutions (a)–(d) in Example 2.1 are approximation consistent.

A solution β on $\mathscr{P}$ is *weakly approximation consistent* if for all $(A, u) \in \mathscr{P}$ the following statement is true:

if $a \in \beta(A, u)$ and $a' \in A$ is such that $u(a') \geq u(a)$ then $a' \in \beta(A, u)$

So, if a solution β is weakly approximation consistent, selection by β of an alternative $a \in A$ in some problem $(A, u) \in \mathscr{P}$, induces selection by β of all 'non-worse' alternatives in the same problem. In the sequel of this paper we will also make use of the following axioms.

A solution β on $\mathscr{P}$ satisfies *non-emptiness* if for every $P = (A, u) \in \mathscr{P}$ we have:

$$\beta(P) \neq \emptyset$$

For every optimization problem $P = (A, u)$ and for every $t \in \mathbb{R}$ the optimization problem (A, v), defined by $v(a) = t + u(a)$ for every $a \in A$, will be denoted by $t + P$. The collection $\mathscr{P}$ is *closed under translation* if for every $P \in \mathscr{P}$ and $t \in \mathbb{R}$ we have

$t + P \in \mathcal{P}$. A solution β on $\mathcal{P}$, which is closed under translation, satisfies *translation invariance* if for every $P \in \mathcal{P}$ and $t \in \mathbb{R}$ we have:

$$\beta(t + P) = \beta(P)$$

For every optimization problem $P = (A, u)$ and for every $\lambda > 0$ the optimization problem (A, v), defined by $v(a) = \lambda u(a)$ for every $a \in A$, will be denoted by λP. The collection $\mathcal{P}$ is *closed under multiplication* if for every $P \in \mathcal{P}$ and $\lambda > 0$ we have $\lambda P \in \mathcal{P}$. A solution β on $\mathcal{P}$, which is closed under multiplication, satisfies *multiplication invariance* if for every $P \in \mathcal{P}$ and $\lambda > 0$ we have:

$$\beta(\lambda P) = \beta(P)$$

The translation and multiplication invariance axioms are desirable axioms in many situations. Consider, for example, the example above where $\mathcal{P}$ denotes a collection of optimization problems dealing with profit maximization and different monetary systems can be used to describe one and the same problem.

We shall use a specific example[1] to illustrate the basic tension between approximation consistency on one side and the invariance axioms on the other side. Consider two optimization problems (A, u) and (A, v) with the same set of alternatives, namely rooms with various degrees of hotness. In both problems, room a is ideal. In problem (A, u), $u(b)$ is the absolute value of the temperature difference between rooms a and b measured in degrees Fahrenheit, while in problem (A, v), $v(b)$ is the absolute value of the temperature difference between rooms a and b measured in degrees Celsius. Suppose that in problem (A, u) any room whose temperature is within 9°F of room a is satisfactory. It then follows from approximation consistency that in problem (A, v), any room whose temperature is within 9°C of room a is satisfactory. Using the invariance axioms, it then follows that in problem (A, u) any room whose temperature is within 16.2°F of room a is satisfactory. Continuing in this way we conclude that every room is satisfactory.

What do we learn from this example? The answer is that approximation consistency clearly rules out the possibility of scale having a meaning. To be more precise, if one wants to use approximation consistency in optimization problems and information about the scale is available, then one can not change the scale. However, we want to stress that using the Celsius scale in the example (A, v) above is using more information than we are considering in our setting. Nothing wrong with this, but such additional information is not always available. Consider for example a decision maker who has to choose some alternative from a set A, and let $u: A \to \mathbb{R}$ be a von Neumann–Morgenstern utility function describing the preferences of the decision maker. Such a description, which is not rare, does not convey any kind of information about any kind of scale. However, any other von Neumann–Morgenstern utility function v describing the same preferences has the form $v = \alpha u + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$. In this situation we want to see whether we

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[1] Suggested by a referee.

can use the axiom of approximation consistency in a meaningful way. Basically this is the issue addressed in Section 4.

3. Axioms and examples

For approximation consistent solutions we have the following proposition.

Proposition 3.1. *Let β be an approximation consistent solution on $\mathcal{P}$ and let (A, u), $(B, v) \in \mathcal{P}$ be such that $u(A) = v(B)$. Then there is a subset T of $u(A)(= v(B))$ such that $\beta(A, u) = u^{-1}(T)$ and $\beta(B, v) = v^{-1}(T)$.*

Proof. Take $T := u(\beta(A, u))$.

If $a \in u^{-1}(T)$ then $u(a) = u(a')$ for some $a' \in \beta(A, u)$ and hence, by approximation consistency, $a \in \beta(A, u)$. So, $u^{-1}(T) \subseteq \beta(A, u)$. The inclusion $\beta(A, u) \subseteq u^{-1}(T)$ is obvious.

If $b \in v^{-1}(T)$ then $v(b) = u(a')$ for some $a' \in \beta(A, u)$. Since (A, u) and (B, v) are sup-equivalent we get, by approximation consistency, $b \in \beta(B, v)$. So, $v^{-1}(T) \subseteq \beta(B, v)$. If $b \in \beta(B, v)$ then, since $u(A) = v(B)$, there is an $a \in A$ such that $u(a) = v(b)$ and hence, by approximation consistency, $a \in \beta(A, u)$. Therefore, $v(b) = u(a) \in T$ and hence $b \in v^{-1}(T)$. So, $\beta(B, v) \subseteq v^{-1}(T)$. $\square$

Proposition 3.1 shows that, if β is approximation consistent, the set $\beta(A, u)$ only depends on the range $u(A)$ of u. Although much information about the optimization problem (A, u) is lost by considering only the values of u, this approach is an extremely common one. So, if we are interested in approximation consistent solutions only, we may identify an optimization problem (A, u) with $u(A)$, the range of u, which is a subset of $\mathbb{R}$. In this and the following sections we focus on this approach.

Let $\mathcal{S}$ be a non-empty collection of non-empty subsets of $\mathbb{R}$. A *solution* σ on $\mathcal{S}$ is a map which assigns to every $S \in \mathcal{S}$ a subset $\sigma(S)$ of S.

A solution σ on $\mathcal{S}$ satisfies (MON) (*monotonicity*) if for every $S_1, S_2 \in \mathcal{S}$ with sup $S_1 \leq$ sup S_2 the following statement is true:

if $s_2 \in \sigma(S_2)$ and $s_1 \in S_1$ is such that $s_1 \geq s_2$ then $s_1 \in \sigma(S_1)$

A solution σ on $\mathcal{S}$ satisfies (AC) (*approximation consistency*) if for every $S_1, S_2 \in \mathcal{S}$ with sup $S_1 =$ sup S_2 the following statement is true:

if $s_2 \in \sigma(S_2)$ and $s_1 \in S_1$ is such that $s_1 \geq s_2$ then $s_1 \in \sigma(S_1)$

The reason for introducing (AC) is given by Proposition 3.1: it is immediate to see that, if a solution β on $\mathcal{P}$ is approximation consistent, as defined in the previous section, then the induced solution σ on the family $\mathcal{S}$ of ranges $u(A)$ $((A, u) \in \mathcal{P})$, satisfies (AC). Conversely, a solution σ on $\mathcal{S}$, satisfying (AC), induces, for every $\mathcal{P}$ with ranges in $\mathcal{S}$, an approximation consistent solution β.

A solution σ on $\mathcal{S}$ satisfies (WAC) (*weak approximation consistency*) if for every $S \in \mathcal{S}$ the following statement is true:

if $s \in \sigma(S)$ and $s' \in S$ is such that $s' \geq s$ then $s' \in \sigma(S)$

The axioms of non-emptiness, translation invariance and multiplication invariance, as defined for solutions on a class $\mathcal{P}$ of optimization problems in Section 2, can be extended in a straightforward way to solutions on a collection $\mathcal{S}$ of non-empty subsets of $\mathbb{R}$.

A solution σ on $\mathcal{S}$ satisfies (NEM) (*non-emptiness*) if for every $S \in \mathcal{S}$ we have:

$$\sigma(S) \neq \emptyset$$

The collection $\mathcal{S}$ is *closed under translation* (CL+) if for every $S \in \mathcal{S}$ and $t \in \mathbb{R}$ we have $t + S := \{t + s: s \in S\} \in \mathcal{S}$. A solution σ on $\mathcal{S}$, obeying (CL+), satisfies (TI) (*translation invariance*) if for every $S \in \mathcal{S}$ and $t \in \mathbb{R}$ we have:

$$\sigma(t + S) = t + \sigma(S)$$

The collection $\mathcal{S}$ is *closed under multiplication* (CL*) if for every $S \in \mathcal{S}$ and $\lambda > 0$ we have $\lambda S := \{\lambda s: s \in S\} \in \mathcal{S}$. A solution σ on $\mathcal{S}$, obeying (CL*), satisfies (MI) (*multiplication invariance*) if for every $S \in \mathcal{S}$ and $\lambda > 0$ we have:

$$\sigma(\lambda S) = \lambda \sigma(S)$$

In this paper we will also characterize solutions on $\mathcal{S}$ making use of the axiom (CCA) (*Chernoff's Choice Axiom*) (see Chernoff, 1954). This axiom is defined as follows: a solution σ on $\mathcal{S}$ satisfies (CCA) if for every $S, T \in \mathcal{S}$ with $S \subseteq T$ one has:

$$\sigma(T) \cap S \subseteq \sigma(S)$$

So, if σ satisfies (CCA), selection by σ of an element $s \in T$, implies selection by σ of s in any subset S of T with $s \in S$. The axiom (CCA) is weaker than the independence of irrelevant alternatives axiom used, for example, in Kaneko (1980) and Peters (1992).

Example 3.1. For the following examples suppose that $\mathcal{S}$ is the collection of all non-empty subsets of $\mathbb{R}$.

(a) The solution σ_{mix} defined by:

$$\sigma_{\text{mix}}(S) := \begin{cases} \{s \in S: s \geq \sup S - 1\} & \text{if } \sup S \leq 0 \\ \{s \in S: s > \sup S - 1\} & \text{if } \sup S \in (0, +\infty) \\ \{s \in S: s \geq 22\} & \text{if } \sup S = +\infty \end{cases}$$

satisfies (AC) and (NEM).

(b) The solution σ_{rat}, defined by:

$$\sigma_{rat}(S) := \begin{cases} S & \text{if } S \subseteq \mathbb{Q} \\ \{s \in S: s \geq 22\} & \text{if } S \not\subseteq \mathbb{Q} \end{cases}$$

satisfies (WAC) and (CCA).

(c) The solution σ_{tot}, defined by:

$$\sigma_{tot}(S) := S$$

satisfies (MON), (NEM), (TI), (MI) and (CCA).

(d) The solution σ_{max}, defined by:

$$\sigma_{max}(S) := \{s \in S: s \geq s' \text{ for every } s' \in S\}$$

satisfies (MON), (TI), (MI) and (CCA).

(e) The solution σ_{ε} (where $\varepsilon > 0$), defined by:

$$\sigma_{\varepsilon}(S) := \{s \in S: s \geq \sup S - \varepsilon\}$$

satisfies (MON), (TI) and (CCA).

(f) The solution σ^k (where $k \in \mathbb{R}$), defined by:

$$\sigma^k(S) := \{s \in S: s \geq k\}$$

satisfies (MON) and (CCA).

(g) The solution $\sigma_{\varepsilon,k}$ (where $\varepsilon > 0$, $k \in \mathbb{R}$), defined by:

$$\sigma_{\varepsilon,k}(S) := \begin{cases} \sigma_{max}(S) & \text{if } \sigma_{max}(S) \neq \emptyset \\ \sigma_{\varepsilon}(S) & \text{if } \sigma_{max}(S) = \emptyset \text{ and } \sigma_{\varepsilon}(S) \neq \emptyset \\ \sigma^k(S) & \text{otherwise} \end{cases}$$

satisfies (WAC) and (NEM).

(h) The solution $\hat{\sigma}_{\varepsilon,k}$ (where $\varepsilon > 0$, $k \in \mathbb{R}$), defined by:

$$\hat{\sigma}_{\varepsilon,k}(S) := \begin{cases} \sigma_{\varepsilon}(S) & \text{if } \sup S \leq k + \varepsilon \\ \sigma^k(S) & \text{if } \sup S > k + \varepsilon \end{cases}$$

satisfies (MON), (NEM) and (CCA). Notice that $\hat{\sigma}_{\varepsilon,k}(S) = \sigma_{\varepsilon}(S) \cup \sigma^k(S)$.

(i) The solution $\sigma_{pro(\alpha,\beta)}(S)$ (where $\alpha > 1$, $\beta < 1$), defined by:

$$\sigma_{pro(\alpha,\beta)}(S) := \begin{cases} \sigma^{\alpha s}(S) & \text{if } \sup(S) = s \in (-\infty, 0) \\ S & \text{if } \sup(S) = 0 \\ \sigma^{\beta s}(S) & \text{if } \sup(S) = s \in (0, +\infty) \\ S & \text{if } \sup(S) = +\infty \end{cases}$$

satisfies (AC), (NEM) and (MI).

The following table summarizes the statements above.

	(MON)	(AC)	(WAC)	(NEM)	(TI)	(MI)	(CCA)
σ_{mix}	−	+	+	+	−	−	−
σ_{rat}	−	−	+	−	−	−	+
σ_{tot}	+	+	+	+	+	+	+
σ_{max}	+	+	+	−	+	+	+
σ_{ε}	+	+	+	−	+	−	+
σ^{k}	+	+	+	−	−	−	+
$\sigma_{\varepsilon,k}$	−	−	+	+	−	−	−
$\hat{\sigma}_{\varepsilon,k}$	+	+	+	+	−	−	+
$\sigma_{pro(\alpha,\beta)}$	−	+	+	+	−	+	−

4. Characterizations for translation and multiplication invariant solutions

Let $\mathscr{S}$ be a collection of non-empty subsets of $\mathbb{R}^*$. We write $\mathscr{S} = \bigcup_{k \in \mathbb{R}^*} \mathscr{S}_k$ where $\mathscr{S}_k := \{S \in \mathscr{S} : \sup S = k\}$ for every $k \in \mathbb{R}^*$. The collection $\mathscr{S}$ is *complete* if all intervals belong to $\mathscr{S}$. Recall that elements of $\mathscr{S}$ can be considered as ranges $u(A)$, which are intervals if, for instance, A is connected and u continuous. So, $\mathscr{S}$ is complete if the underlying collection of optimization problems $\mathscr{P}$ is 'rich' enough.

A solution σ on $\mathscr{P}$ is *closed* if $\sigma(S)$ is a closed subset of S for every $S \in \mathscr{S}$. For a function $a : \mathbb{R}^* \to \overline{\mathbb{R}}$ we define the closed solution σ_a on $\mathscr{S}$ by:

$$\sigma_a(S) := [a(k), k] \cap S$$

where $k = \sup S$ (in particular $\sigma_a(S) = \emptyset$ if $a(k) > k$ or $a(k) = k$ and $k \notin S$ and $\sigma_a(S) = S$ if $a(k) = -\infty$). So, σ_a selects, for every $S \in \mathscr{S}_k$, the elements $s \in S$ with $s \geq a(k)$. Clearly, σ_a satisfies (AC). The following proposition shows that the closed solutions satisfying (AC) are precisely the solutions σ_a, provided that $\mathscr{S}$ is complete. Note that the solutions, defined in Example 3.1(c)–(f), (h), (i), satisfy (AC) and are, in fact, σ_a for some suitably chosen a.

Proposition 4.1. *Let $\mathscr{S}$ be a complete collection of non-empty subsets of $\mathbb{R}$ and let σ be a closed solution on $\mathscr{S}$. The solution σ satisfies (AC) if and only if $\sigma = \sigma_a$ for some function a.*

Proof. We only prove the only-if-part. So, assume that σ satisfies (AC). Let $k \in \mathbb{R}^*$. Define $S_k \in \mathscr{S}$ by $S_k := (-\infty, k]$ if $k \in \mathbb{R}$ and $S_k := \mathbb{R}$ if $k = +\infty$. Since, by (AC) and the fact that σ is closed, $\sigma(S_k)$ is a closed interval there is an $a(k) \in \overline{\mathbb{R}}$ such that $\sigma(S_k) = \{x \in S_k : x \geq a(k)\}$. By (AC) we get $\sigma(S) = [a(k), k] \cap S = \sigma_a(S)$ for every $S \in S_k$. □

If we impose some feasibility condition upon the function a we get closed solutions which are characterized by (AC) and (NEM).

Proposition 4.2. *Let $\mathscr{S}$ be a complete collection of non-empty subsets of $\mathbb{R}$ and let σ be a closed solution on $\mathscr{S}$. The solution σ satisfies (AC) and (NEM) if and only if $\sigma = \sigma_a$ for some function $a\colon \mathbb{R}^* \to \overline{\mathbb{R}}$ satisfying $a(k) < k$ for every $k \in \mathbb{R}^*$.*

Proof. Straightforward by using the fact that $\sigma((-\infty, k]) \neq \emptyset$ for every $k \in \mathbb{R}^*$ and Proposition 4.1. □

The next theorem describes the class of solutions, which are characterized by (AC), (NEM) and (TI). It turns out that these solutions, restricted to the collection of upper bounded sets, coincide with the collection of 'ε-optimal' solutions for some $\varepsilon \in (0, +\infty]$.

Proposition 4.3. *Let $\mathscr{S}$ be a complete collection of non-empty subsets of $\mathbb{R}$ which satisfies (CL +) and let σ be a closed solution on $\mathscr{S}$. The solution σ satisfies (AC), (NEM) and (TI) if and only if there is an $\varepsilon \in (0, +\infty]$ such that $\sigma = \sigma_a$, where $a\colon \mathbb{R}^* \to \overline{\mathbb{R}}$ is defined by:*

$$
\begin{cases}
a(k) := k - \varepsilon & \text{for every } k \in \mathbb{R} \\
a(+\infty) = -\infty
\end{cases}
\tag{1}
$$

Proof. Clearly, $\sigma = \sigma_a$ satisfies (AC), (NEM) and (TI) if a is defined by (1). In order to prove the only-if-part suppose that σ satisfies (AC), (NEM) and (TI). By (AC) and (NEM) we know, according to Proposition 4.2, that $\sigma = \sigma_a$, for some function $a\colon \mathbb{R}^* \to \overline{\mathbb{R}}$ satisfying $a(k) < k$ for every $k \in \mathbb{R}^*$. Take $\varepsilon = -a(0)$. For every $k \in \mathbb{R}$ we have, by (TI):

$$
\begin{aligned}
[a(k), k] &= \sigma((-\infty, k]) &&= \sigma(k + (-\infty, 0]) \\
&= k + \sigma((-\infty, 0]) &&= k + [a(0), 0] \\
&= [k + a(0), k]
\end{aligned}
$$

and hence $a(k) = k + a(0) = k - \varepsilon$. The only thing which remains to be shown is that $a(+\infty) = -\infty$. Since $\sigma(\mathbb{R}) \neq 0$ we can choose $s \in \sigma(\mathbb{R})$. Then for every $t \in \mathbb{R}$ we have $t + \mathbb{R} = \mathbb{R}$ and hence, by (TI), $t + s \in \sigma(t + \mathbb{R}) = \sigma(\mathbb{R})$. Therefore, $\sigma(\mathbb{R}) = \mathbb{R}$ and hence $a(+\infty) = -\infty$. □

The solution $\hat{\sigma}_{\varepsilon,k}$ of Example 3.1 satisfies (AC) and (NEM) but not (TI), the solution σ_ε satisfies (AC) and (TI) but not (NEM) and one easily constructs a solution satisfying (NEM) and (TI) but not (AC) (simply by defining the solution for S with sup $S \in \{0, +\infty\}$ in an arbitrary but not approximation consistent way and extending this solution by translation invariance). Therefore, the axioms (AC), (NEM) and (TI) are logically independent.

In the following proposition we characterize the 'proportional' solutions by (AC), (NEM) and (MI).

Proposition 4.4. *Let $\mathscr{S}$ be a complete collection of non-empty subsets of $\mathbb{R}$ which satisfies (CL*) and let σ be a closed solution on $\mathscr{S}$. The solution σ satisfies (AC),*

(NEM) and (MI) if and only if there are $\alpha > 1$ and $\beta < 1$ such that $\sigma = \sigma_a$, where a: $\mathbb{R}^* \to \overline{\mathbb{R}}$ *is defined by:*

$$
\begin{cases}
a(k) := \alpha k & \text{for every } k < 0 \\
a(0) = -\infty \\
a(k) := \beta k & \text{for every } k \in (0, \infty) \\
a(+\infty) \in \{-\infty, 0\}
\end{cases}
\tag{2}
$$

Proof. Clearly, $\sigma = \sigma_a$ satisfies (AC), (NEM) and (MI) if a is defined by (2). In order to prove the only-if-part suppose that σ satisfies (AC), (NEM) and (MI). By (AC) and (NEM) we know, according to Proposition 4.2, that $\sigma = \sigma_a$, for some function a: $\mathbb{R}^* \to \overline{\mathbb{R}}$ satisfying $a(k) < k$ for every $k \in \mathbb{R}^*$. Take $\alpha = -a(-1)$ and $\beta = a(1)$. For every $k < 0$ we have, by (MI):

$$
\begin{aligned}
[a(k), k] &= \sigma((-\infty, k]) & &= \sigma(-k(-\infty, -1]) \\
&= -k\sigma((-\infty, -1]) & &= -k[a(-1), -1] \\
&= [-ka(-1), k]
\end{aligned}
$$

and hence $a(k) = -ka(-1) = \alpha k$. In the same way one can prove that $a(k) = \beta k$ for every $k \in (0, +\infty)$. For every $\lambda \in (0, \infty)$ we have:

$$
\begin{aligned}
[a(0), 0] &= \sigma((-\infty, 0]) & &= \sigma(\lambda(-\infty, 0]) \\
&= \lambda\sigma((-\infty, 0]) & &= \lambda[a(0), 0] \\
&= [\lambda a(0), 0]
\end{aligned}
$$

Therefore, $a(0) = \lambda a(0)$ for every $\lambda \in (0, +\infty)$. Since $a(0) < 0$ we get $a(0) = -\infty$. In a similar way one can prove that $a(+\infty) \in \{-\infty, 0\}$. $\square$

The solution $\hat{\sigma}_{\varepsilon,k}$ of Example 3.1 satisfies (AC) and (NEM) but not (MI), the solution $\sigma_{\max}$ satisfies (AC) and (MI) but not (NEM) and one easily constructs a solution satisfying (NEM) and (MI) but not (AC) (simply by defining the solution for S with sup $S \in \{-1, 0, 1, +\infty\}$ in an arbitrary but not approximation consistent way and extending this solution by multiplication invariance). Therefore, the axioms (AC), (NEM) and (MI) are logically independent.

Clearly, the trivial solution σ_{tot} satisfies (AC), (NEM), (TI) and (MI). In the following proposition we show the impossibility of finding another solution, satisfying these four properties.

Theorem 4.1. *Let $\mathscr{S}$ be a complete collection of non-empty subsets of $\mathbb{R}$, which satisfies (CL+) and (CL*) Let σ be a closed solution on $\mathscr{S}$. The solution σ satisfies (AC), (NEM), (TI) and (MI) if and only if $\sigma = \sigma_{\text{tot}}$.*

Proof. Again we only prove the only-if-part. Suppose σ satisfies (AC), (NEM), (TI) and (MI). By Proposition 4.4 we get $\sigma((-\infty, 0]) = (-\infty, 0]$ and hence, by (TI):

$$
\sigma((-\infty, k]) = k + \sigma((-\infty, 0]) = k + (-\infty, 0] = (-\infty, k]
$$

for every $k \in \mathbb{R}$. Moreover, by Proposition 4.3, we get $\sigma(\mathbb{R}) = \mathbb{R}$. Using (AC) we may conclude that $\sigma(S) = S$ for every $S \in \mathscr{S}$. □

Let us notice that, in the context of decision making under risk, u and v are von Neumann–Morgenstern utility functions representing the same preferences iff $v = cu + d$, with $c > 0$ and $d \in \mathbb{R}$. So, if one wants to stress the point of view that only preferences have a true meaning, one should use a 'solution rule' for optimization problems that takes this fact into account. But Theorem 4.1 just shows that it is impossible to do this in a non-trivial way. Stated otherwise: for von Neumann–Morgenstern preferences there is no sensible concept of approximate optimum! If one wants to talk in a meaningful way of approximate optimization, an escape route could be the addition of further details that allow for some 'absolute' reference point (for example: how do we decide whether the oscillations of last week at the New York Stock Exchange were wild or not? Maybe we refer to the previous history as a benchmark). The interesting question is whether it can be done in a consistent way, without resorting to an 'absolute' utility function.

Remark. *If a solution σ on $\mathscr{S}$ satisfies (AC) then, for every $S \in \mathscr{S}$, $\sigma(S)$ can be described as $\{s \in S: s \geq \gamma\}$ or $\{s \in S: s > \gamma\}$ for some γ (depending on S). However, we can have strict or weak inequality, depending on the value of sup S, as can be seen in Example 3.1(a). In order to get rid of these kind of approximate solutions we have added the requirement that $\sigma(S)$ is a closed subset of S. However, this addition does not 'force' the parentheses to be closed. Consider, for example, $\mathscr{S} = \{\{0, 1\}\} \cup \{[\alpha, 1]: \alpha \in (0, 1)\}$ and let σ be the solution on $\mathscr{S}$, defined by $\sigma(\{0, 1\}) := \{1\}$ and $\sigma([\alpha, 1]) := [\alpha, 1]$ for every $\alpha \in (0, 1)$. The solution σ satisfies (AC) and $\sigma(S)$ is a closed subset of S for every $S \in \mathscr{S}$, but there is no $\gamma \in \overline{\mathbb{R}}$ such that $\sigma(S) = \{s \in S: s \geq \gamma\}$ for every $S \in \mathscr{S}$. The problems in this example are due to the fact that the collection $\mathscr{S}$ is rather 'poor'. For this reason we added the requirement that $\mathscr{S}$ should be complete, i.e. contains all intervals.*

However, the Propositions 4.1–4.4 and Theorem 4.1 can also be shown to be true for collections $\mathscr{S}$ which are not complete. In this situation however, the proofs become a little bit more technical and an appropriate strengthening of (AC) is needed: a solution σ on $\mathscr{S}$ satisfies (SAC) (strong approximation consistency) if for every $S, S_1, S_2, \ldots \in \mathscr{S}$ with sup $S = $ sup S_i for every $i \in \mathbb{N}$ the following statement is true:

if $s_i \in \sigma(S_i)$ for every $i \in \mathbb{N}$ and $s \in S$ is such that $s \geq \liminf_{i \to \infty} s_i$ then $s \in \sigma(S)$

One easily verifies that (SAC) induces (AC). Moreover, (SAC) implies that $\sigma(S)$ is a closed subset of S for every $S \in \mathscr{S}$. In fact, if $\mathscr{S}$ is a collection of intervals, then σ satisfies (SAC) if and only if σ satisfies (AC) and $\sigma(S)$ is a closed subset of S for every $S \in \mathscr{S}$.

The impossibility result of Theorem 4.1 is caused by the fact that the domain S contains elements without maximum. Let us notice, in particular, that whenever we have at least a couple of problems in $\mathscr{S}$ with the same supremum, but one of which has a maximum and the other not, the (AC) axiom prevents the possibility of choosing the

maximum only, whenever it exists. In fact, a solution like $\sigma_{\varepsilon,k}$ in Example 3.1(g), which tries to capture this kind of idea, violates (AC), as can easily be checked, whenever in $\mathcal{S}$ there are sets with maximum and others without. If we restrict our attention to domains $\mathcal{S}$ containing only elements S for which max S is well-defined then it turns out that $\sigma_{\max}$ is the only non-trivial solution satisfying (AC), (NEM), (TI) and (MI).

Theorem 4.2. *Let $\mathcal{S}$ be a collection of non-empty subsets of $\mathbb{R}$ such that every element of $\mathcal{S}$ has a maximum. Suppose, moreover, that $\mathcal{S}$ satisfies (CL +) and (CL*). Let σ be a solution on $\mathcal{S}$. The solution σ satisfies (AC), (NEM), (TI) and (MI) if and only if $\sigma = \sigma_{\mathrm{tot}}$ or $\sigma = \sigma_{\max}$.*

Proof. Again we only prove the only-if-part. Suppose σ satisfies (AC), (NEM), (TI) and (MI) and suppose that $\sigma \neq \sigma_{\max}$. Then there is some $S^* \in \mathcal{S}$ with $\sigma(S^*) \neq \sigma_{\max}(S^*) = \{\max S^*\}$. By (NEM) we may conclude that there is an $s^* \in \sigma(S^*)$ such that $s^* < \max S^*$. In order to show that $\sigma = \sigma_{\mathrm{tot}}$ take $S \in \mathcal{S}$ and $s \in S$. It suffices to prove that $s \in \sigma(S)$. Choose $\alpha > 0$ such that $\alpha(s^* - \max S^*) + \max S \leq s$ and define $\beta := -\alpha \max S^* + \max S$. Define, moreover, $S' := \alpha S^* + \beta$. By (CL+) and (CL*) we have $S' \in \mathcal{S}$ and by (TI) and (MI) we have $\alpha s^* + \beta \in \sigma(S')$. Since $\alpha s^* + \beta = \alpha(s^* - \max S^*) + \max S \leq s$ and $\max S' = \max S$ we infer by (AC) that $s \in \sigma(S)$. $\square$

5. Characterizations under CCA

Proposition 4.3 provides a characterization of solutions σ on collections of *upper bounded* subsets of $\mathbb{R}$: there is an $\varepsilon \in (0, +\infty]$ such that σ selects all 'ε-optimal' elements. In order to get a nice characterization, which takes also the *unbounded* subsets of $\mathcal{S}$ into account, we can make use of the axiom (CCA) instead of (TI).

Proposition 5.1. *Let $\mathcal{S}$ be a complete collection of non-empty subsets of $\mathbb{R}$ and let σ be a closed solution on $\mathcal{S}$. The solution σ satisfies (AC), (NEM) and (CCA) if and only if $\sigma = \sigma_a$ for some non-decreasing function $a: \mathbb{R}^* \to \overline{\mathbb{R}}$ satisfying $a(k) < k$ for every $k \in \mathbb{R}^*$.*

Proof. We only prove the only-if-part. Suppose σ satisfies (AC), (NEM) and (CCA). By (AC) and (NEM) we may conclude, according to Proposition 4.2, that $\sigma = \sigma_a$ for some function $a: \mathbb{R}^* \to \overline{\mathbb{R}}$ satisfying $a(k) < k$ for every $k \in \mathbb{R}^*$. In order to show that a is non-decreasing let $k, l \in \mathbb{R}^*$ be such that $k \leq l$. Suppose that $a(k) > a(l)$. Then there is an $s \in \sigma((-\infty, l))$ such that $s < a(k) < k$. Since also $s \in (-\infty, k)$ we get, by (CCA), $s \in \sigma((-\infty, k))$, and hence $s \geq a(k)$. Contradiction. $\square$

One easily verifies that the solutions, characterized in Proposition 5.1 by (AC), (NEM) and (CCA), satisfy (MON). However, since (MON) implies (AC) and (CCA), these solutions can also be characterized by (MON) and (NEM).

A result similar to that in Theorem 4.1 can be obtained by using (CCA) and (WAC) instead of (AC). Note that (AC) does not imply (CCA) [see, for example, Example

3.1(a)] and that (CCA) and (WAC) does not imply (AC) [see, for example, Example 3.1(b)]. Of course, (WAC), (NEM), (TI), (MI) and (CCA) do imply (AC), according to the following theorem.

Theorem 5.1. *Let $\mathscr{S}$ be a complete collection of non-empty subsets of $\mathbb{R}$, which satisfies (CL +) and (CL*). Let σ be a closed solution on $\mathscr{S}$. The solution σ satisfies (WAC), (NEM), (TI), (MI) and (CCA) if and only if $\sigma = \sigma_{\text{tot}}$.*

Proof. Again, we only prove the only-if-part. Suppose σ satisfies (WAC), (NEM), (TI), (MI) and (CCA). Let $k \in \mathbb{R}$ and $S_k := (-\infty, k)$. By (NEM) we know that $\sigma(S_k) \neq \emptyset$, so we can choose an $s \in \sigma(S_k)$. Of course, $s < k$. We will show that $\sigma(S_k) = S_k$. Therefore, let $t \in S_k$. Define $\lambda := (k - t)/(k - s) > 0$ and $\mu := (1 - \lambda)k \in \mathbb{R}$. Then $t = \lambda s + \mu$, $S_k = \lambda S_k + \mu$ and hence, by (MI) and (TI), $t \in \sigma(S_k)$. Therefore, $\sigma(S_k) = S_k$. By (CCA) we get $\sigma(S) = S$ for every $S \in \mathscr{S}_k$. By (NEM) and (TI) one easily infers that $\sigma(\mathbb{R}) = \mathbb{R}$. Hence, by (CCA), we get that $\sigma(S) = S$ for every $S \in \mathscr{S}_{+\infty}$. $\square$

Remark. *If the collection $\mathscr{S}$ is not complete, then Proposition 5.1 need not be valid anymore, as the following example shows: consider the class $\mathscr{S}$ of all non-empty and upper bounded subsets $\mathscr{S}$ which satisfy the condition that there exists a $t \in [0,1)$ such that $S \subseteq t + \mathbb{Z}$. Define the solution σ_a by:*

$$a(k) := \begin{cases} k - 22 & \text{if } k \in \mathbb{Z} \\ k - 37 & \text{otherwise} \end{cases}$$

Clearly σ_a satisfies (AC) and (NEM). It also satisfies (CCA): this is due to the fact that for $S, T \in \mathscr{S}$ with $S \subseteq T$ both are contained in the same $t + \mathbb{Z}$ for some $t \in [0, 1)$. In fact, one can prove that any σ_a, with a feasible function a which is non-decreasing on $t + \mathbb{Z}$ for every $t \in [0, 1)$, satisfies (AC), (NEM) and (CCA). In fact, $\mathscr{S}$ can be partitioned into several subcollections such that sets belonging to different subcollections are not related by inclusion. These problems do not occur if $\mathscr{S}$ is complete. A characterization of the solutions in Proposition 5.1 by (MON) and (NEM) on non-complete collections $\mathscr{S}$ can still be obtained by using an appropriate strengthening of (MON), similar to the strengthening (SAC) of (AC), mentioned in the remark at the end of Section 4.

6. Conclusions

The main purpose of this paper is to investigate whether an axiomatic analysis can be carried through for approximate solutions of optimization problems. Fundamental axioms in these characterizations are approximation consistency (reflecting the bounded accuracy of the optimizers), translation and multiplication invariance (reflecting the fact that the solution should be independent from the scale which is chosen) and non-emptiness. It turns out that, if an optimum exists in every optimization problem under consideration, the only non-trivial solution satisfying these axioms is the solution selecting the optimum in every problem. If, however, at least one problem does not have an optimum (and one is really obliged to look for approximate solutions) then such a non-trivial solution does not exist anymore. So, in order to find a non-trivial solution

satisfying some desirable properties one is forced to remove some axiom or to weaken the axioms. If we remove one of the invariance axioms, and consider bounded problems only, we are lead to the class of 'ε-optimal' solutions, respectively, the class of 'proportional' solutions. For unbounded problems such an approach yields unsatisfactory results. If we replace both invariance axioms with Chernoff's Choice Axiom, we get the class of 'satisficing' solutions.

In this paper we did not address one important question: are approximate solutions close to the true solution(s) or, stated otherwise, is the optimization problem under consideration Tikhonov well-posed (see Dontchev and Zolezzi, 1993; Patrone, 1987)? Clearly, to answer this question one needs to have some additional (topological) structure on the set of alternatives. Instead we focused on the values of the objective functions, but we consider issues related to Tikhonov well-posedness as quite important. We believe that characterizing "axiomatically" approximate optima, taking into account at the same time values, domains and objective functions is a formidable task, which should be investigated, perhaps, first in some specific contexts (e.g. maxima of concave functions on convex subsets of euclidean spaces).

Acknowledgements

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