Endogenous coalition formation and bargaining

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ter verkrijging van de graad van doctor aan de Katholieke Universiteit Brabant, op gezag van de rector magnificus, prof. dr. F.A. van der Duyn Schouten, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op woensdag 28 juni 2000 om 16.15 uur door

MARÍA DEL PILAR MONTERO GARCÍA

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Contents

Chapter 1

Coalition Formation and Game Theory

1.1 Coalition formation

Coalition formation is a very frequent activity in everyday life. Firms merge, states sign treaties, authors write papers together and parties form governments. All these situations are characterized by the profitability of cooperation together with the conflict over which coalitions should form and how the gains from cooperation should be divided. The questions of which coalitions will form and how the gains from cooperation will be divided have received attention since the beginning of game theory, but a widely accepted answer has not been provided yet. This thesis attempts to contribute to our understanding of the problem.

Consider the following situation. Alice, Bob and Chris (to whom we will refer as the players) work in a small university. Alice is a macroeconomist, Bob is a microeconomist and Chris is a computer programmer. Alice and Chris can write a book on macroeconomics (Alice would do the theory and Chris would do some simulations); Bob and Chris can write a book on microeconomics (Bob would do the theory and Chris would program some experiments); Alice and Bob can write a book on macroeconomics with microfoundations. The first book is expected to produce a revenue of 60, the second book is expected to produce a revenue of 80 and the third book is expected to produce a revenue of 40. Each individual has time to work on only one book.

Which coalition will form? All three coalitions are profitable, but players have time to participate in one coalition only, thus participating in a coalition has an opportunity cost. There is a conflict over which coalition should form (no individual wants to be left out). Furthermore, even if two individuals agree on forming a coalition, each of them would want to get as much as possible.

The solution to this problem seems easy if we assume that coauthors divide revenues equally. Alice and Chris would get 30 each from the first book, Bob and Chris would get 40 each from the second one, and Alice and Bob would get 20 each from the third one. One should then expect that Bob and Chris would form a coalition, since each of them prefers this alternative to forming a coalition with Alice.

	Alice	Bob	Chris
Alice-Chris	30	-	30
Bob-Chris	-	40	40
Alice-Bob	20	20	-

Table 1.1: Equal Payoff Division in Coalitions

If we look at the above situation more closely, however, we see that the assumption that coalition payoffs will be split equally is problematic. After all, Chris is the most valuable player (coalitions including her earn a higher per capita revenue). Why should she be ready to accept only half of the revenue?

Indeed, while Bob and Chris are considering forming a coalition and splitting the revenues equally, Alice can approach Chris and offer her 41. Then Bob is left out and may offer her 42, and so on. How high can these offers get? Suppose Alice and Bob each offer 55 to Chris. If Chris accepts Bob's offer, Bob gets 25. If Chris accepts Alice's offer, Alice gets 5. It is then better for Bob and Alice to form a coalition themselves and get, for example, 30 and 10. Hence, it seems that the competition between Alice and Bob will not drive the offers to Chris up to 55.

The reader can check that, whatever coalition and payoff division we start with, two of the players could form another coalition with a payoff division that makes them both better-off. In the language of game theory, the *core* of the game is empty.

Does this mean that we cannot make any prediction about the outcome of this situation? Is any proposed payoff division equally unstable? Suppose that Alice demands 10 for her participation in a coalition, Bob demands 30 and Chris demands 50. These payoffs have the property that players are indifferent among possible coauthors. If Alice demands more, none will want to form a coalition with her. If she demands less, the other two players will want to form a coalition with her, and competition between them will drive Alice's payoff up to 10. The values 10, 30 and 50 have the property that competition between the players to be in a coalition tends to drive the payoff demands towards them. In the language of game theory, these values are the *quotas* of the players.

If we accept the quotas as reasonable payoff demands, the question of which coalition will form is left undetermined. Even though players have very different strengths, each makes as good a coalition partner as anybody else.

The assumption of equal payoff division leads to the conclusion that Bob and Chris will form a coalition; on the other hand, assuming competition allows us to predict the "competitive" payoff division within each coalition if it forms, but it leaves the question of which coalition will form unanswered.

Let us now come back to the starting point of this reasoning. Suppose the coalition between Bob and Chris has been proposed with a payoff division of 40 for each. Should Chris argue that she is more valuable than Bob and therefore she should get more or should she be "generous" and accept the equal payoff division? Is it better to get 40 for sure or to get 50 only if one manages to get into a coalition? Notice also that a division of $(40, 40)$ is very safe for Chris, since Alice cannot profit from trying to lure Bob into a coalition with her. Non-competitive behavior in order to heighten one's chance of getting in a coalition is frequent in real life and has also been observed in experiments.

1.2 Game theory

This thesis attempts to give a prediction for situations like the one described above using game theory. Game theory is a set of analytical tools designed to help us understand the phenomena we observe when decision-makers interact (see Osborne and Rubinstein (1994)); coalition formation is one of those phenomena. The basic assumptions that underlie the theory are that decision makers pursue well-defined objectives (they are *rational*) and take into account their knowledge of the rationality of others and form expectations of other decision makers' behavior accordingly (they *reason strategically*).

Game theory is often divided into two different branches, called cooperative game theory and noncooperative game theory. Cooperative game theory focuses on the "big issues" like the payoffs players can achieve together, and not on the concrete strategies players follow in order to achieve those payoffs; in noncooperative game theory all the "practical details" of the game, like the exact set of actions available to the agents and the order of moves, are described in full. Aumann (1997) puts it in the following way

> The noncooperative theory is strategy oriented. It studies what we expect the players to *do* in the game.(...) In the cooperative approach we look directly at the set of *outcomes*, not the nitty-gritty of how one gets there. The noncooperative theory is a kind of micro theory; it involves precise descriptions of what happens (...). The cooperative approach studies games from a macro point of view.

The names of cooperative and noncooperative refer to the fact that cooperative game theory often assumes that players have enforceable contracts available to them, whereas noncooperative game theory often allows for no enforceable contracts at all. However, this distinction is not sharp: one can find cooperative models without binding agreements (e.g. Chwe (1994)) and noncooperative models with the possibility of binding agreements (we will study some of them in this thesis).

Both approaches have advantages as well as drawbacks. Cooperative games are very general, since they abstract from many practical details. On the other hand, there are several solution concepts for cooperative games. Each of them tries to reproduce the arguments players may make in face-to-face communication; it defines a set of arrangements that are immune to deviations of a certain sort by groups of players. Since there are many sorts of arguments players may make during negotiations, there are also many cooperative solution concepts.

Noncooperative game theory tries to pin down those arguments. The rules of the game severely restrict the type of arguments players can make. There is a widely accepted solution concept: the Nash equilibrium (Nash (1951)). This solution concept represents an equilibrium in the sense that each player correctly anticipates what the other players are going to do and acts rationally on the basis of this anticipation. Other solution concepts are merely refinements or coarsenings of the basic concept of Nash equilibrium.

Coalition formation has been studied both by cooperative and by noncooperative game theory. The first game-theoretic analysis of coalition formation, due to von Neumann and Morgenstern (1944), is cooperative. They point out that players may profit from coordinating their actions in the play of a noncooperative game. They view the formation of coalitions as previous to the play of the noncooperative game. In the context of a noncooperative three-player majority game, where each player has to name another player uninformed about the choices of the other two players and two players that name each other win the game, they make the following point.

> Since each player makes his personal move in ignorance of those of the others, no collaboration of the players can be established during the course of the play. Two players who wish to collaborate must get together on this subject before the play, i.e., outside the game.

The formation of coalitions is therefore not the outcome of a procedure with detailed rules (of a noncooperative model). Instead, the set of coalitions and payoffs that can be stable is determined by a cooperative solution concept.

Nash (1953), on the other hand, argues that the negotiation process can be modelled as a noncooperative game.

> One makes the players' steps of negotiation in the cooperative game become moves in the non-cooperative model. Of course, one cannot represent all possible bargaining devices as moves in the non-cooperative game. The negotiation process must be formalized and restricted, but in such a way that each participant is still able to utilize all the essential strengths of his position.

If one takes Nash's approach, the formation of coalitions by players in order to coordinate their moves in a game against other players and the game itself are not substantively different, but stages of the same process.

The main lesson from the noncooperative approach is that the predictions of noncooperative games are very sensitive to "institutional" details such as the order in which players move. The cooperative approach does not offer a complete model of how negotiations proceed, and is therefore more general. Indeed, while the number of cooperative models one can write about the same situation is limited, there are many noncooperative models representing the same situation. While there is a multiplicity of models in noncooperative game theory, there is a multiplicity of solutions in cooperative game theory.

Both approaches are complementary and, in my view, none of them is more fundamental than the other. While cooperative models focus on the essentials of the problem, noncooperative models are more explicit in their assumptions, and this makes it easier to choose between noncooperative models than between cooperative solution concepts. Because of being more explicit in their detail, noncooperative models may shed light over cooperative concepts. An example is how the work by Rubinstein (1982) and Binmore et al. (1986) shed light over when and how to use the Nash bargaining solution.

1.3 Overview of the thesis

This thesis is about coalition formation. Chapter 1 gives an introduction to the subject. Chapter 2 contains technical preliminaries and a review of the literature. Part I (comprising chapters 3, 4, 5 and 6) focuses on noncooperative game theory, and part II (chapters 7 and 8) on cooperative game theory.

Chapter 3 studies noncooperative models of coalition formation in games with externalities. In a game with externalities the formation of a coalition may affect the payoff of other coalitions. This possibility is present in most practical situations: a merger of some firms may hurt competitors, a treaty between some states to reduce pollution may benefit the countries that stayed outside the treaty, and so on. The chapter gives sufficient conditions for the existence of an efficient equilibrium (in the sense of total payoff maximization). The conclusion of this chapter is that efficiency is difficult to achieve even though there is perfect information and players can make binding agreements. The chapter also compares the assumptions and results of different noncooperative games. This chapter is based on Montero (1999b).

Chapter 4 studies the particular case of constant-sum games with externalities, with special attention to the three-player case. Constant-sum games are special since there are no aggregate gains from cooperation: the profit of forming a coalition must come at the expense of the players outside the coalition. For the three-player case, expected payoffs are fairly robust to changes in the details of the basic extensive form game. Players are indifferent as to which partner to choose, so that the fact that one player brings a lot of value to a coalition makes no difference for his desirability as coalition partner *in equilibrium.*

Chapter 5 studies a classical economic application: the one-seller-two-buyers situation. In this situation, a seller can sell an object to one of two buyers. All players know that one of the buyers is prepared to pay more than the other for the good. The presence of a second buyer, however, creates competition and may drive the price up. The chapter focuses on the situation of the second buyer, who brings no value to the game but affects the distribution of the gains from trade between the other two players; we argue that this player can exploit his position and obtain a positive payoff. We also show that the one-seller-two-buyers game is a particular example of the three-person constant-sum games with externalities in chapter 4. Thus, the game illustrates how externalities can arise in games that *a priori* exhibit no external effects. Chapter 5 is based on Montero (1998).

Chapter 6 studies another classical game: the apex game. An apex game is a weighted majority game with one major party, called the apex player, and several (at least three) minor parties, called the base players. The apex player needs only one of the minor players to form a majority; the only coalition that can obtain a majority without the apex player is the coalition of *all* base players. A base player must decide whether to try to cooperate with the remaining base players or to compete with them for the favor of the apex player. The results in this chapter are similar to the results in chapter 4. In equilibrium, a base player is indifferent between forming a coalition with the apex player and forming a coalition with the other base players. Expected equilibrium payoffs coincide with the predictions of a cooperative solution concept, the *kernel*. They are moreover robust to changes in the rules of the game: if the rules of the game give more power to the apex player, the minor players join against him more often, so that expected payoffs remain constant. This chapter is based on Montero (1999a).

Chapter 7 is an overview of some cooperative solution concepts known as *bargaining sets*. The idea behind the concept of bargaining set is that, while a given organization of players into coalitions (a *coalition structure*) is under consideration and a given proposal for payoff division is on the table, a player (or a group) who can get a higher payoff by departing from the proposed coalition structure and forming another coalition may threaten to do so unless he gets a higher share of the payoffs in the current coalition structure. This threat however will not be effective if the players who are asked to give up part of their payoffs can protect their shares by forming a coalition that excludes the objecting player. A coalition structure is in the bargaining set if any threat (also called *objection*) can be deterred by a counterthreat (also called *counterobjection*). If one coalition structure can find a payoff division that is stable in the sense of the bargaining set, and other coalition structure can not, the interpretation is that the bargaining set predicts that the first coalition structure may form (and if it forms, it will choose a payoff division belonging to the bargaining set) and the second will not.

There are several bargaining sets, based on different concepts of objection and counterobjection. In order to construct a bargaining set, one has to answer questions like: Who can object and against whom? Should objections always be directed against somebody? Can several threats be made at the same time? Who can counterobject? What is required for a counterobjection to actually deter the objection? Chapter 7 discusses these questions with the help of examples.

Finally, chapter 8 is devoted to a new bargaining set, called the *stable demand set* (SDS), introduced by Morelli (1998). This bargaining set is based on the idea that a proposed outcome can be viewed as a pair formed by a coalition structure and a demand vector. This demand vector is not necessarily realized in the coalition structure: in our introductory example, a proposed outcome could be Alice forming a coalition with Bob together with the demand vector $(20, 20, 50)$. Only Alice and Bob would get their demands if this proposal is executed; Chris does not get any payoff but it is understood that she charges a price of 50 from participating in a coalition. Consider the following threat: Bob and Chris may threaten to depart from this proposal by forming a coalition and getting 40 each. This is beneficial for Bob, and also for Chris since by "lowering her demand to 40" she would get 40 rather than 0. Given this objection, a counterobjection has to show that the original demand vector is stable: in this example, Alice would have to prove that the demand vector is stable by forming a coalition with Chris in which both of them receive their *original* demands. This is not possible since the coalition of Alice and Chris is only worth 60 and their demands add up to 70. The demand vector we put forward in the introduction, $(10, 30, 50)$, is the only stable demand vector.

Chapter 8 discusses the properties of the stable demand set and its relation with other bargaining sets. It also contains a characterization of the SDS for constant-sum weighted majority games (these are majority games with the property that, for any distribution of the players into two coalitions, one of the two coalitions has a majority, thus ties are not possible). We argue that the SDS makes sharp and reasonable predictions for some games, though at the price of being empty for others. This chapter is based on Morelli and Montero (2000).

Chapter 2

Preliminaries

This chapter includes some basic definitions and a review of the literature on games with externalities. The definitions are taken from Mas-Colell, Whinston and Green (1995), Myerson (1991), and Osborne and Rubinstein (1994). Some of the definitions are only informal; the reader is referred to these textbooks for more details.

2.1 Preferences and utility

Consider a set of alternatives X. A *preference relation* $\succcurlyeq$ is a binary relation on the set of alternatives X, allowing the pairwise comparison of alternatives in X. Given x, $y \in X$, $x \succcurlyeq y$ means that x is at least as good as y. We can derive from a preference relation $\succcurlyeq$ the corresponding *strict preference relation* $\succ$ ($x \succ y \Leftrightarrow x \succcurlyeq y$ and not $y \succcurlyeq x$) and the corresponding *indifference relation* $\sim$ ($x \sim y \Longleftrightarrow x \succcurlyeq y$ and $y \succcurlyeq x$).

Game theory assumes that players have rational preferences. A preference relation is *rational* if it possesses the following two properties: *completeness* (for all $x, y \in X$, we have $x \succcurlyeq y$ or $y \succcurlyeq x$) and *transitivity* (for all x, y, $z \in X$, $x \succcurlyeq y$ and $y \succcurlyeq z$ implies $x \succcurlyeq z$).

Rational preferences can be represented by *utility functions*. A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succcurlyeq$ if, for all $x, y \in X$, $x \succcurlyeq y \Longleftrightarrow u(x) \geq u(y)$.

Game theory also assumes that alternatives with uncertain outcomes are describable by means of (objective or subjective) probabilities defined on a set of outcomes C (assumed finite for simplicity). These representations of risky alternatives are called *lotteries*. A simple lottery is a list $L = (p_1, ..., p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring. Compound lotteries are lotteries whose outcomes are themselves lotteries. Given a compound lottery, individuals are assumed to care only about the probability of each outcome, so that it is enough to define preferences over simple lotteries. Preferences over lotteries are also assumed to be *continuous* (small changes in the probabilities do not change the ordering between two lotteries) and to satisfy the *independence axiom* (if a simple lottery L is preferred to a simple lottery L', then the compound lottery that assigns probability α to lottery L and probability $1-\alpha$ to lottery L'' is preferred to the lottery that assigns probability α to L' and $1-\alpha$ to L'').

If the preferences over lotteries satisfy the continuity and independence axioms, then these preferences admit a utility representation of the *expected utility form.*

Definition 2.1 A utility function over lotteries U has an **expected utility form** if there is an assignment of numbers $(u_1, ..., u_N)$ to the N outcomes such that for every lottery $L = (p_1, ..., p_N)$ we have $U(L) = \sum_n p_n u_n$.

A utility function with the expected utility form is called a *von Neumann-Morgenstern utility function.* The von Neumann-Morgenstern utility function is a *cardinal utility,* that is, it is unique up to affine transformations.

The theory can be extended to an infinite set of outcomes. We will be interested in the case of outcomes being real numbers; real numbers will be interpreted as amounts of money or other perfectly divisible commodity. In this framework, a lottery is a distribution function $F(\cdot)$ over the set of outcomes. The utility of a lottery can be evaluated by a utility function U of the form $U(F) = \int u(x)dF(x)$.

When the outcomes are real numbers, one can formalize the notion of attitude towards risk. A decision maker is *risk-averse* if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int xdF(x)$ for sure is at least as good as the lottery $F(\cdot)$ itself. A decision maker that has the opposite preferences is called *risk-loving.* If the decision maker is indifferent between these two lotteries for any $F(\cdot)$, we say that he is *risk-neutral.*

The concept of expected utility as a descriptive concept has been severely criticized and alternative theories have been proposed. However, the elegance and the analytical power of the concept of expected utility make it pervasive in game theory.

2.2 Noncooperative game theory

A *game* is a formal representation of a situation of interactive decision making. The results of the actions of one individual depend also on the actions of other individuals, and the actions that are best for him to take may depend on what the others do.

2.2.1 Extensive form games and normal form games

To describe a noncooperative game, we need four elements: the *players*, the *rules* (who moves when? what do they know when they move? what can they do?), the *outcomes* (given each possible set of actions taken by the players, what is the outcome?) and the *payoffs* (what are the players' preferences over the possible outcomes?).

The *extensive form representation* of a game includes all the aspects above. We focus on games with *perfect information*, that is, games in which each player, when making a decision, is informed of all events that have previously occurred. Since extensive form games will play a central role in Part I, we provide a formal definition, taken from Osborne and Rubinstein (1994).

Definition 2.2 An **extensive form game with perfect information** has the following components:

- A set $N = \{1, ..., n\}$ of **players**
- A set H of sequences (finite or infinite) that satisfies the following three properties.
  - The empty sequence $\varnothing$ is a member of H.
  - If $(a^k)_{k=1,...,K} \in H$ (K may be infinite) and $L < K$ then $(a^k)_{k=1,...,L} \in H$.
  - If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1,...,L} \in H$ for every positive integer L then $(a^k)_{k=1}^{\infty} \in H$.

  Each member of H is a **history**; each component of a history is an **action** taken by a player. A history $(a^k)_{k=1,...,K}$ is **terminal** if it is infinite or if there is no a^{K+1} such that $(a^k)_{k=1,...,K,\ K+1} \in H$. The set of terminal histories is denoted Z.
- A function P that assigns to each nonterminal history a member of N.
- For each player $i \in N$ a **preference relation** on lotteries over Z.

After any nonterminal history h, player $P(h)$ chooses an action from the set $A(h) = \{a : (h, a) \in H\}$. A player may choose an action with certainty, or randomize between several actions.

The definition above can be extended to include exogenous uncertainty by adding c (*chance* or *Nature*) as a "player" of the game. The function P can then assign Nature to a nonterminal history. For any h such that $P(h) = c$, the rules of the game specify a probability measure $f_c(\cdot|h)$ (($f_c(a|h)$ is the probability that a occurs after the history h).

The definition can also be extended to allow for simultaneous moves, by making P a set-valued function. Each player in $P(h)$ chooses an action from the set $A_i(h)$ without knowing the actions chosen by other players in $P(h)$; H and P jointly satisfy the condition that for every nonterminal history h there is a collection $\{A_i(h)\}_{i \in P(h)}$ of sets for which $A(h) = \{a : (h, a) \in H\} = \times_{i \in P(h)} A_i(h)$.

The *normal form representation* of a game is a condensed version of the extensive form. It is based on the notion of a player's *strategy.* A strategy is a complete contingent plan, or decision rule, that specifies how the player will act in every possible distinguishable circumstance (at each history) in which he may be called upon to move. The set of *pure* (not randomized) strategies for player i will be denoted by S_i.

Given a strategy combination, the (possibly random) outcome of the game is determined, thus players can associate an expected payoff to each strategy combination.

Randomized strategies are called *mixed strategies*. The set of all mixed strategies is denoted by ΔS_i, with generic element σ. Formally, a mixed strategy is a probability measure over the set of pure strategies. For the class of extensive form games we study, there are two equivalent ways of describing mixed strategies: either players randomize at the beginning of the game (mixed strategies in the proper sense), or players randomize at every history in which they have the move (*behavior strategies*).

Definition 2.3 The **normal form representation** of a game can be denoted by $\Gamma = [N, \{\Delta S_i\}, \{U_i\}]$, where $N = \{1, .., n\}$ is the set of players, ΔS_i is the set of strategies for player i and $U_i(s_1, ..., s_n)$ gives the expected utility levels associated with the lottery arising from strategies $(s_1, ..., s_n)$.

A fundamental postulate of game theory is that all players know the structure of the game, know that the other players know it, know that the other players know that they know it, and so on. This postulate is known as *common knowledge*.

2.2.2 Nash equilibrium

We define the concept of Nash (1951) equilibrium for the normal form of a game.

Definition 2.4 Given a normal form game $\Gamma = [N, \{\Delta S_i\}, \{u_i\}]$, a **Nash equilibrium** is a strategy profile σ^* such that for every player $i \in N$ we have

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*) \text{ for every strategy } \sigma_i \text{ of player } i.$$

In a Nash equilibrium, each player is doing the best he can do given the strategies of the other players. A Nash equilibrium is played if each player has correct conjectures about the strategies the other players are going to choose and chooses (one of) his best strategy given these conjectures.

2.2.3 Some refinements of the Nash equilibrium concept

There are many refinements of the Nash equilibrium concept. We will mention only some of them.

Subgame perfect Nash equilibrium

The concept of Nash equilibrium ignores the dynamic structure of the extensive form game. A strategy combination σ^* may be a Nash equilibrium of the extensive form game while at the same time a player i would wish to change his strategy if he reconsidered it after a certain history h. The reason why the player is still playing a Nash equilibrium is that history h is never reached given σ^*. It may be the case that h is never reached *precisely* because of the action i takes at history h. The interpretation is that, at history h, player i threatens to take a certain action. Since it would not be in his interest to take this action were h to be reached, the strategy combination is "unreasonable" even though it is a Nash equilibrium. Selten (1965) addresses this problem by requiring that strategies are a Nash equilibrium after *any* history. This refinement, originally called perfect equilibrium, currently receives the name of *subgame perfect Nash equilibrium* (a subgame is the game that starts after a given history).

Refinements allowing for deviations by groups of players

Since they allow players to coordinate their actions, these refinements are on the frontier between noncooperative and cooperative game theory.

Strong Nash equilibrium Aumann (1959) introduces a refinement of the Nash equilibrium concept that allows for joint deviations by groups of players. Formally, a strategy combination σ^* is a *strong Nash equilibrium* if there is no subset C of N with a strategy combination $\sigma_C \in \times_{i \in C} \Delta(S_i)$ such that $U_i(\sigma^*_C, \sigma^*_{-C}) < U_i(\sigma_C, \sigma^*_{-C})$ for all $i \in C$.

Coalition-proof Nash equilibrium The concept of coalition-proof Nash equilibrium allows only for coalitional deviations that are immune to further deviations by subcoalitions. The formal definition (Bernheim, Peleg and Whinston (1987)) is quite involved. Restricting further deviations to subsets of the deviating coalitions, while allowing for an elegant recursive definition, is subject to controversy.

2.3 Cooperative game theory

2.3.1 Characteristic function games

A *characteristic function game* (or a game in characteristic function form, with transferable payoff or with side-payments) consists of two elements:

- the set of players $N = \{1, 2, ..., n\}$.
- the *characteristic function* $v : 2^N \to \mathbb{R}$ which associates to every subset S of N a real number such that $v(\varnothing) = 0$.

Nonempty subsets of N are called *coalitions*. For each *coalition* S, $v(S)$ is interpreted as a certain amount of a perfectly divisible commodity (usually taken to be money) that is available for division among the members of S if S forms.

We will economize on brackets and denote $v(\{i, j..., k\})$ by $v(i, j, ..., k)$.

Transferable utility versus nontransferable utility

Characteristic function games are sometimes called *transferable utility* (TU) games. We say that utility is transferable if any redistribution of the total payoff $v(S)$ among the players in S results in increments and decrements of individual utilities which sum to zero according to some specific set of utility scales for the players (see Luce and Raiffa (1957), p. 168). This will be the case if the players' utility functions are separable and linear in money. In general, utilities are neither separable nor linear in money. It is then difficult to find a satisfactory standard of interpersonal comparison of utility, and, even if one could find such standard, redistribution of the payoff would not generally lead to increments and decrements of utilities that add up to zero. If payoffs cannot be freely redistributed inside a coalition, or if the redistribution of payoffs does not lead to increments and decrements of utility that sum up to zero, the situation can be modelled as a *game without transferable utility* (or *NTU* game). A game without transferable utility assigns to each coalition S a $|S|$-dimensional set containing the utility vectors that are achievable by the coalition if it forms.

The case for using transferable payoff games to model a situation is strongest if utility is transferable. The assumption of transferable utility is however not necessary for transferable payoff games to make sense. An alternative interpretation of the characteristic function is that, precisely because the utilities of different players are *not* comparable (there is generally no obvious choice for origin and units) players may choose to restrict themselves to arguments over monetary payoffs. Maschler (1992) suggests the possibility of dispensing with the transferable utility assumption in the context of arbitration:

> If you and I find a $100 bill and go to an arbitrator to decide how to split it, I am quite sure that most arbitrators will not care about our utilities for money and will suggest that we share the dollars equally.

Taking this argument a bit further, if you and I find a $100 bill and bargain over how to divide it, we will end up dividing it equally since we have no objective way of comparing our utilities for money.

Another possible argument in favor of the use of TU games is that monetary payoffs may be more salient than utility functions, and thus outcomes such as equal split of monetary payoffs may act as "focal points" in bargaining (see Schelling (1960)).

Superadditivity

If a game is superadditive, coalitions get at least as much by acting together as they would get acting separately.

Definition 2.5 A game in characteristic function form (N, v) is **superadditive** if $v(S \cup T) \geq v(S) + v(T)$ for any $S, T \subseteq N$ s.t. $S \cap T = \varnothing$.

Superadditivity is a very compelling property in applications. Nevertheless, Aumann and Drèze (1974) point out some cases in which superadditivity may not arise: acting together may be illegal, or it may change the nature of the game because of *moral hazard* (if players decide to form a coalition and divide the procedures equally, they may work with less care than if they were singletons, and the total payoff may actually be smaller).

Orthogonal coalitions versus externalities

The characteristic function is an appropriate tool for games in which, once a coalition forms, it does not interact with other coalitions in any way. If this is the case, the payoff of a coalition does not depend on how the players outside the coalition are organized. Shubik (1982) calls these games *games of consent or games with orthogonal coalitions*: in these games, players can only cooperate or ignore each other, but they cannot actively hurt each other. An example would be an economy where players form coalitions to produce goods, trade only with other players inside the coalition, and production does not impose externalities on other coalitions.

In many cases, a coalition forms and then interacts with other coalitions. For example, in von Neumann and Morgenstern (1944), the purpose of forming a coalition is to coordinate strategies in a noncooperative game played with other players (who may also divide themselves in groups that coordinate their strategies). In this context, two questions arise:

1. Given a certain partition of the players into coalitions, how do these coalitions interact with each other?

2. Once a coalition forms, what are the expectations of its members concerning the organization of the remaining players?

A characteristic function can be constructed by assigning to a coalition S the *maximin* value of the game played between S and $N\backslash S$. This characteristic function assumes that coalitions give a pessimistic answer to both questions: when a coalition S forms, it expects that the remaining players will form a complementary coalition and act so as to minimize the payoff of players in S. Such expectations may be justified in the context of a zero-sum game (see von Neumann and Morgenstern (1944)): the players in the complement maximize their joint payoffs by uniting and minimizing the payoff of coalition S. However, it will not generally be in the interest of the players in $N\backslash S$ to react so as to minimize the payoffs of players in S, since this may also hurt $N\backslash S$.

A more general approach would allow the payoff of a coalition to depend on how the remaining players are organized. This is the subject of the following subsection.

2.3.2 Partition function games

If coalitions interact with each other, the payoff of a coalition will generally depend on how the rest of players are organized. A function that assigns a payoff to a coalition depending on the whole coalition structure is called a *partition function.*

Given the set of players N, a pair formed by a coalition S and a partition π of N to which S belongs is called an *embedded coalition.* Denote the set of embedded coalitions by $E(N)$. A partition function φ assigns to each embedded coalition a real number that represents the payoff S gets if coalition structure π forms.

A *partition function game* (or a game in partition function form) consists of two elements:

- the set of players $N = \{1, 2, ..., n\}$.
- the *partition function* $\varphi : E(N) \rightarrow \mathbb{R}$, which associates a real number to each embedded coalition $(S, \pi) \in E(N)$.

The partition function was introduced by Thrall and Lucas (1963) as a generalization of the characteristic function. A key assumption they make is that the partition function is not necessarily superadditive (players may be better-off if they split into smaller groups). Given a partition function, Thrall and Lucas derive a characteristic function in the following way:

$$v(S) = \min_{\pi \ni S} \varphi(S, \pi).$$

Whereas the classical characteristic function is constructed such that the worst case for a coalition is that the complement forms, Thrall and Lucas allow for the worst case to correspond to some other partition of the players.

In the context of the partition function, we can come back to the two questions made in the precious subsection.

1. Given a certain partition of the players into coalitions, how do these coalitions interact with each other? In other words: where does the partition function come from?

   Thrall and Lucas are not very explicit about this question. The most recent literature on partition function games (see Bloch (1996) and Ray and Vohra (1999)) assumes that the partition function reflects the Nash equilibrium of a game played among coalitions acting as single players (for example, a merged entity plays an oligopoly game as if it was a single firm). Thus, $\varphi(S, \pi)$ is the equilibrium payoff of S in the game against the remaining coalitions in π. A similar approach is taken by Ichiishi (1981).[1] Exceptions are Bloch (1995) and Espinosa and Macho-Stadler (2000) for oligopoly games: they assume that individual players cooperate with other players in the same coalition (by associating to do research and by sharing revenue, respectively) while at the same time they take some of the decisions noncooperatively (they take their product market decisions noncooperatively, so that the partition function is derived from the equilibrium of the noncooperative game played by *individual players*; this game in turn depends on the coalition structure since research joint ventures affect the costs of the players and revenue sharing schemes affect their incentives).

2. Once a coalition forms, what are the expectations of its members concerning the organization of the remaining players?

   Thrall and Lucas assume that coalitions are pessimistic. However, pessimism is not always justified outside zero-sum games. Ideally, coalitions should make consistent conjectures about the behavior of the complement. That is, the expectations of a coalition about the behavior of the complement should not be unduly pessimistic (or optimistic) but rather reflect what the complement would actually do if the coalition forms. In the absence of the possibility of commitment, players outside S will do whatever is in their interest to do. The problem with this approach is that what is in the interest of players outside S is not obvious and may depend on the concrete model of coalition formation being used.

   There are some cooperative models that assume players make consistent predictions: Ray and Vohra (1997) study the stability of (proposed) coalition structures assuming that prospective coalitions may only break into smaller subsets and that players

[1] Bloch (1996) and Ray and Vohra (1999) follow a two-stage approach. Once coalitions form, they cannot dissolve anymore. Ichiishi (1981) follows a one-stage approach. He also assumes that coalitions play a Nash equilibrium of the game with other coalitions, but the decision of forming coalitions and the decision of choosing what strategies to play are not separated. The definition of an equilibrium in Ichiishi is related to the concept of strong Nash equilibrium: it requires that coalitions play Nash against each other, but also that no coalition has an incentive to form, and choose another strategy vector, taking the strategies of the remaining players as given. In Bloch (1996) and Ray and Vohra (1999) a coalition that deviates from equilibrium would still play Nash against the rest of the coalitions in the "new" coalition structure.

anticipate further deviations. Huang and Sjöstrom (1999) propose that, given a solution concept, a coalition predicts the reaction of the outsiders by applying the solution concept to the "reduced society" consisting of the non-members.

2.3.3 The Nash bargaining solution

The two-player case

Nash (1950) defines a two-person bargaining problem to consist of a pair (F, d) where F is a closed convex subset of $\mathbb{R}^2$, $d = (d_1, d_2)$ is a vector in $\mathbb{R}^2$, and the set $F \cap \{(x_1, x_2)| \; x_1 \geq d_1 \text{ and } x_2 \geq d_2\}$ is nonempty and bounded. F represents the set of feasible utility allocations and d the disagreement utility allocation. A comprehensive theory of two-player negotiation would assign to each bargaining problem (F, d) an allocation vector in $\mathbb{R}^2$ denoted by $\phi(F, d)$. Nash imposes some desirable axioms on the allocation rule and identifies a unique allocation rule characterized by those axioms.

Nash's axioms are the following:

- **Strong efficiency**. If $\exists\, x \in F$: $x \geq \phi(F, d)$ then $x = \phi(F, d)$.
- **Individual rationality**. $\phi(F, d) \geq d$.
- **Scale covariance**. For any numbers λ_1, λ_2, γ_1 and γ_2 such that $\lambda_1 > 0$ and $\lambda_2 > 0$, if

  $G = \{(\lambda_1 x_1 + \gamma_1,\; \lambda_2 x_2 + \gamma_2) \;|(x_1, x_2) \in F\}$

  and $w = (\lambda_1 d_1 + \gamma_1,\; \lambda_2 d_2 + \gamma_2)$

  then

  $\phi(G, w) = (\lambda_1 \phi_1(F, d) + \gamma_1,\; \lambda_2 \phi_2(F, d) + \gamma_2)$.
- **Independence of irrelevant alternatives**. For any closed convex set G, if $G \subseteq F$ and $\phi(F, d) \in G$, then $\phi(G, d) = \phi(F, d)$.
- **Symmetry**. If $d_1 = d_2$ and $\{(x_2, x_1) \mid (x_1, x_2) \in F\} = F$, then $\phi_1(F, d) = \phi_2(F, d)$.

Nash (1950) proves that there is a unique solution function $\phi(\cdot, \cdot)$ that satisfies these axioms. This solution satisfies, for every bargaining problem (F, d),

$$\phi(F, d) \in \arg \max_{x \in F,\; x \geq d} \; (x_1 - d_1)(x_2 - d_2).$$

Rubinstein (1982) approaches the two-person bargaining problem by modelling negotiations as a noncooperative bargaining game. In this game, players alternate making offers until one is accepted, and future payoffs are discounted by a factor $\delta < 1$. Rubinstein shows that the unique subgame perfect equilibrium of this game leads to the Nash bargaining solution in the limit when discounting vanishes.

Binmore et al. (1986) build on the work of Rubinstein (1982). They try to clarify the interpretation of the Nash bargaining solution by distinguishing between two possible sources of friction in the bargaining process that may induce players to reach an agreement: the impatience of the players and the exogenous risk that the negotiations break down without an agreement. They conclude that in both cases the subgame perfect equilibrium of the noncooperative game with frictions converges to the Nash bargaining solution as frictions vanish. However, the appropriate choice of the pair (F, d) to which to apply the Nash bargaining solution depends on the source of friction: if the source of friction is the exogenous risk of breakdown, one should use a von Neumann-Morgenstern utility function and choose as the disagreement point the utilities players get if breakdown occurs. In the case of discounting of future payoffs, one should use a utility function that reflects the time preferences of the players (see Fishburn and Rubinstein (1982)) and use as a disagreement point the no loss-no gain outcome, that is, the outcome that gives the parties the same income streams they are receiving during the dispute.

The n-player case

The axioms can be extended to apply to bargaining problems with n players, and it can be shown that the unique bargaining solution that satisfies the extended axioms is the function that associates with each problem (F, d) (where F is now a subset $\mathbb{R}^n$ and d a vector in $\mathbb{R}^n$) the vector of utilities

$$\arg \max_{x \in F,\ x \geq d} \prod_{i=1}^{n} (x_i - d_i).$$

The Nash bargaining solution is clearly appropriate for n-person games in which agents have only two options: either to cooperate with *all* other players or to remain singletons. In many cases, partial cooperation is also profitable. Even if eventually all players cooperate, the payoffs from partial cooperation, and not only the payoffs players get if they do not cooperate, may influence the outcome.

There have been some attempts to determine the disagreement payoffs endogenously: the disagreement payoff is then determined by the opportunities of the players in other coalitions (which may or may not be subsets of the coalition under consideration). Binmore (1985) studies the three-player-three-cake problem. This is a problem in which only partial cooperation is profitable: each pair of players can divide some "cake" among themselves, and only one of the three cakes can be divided. The cake that is eventually divided and how it is divided would depend on the division of the remaining two cakes. Bennett (1997) considers the general case of n players and a characteristic function. She takes the reference point d to be the best payoff the agent can get in an alternative coalition.[2] Gerber (1996) chooses instead the average of the possibilities the player has in the "reduced game" obtained by making the coalition under consideration unfeasible.

One may also conjecture that the Nash bargaining solution is still valid in the presence of gains from partial cooperation provided that those gains are small in comparison with the gains from full cooperation. By small we mean the following: compute the Nash bargaining solution as if partial cooperation is not possible. Compare what the players get in the Nash bargaining solution and what they can get by partial cooperation. If players are better-off in the Nash bargaining solution, we can disregard the possibility of partial cooperation. In other words, forming subcoalitions is not credible. This reasoning is in the spirit of the outside option principle (Binmore et al. (1989)). We will come back to this point in the next chapter.

2.3.4 The core

Let (N, v) be a characteristic form game with side payments. A *payoff allocation* is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^n$, where x_i is interpreted as the payoff to player i.

Definition 2.6 We say that an allocation y is **feasible** for a coalition S if

$$\sum_{i \in S} y_i \leq v(S).$$

If an allocation is feasible for S, players in S can achieve their components of y if they get together.

Definition 2.7 We say that an allocation x is **dominated** if

$$\exists S \subseteq N : v(S) > \sum_{i \in S} x_i.$$

[2] We refer to the "reference point" and not to the "disagreement outcome" because the reference point may be unfeasible in case of disagreement, that is, it may not be possible for all players in the coalition under consideration to simultaneously realize their best alternative opportunity (see Bennett (1997)).

Definition 2.8 An allocation x is in the **core** if it is feasible for N and not dominated. That is, x is in the core if

$$\sum_{i\in N} x_i = v(N) \text{ and } \sum_{i\in S} x_i \geq v(S) \ \ \forall S \subseteq N.$$

The core is probably the most solid solution concept in cooperative game theory. However, it is not without drawbacks. It is empty for many games (for example, it is empty for the three-person game presented in the introduction), and very large for others. In some games where it is nonempty, its predictions are very extreme. A classical example is the so-called *glove game*.

Consider a game in which there are 2,000,001 players, among whom 1,000,000 players can each supply one left glove, and 1,000,001 players can each supply one right glove. Let N_L denote the set of left-glove suppliers and let N_R denote the set of right-glove suppliers. The worth of each coalition is defined to be the number of matched pairs (one left glove and one right glove) that it can make. That is,

$$v(S) = \min\{\ |S \cap N_L|,\ |S \cap N_R|\ \}.$$

The core of this game consists of the unique allocation x such that

$$x_i = \begin{cases} 1 \text{ if } i \in N_L \\ 0 \text{ if } i \in N_R \end{cases}$$

Any outcome that gives a positive payoff to one of the right-glove owners cannot be in the core. The reason is that then the remaining $2,000,000$ players are getting less than $1,000,000$ and they could get a higher payoff by excluding that right-glove owner.

While this result makes economic sense (the price of right gloves is zero because they are in excess supply) it is very sensitive to the number of agents in the market. By adding just two left-glove suppliers, the result would be reversed. Moreover, a right-glove supplier would not lose anything by refusing to cooperate with the left-glove suppliers.

The core has also been criticized on the grounds that it assumes myopic behavior of the players. The definition of the core seems to imply that if a coalition S can improve over a certain allocation x, then S will form and thus x will not be realized. However, the formation of S with a given payoff division may itself be subject to further deviations that would make S worse-off, and thus S may decide not to depart from x after all. There are many solution concepts that require that a deviation should be immune to further deviations. The bargaining sets (see part II) and the farsighted solution concepts (see Harsanyi (1974), Chwe (1994) and Xue (1998)) are examples in this line.

Notice that the concept of strong Nash equilibrium is based on the same idea as the core, whereas the concept of coalition-proof Nash equilibrium imposes that deviations should be immune to further deviations.

2.3.5 The Shapley value

The idea of the Shapley value is to identify, for any characteristic function game, a unique expected payoff allocation for the players. That is, one would like to identify a function ϕ that assigns to each game (N, v) an expected payoff vector $(\phi_i(v))_{i \in N}$.

Shapley (1953a) approached this problem axiomatically. He proposed the following axioms

- **Symmetry**. This axiom asserts that only the role of a player in the game should matter, not his specific label in the set N.

  Formally, we can rename the players by taking a permutation of N, that is, a function $\pi : N \to N$ such that for any j in N there is exactly one player i in N such that $\pi(i) = j$. One can also define the game that results after relabelling the players, πv, in the following way:

  $$\pi v(\{\pi(i) | i \in S\}) = v(S), \forall S \subseteq N.$$

  Axiom 2.1 *(Symmetry)* For any characteristic function v, any permutation $\pi : N \to N$, and any player i in N, $\phi_{\pi(i)}(\pi v) = \phi_i(v)$.

- **Carrier**. We say that a coalition R is a carrier of a coalitional game v if

  $$v(S \cap R) = v(S), \forall S \subseteq N.$$

  If R is a carrier, the players who are not in R are called *dummies*, because their entrance in a coalition does not change its worth. The carrier axiom requires that players in the carrier should divide the worth of the grand coalition among themselves.

  Axiom 2.2 *(Carrier)* For any characteristic function v, and any coalition R, if R is a carrier of v, then $\sum_{i \in R} \phi_i(v) = v(R)$.

- **Linearity.**[3] Given two characteristic functions, v and w, denote by $pv + (1-p)w$ the game such that, for each coalition S,

$$(pv + (1-p)w)(S) = pv(S) + (1-p)w(S).$$

The interpretation of this game is that players will play game v with probability p and game w with probability $1-p$. If players bargain over how to divide the payoff before the uncertainty is resolved, player i's expected payoff is $\phi_i(pv + (1-p)w)$. If players bargain only after it becomes known whether they will play v or w, player i's expected payoff is $p\phi_i(v) + (1-p)\phi_i(w)$. The axiom of linearity asserts that expected payoffs should be the same in both cases.

Axiom 2.3 *(Linearity)* For any two characteristic functions v and w, any number p such that $0 \leq p \leq 1$ and any player i in N,

$$\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi_i(w).$$

Shapley (1953a) shows that there is exactly one function that satisfies the axioms above. This function satisfies the following equation

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup \{i\}) - v(S)).$$

Shapley (1953a) provides the following interpretation of his value. Suppose that the players in N enter a room randomly so that each of the $|N|!$ possible orders is equally likely. On entering the room, player i finds a (possibly empty) coalition S already there. Player i gets his *marginal contribution* to coalition S, $v(S \cup \{i\}) - v(S)$. The probability that player i finds exactly coalition S in the room is $\frac{|S|!(|N|-|S|-1)!}{|N|!}$, since all possible orders in which players in S and players in $N \setminus S$ enter the room are equivalent, provided that players in S enter before i and players in $N \setminus S$ after i.

Roth (1977) shows that there are two conditions for the Shapley value to be a von Neumann-Morgenstern utility function: players have to be neutral to "ordinary" risk (risk derived from lotteries) and also to strategic risk (risk derived from negotiation with other players). Roughly speaking, neutrality to strategic risk means that a player is indifferent between having a unit of utility all for himself and having to bargain with $n-1$ other players over how to divide n units of utility.

[3] The original axiom of Shapley, additivity, is stronger. See Myerson (1991).

Extensions of the Shapley value to games with coalition structures

The original Shapley value allows players to evaluate their prospects from playing a game. The extensions of the Shapley value to games with coalition structures allow players to evaluate their prospect from playing a game given that a certain coalition structure forms.

The Aumann and Drèze extension Harsanyi (1963) and Aumann and Drèze (1974) suggested that, given a coalition structure, the expected payoff for player i should be the Shapley value of the game restricted to subsets of the coalition S to which i belongs in the coalition structure, that is,

$$\phi_i(v,S) = \sum_{T \subseteq S \setminus \{i\}} \frac{|T|!(|S|-|T|-1)!}{|S|!}(v(T \cup \{i\}) - v(T))$$

Notice that the expected payoff of player i does not depend on how players outside i are organized.

Myerson (1980) shows that this Shapley value has the following property, called the *balanced contributions* property:

$$\phi_i(v,S) - \phi_i(v,S\setminus\{j\}) = \phi_j(v,S) - \phi_j(v,S\setminus\{i\}).$$

The balanced contributions property implies that the loss or gain for player i if player j would withdraw from coalition S equals the loss or gain of j if i would withdraw from coalition S. This is a very interesting property of the Shapley value, which relates it to the concepts we will study in chapter 7.

The Owen extension Owen (1977) proposes a different extension of the Shapley value. This extension has also been studied by Hart and Kurz (1983).

The idea of the Owen value is as follows. Coalitions form and then bargain with other coalitions over the division of the value of the grand coalition. The bargaining process between coalitions is such that each coalition gets the Shapley value of the game in which coalitions are players. The share each player gets is computed in the following way: as in the original Shapley value, we compute the average marginal contribution of the players. However, we only consider orderings that are *consistent* with the coalition structure in the sense that for any two players i and j who are together in a coalition S, all players that appear between i and j in the ordering must be also in S. In terms of the story of random entrance in a room, players enter randomly but players in the same coalition must enter successively, so that the probability of each order given coalition structure $\pi = (S_1, ..., S_m)$ is $(m!|S_1|!|S_2|!...|S_m|!)^{-1}$. The Owen value is then given by

$$\phi^i(v, \pi) = E\left[v(\mathcal{P}^i \cup \{i\}) - v(\mathcal{P}^i)\right]$$

where the expectation E is over all random orders on N that are consistent with π, and $\mathcal{P}^i$ denotes the (random) set of predecessors of i.

Notice that the Owen value allows for externalities among coalitions: if some of the players reorganize themselves, this may affect the Owen value of players who are still in the same coalition. The interpretation is that the formation of coalitions affects the bargaining power of the players. Furthermore, since the Owen value assumes that the final outcome of the game will always be efficient *regardless of the coalition structure*, the formation of coalitions does not affect the total payoff to be divided but only the distribution of this payoff, and the only possible reason why a player would want to enter a coalition is to improve his bargaining position vis-a-vis other players.

The Owen value introduces externalities in a setting (the characteristic function) that did not exhibit any externalities a priori. Chapters 4 and 5 of this thesis are very much in the spirit of the Owen value.

2.4 Noncooperative models of coalition formation

The first part of this thesis focuses on noncooperative models of coalition formation in partition function games. This section contains an introduction to the literature. It is partly based on the survey by Bloch (1997).

Bloch (1997) divides the noncooperative models of coalition formation in two groups: *simultaneous games* (in which all players make announcements simultaneously) and *sequential games* (players move in turn and each player can observe the moves made so far).

2.4.1 Simultaneous games

In the simultaneous games we discuss below, players can announce their willingness to participate in a coalition, but they cannot argue over payoff division. Payoff division is assumed to be given by an allocation rule or valuation ϕ: $\Pi(N) \rightarrow \mathbb{R}^n$. Possible valuations are the Aumann and Drèze (1974) or the Owen (1977) extension of the Shapley value to games with coalition structure.

There are two kinds of simultaneous games: open membership games and exclusive membership games.

Open membership games

In *open membership* games, players can only announce their willingness to be in a coalition, but they cannot specify the composition of the coalition to which they want to belong. Examples are the cartel formation game of d'Aspremont et al. (1983) and the general open membership game proposed by Yi and Shin (1995). In the cartel formation game players have only two strategies: *yes* and *no*. All players that say *yes* will be together in a coalition; players that say *no* remain singletons. In the general open membership game, each player chooses a message from a given set; players announcing the same message will be together in a coalition (this game can be extended by adding a message that commits the players to stay alone).

Exclusive membership games

In *exclusive membership* games, players are not free to join any existing coalition. Each player announces the composition of the coalition to which he wants to belong, and only coalitions containing announced players can form. There are two variants of the exclusive membership game: the game Γ and the game Δ. In these games the strategy space of a player is the set of coalitions to which he belongs

$$S_i = \{C \subseteq N, i \in C\}.$$

The game Γ was introduced by von Neumann and Morgenstern (1944, section 26). A coalition C is formed in this game if and only if each member i of C has announced $s_i = C$. Starting from a situation in which *all* players in C have announced C, a unilateral deviation of a member of C implies that the remaining players become singletons.

The game Δ was introduced by Hart and Kurz (1983). In this game, a coalition D is formed by all players i such that $s_i = C$. Clearly, $D \subseteq C$ but it is not required that $C = D$. Thus, if some of the players in C do not want to form a coalition, the remaining players will form the coalition without them. Starting from a situation in which all players in C want to form a coalition, a unilateral deviation by one of them implies that the remaining players stay together.

The rules of the games Γ and Δ are rather ad hoc: when a player leaves a coalition, it is systematically assumed that either the remaining players will split into singletons, or that they will stick together. Which of these models is more reasonable? The answer cannot be found without looking at the specific game. As Hart and Kurz (1983, p. 1060-1061) suggest, none of the games is conceptually more correct than the other. Thoron (2000) argues that, given that a coalition is left by some of its players, the remaining players should break apart or stick together depending on what is in their interest (for symmetric games, all players would agree on which of these two alternatives is more profitable). This approach is problematic when there is more than one coalition in this situation: each of the coalitions that have been abandoned by some of their players would have to decide simultaneously whether to break apart or to stick together. The situation becomes even more complex if we allow for any strategic reaction of the remaining players (not only sticking together or splitting into singletons). Thoron (2000) concludes that an explicit dynamic model is needed.

Notice that Thoron's critique is also applicable to open membership games. All open membership games are Δ-games: when a player abandons a coalition, the remaining players stick together.

The games Γ and Δ model the reaction of the players whose coalitions have lost some members, assuming that the "outside" players (players in coalitions that have not been altered by the deviation) do not react in any way. This assumption does not seem justified in games with externalities: in such games, all players are potentially affected by a deviation, either directly (their coalition is abandoned by some players) or indirectly (their players change because of the change in the coalition structure).

Simultaneous games may have many equilibria. Indeed, for all of them except the open membership game of Yi and Shin, there is always a trivial equilibrium in which all players remain singletons. In order to make sharper predictions, refinements such as coalition-proof Nash equilibrium and strong Nash equilibrium can be used.

Hart and Kurz (1983, 1984) study the games Γ and Δ taking the Owen value as allocation rule and using the Strong Nash equilibrium refinement. Meca-Martínez et al. (1998) and Slikker (1999) consider general allocation rules. Yi (1997) studies the open membership game for symmetric partition function games that systematically exhibit either positive or negative externalities[4] and compares it with the models with exclusive membership of Bloch (1996) and Ray and Vohra (1997). In partition function games with

[4] It is worth noticing that the definitions of positive and negative externalities in Yi (1997) are more demanding than the usual definitions. In particular, they not only concern the *external* effects of a change in the coalition structure, but also the *internal* effects. See Yi (1997) for more details.

negative externalities, the open membership rule achieves the efficient outcome (the grand coalition); the results are generally not efficient for the models with exclusive membership of Bloch (1996) and Ray and Vohra (1997). For positive externalities, the two models with exclusive membership support more concentrated coalition structures than the open membership game but the grand coalition is seldom an equilibrium due to free-riding problems. Applications to games with negative externalities include Yi (1996) for the case of custom unions, Zissimos and Vines (1999) for trading blocks and Yi (1998) for research joint ventures; Yi and Shin (2000) analyze research coalitions with positive spillovers.

2.4.2 Sequential games

Simultaneous games have drawbacks. The set of equilibria is often very large, when using refinements one may get an empty set (as it is the case with strong Nash equilibrium) and moreover, players are not allowed to react optimally to deviations of other players: the coalition that is abandoned reacts in an *ad hoc* way, and the possibility of reaction by other coalitions is not even discussed.

In sequential games of coalition formation, coalitions are formed one at a time and once a coalition forms its players are set aside, thus coalitions cannot be enlarged. The players who have not formed a coalition yet can react optimally, whereas the players who have already left the game cannot react in any way.

Choosing the extensive form

All sequential games of coalition formation (with or without externalities among coalitions) build on Rubinstein's (1982) seminal work on bargaining. In the original game, there are only two players. A player (selected by the rules of the game) starts by proposing a division of the payoff of the two-player coalition; the other player can accept or reject. If he rejects, the game proceeds to the next period and he can make a counteroffer; the other player can accept or reject, and so on. Players discount future payoffs.

In generalizing this game to n players, some choices have to be made. As in the two-player version, the rules of the game will select a player to be the first proposer. The first choice that has to be made is what will be the strategy space of the proposer. What coalitions can he propose? Only the grand coalition or any coalition including him? The first option would mean that players have agreed, at least "in principle", to form the grand coalition, and they are now discussing the payoff division. Other possible coalitions will enter the game in other ways (for example, as "outside options" that players can exercise at some given points in the game) but not as part of the proposal. The second option would mean that no coalition is a priori more likely to form than others (no coalition is favored by the rules of the game) and thus the coalition structure and payoff division will be determined endogenously and simultaneously. We will be more interested in the second approach.

Suppose now that the proposer proposes a coalition S and a (feasible) division of $v(S)$. The rules of the game must specify the order in which players accept or reject the proposal. Two choices can be made: either the order of response is exogenously determined by the rules of the game (in a deterministic or in a random way), or the order is the result of the strategic choices of the players (for example, the proposer may choose the order in which players respond, or at least he can choose the first player to respond, who in turn will choose the second, etc.). We will refer to the rule specifying the order in which players accept or reject as the *rule of order*.

The third choice that must be made is which players must accept in order for the proposal to be approved. Without loss of generality, we can restrict ourselves to the unanimity rule (all players to whom the proposal is addressed must accept). If the consent of all players is not needed, this can be modelled through the characteristic function. As an example, consider the division of a dollar between three players by majority rule. Bargaining in this game can be modelled as a variant of the Rubinstein game with unanimity. The majority requirement does not enter the extensive form game but it is included in the characteristic function. If the votes in favor of any two players i and j are enough to approve a proposal, we will say that $v(N) = v(i, j) = 1$ for all i, j, and the original situation in which all three players accept or reject and a majority of acceptances is sufficient for the proposal to be approved is modelled as a situation in which i can propose to j only and unanimity in the proposed coalition is required.

Suppose now that at least one of the responders rejects the proposal. The rules of the game must select what happens next. It may be the case that the game is over (at least with a certain probability) or that a new proposer is selected. We will concentrate on the second option for now. How is the next proposer selected? As in the case of selecting the order of the responders, two groups of models arise: in the first of them, the next proposer is selected *exogenously* by the rules of the game (by exogenously we mean as a function of the set of players left, and not of elements of history such as which coalitions have been proposed or which player has rejected the proposal); in the second, the next proposer depends on the strategic choices of the players. The natural generalization of the Rubinstein game is of the second sort: it assumes that the first player to reject a proposal becomes the next proposer. We will call the rule that selects the next proposer after a proposal has been rejected the *continuation rule.*

If the continuation rule is such that the next proposer is determined exogenously, the rule of order is not relevant to the game. That is, since the continuation rule selects the next proposer independently of who rejected the proposal, the order in which players respond to a proposal is not relevant to the outcome of the game.[5] If the continuation rule determines the next proposer endogenously, then the rule of order becomes very important. The subsequent developments of the game may be very sensitive to who was the first player to reject the proposal, and this in turn crucially depends on the rule of order. One could think that who is going to reject the proposal would not depend on the rule of order, but only on the proposed payoff division. Suppose however, that the last player (say, i) who has to accept or reject is going to reject the proposal for sure because it offers him a very low payoff. The last-but-one player (j), instead, is offered a very high payoff and he would certainly want the proposal to be approved. However, given that i is going to reject, j may want to reject himself in order to take the initiative afterwards. Thus, whether it will be i or j who become the next proposer will depend on the rule of order. Because the rule of order is only relevant to the case in which the next proposer depends on history, such games may be called *games with a rule of order.* In other games it is just assumed that there is *some* rule of order, but it is not modelled explicitly since it is irrelevant to the results.

The last choice to be made is not specific to n-person games, but it concerns 2-person games as well. It is the introduction of some friction in the bargaining process. The introduction of frictions in the bargaining process makes the model more realistic and shrinks the range of equilibria (Rubinstein (1982) finds a unique subgame perfect equilibrium of the game thanks to the introduction of frictions). Bargaining may be costly because players directly spend resources in it (for example, the proposer has to phone all other players in order to make a proposal), because players are impatient (time elapses during bargaining and players discount future payoffs, so that agreement in the future is less valuable than agreement today), or because there is some exogenous risk of breakdown of the negotiations (for example, players may "get upset" and leave the negotiation table, though this could perhaps better be modelled as a strategic choice; alternatively, while players bargain over how to divide the profits from exploiting some business op-

[5] That is, it is relevant to the game only if players choose to condition their actions on this information. We will exclude this possibility by using a refinement of the Nash equilibrium concept that allows players to condition their choices only on information that is directly relevant.

portunity, the conditions in the market may change and this opportunity may become unprofitable). Most games have focused on the impatience of the players as the driving force of agreements, a few have focused on the risk of breakdown, and virtually none on constant costs of bargaining, partly because of the unpromising results (see Rubinstein (1982)). In games with impatience, it is assumed that after a proposal is rejected a time period elapses and players discount future payoffs; in games with breakdown probability, it is assumed that after a proposal is rejected, breakdown of the negotiations may occur with some exogenous probability and players then receive some exogenously given payoffs. We will consider both sources of friction in chapter 3.

Rubinstein (1982) was able to select a unique equilibrium of his game using the widely accepted refinement of subgame perfect equilibrium. In games with more than two players, additional refinements are needed (otherwise almost any outcome is possible, see e.g. Baron and Ferejohn (1989) or Chatterjee et al. (1993)). A frequently used refinement is the concept of *stationary subgame perfect equilibrium* (or stationary perfect equilibrium). In a stationary perfect equilibrium, the strategies of the players only depend on the *state* of the game. By the state of the game we mean the set of variables that are directly payoff relevant. For example, in a bargaining game, the strategy of a player would be allowed to depend on the set of players that remain in the game (after some players have formed coalitions, the payoff opportunities for the remaining players may be very different) and the current proposal, but not on the history of acceptances and rejections, that are not directly relevant for payoffs (unless players choose otherwise by playing strategies that reward or punish past behavior, that is, by playing non-stationary strategies). Notice that the requirement of stationarity is added to the requirement that strategies are an equilibrium (usually also a subgame perfect equilibrium), that is, a stationary perfect equilibrium is also immune against non-stationary deviations. While the concept of stationary equilibrium assumes a lot of forgiveness of the players and excludes the possibility of threats (not only incredible threats, but also credible -subgame perfect- threats), it may be defended on grounds of its simplicity (Baron and Kalai (1992); Chatterjee and Sabourian (1997)) or its saliency (as argued in Chatterjee et al. (1993)).

Proposal-making games with a rule of order

Selten (1981) considers a bargaining game for TU games with n players. In this game a player makes a proposal to some coalition, and chooses a responder from this coalition to respond first. The responder may agree and select the next responder (and so on until all responders accept or one rejects), or he may reject. If he rejects, he becomes the next proposer. Selten assumes that the characteristic function is *zero-normalized* (that is, $v(i) = 0$ for all i in N), superadditive, *essential* (that is, for zero-normalized games, $v(N) > 0$) and has the *one-stage property*, (that is, for zero-normalized, essential and superadditive games, $v(S) > 0$ implies $v(N \backslash S) = 0$). Because of the last property, the game ends after one coalition has been formed, since there are no profitable coalitions left.

This game does not have any frictions, and thus the set of stationary perfect equilibria is generally large (in the two-player case, any payoff division preferred by both players to remaining singletons would be an equilibrium). Selten (1981) shows that every subgame perfect equilibrium is connected to an aspiration demand vector (see chapter 7). This game has been extensively played in the laboratory (see Uhlich (1990)).

Binmore's (1985) three-player telephone bargaining is along the same lines but with discounting. A central feature of this game is that the discount factor is not necessarily the same for all players. Binmore shows that strength may easily become weakness, that is, being patient, while being a source of strength in the two-player game, is a source of weakness in the three-player game.

Selten's game was extended to NTU games by Moldovanu (1992) (for three players) and Moldovanu and Winter (1995). In Moldovanu and Winter (1995) more than one profitable coalition may form simultaneously.

A slightly different version of Selten's game can be found in Chatterjee et al. (1993). In this game the proposer cannot choose the first responder, but the order in which players accept or reject proposals is exogenously given by the rules of the game.[6] Players share the same discount factor and several coalitions can be formed.

The proposal-making model with a rule of order has been extended to partition function games by Bloch (1996) and Ray and Vohra (1999). In Bloch (1996) proposals consist of coalitions only, whereas payoff division is given by an allocation rule; special attention is paid to symmetric games with equal payoff division. Ray and Vohra (1999) is a direct extension of Chatterjee et al. (1993) to partition function games. Proposals consist of coalitions and conditional statements over how to divide the payoff depending on the partition that eventually forms. The results of Ray and Vohra for symmetric games with endogenous payoff division vindicate the assumptions of Bloch (1996): indeed, symmetric

[6]This is not an essential difference: the order in which players respond only matters if the proposal is going to be rejected; in this case the proposer can still pass the initiative to a player of his choice by making an unacceptable proposal to him only.

players will split the payoff equally. While this is a very reasonable result, it is by no means obvious: the games we will study in chapter 3 do *not* have this property.

Applications of the game with a rule of order to symmetric oligopoly games include Bloch (1995) for R&D joint ventures (a joint venture reduces own cost but it also reduces the cost of those competitors that participate in the joint venture, triggering more aggressive behavior on their part), Ray and Vohra (1999) for the symmetric Cournot game (a merger increases market power but it also triggers an expansion of the quantity produced by outsiders) and Espinosa and Macho-Stadler (2000) for the symmetric Cournot game with moral hazard (costs are incurred privately whereas revenue is divided equally). In all these applications, the resulting coalition structure is both asymmetric and inefficient.

Proposal-making games with random proposers

The distinctive feature of games with random proposers is that, once a proposal is rejected, the rules of the game select the next proposer conditional on the state of the game (that is, on the payoff relevant information, such as the set of players who have not formed coalitions yet). While it would be perfectly possible to chose the next player deterministically, this is not frequent in the literature (an exception is the work by Morelli and Penelle (1997) on economic integration: in their model, an existing union makes an offer to a subset of countries every period). Most of the existing work concerns games in which the next proposer is selected randomly.

The first game with random proposers was introduced by Binmore (1987) for the two-player case. This game was extended by Baron and Ferejohn (1989) to bargaining in simple majority games and by Okada (1996) to general *TU* games. There has not been a systematic study of *NTU* games with random proposers so far (we will consider a very simple example in chapter 3).

An extension of the proposal-making model with random proposers to partition function games can be found in Montero (1998, 1999b). Chapters 3, 4 and 5 are devoted to this extension.

The games we have described so far share some characteristics. First, the rules of the game determine whose turn is to move (deterministically or randomly, perhaps by designating a player who will choose who moves next). Second, the actions players can take are proposals of a coalition with a given payoff division, and acceptance or rejections of these proposals. Third, time is discrete. These games are the object of study in Part I. We will briefly mention some sequential games that are constructed under other assumptions.

Other sequential proposal-making games

Perry and Reny (1994) study a game in which time is continuous and players decide when to make proposals.

Games in which players cannot choose to whom they make proposals, but they are randomly matched in pairs can be found in Gul (1989), and in the book by Osborne and Rubinstein (1990).

Games in which only one coalition can be proposed are considered by Krishna and Serrano (1996) and Hart and Mas-Colell (1996). Proposals must address the grand coalition of the players that remain in the game. Players can exit the game in both models, but because of different reasons. In Krishna and Serrano (1996) some players can exit the game with their shares when a proposal is not accepted by all players. In Hart and Mas-Colell (1996), there is the possibility of partial breakdown: if a proposal is rejected the proposer is "thrown out of the game" with a certain probability, and bargaining continues among the remaining players.

Demand games

In a demand game, players do not make proposals to a concrete coalition; instead, each player can make a payoff demand that applies to any possible coalition including him. Demands and acceptances are not different stages of the game as in the proposal-making game: at any moment, a player i whose turn is to move can either make a demand or accept the demands of players in any coalition $S\backslash\{i\}$ (if they have already formulated them) in order to form S.

In the simpler version of a demand game (Binmore, 1985), there are three players and only two-player coalitions can form. The players move in a fixed order. Player 1 announces a demand (a payoff he requires in order to participate in a coalition), then player 2 can accept this demand or reject it and make a demand himself. If he accepts, coalition $\{1,2\}$ forms, 1 gets his demand, 2 gets $v(1,2)$ less player 1's demand and the game is over. If he rejects, player 3 can choose to accept player 1's demand, player 2's demand or reject both and make his own demand. If he rejects, now it's player 1 who has two demands to choose from, and so on. Players discount future payoffs. If no player is dominant (in the sense that coalitions including him are much more valuable than the remaining coalition) in equilibrium players are indifferent between the two coalitions they can form.

Bennett and van Damme (1991) consider a finite extensive form game in which players make demands only once and a player that makes a demand selects the next mover. They introduce a refinement of subgame perfection, called *credible subgame perfect equilibrium*, based on the idea that players can strategically exploit their indifferences. By applying this refinement, they are able to make sharp predictions for apex games.

Selten (1992) considers a demand game with two rounds (thus players may change their demands from one round to the next). He assumes that players incur a fixed cost from making a (new) demand as well a cost from forming coalitions that is increasing in the number of players involved. This game has been played in the laboratory for essentially the same situation we described in chapter 1, with mixed results: while the quotas influence the outcome, they do not determine it completely (see Selten and Kuon, (1993)).

A demand bargaining model for three-player partition function games can be found in Cornet (1998). The results are consistent with those in Binmore (1985): for a wide range of the parameters, the equilibrium of the game is such that players are indifferent between prospective coalition partners. We will find the same property in chapter 4.

Renegotiation

All the games we have discussed so far assume that, when a coalition forms, it leaves the game and it is committed not to come back to the "negotiation table". The results of those games are often very inefficient, since by creating inefficiencies players may get a higher payoff. An example could be a game with three players where each pair can get 60 and the three players together can get 72; even though the grand coalition maximizes the joint payoffs of the three players, each player prefers to be in a two-player coalition since he would get 30 rather than 24. It is then reasonable to assume that players can renegotiate the inefficient agreements and enlarge the coalitions. In the example, after forming a two-player coalition, the two players may wish to approach the third player and divide the 12 units in some way; notice though that if the rules of the game constrain the players to divide payoffs equally this will not happen.

Seidmann and Winter (1998) have considered the possibility of renegotiation in games with a rule of order. They make a distinction between reversible and irreversible actions. If actions are reversible, players may continue bargaining with other players after implementing an agreement. They show that renegotiation will result in the formation of the grand coalition if actions are reversible. Okada (1999) introduces renegotiation in the random proposal game and shows that the possibility of renegotiation changes the incentives of the players, that is, players may form different coalitions if they anticipate the possibility of renegotiation. Montero (1998) allows for the possibility of renegotiation in the one-seller-two-buyer game: if the seller and the weak buyer form a coalition in which the weak buyer gets the good, the good can still be resold to the strong buyer; similarly to the results of Okada (1999) the possibility of reselling makes the coalition of the seller and the weak buyer more attractive. Gomes (1999) considers coalition formation with externalities and the possibility of renegotiation for general three-player games with an extended strategy space (when addressing a coalition, the proposer can choose between making individual offers to players or making an offer to a coalition that needs to be approved by all players).

2.5 Cooperative concepts of stability for games with externalities

2.5.1 Core stability

In characteristic function games the natural stability concept for coalition structures is the *coalition structure core* (see Greenberg (1994)). Given a coalition structure $\pi = \{S_1, ..., S_m\}$, the coalition structure core is the set of payoff vectors $(x_i)_{i \in N}$ that are feasible for π (that is, $\sum_{i \in S_j} x_i \leq v(S_j)$ for all $S_j \in \pi$) and that are not dominated ($\nexists T \subseteq N : v(T) > \sum_{i \in T} x_i$). A coalition structure is then *stable* if its core is nonempty, that is, if there is a payoff vector feasible for this coalition structure such that no group of players can deviate and all be better-off. For games with fixed payoff division given by an allocation rule, a coalition structure is stable if $\nexists T \subseteq N : \phi_i(T) > \phi_i(\pi)$ for all $i \in T$.[7]

In characteristic function games, it suffices to consider deviations by a group of players to form a single coalition. In partition function games, we have to consider deviations by a group of players to form one or more coalitions. Since in partition function games the payoff of a coalition depends on how other players are organized, one has to make assumptions about the conjectures of the deviating players concerning the reaction of the others. There are several definitions of *core stability* depending on those conjectures.

[7]In order for $\phi_i(T)$ to be well-defined, the allocation rule has to be such that $\phi_i(T)$ depends only on T; if ϕ depends on the whole coalition structure, we need to know the new coalition structure induced by the deviation of T (see below).

The deviating players may assume the best possible scenario and deviate provided that there is at least one partition of the remaining players such that they are all better-off (see Shenoy (1979)), or they may assume the worst possible scenario, as in the α-stability concept, and only deviate if they are better-off regardless of the reaction of the outsiders. A bit less pessimistic is the concept of β-stability: a group of players S will deviate if for every partition of the remaining players there is a partition of S that makes all players in S better-off.[8] Intermediate stability requirements are given by the γ-stability concept and the δ-stability concept. These concepts assume that: a) coalitions that have not been broken by the deviation do not react at all; b) coalitions that have been broken react always in the same way: either breaking into singletons (γ-stability) or sticking together (δ-stability).

The concepts of α-, β-, γ- and δ-stability can be defined for a fixed allocation rule, or without using any allocation rule. For a fixed allocation rule ϕ, a coalition structure π will be core stable if there is no group of players T who can organize themselves in one or more coalitions $(T_1, ..., T_l)$ and be all better-off given the new coalition structure $\pi' \supseteq \{T_1, ..., T_l\}$ (that is, $\phi_i(\pi') > \phi_i(\pi)$ for all $i \in \cup_{j=1}^{l} T_j$). If no allocation rule is used, a payoff vector x will be in the coalition structure core corresponding to a given partition π if there is no group of players T who can organize themselves in one or more coalitions $(T_1, ..., T_l)$ and be all better-off given the new coalition structure $\pi' \supseteq \{T_1, ..., T_l\}$ (that is, $\varphi(T_j, \pi') > \sum_{i \in T_j} x_i$ for all $j = 1, ..., l$); coalition structure π will then be stable if its core is non-empty. In both cases, the new coalition structure π' will depend on the stability concept.[9]

Even in games in characteristic function form, one may need to make conjectures about the behavior of players outside the coalition. This is the case if the allocation rule depends on the whole coalition structure, as in Hart and Kurz (1983).

Hart and Kurz (1984) show that there are games and allocation rules such that no outcomes are even α-stable.

The stability concepts above assume that the deviating players expect outsiders to react in an ad hoc way. A concept of ω-stability (Thoron (2000)) that would assume that players react optimally, is generally difficult to define. Extensive form games, on the

[8] These two stability concepts are in the spirit of the α- and β- cores introduced by Aumann (1967). The difference is that Aumann uses pessimistic conjectures in order to construct a characteristic function, whereas the concepts of α- and β- stability are defined for an exogenously given partition function.

[9] Notice that, for a fixed allocation rule, the concepts of γ- and δ-stability are equivalent to the coalition structure being a strong Nash equilibrium outcome in the games Γ and Δ respectively.

other hand, make players expectations consistent, though paying the price of having to choose a given extensive form game (and it may not be clear which one to choose). One may wonder whether there is any relation between the noncooperative games and the cooperative stability concepts. Are the equilibrium coalition structures of the extensive form games stable in any of the senses above? Bloch (1996) addresses this question. In general, a coalition structure may be an equilibrium of an extensive form game without being α-, β-, γ- or δ-stable and the reverse. Since the reaction of outsiders in an extensive form game will be an optimal reaction and this reaction may be more favorable to the deviating players than the reaction prescribed by the stability concept, it is clear that a stable coalition structure may not be an equilibrium outcome of the extensive form game. Furthermore, since the extensive form games require coalitions that form to "leave the room", these coalitions cannot react to deviations by other players. On the other hand, extensive-form games do not allow several coalitions to form at the same time, so a deviating group of players is restricted to organize itself in a single coalition.

2.5.2 Farsighted models

Ray and Vohra (1997) develop the concept of *equilibrium binding agreements.* This concept is defined recursively. A key element is that only deviations to finer coalition structures are allowed. Moreover, deviations are required to be consistent in the sense that they should be immune to deviations to even finer partitions. A variant of this model has recently been used to study the optimality of international agreements (see Conconi and Perroni (2000)).

Chwe (1994) defines some solution concepts based on the idea of *indirect domination* (see also Harsanyi (1974)). The indirect domination concept captures the idea that deviating coalitions anticipate further deviations. Chwe's analysis could be applied to the stability of coalition structures, though in the case of variable payoff division this seems very difficult. Moreover, Chwe's solution concept, the *largest consistent set,* is often very large. A refinement of the largest consistent set has proved useful to predict the stability of coalition structures in the context of Cournot oligopoly (see Mauleón and Vannetelbosch (1999)).

2.5.3 Membership rights

Stability concepts inspired by the interpretation of coalitions as jurisdictions restrict the set of players that can deviate: only one individual at a time can deviate and he may need the consent of the coalition that he leaves or joins. Such are the concepts of *individual stability* (the coalition that is joined by the individual must agree to this move) and *contractual stability* (both the coalition that is left and the coalition that is joined must agree). Those concepts are discussed in Greenberg (1994).

Part I

Noncooperative Games of Coalition Formation

Chapter 3

Noncooperative Bargaining in Partition Function Games

In most economic situations, the payoff of a coalition depends on what other coalitions form, that is, there are externalities between coalitions. A function assigning to each coalition a payoff depending on the whole coalition structure is called a *partition function* (Thrall and Lucas (1963)). In spite of the empirical relevance of externalities, standard cooperative game theory is not based on the partition function but on the characteristic function, which assigns a payoff to each coalition regardless of how outsiders are organized. In order to derive a characteristic function from a situation with externalities, players are assumed to have conjectures about how the rest of the players will organize themselves given that a coalition forms. These conjectures are usually pessimistic (players fear the worst) or optimistic (players expect the best).[1] Thus, when a coalition forms, the rest of the players are expected to partition themselves so as to either minimize or maximize the payoff of the coalition, regardless of their own interest.

An alternative to indiscriminated optimism or pessimism is to use an extensive form game of coalition formation (together with the concept of subgame perfection) in order to allow a coalition to predict the reaction of the outsiders as an equilibrium reaction, so that conjectures are consistent (see Bloch (1996) and Ray and Vohra (1999)). Of course, extensive form games have the drawback that the reaction of the outsiders may heavily depend on the details of the extensive form game.

Because the details of the extensive form game matter, it is useful to compare different extensive form games. This chapter coincides with Bloch (1996) and Ray and Vohra

[1]For a recent illustration of this approach, see Funaki and Yamato (1999).

(1999) in using an extensive form game to predict coalition formation, but it differs in the concrete extensive form.

Ray and Vohra (1999) extend the model of coalitional bargaining of Chatterjee et al. (1993) to games with externalities. Both models are natural generalizations of the Rubinstein (1982) two-player alternating offers bargaining game.[2] A distinctive feature of these models is that players respond to proposals according to a predetermined *rule of order* and the first player to reject a proposal automatically becomes the next proposer. Competition between the players is then very limited, as two players may continuously make offers and counteroffers to each other without any other player having the opportunity to step in.

The assumption that players, by making offers to each other, may exclude other players from the negotiations may not always be appropriate. Muthoo (1999) refers to this question in his book:

> In modelling this three player (one seller-two buyers) bargaining situation, one should allow for the possibility that the two buyers may consider forming a coalition, and then bargain with the seller over the sale (...). At the same time, perhaps while the buyers are negotiating the terms and structure of their coalition, the seller may have the opportunity to approach one of the two buyers and negotiate to sell the house to her - a strategy intended to prevent the formation of the buyers' coalition. Clearly, such factors need to be considered when modelling bargaining situations with three or more players.

The present chapter attempts to address this issue by considering an alternative model of coalitional bargaining. This model has been studied by Binmore (1987) for two players, Baron and Ferejohn (1989) for symmetric majority games, and Okada (1996) for characteristic function games. This chapter considers the extension to partition function bargaining. The distinctive feature of this model is that a player who rejects an offer does not automatically become the next proposer. Instead, proposers are selected randomly by Nature. By giving less power to the responders, this model incorporates competition between the players: any player may have an opportunity to "step in" with a proposal during the negotiations.

[2] Bloch (1996) also belongs to this family of models, even though it is not exactly an extension of Rubinstein (1982) and Chatterjee et al. (1993) to games with externalities, since it assumes exogenous rather than endogenous payoff division.

As it is common in coalitional bargaining games, we assume that binding contracts can be written over the division of the coalitional payoff, so that both coalition structure and payoff division are determined endogenously. A player that has proposed a coalition and a division of payoffs is committed to his proposal, and no other forward commitments are possible.

Since the payoff of a coalition depends on the whole coalition structure, players may have to take their coalition formation decisions without knowing what the payoff of the coalition will be. This is not a problem since in equilibrium the probabilities of each coalition structure can be calculated from the (equilibrium) strategies of the players and thus the expected coalitional payoff is correctly anticipated.

In principle, the proposal could assign a different payoff division to each possible coalition structure. Players could then calculate their expected payoffs given the equilibrium probabilities of each coalition structure, and decide whether to accept or reject the proposal on the basis of these expected payoffs. Since it is only expected payoffs that matter, we adopt the simplifying convention that the proposer offers a fixed payoff to the rest of players in the coalition (he "buys the right to represent them") and thus becomes the residual claimant.

The partition function is assumed to be fully cohesive, that is, a merger of two or more coalitions is always (weakly) profitable for a fixed partition of the remaining players. Two possible extensive form games are considered, differing on the source of friction in the bargaining process: a model with discounting of payoffs, and a model with an exogenous probability of breakdown after a proposal is rejected. We will focus on the limiting case in which frictions are arbitrarily small. The solution concept is stationary perfect equilibrium.

The main focus of the analysis is on the efficiency (in the sense of aggregate expected payoff maximization) of the outcome. Two relevant properties are immediate agreement and formation of the grand coalition. These properties are sufficient for efficiency given the assumptions, and also necessary if players discount the future and the grand coalition is the only coalition structure that maximizes aggregate payoffs.

Agreement is always immediate in the model with discounting. Since time itself is not valuable in the model with breakdown probability, it is not surprising that delay can arise in equilibrium. Delay in this context does not imply a loss of efficiency; furthermore, we show that a delayed agreement can only occur if it maximizes aggregate payoffs. We also show that delay is not a generic phenomenon and that a player who creates delay in equilibrium by making an unacceptable proposal is playing a "weak" strategy in the sense that he could get the same payoff by making an acceptable proposal to the grand coalition. One can derive the following practical rule from this result: if the grand coalition cannot arise in equilibrium, neither can agreement be delayed.

Two natural sufficient conditions are derived for the grand coalition to arise in equilibrium given that all players have the same probability of becoming proposers. For the model with discounting, the grand coalition must have the highest per capita payoff over all coalition structures. For the model with probability of breakdown, the grand coalition must have the highest per capita gain (with respect to the situation with no coalitions) over all coalition structures.

Given an equilibrium of the extensive form game, one can construct a characteristic function game by assigning to each coalition its expected *equilibrium* payoff. We show that, for any equilibrium such that the grand coalition forms immediately and with probability one, expected equilibrium payoffs must lie in the core of this characteristic function game.

Finally, the game with random proposers is compared to the game with a rule of order studied by Ray and Vohra (1999). The game with random proposers guarantees immediate agreement for fully cohesive games, unlike the game with a rule of order. In general, the two games cannot be ranked in terms of efficiency. However, in very specific cases, like symmetric games without externalities in which only one coalition with positive value can form, the outcome of the game with random proposers is at least as efficient as the outcome of the game with a rule of order. In this particular case, the higher efficiency of the random proposers game is achieved due to the advantage of the proposer (which does not disappear in the limit when the frictions of the bargaining process vanish), thus there is a trade-off between efficiency and distribution.

The rest of the chapter is organized as follows. Section 3.1 presents the two versions of the random proposers model. Sections 3.2 and 3.3 are devoted to the efficiency properties of the equilibrium. Section 3.4 relates efficient equilibria to the core of the characteristic function game derived from the equilibrium strategies. Section 3.5 compares the game with random proposers studied in this chapter to the game with a rule of order in Ray and Vohra (1999). Section 3.6 concludes.

3.1 The model

3.1.1 The partition function

Let $N = \{1, 2, ..., n\}$ be the set of players. The non-empty subsets of N are called *coalitions*.

Definition 3.1 A **coalition structure** $\pi := \{S_1, ..., S_m\}$ is a partition of N into coalitions, hence it satisfies

$$S_j \cap S_k = \varnothing \text{ if } j \neq k, \quad \cup_{j=1}^{m} S_j = N. \tag{3.1}$$

The set of all coalition structures is denoted by $\Pi(N)$. For any subset T of N, the set of partitions of T is denoted by $\Pi(T)$ with typical element π_T.

Definition 3.2 An **embedded coalition** is a pair (S, π) with $S \in \pi$.

The set of all embedded coalitions is denoted by $E(N)$.

Definition 3.3 A **partition function** φ assigns a real number to each embedded coalition (S, π), thus $\varphi : E(N) \longrightarrow R$.

The value $\varphi(S, \pi)$ represents the payoff of coalition S given that coalition structure π forms.

Notation 3.1 Given a coalition structure $\pi = \{S_1, ..., S_m\}$ and a partition function φ, we will denote the m-dimensional vector $(\varphi(S_i, \pi))_{i=1}^{m}$ by $\overline{\varphi}(S_1, ..., S_m)$. It will sometimes be convenient to write down the partition function in terms of $\overline{\varphi}$. We will also denote $\varphi(N, \{N\})$ by $\varphi(N)$ and the partition of N into singletons by $\langle N \rangle$. Analogously, the partition of any set $T \subset N$ into singletons will be denoted by $\langle T \rangle$. We will economize on brackets and suppress the commas between elements of the same coalition. Thus, we will write $\varphi(\{i\}, \{\{i\}, \{j, k\}\})$ as $\varphi(i, \{i, jk\})$ and $\overline{\varphi}(\{i\}, \{j, k\})$ as $\overline{\varphi}(i, jk)$.

Definition 3.4 A **cooperative game in partition function form** (a partition function game) is a pair (N, φ), where N denotes the set of players and $\varphi : E(N) \longrightarrow R$.

Definition 3.5 A partition function game (N, φ) is **positive** if

$$\varphi(S, \pi) \geq 0 \text{ for all } (S, \pi) \text{ and } \varphi(S, \pi) > 0 \text{ for all } (S, \pi),\ |S| \geq 2.$$

Definition 3.6 A partition function game (N, φ) is **superadditive** if for all $\pi \in \Pi(N)$, S_i, $S_j \in \pi$, $S_i \neq S_j$ it holds that

$$\varphi(S_i \cup S_j, (\pi \backslash \{S_i, S_j\}) \cup \{S_i \cup S_j\}) \geq \varphi(S_i, \pi) + \varphi(S_j, \pi).$$

Superadditivity means that a merger of any two coalitions is weakly profitable for a given partition of the remaining players.

Definition 3.7 A partition function game (N, φ) is **cohesive** if

$$\varphi(N) \geq \sum_{S \in \pi} \varphi(S, \pi) \text{ for all } (S, \pi) \in E(N).$$

Cohesiveness means that total payoffs are maximized when players form the grand coalition. Thus, starting from an arbitrary partition, a merger of all coalitions to form the grand coalition is always weakly profitable.

If any merger of coalitions is weakly profitable for any given partition of the remaining players, the partition function is called fully cohesive.

Definition 3.8 A partition function game (N, φ) is **fully cohesive** if for all $(S, \pi) \in E(N)$ and for all $\pi_S \in \Pi(S)$ it holds that

$$\varphi(S, \pi) \geq \sum_{T \in \pi_S} \varphi(T, (\pi \backslash \{S\}) \cup \pi_S).$$

Notice that cohesiveness and superadditivity are independent properties, both of them weaker than full cohesiveness.[3] The following examples illustrate these three properties.

Example 3.1 A game that is superadditive but not cohesive

$N = \{1, 2, 3\}$

$\overline{\varphi}(1, 2, 3) = (3, 3, 3)$

$\overline{\varphi}(ij, k) = (7, 0)$

$\overline{\varphi}(123) = 8$

Even though any merger of two coalitions is profitable, total payoffs are maximized when all players remain singletons. The externality that a two-player merger imposes on the outsider is stronger than the internal gain, thus the game is not cohesive.

[3]For the particular case of games in characteristic function form, superadditivity and full cohesiveness are equivalent.

Example 3.2 A game that is cohesive but not superadditive

$N = \{1, 2, 3\}$

$\overline{\varphi}(1, 2, 3) = (1, 1, 1)$

$\overline{\varphi}(ij, k) = (0, 3)$

$\overline{\varphi}(N) = 5$

Example 3.3 A game that is cohesive and superadditive but not fully cohesive

$N = \{1, 2, 3, 4\}$

$\overline{\varphi}(1, 2, 3, 4) = (3, 3, 3, 3)$

$\overline{\varphi}(ij, k, l) = (7, 0, 0)$

$\overline{\varphi}(ijk, l) = (8, 0)$

$\overline{\varphi}(ij, kl) = (2, 2)$

$\overline{\varphi}(N) = 15$

The merger of any two coalitions is profitable and the grand coalition achieves the highest payoff, but the merger of three singletons is unprofitable.

Example 3.4 A game that is fully cohesive

$N = \{1, 2, 3, 4\}$

$\overline{\varphi}(1, 2, 3, 4) = (1, 1, 1, 1)$

$\overline{\varphi}(ij, k, l) = (3, 0, 0)$

$\overline{\varphi}(ijk, l) = (3, 0)$

$\overline{\varphi}(ij, kl) = (1, 1)$

$\overline{\varphi}(N) = 4$

This example shows that full cohesiveness does not imply that going from a finer to a coarser partition will always improve payoffs. Starting from the situation where all players are singletons, full cohesiveness implies that any two of them would profit from merging. Moreover, starting from a partition of the type $\{ij, k, l\}$, full cohesiveness implies that k and l would profit from merging. Nevertheless, starting from the partition $\{i, j, k, l\}$ players need not benefit by forming the coarser partition $\{ij, kl\}$, as the example shows.

Example 3.4 also shows that a partition function may be fully cohesive while at the same time the formation of coalitions can only reduce aggregate payoffs.

Assumption 3.1 We will assume in most of this chapter that the partition function is fully cohesive; for some of the results we will also assume it to be positive.

3.1.2 The noncooperative bargaining game

The noncooperative bargaining procedure described in this section was introduced by Binmore (1987) in the context of two-player bargaining games, and later extended by Baron and Ferejohn (1989) to symmetric majority games, and by Okada (1996) to characteristic function games. This chapter considers the extension to partition function games.

We will consider two versions of the bargaining game, depending on the source of friction: the game with discounting and the game with breakdown probability.

The game with discounting

Time is discrete and indexed by $t = 1, 2...$ Given the underlying partition function game (N, φ), bargaining proceeds as follows:

- Nature selects a player randomly to be the *proposer* according to the probability vector $\theta := (\theta_i)_{i \in N}$, where $\theta_i > 0$ for all $i \in N$ and $\sum_{i \in N} \theta_i = 1$. This probability vector is called a *protocol*. If $\theta_i = \frac{1}{n}$ for all i it is called *symmetric* or *egalitarian* protocol, denoted by θ^E.

- The proposer i makes a *proposal* $(S, x^{S\backslash\{i\}})$ where S is a coalition to which i belongs and $x^{S\backslash\{i\}} = (x_j^{S\backslash\{i\}})_{j \in S\backslash\{i\}}$ is a payoff vector. The jth component $x_j^{S\backslash\{i\}}$ represents the payoff player j will receive provided that all players in S accept the proposal and that a coalition structure is formed. Thus, we can think of i as "buying the right to represent coalition S ".

- Given a proposal, the rest of players in S (called *responders*) accept or reject sequentially (the order does not affect the results).

- If all players in S accept, S is formed and bargaining continues between players in $N\backslash S$ provided $|N\backslash S| > 1$. A player j will then be selected to be the proposer with probability 0 if he belongs to S and with probability $\frac{\theta_j}{\sum_{k \in N\backslash S} \theta_k}$ if he does not.

- If at least one player in S rejects, the game proceeds to the next period in which Nature selects a new proposer according to θ.

- Players are risk-neutral and share a discount factor $\delta < 1$. We will think of δ as being close to 1.

- When a coalition structure π is formed[4], each coalition $S \in \pi$ gets $\varphi(S, \pi)$. Coalitions of more than one player share this payoff in the following way: the player i whose proposal to form S was accepted pays $x_j^{S\backslash\{i\}}$ to each responder j in $S\backslash\{i\}$ and gets the residual payoff $\varphi(S, \pi) - \sum_{j \in S\backslash\{i\}} x_j^{S\backslash\{i\}}$. This payoff may be negative (we assume players are not financially constrained). Thus, if a coalition structure is formed at time t, each player in S other than i receives $\delta^{t-1} x_j^{S\backslash\{i\}}$ and i receives $\delta^{t-1}[\varphi(S, \pi) - \sum_{j \in S\backslash\{i\}} x_j^{S\backslash\{i\}}]$ (we refer to payoffs evaluated at $t = 1$). If no coalition structure is formed, all players receive zero.

We assume a period elapses after a proposal is rejected, but not after a coalition is formed. One may alternatively assume that a period elapses in both cases without affecting the results in any essential way.

Definition 3.9 A (pure) **strategy** for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy of player i, prescribes

(i) A **proposal** $(S, x^{S\backslash\{i\}})$.

(ii) A **response function** assigning "yes" or "no" to all possible proposals of the other players.

Notice that since no time elapses after a coalition is formed there may be several "stages" at time t, each of them with a smaller set of remaining players than the previous, and each player taking an action in at most one of these stages.

The solution concept is *stationary perfect equilibrium.*

Definition 3.10 **A stationary perfect equilibrium** is a subgame perfect equilibrium with the property that the strategies of the players depend only on the set of coalitions that have already formed, $\pi_{N\backslash T}$, and the current proposal.

Definition 3.11 An equilibrium is **efficient** if it maximizes the aggregate payoffs of the players.

Efficiency in the game with discounting implies two requirements: immediate agreement and formation of a coalition structure with maximal aggregate payoffs.

[4]Formally, let R_t be the set of players who have not yet formed a coalition at the end of period t. If $|R_t| \in \{0, 1\}$ and $|R_{t-1}| > |R_t|$, we say that a coalition structure has formed at time t.

Notation 3.2 We will denote the extensive form game described above by $G(N, \varphi, \theta, \delta)$.

We will also introduce notation for the *reduced games*.

Definition 3.12 A **reduced game** is a subgame starting immediately after a coalition is formed.

Once a certain set of players $N \backslash T$ have formed coalitions, bargaining continues among the players in T. Players in T face the partition function game $(T, \varphi^{\pi_{N\backslash T}})$, where the partition function $\varphi^{\pi_{N\backslash T}} : E(T) \longrightarrow R$ is obtained from φ by fixing $\pi_{N\backslash T}$.

Notation 3.3 We will denote the reduced game arising after the partition $\pi_{N\backslash T}$ has formed by $G(T, \varphi^{\pi_{N\backslash T}}, \theta, \delta)$.

The game with breakdown probability

This game is identical to the game with discounting, except in the source of friction in the bargaining process. In a game with breakdown probability:

- Players do not discount future payoffs.
- After a proposal is rejected Nature selects a new proposer with probability p (we will again think of p as being close to 1) and a breakdown of the negotiations occurs with probability $1 - p$.
- If a breakdown of the negotiations occurs, all players that did not form a coalition yet remain singletons. Thus, if we denote the set of remaining players by T and the set of coalitions already formed by $\pi_{N\backslash T}$, a breakdown of the negotiations induces the partition $\pi_{N\backslash T} \cup \langle T \rangle$.
- Payoffs are as in the previous game but undiscounted. Thus, when a coalition structure π is formed, coalitions containing more than one player share the coalitional payoff in the following way: player i whose proposal to form S was accepted pays $x_j^{S\backslash\{i\}}$ to responder j and gets the residual payoff $\varphi(S, \pi) - \sum_{j \in S\backslash\{i\}} x_j^{S\backslash\{i\}}$, and each singleton $\{k\}$ receives $\varphi(k, \pi)$.

Unlike in the game with discounting, delay by itself does not imply a loss of efficiency.

The possibility of breakdown implies that a coalition structure forms with probability 1 in this game no matter what strategies are played (either players agree to it, or breakdown will eventually occur).

Notation 3.4 We will denote the extensive form game with breakdown probability by $G(N, \varphi, \theta, p)$ and the reduced game arising after partition $\pi_{N\backslash T}$ has formed by $G(N, \varphi^{\pi_{N\backslash T}}, \theta, p)$.

Expected payoffs and continuation values

Let σ be a combination of stationary strategies. Suppose that no coalitions have formed yet. Thus, we are at the beginning of the game or at a subgame that is equivalent to the beginning of the game. We will denote the *expected payoff* of player i given σ by $w_i(\sigma)$. This expectation is computed before Nature draws the proposer. We will denote by $w_i^j(\sigma)$ the expected payoff for player i given that player j has been selected to be the proposer.

Suppose now a proposal has been made to player i. The expected payoff of player i if he rejects a proposal is called the *continuation value* of player i and will be denoted by $z_i(\sigma)$. This is also i's expected payoff if somebody else other than i rejects a proposal.

In the game with discounting, it holds that $z_i(\sigma) = \delta w_i(\sigma)$. In the game with probability of breakdown, $z_i(\sigma)$ is a convex combination of $w_i(\sigma)$ (with weight p) and $\varphi(i, \langle N \rangle)$ (with weight $1 - p$).

Because the strategies are stationary, these values are the same at any subgame provided that no coalitions have formed yet. The definitions are analogous for a reduced game. We will denote player i's expected payoff in the reduced game arising after $\pi_{N\backslash T}$ has formed by $w_i^{\pi_{N\backslash T}}(\sigma)$, and his continuation value by $z_i^{\pi_{N\backslash T}}(\sigma)$. We will drop σ from the notation when no confusion can arise.

The ordering of the responders and the identity of the proposer

The order in which the responders accept or reject a proposal need not be specified, since it has no practical relevance. In any subgame perfect equilibrium, a player accepts any proposal that gives all the responders at least their continuation values. If a proposal gives one of the responders less than his continuation value, it will be rejected (possibly by another responder). Since the consequences of a rejection are the same regardless of which player rejected the proposal, the order in which players respond to a proposal does not affect the results.

The identity of the proposer will also be of little relevance in the following sense: whether a payoff vector $x^{S\backslash\{i\}}$ will be accepted or rejected does not depend on the identity of the proposer i. What matters is how this payoff vector compares with the continuation values of the players in $S\backslash\{i\}$, and stationarity implies that the continuation values are independent of who made the last rejected proposal.

Buying the right to represent a coalition

We assume that the proposer offers a fixed payoff to the responders, rather than a payoff contingent on the coalition structure that eventually forms. This assumption is without loss of generality since players are risk-neutral, not financially constrained and the solution concept is stationary perfect equilibrium. Given a proposal to form S with a contingent payoff division, players can compute expected payoffs using the probability that each coalition structure will form (stationarity implies that these probabilities are independent of the payoff division in S), and they will accept or reject on the basis of expected payoffs.

3.2 No-delay results

Given a strategy combination, we will say that a coalition structure forms without delay in equilibrium if equilibrium strategies are such that all proposals that are made with positive probability will be accepted. In all other cases (thus also in the extreme case when players never make acceptable proposals), we will speak of *possible delay.*

3.2.1 The game with discounting

Theorem 3.1 states that a coalition structure will form without delay if the underlying partition function is fully cohesive and positive. This theorem is an extension of theorem 1 in Okada (1996) to partition function games. The proof rests on the following lemma:

Lemma 3.1 *Consider a positive and fully cohesive game* (N, φ). *In any stationary perfect equilibrium* σ^* *of the game* $G(N, \varphi, \theta, \delta)$ *it holds that*

(i) $\sum_{i \in N} w_i(\sigma^*) \leq \varphi(N)$.

(ii) $\sum_{i \in T} w_i^{\pi_{N \setminus T}}(\sigma^*) \leq \varphi^{\pi_{N \setminus T}}(T)$.

Proof. Since the game is positive and fully cohesive, the maximum aggregate payoff for the players is achieved if the grand coalition is formed immediately. Delay of the agreement or formation of subcoalitions can only reduce aggregate payoffs.[5] The same reasoning applies to any reduced game. ■

[5]Notice that this result holds for any strategy combination, not necessarily an equilibrium.

Theorem 3.1 *Consider a positive and fully cohesive game (N, φ). In any stationary perfect equilibrium σ^* of the game $G(N, \varphi, \theta, \delta)$, a coalition structure is formed without delay.*

Proof. Consider any stationary perfect equilibrium σ^*. Let $\mu(\pi^t|(\sigma^*, S))$ be the probability that coalition structure π is formed at time t given that players follow σ^* and that S is the first coalition to form. For every $i = 1, ..., n$, let $m_i(\sigma^*)$ be the maximum value of the following maximization problem

$$\begin{array}{ll} \max\limits_{S, y^{S\setminus\{i\}}} & \sum\limits_{\substack{\pi\in\Pi(N)\\ S\in\pi}} \sum\limits_{t=1}^{\infty} \mu(\pi^t|(\sigma^*, S))\delta^{t-1} \left[\varphi(S,\pi) - \sum\limits_{j\in S\setminus\{i\}} y_j^{S\setminus\{i\}}\right] \\ s.t. & i \in S \subseteq N \\ & \sum\limits_{\substack{\pi\in\Pi(N)\\ S\in\pi}} \sum\limits_{t=1}^{\infty} \mu(\pi^t|(\sigma^*, S))\delta^{t-1}\, y_j^{S\setminus\{i\}} \geq \delta w_j(\sigma^*) \end{array}$$

We will show that $w_i^i(\sigma^*) = m_i(\sigma^*)$, that is, the expected payoff for player i given that he is selected to be the proposer and follows his equilibrium strategy equals the expected payoff for player i given that he makes the proposal that solves the maximization problem above.

Subgame perfection implies $w_i^i(\sigma^*) \geq m_i(\sigma^*)$. In a subgame perfect equilibrium all responders accept any proposal that gives each responder j at least $\delta w_j(\sigma^*)$ in expected terms, thus player i can get at least $m_i(\sigma^*)$.

On the other hand, player i cannot get more than $m_i(\sigma^*)$ in equilibrium. If player i proposes $(S, y^{S\setminus\{i\}})$ with $\sum_{\pi\in\Pi(N),\, S\in\pi} \sum_{t=1}^{\infty} \mu(\pi^t|(\sigma^*, S))\delta^{t-1}[\varphi(S,\pi) - \sum_{j\in S\setminus\{i\}} y_j^{S\setminus\{i\}}] > m_i(\sigma^*)$, the proposal will be rejected (otherwise at least one responder j is getting less than $\delta w_j(\sigma^*)$ and could do better by rejecting the proposal) and i will get $\delta w_i(\sigma^*)$. Lemma 3.1 states that $\sum_{i\in N} w_i(\sigma^*) \leq \varphi(N)$. This inequality implies that player i could have proposed the grand coalition, paid every other player j $w_j(\sigma^*)$ and kept at least $w_i(\sigma^*)$ for himself. Thus, $m_i(\sigma^*) \geq w_i(\sigma^*) \geq \delta w_i(\sigma^*)$.

Since player i can always get $m_i(\sigma^*)$ (by making the optimal acceptable proposal) and he cannot get more than $m_i(\sigma^*)$ (unacceptable proposals yield at most $m_i(\sigma^*)$), it follows that $w_i^i(\sigma^*) = m_i(\sigma^*)$.

To prove that player i strictly prefers to make acceptable proposals, we must prove $\delta w_i(\sigma^*) < m_i(\sigma^*)$. Since $w_i(\sigma^*) \leq m_i(\sigma^*)$ and $\delta < 1$, $\delta w_i(\sigma^*) = m_i(\sigma^*)$ would imply

$w_i(\sigma^*) = m_i(\sigma^*) = 0$. Since $\sum_{j \in N} w_j(\sigma^*) \leq \varphi(N)$, it follows that $m_i(\sigma^*) \geq (1-\delta)\varphi(N) > 0$ (any player can get a strictly positive payoff as a proposer by exploiting the cohesiveness of the partition function and the impatience of the other players).

Notice that the same reasoning applies to any reduced game, so that the whole coalition structure forms without delay. ■

We have assumed that the proposer pays the responders after a coalition structure has formed. If instead the proposer pays the responders immediately after S is formed, the objective function of the proposer becomes $\sum_{\pi \in \Pi(N), S \in \pi} \sum_{t=1}^{\infty} \mu(\pi^t|(\sigma^*, S))\delta^{t-1}\varphi(S,\pi) - \sum_{j \in S\setminus\{i\}} y_j^{S\setminus\{i\}}$, and the constraint becomes $y_j^{S\setminus\{i\}} \geq \delta w_j(\sigma^*)$. Notice that the reasoning of the proof is also valid for this case.

Corollary 3.1 *Consider a positive and fully cohesive game (N, φ). In any stationary perfect equilibrium σ^* of the game $G(N, \varphi, \theta, \delta)$, every player i in N proposes a solution of the following maximization problem*

$$\max_{S, y^{S\setminus\{i\}}} \sum_{\substack{\pi \in \Pi(N) \\ S \in \pi}} \mu(\pi|(\sigma^*, S)) \left[\varphi(S,\pi) - \sum_{j \in S\setminus\{i\}} y_j^{S\setminus\{i\}} \right] \tag{3.2}$$
$$\text{s.t. } i \in S \subseteq N$$
$$y_j^{S\setminus\{i\}} \geq \delta w_j(\sigma^*)$$

The proposer makes a proposal that maximizes his own payoff subject to the proposal being accepted. Since players are risk-neutral, what matters for the proposer is his expected payoff. Player i's payoff as a proposer depends on the equilibrium strategies of the other players through the probabilities $\mu(\pi|(\sigma^*, S))$ and the expected payoff vector $(w_j(\sigma^*))_{j \in N\setminus\{i\}}$. The expression for player i's expected payoff incorporates the fact that there is no delay in equilibrium.

Corollary 3.2 *Consider a positive and fully cohesive game (N, φ). In any stationary perfect equilibrium of the game $G(N, \varphi, \theta, \delta)$, every player i in N has a strictly positive expected payoff.*

As we have seen in the proof of theorem 3.1, a player can get a strictly positive expected payoff as a proposer by exploiting the impatience of the other players together with the cohesiveness of the partition function. As a responder, he can guarantee himself a zero payoff by rejecting all proposals and possibly becoming a singleton.

Corollary 3.3 *Consider a positive and fully cohesive game (N, φ). Every player i gets a higher payoff as a proposer than as a responder.*

This follows from the fact that $m_i(\sigma^*) > \delta w_i(\sigma^*)$.

Remark 3.1 *From the proof of theorem 3.1 we can rank the three following payoffs: the payoff a player gets as a proposer, his expected payoff before the proposer is selected, and his payoff as a responder, namely*

$$m_i(\sigma^*) = w_i^i(\sigma^*) > w_i(\sigma^*) > w_i^j(\sigma^*) = \delta w_i(\sigma^*).$$

We have assumed that the partition function is positive. This assumption plays an essential role in the proof of theorem 3.1. First, the grand coalition must have a strictly positive payoff or players would (weakly) prefer to bargain forever. Second, to make sure that the whole coalition structure is formed without delay, all subcoalitions must have a strictly positive payoff so that the argument can apply to every possible reduced game. Singletons are an exception since they are not always formed voluntarily (that is, if only one player is left the singleton coalition is formed automatically). The arguments in the proof would still go through if singletons could have negative payoffs, but we have chosen to keep the partition function positive.

We have limited the analysis to positive partition functions mostly for reasons of interpretation. Since the extensive form game is such that players do not receive payoffs until a coalition structure is formed, the use of this game to model partition function bargaining seems only appropriate for situations in which coalitions get positive payoffs. Imagine a situation of conflict: once a coalition is formed, it may be that the players outside the coalition can only get negative payoffs. Then these players would prefer not to reach an agreement, and no payoffs would be realized since a coalition structure has not been formed. This does not seem to be a feasible alternative in actual conflicts.[6] This example shows that "normalization" of payoffs is not without loss of generality in this game (only normalization by multiplying by a positive constant is innocuous).[7]

We include a counterexample that shows that delay may arise if the partition function is not positive. A counterexample that shows that delay may arise if the partition function is not fully cohesive can be found in subsection 3.2.3.

[6] One can assign utilities different from zero to the outcome of perpetual disagreement. However, this seems paradoxical: when are these payoffs realized?

[7] This fact was first pointed out by Fishburn and Rubinstein (1982).

Example 3.5 Delay with a partition function that is not positive

$N = \{1, 2, 3\}$

$\overline{\varphi}(1, 2, 3) = (-1, -1, -1)$

$\overline{\varphi}(ij, k) = (-2, -2)$

$\overline{\varphi}(123) = -3$

This partition function is fully cohesive but not positive. There are many stationary perfect equilibria, all of them ending in perpetual disagreement. For example, all players may make unacceptable proposals (offering to each responder a payoff of less than zero). It may also be that the first player to be selected forms a singleton coalition, and then the other two players never reach an agreement.

3.2.2 The game with breakdown probability

In contrast to the game with discount factor, delay is not completely excluded in the game with breakdown probability. However, delay will not be a generic phenomenon. Even though players do not discount payoffs, the possibility of a breakdown will induce players to reach an agreement unless players have nothing to lose by the breakdown, which generically is not the case in fully cohesive games. Furthermore, possible delay implies efficiency in the subgame where it arises.

Notice that the eventual formation of a coalition structure is guaranteed by the structure of the game.

Remark 3.2 *In any stationary perfect equilibrium of the game $G(N, \varphi, \theta, p)$ a coalition structure is eventually formed.*

Because a breakdown of the negotiations occurs with positive probability every time a proposal is rejected, a coalition structure will eventually form even if players never make acceptable proposals.

Notice that, unlike in the game with discounting, players cannot "escape" negative payoffs by making unacceptable proposals. This is the reason why in this section we assume only that the game is fully cohesive, and not that the game is positive.

Lemma 3.2 *Consider a fully cohesive game* (N, φ). *In any stationary perfect equilibrium* σ^* *of the game* $G(N, \varphi, \theta, p)$ *it holds that*

(i) $\sum_{i \in N} z_i(\sigma^*) \leq \varphi(N)$.

(ii) $\sum_{i \in T} z_i^{\pi_{N \setminus T}}(\sigma^*) \leq \varphi^{\pi_{N \setminus T}}(T)$.

Proof. Since the game is fully cohesive and a coalition structure always forms, the maximum total payoff for the players is achieved if the grand coalition is formed. The formation of subcoalitions (or the breakdown of the negotiations) can only reduce total payoffs. The same reasoning applies to any reduced game $G(T, \varphi^{\pi_{N \setminus T}}, \theta, p)$. ∎

Theorem 3.2 *Consider a fully cohesive game* (N, φ). *Suppose there is a stationary perfect equilibrium* σ^*, *a reduced game* $G(T, \varphi^{\pi_{N \setminus T}}, \theta, p)$ *and a player* i *in* T *such that player* i *makes unacceptable proposals with positive probability in* $G(T, \varphi^{\pi_{N \setminus T}}, \theta, p)$. *Then*

(i) Player i *could get the same payoff by making an acceptable proposal to the grand coalition (of the remaining players)* T.

(ii) $\sum_{i \in T} z_i^{\pi_{N \setminus T}}(\sigma^*) = \varphi^{\pi_{N \setminus T}}(T)$.

Proof. For every $i = 1, ..., n$, recall that $w_i^j(\sigma^*)$ denotes player i's equilibrium expected payoff conditional on player j becoming the proposer at time 1. Analogously to the game with discounting, we denote by $m_i(\sigma^*)$ the maximum payoff the proposer can get *given that he makes an acceptable proposal.* In the game with breakdown probability, $m_i(\sigma^*)$ is the value of the following maximization problem

$$\begin{aligned} \max_{S, y^{S \setminus \{i\}}} \; & \sum_{\pi \in \Pi(N)} \mu(\pi | (\sigma^*, S)) \varphi(S, \pi) - \sum_{j \in S \setminus \{i\}} y_j^{S \setminus \{i\}} \\ s.t. \; & i \in S \subseteq N \\ & y_j^{S \setminus \{i\}} \geq z_j(\sigma^*) \end{aligned}$$

where $\mu(\pi | (\sigma^*, S))$ is the probability that coalition structure π is eventually formed given that players follow σ^* and that S is the first coalition to form, and $z_j(\sigma^*)$ is the continuation value for player j, that is, the expected payoff for player j when he rejects a proposal. Notice that since payoffs are not discounted, the time at which a coalition structure is formed does not matter for payoffs. We show that $w_i^i(\sigma^*) = m_i(\sigma^*)$.

Subgame perfection implies $w_i^i(\sigma^*) \geq m_i(\sigma^*)$. In a subgame perfect equilibrium all responders accept any proposal that gives each responder j at least $z_j(\sigma^*)$ in expected terms, thus player i can get at least $m_i(\sigma^*)$.

On the other hand, player i cannot get more than $m_i(\sigma^*)$. If player i proposes $(S, y^{S\setminus\{i\}})$ with $\sum_{\pi\in\Pi(N)} \mu(\pi|(\sigma^*, S))\varphi(S,\pi) - \sum_{j\in S\setminus\{i\}} y_j^{S\setminus\{i\}} > m_i(\sigma^*)$, the proposal will be rejected (otherwise at least one responder j is getting less than $z_j(\sigma^*)$ and could do better by rejecting the proposal) and i will get $z_i(\sigma^*)$. Since the game is cohesive, lemma 3.2 implies that $\sum_{j\in N} z_j(\sigma^*) \leq \varphi(N)$. Therefore, player i could have proposed the grand coalition, paid every other player j $z_j(\sigma^*)$ and kept at least $z_i(\sigma^*)$ for himself. Thus, $m_i(\sigma^*) \geq z_i(\sigma^*)$, and $m_i(\sigma^*) = z_i(\sigma^*)$ only if $\sum_{j\in N} z_j(\sigma^*) = \varphi(N)$.

The previous reasoning applies to delay in the formation of a coalition given that the set of remaining players is N. Since the game is fully cohesive, the same reasoning applies to the occurrence of delay in any reduced game. ■

Since player i weakly prefers to propose the grand coalition (of the remaining players) and offer each of the other players his continuation value, a stationary perfect equilibrium in which agreement is delayed is "weak". Theorem 3.2 also implies that if players would have lexicographic preferences (preferring to agree earlier than later holding everything else constant) then all stationary perfect equilibria would exhibit immediate agreement.

Since players do not discount the future, delay is not a source of inefficiency. Furthermore, possible delay in a subgame *implies* efficiency in the subgame as the following corollary shows.

Corollary 3.4 *Consider a fully cohesive game (N, φ). Suppose there is a reduced game $G(T, \varphi^{\pi_{N\setminus T}}, \theta, p)$ exhibiting possible delay. Then*

(i) The final outcome is always efficient for this reduced game.

(ii) $\sum_{i\in T} \varphi^{\pi_{N\setminus T}}(i, \langle T\rangle) = \varphi^{\pi_{N\setminus T}}(T)$.

Proof. We make the reasoning for the case of possible delay in the formation of the first coalition. The same reasoning applies to any reduced game.

The sum of the continuation values, $\sum_{i\in N} z_i$, is a convex combination of the sum of the expected payoffs given that Nature continues the game (with weight p) and the sum of expected payoffs if breakdown occurs (with weight $1-p$). Each of these sums is at most $\varphi(N)$. Since possible delay implies $\sum_{i\in N} z_i = \varphi(N)$, in an equilibrium with delay *both* sums must be equal to $\varphi(N)$. Given that players can never get more than $\varphi(N)$, $\sum_{i\in N} z_i = \varphi(N)$ implies that after a proposal is rejected equilibrium strategies are such that players *always* (and not only on average) get $\varphi(N)$. This must also be the case at the beginning of the game because of stationarity, thus efficiency is always achieved. In particular, the sum of expected payoffs if breakdown occurs, $\sum_{i\in N} \varphi(i, \langle N\rangle)$, equals $\varphi(N)$. ■

Corollary 3.4 implies that delay will not be a generic phenomenon, since it requires $\sum_{i\in T} \varphi^{\pi_{N\setminus T}}(i, \langle T\rangle) = \varphi^{\pi_{N\setminus T}}(T)$. This condition means that the total payoff of the players that remain in the game can never be higher than the total payoff they would obtain as singletons, so that the formation of coalitions can only bring inefficiency (though not in equilibrium).

Lemma 3.2 and theorem 3.2 suggest the following practical rule:

Corollary 3.5 *Let σ^* be a stationary perfect equilibrium. Let $(z_i(\sigma^*))_{i\in N}$ be the corresponding continuation value vector. Consider the situation of player j as a proposer. If proposing the grand coalition is not a best response for j against σ^*_{-j}, making unacceptable proposals cannot be a best response either.*

This corollary follows directly from the fact that in any stationary perfect equilibrium, any player weakly prefer to propose the grand coalition rather than to make unacceptable proposals.

Corollary 3.5 will be of use while looking for a stationary perfect equilibrium (if the grand coalition is excluded as an equilibrium proposal, unacceptable proposals are automatically excluded as well) or while testing a candidate stationary perfect equilibrium against deviations (if deviation to proposing the grand coalition is not profitable, deviation to a proposal that will not be accepted cannot be profitable either).

3.2.3 An example of possible delay with a partition function that is not fully cohesive

This subsection provides an example of a partition function that is superadditive but not fully cohesive, and shows that delay is possible in equilibrium. Since the game with discounting and the game with breakdown probability are equivalent provided that $\varphi(i, \langle N\rangle) = 0$ for all $i \in N$, example 3.6 is valid for the game $G(N, \varphi, \theta, p)$ as well as for

$G(N, \varphi, \theta, \delta)$. The proof uses the notation of the game with discounting (rephrasing the proof in the terms of the game with breakdown probability is straightforward).

Example 3.6 $N = \{1,2,3,4,5,6,7,8,9\}$, $\theta_i = \frac{1}{9}$ for all i.

We are going to consider a symmetric game. Since the number of players is large, we will depart from our usual notation and denote each coalition by its size.

$\overline{\varphi}(4,2,3) = (16,12,9)$

$\overline{\varphi}(6,3) = (28,2)$

$\overline{\varphi}(2,5,2) = (1,20,1)$

$\overline{\varphi}(5,4) = (21,3)$

$\overline{\varphi}(2,7) = (2,25)$

$\overline{\varphi}(9) = 30$

For the other coalition structures, it holds that, regardless of how the rest of the players are distributed, singletons get 0 and coalitions of size $s > 1$ get a payoff of s. Thus, for example, $\overline{\varphi}(6,2,1) = (6,2,0)$. Notice that this partition function is superadditive.[8]

This example is based in the following idea: a coalition structure of the type $(4,2,3)$ is going to arise in equilibrium. In order for this to be the case, coalitions have to form in a given order: a coalition of size 2 is the most attractive in terms of per capita payoffs, but it cannot form first because then a coalition of size 5 would follow and the coalition of size 2 would have a low payoff. A coalition of size 4 would give rise to the coalition structure $(4,2,3)$, but it cannot form immediately, because then it would be more profitable in expected terms to wait until someone else forms the coalition of 4 and then be in the coalition of 2 or in the coalition of size 3 (the lottery of being either in a coalition of size 2 or in a coalition of size 3 is more attractive than being in the coalition of size 4 for sure). There is an asymmetric equilibrium in which some players form a coalition of 4 and others wait, thus delay is possible.

Consider the partition $(4,2,3)$ as a candidate equilibrium partition (in the calculations below, all the numerical values are computed for $\delta \to 1$). We will first consider fully symmetric equilibria (that is, equilibria in which all players propose a coalition of the same

[8]The game in this example is not cohesive. It is easy to construct an example of possible delay in a cohesive game by using this example as a subgame. For example, consider a symmetric game with 20 players such that the first proposer will propose a coalition of size 11. Given that this coalition of 11 has formed, the subgame with 9 players is identical to the game in example 3.6. To make the game cohesive and to ensure that the coalition of 11 players will form, set $\overline{\varphi}(11,4,2,3) = (1100,16,12,9)$, and $\varphi(N) = 1137$ (and $\overline{\varphi}(15,2,3) = (1116,2,3)$, $\overline{\varphi}(11,6,3) = (1100,28,2)$...).

size and, in doing so, they propose to each of the other players with equal probability. This implies that, for any reduced game, all the players have the same expected payoff).

We first check that coalition structure $(4, 2, 3)$ cannot form in equilibrium starting from a coalition of size 2.

Suppose the formation of a coalition of size 2 is followed by a coalition of size 4 and then by a coalition of size 3. Then the expected payoffs[9] (after the formation of a coalition of size 2) would be given by

$$w = \frac{1}{7}(16 - 3\delta w) + \frac{6}{7}\left[\frac{3}{6}\delta w + \frac{3}{6}3\right].$$

This equation reflects the fact that, when a player is selected to be the proposer (with probability $\frac{1}{7}$) he proposes a coalition of size 4. If the player is not selected to be the proposer (with probability $\frac{6}{7}$) he will be a responder with probability $\frac{3}{6}$ (since the proposer will form a coalition of size 4 and all the remaining players will have the same probability to be responders) and he will be left out with probability $\frac{3}{6}$. In this case, a coalition of the three remaining players would form without delay, so that each of the three players will have an expected payoff of 3.

The solution to the equation is $w = \frac{25}{7}$. The proposer gets then about $\frac{37}{7}$. However, he could do better by proposing a coalition of size 5 and, counting on the fact that the remaining two players will form a coalition, get $20 - 4 * \frac{25}{7} = \frac{40}{7}$.

If a coalition of size 3 were to follow the coalition of size 2, this would not be an equilibrium since the players would rather wait than form the coalition of size 3. Expected payoffs (taking into account that a coalition of 4 will form after the coalition of 3) would be

$$w = \frac{1}{7}(9 - 2\delta w) + \frac{6}{7}\left[\frac{2}{6}\delta w + \frac{4}{6}4\right].$$

This yields $w = \frac{25}{7}$. The proposer would get $9 - 2 * \frac{25}{7} = \frac{13}{7} < \frac{25}{7}$. He would then prefer to wait, hoping to be in a coalition of size 4 later on.[10]

[9] We will denote all the expected payoffs by w (without subindex since it will be the same for all players and without superindex to simplify notation, hoping that no confusion will arise). We will also omit from the notation the strategy combination with respect to which expected payoffs are computed.

[10] In general, if the partition function and the protocol are symmetric and the equilibrium is such that players are to be divided into two coalitions with different per capita payoffs, the coalition with the highest per capita payoff should form first (or players would prefer to wait, hoping to be in the second coalition). See also subsection 3.5.4 on the role of per capita payoffs.

Suppose a coalition of 4 forms first, and this is followed by a coalition of size 2 and then by a coalition of size 3. It is clear that a coalition of size 3 would form, given that the two other coalitions have formed. It is also easy to check that a coalition of size 2 would follow a coalition of size 4. Thus, $(4,2,3)$ would be an equilibrium coalition structure *if a coalition of size* 4 *forms in the first place.* But will this be the case? Suppose a coalition of size 4 forms without delay. Then the expected payoff at the beginning of the game, w, would be given by the following equation:

$$w = \frac{1}{9}\left[16 - 3\delta w\right] + \frac{8}{9}\left[\frac{3}{8}\delta w + \frac{5}{8}\left(\frac{2}{5}6 + \frac{3}{5}3\right)\right].$$

This yields $w = \frac{37}{9} > 4$. This cannot be an equilibrium for high values of δ since the proposer would get a higher payoff by making an unacceptable proposal.

The reason why forming a coalition of size 4 cannot be an equilibrium is that waiting (hoping to get into the coalition of size 2 later on) is a more attractive alternative.

Analogously, coalition structure $(4,2,3)$ cannot form starting by a coalition of size 3 because, even though a coalition of 2 and then a coalition of 4 would follow a coalition of 3, the coalition of 3 would not form in the first place.

There is an equilibrium in which coalitions of sizes $4, 2,$ and 3 form in this order, but this equilibrium exhibits possible delay. Suppose four of the players make acceptable proposals to each other (of forming a coalition of size 4) and the other five "wait" in the hope of getting into a coalition of size 2. The two groups of players may now have different expected payoffs, so we will denote them by w_l and w_h respectively.

$$\begin{aligned} w_l &= \frac{1}{9}\left[16 - 3\delta w_l\right] + \frac{8}{9}\delta w_l \\ w_h &= \frac{5}{9}\delta w_h + \frac{4}{9}\left(\frac{2}{5}6 + \frac{3}{5}3\right) \end{aligned}$$

This system yields $w_l = \frac{16}{9-5\delta}$ and $w_h = \frac{84}{5(9-5\delta)}$.

We only have to check that a proposer whose continuation value is w_l prefers to propose the coalition of 4 (and get approximately 4). Alternatively, he could propose a coalition of 3 and get approximately $9 - 8 = 1$, a coalition of size 2 and get a negative payoff (since a coalition of 5 will follow a coalition of 2), a coalition of 1 and get 0, a coalition of 5 or 6 and get a negative payoff (since this would be followed by a coalition of 3 and 2 respectively), a coalition of 7 and get 0.4, a coalition of 8 and get a negative payoff, or the grand coalition and get -3.

In fact, coalitions of size 4, 2 and 3 have to form in this order. A coalition of size 3 cannot be the first coalition to form, even with delay, because then players would prefer to form a coalition of 4 first.

One may also wonder whether there are asymmetric equilibria without delay. This does not seem to be the case. The reason is the following: we have managed to construct an equilibrium where some players form a coalition of 4 and some others wait. Thus, some players will be in a coalition of size 4 with probability 1 in equilibrium, and this is what makes their expected payoffs close to $\frac{16}{4} = 4$. If we want both a coalition of 4 players forming first *and* no delay, *all* players have to propose a coalition of 4, but this implies that some player will be in the coalition of 4 with probability less than 1 and thus have expected payoffs larger than 4 (since the expected payoff from being left out of the size 4 coalition, $\frac{2}{5}6 + \frac{3}{5}3 = \frac{21}{5}$, is larger than the expected payoff from being in it[11], $\frac{16}{4}$) and players would prefer to make unacceptable proposals rather than follow their prescribed strategies.

3.3 Formation of the grand coalition

In this section, we will provide *sufficient* conditions for the existence of a no-delay stationary perfect equilibrium in which the grand coalition forms with probability 1 (we will say that those equilibria exhibit *immediate formation of the grand coalition*). These conditions are different for the game with discounting and the game with breakdown probability.

We will assume in this section that the partition function is cohesive (for the game with discounting, we will also assume it to be positive). Immediate formation of the grand coalition implies efficiency for cohesive partition function. For strictly cohesive partition functions[12] it is also the only efficient possibility.[13] The question of existence of an equilibrium with immediate formation of the grand coalition is then most interest-

[11]We use the fact that all players must have the same expected payoffs in the reduced game.

[12]A partition function game is strictly cohesive if

$$\varphi(N) > \sum_{S\in\pi} \varphi(S,\pi) \text{ for all } (S,\pi) \in E(N),\ S \subset N.$$

[13]In the game with breakdown probability, the outcome may be *ex post* efficient even if the agreement is not immediate, but in expected terms there is an inefficiency due to the possibility that breakdown occurs.

ing for strictly cohesive games. However, we will provide examples showing that strict cohesiveness is neither necessary nor sufficient for immediate formation of the grand coalition.

3.3.1 The game with discounting

The following lemma will be useful

Lemma 3.3 *Let (N,φ) be a partition function game. Suppose there is a stationary perfect equilibrium of the game $G(N,\varphi,\theta,\delta)$ with immediate formation of the grand coalition. Then the expected payoff for player i equals*

$$w_i = \theta_i \varphi(N). \tag{3.3}$$

Proof. If all players make acceptable proposals to form the grand coalition, the expected payoff for player i can be found from the following equation

$$w_i = \theta_i \left[\varphi(N) - \delta \sum_{j \in N \setminus \{i\}} w_j \right] + (1-\theta_i)\delta w_i.$$

Re-arranging terms yields

$$w_i = \theta_i \left[\varphi(N) - \delta \sum_{j \in N} w_j \right] + \delta w_i.$$

Using $\sum_{j \in N} w_j = \varphi(N)$, we obtain $w_i = \theta_i \varphi(N)$. ∎

Theorem 3.3 *Let (N,φ) be a positive and cohesive partition function game. There exists a stationary perfect equilibrium of the game $G(N,\varphi,\theta,\delta)$ with immediate formation of the grand coalition if*

$$\sum_{j \in S} \theta_j \varphi(N) \geq \varphi(S,\pi) \text{ for all } (S,\pi) \in E(N). \tag{3.4}$$

Proof. Suppose all players make acceptable proposals to form the grand coalition. Expected payoffs are then given by (3.3). This strategy combination is an equilibrium if no proposer can do better by proposing some other coalition S. If the proposer proposes to form the grand coalition, his payoff is $\varphi(N) - \delta \sum_{j \in N \setminus \{i\}} \theta_j \varphi(N)$.

If he proposes some other coalition S instead, his payoff depends on the coalition structure that eventually forms. A sufficient condition for a deviation not to be profitable is that it is not profitable for *any* coalition structure, even in the most favorable case of this coalition structure forming immediately (delay would only reduce the payoff of the proposer), thus

$$\varphi(N) - \delta \sum_{j \in N \setminus \{i\}} \theta_j \varphi(N) \geq \varphi(S, \pi) - \delta \sum_{j \in S \setminus \{i\}} \theta_j \varphi(N) \text{ for all } (S, \pi) \in E(N).$$

Re-arranging terms, we obtain

$$\varphi(N) - \delta \sum_{j \in N \setminus S} \theta_j \varphi(N) \geq \varphi(S, \pi) \text{ for all } (S, \pi) \in E(N).$$

If we want this condition to hold for all δ, it is sufficient that it holds for $\delta = 1$. Substituting $\delta = 1$ and re-arranging, we obtain condition (3.4).

So far we have considered deviations to acceptable proposals only. The fact that the partition function is positive and cohesive ensures that players prefer to propose the grand coalition rather than make unacceptable proposals. ■

Corollary 3.6 *If all players are selected to be the proposers with the same probability, the sufficient condition (3.4) becomes*

$$\frac{\varphi(N)}{|N|} \geq \frac{\varphi(S, \pi)}{|S|} \textit{ for all } (S, \pi) \in E(N). \tag{3.5}$$

Condition (3.5) means that the grand coalition has the highest per capita payoff of all possible embedded coalitions. It also implies that the partition function game is cohesive (but not necessarily fully cohesive). The assumption of cohesiveness in theorem 3.3 is in this sense redundant, but we have preferred to keep it explicit.

Notice that the egalitarian division $\frac{\varphi(N)}{|N|}$ is predicted by the Nash bargaining solution for the unanimity game in which only the grand coalition can form. The interpretation is that the equilibrium of the game is robust to the introduction of "small" gains from partial cooperation. The possibility of forming subcoalitions does not affect the outcome if they are not profitable compared to what players get in the grand coalition. In order words, the formation of subcoalitions is not credible.

3.3.2 The game with probability of breakdown

Lemma 3.4 *Suppose there is a stationary perfect equilibrium of the game $G(N, \varphi, \theta, p)$ with immediate formation of the grand coalition. Then the expected payoff for player i equals*

$$w_i = \varphi(i, \langle N \rangle) + \theta_i \left[\varphi(N) - \sum_{j \in N} \varphi(j, \langle N \rangle) \right] \tag{3.6}$$

and his continuation value equals

$$z_i = \varphi(i, \langle N \rangle) + p\theta_i \left[\varphi(N) - \sum_{j \in N} \varphi(j, \langle N \rangle) \right]. \tag{3.7}$$

Proof. If all players propose the grand coalition and these proposals are accepted, the continuation value for player i can be found from the following equation:

$$z_i = p\theta_i \left[\varphi(N) - \sum_{j \in N \setminus \{i\}} z_j \right] + p(1 - \theta_i) z_i + (1 - p)\varphi(i, \langle N \rangle).$$

Re-arranging terms yields

$$z_i = p\theta_i \left[\varphi(N) - \sum_{j \in N} z_j \right] + p z_i + (1 - p)\varphi(i, \langle N \rangle).$$

Substituting for $\sum_{j \in N} z_j = p\varphi(N) + (1 - p) \sum_{j \in N} \varphi(j, \langle N \rangle)$ and solving for z_i, we obtain (3.7). Using $z_i = p w_i + (1 - p)\varphi(i, \langle N \rangle)$, we obtain (3.6). ■

Theorem 3.4 *Let (N, φ) be a cohesive partition function game. There exists a stationary perfect equilibrium of the game $G(N, \varphi, \theta, p)$ with immediate formation of the grand coalition if*

$$\sum_{j \in S} \theta_j \left[\varphi(N) - \sum_{j \in N} \varphi(j, \langle N \rangle) \right] \geq \varphi(S, \pi) - \sum_{j \in S} \varphi(j, \langle N \rangle) \text{ for all } (S, \pi) \in E(N). \tag{3.8}$$

Proof. Suppose all players make (acceptable) proposals to form the grand coalition. Then the payoff a player gets as a responder is given by (3.7) and the payoff he gets as a proposer is given by $\varphi(N) - p \sum_{j \in N \setminus \{i\}} \theta_j \left[\varphi(N) - \sum_{j \in N} \varphi(j, \langle N \rangle) \right] - \sum_{j \in N \setminus \{i\}} \varphi(j, \langle N \rangle))$.

If player i proposes another coalition S, his payoff will depend of the coalition structure that eventually forms. Again, it is sufficient that a deviation will not be profitable for any coalition structure, that is, for all (S,π) in $E(N)$

$$\varphi(N) - p\sum_{j\in N\setminus\{i\}}\theta_j\left[\varphi(N) - \sum_{j\in N}\varphi(j,\langle N\rangle)\right] - \sum_{j\in N\setminus\{i\}}\varphi(j,\langle N\rangle)) \geq$$
$$\varphi(S,\pi) - p\sum_{j\in S\setminus\{i\}}\theta_j\left[\varphi(N) - \sum_{j\in N}\varphi(j,\langle N\rangle)\right] - \sum_{j\in S\setminus\{i\}}\varphi(j,\langle N\rangle)).$$

Re-arranging terms and substracting $\varphi(i,\langle N\rangle)$ from both sides, we obtain

$$\varphi(N) - p\sum_{j\in N\setminus S}\theta_j\left[\varphi(N) - \sum_{j\in N}\varphi(j,\langle N\rangle)\right] - \sum_{j\in N\setminus}\varphi(j,\langle N\rangle)) \geq$$
$$\varphi(S,\pi) - \sum_{j\in S}\varphi(j,\langle N\rangle)) \text{ for all } (S,\pi)\in E(N).$$

Notice that it is sufficient that this inequality holds for $p=1$. Substituting $p=1$ and re-arranging we get expression (3.8).

Finally, for cohesive games, proposing the grand coalition is always weakly preferred to making unacceptable proposals. ■

Corollary 3.7 *If each player is selected to be the proposer with the same probability, the sufficient condition (3.8) becomes*

$$\frac{\varphi(N) - \sum_{i\in N}\varphi(i,\langle N\rangle)}{|N|} \geq \frac{\varphi(S,\pi) - \sum_{i\in S}\varphi(i,\langle N\rangle)}{|S|} \quad \textit{for all } (S,\pi)\in E(N). \tag{3.9}$$

Condition (3.9) means that the grand coalition has the maximum per capita gain with respect to the situation in which all players are singletons. It also implies that the partition function φ is cohesive (not necessarily fully cohesive).

Consider the partition function game φ_g defined in terms of gains with respect to the coalition structure $\langle N\rangle$, that is, $\varphi_g(S,\pi) = \varphi(S,\pi) - \sum_{i\in S}\varphi(i,\langle N\rangle)$. Then conditions (3.8) and (3.9) can be rephrased in terms of φ_g so that they look analogous to (3.4) and (3.5).

One could be tempted to substitute φ by φ_g in the extensive form, so that the game with breakdown probability $G(N,\varphi_g,\theta,p)$ would have breakdown payoffs of zero, and thus be equivalent to a game with discounting. The first obvious difficulty is that the breakdown outcome in the reduced games would not necessarily be 0. Furthermore, the equilibrium of the game with discounting is not invariant to this sort of transformation. We will discuss this issue in section 3.3.4 below. We will also argue that there are good reasons why the equilibrium outcome does *not* have to be invariant to this sort of transformation.

Similarly to the game with discounting, expected payoffs with an egalitarian protocol coincide with the Nash bargaining solution, this time taking the vector $(\varphi(i, \langle N \rangle)_{i \in N}$ as disagreement point. The possibility of partial cooperation does not affect the outcome provided that partial cooperation yields lower payoff for the players than forming the grand coalition.

3.3.3 Examples

The examples in this subsection show that neither of the two extensive form games considered so far promotes efficiency better than the other. This is true not only for a fixed protocol but also if the protocol can be chosen so as to maximize the efficiency of the outcome. The examples also show that strict cohesiveness is neither necessary nor sufficient for the grand coalition to form with probability 1.

Example 3.7 *Consider the following partition function with* $N = \{1, 2, 3\}$

$\overline{\varphi}(1, 2, 3) = (1, 2, 3)$
$\overline{\varphi}(12, 3) = (4, 0)$
$\overline{\varphi}(1, 23) = (0, 6)$
$\overline{\varphi}(13, 2) = (5, 0)$
$\overline{\varphi}(123) = 8$

We may think of the players as three firms differing in efficiency. A coalition can then be interpreted as a merger, a (binding) agreement to collude, a research joint venture, etc. The (internal) profit gain from any merger of two firms is 1, whereas the profit gain form forming the grand coalition is 2. The game is fully cohesive and the only efficient coalition structure is the grand coalition.

Suppose that each of the players is selected to be the proposer with the same probability, thus $\theta_i = \frac{1}{3}$ for $i = 1, 2, 3$. The sufficient condition (3.9) holds (thus always forming the grand coalition is an equilibrium in the game with probability of breakdown), whereas the sufficient condition (3.5) does not hold. We now check that forming the grand coalition with probability 1 is an equilibrium in the game with probability of breakdown, but not in the game with discounting.

Suppose all players propose the grand coalition. In the game with probability of breakdown, each player's expected payoff equal his payoff in the event of breakdown plus an equal share of the gain from forming the grand coalition (equation (3.6)). Thus, each player gains $\frac{2}{3}$ with respect to the situation in which all players are singletons: $w_1 = 1 + \frac{2}{3}$, $w_2 = 2 + \frac{2}{3}$, and $w_3 = 3 + \frac{2}{3}$.

In the limit when p tends to 1, each player receives w_i regardless of whether he was the proposer or the responder. Deviations to other strategies will not be profitable: any pair of players gains more ($\frac{4}{3}$) by forming the grand coalition than by forming a two-player coalition (1).

Things are different in the game with discounting. Equation (3.3) prescribes equal shares of the value of the grand coalition, thus expected payoffs (and actual payoffs in the limit when $\delta \to 1$) equal $\frac{8}{3}$. Unlike in the game with breakdown probability, all players proposing the grand coalition is not an equilibrium. Player 2 prefers to propose to player 3, offer him $\frac{8}{3}$ and keep for himself $6 - \frac{8}{3} > \frac{8}{3}$.

Since expected payoffs given that the grand coalition forms are sensitive to the probabilities of being a proposer, we may find a different probability vector that, by favoring players 2 and 3, will give them an incentive to stick to the grand coalition.

Consider, for example, $\theta = (\frac{1}{8}, \frac{3}{8}, \frac{4}{8})$. If all players propose the grand coalition, expected payoffs equal $w_1 = 1$, $w_2 = 3$ and $w_3 = 4$. Always forming the grand coalition is now an equilibrium.

Example 3.8 Consider the following partition function

$\overline{\varphi}(1, 2, 3) = (5, 0, 0)$
$\overline{\varphi}(12, 3) = (5, 0)$
$\overline{\varphi}(1, 23) = (0, 4)$
$\overline{\varphi}(13, 2) = (5, 0)$
$\overline{\varphi}(123) = 8$

Player 1 is a very productive player when all players are alone. However, the cooperation of player 1 with one of the other two players does not bring any additional value. Cooperation of the other two players against player 1 is very profitable (they earn four units more than they were earning separately) and cooperation of the three players is a bit less profitable (they earn three units).

Suppose all players are selected to be proposers with the same probability. None of the sufficient conditions is satisfied, so that it is a priori unclear whether formation of the grand coalition will be an equilibrium. We will check that this is the case in the game with discounting, but not in the game with possible breakdown.

Suppose the grand coalition always forms in the game with discounting. This implies that each player has an expected payoff (and actual payoff in the limit when δ tends to

1) of $\frac{8}{3}$. This is an equilibrium even though player 1 receives less than he got when all players were alone. The reason is that player 1 cannot secure 5 for himself: if he decides to stay alone players 2 and 3 will form a coalition and player 1 will get zero. On the other hand, no player will profit from proposing a two-player coalition since any two players get $\frac{16}{3} > 5$.

All players proposing the grand coalition cannot be an equilibrium in the game with breakdown probability *for any protocol.* The reason is that player 1 must receive at least 5. Player 2 and 3 together get then no more than 3, so that any of them has an incentive to propose coalition $\{2,3\}$ instead of the grand coalition.

It seems that the game with impatience promotes efficiency better if we are allowed to choose the protocol, since we can induce any division of $\varphi(N)$ by manipulating the vector θ, whereas in the game with breakdown probability we are constrained by the fact that each player must receive at least $\varphi(i, \langle N \rangle)$. This is however not the case since the payoff a coalition can expect when deviating from the grand coalition may be different for the two games, so that temptation to defect may be higher in the game with discounting. The following example illustrates this fact.

Example 3.9 $N = \{1,2,3,4,5\}$

$\overline{\varphi}(123,4,5) = (35,1,5)$

$\overline{\varphi}(123,45) = (30,8)$

$\overline{\varphi}(ijkl,m) = (40,0)$

$\varphi(N) = 50$

$\overline{\varphi}(\pi) = (0,...,0)$ for all other $\pi \in \Pi(N)$.

An efficient equilibrium is not possible in the game with discounting for any protocol. In order to achieve an efficient outcome, the protocol has to be symmetric, since each four-player coalition can get a payoff of 40, and this together with $\varphi(N) = 50$ implies that each player has to get exactly 10. The only candidate protocol for an efficient equilibrium is then the egalitarian protocol.

Consider the game with discounting. If coalition $\{1,2,3\}$ forms, players 4 and 5 will not form a two-player coalition for sure because this would imply a payoff of 4 for each player and then player 5 would prefer to form a singleton. Thus, players 1, 2 and 3 must receive more than 30 in an efficient equilibrium, but this is not feasible with an egalitarian protocol.

In the game with breakdown probability, the symmetric protocol achieves an efficient outcome. Coalition $\{1, 2, 3\}$ cannot receive more than 30, since the formation of this coalition will be followed by the formation of coalition $\{4, 5\}$.

Examples 3.7 and 3.8 show that strict cohesiveness (even strict full cohesiveness[14]) is not sufficient to ensure formation of the grand coalition. The following example shows that it is not necessary either.

Example 3.10 $N = \{1, 2, 3, 4, 5\}$, $\varphi(5) = 10$, $\overline{\varphi}(2, 3) = (12, 1)$, $\overline{\varphi}(2, 1, 1, 1) = (0, 1, 1, 1)$, $\varphi(S, \pi) = 0$ for all other π. This game is not cohesive.

With the egalitarian protocol, the grand coalition forms with probability 1 both in the game with discounting and in the game with breakdown probability. Each player then has an expected payoff of 2. The only coalition that could profit from deviating is a coalition of size 2, but this would only be the case if the coalition of size 2 is followed by a coalition of size 3. Since a coalition of size 2 would be followed by three singletons, no player has an incentive to deviate.

3.3.4 Comment on strategic equivalence and the outside option principle

Strategic equivalence refers to the idea that games with different payoffs may nevertheless lead to the same strategic considerations in the part of the players. For characteristic function games, it is argued that any characteristic function v is equivalent to the characteristic function v' defined in the following way:

$$v'(S) = \frac{v(S) - \sum_{i \in S} v(i)}{v(N) - \sum_{i \in N} v(i)}.$$

The function v' is then called the 0,1 normalization of v (notice that it does not exist if $v(N) = \sum_{i \in N} v(i)$!). This transformation amounts to observing that multiplying the value of all coalitions by a constant or adding a constant to the payoffs of all coalitions containing a certain player does not change the strategic incentives.

[14] We say that a partition function is strictly fully cohesive if, for all $(S, \pi) \in E(N)$ and for all $\pi_S \in \Pi(S)$, $\pi_S \neq \{S\}$, it holds that

$$\varphi(S, \pi) > \sum_{T \in \pi_S} \varphi(T, (\pi \backslash \{S\}) \cup \pi_S).$$

For partition function games, the $0, 1$ normalization would amount to

$$\varphi'(S, \pi) = \frac{\varphi(S, \pi) - \sum_{i \in S} \varphi(i, \langle N \rangle)}{\varphi(N) - \sum_{i \in N} \varphi(i, \langle N \rangle)}.$$

The argument that multiplying by a positive constant should not matter is very sound, since it amounts to a change of units. The argument that adding constants should not matter is more controversial. Luce and Raiffa (1957, p.186) make the following argument in favor of this normalization:

> Suppose that we have a game with characteristic function v and suppose that, in one way or another, each player i is paid (or is caused to pay, depending upon the sign) an amount a_i, prior to the play of the game. Certainly these payments do not alter the strategic considerations of the game, and so they should not have an effect upon the rational selection of strategies nor on the outcomes of the game. But, if this is done, then the total payment to a coalition S is not just the $v(S)$ of the game, for that ignores the payments a_i. The fixed payments to (or from) the coalition S are $\sum_{i \in S} a_i$, so that the total payment to S is $v(S) + \sum_{i \in S} a_i$.

It is not clear whether adding a constant to the payoff of the coalitions containing a certain player will make a difference. Consider two games, one with $v(1) = 2$, $v(2) = 0$, $v(1,2) = 12$, and other with $v'(1) = v'(2) = 0$, $v'(1,2) = 10$. The argument made by Luce and Raiffa rests on the idea that players are paid upfront, before playing the game. However, the usual interpretation of the characteristic function is that player i will get $v(i)$ *only after he forms a singleton.* It may well be the case that players 1 and 2 split the payoff equally in game v even though player 1 can get 2 by himself and player 2 cannot get anything. In other words, the threat of player 1 to form a singleton may not be credible.

Players 1 and 2 splitting the payoff equally in game v is an example of the so-called *outside option principle.* The outside option principle states that, starting from an equilibrium in which the grand coalition always forms, making some subcoalition more attractive does not affect the equilibrium if players would get a lower payoff in the subcoalition than they got in the original equilibrium (see Binmore et al. (1989)). In such cases, forming subcoalitions is not a credible threat. The outside option principle is then incompatible with strategic equivalence.

The equilibrium of the game with discounting described in section 3.3.1 is consistent with the outside option principle. As for the equilibrium of the game with breakdown probability, it is consistent with the outside option principle except for changes in the payoffs for the all-singletons coalition structure. Changing those payoffs influences the equilibrium regardless of whether they are more attractive than the equilibrium payoffs of the original game. The reason is that the payoffs of the players as singletons are also the breakdown outcome. Since breakdown is forced on the players, the issue of credibility does not arise. The fact that the payoffs for the all-singleton coalition structure act as the breakdown outcome allows the equilibrium of the game with breakdown probability to satisfy strategic equivalence.

An additional problem appears in noncooperative games with discounting: the sign of the payoffs plays a central role in these games (it determines whether the players prefer to get the payoffs now or later). Adding or substracting constants may reverse the signs of the payoffs.

3.4 Immediate formation of the grand coalition and the core

Chatterjee et al. (1993) show that the limit payoff vector of any equilibrium in which the grand coalition forms without delay and with probability 1 must belong to the core of the characteristic function game. This implies that no efficient equilibrium can arise (for high discount factors) if the underlying game is strictly superadditive and has an empty core. In this section, we find analogous results for partition function games.

Given a partition function, there are several characteristic functions that can be associated to it. Two well-known possibilities are the *optimistic* characteristic function and the *pessimistic* characteristic function.

Definition 3.13 Given a partition function (N, φ), we define the **optimistic characteristic function** v^+ in the following way

$$v^+(S) \equiv \max_{\substack{\pi \in \Pi(N) \\ \pi \ni S}} \varphi(S, \pi) \text{ for all } S \subseteq N.$$

Definition 3.14 Given a partition function (N, φ), we define the **pessimistic characteristic function** v^- in the following way

$$v^-(S) \equiv \min_{\substack{\pi \in \Pi(N) \\ \pi \ni S}} \varphi(S, \pi) \text{ for all } S \subseteq N.$$

By replacing φ by φ^g, one can define the optimistic and pessimistic games in terms of gains, denoted by v_g^+ and v_g^- respectively.

In the previous section we saw two sufficient conditions for the existence of an equilibrium with immediate formation of the grand coalition. These sufficient conditions can be rephrased in terms of the core of the optimistic game.

Remark 3.3 *The sufficient condition for immediate formation of the grand coalition in the game with discounting (3.4) is equivalent to $(\theta_i\varphi(N))_{i\in N}$ being in the core of the optimistic game.*

Remark 3.4 *The sufficient condition for immediate formation of the grand coalition in the game with breakdown probability (3.8) is equivalent to $(\theta_i\varphi(N))_{i\in N}$ being in the core of the optimistic game defined in terms of gains v_g^+.*

The following theorems establish *necessary* conditions for immediate formation of the grand coalition: this is only possible if the expected payoff vector found in lemma 3.3 (game with discounting) or 3.4 (game with probability of breakdown) is in the core of a characteristic function game that assigns to each coalition its expected payoff given the *equilibrium* strategies. We will denote this characteristic function simply by v^*. Notice however that v^* depends on the extensive form game and on the (equilibrium) strategy vector σ^*, so that a complete notation would be $v^*_{G(N,\varphi,\theta,\delta),\sigma^*}$ for the game with discounting and $v^*_{G(N,\varphi,\theta,p),\sigma^*}$ for the game with breakdown probability.

Theorem 3.5 *Let (N,φ) be a partition function game. Suppose there is a sequence $\delta^k \to 1$ and a corresponding sequence of stationary perfect equilibria $\sigma^*(\delta^k)$ of the game $G(N,\varphi,\theta,\delta^k)$ with immediate formation of the grand coalition. Then the expected payoff vector $w = (w_i(\sigma^*(\delta^k)))_{i\in N}$ does not depend on δ^k and is in the core of the characteristic function game (N,v^*), where*

$$v^*(S) = \lim_{\delta\to 1} \sum_{\substack{\pi\in\Pi(N)\\ S\in\pi}} \sum_{t=1}^{\infty} \delta^{t-1}\mu(\pi^t|(\sigma^*(\delta),S))\varphi(S,\pi).$$

Proof. Immediate formation of the grand coalition implies that expected payoffs are $w_i = \theta_i\varphi(N)$ for all $i \in N$. Since agreement is immediate, expected payoffs do not depend on δ.

Suppose now that player i is selected to be the proposer. If he sticks to his prescribed strategy and proposes to form the grand coalition, he offers δw_j to each player $j \in N\backslash\{i\}$

and keeps $\varphi(N) - \sum_{j \in N \setminus \{i\}} \delta w_j$ for himself. If instead he proposes coalition $S \subset N$, he offers δw_j to each player j in $S \setminus \{i\}$. The expected payoff for i will then depend on the payoff coalition S gets in the game. Recall that $\mu(\pi^t | (\sigma^*(\delta), S))$ denotes the probability that coalition structure π forms at time t given that players follow strategy $\sigma^*(\delta)$ and that S is the first coalition to form. Player i will only propose the grand coalition if

$$\sum_{\substack{\pi \in \Pi(N) \\ S \in \pi}} \sum_{t=1}^{\infty} \mu(\pi^t | (\sigma^*, S)) \left[\delta^{t-1} \varphi(S, \pi) - \sum_{j \in S \setminus \{i\}} \delta w_j \right] \leq \varphi(N) - \sum_{j \in N \setminus \{i\}} \delta w_j \text{ for all } S \ni i. \tag{3.10}$$

Since any player may be selected to be the proposer, condition (3.10) must be satisfied for each $i \in N$.

In the limit when $\delta \to 1$ the advantage of the proposer disappears and each player i gets w_i regardless of whether he is a proposer or a responder. Taking into account that $w_j = \theta_j \varphi(N)$, condition (3.10) becomes

$$v^*(S) \leq \sum_{i \in S} w_i \text{ for all } S \subset N.$$

that is, the vector w must be in the core of the game (N, v^*). ■

Theorem 3.5 illustrates the assumptions we make in this chapter about the reaction of the complement of S if S forms. If S forms, the complement of S does not necessarily react in such a way that the payoff of S is minimized. The coalition structure that forms given S is an equilibrium coalition structure (there may be several possible coalition structures if the equilibrium is in mixed strategies) and the payoff S receives is a subgame perfect equilibrium payoff. Thus, no incredible threats on the part of $N \setminus S$ are assumed.

A similar theorem holds for the game $G(N, \varphi, \theta, p)$. The difference is in the expected payoff vector w and in the equilibrium strategy vector σ^*.

Theorem 3.6 *Let (N, φ) be a partition function game. Suppose there is a sequence $p^k \to 1$ of continuation probabilities and a corresponding sequence of stationary perfect equilibria $\sigma^*(p^k)$ of the game $G(N, \varphi, \theta, p^k)$ with immediate formation of the grand coalition and expected payoff vector $w = (w_i(\sigma^*(p^k)))_{i \in N}$. Then w is independent of p^k and lies in the core of the characteristic function game (N, v^*), where*

$$v^*(S) = \lim_{p \to 1} \sum_{\substack{\pi \in \Pi(N) \\ S \in \pi}} \mu(\pi | (\sigma^*(p), S)) \varphi(S, \pi).$$

Proof. Immediate formation of the grand coalition implies the following continuation values

$$z_i = \theta_i p[\varphi(N) - \sum_{j \in N} \varphi(j, \langle N \rangle)] + \varphi(i, \langle N \rangle)$$

and expected payoffs

$$w_i = \theta_i[\varphi(N) - \sum_{j \in N} \varphi(j, \langle N \rangle)] + \varphi(i, \langle N \rangle).$$

Consider the situation of player i when he is selected to be the proposer. He will only propose the grand coalition if

$$\sum_{\substack{\pi \in \Pi(N) \\ S \in \pi}} \mu(\pi | (\sigma^*(p), S)) \varphi(S, \pi) - \sum_{j \in S \setminus \{i\}} z_j \leq \varphi(N) - \sum_{j \in N \setminus \{i\}} z_j \text{ for all } S \ni i.$$

Immediate formation of the grand coalition requires this condition to be satisfied for all players i in N.

In the limit when p tends to 1, the advantage of the proposer disappears and each of the players gets w_i. For the strategy combination σ^* to be an equilibrium, we then need

$$v^*(S) \leq \sum_{i \in S} w_i \text{ for all } S \subset N.$$

that is, the vector w must be in the core of the game (N, v^*). ■

Nonemptiness of the core of v^* is not sufficient for the game to have an equilibrium with immediate formation of the grand coalition. This is easy to see for the game with breakdown probability, since the payoff for a player i in an efficient equilibrium is constrained to be at least $\varphi(i, \langle N \rangle)$, a value that may not be relevant for the computation of v^*. The following example illustrates this point.

Example 3.11 We compute the function v^* for the partition function in example 3.8. We find that $v^*(123) = 8; v^*(12) = v^*(13) = 5; v^*(23) = 4$ and $v^*(i) = 0$ for all i.

The function is the same for both extensive form games, but this is not the case in general.[15] Notice that the formation of the grand coalition or a two-player coalition completely determines the coalition structure. For singleton coalitions, one has to consider

[15] See example 3.9, in which $v^*(123)$ is larger for the game with discounting given an egalitarian protocol.

the reaction of the players external to the coalition. If player 3 forms a singleton, players 1 and 2 will form a two-player coalition, thus $v^*(3) = 0$. If players 1 or 2 form a singleton, their payoff is 0 regardless of whether the other two players form a coalition, thus $v^*(1) = v^*(2) = 0$.

The core of this game is nonempty but there is no efficient equilibrium in the game $G(N, \varphi, \theta, p)$ for any θ. If there was an efficient equilibrium for some θ, w_1 would be at least 5, but no payoff vector that gives player 1 at least 5 lies in the core.

Nonemptiness of the core is sufficient for the existence of an efficient equilibrium in the game with discounting *provided that the game v^* does not change with the protocol.* However, it will generally be the case that v^* changes with the protocol, and then nonemptiness of the core for a given protocol does not imply the existence of an efficient equilibrium. The following example illustrates this point.

Example 3.12 $N = \{1, 2, 3, 4\}$

$\overline{\varphi}(14, 2, 3) = (1, 0, 0)$
$\overline{\varphi}(13, 2, 4) = (1, 7, 0)$
$\overline{\varphi}(2, 134) = (0, 1)$
$\overline{\varphi}(3, 124) = (0, 1)$
$\overline{\varphi}(4, 123) = (0, 8)$
$\overline{\varphi}(12, 34) = \overline{\varphi}(14, 23) = (1, 20)$
$\overline{\varphi}(13, 24) = (1, 7)$
$\overline{\varphi}(N) = 24$
$\overline{\varphi}(\pi) = (0, .., 0)$ for all other π.
Note that this game is fully cohesive.

We now calculate the function v^* for the game $G(N, \varphi, \theta, \delta)$ with $\theta_i = \frac{1}{4}$ for all i. Since the coalition structure is completely determined by the formation of a three-player coalition or the grand coalition, $v^*(N) = 24$, $v^*(123) = 8$, $v^*(124) = v^*(134) = 1$ and $v^*(234) = 0$ regardless of the extensive form game. Moreover, players 1, 3 and 4 get a payoff of 0 in all coalition structures where they are singletons, thus $v^*(1) = v^*(3) = v^*(4) = 0$ regardless of the extensive form game. If player 2 forms a singleton, the other three players find themselves in a situation where only coalitions including player 1 are profitable. It is easy to check that players 3 and 4 will propose to player 1 and that, given that the protocol is egalitarian, player 1 will propose to each of the other two players with equal probability, therefore coalitions $\{1, 3\}$ and $\{1, 4\}$ form with probability $\frac{1}{2}$ each, and $v^*(2) = 3.5$. If coalition $\{1, 3\}$ forms, it gets 1 regardless of the coalition structure. Any other two-player coalition will trigger the coalition of the complement, thus $v^*(12) = v^*(14) = 1$, $v^*(24) = 7$, and $v^*(23) = v^*(34) = 20$.

The core of this game is nonempty for the egalitarian protocol: for example, $(0, 4, 17, 3)$ is in the core.

A necessary condition for the game to have an efficient equilibrium for some protocol is that the game v^* associated to the protocol has a nonempty core. Notice that $v^*(23)$ and $v^*(34)$ do not depend on the protocol. Thus, given any protocol, a payoff vector has to satisfy the following necessary conditions in order to be in the core of the corresponding v^*:

$$w_2 + w_3 \geq 20 \text{ (which implies } w_4 \leq 4) \tag{3.11}$$

$$w_4 + w_3 \geq 20 \text{ (which implies } w_2 \leq 4). \tag{3.12}$$

Thus, $w_3 \geq 16$ in any efficient equilibrium. This in turn implies $\theta_3 \geq \frac{16}{24} = \frac{2}{3}$, but $\theta_3 \geq \frac{2}{3}$ implies $v^*(2) \geq \frac{14}{3} > 4$, which, together with (3.12) means that we cannot find any protocol θ such that the associated expected payoff vector w is in the core of the associated v^*.

A weaker necessary condition for an equilibrium with immediate formation of the grand coalition to exist is that the expected payoffs in lemmas 3.3 and 3.4 must be in the core of the pessimistic game. If the core of the pessimistic game is empty, then no equilibrium with immediate formation of the grand coalition can exist. This condition is weaker than the conditions referring to the core of v^*, but it is also easier to check.

In games without externalities (that is, in characteristic function games) the optimistic and the pessimistic game coincide with the original characteristic function game. Thus, it is both necessary and sufficient that the payoff vector we found in lemma 3.3 (for the game with discounting) and 3.4 (for the game with breakdown probability) is in the core of the characteristic function game (see also Okada (1996), theorem 3).

3.5 Random proposers versus rule of order

Ray and Vohra (1999) study a noncooperative game in which the first player to reject a proposal becomes the next proposer. This implies that the order in which players accept or reject proposals may affect the results, so that one has to specify a *rule of order* selecting not only proposers but also responders. Ray and Vohra assume that players discount payoffs, so that the comparison will be made between their game and the game $G(N, \varphi, \theta, \delta)$. We will assume $\theta_i = \frac{1}{n}$ unless otherwise specified.

The game with a rule of order does not guarantee immediate agreement for fully cohesive games. The conditions for the formation of the grand coalition are robust to changes in the rule of order, unlike in the game with random proposers, where we obtained different conditions for each protocol. If players are symmetric, the game with a rule of order ensures that payoff division inside a coalition is symmetric as well, unlike in the game with random proposers. If players are asymmetric, the game with a rule of order may give too much power to the responders, so that competition between players is not reflected in the outcome. As for the efficiency of the outcome, the two procedures cannot be ranked in general. The two procedures are equivalent for symmetric games with (symmetric) fixed payoff division.

3.5.1 Riskless versus risky bargaining

By just looking at the procedures, one can see that the game with a rule of order is "safer" for the responders: if a responder rejects a proposal, he is sure of making the next one. Instead, in the game with random proposers, it may be that somebody else is selected and that the responder who rejected is left out of a desirable coalition. These features are best illustrated by an example. We will take a game without transferable payoff to make things more transparent. This game was proposed by Roth (1980).

Example 3.13 $N = \{1, 2, 3\}$. If 1 and 2 form a coalition, each gets $\frac{1}{2}$. If 1 and 3, or 2 and 3 form a coalition, 3 gets $1 - p$ and the other player gets p $(0 < p < \frac{1}{2})$.

Roth (1980) argues that the outcome of the game must be $(\frac{1}{2}, \frac{1}{2}, 0)$.

> This is because, when $p < \frac{1}{2}$, the outcome $(\frac{1}{2}, \frac{1}{2}, 0)$ is *strictly* preferred by *both* players 1 and 2 to every feasible outcome, and because the rules of the game permit players 1 and 2 to achieve this outcome without the cooperation of player 3. So (...) there is really no conflict between players 1 and 2.

Aumann (1985) argues that $(\frac{1}{2}, \frac{1}{2}, 0)$ will not necessarily be the outcome, because players 1 and 2 may accept an offer of player 3 out of security considerations.

> Suppose the players and the rules have just been announced on television. The amount 1 to be shared may be fairly large, so the players are rather excited.

> Suddenly the phone rings in 1's home; 3 is on the line with an offer. At first 1 is tempted to dismiss it. But then he realizes that if he does so, and if 3 manages to get in touch with 2 before he (1) does, then he won't get anything at all out of the game, unless 2 also rejects 3's offer. "But wait a minute", 1 says now to himself; "2 will only reject 3's offer if he thinks that I will reject it. When he gets 3's phone call, he will go through the agonizing that I am going through now, and will realize that in this situation I would also agonize. (...) I'm beginning not to like this one bit".

Aumann also offers some noncooperative bargaining games in which coalition $\{1,2\}$ does not necessarily form. In one of them, a player i is picked at random and given the "initiative". That is, i chooses another player j, and makes him an offer. If j rejects the offer, i makes an offer to the remaining player k, but k does not know of the previous offer to j. If k also rejects i's offer, coalition $\{j,k\}$ forms. In this game, players 1 and 2 always forming a coalition is an equilibrium, but players accepting any offer they receive is also an equilibrium. Aumann conjectures that, for a large p, safety considerations are more important (so that players will rush to form coalitions), and for a small p, payoffs are more important (players 1 and 2 are willing to take some risk in the hope of getting a much higher payoff).

The games Aumann considers are finite games with imperfect information. The games with random proposers and breakdown probability we have described are infinite games with perfect information.

Suppose for simplicity that players do not discount future payoffs. If we take the game with a rule of order, coalition $\{1,2\}$ will always form. If the protocol is such that player 3 has the initiative and he proposes to player 2, player 2 can safely reject the proposal, knowing that he will be able to address player 1 before player 3 does. In the game with random proposers, rejecting the proposal is risky: it may be that nature selects 3 again and that 3 proposes to 1; if 1 accepts this proposal, 2 will get nothing.

Consider the game with random proposers with an egalitarian protocol. Consider the following strategies: players 1 and 2 propose to each other, player 3 proposes to each of players 1 and 2 with probability $\frac{1}{2}$ and all proposals are accepted. Suppose now that player 1 receives an offer of player 3. If he accepts, he receives p. If he rejects, he will get $\frac{2}{3}\frac{1}{2}+\frac{1}{6}p$. Thus, the strategy combination we have described is an equilibrium if $p \geq \frac{2}{5}$. Of course, players 1 and 2 proposing to each other and ignoring the proposals of player 3 is also an equilibrium. Thus, the arguments of Aumann, namely that players 1 and 2 may want to accept player 3's offer, are supported by the results of using the extensive form game with random proposers, while the arguments of Roth are supported by the results of using an extensive form game with a rule of order.

3.5.2 No-delay result

In the context of characteristic function games, Okada (1996) proves that superadditivity of the underlying characteristic function implies no delay in the game with random proposers, unlike in the game with a rule of order considered by Chatterjee et al. (1993). Analogously, one can check that full cohesiveness does *not* guarantee immediate agreement in the game considered by Ray and Vohra (1999). Indeed, any superadditive characteristic function is a fully cohesive partition function, so that the result follows immediately.

Consider the following example, quoted in Chatterjee et al. (1993) and Okada (1996) and originally due to Bennett and van Damme:

Example 3.14 $N = \{1, 2, 3, 4\}, v(i) = 0$ for all i in N, $v(1, j) = 50$, $j = 2, 3, 4$; $v(i, j) = 100$, $i, j = 2, 3, 4$, $i \neq j$; $v(S) = 100$, $|S| = 3$ and $v(N) = 150$.

This example can be rephrased in terms of a partition function, where $\overline{\varphi}(\langle N\rangle) = (0,0,0,0)$, $\overline{\varphi}(1i, jk) = (50, 100)$, $\varphi(1i, j, k) = (50, 0, 0)$, $\overline{\varphi}(1, i, jk) = (0, 0, 100)$, $\overline{\varphi}(ijk, l) = (100, 0)$ and $\overline{\varphi}(N) = 150$.

There are three strong players and a weak player in this example (in fact, the core consists of one single point: $(0, 50, 50, 50)$). If the rule of order is such that player 1 starts the game, he will make an unacceptable proposal. The reason is as follows: if a player other than player 1 is selected to be the proposer, he will propose to another "strong" player. It is easy to see that all strong players have the same continuation value, $\frac{100\delta}{1+\delta}$ (that is, almost 50 for large δ). If player 1 makes his best acceptable proposal (the grand coalition) he gets $150 - \frac{300\delta}{1+\delta}$, close to 0 for large δ. If instead he makes an unacceptable proposal to a strong player, this player will reject and form a coalition with another strong player. This leaves player 1 and the remaining strong player in a symmetric situation, so that player 1 can get a payoff of about 25 rather than a payoff close to 0.

In the game with random proposers, player 1 will make an acceptable proposal to the grand coalition. Expected payoffs (for the case $\theta_i = \frac{1}{4}$ for all i) are then given by the following system of equations, where w_s represent the expected payoff for a strong player[16]

$$\begin{aligned} w_1 &= \tfrac{1}{4}(150 - 3\delta w_s) + \tfrac{3}{4}25 \\ w_s &= \tfrac{1}{4}(100 - \delta w_s) + \tfrac{1}{2}\delta w_s + \tfrac{1}{4}25. \end{aligned}$$

[16]It is easy to see that all strong players must have the same expected payoff in equilibrium.

The solution to this system is (in the limit when $\delta \to 1$) $w_1 = 25$ and $w_s = \frac{125}{3}$. The reason why there is no delay is that the game with random proposer gives less power to the responders. When a strong player rejects an offer, he is not sure of being the next proposer; the continuation values of the strong players reflect this risk, so that it is profitable for the weak player to make an acceptable proposal rather than to wait.

3.5.3 Formation of the grand coalition

The sufficient conditions we found in section 3.3 were sensitive to the protocol. However, in the game with a rule of order, one obtains condition (3.5) regardless of the rule of order. Thus, the game with a rule of order yields more "robust" results, while the game with random proposers is more "flexible". In example 3.7, one could obtain an efficient outcome by changing the protocol, whereas the game with a rule of order does not have an efficient equilibrium for any rule of order.

3.5.4 Formation of the coalition with highest per capita payoff in symmetric games

Ray and Vohra show that, for symmetric games and provided that the equilibrium exhibits no delay, players form the coalition that maximizes the expected per capita payoff given the reaction of outsiders.

This is not always the case in the game $G(N, \varphi, \theta, \delta)$ (even with a symmetric protocol), as the following example shows

Example 3.15 $N = \{1,2,3,4,5\}$, $\overline{\varphi}(4,1) = (18,0)$, $\overline{\varphi}(3,2) = \overline{\varphi}(3,1,1) = (14,0)$, $\overline{\varphi}(\pi) = 0$ for all other π.

This is a game in characteristic function form, so we do not need to consider the reaction of outsiders.

The coalition of size 3 has the highest per capita payoff. Suppose however that players always propose a coalition of size 3 in the game with random proposers. Then the expected payoff (given a symmetric protocol) is given by[17]

$$w = \frac{1}{5}(14 - 2\delta w) + \frac{4}{5}\frac{1}{2}\delta w.$$

[17] A more direct way of computing expected payoffs is to notice that the total payoff is always 14 and the equilibrium is symmetric, thus $w = \frac{14}{5} = 2.8$.

This yields $w = 2.8$. The payoff of the proposer is then approximately 8.4. However, he would get an even higher payoff by proposing a coalition of four $(18 - 3 * 2.8 = 9.6)$.

In equilibrium, only coalitions of four players form. Expected payoffs are then $\frac{18}{5} = 3.6$, so that a deviation to proposing a coalition of three would not be profitable.

The reason why larger coalitions that have lower per capita payoffs instead of smaller coalitions that have higher per capita payoffs form is that a player who rejects a proposal will not be the next proposer for sure. Thus, he will be left out of the coalition that eventually forms with positive probability, and this negatively affects his continuation value. Since the responders are "underpaid" because of the risk they have of being left out if they reject the proposal, it may pay to form larger coalitions. Example 3.15 thus points to a trade-off between efficiency and distribution.

In the game with a rule of order, players in the same coalition always split the payoff equally (see Ray and Vohra (1999)). Thus, responders are never "underpaid".

Per capita payoffs also play some role in the game with random proposers. For example, suppose in a symmetric game that the equilibrium is such that exactly two coalitions will form without delay. Then, it is still true that the coalition with the higher per capita payoff must form first, or, if both coalitions have the same per capita payoff, the largest coalition must form first (otherwise players would prefer to wait instead of forming the coalition with the smallest per capita payoff).

3.5.5 Competition between responders

As we have seen in the previous subsection, responders are never "exploited" in symmetric games with a rule of order. Indeed, when players are asymmetric, responders may have "too much power", as in the following market game (cf example 2 in Chatterjee et al.)

Example 3.16 $N = \{1, 2, ..., n\}$. Player 1 is a seller who owns a unit of some good and players $2, ..., n$ are potential buyers whose reservation price is u. The characteristic function is such that $v(S) = 1$, $|S| \geq 2$, $1 \in S$, and $v(S) = 0$ otherwise.

We now calculate the continuation value of a buyer in the game with a rule of order. After rejecting a proposal, he will make a proposal to the seller and offer him his continuation value. Thus, $z_i = \delta(1 - z_1)$. This continuation value is the same for all buyers regardless of the strategy of the seller. Let us denote it by z_b. No matter how the seller randomizes between the potential buyers, his continuation value will be $z_1 = \delta(1 - z_b)$.

Therefore, $z_1 = z_b = \frac{\delta}{1+\delta}$, or the seller cannot benefit from the competition between the buyers.

Instead, in a game with random proposers, the equilibrium would be much more competitive: two buyers are enough to drive the price up to 1 (in the limit when δ tends to 1). Expected payoffs would be then given by the following system of equations

$$w_1 = \tfrac{1}{n}(1-\delta w_b) + \tfrac{n-1}{n}\delta w_1$$
$$w_b = \tfrac{1}{n}(1-\delta w_1) + \tfrac{1}{n}\tfrac{1}{n-1}\delta w_b$$

These equations take into account that in equilibrium both buyers must have the same continuation value and thus the seller must propose to each of the potential buyers with the same probability. The reason is that, on the one hand, the seller would like to propose to the buyer with the lowest continuation value, whereas, on the other hand, the expected payoff of a buyer is larger the larger is the probability that he receives a proposal from the seller.

Solving the system above, we find $w_1 = \frac{n-1-\delta}{n(n-1)-\delta(n^2-2n+2)}$ and $w_b = \frac{(n-1)(1-\delta)}{n(n-1)-\delta(n^2-2n+2)}$.

The limit of these expected payoffs is $w_1 = 1$, $w_b = 0$. Thus, even with only two players, the price converges to 1 in the limit when $\delta \to 1$.

3.5.6 Efficiency of the outcome

Theorem 3.1 and example 3.15 seem to point in the direction of higher efficiency for the game with random proposers. This is indeed the case for very specific games (like three-person quota games with the grand coalition and symmetric games without externalities where only one profitable coalition can form at a time), but not in general (see example 3.17 below).

Three-person quota games with the grand coalition

Consider the following cooperative game. $N = \{1,2,3\}, v(i) = 0\ \forall i \in N$, $v(1,2) = a+b$, $v(1,3) = a+c$, $v(2,3) = b+c$, $v(1,2,3) = a+b+c$, $a > b > c$.

The equilibrium of the game with a rule of order is as follows. Players 1 and 2 propose to each other and split nearly equally, and player 3 proposes to player 1. Continuation values are $z_1 = z_2 = \delta\frac{a+b}{1+\delta}$, $z_3 = \delta c + \delta\frac{a-\delta b}{1+\delta}$. Given these continuation values, nobody wishes to propose the grand coalition, since this would imply adding a responder whose continuation value is larger than his quota.[18]

As for the game with random proposers, notice the following. As in the game with a rule of order, each proposer wishes to include all responders whose continuation value is less than their quota. Notice also that the sum of continuation values is strictly smaller than the sum of the quotas (that is, $\delta \sum_{i \in N} w_i < v(N) = a + b + c$). Thus, at least one player has a continuation value of less than his quota. This player must be a responder with probability 1. On the other hand, it cannot be the case that *all* players have a continuation value smaller than their quotas, since this would imply the grand coalition forming with probability 1 and all players splitting equally, contradicting $\delta w_3 < c$. Thus, at least one player and at most two must have a continuation value smaller than their quotas. Suppose only one player (player 1) satisfies this property. It can be checked that this is not possible when the other two players have continuation values that are larger than their quotas. It is possible to find an equilibrium in which the other two players have a continuation value that exactly equals their quota. Consider the following strategies: player 1 always proposes the grand coalition. Players 2 and 3 randomize between proposing the grand coalition and proposing to player 1. This strategy combination constitutes an equilibrium. In the limit when δ tends to 1, the probability of the grand coalition being formed tends to 1 as well. Thus, efficiency is higher in the game with random proposers.

Symmetric games without externalities and with the one-stage property

Consider a symmetric, zero-normalized and essential characteristic function game with the one-stage property, that is, such that $v(S) > 0$ implies $v(T) = 0$ for all $T \subseteq N \backslash S$. Let k be the cardinality of the coalition with the highest per capita payoff (or, if there are several, of the largest coalition). This is the coalition that will form in the game with a rule of order. We now show that a coalition with a smaller total payoff cannot form with positive probability in the game with random proposers.

Suppose we have an equilibrium of the game with random proposers. Symmetry of the game and of the protocol imply that all players must have the same continuation value. Recall that, in the game with random proposers, the continuation value of a player is δ times his expected payoff. If we represent the value of a coalition of cardinality m by $v(m)$, and the probability of a coalition of size m to be formed given the (equilibrium) strategies of the players by λ_m, expected payoffs are $\frac{\sum_{m=1}^{n} \lambda_m v(m)}{n} \leq \frac{v(k)}{k}$. Since $0 < w \leq \frac{v(k)}{k}$, it

[18] Experimental evidence supporting this outcome (though using a very different bargaining procedure) has been recently found by Bolton et al. (1999).

follows that $\delta w < \frac{v(k)}{k}$.

A coalition of cardinality l with $v(l) < v(k)$ cannot form in equilibrium in the game with random proposers. The reasons are obvious for $l > k$. For $l < k$, the proposer would always want to enlarge the coalition to k players, since doing so will increase the value of the coalition by at least $(k-l)\frac{v(k)}{k}$, while he will only have to pay $\delta(k-l)w$.

Things are very different in games where more than one coalition with positive value can form. Consider the following game

Example 3.17 $N = \{1,2,3,4,5\}; v(1) = 0,\ v(2) = 5,\ v(3) = 14,\ v(4) = 18,\ v(5) = 19.$

In the game with a rule of order, a coalition of size 3 will form, followed by a coalition of size 2. Total payoffs are then 19. However, in the game with random proposers, a four-player coalition is formed with probability 1, so that total payoffs are only 18.

Thus, one can conclude that, except for very specific situations (like three-person quota games with the grand coalition or games in which only one coalition is formed), neither of the procedures (rule of order or random proposers) yields more efficient results in general.

3.5.7 Random proposers versus rule of order with fixed payoff division

A distinctive feature of the game with random proposers is that it puts responders in a weak position, and this induces them to accept lower payoffs than in the game with a rule of order. This suggests that in a game with fixed payoff division (such as the games considered by Bloch (1996)) it should not make a difference how proposers are selected.

Consider the following example:

Example 3.18 $N = \{1,2,3\},\ \overline{\varphi}(i,j,k) = (0,0,0),\ \overline{\varphi}(ij,k) = (12,0),\ \overline{\varphi}(N) = 15.$

Suppose payoff division is restricted to be egalitarian. Both in the game with a rule of order and in the game with random proposers, a two-player coalition forms in any stationary perfect equilibrium. The differences between the two games are limited to subgames that are off the equilibrium path. Should the grand coalition be proposed in the game with a rule of order, it would be rejected, since the rejector can earn $6 > 5$ by proposing a two-player coalition. In the game with random proposers, however, the grand coalition would be accepted, since, by rejecting the grand coalition, the responder is running the risk of being left out, and thus his expected payoff from rejecting the proposal is only (given a symmetric protocol) $\frac{2}{3}6 = 4 < 5$. Nevertheless, the grand coalition will not be proposed in equilibrium, so that selecting proposers at random makes no practical difference. As in the game with a rule of order, the proposer chooses the coalition with the highest per capita payoff.

Drawing the proposer at random can make a difference in symmetric games if the (fixed) payoff division is not egalitarian (see example 3.13).

3.6 Conclusion

We have studied two games of coalition formation with externalities and random proposers, differing in the source of friction. There are two potential sources of inefficiency in those games: delay of the agreement (only in the game with discounting) and formation of subcoalitions. The sufficient conditions we have found for efficiency are rather demanding. Analogously to the case of characteristic function games, the possibility of making binding agreements and the fact that there are gains from merging do not guarantee the formation of the grand coalition, even with perfect information.

We have also compared the model with random proposers to the model with a rule of order. These two models differ in the power of the responder. Giving less power to responders makes immediate agreement easier, though the final outcome is not necessarily more efficient. When players are symmetric, the game with a rule of order yields a symmetric payoff division whereas the game with random proposers typically yields an asymmetric distribution. In some cases, this asymmetric distribution is needed to achieve efficiency, so that there is a trade-off between efficiency and distribution. The same feature of the game with a rule of order that is attractive for symmetric games (the large power of the responders) may be undesirable when players are asymmetric, since competition between players is not reflected in the outcome.

We have assumed in the whole chapter that the contracts signed by the players do not affect the partition function in any way. That is, given that a partition π forms, the payoff of a coalition $S \in \pi$, $\varphi(S, \pi)$, does not depend on the payoff division rule players use. This may not be the case in some applications. For example, suppose that players in S are a team that engages in joint production by using some assets. The cost of production is incurred privately by each player, and the only verifiable variable is output (that is, binding contracts can be written over the division of the output only). If contracts are such that each player gets a share of the output, there is a moral hazard problem: players will produce less than they would if costs as well as revenues were shared by all players. If one of the players could buy the assets of the other ones, the problem of moral hazard would disappear. Such a possibility cannot be taken for granted: it may be the case that the asset players have is their labor! The formation of coalitions in the presence of moral hazard has been studied by Espinosa and Macho-Stadler (2000) in the context of a Cournot oligopoly using a game with a rule of order.

Another assumption we have kept through this chapter is that the frictions of the bargaining process are arbitrarily small (δ and p tend to 1). The games $G(N, \varphi, \theta, \delta)$ and $G(N, \varphi, \theta, p)$ may have very different equilibria depending on the discount factor and indeed the assumptions have played a role when we were solving the examples. However, the no-delay results and the sufficient conditions for the formation of the grand coalition in section 3.3 do not depend on this assumption, as it is clear from inspection of the proofs. The same can be said about the assumption that Nature uses the same probability vector (renormalized to account for the fact that some players have already formed coalitions) in all reduced games. Assuming that the probability distribution used by Nature is a function θ of the set of players T that did not form a coalition yet such that $\theta_i > 0$ for all $i \in T$ and $\sum_{i \in T} \theta_i = 1$ would make no difference for the no-delay results and the sufficient conditions for the formation of the grand coalition.

We have also assumed that coalitions cannot be enlarged. This is the approach usually taken in the literature. Exceptions are Seidmann and Winter (1998) for characteristic function games where bargaining follows a rule of order, and several papers that assume random proposers: Montero (1998) for the one-seller-two-buyers game (see also Chapter 5), Okada (1999) for characteristic function games with the one-stage property, and Gomes (1999) for the general three-player case. Since the outcome of the game is often inefficient, players may be willing to renegotiate and enlarge the coalitions. In such a renegotiation stage, it is reasonable to assume that coalitions can only merge but cannot be dissolved. The current agreement matters either by assuming that players negotiate over payoff increases only, or by assuming that the current agreement is the breakdown outcome. Allowing for the possibility of renegotiation as long as the outcome is not efficient will eventually bring an efficient outcome. On the other hand, the possibility of renegotiation may change the incentives of he players: forming intermediate coalitions may become more attractive, since it improves the members' bargaining position without giving up the efficiency of the final outcome (see Okada (1999)).

The present approach, in which coalitions cannot be enlarged, would suggest that enlarging coalitions brings substantial transaction costs to the players. Clearly, this is not generally true. Thus, the present model should be seen as one of many: it predicts coalition formation in rather competitive and risky situations, where coalitions cannot be enlarged because of substantial transaction costs. One may argue that this is the most interesting setting to study coalition formation: if the grand coalition is eventually going to form, intermediate coalitions have no influence on the final outcome. Of course, the questions of how to increase one's bargaining power through intermediate coalitions, and which payoffs players get through the use of intermediate coalitions to their advantage are still very interesting. The next two chapters can be seen as a contribution to answering this question.

Chapter 4

Constant-Sum Games

This chapter is devoted to the study of coalition formation in constant-sum partition function games. Constant-sum games are special because there are no aggregate gains from forming the grand coalition, therefore one would not expect the grand coalition to form. The question is then which subcoalitions will form and what will be the payoffs for the players.

Constant-sum games arise naturally in the context of efficient bargaining. Suppose there is a cake of fixed size to be divided between n players by bargaining. Prior to the bargaining process, players may find it profitable to form coalitions in order to improve their bargaining position. A coalition is then understood as a *bargaining unit*. Since the share of the cake a coalition gets may depend on how other players organize themselves, it will be given by a *partition function*.

In the process of bargaining between coalitions, the payoff coalitions can get by themselves (that is, without cooperating with other coalitions) may play an important role (in fact, this payoff may be the source of the improved bargaining position). If a coalition does not interact with other players it is reasonable to assume that its payoff does not depend on how the other players are organized, thus the payoff a coalition can get by itself will be given by a *characteristic function*.

For a coalition S, we will denote by $v(S)$ the payoff the coalition can get by itself; we will also call this payoff *autarchy payoff*. We will denote by $\varphi(S, \pi)$ the (expected) part of the cake coalition S gets if players partition themselves according to π and then bargain over the division of the cake. The partition function φ will be related to the characteristic function v. A partition function can be generated starting from a characteristic function either axiomatically (the Owen value is an example) or strategically (see the next chapter for an illustration).

The idea that subcoalitions can form and then bargain with other coalitions over the division of the value of the grand coalition can be traced back to Maschler (1963) and Hart and Kurz (1983).

Maschler (1963) argues that the characteristic function is not an appropriate measure of the actual payoff a coalition gets if it forms. The characteristic function represents a lower bound, but the actual payoff will generally be larger.

> Obviously, in addition to what a coalition can make by itself, it may gain some more "strength" by threatening not to cooperate with other players.

Hart and Kurz (1983) put it in the following way

> Our view is that the reason coalitions form is not in order to get their worth, but to be in a better position when bargaining with the others on how to divide the maximal amount available.

The formation of subcoalitions in order to improve bargaining position has also been observed in experiments (see, for example, Kalisch et al. (1954) and Maschler (1978)).

The first question we investigate in this chapter is whether it is true that the grand coalition can never arise in essential constant-sum partition function games. Perhaps surprisingly, this turns out to be the case if the partition function is positive and fully cohesive, but not in general.

Assuming that the partition function is fully cohesive, we turn to the question of what coalition structure will form in the simplest case of three-player games. The answer turns out to depend exclusively on the protocol, and not on the parameters of the partition function. Thus, whether a player brings a lot of value to a coalition or whether he is nearly a dummy, his presence in a coalition will only depend on the protocol! Expected payoffs depend both on the parameters of the partition function and on the protocol, but they are fairly robust to changes in the protocol. The reason for this is that the equilibrium strategies can adjust to (and often completely compensate) changes in the protocol. For a wide range of protocols, expected payoffs coincide with the kernel of the grand coalition. Extensions of the results above to four-player games are not possible without further assumptions.

The chapter also includes a brief discussion of two natural ways of generating the partition function axiomatically (namely by using the Owen value and by using the Nash bargaining solution), and a review of the related literature.

4.1 Conditions for the formation of the grand coalition

In this section, we define constant-sum games and we study the possibility of formation of the grand coalition in such games.

4.1.1 Preliminaries

The definition we introduce in this section is stronger than the usual definition for characteristic function games: we will require that total payoff remains constant for *any* partition of the set of players, whereas the definition for characteristic function requires it only for partitions into two sets (say, S and $N \backslash S$).

Definition 4.1 A partition function game (N, φ) is **constant-sum** if

$$\sum_{S \in \pi} \varphi(S, \pi) = K \text{ for all } \pi \in \Pi(N). \tag{4.1}$$

Constant-sum games arise naturally in the context of bargaining. Suppose there is a cake of size K to be divided between the players. Before the cake is divided, players may form coalitions (bargaining units) in order to improve their bargaining position. Assuming efficient bargaining, players will agree to a division of the cake for any organization of the players in bargaining units. The distribution of the cake may nevertheless be affected by the formation of bargaining units. The partition function φ assigns to each coalition S and partition of the players π to which S belongs the payoff of coalition S given that π forms. Thus, the partition function is a "reduced form" that captures the outcome of the bargaining between coalitions.

Intuitively, there is no reason for the formation of the grand coalition in constant-sum games, since there are no aggregate gains from cooperation. This intuition is confirmed for fully cohesive games. Indeed, given any of the two extensive form games we studied in the previous chapter, $G(N, \varphi, \theta, \delta)$ or $G(N, \varphi, \theta, p)$, the grand coalition can arise with probability 1 in equilibrium only if (N, φ) is inessential, as we will show below.

Definition 4.2 A partition function game (N, φ) is **inessential** if

$$\varphi(S, \pi) = \sum_{i \in S} \varphi(i, \langle N \rangle) \text{ for all } (S, \pi) \in E(N). \tag{4.2}$$

A game is inessential when there is no gain or loss from acting together: if some players form a coalition, the payoff they obtain equals the sum of payoffs they got when they were acting separately. Moreover, the formation of a coalition does not change the distribution of payoffs among the remaining players.

Definition 4.3 A partition function game (N, φ) is **essential** if

$$\exists\ (S, \pi) \in E(N) : \varphi(S, \pi) \neq \sum_{i \in S} \varphi(i, \langle N \rangle). \tag{4.3}$$

Remark 4.1 *All two-player constant-sum games are inessential.*

Notation 4.1 In what follows we will often refer to the payoff of a coalition S when its complement forms, $\varphi(S, \{S, N \backslash S\})$. In order to simplify notation, we will denote this payoff by $\varphi^c(S)$.

4.1.2 Results

Assumption 4.1 We will assume that the partition function (N, φ) is constant-sum, positive, and fully cohesive. Notice though that the positiveness assumption is only required for the results concerning the game with discounting.

Lemma 4.1 *If (N, φ) is fully cohesive and constant-sum, then*

$$\varphi(S, \pi) \geq \varphi^c(S) \text{ for all } (S, \pi) \in E(N). \tag{4.4}$$

Proof. Starting from the partition π, full cohesiveness implies that $N \backslash S$ weakly profits from the merger, that is, $\varphi^c(N \backslash S) \geq \sum_{T \in \pi \backslash \{S\}} \varphi(T, \pi)$. Since φ is constant-sum, this implies that S cannot be better-off: given that $\varphi(S, \pi) = \varphi(N) - \sum_{T \in \pi \backslash \{S\}} \varphi(T, \pi)$ and $\varphi^c(S) = \varphi(N) - \varphi^c(N \backslash S)$, (4.4) follows. ∎

Lemma 4.1 states that the worst that can happen to a coalition is that its complement forms. Of course, this is not necessarily the case for games that are not fully cohesive (see example 4.1).

Lemma 4.2 *Let (N, φ) be a positive and fully cohesive constant-sum game. Suppose there is an equilibrium of the game $G(N, \varphi, \theta, \delta)$ or $G(N, \varphi, \theta, p)$ in which the grand coalition forms with probability 1. Let w_i be the expected equilibrium payoff for player i. Then*

$$w_i = \varphi(i, \langle N \rangle) \text{ for all } i \in N. \tag{4.5}$$

Proof. Recall that, in any equilibrium such that the grand coalition forms with probability 1, the (limit) payoff of any player is w_i regardless of whether he is the proposer or the responder.

Consider the coalition structures of the form $(i, N\backslash\{i\})$. If the grand coalition forming with probability 1 is an equilibrium, it must be the case that $w_i = \varphi^c(i)$ for all $i \in N$. If $w_i > \varphi^c(i)$, any player in $N\backslash\{i\}$ would prefer to propose $N\backslash\{i\}$ instead of N. If $w_i < \varphi^c(i)$, player i would prefer forming a singleton rather than proposing the grand coalition (as the partition function is fully cohesive and constant-sum, lemma 4.1 implies that player i will get at least $\varphi^c(i)$[1]). Thus, $w_i = \varphi^c(i)$ for all i.

Lemma 4.1 implies $\varphi^c(i) \leq \varphi(i, \langle N \rangle)$ for all i. Since $w_i = \varphi^c(i)$, it follows that

$$w_i \leq \varphi(i, \langle N \rangle) \text{ for all } i. \tag{4.6}$$

Since φ is constant-sum, $\sum_{i\in N} \varphi(i, \langle N \rangle) = \varphi(N)$. Immediate agreement to form the grand coalition implies $\sum_{i\in N} w_i = \varphi(N)$. These two equalities can only be compatible with (4.6) if (4.5) holds. ∎

Theorem 4.1 *Let (N, φ) be a positive and fully cohesive constant-sum game. Suppose there is an equilibrium of the game $G(N, \varphi, \theta, \delta)$ or $G(N, \varphi, \theta, p)$ in which the grand coalition forms with probability 1. Then (N, φ) is an inessential game.*

Proof. Suppose there is a coalition structure $\overline{\pi}$ and a coalition $C \in \overline{\pi}$ such that $\varphi(C, \overline{\pi}) \neq \sum_{i\in C} w_i$. Because φ is constant-sum, this implies that there is a coalition $S \in \overline{\pi}$ such that $\varphi(S, \overline{\pi}) < \sum_{i\in S} w_i$, therefore $\sum_{T\in\overline{\pi}\backslash\{S\}} \varphi(T, \overline{\pi}) > \sum_{i\in N\backslash S} w_i$. If we merge all coalitions other than S, full cohesiveness implies $\varphi^c(N\backslash S) > \sum_{i\in N\backslash S} w_i$. Lemma 4.1 (together with theorem 3.1 for the game with discounting) implies that, if $N\backslash S$ forms, its payoff will be at least $\varphi^c(N\backslash S)$. A player in $N\backslash S$ who is selected to be the proposer would then profit from proposing coalition $N\backslash S$ instead of N, a contradiction. Thus, if the grand coalition forming with probability 1 is an equilibrium, it must be the case that $\varphi(S, \pi) = \sum_{i\in S} w_i$ for all $(S, \pi) \in E(N)$, that is, that is (using lemma 4.2) $\varphi(S, \pi) = \varphi(i, \langle N \rangle)$ for all $(S, \pi) \in E(N)$, thus φ is inessential. ∎

It is well known that the core of any essential constant-sum, characteristic function game is empty. This result generalizes to positive fully cohesive constant-sum partition function games in terms of the equilibrium characteristic function v^*.

[1]Recall that no delay can occur in the game with discounting for positive and fully cohesive games (theorem 3.1), so that the payoff of player i will not be reduced by delay. As for the game with breakdown probability, delay does not reduce payoffs and eventual formation of a coalition structure is guaranteed.

Proposition 4.1 *Let (N, φ) be an essential, positive and fully cohesive constant-sum partition function game. Let v^* be the equilibrium characteristic function corresponding to any of the strategic form games $G(N, \varphi, \theta, \delta)$ or $G(N, \varphi, \theta, p)$. Then the core of v^* is empty.*

Proof. Consider any payoff vector $(x_i)_{i \in N}$ such that $\sum_{i \in N} x_i = \varphi(N)$. Since the game is essential, there must be an embedded coalition (S, π) such that $\sum_{i \in S} x_i > \varphi(S, \pi)$. But then, since the game is positive, constant-sum and fully cohesive, $\sum_{i \in N \setminus S} x_i < \varphi^c(N \setminus S) \leq v^*(N \setminus S)$, and thus $Core(v^*) = \varnothing$. ∎

Unlike in characteristic function games, the grand coalition may form (and thus $Core(v^*) \neq \varnothing$) in constant-sum games that are not fully cohesive. A trivial example is a three-player game with $\varphi(i, j) < \varphi(i, \langle N \rangle) + \varphi(j, \langle N \rangle)$ for all i, j. In this trivial example, both the grand coalition and the all-singletons structure can arise in equilibrium. It is also possible to find constant-sum games in which players strictly prefer to form the grand coalition, as the following example of a symmetric game with five players shows

Example 4.1 $N = \{1, 2, 3, 4, 5\}$

$\varphi(1, \langle N \rangle) = \frac{K}{5}$

$\overline{\varphi}(2, 1, 1, 1) = (K, 0, 0, 0)$

$\overline{\varphi}(2, 2, 1) = (\frac{K}{3}, \frac{K}{3}, \frac{K}{3})$

$\overline{\varphi}(2, 3) = (\frac{K}{2}, \frac{K}{2})$

$\overline{\varphi}(3, 1, 1) = (K, 0, 0)$

$\overline{\varphi}(4, 1) = (\frac{K}{2}, \frac{K}{2})$

$\varphi(N) = K.$

This game is not fully cohesive.

Consider the game $G(N, \varphi, \theta, \delta)$ or the game $G(N, \varphi, \theta, p)$ with $\theta_i = \frac{1}{5}$ for all i. If the grand coalition forms with probability 1, expected payoffs are $\frac{K}{5}$ for each player. For this to be an equilibrium, the vector $(\frac{K}{5})_{i \in N}$ must be in the core of the game v^* (see theorems 3.5 and 3.6). It is easy to see that $v^*(1) = 0$ (a coalition of 3 players would follow), $v^*(2) = \frac{K}{3}$ (a singleton and then a coalition of two players would follow), $v^*(3) = \frac{K}{2}$, $v^*(4) = \frac{K}{2}$ and $v^*(N) = K$ for both $G(N, \varphi, \theta, \delta)$ and $G(N, \varphi, \theta, p)$, so that forming the grand coalition with probability 1 is an equilibrium of both games. Moreover, this equilibrium is strict.

For characteristic function games, it is usually claimed that the grand coalition is more likely to form if the game is superadditive. This claim cannot be made for partition function games. Indeed, in example 4.1 it is precisely the lack of superadditivity of the partition function game that allows the grand coalition to form.

4.2 The three-player case

We have established that the grand coalition cannot form with probability 1 in any essential fully cohesive constant-sum game. We now turn to the question of what subcoalitions will form and what will be the expected payoffs for the players for essential and fully cohesive constant-sum games with three players.

4.2.1 Preliminaries

Notation 4.2 For three-person games, the formation of a two-player coalition completely determines the coalition structure. We will therefore denote the value of a two-player coalition simply by $\varphi(i,j)$ rather than $\varphi(ij, \{ij, k\})$.

In the context of three-person constant-sum games, full cohesiveness is equivalent to the following condition:

Definition 4.4 A partition function satisfies **profitability of two-player coalitions** if

$$\varphi(i,j) \geq \varphi(i, \langle N \rangle) + \varphi(j, \langle N \rangle) \text{ for all } i,j \in N, i \neq j. \tag{4.7}$$

Thus, profitability of two-player coalitions means that, starting from a situation in which all players are alone, a merger of two of the players does not reduce their aggregate payoff.

Consider a three-player constant-sum partition function game satisfying profitability of two-player coalitions. The partition function game is essential if and only if at least one of the inequalities in (4.7) is strict.

Remark 4.2 *Profitability of two-player coalitions implies the following inequality for constant-sum, essential games*

$$\varphi(1,3) + \varphi(2,3) + \varphi(1,2) > 2K \tag{4.8}$$

Inequality (4.8) is obtained by adding up the three inequalities in (4.7) and using the fact that $\sum_{i \in N} \varphi(i, \langle N \rangle) = K$ and that at least one of the three inequalities is strict.

Assumption 4.2 The partition function game is constant-sum, fully cohesive and essential, that is, (4.1), (4.7) and (4.8) are satisfied.

In what follows, we will focus on the extensive form game with breakdown probability $G(N, \varphi, \theta, p)$. This game has the following convenient property:

Remark 4.3 *If the partition function is such that $\varphi(j,k) > \varphi(j, \langle N \rangle) + \varphi(k, \langle N \rangle)$, then the formation of coalition $\{i\}$ will be followed by the formation of coalition $\{j,k\}$ with probability* 1.

The game with discounting has this property only in the limit when δ tends to 1.

4.2.2 Eliminating some candidate equilibria

In order to answer the question of what coalitions will form and what will be the payoffs for the players, we will consider all possible strategy combinations in turn. Some of these strategy combinations will be completely discarded, and others will be shown to be equilibria under certain conditions. Considering all possible strategy combinations may seem difficult, but this is not the case when we are looking for a stationary perfect equilibrium without delay.

Remark 4.4 *In order to describe a stationary perfect equilibrium without delay, all we need to know about the strategy of a player is the probability distribution over the coalitions he proposes.*

In a stationary perfect equilibrium without delay, each responder j will accept any offer that gives him at least his continuation value z_j and will be offered exactly z_j. This fact together with the probability distribution used by the proposers determines $(z_j)_{j \in N}$, therefore we need to specify neither the payoffs offered to the responders nor the set of proposals players accept.

The following lemmas exclude certain strategy combinations from being a (stationary perfect) equilibrium.

Lemma 4.3 *In a stationary perfect equilibrium, $z_i \geq K - \varphi(j,k)$ for all i in N.*

Proof. A player can always reject all proposals that come to him and propose to stay alone if he is selected to be the proposer. The continuation value associated to this strategy would be at least $K - \varphi(j,k)$.[2] ■

Lemma 4.4 *No player proposes to stay alone in equilibrium.*

Proof. Suppose there is an equilibrium in which (without loss of generality) player 1 proposes to stay alone with positive probability. This implies $z_1 = K - \varphi(2,3)$ (if $z_1 > K - \varphi(2,3)$, player 1 would prefer to make an unacceptable proposal rather than to stay alone; $z_1 < K - \varphi(2,3)$ would contradict lemma 4.3). Since $\sum_{i \in N} z_i = K$, it follows that $z_2 + z_3 = \varphi(2,3)$.

It must be the case that player 1 cannot get a strictly higher payoff by proposing two-player coalitions, that is,

$$K - \varphi(2,3) \geq \varphi(1,2) - z_2 \tag{4.9}$$
$$K - \varphi(2,3) \geq \varphi(1,3) - z_3 \tag{4.10}$$

Adding up inequalities (4.9) and (4.10), we get

$$2K - 2\varphi(2,3) \geq \varphi(1,2) + \varphi(1,3) - z_2 - z_3.$$

If we take into account that $z_2 + z_3 = \varphi(2,3)$, it follows that $2K \geq \varphi(1,2) + \varphi(1,3) + \varphi(2,3)$, contradicting (4.8). ■

Lemma 4.5 *None of the players proposes to form the grand coalition in equilibrium.*

Proof. Suppose player 1 proposes the grand coalition in equilibrium. Since he might as well have proposed a two-player coalition, it must be the case that $K - z_3 - z_2 \geq \varphi(1,2) - z_2 \Rightarrow z_3 \leq K - \varphi(1,2)$; analogously $z_2 \leq K - \varphi(1,3)$. As a proposer, player 2 can get at least $\varphi(2,3) + \varphi(1,2) - K$ (by proposing to 3). In all other cases, player 2 can get no less than $K - \varphi(1,3)$. Condition (4.8) implies $\varphi(2,3) + \varphi(1,2) - K > K - \varphi(1,3)$, thus $z_2 > K - \varphi(1,3)$, a contradiction. ■

[2] If the first coalition to form is the singleton $\{i\}$, then either $\varphi(j,k) > \varphi(j,\langle N\rangle) + \varphi(k,\langle N\rangle)$, implying that j and k will form a coalition, or $\varphi(j,k) = \varphi(j,\langle N\rangle) + \varphi(k,\langle N\rangle)$, implying $\varphi(i,\langle N\rangle) = K - \varphi(j,k)$, so that the payoff for player i is $K - \varphi(j,k)$ regardless of whether players j and k form a coalition.

A coalition may form or breakdown may occur before i gets to be the proposer. The only coalition that can form is $\{j,k\}$, since player i would reject any proposal including him. Then the payoff for player i is $K - \varphi(j,k)$. If breakdown occurs, i receives $\varphi(i,\langle N\rangle)$, and this is larger or equal than $K - \varphi(j,k)$ because of (4.7).

Notice that the proof of lemma 4.5 excludes the grand coalition not only from being *proposed* in equilibrium, but also from being an *optimal* proposal in equilibrium.

Remark 4.5 *Since proposing the grand coalition cannot be optimal in equilibrium, corollary 3.5 excludes any equilibria with possible delay.*

Lemma 4.6 *There is no stationary perfect equilibrium* in pure strategies[3] *in which a coalition is proposed by all its members with probability 1.*

Proof. Since we have proven that no player proposes the grand coalition or to stay alone, only two-player coalitions can be proposed in equilibrium. Without loss of generality, suppose there is a stationary perfect equilibrium in which players 1 and 2 propose to each other. Since the equilibrium is in pure strategies, the third player has to propose to one of the other two, say, to player 1. If 3 proposes to 1, the continuation values are given by the following system of equations

$$\begin{aligned}
z_1 &= p\theta_1\left(\varphi(1,2) - z_2\right) + p(\theta_2+\theta_3)z_1 + (1-p)\varphi(1,\langle N\rangle)\\
z_2 &= p\theta_2\left(\varphi(1,2) - z_1\right) + p\theta_1 z_2 + p\theta_3(K - \varphi(1,3)) + (1-p)\varphi(2,\langle N\rangle)\\
z_3 &= p\theta_3\left(\varphi(1,3) - z_1\right) + p(\theta_1+\theta_2)(K - \varphi(1,2)) + (1-p)\varphi(3,\langle N\rangle).
\end{aligned}$$

The solution of this system (in the limit when $p \to 1$) is $z_1 = \varphi(1,2) + \varphi(1,3) - K$, $z_2 = K - \varphi(1,3)$, $z_3 = K - \varphi(1,2)$. For this strategy combination to be an equilibrium, player 3 should prefer to propose to player 1 rather than to player 2. This implies $\varphi(2,3) + \varphi(1,3) - K \leq K - \varphi(1,2)$, contradicting (4.8). ■

The only possible equilibrium in pure strategies that we have not considered yet is that players propose two-player coalitions, but no two players propose to each other. We will show that, for such a strategy combination to be an equilibrium, each player has to be indifferent between proposing to each of the other two players. Since the only essential difference between pure and mixed strategies is that mixed strategies require an indifference condition, we can treat those pure strategy combinations as "corner" mixed strategies.

Lemma 4.7 *If there is a stationary perfect equilibrium in pure strategies, it must be the case that each player is indifferent between proposing to any of the other two.*

[3]Notice that this lemma only excludes the possibility of *all three players* playing pure strategies. It leaves open the possibility that *some but not all* of the players play pure strategies.

Proof. The only case that remains to be checked is the case in which no two players propose to each other. Without loss of generality, consider the case in which player 1 proposes to player 2, player 2 proposes to player 3, and player 3 proposes to player 1. Then the following inequalities must hold

$$\varphi(1,2) - z_2 \geq \varphi(1,3) - z_3 \tag{4.11}$$

$$\varphi(2,3) - z_3 \geq \varphi(1,2) - z_1 \tag{4.12}$$

$$\varphi(1,3) - z_1 \geq \varphi(2,3) - z_2 \tag{4.13}$$

It follows from (4.11) that $z_2 \leq \varphi(1,2) - \varphi(1,3) + z_3$ and from (4.13) that $z_2 \geq \varphi(2,3) - \varphi(1,3) + z_1$. Thus, $\varphi(2,3) - z_3 \leq \varphi(1,2) - z_1$, which together with (4.12) implies $\varphi(2,3) - z_3 = \varphi(1,2) - z_1$. Thus, player 2 must be indifferent in equilibrium (the same reasoning applies to players 1 and 3). ■

Lemma 4.8 *Any strict equilibrium with player i using a mixed strategy and players j and k using a pure strategy must have players j and k proposing to each other.*

Proof. We can easily eliminate the other two possibilities, namely that players j and k both propose to i and that j proposes to i while k proposes to j.

If both j and k propose to i, expected payoffs are (in the limit when p tends to 1) $z_i = \varphi(i,j) + \varphi(i,k) - K$, $z_j = K - \varphi(i,k)$ and $z_k = K - \varphi(i,j)$. Condition (4.8) then implies that players j and k would prefer to propose to each other rather than to player i.

If j proposes to i whereas k proposes to j, the conditions for j and k to play a best response ($\varphi(i,j) - z_i \geq \varphi(j,k) - z_k$ and $\varphi(j,k) - z_j \geq \varphi(i,k) - z_i$) together with the indifference condition of player i ($\varphi(i,j) - z_j = \varphi(i,k) - z_k$) imply that j and k are indifferent as well, so that the equilibrium is not strict and we can consider it as a "corner" equilibrium in mixed strategies. ■

The next lemma shows that equilibria in which two players are playing mixed strategies can be reduced to corner cases of all three players playing mixed strategies.

Lemma 4.9 *If there is a stationary perfect equilibrium in which two players are randomizing between two-player coalitions, it must be the case that each player is indifferent between proposing to any of the other two.*

Proof. Without loss of generality, suppose players 1 and 2 are randomizing between two-player coalitions. This implies the following indifference conditions

$\varphi(1,2) - z_2 = \varphi(1,3) - z_3$

$\varphi(1,2) - z_1 = \varphi(2,3) - z_3$.

Solving for z_3 in the two previous equations and re-arranging, we obtain

$\varphi(1,3) - z_1 = \varphi(2,3) - z_2$.

Thus, player 3 must be indifferent between proposing to 1 and proposing to 2. ■

We are now ready to describe the equilibria of the game. We have shown that in equilibrium only two-player coalitions are proposed. Moreover, we have reduced the possible strategy combinations to two groups: the first possibility is that two players propose to each other and the third randomizes; the second possibility is that all players are indifferent between the two-player coalitions they can propose (that is, all players are playing -possibly degenerated- mixed strategies). The equilibrium will turn out to depend crucially on the protocol, thus we will distinguish two kinds of protocols: those in which a player is selected more than half of the time to be proposer (called *protocols with a dominant player*), and those in which none of the players is selected more than half of the time to be proposer (called *protocols without a dominant player*).

4.2.3 Protocols with a dominant player

The next theorem shows that two players proposing to each other and the third player randomizing is an equilibrium provided that the third player is selected to be the proposer more than half of the time. Thus, if the protocol makes one of the players very powerful, the other two will unite against him.

Theorem 4.2 *Let (N, φ) be a three-player constant-sum, fully cohesive and essential partition function game. Consider the coalition formation game $G(N, \varphi, \theta, p)$. Suppose there is a player i with $\theta_i > \frac{1}{2}$. Then there is a stationary perfect equilibrium where players j and k propose to each other and player i randomizes.*

Proof. Without loss of generality, suppose players 1 and 2 propose to each other, and 3 proposes to 1 with probability γ, and to 2 with probability $1-\gamma$. Then the continuation values are given by the following system of equations (we denote $\varphi(i, \langle N\rangle)$ by $\varphi_i^{\langle N\rangle}$ for reasons of presentation):

$$z_1 = p\theta_1(\varphi(1,2) - z_2) + p(\theta_2 + \theta_3\gamma)z_1 + p\theta_3(1-\gamma)[K - \varphi(2,3)] + (1-p)\varphi_1^{\langle N \rangle}$$
$$z_2 = p\theta_2(\varphi(1,2) - z_1) + p(\theta_1 + \theta_3(1-\gamma))z_2 + p\theta_3\gamma[K - \varphi(1,3)] + (1-p)\varphi_2^{\langle N \rangle}$$
$$z_3 = p\theta_3[\gamma(\varphi(1,3) - z_1) + (1-\gamma)(\varphi(2,3) - z_2)] + p(\theta_1 + \theta_2)[K - \varphi(1,2)] + (1-p)\varphi_3^{\langle N \rangle}.$$

together with the indifference condition for player 3

$$\varphi(1,3) - z_1 = \varphi(2,3) - z_2.$$

The solution to this system of equations is (in the limit when $p \to 1$)

$$z_1 = \frac{\theta_3 K + (1-\theta_3)[\varphi(1,2) + \varphi(1,3)] - \varphi(2,3)}{2-\theta_3} \tag{4.14}$$
$$z_2 = \frac{\theta_3 K + (1-\theta_3)[\varphi(1,2) + \varphi(2,3)] - \varphi(1,3)}{2-\theta_3} \tag{4.15}$$
$$z_3 = K - \varphi(1,2) + \frac{\theta_3[\varphi(1,2) + \varphi(1,3) + \varphi(2,3) - 2K]}{2-\theta_3} \tag{4.16}$$
$$\gamma = \frac{\theta_2}{\theta_1 + \theta_2}. \tag{4.17}$$

For this strategy combination to be an equilibrium, players 1 and 2 must indeed prefer to propose to each other rather than to player 3, that is

$$\varphi(1,2) - z_2 \geq \varphi(1,3) - z_3$$
$$\varphi(1,2) - z_1 \geq \varphi(2,3) - z_3.$$

Substituting for the z_i's, one can check that these two inequalities are satisfied provided that $\theta_3 > \frac{1}{2}$: indeed, the difference between the left-hand side and the right-hand side is $\frac{(2\theta_3-1)(\varphi(2,3)+\varphi(1,3)+\varphi(1,2)-2K)}{2-\theta_3}$ in both cases. Because of assumption (4.8), the two inequalities are satisfied provided that $\theta_3 > \frac{1}{2}$.

One can check as well that no player would deviate to staying alone, proposing the grand coalition or making unacceptable proposals. ■

We conclude this subsection with some remarks about the expected payoffs for the players and the probability of each coalition to form. All results correspond to $p \to 1$.

Remark 4.6 *If $\theta_i > \frac{1}{2}$ for some i, the expected payoff of player i is increasing in θ_i, whereas the expected payoffs of players j and k are decreasing in θ_i.*

This is easy to see by inspecting expressions (4.14), (4.15) and (4.16)[4] and using (4.8). In particular, notice that (4.8) implies $\varphi(i,j)+\varphi(j,k)>K$ for any (distinct) i, j and k.

Remark 4.7 *If i is the player with $\theta_i > \frac{1}{2}$, expected payoffs are a function of θ_i only.*

The reason is that, if θ_j increases at the expense of θ_k, player i will propose to player k more often, so that expected payoffs remain unchanged.

The probabilities of the two-player coalitions depend only on the protocol and on the strategies of the players, not on the parameters of the partition function.

Remark 4.8 *If $\theta_i > \frac{1}{2}$, the probabilities of coalitions $\{i,j\}$, $\{i,k\}$ and $\{j,k\}$ are $\frac{\theta_i\theta_k}{\theta_j+\theta_k}$, $\frac{\theta_i\theta_j}{\theta_j+\theta_k}$ and $\theta_j+\theta_k$, respectively.*

These probabilities reflect the fact that the two weak players always join forces against the strong player, whereas the strong player proposes more often to the weakest player.

4.2.4 Protocols without a dominant player

We now show that an equilibrium in mixed strategies is possible if no player is selected to be proposer with probability more than half. The equilibrium will be unique in terms of payoffs, but not in terms of strategies.

Theorem 4.3 *Let (N,φ) be a three-player constant-sum, fully cohesive and essential partition function game. Consider the coalition formation game $G(N,\varphi,\theta,p)$. If $\theta_i < \frac{1}{2}$ $\forall i \in N$, there is a family of stationary perfect equilibria where each player is indifferent between proposing to each of the other two. The expected payoff for player i is constant across this family of equilibria and equals (in the limit when $p \longrightarrow 1$)*

$$\frac{K-2\varphi(k,j)+\varphi(i,j)+\varphi(i,k)}{3}. \tag{4.18}$$

Proof. We will construct the family of mixed-strategy equilibria described in the theorem and check there is no profitable deviation.

[4] These are the limit expressions for both expected payoffs and continuation values.

Suppose all players follow a mixed strategy, and they propose only two-player coalitions. We will use the following notation for the probabilities

	1	2	3
1	$-$	α	$1-\alpha$
2	β	$-$	$1-\beta$
3	γ	$1-\gamma$	$-$

The row indices indicate the proposers and the column indices indicate the responders. Thus, player 1 proposes to 2 with probability α and to 3 with probability $1-\alpha$, player 2 proposes to 1 with probability β, and so on. Using this notation, continuation values are given by (again we denote $\varphi(i, \langle N\rangle)$ by $\varphi_i^{\langle N\rangle}$ for the sake of presentation)

$$
\begin{aligned}
z_1 &= p\,[\theta_1[\varphi(1,2) - z_2] + [\theta_2\beta + \theta_3\gamma]z_1 + [\theta_2(1\text{-}\beta) + \theta_3(1\text{-}\gamma)][K - \varphi(2,3)]] + (1\text{-}p)\varphi_1^{\langle N\rangle} \\
z_2 &= p\,[\theta_2\,[\varphi(1,2) - z_1] + [\theta_1\alpha + \theta_3(1\text{-}\gamma)]z_2 + [\theta_1(1\text{-}\alpha) + \theta_3\gamma][K - \varphi(1,3)]] + (1\text{-}p)\varphi_2^{\langle N\rangle} \\
z_3 &= p\,[\theta_3[\varphi(1,3) - z_1] + [\theta_1(1\text{-}\alpha) + \theta_2(1\text{-}\beta)]z_3 + [\theta_1\alpha + \theta_2\beta][K - \varphi(1,2)]] + (1\text{-}p)\varphi_3^{\langle N\rangle}.
\end{aligned}
$$

Consider for example player 1. If a proposal is rejected, Nature will choose a new proposer with probability p (and then player i will be selected to be the proposer with probability θ_i) and a breakdown of negotiations will occur with probability $1-p$. The continuation value of player 1 incorporates the following four possible payoffs:

1. The payoff he gets as a proposer. Since player 1 proposes to player 2 with probability α and to player 3 with probability $1-\alpha$, we can write the expected payoff for player 1 when he becomes a proposer as $\alpha(\varphi(1,2)-z_2)+(1-\alpha)(\varphi(1,3)-z_3)$. Since player 1 is playing a mixed strategy, he must be indifferent between proposing to player 2 or to player 3, thus $\varphi(1,2) - z_2 = \varphi(1,3) - z_3$, and we can write the payoff as $\varphi(1,2) - z_2$ or $\varphi(1,3) - z_3$ indistinctly.

2. The payoff he gets as a responder. Given the protocol and the strategies of the other players, player 1 will get a proposal from player 2 with probability $\theta_2\beta$ and from player 3 with probability $\theta_3\gamma$; in both cases player 1's payoff will be z_1.

3. The payoff he gets when he is left out of the coalition that forms. With probability $\theta_2(1-\beta)$ player 2 is selected to be the proposer and proposes to 3. With probability $\theta_3(1-\gamma)$ player 3 is selected to be the proposer and proposes to 2. In both cases player 1 receives $K - \varphi(2,3)$.

4. The payoff he gets when breakdown occurs. This happens with probability $1-p$. All players remain singletons and player 1 gets $\varphi(1, \langle N \rangle)$.

As argued above, all three players must get the same payoff from all coalitions they propose in equilibrium. Thus it must be the case that

$$\begin{aligned}
\varphi(1,2) - z_2 &= \varphi(1,3) - z_3 \text{ (indifference condition for player 1)} \\
\varphi(1,2) - z_1 &= \varphi(2,3) - z_3 \text{ (indifference condition for player 2)} \\
\varphi(1,3) - z_1 &= \varphi(2,3) - z_2 \text{ (indifference condition for player 3).}
\end{aligned}$$

These three equations are not linearly independent since any two of them imply the third. If we take two of these equations together with the three equations above describing expected payoffs, we have a system of five equations and six unknowns. We will assume without loss of generality that $\theta_1 \leq \theta_2 \leq \theta_3$.

Taking γ as a parameter, the solution of the system is (for α and β we have taken the limit when $p \to 1$)

$$z_1 = \frac{K + \varphi(1,2) + \varphi(1,3) - 2\varphi(2,3)}{3} \tag{4.19}$$

$$z_2 = \frac{K + \varphi(1,2) + \varphi(2,3) - 2\varphi(1,3)}{3} \tag{4.20}$$

$$z_3 = \frac{K + \varphi(1,3) + \varphi(2,3) - 2\varphi(1,2)}{3} \tag{4.21}$$

$$\alpha = \frac{\gamma\theta_3 + \theta_1 - \theta_2}{\theta_1} \tag{4.22}$$

$$\beta = \frac{(1-\gamma)\theta_3 + \theta_2 - \theta_1}{\theta_2}. \tag{4.23}$$

Notice that the continuation values in (4.19), (4.20) and (4.21) follow directly from the indifference conditions together with assumption (4.1).

For α and β to be between 0 and 1, we need

$$\gamma\theta_3 < \theta_2 < \gamma\theta_3 + \theta_1 \tag{4.24}$$

$$(1-\gamma)\theta_3 < \theta_1 < (1-\gamma)\theta_3 + \theta_2. \tag{4.25}$$

These two inequalities imply that $\theta_3 < \theta_1 + \theta_2$. Since $\theta_3 \geq \theta_2 \geq \theta_1$, an equilibrium in mixed strategies can only exist if $\theta_i < \frac{1}{2}$ for all i.

We now check that we can find values for γ that make α and β between 0 and 1 for any θ with $\theta_3 < \theta_1 + \theta_2$.

We can rewrite inequalities (4.24) and (4.25) as

$$\frac{\theta_2 - \theta_1}{\theta_3} < \gamma < \frac{\theta_2}{\theta_3}$$
$$\frac{\theta_3 - \theta_1}{\theta_3} < \gamma < \frac{\theta_3 - \theta_1 + \theta_2}{\theta_3}.$$

Since $\theta_3 \geq \theta_2 \geq \theta_1$, it is enough that $\frac{\theta_3-\theta_1}{\theta_3} < \gamma < \frac{\theta_2}{\theta_3}$. Since $0 \leq \frac{\theta_3-\theta_1}{\theta_3}, \frac{\theta_2}{\theta_3} \leq 1$ and $\theta_3 < \theta_1 + \theta_2$, we can always find γ between 0 and 1 satisfying (4.24) and (4.25). Furthermore, we can find a continuum of possible values for γ.

It is easy to check that any of these strategy combinations (parametrized by γ) is an equilibrium. Responders do not have an incentive to reject a proposal that gives them their continuation value. As for proposers, since they are by construction indifferent between the two-player coalitions they can propose, a profitable deviation can only be towards proposing the grand coalition, staying alone or making unacceptable proposals. Condition (4.8) ensures that none of these alternatives is profitable. We will consider the case of player 1, the other two cases being analogous.

If player 1 proposes the grand coalition, he has to offer both other players their continuation values. Player 1 then gets $K - z_2 - z_3$. If instead he follows his prescribed strategy, he gets $\varphi(1,2) - z_2$. Player 1 will prefer to propose the grand coalition if $K - z_3 > \varphi(1,2)$. Substituting for z_3 from equation (4.21) and re-arranging terms, this would imply $2K - \varphi(1,3) - \varphi(2,3) - \varphi(1,2) > 0$, contradicting (4.8). Lemma 3.2 implies that player 1 cannot profit from deviating to unacceptable proposals either.

If player 1 decides to stay alone, he gets $K - \varphi(2,3)$. For this deviation to be profitable, it must be the case that $K - \varphi(2,3) > \varphi(1,2) - z_2$, implying $2K - \varphi(1,3) - \varphi(2,3) - \varphi(1,2) > 0$, and again contradicting (4.8).

So far we have calculated the continuation values, but not the expected payoffs. The expected payoff for player i equals[5]

$$\frac{K + \varphi(i,j) + \varphi(i,k) - 2\varphi(j,k) - (1-p)3\varphi(i,\langle N\rangle)}{3p} \tag{4.26}$$

In the limit when p tends to 1, expected payoffs and continuation values coincide. ■

[5] To see this, notice that $z_i = pw_i + (1-p)\varphi(i,\langle N\rangle)$. Substituting for the equilibrium value of z_i yields (4.26).

Corollary 4.1 *Let (N, φ) be a three-player constant-sum, fully cohesive and essential partition function game. Consider the coalition formation game $G(N, \varphi, \theta^E, p)$. There is a family of stationary perfect equilibria where each player is indifferent between proposing to each of the other two. In the limit when p tends to 1, each two-player coalition forms with probability $\frac{1}{3}$.*

Proof. If $\theta_i = \frac{1}{3}$ for all i, the expressions for α and β that solve the system of equations in the proof of theorem 4.3 are

$$\alpha = \gamma + (1-p)\frac{3[K - 3\varphi(2, \langle N \rangle) + \varphi(1,2) + \varphi(2,3) - 2\varphi(1,3)]}{p[\varphi(1,2) + \varphi(1,3) + \varphi(2,3) - 2K]} \tag{4.27}$$

$$\beta = 1 - \gamma + (1-p)\frac{3[K - 3\varphi(1, \langle N \rangle) + \varphi(1,2) + \varphi(1,3) - 2\varphi(2,3)]}{p[\varphi(1,2) + \varphi(1,3) + \varphi(2,3) - 2K]} \tag{4.28}$$

In the limit when p tends to 1 we have $\alpha = \gamma$ and $\beta = 1 - \gamma$. This implies that each two-player coalition forms with probability $\frac{1}{3}$.[6]

As we saw in the proof of theorem 4.3, there is a continuum of values of γ that induce an equilibrium. If $\theta_i = \frac{1}{3}$ for all i, any $0 < \gamma < 1$ satisfies the inequalities. For $\gamma = 0$ or $\gamma = 1$, it may be the case that the corresponding α or β are not between 0 and 1 for $p < 1$, depending on the parameters of the partition function. If the partition function is symmetric, $\gamma = 0$ and $\gamma = 1$ induce an equilibrium. ■

We now turn to comment on the expected payoffs of the players and the probabilities of the two-player coalitions.

Remark 4.9 *The equilibrium payoffs are robust to changes in the protocol.*

The reason is that equilibrium strategies adjust so as to compensate changes in the protocol: if the protocol selects one of the players more often to be the proposer, equilibrium strategies adjust so that he becomes a responder less often, and equilibrium payoffs remain unchanged. If the player is selected with probability more than one half, he is a responder with probability zero, so that the strategies cannot adjust any further and his payoff increases the more often he is selected to be the proposer. Notice that, even in this case, payoffs only depend on the largest value of θ, so strategies still adjust to changes in the other two probabilities so as to keep payoffs constant.

[6] To check this, take for example coalition $\{1, 3\}$. This coalition is formed if player 1 is selected to be the proposer and proposes to player 3 (this happens with probability $\frac{1}{3}(1-\alpha)$) or player 3 is selected and proposes to 1 (with probability $\frac{1}{3}\gamma$). Since in the limit $\alpha = \gamma$, the sum of these two probabilities is $\frac{1}{3}$.

Remark 4.10 *The probability that coalition $\{i, j\}$ will form in equilibrium equals θ_k in the limit when p tends to* 1.

Thus, the more often a player is selected to be the proposer, the more often the other two players will unite against him.

Remark 4.11 *The probability that a coalition forms depends exclusively on the protocol (in the limit when p tends to* 1*), whereas the expected payoffs depend exclusively on the parameters of the partition function.*

Thus, for protocols without a dominant player, we can completely separate the issues of what coalition will form and what will be the expected payoffs for the players!

Remark 4.12 *A player has a higher expected payoff the larger the value of the two-player coalitions to which he belongs and the smaller the value of the two-player coalition excluding him.*

This is a very reasonable result. It may be surprising that it is the absolute values that matter (and not the marginal contributions) but recall that the all-singleton coalition structure is not a real alternative, but the alternative to forming a two-player coalition is forming another two-player coalition.

Remark 4.13 *The result in corollary 4.1 vindicates the assumption of random matching made in the literature (see Gul (1989) or Osborne and Rubinstein (1990)). Letting players choose their partners may not make any difference!*

Gul (1989) and Osborne and Rubinstein (1990) consider coalition formation games with random matching. Players cannot freely choose their partners: instead, each pair can meet with probability $\frac{1}{3}$ and decide whether to form a coalition. In this chapter, players can freely choose to address their proposals to any of the other two players; however, the result is that the egalitarian protocol yields each coalition forms with probability $\frac{1}{3}$, the same as in a game with random matching (if the equilibrium of the latter is such that all players who meet form a coalition).

The results for the three-player constant-sum game differ from the results for the standard two-person bargaining game with probability of breakdown. First, the breakdown outcome does not make any difference in the limit when p tends to 1. Second, for $p < 1$, one gets a "strength is weakness" result: the higher the payoff in case of breakdown, the lower the expected payoff. Of course, one could change the extensive form game by allowing breakdown also before the first proposal is made, so that expected payoffs and continuation values coincide; this would eliminate the "strength is weakness" effect.

4.2.5 Properties of the equilibrium payoffs for protocols without a dominant player

In theorems 3.5 and 3.6 we defined the characteristic function v^*. This characteristic function assigns to a coalition S its equilibrium payoff given that S is the first coalition to form. The following proposition relates the equilibrium payoffs of the three-player game and the kernel[7] of the game v^*.

Proposition 4.2 *Under the assumptions of theorem 4.3, the continuation values (and the limit expected payoffs) coincide with the kernel of the game v^*.*

Proof. We show that the continuation values found in theorem 4.3 belong to the kernel. This result is linked to the fact that the equilibrium is in mixed strategies. Consider, for example, players 1 and 2. The excess of player 1 over player 2 must equal the excess of player 2 over player 1. Given the continuation values in theorem 4.3, the best coalition 1 can form without 2 is $\{1,3\}$ and the best coalition 2 can form without 1 is $\{2,3\}$. If $(z_i)_{i\in N}$ is in the kernel, then the excess of 1 over 2 must equal the excess of 2 over 1, that is

$$\varphi(1,3) - z_1 - z_3 = \varphi(2,3) - z_2 - z_3 \tag{4.29}$$

which implies

$$\varphi(1,3) - z_1 = \varphi(2,3) - z_2. \tag{4.30}$$

In other words, player 3 must be indifferent between forming a coalition with player 1 or with player 2.

Notice also that since $\sum_{i\in N} z_i = \varphi(N)$ and $z_i \geq v^*(i) = K - \varphi(j,k)$ for all i (see lemma 4.3), $(z_i)_{i\in N}$ is an imputation.

We have shown that the vector z from theorem 4.3 is in the kernel. Since the kernel of a three-person game contains a unique payoff vector (Davis and Maschler, 1965), no other payoff vector can belong to the kernel. ■

Expected payoffs also coincide with the Shapley value of v^*.

Proposition 4.3 *Under the assumptions of theorem 4.3, the limit expected payoffs for the players coincide with the Shapley value of the game v^*.*

[7] A formal definition of the kernel can be found in chapter 7.

Proof. Because of the simplicity of three-person games, in order to construct the game v^* we only need to know the payoff of a player i if he decides to form a singleton. This payoff is $K - \varphi(j,k)$. For the rest of the coalitions, we have $v^*(S) = \varphi(S)$. It is easy to check that the Shapley value of v^* coincides with the equilibrium payoffs in proposition 4.3. ∎

In games with more than three players, expected payoffs will rarely coincide with the Shapley value of v^*: the Shapley value takes all coalitions into account, whereas expected payoffs depend only on which coalitions form in equilibrium. The property that expected payoffs coincide with the kernel of the grand coalition may generalize to games with more than three players, but one would need additional assumptions besides the game being constant-sum, essential and fully cohesive (see section 4.4).

4.3 Constructing a partition function from a characteristic function: the axiomatic approach

4.3.1 Two partition functions based on axiomatic solution concepts

As we have argued in the introduction, constant-sum partition function games may reflect the outcome of efficient bargaining among coalitions. Consider a given coalition structure $\pi := \{S_1, ..., S_m\}$. Given this coalition structure, if the coalitions reach an agreement to cooperate payoffs are given by $\overline{\varphi}(\pi)$. If coalitions do not cooperate with each other, the payoffs they get are presumably given by a characteristic function. Call this characteristic function v. The values of φ may be a function of the values of v (the *autarchy* or *disagreement payoffs*). In this section, we briefly discuss two possible ways to construct a partition function φ from a characteristic function v. These two ways were already suggested by Maschler (1963) and called by him *standards of fairness*.

- The first possibility is to assume that coalitions will split the gains from cooperation between themselves by using the **Nash bargaining solution**. This partition function will be denoted by φ_v^N. Thus, if there are m coalitions in π

$$\varphi_v^N(S_j, \pi) = v(S_j) + \frac{v(N) - \sum_{k=1}^m v(S_k)}{m} \text{ for all } S_j \in \pi.$$

- The second possibility is to take the **Shapley value** of the game in which the coalitions in π act as players, that is, the **Owen value** for the game v with coalition structure π. This partition function will be denoted by φ_v^{Sh}. If there are m coalitions in π, φ_v^{Sh} is given by

$$\varphi_v^{Sh}(S_j, \pi) = \sum_T \frac{(t-1)!(m-t)!}{m!}[v(T) - v(T \backslash S_j)] \text{ for all } S_j \in \pi.$$

  where the T's are all possible unions of coalitions in π and t is the number of coalitions included in T.

The use of the Nash bargaining reflects the fact that partial agreements among the coalitions in π are not viable or not profitable, whereas the use of the Shapley value reflects the possibility of partial cooperation among the coalitions in π. Both are *axiomatic approaches* to the problem of determining coalitional payoffs given a coalition structure and the disagreement payoffs given by v.

Table 4.1 gives the partition functions φ_v^N and φ_v^{Sh} for three-player games. Notice that they only differ in their values for coalition structure $\langle N \rangle$.

	φ_v^N	φ_v^{Sh}
$\varphi(i, \langle N \rangle)$	$\frac{v(N)+2v(i)-v(j)-v(k)}{3}$	$\frac{2v(N)+2v(i)-v(j)-v(k)+v(i,j)+v(i,k)-2v(j,k)}{6}$
$\varphi(i,j)$	$\frac{v(N)+v(i,j)-v(k)}{2}$	$\frac{v(N)+v(i,j)-v(k)}{2}$
$\varphi^c(i)$	$\frac{v(N)-v(i,j)+v(k)}{2}$	$\frac{v(N)-v(i,j)+v(k)}{2}$
$\varphi(N)$	$v(N)$	$v(N)$

Table 4.1: The partition functions φ_v^N and φ_v^{Sh} for three-player games

4.3.2 Properties of φ_v^N and φ_v^{Sh} in three-player games

One can wonder whether φ_v^N and φ_v^{Sh} satisfy the assumptions of theorem 4.3. Suppose $N = \{1, 2, 3\}$ and the function v reflecting the autarchy payoffs is superadditive and essential. Will φ_v^N and φ_v^{Sh} be fully cohesive, essential and constant-sum? Since the Nash bargaining solution and the Shapley value are efficient, φ_v^N and φ_v^{Sh} are constant-sum. It is also clear that φ_v^N and φ_v^{Sh} will be essential whenever the underlying characteristic function is essential. However, v being superadditive does not guarantee that φ_v^N and φ_v^{Sh} will be fully cohesive. In order for a constant-sum partition function to be fully cohesive, we only need to check that $\varphi(i,j) \geq \varphi(i, \langle N \rangle) + \varphi(j, \langle N \rangle)$ for all i, j. Thus, for φ_v^N to be fully cohesive, the following condition must be satisfied for all (distinct) i, j and k:

$$\frac{v(N)+v(ij)-v(k)}{2} \geq \frac{v(N)+2v(i)-v(j)-v(k)}{3} + \frac{v(N)+2v(j)-v(i)-v(k)}{3}$$

or, re-arranging

$$v(ij)-v(i)-v(j) \geq \frac{v(N)-v(ij)-v(k)}{2}. \tag{4.31}$$

In order for φ_v^{Sh} to be fully cohesive, we need for all (distinct) i, j and k:

$$\frac{v(N)+v(ij)-v(k)}{2} \geq \frac{2v(N)+2v(i)-v(j)-v(k)+v(ij)+v(ik)-2v(jk)}{6} + \frac{2v(N)+2v(j)-v(i)-v(k)+v(ij)+v(jk)-2v(ik)}{6}$$

Re-arranging terms we obtain

$$v(N)+v(1)+v(2)+v(3) \leq v(12)+v(13)+v(23). \tag{4.32}$$

Condition (4.31) is more demanding than condition (4.32). Indeed, adding up the three inequalities in (4.31) and re-arranging (taking into account that at least one is strict), one gets (4.32). One can easily find games for which (4.32) is satisfied but (4.31) is not.

Example 4.2 $N = \{1,2,3\}$, $v(1) = 4$, $v(2) = v(3) = 0$, $v(12) = v(13) = v(23) = 5$ and $v(N) = 10$.

$\overline{\varphi}_v^N(1,2,3) = (6,2,2)$, whereas $\overline{\varphi}_v^{Sh}(1,2,3) = (\frac{28}{6}, \frac{16}{6}, \frac{16}{6})$. $\varphi_v^N(1,2) = \varphi_v^{Sh}(1,2) = 7.5$. φ_v^N is not fully cohesive since players 1 and 2 would lose by merging (they would get 7.5 instead of 8). φ_v^{Sh} is fully cohesive: for example, if 1 and 2 merge, they get 7.5 instead of $\frac{44}{6} \approx 7.33$.

If φ_v^{Sh} and φ_v^N are fully cohesive and essential, theorem 4.3 applies and expected payoffs are given by (4.18). Moreover, expected payoffs coincide with the Shapley value of v.

Proposition 4.4 *Let $N = \{1,2,3\}$ and φ a constant-sum, fully cohesive and essential partition function based on an underlying characteristic function v. Consider the game $G(N,\varphi,\theta,p)$ with $\theta_i < \frac{1}{2}$ for all i. Expected equilibrium payoffs for the players in this game coincide with the Shapley value of v provided that*

$$\varphi(ij) = \frac{v(N) + v(ij) - v(k)}{2} \text{ for all distinct } i,j,k \in N \tag{4.33}$$

Proof. Substitute (4.33) in (4.18). ∎

In games with more than three players expected equilibrium payoffs of the game $G(N,\varphi_v^N,\theta,p)$ or $G(N,\varphi_v^{Sh},\theta,p)$ will not generally coincide with the Shapley value of v, even if φ_v^{Sh} and φ_v^N satisfy the conditions in theorem 4.5. The Shapley value is a function of the values of all coalitions, whereas the expected payoffs in the game $G(N,\varphi,\theta,p)$ are generally a function of the values of only some coalitions.

4.4 Random matching results for the four-player case

In this section we want to explore whether the results for three-player constant-sum games are highly specific to the three-player situation or whether they can be generalized to four players. We will not analyze all possible strategy combinations as we did for the three-player case, but we will directly turn to the question of whether, given an egalitarian protocol, we can get a family of equilibria analogous to the one described in theorem 4.3 and corollary 4.1. In those equilibria, all coalitions of size two could form in equilibrium and players where indifferent between possible coalition partners. In the four player case, we can investigate whether there are equilibria in which all coalitions of size two can form (each followed by another coalition of size two), and whether there are equilibria in which all coalitions of size three can form. We will see that there are no equilibria in which all pairs can form unless the game is inessential; whether there are equilibria in which all trios can form depends on the parameters. The conclusion emerging from this section is that generalization of the results for three-person games to four-player games is not possible without making further assumptions on the partition function.

Theorem 4.4 *Consider the game $G(N,\varphi,\theta^E,p)$ where $N = \{1,2,3,4\}$, and (N,φ) is a fully cohesive constant-sum partition function game. Suppose we have a stationary perfect equilibrium of the game $G(N,\varphi,\theta^E,p)$ where only pairs are formed and players are indifferent between all pairs they can form. Then (N,φ) is inessential.*

Proof. We first notice that a pair $\{i, j\}$ always gets a payoff $\varphi^c(i, j)$. The reason is that, since the game is fully cohesive, the other two players weakly gain from forming the complementary coalition. If this gain is strict, then the other two players form a coalition. If not, it does not matter whether they do.

We now turn to the question of what would be the payoffs in such equilibrium. Since each player should be indifferent between proposing to any of the other three, we get eight equations of the form $\varphi^c(i, j) - z_j = \varphi^c(i, k) - z_k$ (player i is indifferent between j and k for all (distinct) $i, j, k \in \{1, 2, 3, 4\}$) together with the requirement that $\sum_{i\in N} z_i = \varphi(N)$. Solving this system of equations (of which only four are linearly independent) we get

$$z_i = \frac{\varphi^c(i, j) + \varphi^c(i, k) + \varphi^c(i, l) - K}{2}.$$

These continuation values are such that $z_i + z_j = \varphi^c(i, j)$. Player i then gets z_i regardless of whether he is a proposer or a responder and players are indifferent between following their prescribed strategies and proposing to form the grand coalition. If no player is to deviate to proposing a three-player coalition, it must be the case that $\varphi(i, j, k) \leq z_i + z_j + z_k$, which implies that the game is inessential. Indeed, consider coalition structures of the type (ij, k, l). Because the game is fully cohesive and constant-sum, the payoff coalition $\{i, j\}$ gets in this coalition structure is at least z_i+z_j. Since $\varphi(i, j, k) \leq z_i + z_j + z_k$, it must be the case that the payoff k gets in coalition structure (ij, k, l) is at least z_k and l obtains at least z_l. This is only possible if $\overline{\varphi}(ij, k, l) = (z_i + z_j, z_k, z_l)$ and $\varphi(i, j, k) = z_i + z_j + z_k$. It is easy to see that this implies $\varphi(i, \langle N\rangle) = z_i$. ∎

We now turn to the possibility that an equilibrium exist in which only three-player coalitions may form and all three-player coalitions are possible. We will use the following property

Remark 4.14 *If $N = \{1, 2, 3, 4\}$ and (N, φ) is fully cohesive, it holds that*

$$\varphi(1, 2, 3) + \varphi(1, 2, 4) + \varphi(1, 3, 4) + \varphi(2, 3, 4) \geq 3K \tag{4.34}$$

This result is established by using the fact that $\varphi(i, j, k) \geq \varphi(i, \langle N\rangle) + \varphi(j, \langle N\rangle) + \varphi(k, \langle N\rangle)$ and $\sum_{i\in N} \varphi(i, \langle N\rangle) = K$. Notice that, if the game is essential, the inequality in (4.34) is strict.

Theorem 4.5 *Let (N, φ) be a four-player constant-sum, fully cohesive partition function game exhibiting negative externalities*[8] *and satisfying the following conditions*

$$\varphi^c(i,j) \leq \frac{3\varphi(i,j,k) + 3\varphi(i,j,l) - \varphi(i,k,l) - \varphi(j,k,l) - K}{4} \tag{4.35}$$

$$\varphi(i, \langle N \rangle) \leq \frac{\varphi(i,j,k) + \varphi(i,j,l) + \varphi(i,k,l) - \varphi(j,k,l) - K}{2}. \tag{4.36}$$

Consider the coalition formation game $G(N, \varphi, \theta^E, p)$. There is a family of stationary perfect equilibria where only three-player coalitions are proposed and each player is indifferent between the three-player coalitions he can propose. In the limit when p tends to 1, each three-player coalition forms with probability $\frac{1}{4}$ and the expected value for player i equals (in the limit when $p \to 1$)

$$\frac{K + \varphi(i,j,k) + \varphi(i,j,l) + \varphi(i,k,l) - 3\varphi(j,k,l)}{4}. \tag{4.37}$$

Proof. Suppose we have an equilibrium with the property that only three-player coalitions form and that players are indifferent between the three-player coalitions they can propose. Knowing this, the continuation values can be easily calculated from the indifference conditions of the players together with the fact that the game is constant-sum.

The indifference conditions for player i are of the form

$$\varphi(i,j,k) - z_j - z_k = \varphi(i,j,l) - z_j - z_l;$$

where i, j, k and l are distinct players. The game being constant-sum implies

$$\sum_{i \in N} z_i = K.$$

Solving the system of (not linearly independent) nine equations we get that the continuation values $(z_i)_{i \in N}$ as well as the limit expected payoffs are given by (4.37).

If only three-player coalitions are formed in equilibrium, it must be the case that players would not profit from proposing other coalitions. Since players are by construction indifferent between the three-player coalitions they can propose, we only have to check possible deviations to proposing the grand coalition, a two-player coalition or staying alone.

[8] A partition function games exhibits negative externalities if mergers of coalitions have no beneficial effects on outsiders, that is, $\varphi(S, \pi) \leq \varphi(S, \{\pi \backslash T\} \cup \pi_T)$ for all S, $T \in \pi$, for all $\pi \in \Pi(N)$.

Consider the situation of player 1. His payoff as a proposer equals $\varphi(1,2,3) - z_2 - z_3$, that is (substituting for z_2 and z_3 from (4.37))

$$\frac{\varphi(1,2,3) + \varphi(1,2,4) + \varphi(1,3,4) - \varphi(2,3,4) - K}{2}.$$

If instead player 1 would propose the grand coalition, his payoff would be $\varphi(N) - z_2 - z_3 - z_4$. Player 1 will not profit from proposing the grand coalition provided that $z_4 \geq \varphi(N) - \varphi(1,2,3)$. Using (4.37), one can see that $z_4 - [\varphi(N) - \varphi(1,2,3)] = \frac{\varphi(1,2,3)+\varphi(1,2,4)+\varphi(1,3,4)+\varphi(2,3,4)-3K}{4}$, and this is nonnegative because of (4.34). Thus, player 1 (or, in general, player i) does not have an incentive to deviate. Corollary 3.5 implies that making unacceptable proposals is not a profitable deviation either.

If player 1 proposes a two-player coalition (say, $\{1,2\}$), the payoff of the coalition will be $\varphi^c(12)$. Thus, player 1 will not profit from proposing a two-player coalition if $\varphi^c(1,2) - z_2 \leq \frac{\varphi(1,2,3)+\varphi(1,2,4)+\varphi(1,3,4)-\varphi(2,3,4)-K}{2}$. Substituting for z_2 from (4.37), one can easily see that this implies

$$\varphi^c(1,2) \leq \frac{3\varphi(1,2,3) + 3\varphi(1,2,4) - \varphi(1,3,4) - \varphi(2,3,4) - K}{4}.$$

More generally, player i will not profit from proposing coalition $\{i,j\}$ provided that

$$\varphi^c(i,j) \leq \frac{3\varphi(i,j,k) + 3\varphi(i,j,l) - \varphi(i,k,l) - \varphi(j,k,l) - K}{4}.$$

This is precisely condition (4.35).

Suppose player 1 forms a singleton. His expected payoff cannot be determined solely from the fact that the game is fully cohesive and constant-sum. Depending of the parameters, he may face a three-player coalition or a two-player coalition and a singleton. Assuming negative externalities (that is, a player never profits from other players forming a coalition), player 1's payoff is maximized when all the other players remain singletons, and we can find the following *sufficient* condition

$$\varphi(1,\langle N\rangle) \leq \frac{\varphi(1,2,3) + \varphi(1,2,4) + \varphi(1,3,4) - \varphi(2,3,4) - K}{2}.$$

More generally, player i will not profit from forming a singleton if

$$\varphi(i,\langle N\rangle) \leq \frac{\varphi(i,j,k) + \varphi(i,j,l) + \varphi(i,k,l) - \varphi(j,k,l) - K}{2}.$$

This is precisely condition (4.36). Notice that a necessary and sufficient condition would be

$$v^*(i) \leq \frac{\varphi(i,j,k) + \varphi(i,j,l) + \varphi(i,k,l) - \varphi(j,k,l) - K}{2}.$$

However, this condition is not very helpful because $v^*(i)$ depends on the concrete partition function we are considering.

So far we have checked that, given the expected payoffs found from the indifference conditions of the players, players will not have an incentive to propose coalitions other than the ones prescribed. It remains to find mixed strategies inducing the expected payoffs in (4.37). We will use the following notation for the probabilities

	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$
1	α	β	$1-\alpha-\beta$	$-$
2	γ	ρ	$-$	$1-\gamma-\rho$
3	λ	$-$	μ	$1-\lambda-\mu$
4	$-$	ϕ	ψ	$1-\phi-\psi$

(4.38)

Thus, player 1 proposes coalition $\{1,2,3\}$ with probability α, $\{1,2,4\}$ with probability β, and so on.

Given this notation, the continuation values are given by a system of four equations (we only include the equation corresponding to player

$$\begin{aligned} z_1 &= \tfrac{p}{4}(\varphi(1,2,3)-z_2-z_3)+\tfrac{p}{4}(\gamma+\rho+\lambda+\mu+\phi+\psi)z_1+ \\ &+\tfrac{p}{4}(3-\gamma-\rho-\lambda-\mu-\phi-\psi)[K-\varphi(2,3,4)]+(1-p)\varphi(1,\langle N\rangle). \end{aligned}$$

If we substitute for the z_i's from (4.37), we get four equations with eight unknowns (the probabilities in table (4.38)). As in the three-player case, there is a family of equilibria. Taking ρ, λ, μ, ϕ and ψ as parameters and solving for α, β and γ, one gets (in the limit when p tends to 1) the following expressions

$$\begin{aligned} \alpha &= \rho+\mu+\phi+\psi-1 \\ \beta &= 1-\rho-\phi \\ \gamma &= 2-\rho-\lambda-\mu-\phi-\psi. \end{aligned}$$

These values imply that each four-player coalition forms with probability $\frac{1}{4}$.[9]
Notice that setting $\rho=\lambda=\mu=\phi=\psi=\frac{1}{3}$ we obtain $\alpha=\beta=\gamma=\frac{1}{3}$. ■

Theorem 4.5 implies that, analogously to the three-player case, the relative strength of the players as measured by the partition function has no influence on coalition frequencies. However, the result does not follow for all four-player constant-sum, essential and fully cohesive games but it requires additional assumptions.

[9]Consider, for example, coalition $\{1,2,3\}$. The (limit) probability of this coalition forming equals $\frac{1}{4}(\alpha+\gamma+\lambda)=\frac{1}{4}$.

Remark 4.15 *Conditions (4.35) and (4.36) above hold for all symmetric four-player constant-sum fully cohesive games.*

Proof. Denote each coalition by its size. It follows from symmetry that $\varphi^c(2) = \frac{K}{2}$ and $\varphi(1, \langle N \rangle) = \frac{K}{4}$. For symmetric games, conditions (4.35) above becomes

$$\frac{K}{2} \leq \frac{4\varphi(3) - K}{4}.$$

Re-arranging terms we get

$$3K \leq 4\varphi(3). \tag{4.39}$$

Expression (4.39) is expression (4.34) for symmetric games.

Analogously, condition (4.36) becomes

$$\frac{K}{4} \leq \frac{2\varphi(3) - K}{2}. \tag{4.40}$$

This expression is also equivalent to (4.34) for symmetric games. ∎

Thus, for any symmetric four-player constant-sum fully cohesive game with negative externalities, one can construct a family of equilibria as described in theorem 4.5. Conditions (4.35) and (4.36) may not hold for asymmetric games. We will illustrate the possibility (or impossibility) of an equilibrium in which each three-player coalition form with probability $\frac{1}{4}$ with an example.

Example 4.3 $N = \{1, 2, 3, 4\}$, $\varphi(N) = 100 = \varphi(1, 2, 3)$, $\varphi(1, 2, 4) = 85$, $\varphi(1, 3, 4) = 80$, $\varphi(2, 3, 4) = 65$.

Notice that this is a *quota game* in the following sense: there are numbers $q_1, ..., q_4$ such that $\varphi(i, j, k) = q_i + q_j + q_k$ for any distinct i, j and k. The quotas of the players are $q_1 = 45$, $q_2 = 30$, $q_3 = 25$, $q_4 = 10$. Notice also that players 1, 2 and 3 can obtain the whole pie by uniting against 4. We can try to construct an equilibrium in which players only propose three-player coalitions and they are indifferent between them. We know from (4.37) that expected payoffs in such an equilibrium would be $w_1 = 42.5$, $w_2 = 27.5$, $w_3 = 22.5$, $w_4 = 7.5$ (each player gets 2.5 less than their quotas[10]). Thus, when player i is selected to be a proposer, his payoff is $q_i + 5$. In order to

[10] That is, each player loses $\frac{q_1+q_2+q_3+q_4-\varphi(N)}{4}$.

check whether this is an equilibrium, we need the rest of the parameters of the partition function.

Suppose $\varphi^c(1,2) = 80$. Then we do not have an equilibrium because player 1 would rather propose coalition $\{1,2\}$ and get $80 - 27.5 = 52.5$ rather than proposing a three-player coalition and getting 50. The rest of the values of the partition function could be $\overline{\varphi}(12,3,4) = (80,20,0)$, $\overline{\varphi}(14,2,3) = (65,20,15)$, $\overline{\varphi}(13,2,4) = (70,20,10)$, $\overline{\varphi}(23,1,4) = (65,35,0)$, $\overline{\varphi}(24,1,3) = (50,35,15)$, $\overline{\varphi}(34,1,2) = (45,35,20)$, $\overline{\varphi}(13,24) = (60,40)$, $\overline{\varphi}(14,23) = (55,45)$, $\overline{\varphi}(1,2,3,4) = (40,20,20,20)$. This game is fully cohesive and essential but there are no equilibria in which each player is indifferent between all three-player coalitions containing him.

Suppose instead we have the same partition function but with $\overline{\varphi}(12,34) = (65,35)$. Do we have an equilibrium? First, we have to check that none of the players prefers to form a two-player coalition, that is, $\varphi^c(i,j) \leq q_i + q_j + 2.5$ for all $\{i,j\}$. This is indeed the case. It remains to check whether a player would have an incentive to form a singleton. This is clearly not the case of players 1, 2 and 3. Player 4 would form a singleton, provided that all other players would form singletons as well. Thus, the sufficient condition (4.36) is not satisfied for player 4. However, coalition structure $\langle N \rangle$ will never arise after a deviation by player 4 since, given that player 4 and another player form singletons, the remaining two players would always form a two-player coalition. Finally, we know from the proof of theorem 4.5 that deviations to proposing the grand coalition are not profitable: indeed, proposing the grand coalition would give player i $q_i - 2.5$ rather than $q_i + 5$.

In order to describe the equilibrium strategies, we need to know the probability of a given player proposing a given (three-player) coalition. We can construct an equilibrium as follows: using the notation from the proof of theorem 4.5, we can set $\rho = \lambda = \mu = \phi = \psi = \frac{1}{3}$; the other three probabilities can be found from the system of equations representing the continuation values as functions of the probabilities. In this example, we obtain $\alpha = \frac{16-15p}{3p}$, $\beta = \frac{5p-4}{3p}$ and $\gamma = \frac{4-3p}{3p}$.

4.5 Related literature

The idea that the value of a coalition as given by the characteristic function may not be an adequate representation of the actual payoff of the coalition can be traced back to Luce and Raiffa (1957). They made the point that the use of a characteristic function to represent a situation originally given by a *normal form* game is not without cost. Maschler (1963) argues that the characteristic function may be a poor measure of the payoff of a coalition *even if the game is given only in characteristic function form.* The reason is that coalitions may cooperate with other coalitions and split the gains from cooperation.

Maschler postulates that the outcome of any game will be an imputation (that is, the value of the grand coalition will always be realized) but subcoalitions may appear in intermediate stages of the negotiations. When an intermediate coalition is formed, it may partition itself into subcoalitions or *negotiation groups.* Given a partition of N into negotiation groups, a *standard of fairness* determines what the payoff of each negotiation group. The standard of fairness is such that each negotiation group gets at least its value and the sum of the payoffs for all negotiation groups is the value of the grand coalition. The standard of fairness is therefore a *constant-sum partition function.* The standard of fairness is assumed to depend on factors such as bargaining abilities of the players and accepted distributional rules and may thus vary depending on the concrete game being played. Maschler, however, suggests two possible standards of fairness: the *cooperative standard of fairness* (which assumes that each negotiation group gets an equal share of the difference between the value of the grand coalition and the sum of the values of the negotiation groups) and the *standard of fairness based on the Shapley value* (which assigns to each negotiation group the Shapley value of a game in which players are the negotiation groups, that is, each negotiation group receives its Owen value). The payoff of a coalition will then be the sum of the payoffs obtained by its negotiation groups. Since the partition function is constant-sum, Maschler assumes that if a coalition forms its complement also forms; the power of a coalition is then defined to be the von Neumann minimax value of the game between the coalition and its complement, where the pure strategies of a coalition are the partitioning into negotiation groups. In this way one obtains a new game (the *derived game*), to which all traditional solution concepts can be applied.

The spirit of the chapter is very close to Maschler (1963). However, there are some important differences. The first difference is the concept of coalition. Maschler assumes that a coalition may partition itself in several negotiation groups (thus payoff is transferable across negotiation groups belonging to the same coalition), whereas in this chapter a coalition is understood as a bargaining unit, and thus it is forced to act as a single negotiation group. We argue that dividing a coalition into negotiation groups may demand too much commitment from the players.[11] Notice however that whether splitting a coalition into negotiation groups is allowed or not is irrelevant in games where splitting is not profitable. Full cohesiveness of the partition function ensures this. Indeed, the bargaining set for the derived game (called $\mathcal{M}_1^{(im)}$) computed by Maschler for three-person games for both the cooperative standard of fairness and the standard of fairness based on the Shapley value assigns the payoffs in (4.18) to the grand coalition for the case in which it is not profitable for coalitions to split themselves into several negotiation groups. The second difference between this chapter and Maschler (1963) is that Maschler assumes that when a coalition forms its complement also forms; the game in this chapter allows for simultaneous formation of any number of coalitions. Equilibria with more than two coalitions are possible in our game, even though we have focused on conditions under which only two coalitions form. The third difference is that Maschler does not model the process of coalition formation.

New versions of the bargaining set $\mathcal{M}_1^{(im)}$ due to Maschler can be found in chapter 5 of Kahan and Rapoport (1984). The first version, called the *modified bargaining set* $\mathcal{M}^M$, keeps objections and counterobjections in the modified game, but actual payoffs must be in the original game. Thus, there is a separation of the bargaining space and the payoff space. A simplified version of $\mathcal{M}^M$, in which any coalition that forms as well as its complement are restricted to act as a single negotiation group, is called the *power bargaining set* $\mathcal{M}^P$ (see also Rapoport and Kahan (1979) and (1982)). This bargaining set requires less cognitive ability from the players (as it restricts the number of partitions that need to be considered) and it makes no use of the minimax theorem. The modified bargaining set and the power bargaining set abandon the idea that the payoff of a coalition can be greater than its value, and the derived game is only used to determine the distribution of the value of a coalition among its members.

[11] Maschler provides the following example: $N = \{1,2,3\}$, $v(12) = v(13) = 100$, $v(123) = 120$, and $v(S) = 0$ otherwise.

> The coalition $\{2,3\}$ may act as one player, and if it does the following outcome $(60,30,30)$ may result. However, coalition $\{2,3\}$ can still do better by forming an intermediate coalition, deciding not to listen to any offer from player 1 to one of them but enter the game as two players. In this case the outcome will be $(40,40,40)$.

One may well argue that deciding not to listen to any offer from player 1 to one of them while at the same time remaining two autonomous players may require too much commitment from players 2 and 3.

A theory of coalition formation based on the possibility of bargaining between coalitions is the *central union theory* (Michener and Au (1994)). This theory distinguishes two kinds of alliances among players: unions and coalitions. A union is an agreement among a set of players U that none of them will establish any coalitional agreement with any of the players inside U unless all of them are included. Unions form in order to improve the bargaining power of the players. Thus, the concept of union coincides with what we have called *(intermediate) coalition* or *bargaining unit* in this chapter. The central union theory also distinguishes between value v and power p. Players evaluate alternative unions in terms of their power. In order to calculate the power of a union U, it is assumed that the power of a union equals its value plus a share of the *excess* $v(N) - v(U) - v(N/U)$. Thus, implicitly players assume that the union would bargain with $N\backslash U$, as in Rapoport and Kahan (1982). Players evaluate alternative unions on the basis of their power. A union preferred by all its members according to this criterion is called a *central union* (there is always at least one). The theory predicts that the final coalition structure will always contain a central union or a superset of a central union. Actual payoffs, however, are in v, and the (rather complicated) rules of payoff division use only information provided by v and not by p.

This chapter shares the interpretation of coalitions as bargaining units with Hart and Kurz (1983). Hart and Kurz use the Owen value to evaluate not only the prospects of a coalition as a whole, but also the payoffs of the individual players. They do not have a dynamic model of coalition formation but rather look at which coalition structures are (α-, β-, γ-, or δ-) stable. For three-player games, a player has the same Owen value in any two-player coalition, that is, any of the other players makes as good a partner as the other. This is also a characteristic of the equilibrium of our game: each player is indifferent between proposing to any of the other two players.

Our extensive form game does not implement the Shapley value of either v or v^*. Hart and Mas-Colell (1996) have shown that the Shapley value can be implemented by a game with random proposers. Hart and Mas-Colell assume that unanimity is a requirement for any agreement (thus, intermediate coalitions are not allowed) and that, if a proposal is rejected, the possibility of breakdown only affects the proposer: with positive probability the proposer is "expelled" from the game and bargaining continues among the rest of players.

Gul (1989) implements the Shapley value using a bargaining game with random matching. We are able to endogeneize the random matching result and still obtain the Shapley value as the equilibrium payoff vector, though only for three players.

4.6 Concluding remarks

We have learned from the three-player case that the frequency of each two-player coalition depends on the protocol rather than on the partition function. One might expect that the relative strength of the players, as described by the partition function, should influence the frequency with which players are part of a coalition. However, we have seen that under relatively general conditions (the game being constant-sum and fully cohesive) this is not the case. If the protocol gives a lot of power to one of the players, the other two unite against him. On the other extreme, if the protocol is completely egalitarian, each coalition forms with probability $\frac{1}{3}$ regardless of how weak or strong each player is in terms of the partition function. For the case of four-person constant-sum games, one cannot get the same kind of results without making additional assumptions on the parameters.

Full cohesiveness requires two-player coalitions to be weakly profitable only. If the game is essential, at least one of the inequalities is strict. It may seem strange that players are indifferent in equilibrium between a coalition that is weakly profitable and a coalition that is strictly profitable. Nevertheless, one should take into account that profitability is defined with respect to coalition structure $\langle N \rangle$, which never arises in equilibrium. The actual alternative to a given two-player coalition is not $\langle N \rangle$ but another two-player coalition.

The limit results we have obtained for the game with breakdown probability can be obtained for positive games with discounting as well. The results can also be extended to some games that are not constant-sum: what matters is that the sum of the values of the coalitions in coalition structures of the type $\{\{i, j\}, \{k\}\}$ (in three-player games) and $\{\{i, j, k\}, \{l\}\}$ (in four-player games) is the value of the grand coalition.

We have argued that a characteristic function game becomes a partition function game if we allow coalitions to cooperate with each other. This may change the coalition structure that arises. Suppose we have a symmetric characteristic function game. Consider the partition function based on the Nash bargaining solution φ_v^N and suppose that players in a given coalition have to split the payoff equally. If we find the equilibrium of the game with random proposers and fixed payoff division, we know from the previous chapter that players will form the coalition with the highest per capita value of v^*. This is not necessarily the coalition that had the highest per capita payoff in the characteristic function, as the following example illustrates:

Example 4.4 $N = \{1, 2, 3, 4\}$, $v(i) = 0$ for all $i \in N \in$, $v(S) = 28$ for all S with $|S| = 2$, $v(S) = 45$ for all S with $|S| = 3$, $v(N) = 100$.

If a coalition of two players forms, another coalition of two players will follow (bringing each of the members 25 rather than $\frac{100-28}{3} = 24$). Each player will then get 25. If instead a three-player coalition forms, it will get $\frac{100+45}{2} = 72.5$, which gives to each player in the three-player coalition less than 25. Thus, the coalition with highest per capita payoff according to v^* (a two-player coalition) is different from the coalition with highest per capita payoff according to v (a three-player coalition).

Chapter 5

The Strategic Approach to Deriving a Partition Function: The One-Seller-Two-Buyers Game

In the previous chapter, we argued that a characteristic function game can give rise to a partition function game if coalitions cooperate with each other. In this chapter, we turn attention to a classical economic application: the one-seller-two-buyers game. We will use this game to show how a partition function can be derived from a characteristic function, and also to illustrate the assumptions and predictions of the random proposer model.

The basic situation involves three agents. A seller owns a unit of some good and derives zero utility from keeping it. He can sell the object to one of two buyers with different reservation prices. The allocation of the good and payments to be made are determined by bargaining among the players. All players are risk-neutral and the reservation values are common knowledge.

An interesting question is whether the buyer with the lower reservation price can expect a positive payoff in this situation. Efficient bargaining implies that this buyer cannot expect to get the object; however, his presence in the market benefits the seller, who can presumably ask a higher price when there are two buyers, whereas his exit from the market would benefit the other buyer, who would then pay a lower price.

If coalitions can be formed prior to the bargaining process, the weak buyer may benefit from his influence on the price. He can negotiate with the seller and get paid to be in the market, or he can negotiate with the other buyer and get paid to be out of the market. The question is then what coalition will form and what are the expected payoffs for the players (will the weak buyer "capture" the whole price difference or not?).

The one-seller-two-buyer case is an example of a more general situation in which a player does not add value to the game but his presence or absence in the game modifies the value added of other players. Brandenburger and Nalebuff (1995) provide some real-life examples. They come to the following conclusion:

> Sometimes the most valuable service you can offer is creating competition, so don't give it away for free.

The one-seller-two-buyer situation fits into the general framework of chapter 4. Chapter 4 assumes that there is a given value to be divided between the players according to some bargaining procedure. Prior to this procedure, players may form coalitions; if a coalition forms, it will act as a single unit in the bargaining procedure. There are then two stages in the game: coalition formation and bargaining. The outcome of the bargaining process will depend on how players are divided in coalitions, thus it will be described by a partition function.

In the case of the one-seller-two-buyers situation, the amount to be divided is the highest reservation price. We will assume that the allocation of the good and payments to be made are determined by noncooperative bargaining. The players may benefit from forming coalitions prior to bargaining over the allocation of the good: the most salient case is the two buyers forming a cartel, but the weak buyer may also get paid by the seller to create competition in the market. We will model the two stages of the game (coalition formation and bargaining) in a similar way: in the bargaining stage, a coalition is selected randomly to make a proposal over the allocation of the good and payments; in the coalition formation stage, a player is selected randomly to propose a coalition and a division of the payoff the coalition will attain in the second stage.

The solution to the second stage leads to a partition function. Once we have derived the partition function, we can apply theorem 4.3 and obtain the following results: the weak buyer has a positive expected payoff, but he cannot capture the whole price difference, as the seller and the strong buyer would then form a coalition themselves. For any values of the parameters, each two-player coalition forms with probability one third. No matter how weak the weak buyer is, he will be part of a coalition two thirds of the time! His expected payoff will be affected by how weak he is, but not his probability of being in a coalition. The players' expected payoffs correspond to the kernel and the Shapley value of a related cooperative game, and the possibility of forming coalitions generally makes the two buyers better-off.

5.1 The model

The set of agents is $N = \{1, 2, 3\}$. Agent 1 is a potential seller who owns one unit of some good and derives zero utility from keeping it; agents 2 and 3 are potential buyers, whose reservation prices for the good are respectively u_2 and u_3, $0 < u_2 < u_3$. All players are risk-neutral and valuations are common knowledge. The characteristic function associated to this situation is then $v(1,2,3) = v(1,3) = u_3$, $v(1,2) = u_2$, $v(2,3) = v(i) = 0$ for $i = 1,2,3$. The characteristic function assigns to each coalition the payoff it can get *if it remains isolated*: if the three players remain isolated, all of them receive zero (players 2 and 3 want the good but do not have it; player 1 has the good but does not want it); if the two buyers form a coalition, they still receive zero since they do not own the good; if players 1 and 2 form a coalition they can trade the good and make a total gain of u_2; if players 1 and 3 form a coalition they can trade and make a total gain of u_3. The value of the grand coalition is the same as the value of coalition $\{1,3\}$; since there is only one unit of the good and player 3 values it most, the presence of player 2 does not affect the *total* value that can be created (though it affects the *marginal* contribution of player 3 to the game).

The question we investigate is whether player 2 can expect a positive payoff in this situation. On the one hand, if the outcome of bargaining is efficient, he cannot expect to buy the good, as for any price player 2 is prepared to pay, player 3 is prepared to pay more. On the other hand, player 2 may affect the price of the good: if player 2 were not in the game, the "intuitive" outcome would be player 3 buying the good for $\frac{u_3}{2}$; if player 2 is in the game, the price cannot be lower than u_2. Player 2 may then exploit this power and, say, form a cartel with player 3 and get a share of the gains derived from the cartel.

Different cooperative solution concepts give different answers to this question. The core of v contains all allocations of the form $(x, 0, u_3 - x)$ with $x \geq u_2$; thus, in any core allocation player 2 must receive zero. On the other hand, the Shapley value of v is $(\frac{3u_3+u_2}{6}, \frac{u_2}{6}, \frac{3u_3-2u_2}{6})$; it gives a payoff to player 2 that is proportional to his reservation price.

To address the question of whether player 2 can expect a positive payoff, we model the situation described above as a two-stage game. The allocation of the good and payments are determined in a bargaining process with random proposers and exogenous probability of breakdown. Prior to this bargaining players may form coalitions: if two players form a coalition, they will bargain with the third player as a single unit. It is assumed that contracts specifying the division of the coalitional payoff can be enforced. The game is now described in more detail.

5.1.1 The coalition formation stage

This stage is just a concrete case of the game with breakdown probability $G(N, \varphi, \theta^E, p)$, where φ is derived from the equilibrium of the second stage. The outcome of the coalition formation stage is a coalition structure π (interpreted as the distribution of players in bargaining units to play the second stage) and an agreement of the players over side payments.

The game starts with Nature selecting a proposer; each of the three players is selected with probability $\frac{1}{3}$. A proposal consists of a coalition S to which the proposer belongs and a payment to the remaining players in S.

Once a proposal is made, the rest of the players in S accept or reject the proposal sequentially. If the proposal is accepted the coalition forms, and the game proceeds to the next stage unless the coalition formed was a singleton, in which case Nature chooses a proposer among the remaining two players. If the proposal is rejected, Nature chooses a new proposer with probability p, and breakdown occurs with probability $1-p$. If breakdown occurs, the coalition structure $\langle N \rangle$ forms.

The coalition formation stage lasts until a coalition structure is formed by breakdown of the negotiations or by agreement of the players.

There are two cases in which the game ends after the first stage (or, alternatively, the second stage is trivial): if the grand coalition forms, the division of the value of the grand coalition has already been decided (as a proposal to form a coalition includes a payoff division) and nothing remains to be settled; if coalition $\{1,3\}$ forms, it can achieve a payoff of u_3 by itself.

5.1.2 The bargaining stage

In the bargaining game between coalitions each coalition acts as a unit (say, each coalition is represented by its proposer). Depending on the outcome of the first stage, bargaining takes place between a two-player coalition and a single player, or among three single players.

The coalitional payoff is determined by the equilibrium of this stage. It can be a monetary payment (e.g., the payoff for coalition $\{1\}$ would be the price), a consumer's surplus (e.g., for coalition $\{3\}$ the payoff would be the difference between his reservation price and the price actually paid) or a sum of payments and consumer's surplus (for coalition $\{1,3\}$ the payoff would be the total value to be created, u_3).

The bargaining process runs as follows: first, a coalition is chosen by Nature to be the proposer (all coalitions are chosen with equal probability). This coalition makes a proposal about the allocation of the good and transfers between coalitions. The coalitions affected by the proposal accept or reject sequentially. If a proposal is rejected, nature selects a new proposer with probability p. With probability $1-p$, a breakdown of the negotiations takes place and each coalition gets the payoff it can get by itself, $v(S)$. We will think of p as being close to 1.

Notice the difference between $v(S)$ and the expected payoff of a coalition in this bargaining process, $\varphi(S,\pi)$. The value of the characteristic function, $v(S)$, represents the payoff the coalition gets "by itself", i.e., when it is isolated from other players. However, the coalition will often be able to do better than this, and can improve upon $v(S)$ by reaching an agreement with other coalitions. For example, $v(2,3)=0$, as neither 2 nor 3 own the good, but $\varphi(2,3)>0$, as the coalition $\{2,3\}$ can act as a single buyer and reach an agreement with 1 about the price of the good.

Once an agreement is reached, it is implemented and the game ends, except in one case: if the three players are playing the bargaining game as single units, and player 1 agrees to sell the good to player 2, player 2 may resell the good to player 3. The game continues then until players 2 and 3 reach an agreement or breakdown occurs. If breakdown occurs, player 2 keeps the good. This responds to the idea of allowing for bargaining between the players until all gains from contracting are exhausted.

The division of the payoff of the coalition is implemented as follows: once the game has ended (either with agreement or with disagreement) the proposer pays the responders the amount agreed at stage 1 and keeps the (positive or negative) residual payoff. Suppose for example that coalition $\{1,2\}$ forms with the agreement that 1 will pay $\frac{u_2}{3}$ to player 2. If the game ends without agreement with player 3, this means that player 1 sells the good to player 2 for $\frac{2u_2}{3}$. If the coalition sold the good for, say, $u_2+\epsilon$, player 1 will pay $\frac{u_2}{3}$ to player 2 and keep $\frac{2u_2}{3}+\epsilon$ for himself.

The coalition formation stage and the bargaining stage are formally very similar. In both stages players move sequentially, proposers are selected randomly and there is an exogenous probability of breakdown. However there are three differences: the stages differ in the players (the first stage is played among individual players, whereas the second is played among coalitions), in the content of proposals (in the first stage, proposals include a coalition and payoff division within the coalition, whereas in the second stage a proposal consists of a payoff division among coalitions) and in the consequences of breakdown (if breakdown occurs in the first stage, no coalitions form and the game proceeds to the second stage; if breakdown occurs in the second stage, the game ends).

As in the previous chapters, we will focus on *stationary perfect equilibria.*

The game can be solved by backward induction: first, the equilibrium of the bargaining stage is found for each possible coalition structure; this determines the expected payoffs (the function $\varphi(S,\pi)$) that are used as an input to solve the coalition formation stage.

5.2 Solving the bargaining stage

The bargaining process in the second stage depends on the outcome of the first stage. If either $\{1,2,3\}$ or $\{1,3\}$ have been formed, nothing remains to be settled as the total value has been divided. Thus, there are three bargaining processes to be considered, corresponding to coalition structures $(1,23)$, $(12,3)$ and $(1,2,3)$.

5.2.1 The second stage with a buyer cartel

If players 2 and 3 form a "buyer cartel", there is effectively only one buyer in the market. There is no asymmetry between the two coalitions, as both of them receive a payoff of zero if a breakdown occurs, so we can expect both coalitions to receive the same payoff. As the total value to be created is u_3, each coalition gets $\frac{u_3}{2}$. This is also the expected outcome of bargaining between the two coalitions with random proposers and exogenous probability of breakdown.

We will call $Z_{\{1\}}$ (analogously, $Z_{\{2,3\}}$) the continuation value for coalition $\{1\}$ (i.e., the expected payoff for player 1 given that he rejects an offer). We can find the continuation values from the following system of equations:

$$\begin{aligned} Z_{\{1\}} &= \frac{p}{2}[u_3 - Z_{\{2,3\}}] + \frac{p}{2}Z_{\{1\}} \\ Z_{\{2,3\}} &= \frac{p}{2}[u_3 - Z_{\{1\}}] + \frac{p}{2}Z_{\{2,3\}} \end{aligned}$$

The solution to this system is $Z_{\{1\}} = Z_{\{2,3\}} = \frac{p}{2}\,u_3$. The expected payoffs for the coalitions are $W_{\{1\}} = W_{\{2,3\}} = \frac{u_3}{2}$.[1]

[1]Notice that $W_{\{1\}}$ is an expected price, whereas $W_{\{2,3\}}$ is the expected consumer surplus for player 3

The outcome with a buyer cartel also reflects the situation that would occur if player 2 were not in the game: then the only buyer (player 3) would buy the good for $\frac{u_3}{2}$.

5.2.2 The second stage with no prior coalitions

If the coalition structure resulting from the first stage is $(1, 2, 3)$, bargaining takes place among individual players. This is the benchmark case, and the only possible case if agreements to collude cannot be enforced. Bargaining starts by a chance move that determines the first proposer; each player is selected with probability $\frac{1}{3}$.

If the seller is selected, he can offer the good to one of the buyers for a price. If a buyer is selected, he can propose a price to the seller. Players may also propose a "global agreement" in which 3 gets the good and pays something to the other two players.

In this subgame it makes a difference whether reselling is feasible.

Unfeasible reselling

If reselling is not feasible, the outcome of the bargaining procedure may not be efficient, as it is possible that player 2 buys the good from 1 instead of proposing a global agreement in which player 3 gets the good and pays transfers to both 1 and 2.

It seems reasonable to expect that the presence of a second buyer will not affect the price if he is not prepared to pay more than $\frac{u_3}{2}$ (the "threat" player 1 has to sell the good to player 2 is not credible). On the other hand, if $u_2 > \frac{u_3}{2}$, we can expect that the seller sells the good to player 3 at a price equal to u_2. Proposition 5.1 states that this is indeed the case.

In order to find the equilibrium of the three-player bargaining stage, notice that player 1 has three meaningful alternatives: he can either propose to sell the good to player 2, to player 3 or randomize; player 2 can buy the good from player 1, propose a global agreement or randomize; 3 can only propose to buy the good from player 1. There are then nine candidate equilibria.

We will denote expected payoffs and continuation values in this subgame with capital letters (thus Z_1 is the continuation value of player 1 and W_1 his expected payoff) in order to distinguish them from the expected payoffs and continuation values of the whole game.

(gross of payment to player 2). If the players reach an agreement over the price, player 3 keeps the good and player 2 receives a monetary payment that depends on the agreement 2 and 3 reached in the first stage.

Proposition 5.1 *In the bargaining stage with coalition structure* $(1, 2, 3)$ *and without the possibility of reselling, the following strategies constitute the unique equilibrium for* p *close to 1:*

a) If $u_2 \leq \frac{u_3}{2}$*, player 1 offers the good to player 3, player 3 buys the good from player 1, and player 2 proposes a global agreement. Further, player* i *accepts any proposal that gives him at least his continuation value.*

Expected payoffs when p *tends to 1 are* $(\frac{u_3}{2}, 0, \frac{u_3}{2})$.

b) If $u_2 > \frac{u_3}{2}$ *, 1 offers the good to 3, 3 buys the good from 1, and 2 randomizes between buying the good from 1 and proposing a global agreement; in the limit when* p *tends to 1, he proposes a global agreement with probability 1. Player* i *accepts any proposal that gives him at least his continuation value.*

Expected payoffs when p *tends to 1 are* $(u_2,\ 0,\ u_3 - u_2\ )$.

Proof. a) The continuation values can be found from the following system of equations

$$\begin{aligned} Z_1 &= \frac{p}{3}(u_3 - Z_3) + \frac{2p}{3}Z_1 \\ Z_2 &= \frac{p}{3}(u_3 - Z_1 - Z_3) \\ Z_3 &= \frac{p}{3}(u_3 - Z_1) + \frac{2p}{3}Z_3. \end{aligned}$$

This implies

$$Z_1 = Z_3 = \frac{pu_3}{3-p} \text{ and } Z_2 = \frac{p(1-p)u_3}{3-p}.$$

We prove now that both players 1 and 2 stick to these strategies provided that $u_2 \leq \frac{u_3}{2}$.

Player 1 gets $u_3 - Z_3$ if he follows his prescribed strategy, whereas he gets $u_2 - Z_2$ if he proposes to sell the good to player 2. The difference between the two payoffs is $\frac{3-p^2-p}{3-p}u_3 - u_2$, larger or equal than zero for $u_3 \geq \frac{3-p}{3-p^2-p}u_2 = \Psi(p)u_2$. Because $\Psi'(p) > 0$, the condition is more restrictive for higher values of p. $\Psi(1) = 2$, thus we need $u_3 \geq 2u_2$ for the inequality to be satisfied for p arbitrarily close to 1.

Player 2 gets $u_3 - Z_1 - Z_3$ if he proposes a global agreement, and $u_2 - Z_1$ if he buys the good for himself. The difference between these two payoffs is $\frac{3-2p}{3-p}u_3 - u_2$, larger or equal than zero for $u_3 \geq \frac{3-p}{3-2p}u_2$. Again the inequality is more restrictive for higher values of p, and it holds for p arbitrarily close to 1 given that $u_3 \geq 2u_2$.

b) Suppose player 2 proposes to buy the good from 1 with probability λ, and a global agreement with probability $1-\lambda$. The continuation values can be found from the following system of equations, the fourth being an indifference condition for player 2:

$$\begin{aligned}
Z_1 &= \frac{p}{3}(u_3 - Z_3) + \frac{2p}{3} Z_1 \\
Z_2 &= \frac{p}{3}(u_3 - Z_1 - Z_3) \\
Z_3 &= \frac{p}{3}(u_3 - Z_1) + \frac{(2-\lambda)p}{3} Z_3 \\
u_3 - Z_1 - Z_3 &= u_2 - Z_1 \Rightarrow Z_3 = u_3 - u_2.
\end{aligned}$$

The solution to this system is

$$\begin{aligned}
Z_1 &= \frac{pu_2}{3-2p} \\
Z_2 &= \frac{p(1-p)u_2}{3-2p} \\
Z_3 &= u_3 - u_2 \\
\lambda &= \frac{3(1-p)\left[(3-p)u_2 - (3-2p)u_3\right]}{p(u_3-u_2)(3-2p)}.
\end{aligned}$$

$\lambda \in (0,1)$ provided that[2] $u_2 < u_3 < 2u_2$ and p close enough to 1.

We now check that none of the players has an incentive to deviate from his prescribed strategy. First, it is clear that player i should accept any offer that gives him an expected payoff of at least Z_i. Second, it can be easily checked that the payoff a player gets when he makes the prescribed proposal is higher than his continuation value, thus no player has an incentive to deviate to make unacceptable proposals. Moreover, since $Z_2 \geq 0$, players 1 and 3 have no incentive to deviate to proposing the grand coalition. As for player 2, he is by construction indifferent between proposing to buy the good from 1 and proposing a global agreement. Player 1 can stick to his prescribed strategy and get u_2, or he can propose to sell the good to player 2 and get $u_2 - Z_2 \leq u_2$. Thus, we have indeed found an equilibrium.

It can be checked that no other strategies constitute an equilibrium. If player 1 would propose to 2 or randomize, and 2 would propose a global agreement or randomize, this would require $Z_3 \geq u_3 - u_2 + Z_2$ and $Z_3 \leq u_3 - u_2$. We are left with the three

[2] There is a discontinuity of the equilibrium strategies at $u_3 = u_2$. Player 1's strategy changes from a pure strategy to a mixed strategy that places a weight of $\frac{1}{2}$ on each buyer. Continuation payoffs then equal $\frac{2p(1-p)u_3}{(6-5p)}$ for players 2 and 3 and $\frac{p(2-p)u_3}{(6-5p)}$ for player 1, as in the reselling case.

strategy combinations in which player 2 proposes to player 1. All of them would require $Z_3 \geq u_3 - u_2$ but it can be checked that this is not feasible precisely because player 3 never gets a proposal from player 2. ■

For $u_2 > \frac{u_3}{2}$, the outcome of the process may be inefficient, as player 2 receives the good with positive probability. However, this inefficiency vanishes as p tends to 1.

For $u_2 > \frac{u_3}{2}$, the seller benefits from the presence of a second buyer (and the first buyer is hurt to the same extent), but the second buyer himself does not get anything for $p \to 1$.

Feasible reselling

If the good is bought by player 2 and reselling is feasible, it is natural to assume that player 3 will buy the good from 2. The resale price is again determined by bargaining between the two players. Each player is selected with probability $\frac{1}{2}$ to be the proposer, and breakdown occurs with probability $1 - p$ after a proposal is rejected. If breakdown occurs, player 2 keeps the good.

Expected payoffs for players 2 and 3 in this subgame can be found to be $\frac{u_3+u_2}{2}$ (the expected price in this subgame) and $\frac{u_3-u_2}{2}$ respectively.

Remark 5.1 *If player 2 were not in the game, the seller would sell the good to player 3 for $\frac{u_3}{2}$; player 2 can obtain a higher price from player 3 because he has a positive valuation for the object.*

The possibility of reselling may affect the price at which the seller sells the good in the first place. If 1 is selected to be the proposer and proposes to 2, he anticipates that 2 will resell the good, so that the total value to be divided between 1 and 2 is not u_2 but $\frac{u_3+u_2}{2}$. Analogously, if 2 is selected to be the proposer and proposes to buy the good from 1, he anticipates that he will resell it. The continuation value of player 3 takes reselling into account as well: if player 3 rejects a proposal, with a certain probability (determined by the strategies of the players) the good will be sold to player 2. Player 2 will then resell the good at the expected price of $\frac{u_3+u_2}{2}$; this implies an expected payoff for 3 of $\frac{u_3-u_2}{2}$.

Proposition 5.2 describes the (unique) equilibrium of this subgame. The details of the equilibrium strategies depend on the size of u_2 relative to u_3. As for the equilibrium payoffs, a clear picture emerges from this proposition: the presence of player 2 in the game only shifts payoffs from player 3 in favor of player 1; player 2 himself does not benefit.

Proposition 5.2 *In the bargaining stage with coalition structure* $(1,2,3)$ *and possibility of reselling, the following strategies constitute the unique equilibrium for* p *close to 1:*

a) If $u_3 \geq 3u_2$ *player 1 offers the good to player 3, and players 2 and 3 buy the good from player 1. If player 2 gets the good, he resells it to 3. Further, player* i *accepts any proposal that gives him at least his continuation value.*

b) If $u_3 < 3u_2$ *player 1 randomizes between offering the good to players 2 and 3, and players 2 and 3 buy the good from player 1. If player 2 gets the good, he resells it to 3. Further, player* i *accepts any proposal that gives him at least his continuation value.*

For all values of the parameters the limit of the expected payoffs when p *tends to 1 is*

$$\begin{aligned} W_1 &= \frac{u_3+u_2}{2} \\ W_2 &= 0 \\ W_3 &= \frac{u_3-u_2}{2}. \end{aligned}$$

Proof. We will consider the nine candidate equilibria in turn.

a) Suppose 1 always proposes to 3. Then the continuation values are given by

$$\begin{aligned} Z_1 &= \frac{p}{3}[u_3 - Z_3] + \frac{2p}{3} Z_1 \\ Z_2 &= \frac{p}{3}[\frac{u_3+u_2}{2} - Z_1] \\ Z_3 &= \frac{p}{3}[u_3 - Z_1] + \frac{p}{3} Z_3 + \frac{p}{3}\frac{u_3-u_2}{2}. \end{aligned}$$

Notice that the continuation values incorporate the possibility of reselling (if 2 buys the good from 1, he gets $\frac{u_3+u_2}{2} - Z_1$ and player 3 gets $\frac{u_3-u_2}{2}$).

The solution of this system is

$$\begin{aligned} Z_1 &= \frac{p\left[(6-5p)\,u_3 + pu_2\right]}{2\left(p^2-9p+9\right)} \\ Z_2 &= \frac{p(1-p)\left[(3-2p)\,u_3 + 3u_2\right]}{2\left(p^2-9p+9\right)} \\ Z_3 &= \frac{p\left[(9-8p)\,u_3 - (3-2p)\,u_2\right]}{2\left(p^2-9p+9\right)}. \end{aligned}$$

As p tends to 1, Z_1 tends to $\frac{u_3+u_2}{2}$, Z_2 tends to 0, and Z_3 tends to $\frac{u_3-u_2}{2}$.

For these strategies to be an equilibrium, it must be the case that none of the players can be better-off by deviating from his prescribed strategy. First, it is clear that player i should accept any offer that gives him an expected payoff of at least Z_i. Second, it can be easily checked that the payoff a player gets when he makes the prescribed proposal is higher than his continuation value, thus no player has an incentive to deviate to make unacceptable proposals. Moreover, since $Z_2 \geq 0$, players 1 and 3 have no incentive to deviate to proposing the grand coalition.

It remains to be checked that player 1 does not prefer to propose to player 2. Suppose that player 1 is selected to be the proposer. If he sticks to his prescribed strategy, he gets $u_3 - Z_3$, whereas if he proposes to player 2 he gets $\frac{u_3+u_2}{2} - Z_2$. The difference between those expressions equals $\frac{(1-p)\left[9-6p-2p^2)u_3+3(2p-3)u_2\right]}{2(p^2-9p+9)}$. This expression is positive for $u_3 \geq \frac{3(3-2p)}{9-6p-2p^2}u_2 := \Phi(p)u_2$. Notice that $\Phi^{'}(p) > 0$, or the condition is more restrictive for higher values of p. $\Phi(1) = 3$, thus if $u_3 \geq 3u_2$, the inequality is satisfied for all values of p.

We conclude that, for $u_3 \geq 3u_2$, player 1 has no incentive to deviate. As for player 2, he would profit from proposing a global agreement if $Z_3 < \frac{u_3-u_2}{2}$. Given the strategies of the players, $Z_3 - \frac{u_3-u_2}{2} = \frac{3(1-p)[(3-p)u_2-(1-p)3u_3]}{2(p^2-9p+9)} \geq 0$.

b) Suppose player 1 offers the good to player 2 with probability λ and to player 3 with probability 1 - λ. The continuation values are then given by the following system of equations (including the indifference condition for player 1)

$$
\begin{aligned}
Z_1 &= \frac{p}{3}[u_3 - Z_3] + \frac{2p}{3}Z_1 \\
Z_2 &= \frac{p}{3}[\frac{u_3+u_2}{2} - Z_1] + \frac{p}{3}\lambda Z_2 \\
Z_3 &= \frac{p}{3}(u_3 - Z_1) + \frac{p}{3}(1-\lambda)Z_3 + \frac{p}{3}(1+\lambda)\frac{u_3-u_2}{2} \\
u_3 - Z_3 &= \frac{u_3+u_2}{2} - Z_2.
\end{aligned}
$$

The solution of this system is

$$
\begin{aligned}
Z_1 &= \frac{p\left[(3-2p)\,u_3 + u_2\right]}{2\,(6-5p)} \\
Z_2 &= \frac{(1-p)\left[(4p-3)\,u_3 + 3u_2\right]}{2\,(6-5p)} \\
Z_3 &= \frac{(3+2p-4p^2)\,u_3 - (3-2p)\,u_2}{2\,(6-5p)} \\
\lambda &= \frac{(9-6p)\,u_2 - (9-2p^2-6p)\,u_3}{p\left[3u_2 + (4p-3)\,u_3\right]}.
\end{aligned}
$$

Notice that λ increases with p. For $p = 1$, $\lambda = \frac{3u_2 - u_3}{3u_2 + u_3}$, thus we need $3u_2 > u_3$ to keep λ nonnegative.

The limit of the continuation values when p tends to 1 is the same as in the previous case: Z_1 tends to $\frac{u_3+u_2}{2}$, Z_2 tends to 0, and Z_3 tends to $\frac{u_3-u_2}{2}$.

For the limit case $u_2 = u_3$ (identical buyers), λ equals $\frac{1}{2}$ and $Z_2 = Z_3 = \frac{2p(1-p)u_3}{(6-5p)}$, whereas $Z_1 = \frac{p(2-p)u_3}{(6-5p)}$ (thus in the limit when p tends to 1, the seller gets all the surplus).

Player 1 is by construction indifferent between proposing to 2 and to 3. Since $Z_3 - \frac{u_3-u_2}{2} = \frac{(1-p)[(4p-3)u_3+3u_2]}{2(6-5p)} \geq 0$, player 2 has no incentive to deviate either.

Now we proceed to discard the other seven candidate equilibria.

If player 1 always offers the good to player 2, the continuation values can be found from the following system of equations

$$
\begin{aligned}
Z_1 &= \tfrac{p}{3}[u_3 - Z_3] + \tfrac{2p}{3} Z_1 \\
Z_2 &= \tfrac{p}{3}[\tfrac{u_3+u_2}{2} - Z_1] + \tfrac{p}{3} Z_2 \\
Z_3 &= \tfrac{p}{3}(u_3 - Z_1) + \tfrac{2p}{3}\tfrac{u_3-u_2}{2}.
\end{aligned}
$$

The solution to this system is $Z_1 = \frac{p\,(u_3+u_2)\,(3-2p)}{2(p^2-9p+9)}$, $Z_2 = \frac{3p\,(1-p)\,(u_3+u_2)}{2(p^2-9p+9)}$, and $Z_3 = pu_3 + \frac{5\,(u_3+u_2)}{2} - \frac{3\,(15-13p)\,(u_3+u_2)}{2(p^2-9p+9)}$. If player 1 would propose to 3 instead of to 2, he would gain $\frac{(1-p)[2p^2u_3+9(1-p)(u_3-u_2)]}{2(p^2-9p+9)} > 0$ for $p < 1$. Thus, this strategy combination cannot be an equilibrium.

There are still six candidate equilibria. Some of them can be easily discarded.

If player 1 proposes to player 2 or randomizes, this implies $Z_3 \geq \frac{u_3-u_2}{2} + Z_2$. If player 2 proposes a global agreement or randomizes, this implies $Z_3 \leq \frac{u_3-u_2}{2}$. Since $Z_2 > 0$ in any equilibrium, these two inequalities cannot be satisfied simultaneously.

If 1 proposes to 3 and 2 proposes a global agreement, the positions of players 1 and 3 become symmetric. Then $Z_3 = Z_1 = \frac{pu_3}{3-p} > \frac{u_3-u_2}{2}$, and player 2 would not be playing a best response.

The only possibility left is that 1 proposes to 2 and 2 randomizes. This implies $Z_3 = \frac{u_3-u_2}{2}$. Then player 3 gets the same payoff independently of the strategy of player 2. The system of equations describing the continuation values would be the same as in section a), with the additional constraint that $Z_3 = \frac{u_3-u_2}{2}$, and therefore would be incompatible. ■

The results can be summarized as follows: in the limit when $p \to 1$, player 1 always receives a price $\frac{u_3+u_2}{2}$, and player 3 gets the good and enjoys a consumer's surplus $\frac{u_3-u_2}{2}$. Player 2 does not receive anything, either because players 1 and 3 reach an agreement with each other, or because he resells the good at the same price that he bought it. The seller benefits from the presence of a second buyer, as he obtains a price $\frac{u_3+u_2}{2}$ instead of $\frac{u_3}{2}$, but the second buyer himself does not get anything.

Notice that the presence of player 2 in the market is relevant regardless of whether his reservation price is higher or lower than the price when he is not in the market, $\frac{u_3}{2}$.

Remark 5.2 *The presence of a second buyer always results in a higher price, regardless of whether u_2 is greater or smaller than the price when he is not in the market, $\frac{u_3}{2}$.*

The presence of player 2 always influences the price because he can resell the good to player 3 for $\frac{u_3+u_2}{2}$ (player 2 has an advantage over player 1 in a bilateral bargaining with 3 because he has a positive valuation for the good). The reservation price of player 2 becomes in practice $\frac{u_3+u_2}{2}$, which is higher than the price when he is not in the market $\frac{u_3}{2}$. Player 1 should then be able to obtain a price $\frac{u_3+u_2}{2}$, as it is the case.

Remark 5.3 *The seller benefits from the possibility of reselling, as $\frac{u_3+u_2}{2} > \max\left(u_2, \frac{u_3}{2}\right)$.*

5.2.3 The second stage with collusion between the weak buyer and the seller

This case is equivalent to the case of player 2 having bought the good from player 1 and reselling it to player 3. Because of the asymmetry between the two coalitions (coalition $\{1,2\}$ obtains a payoff equal to u_2 in the event of breakdown, whereas coalition $\{3\}$ obtains a payoff of 0) expected price is not $\frac{u_3}{2}$ but $\frac{u_3+u_2}{2}$.

5.2.4 Summing up: the partition function

We have solved the bargaining game between coalitions for all possible coalition structures. The partition function associated with this game assigns a payoff for each coalition in each coalition structure. As it results from the equilibrium of the game, the partition function is given by (we take the limit when p tends to 1)

$$\begin{array}{|lll|}
\hline
\varphi(N) & = u_3 & \\
\overline{\varphi}(13,2) & = (u_3, 0) & \\
\overline{\varphi}(12,3) & = (\frac{u_3+u_2}{2}, \frac{u_3-u_2}{2}) & \\
\overline{\varphi}(1,23) & = (\frac{u_3}{2}, \frac{u_3}{2}) & \\
\overline{\varphi}(1,2,3) & = \begin{cases} (\frac{u_3+u_2}{2}, 0, \frac{u_3-u_2}{2}) & \text{if reselling is feasible.} \\ (u_2, 0, u_3 - u_2) & \text{if reselling is not feasible and } u_2 > \frac{u_3}{2}. \\ (\frac{u_3}{2}, 0, \frac{u_3}{2}) & \text{if reselling is not feasible and } u_2 \leq \frac{u_3}{2}. \end{cases} & \\
\hline
\end{array}$$

5.3 Solving the coalition formation stage

Once we have constructed the partition function φ, we can solve the extensive form game $G(N, \varphi, \theta^E, p)$. Since φ is constant-sum, we can apply the results in the previous chapter. The following proposition is a particular case of theorem 4.3 and corollary 4.1. It applies for all variants of φ we have considered in the previous section, that is, whether reselling is feasible and whether $u_2 > \frac{u_3}{2}$ does not make any difference.

Proposition 5.3 *There is a family of stationary perfect equilibria in which at least two players randomize between two-player coalitions.*

For any equilibrium in this family, the continuation values (and also the limit expected payoffs) of the players are

$$\begin{aligned} z_1 &= \frac{(3u_3 + u_2)}{6} = \lim_{p \to 1} w_1 \\ z_2 &= \frac{u_2}{6} = \lim_{p \to 1} w_2 \\ z_3 &= \frac{(3u_3 - 2u_2)}{6} = \lim_{p \to 1} w_3 \end{aligned}$$

In the limit when $p \to 1$, each two-player coalition forms with probability $\frac{1}{3}$.

There is no other equilibrium.

Particularly appealing equilibria are perhaps the ones in which one of the players proposes to each of the other players with probability $\frac{1}{2}$. When $p \to 1$, these equilibria converge to the same strategy combination in which every player proposes to each of the other two with probability $\frac{1}{2}$.

Remark 5.4 *The size of u_2 has no influence on coalition frequencies.*

The size of u_2 determines the extent to which the presence or absence of player 2 in the market influences price, and can be considered a measure of player 2's bargaining power. This value, however, does not play a role in determining how often player 2 is part of a coalition.

Remark 5.5 *The possibility of reselling does not make a difference for the equilibrium.*

This is a particular instance of the fact that the breakdown outcome does not influence expected equilibrium payoffs.

The expected payoff for player 2 is $\frac{u_2}{6}$. If the weak buyer forms a coalition with the seller, the price of the good is $\frac{u_3+u_2}{2}$; if he forms a coalition with the other buyer, the price is $\frac{u_3}{2}$. Thus, the change in the price player 2 can induce equals $\frac{u_2}{2}$. If coalitions are allowed, player 2 captures exactly $\frac{1}{3}$ of this value.

Remark 5.6 *The possibility of forming coalitions makes a difference to the expected payoffs of the players. If the players bargain individually, the expected payoffs vector lie in the core of v (and thus player 2 must receive zero). If we take into account the possibility of forming coalitions, the expected payoffs vector does not lie in the core of v and coincides with the Shapley value and with the kernel for the grand coalition.*

This section concludes with two more remarks about how the possibility of forming coalitions influences the equilibrium price and the players' expected payoffs.

5.3.1 Feasible reselling

Remark 5.7 *The equilibrium price is lower when players are allowed to form coalitions.*

If players are not allowed to form coalitions, they have to play the bargaining game as single units; the price is then $\frac{u_3+u_2}{2}$. If players are allowed to form coalitions, the expected price of the good depends on the concrete equilibrium being considered. This price equals $\frac{u_3+u_2}{2}$ when coalition $\{1,2\}$ forms and $\frac{u_3}{2}$ when coalition $\{2,3\}$ forms. However, when coalition $\{1,3\}$ forms, the price depends on which of the two players was the proposer, and that depends on the concrete equilibrium being considered (this is related to the fact that players get a higher payoff when they are proposers). Taking this into account, we can conclude that the expected price ranges from $\frac{u_3}{2}+\frac{2u_2}{9}$ (expected price when 3 is always the proposer for coalition $\{1,3\}$) to $\frac{u_3}{2}+\frac{5\,u_2}{18}$ (expected price when 1 is always the proposer for coalition $\{1,3\}$).[3]

Remark 5.8 *The possibility of forming coalitions makes the two buyers better-off, whereas the seller is worse-off.*

Expected payoffs corresponding to the no coalition case are $\frac{u_3+u_2}{2}$ for player 1, 0 for player 2, and $\frac{u_3-u_2}{2}$ for player 3. Comparing these payoffs with the ones in proposition 5.3, we see that both buyers see their payoff increased by $\frac{u_2}{6}$.

The intuition for these results is that the seller cannot get much from the possibility of forming coalitions: if he forms a coalition with 3, there is nothing this coalition can gain from bargaining with player 2; if he forms a coalition with 2, the coalition gets the same payoff player 1 gets in the bargaining process with no coalitions.

5.3.2 Unfeasible reselling

As we have seen in the previous section, the infeasibility of reselling does not change the payoff of a two-player coalition but it affects the payoffs players get when no coalitions are formed. The presence of player 2 in the market is then irrelevant if $u_2 \leq \frac{u_3}{2}$, and leads to a price of only u_2 (instead of $\frac{u_3+u_2}{2}$) if $u_2 > \frac{u_3}{2}$.

Remark 5.9 *The weak buyer benefits from the possibility of forming coalitions even if his presence was irrelevant for the original market (i.e., even if $u_2 \leq \frac{u_3}{2}$).*[4]

Remark 5.10 *The expected equilibrium price may be higher or lower when players are allowed to form coalitions, depending on how large is u_2.*

As in the reselling case, the expected equilibrium price ranges from $\frac{u_3}{2}+\frac{2u_2}{9}$ to $\frac{u_3}{2}+\frac{5\,u_2}{18}$. For a small u_2, this price is always higher than the price when coalitions are not allowed, for high values of u_2 is always lower, and for intermediate values it depends on the concrete equilibrium considered.

[3] This may seem surprising, since proposition 5.3 states that expected payoffs do not depend on the equilibrium considered. However, expected payoffs *conditional on a certain coalition being formed* do depend on the concrete equilibrium, because a player receives a higher payoff when he is the proposer (this proposer's advantage does not vanish as p tends to 1).

[4] In the reselling case, coalition $\{1,2\}$ could be interpreted as "2 gets paid to be in the market", (as opposed to coalition $\{2,3\}$ in which "2 gets paid to be out of the market") since the payoff is the same for coalition $\{1,2\}$ and for player 1 when no coalitions are formed. That interpretation is not possible for the no-reselling case.

Remark 5.11 *The possibility of forming coalitions always makes player 2 better-off. Player 1 is better-off for small values of u_2 relative to u_3 ($u_2 < \frac{3u_3}{5}$) whereas player 3 is better-off for large values of u_2 ($u_2 > \frac{3u_3}{4}$).*

We can then conclude that the results in the no-reselling case are similar to the results in the reselling case provided that u_2 is large enough relative to u_3.

5.4 Discussion

In dynamic models of bargaining it is usually assumed that there is a force that motivates the players to reach an agreement as soon as possible: this force may be players' impatience or an exogenous probability of breakdown. We have chosen the game with breakdown probability because of its "split the difference" properties. However, an exogenous probability of breakdown is perhaps a less natural assumption than players' impatience.

A coalition in this chapter is understood as a *bargaining unit*. This means that when the weak buyer forms a coalition with either the strong buyer or the seller, they play the bargaining game together against the other player. Alternatively, we could assume that the weak buyer can get paid to play the bargaining game by the seller (or not to play it by the other buyer), but that, if he plays this game, he will do it noncooperatively (so he cannot collude with any of the other parties during the bargaining process). This restriction makes a difference if reselling is not allowed (if the weak buyer can get paid to be in the market, but he cannot collude with the seller once he is there, the price will be u_2 or $\frac{u_3}{2}$, whereas if they can act together also during the bargaining process, the price will be $\frac{u_3 + u_2}{2}$).

We have chosen a two-stage approach (coalition formation and bargaining). This two-stage approach is common in the literature: first coalitions form and then, once a coalition structure is determined, payoffs are determined by a rule (say, the Owen value in Hart and Kurz (1983)) or by the equilibrium of a game played among the coalitions (as it is the case in Bloch (1996)). In other papers (for example, Gul (1989)), coalition formation and bargaining are not separated: coalitions form and are enlarged in many steps until the grand coalition is formed. For the one-seller-two-buyer game, both approaches would yield the same results, but we have chosen to keep the decision to collude and the decision to transact over the good separate.

Gul (1989) considers a bargaining model that implements the Shapley value.[5] Agents

[5]Osborne and Rubinstein (1990) explicitly consider Gul's model applied to the one-seller-two-buyer

own valuable resources that can be combined and produce flows of utility according to a characteristic function, and may buy each other's resources. Only pairwise meetings are allowed. Since the utility of a coalition depends on how the remaining resources are partitioned, there are externalities among coalitions. The present chapter is very much related to Gul's approach. It shows that the assumption of random matching may not be so restrictive as it seems. Even though players can choose their partners, coalitional frequencies are the same as if matching was random.

We can compare the results of this strategic model with the results of the axiomatic model of Hart and Kurz (1983). The use the Owen value to assign a payoff *to each player* in each coalition structure. This yields the following payoff vectors for our game:

- $(\frac{2\,u_3+u_2}{4}, \frac{u_2}{4}, \frac{u_3-u_2}{2})$ for coalition structure $(12,3)$
- $(\frac{u_3}{2}, \frac{u_2}{4}, \frac{2\,u_3-u_2}{4})$ for coalition structure $(1,23)$
- $(\frac{2\,u_3+u_2}{4}, 0, \frac{2\,u_3-u_2}{4})$ for coalition structure $(13,2)$
- $(\frac{3\,u_3+u_2}{6}, \frac{u_2}{6}, \frac{3\,u_3-2\,u_2}{6})$ for coalition structures (123) and $(1,2,3)$.

Adding the payoffs of the players in each coalition one gets a partition function based on the Owen value. We have called this partition function φ_v^{Sh}. This partition function is identical to the one we have derived strategically except for $\overline{\varphi}_v^{Sh}(1,2,3)$, which is not equal to $\overline{\varphi}(1,2,3)$ but to (w_1, w_2, w_3).

Hart and Kurz do not present a dynamic process according to which coalition structures form, but they specify which ones will be stable, using the notion of strong equilibrium. A coalition structure is not stable if a group of players can abandon the coalitions to which they belong and organize themselves in a way that makes all of them strictly better-off. All two-player coalitions are stable in this sense in our game.

The results in this chapter are consistent with Hart and Kurz's. With regard to stability of coalition structures, the same coalitions that form in the equilibrium of the extensive form game are stable in the sense of Hart and Kurz. The equilibrium of the extensive form game is characterized by each player being indifferent between his two possible partners; this is also a characteristic of the Owen value. It can be checked that the payoffs for the players *conditional on a coalition structure* coincide (in the limit when p tends to 1) with the Owen value of that coalition structure provided that we take the equilibrium in which

problem in section 9.5 of their book.

(again in the limit when p tends to 1) each player proposes to each of the other two with probability $\frac{1}{2}$.

Consider, for example, coalition structure $(1, 23)$. Given that $\{2, 3\}$ forms, player 1's payoff is $\frac{u_3}{2}$, corresponding to Owen's value. The payoff for coalition $\{2, 3\}$ is $\frac{u_3}{2}$, divided in the following way: with probability $\frac{1}{2}$, player 2 offers to player 3 his continuation value $\frac{3u_3-2u_2}{6}$ and gets $\frac{u_2}{3}$; with probability $\frac{1}{2}$, player 3 offers to 2 his continuation value $\frac{u_2}{6}$ and gets $\frac{3u_3-u_2}{6}$. Expected payoffs coincide then with the Owen value.

We have assumed that proposers are selected randomly. We may instead assume a *rule of order* as in Chatterjee et al. (1993). As we saw in chapter 3 this bargaining procedure does not capture the competition between responders. Thus, if players bargain as single units, the price of the good is always $\frac{u_3}{2}$ regardless of the value of u_2. If reselling is not feasible, expected payoffs depend on whether $2u_2 \geq u_3$. If $u_2 < \frac{u_3}{2}$, expected payoffs are $\frac{u_3}{2}$ for players 1 and 3 and 0 for player 2. If $u_2 \geq \frac{u_3}{2}$, expected payoffs are undetermined. Take the first proposer with probability $\frac{1}{3}$. Then expected payoffs are $(\frac{u_3}{2}, \frac{2u_2-u_3}{6}, \frac{u_3}{3})$. If reselling is feasible and each player is selected with probability $\frac{1}{3}$ to be the first proposer, expected payoffs for this subgame equal $(\frac{u_3}{2}, \frac{u_2}{6}, \frac{3u_3-u_2}{6})$. Notice that, since the rule of order procedure does not reflect the competition between responders, the presence of a second buyer does not help the seller to get a higher price, even though buyers are not colluding and there is perfect information. Imperfect competition allows the buyer with a lower reservation price to get a positive payoff in this subgame. If forming coalitions is possible, the presence of a second buyer becomes valuable for the seller, since his utility for the good makes him a "better seller" who can get a higher price from the strong buyer. The equilibrium of the two-stage game is such that players 1 and 2 propose to each other, and both players benefit from the possibility of forming coalitions. Unlike in the game with random proposers, equilibrium payoffs are a function of breakdown payoffs even in the limit when p tends to 1; thus, equilibrium payoffs depend on whether $u_2 \geq \frac{u_3}{2}$ and whether resale is feasible.

Precisely because breakdown payoffs make a difference, the game with a rule of order has different equilibria in its two versions (two-stage and Gul's type). In a game of Gul's type the breakdown outcome is always the autarchy outcome. The equilibrium of this game is equivalent to the equilibrium of the second stage of the two-stage game with reselling: players 1 and 3 always propose to each other, and player 2 always proposes to player 1. If $\{1, 2\}$ forms, they sell the good to player 3 for $\frac{u_3+u_2}{2}$; otherwise the price is $\frac{u_3}{2}$. Expected payoffs are $(\frac{u_3}{2}, 0, \frac{u_3}{2})$ if $\{1, 3\}$ forms directly, and $(\frac{u_3}{2}, \frac{u_2}{2}, \frac{u_3-u_2}{2})$ if the intermediate coalition $\{1,2\}$ forms.

The one-seller-two-buyer situation was already used by von Neumann and Morgenstern (1944) to illustrate the concept of solution, later called stable set. In any stable set there are imputations where the price lies between u_2 and u_3 and imputations where it lies under u_2. An outcome under u_2 is related to the existence of a coalition between the two buyers. The results in this chapter differ from the results of von Neumann and Morgenstern. They attribute a price smaller than u_2 to a coalition of the two buyers; this is not always the case in the present paper. If coalition $\{1,3\}$ forms, the price of the good is either $\frac{3\,u_3\,+\,2\,u_2}{6}$ or $\frac{3\,u_3\,-\,u_2}{6}$; these values may be smaller than u_2 depending on the parameters. Thus the mere *possibility* of a buyers' coalition may drive the price below u_2.

The present chapter underlines the possibility that the weak buyer forms a coalition with the seller. Commenting on the book of von Neumann and Morgenstern, Stone (1948) argues that the weak buyer can never be of use to the seller. This chapter argues that the weak buyer can be valuable for the seller not only because his entry in the market creates competition, but also because the weak buyer may have a comparative advantage as a seller.

5.5 Conclusion

In a bargaining game with a seller and two buyers, we have studied the position of the "weak" buyer who can not get the good if bargaining is efficient, but who can benefit from his influence in the price by colluding with either the other buyer or the seller. We have found that the weak buyer can indeed get a positive payoff from playing this game, even though he can not capture the whole influence he has on the price.

The results illustrate once more how the characteristic function may not be an accurate representation of the actual payoff of a coalition ($v(23)$ equals zero whereas the actual payoff of coalition $\{2,3\}$ is $\frac{u_3}{2}$). They also illustrate that players who, without adding value to the game, can shift the balance of power between other players in the game, may get paid for "changing the game".

Chapter 6

Noncooperative Bargaining in Apex Games and the Kernel

An apex game is a simple n-player game with one major player (the apex player) and $n-1 \geq 3$ minor players (also called base players). A winning coalition can be formed by the apex player together with at least one of the minor players or by all the minor players together. Apex games can be interpreted as weighted majority games in which the major player has $n-2$ votes, each of the $n-1$ minor players has one vote, and $n-1$ votes are required for a majority.

Since the apex player only needs one of the minor players he can play them off against each other to obtain favorable terms. Each minor player has two options: either try to unite with the other minor players (and run the risk that one of the others yields to an advantageous offer of the apex player) or compete with them for the favor of the apex player. Apex games have received a lot of attention both in theory and in experiments. Rapoport et al. (1979) put it the following way:

> The centrality of the Apex player, which produces the conflict faced by each Base player of whether to cooperate and trust all the remaining $n-2$ Base players or to do his best, negotiating from weakness, against the Apex player, has aroused intense interest in apex games, such that they may become to n-person experimental games what the Prisoner's Dilemma has been to two-person noncooperative games.

This chapter addresses three questions concerning apex games:

1. What coalition(s) are likely to form?

2. How will the gains from cooperation be divided for each possible coalition?

3. What are the *ex ante* expected payoffs for the players?

There are very different answers in the literature to the first question. Some papers (Bennett (1983); Morelli, (1998)) predict that all minimal winning coalitions are possible, whereas others limit the possible outcomes to the coalition of all small players (Aumann and Myerson (1988); Hart and Kurz (1984)) or to coalitions consisting of the major player with a minor player (Chatterjee et al. (1993)).

As for the second question, equal division of gains seems indicated if all minor players form a coalition. If the apex player forms a coalition with a minor player, the division of gains is not so clear-cut. The answers given in the literature point either to the "egalitarian" $\frac{1}{2} : \frac{1}{2}$ split corresponding to the kernel (Davis and Maschler (1965)) or to the "proportional" (to the number of votes) $\frac{n-2}{n-1} : \frac{1}{n-1}$ split that comes from observing that a small player cannot expect more than $\frac{1}{n-1}$ if all the minor players form a coalition. The bargaining set (Aumann and Maschler (1964)) includes these two extremes and all outcomes in between.

Most of the literature has little to say about *ex ante* payoffs. They are either very extreme (as the major player receives a payoff of zero) or undetermined (when several coalitions are possible, *ex ante* expected payoffs depend on the likelihood of each coalition, and the latter is left undetermined). On the other hand, *ex ante* concepts like the Shapley value give no predictions about coalitions or division of gains. This chapter attempts to provide an answer to the three questions simultaneously.

We model bargaining in apex games using a simplified version of the noncooperative procedure with random proposers described in chapter 3. Two types of protocol are examined: the egalitarian protocol in which each player is selected to be the proposer with equal probability, and the proportional protocol, in which each player is selected with a probability proportional to his number of votes.[1] The solution concept is stationary perfect equilibrium with symmetric strategies for the minor players.

Intuitively, the apex player should benefit from a proportional protocol since he is chosen more often to be the proposer. However, we show that this is not the case: for

[1] This would be the case in a parliamentary system where the probability of a party being asked to form a government is proportional to the number of seats it holds. Baron and Ferejohn (1989) use a proportional protocol in one of their examples. Okada (1996) only considers the egalitarian protocol.

both protocols, expected equilibrium payoffs are proportional to the number of votes of the players. The reason is that equilibrium strategies change so as to compensate changes in the protocol: if the protocol selects a player to be the proposer with a higher probability, the other players make offers to him with a lower probability so that his *ex ante* expected payoff remains unchanged.

We also show that all minimal winning coalitions may form, and that the probability of a coalition being formed depends on the protocol (the coalition of all minor players being more frequent under a proportional rule). Expected payoffs conditional on a coalition of the major player and a minor player depend on the protocol: for an egalitarian protocol, the expected division is "close" to the egalitarian division (and converges to it when the number of players tends to infinity); for a proportional protocol, the expected division is proportional.

The rest of the chapter is organized as follows: section 6.1 describes the model and the results, section 6.2 reviews some of the literature, section 6.3 discusses possible extensions and section 6.4 concludes.

6.1 Bargaining with random proposers in apex games

6.1.1 The model

Apex games consist of one major player (the apex player) and $n-1$ minor players. If $N = \{1, 2, ..., n\}$ and 1 is the apex player, then $v(S) = 1$ if either $1 \in S$ and $S \backslash \{1\} \neq \varnothing$, or $S = N \backslash \{1\}$. Apex games can be interpreted as weighted majority games in which the major player has $n-2$ votes, each minor player has 1 vote, and $n-1$ votes are required to obtain a majority.[2]

Bargaining in apex games is modeled following Okada (1996), which in turn extends the model of Baron and Ferejohn (1989). The model has been described in chapter 3 for the more general case of partition function games; here we use a simplified version, adapted to apex games.

Given the underlying cooperative (apex) game (N, v), bargaining proceeds as follows:

[2] This is of course only one of the many possible vectors of weights we can assign to the players. We have chosen a so-called *homogeneous representation*, in which all minimal winning coalitions have the same number of votes.

- At every round $t = 1, 2, ...$ Nature selects a player randomly to be the proposer according to a probability vector (a *protocol*) $\theta := (\theta_i)_{i \in N}$, where $\theta_i > 0 \; \forall i \in N$ and $\sum_{i \in N} \theta_i = 1$.
- This player proposes a coalition $S \subseteq N$ to which he belongs and a division of $v(S)$, denoted by $x^S = (x_i^S)_{i \in S}$. The ith component x_i^S represents a payoff for player i in S.
- Given a proposal, the rest of players in S (called responders) accept or reject sequentially (the order does not affect the results).
- If all players in S accept, the proposal is implemented and the game ends.[3] If at least one player rejects, the game proceeds to the next period in which nature selects a new proposer (always with the same probability distribution).
- Players are risk-neutral and share a discount factor $\delta < 1$.[4]
- If a proposal x^S is accepted by all players in S at time t, each player in S receives a payoff $\delta^{t-1} x_i^S$. A player not in S remains a singleton and receives zero.

A (pure) *strategy* for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy of player i, prescribes

1. A *proposal* (S, x^S).
2. A *response function* assigning "yes" or "no" to all possible proposals of the other players.

The solution concept is *symmetric stationary subgame perfect equilibrium.* Stationarity requires that players follow the same strategy at every round t. Symmetry requires that each minor player proposes coalition $N \backslash \{1\}$ with the same probability. Notice that this concept of symmetry is rather weak, since it does not impose any restriction on the payoffs offered or on the strategy of the major player.

Concerning the probability of players being selected to be proposers, two natural protocols suggest themselves: the *egalitarian protocol* $\theta^E := (\frac{1}{n}, ..., \frac{1}{n})$, which selects each

[3] Ending the game after one coalition has been formed is a simplifying assumption that does not affect the results for apex games.

[4] Alternatively, after a proposal is rejected the game ends with probability $1 - p$.

player with the same probability, and the *proportional protocol* $\theta^P := (\frac{n-2}{2n-3}, \frac{1}{2n-3}, ..., \frac{1}{2n-3})$, which selects each player with a probability proportional to his number of votes.

We denote the noncooperative game described above $G(N, v, \theta, \delta)$. We will think of v as the characteristic function of an apex game, unless otherwise specified.

6.1.2 The equilibrium

We will make use of theorem 1 in Okada (1996). Even though the original theorem assumes the egalitarian protocol θ^E, it can be applied to any protocol θ.

Theorem 6.1 *(Okada, 1996) Consider a zero-normalized, essential ($v(N) > 0$) and superadditive[5] game (N, v). In any stationary subgame perfect equilibrium of the game $G(N, v, \theta, \delta)$, every player i in N proposes a solution of the maximization problem*

$$\max_{S,y^S} \ (v(S) - \sum_{j\in S, j\neq i} y_j^S) \tag{6.1}$$

$$\begin{aligned} s.t. \quad & i \in S \subseteq N \\ & y_j^S \geq \delta w_j \ \forall j \in S\backslash\{i\} \end{aligned}$$

where w_j is the equilibrium expected payoff of player j. Moreover, the proposal is accepted.

Proof. For every $i = 1, ..., n$, let w_i^j be player i's equilibrium expected payoff conditional on player j becoming the proposer at time 1, and let m_i be the maximum value of (6.1). We first show that $w_i^i = m_i$.

Subgame perfection implies $w_i^i \geq m_i$. In a subgame perfect equilibrium any player j must accept any proposal that gives him at least δw_j, thus player i can get at least m_i.

Can player i get more than m_i? If player i proposes (S, y^S) at round 1 with $y_i > m_i$, the proposal will be rejected (otherwise at least one responder j is getting less than δw_j and could do better by rejecting the proposal) and i will get δw_i. Since the characteristic function is superadditive we have $v(N) \geq \sum_{j\in N} w_j$. This implies that the pair (N, w), $w = (w_j)_{j\in N}$ is a feasible proposal and thus we must have $m_i \geq w_i \geq \delta w_i$, therefore $m_i \geq w_i^i$.

Since $w_i^i \geq m_i$ and $m_i \geq w_i^i$, it follows that $w_i^i = m_i$.

To prove that player i makes an acceptable proposal, we must prove $m_i > \delta w_i$. We know that $m_i \geq \delta w_i$. If $\delta w_i = m_i$, then $w_i = m_i = 0$ since $m_i \geq w_i$ and $\delta < 1$. Since $(N, \delta w)$ is a feasible proposal, $m_i \geq (1 - \delta)v(N) > 0$. ■

[5]Notice that this assumptions are slightly different to the ones we made in chapter 3.

The following corollary follows from the proof of theorem 6.1.

Corollary 6.1 *Consider a zero-normalized, essential and superadditive game* (N, v). *In any stationary subgame perfect equilibrium of the game* $G(N, v, \theta, \delta)$, *every player in* N *has a strictly positive expected payoff* ($w_i > 0 \; \forall i \in N$).

Corollary 6.1 has the following implications for simple games.[6]

Corollary 6.2 *Consider a zero-normalized, superadditive and essential simple game* (N, v). *In any stationary subgame perfect equilibrium of the game* $G(N, v, \theta, \delta)$, *all players propose winning coalitions in which each responder is pivotal.*

Proof. Proposing a losing coalition cannot be a solution of (6.1), since it would yield at most zero for the proposer and he can get at least $1 - \delta > 0$ by proposing the grand coalition and offering δw_j to each responder. On the other hand, since by corollary 6.1 $w_i > 0 \; \forall i \in N$, a proposal including a responder who is not pivotal cannot be a solution of (6.1). ■

It does *not* follow from Corollary 6.2 that only minimal winning coalitions are proposed in equilibrium. In fact, a stationary perfect equilibrium may include proposals of coalitions in which the *proposer* is not pivotal (see Section 6.3). For apex games however only minimal winning coalitions are proposed, as Corollary 6.3 shows.

Corollary 6.3 *If* (N, v) *is an apex game, in any stationary subgame perfect equilibrium of the game* $G(N, v, \theta, \delta)$ *all players propose minimal winning coalitions.*

Proof. A winning but not minimal winning coalition in an apex game must contain the apex player and at least two minor players, thus at least two players in the coalition are not pivotal (the two minor players). If such a coalition would be proposed in equilibrium, at least one responder would not be pivotal, contradicting Corollary 6.2. ■

The following proposition shows that if the protocol selects all minor players with equal probability, then all minor players have the same expected equilibrium payoff.

[6]A game (N, v) is a *simple game* if $v(\varnothing) = 0$, $v(N) = 1$, $v(S) = 0$ or 1 for all $S \subseteq N$, and $v(S) = 1$ whenever $v(T) = 1$ for some $T \subseteq S$ (see also chapter 8). A simple game is called *proper if* $v(S) = 1$ implies $v(N \backslash S) = 0$. Notice that a proper simple game is superadditive, and (if there is not player i such that $v(i) = 1$) zero-normalized and essential.

A coalition S in a simple game is called *winning* if $v(S) = 1$ and *losing* if $v(S) = 0$. A player $i \in S$ is called *pivotal* if $v(S) = 1$ and $v(S \backslash \{i\}) = 0$. If all players in S are pivotal S is called a *minimal winning coalition.*

Proposition 6.1 *Consider an apex game (N, v) and a protocol θ such that $\theta_i = \theta_j$ for all $i, j \in N\backslash\{1\}$. Let σ^* be a stationary subgame perfect equilibrium of the game $G(N, v, \theta, \delta)$ and w the associated expected payoff vector. Then $w_i = w_j$ for all $i, j \in N\backslash\{1\}$.*

Proof. Suppose we have an equilibrium σ^* such that two minor players have different expected payoffs. Consider a minor player j that receives proposals from the apex player with positive probability and a minor player i with a different expected value, thus $w_i > w_j$. Denote by θ_m the probability each minor player has of becoming a proposer ($\theta_m > 0$, $(n-1)\theta_m < 1$) and by r_i and r_j the probabilities with which i and j receive proposals. Then the expected payoffs of players i and j are given by

$$w_i = \theta_m w_i^i + r_i \delta w_i \tag{6.2}$$

$$w_j = \theta_m w_j^j + r_j \delta w_j. \tag{6.3}$$

We now note some relations between the parameters in (6.2) and (6.3).

a) We start by showing that $r_j > r_i$.

   Indeed, the apex player proposes to j, but not to i; minor players propose to either both or none of i and j. Could it be that j proposes to i more often than i to j? Suppose j propose to i with positive probability. This means that j proposes the minor player coalition, and thus that $1 - w_1 \leq 1 - w_i - \sum_{k \in N\backslash\{1,i,j\}} w_k$. But then $1 - w_1 < 1 - w_j - \sum_{k \in N\backslash\{1,i,j\}} w_k$, thus i should propose to j with probability 1.

b) It is easy to see that $w_i^i \geq w_j^j$ (both i and j get the same payoff as proposers if they propose to the apex player; if they propose the minor player coalition, player i gets more as a proposer since he has to pay less to j than j has to pay to i).

c) We now show that $w_i^i - w_j^j \leq w_i - w_j$.

   If $w_i^i \neq w_j^j$, it must be the case that $w_i^i > w_j^j$. This means that, either both players propose the minor player coalition, or player i proposes the minor player coalition and player j the coalition with the apex player. If they both propose the minor player coalition, $w_i^i - w_j^j = \delta(w_i - w_j)$, and the result follows. If player j proposes to the apex player, it must be the case that $w_1 \leq w_i + \sum_{k \in N\backslash\{1,i,j\}} w_k$. On the other hand, $w_i^i - w_j^j > w_i - w_j$ would imply $w_1 \geq w_i + \sum_{k \in N\backslash\{1,i,j\}} w_k$, and this is only compatible with j proposing to the apex player if $w_1 = w_i + \sum_{k \in N\backslash\{1,i,j\}} w_k$, but then player j gets the same payoff proposing to the apex and proposing the minor player coalition, thus we are back in the previous case with $w_i^i - w_j^j = \delta(w_i - w_j)$.

Substracting (6.3) from (6.2) and re-arranging, we get

$$w_i - w_j = \theta_m(w_i^i - w_j^j) + r_i\delta(w_i - w_j) - (r_j - r_i)\delta w_j.$$

Substituting $w_i^i - w_j^j$ by $w_i - w_j - \epsilon$ ($\epsilon \geq 0$) and re-arranging, we obtain

$$(1 - \theta_m - r_i\delta)(w_i - w_j) = -\theta_m\epsilon - (r_j - r_i)\delta w_j.$$

Since $1-\theta_m-r_i\delta > 0$, $w_i - w_j > 0$, $\epsilon \geq 0$ and $r_j - r_i > 0$, the two sides of the equation have different signs, a contradiction. ■

Notation 6.1 In the sequel we will denote equilibrium expected payoffs by w_m for a minor player and w_a for the apex player.

Notice that proposition 6.1 applies to any stationary perfect equilibria and not only to symmetric ones. Restricting ourselves to symmetric equilibria, we obtain the following corollary.

Corollary 6.4 *Consider an apex game (N, v) and a protocol θ such that $\theta_i = \theta_j$ for all $i, j \in N\backslash\{1\}$. Let σ^* be a symmetric stationary subgame perfect equilibrium of the game $G(N, v, \theta, \delta)$. Then the apex player proposes to each minor player with equal probability.*

Proof. Recall from the proof of proposition 6.1 that, if i is a minor player, his expected payoff is given by $w_i = \theta_m w_i^i + r_i\delta w_i$. If all minor players have the same expected payoff w_i, they must also have the same payoff when they are proposers, w_i^i. It then follows that they must have the same probability of being responders, r_i. ■

Symmetry of the equilibrium implies that all minor players have the same probability of *receiving a proposal* to form the minor player coalition, thus the apex player must propose to each minor player with equal probability. The requirement of symmetry eliminates the possibility of mixed-strategy equilibria where every minor player has the same probability of being the responder, but some minor players propose the minor player coalition more often than others and receive proposals from the apex player more often than others.

Proposition 6.2 describes the symmetric stationary perfect equilibrium of the game for the egalitarian protocol; proposition 6.3 describes it for the proportional protocol. Both protocols satisfy the assumptions of proposition 6.1 and corollary 6.4, thus equilibrium expected payoff must be the same for all minor players and the apex player must propose to each minor player with equal probability.

Proposition 6.2 *The unique symmetric stationary subgame perfect equilibrium of $G(N, v, \theta^E, \delta)$ is as follows*

a) When selected as a proposer, the apex player proposes to form a coalition with each of the minor players with equal probability.[7]

b) When a minor player is selected as a proposer, he randomizes between proposing a coalition with the apex player (with probability $\frac{\delta + (n-1)\,(n-3)}{\delta\,(n-1)\,(n-2)}$) and with all other minor players.

These proposals are accepted and expected payoffs are $\frac{n-2}{2n-3}$ for the apex player and $\frac{1}{2n-3}$ for each minor player.

Proof. The equilibrium strategy of the apex player follows from Corollary 6.4. Corollary 6.2 leaves three possibilities for the minor players: they may propose a coalition with the apex player, a coalition including all the minor players, or they may randomize. If they randomize, symmetry implies that all of them must use the same probability distribution.

Suppose they propose a coalition to the apex player. The continuation values are then found from the following system of equations, where w_a denotes the continuation value for the apex player and w_m denotes the continuation value for a minor player:

$$\begin{aligned} w_a &= \tfrac{1}{n}\left[1 - \delta w_m\right] + \tfrac{n-1}{n}\delta w_a \\ w_m &= \tfrac{1}{n}\left[1 - \delta w_a\right] + \tfrac{1}{n(n-1)}\delta w_m. \end{aligned}$$

The solution to this system is $w_a = \frac{(n-\delta-1)}{n(n-1)-\delta(n^2-2n+2)}$ and $w_m = \frac{(1-\delta)(n-1)}{n(n-1)-\delta(n^2-2n+2)}$. When δ is close to 1, w_a is close to 1 and w_m is close to 0, thus this strategy combination cannot be an equilibrium (a minor player would prefer to form a coalition with the other minor players and get a payoff close to 1, instead of following his prescribed strategy and get a payoff close to zero).

Suppose each minor player proposes to the rest of the minor players. Then the continuation values are found from the following system of equations:

$$\begin{aligned} w_a &= \tfrac{1}{n}\left[1 - \delta w_m\right] \\ w_m &= \tfrac{1}{n}\left[1 - (n-2)\delta w_m\right] + \tfrac{1}{n(n-1)}\delta w_m + \tfrac{n-2}{n}\delta w_m. \end{aligned}$$

[7] Recall that it suffices to describe the equilibrium strategies of the players by a probability distribution over the coalitions they propose. Proposition 6.1 implies that each responder j will accept any offer that gives him at least δw_j and will be offered exactly δw_j. This fact together with the probability distribution used by the proposers determines $(w_j)_{j\in N}$, therefore we need to specify neither the payoffs offered to the responders nor the set of proposals players accept.

The solution to this system of equations is $w_a = \frac{(n-1-\delta)}{n(n-1)-\delta}$, $w_m = \frac{(n-1)}{n(n-1)-\delta}$. Clearly, $w_a < w_m$, thus these strategies cannot constitute an equilibrium (a minor player would prefer to propose to the apex player).

Suppose a minor player proposes to the apex player with probability λ and to the other minor players with probability $1-\lambda$. The continuation values and λ are found from the following system of equations (the third one being an indifference condition for the minor players):

$$\begin{aligned} w_a &= \tfrac{1}{n}\left[1-\delta w_m\right] + \tfrac{n-1}{n}\lambda\delta w_a \\ w_m &= \tfrac{1}{n}\left[1-\lambda\delta w_a-(1-\lambda)(n-2)\delta w_m\right] + \tfrac{1}{n(n-1)}\delta w_m + \tfrac{n-2}{n}(1-\lambda)\delta w_m \\ w_a &= (n-2)w_m. \end{aligned}$$

The solution to this system of equations is $w_a = \frac{n-2}{2n-3}$, $w_m = \frac{1}{2n-3}$, $\lambda = \frac{\delta+(n-1)(n-3)}{\delta(n-1)(n-2)}$. When δ tends to 1, λ tends to $\frac{n-2}{n-1}$. ■

Proposition 6.3 *The unique symmetric stationary subgame perfect equilibrium of $G(N, v, \theta^P, \delta)$ is as follows*

a) When selected as a proposer, the apex player proposes to form a coalition with each of the minor players with equal probability.

b) When a minor player is selected as a proposer, he randomizes between proposing a coalition with the apex player (with probability $\frac{1}{n-1}$) and with all other minor players.

These proposals are accepted and expected payoffs are $\frac{n-2}{2n-3}$ for the apex player and $\frac{1}{2n-3}$ for a minor player.

Proof. It is easy to check that there is no equilibrium in which the minor players play pure strategies (see the proof of proposition 6.2).

Suppose a minor player proposes to the apex player with probability λ and to the other minor players with probability $1-\lambda$. The continuation values and λ are found from the following system of equations:

$$\begin{aligned} w_a &= \tfrac{n-2}{2n-3}\left[1-\delta w_m\right] + \tfrac{n-1}{2n-3}\lambda\delta w_a \\ w_m &= \tfrac{1}{2n-3}\left[1-\lambda\delta w_a-(1-\lambda)(n-2)\delta w_m\right] + \tfrac{n-2}{(2n-3)(n-1)}\delta w_m + \tfrac{n-2}{2n-3}(1-\lambda)\delta w_m \\ w_a &= (n-2)w_m. \end{aligned}$$

The solution to this system of equations is $w_a = \frac{n-2}{2n-3}$, $w_m = \frac{1}{2n-3}$, $\lambda = \frac{1}{n-1}$. ■

We now turn to comment on the results.

6.1.3 Expected payoffs

The proportional protocol seems more favorable to the apex player than the egalitarian protocol. However, both protocols yield the same expected payoff.

Remark 6.1 *Expected payoffs are the same for both protocols.*

The reason why payoffs are the same for both protocols is that equilibrium strategies change so as to compensate changes in the protocol: if the protocol selects a player to be a proposer more often, equilibrium strategies adjust so that he becomes a responder less often and his expected payoff remains unchanged.

Since in equilibrium a winning coalition always forms without delay, we have

$$w_a + (n-1)w_m = 1.$$

If the minor players follow a *mixed* strategy in equilibrium, this condition together with the indifference condition

$$w_a = (n-2)w_m$$

determines expected payoffs regardless of the protocol. These payoffs are such that players are indifferent between all the minimal winning coalitions they can propose, thus there is always such an equilibrium provided that the corresponding λ is indeed a probability, that is, a number between 0 and 1. If we restrict ourselves to protocols that give the same probability of being the proposer to all minor players, all values of θ_a in the (open) interval $]0, \frac{1}{2}[$ yield the same ex ante payoffs. Outside this interval, the minor players no longer randomize, there is no room for strategies to compensate changes in the protocol (one would need $\lambda < 0$ or $\lambda > 1$) and the intuitive result that a player gets a higher expected payoff if he is more often selected to be the proposer is obtained.

Remark 6.2 *Expected payoffs coincide with the kernel for the grand coalition.*

For both protocols, expected equilibrium payoffs follow from the indifference condition for a minor player together with the fact that players propose winning coalitions and there is no delay in equilibrium.

The indifference condition of the minor player, $w_a = (n-2)w_m$, implies $1 - w_a = 1 - (n-2)w_m$. Substracting w_m from both sides we get $1 - w_a - w_m = 1 - (n-1)w_m$. That is, in the language of the kernel, the surplus of the apex player against a minor player equals the surplus of a minor player against the apex player.

Because there is no delay in equilibrium (and players always propose winning coalitions), the sum of all expected payoffs equals 1, that is $w_a+(n-1)w_m = 1$. In the language of the kernel, $(w_i)_{i\in N}$ is an imputation.

Remark 6.3 *Any stationary perfect equilibrium of the game $G(N,\varphi,\theta^E,\delta)$ or $G(N,\varphi,\theta^P,\delta)$ has the same expected payoffs as the corresponding symmetric equilibrium.*

Indeed, in the proofs of proposition 6.2 and 6.3, we have excluded the possibility of an equilibrium in pure strategies. For equilibria in mixed strategies, expected payoffs are determined by the indifference condition of the minor players together with the no-delay condition.

6.1.4 The advantage of the proposer

The proposer is said to have an advantage if the payoff a player gets as proposer is higher than the payoff he gets as a responder. This advantage is common to most noncooperative bargaining games. In apex games (and indeed in most games) with random proposers, this advantage does not vanish in the limit when δ tends to 1, in contrast with the Rubinstein (1982) game and its natural extensions. The reason is that the impatience of the players is not the only source of advantage for the proposer; a second sort of advantage is the majority rule (as opposed to the unanimity rule). The proposer will offer to each responder δ times his continuation value. Since the proposer only needs to form a minimal winning coalition, he "saves" the expected payoff of players outside the minimal winning coalition. Because expected payoffs for the players are proportional to their number of votes, the proposer gets a payoff of at least one half.[8]

Remark 6.4 *The proposer has an advantage that does not completely disappear in the limit when δ tends to 1.*

6.1.5 Coalition formation

The reason why expected payoffs can be robust to changes in the protocol is that equilibrium strategies adjust in such a way that the players who become proposers more often

[8]This reasoning is not valid for games with a rule of order, since the continuation values and the expected payoffs are not closely related in this game. Indeed, the sum of the continuation values may well exceed the value of the grand coalition.

become responders less often. At first sight, the effect of a change in the protocol on the frequency of the different coalitions seems ambiguous: on the one hand, the players get to make proposals less often with a proportional protocol; on the other hand, *given that a minor player is selected to be proposer*, the minor player coalition is proposed more often. However, it is easy to prove that the net effect must be a higher frequency of the minor player coalition.

Remark 6.5 *The coalition of all minor players forms more often under a proportional protocol.*

As we have argued above, the payoff a player gets as a proposer and the payoff he gets as a responder are the same for both protocols. If the protocol changes so that a player is selected more often to be a proposer, strategies change so that he is selected less often to be a responder to keep expected ex ante payoffs unchanged. Moreover, since a player gets a higher payoff as a proposer rather than as a responder (see remark 6.4), his *total* probability of being in a coalition (that is, the sum of his probability of being a proposer and his probability of being a responder) must decrease as well.

The reasoning above implies that the apex player must be less often in a coalition under a proportional protocol, thus the coalition of all minor players must form more often. We can calculate the concrete probabilities using propositions 6.2 and 6.3.

Under an egalitarian protocol, each minor player proposes the coalition of all minor players with probability $\frac{1}{n-1}$ (in the limit when δ tends to 1). Since one of the minor players is selected to be the proposer with probability $\frac{n-1}{n}$, this implies that the coalition of all minor players forms with probability $\frac{1}{n-1}\frac{n-1}{n} = \frac{1}{n}$. A coalition of the apex player with a given minor player forms with probability $\frac{1}{n}\frac{1}{n-1} + \frac{1}{n}\frac{n-2}{n-1} = \frac{1}{n}$ (the apex player is selected with probability $\frac{1}{n}$ and proposes to each minor player with probability $\frac{1}{n-1}$; a given minor player is selected with probability $\frac{1}{n}$ and proposes to the apex player with probability $\frac{n-2}{n-1}$). Thus, each minimal winning coalition is equally likely.

Under a proportional protocol, each minor player proposes the coalition of all minor players with probability $\frac{n-2}{n-1}$. Since a minor player is selected to be the proposer with probability $\frac{n-1}{2n-3}$, this implies that the coalition of all minor players forms with probability $\frac{n-2}{2n-3}$ (approximately $\frac{1}{2}$ for large n). Analogous computations show that each coalition of the apex player with a minor player forms with probability $\frac{1}{2n-3}$.

6.1.6 Payoff division

For any of the minimal winning coalitions, the payoff division is fixed: the proposer offers the responders their continuation values (approximately equal to their share of the *total* votes) and keeps the rest (always more than half, since he needs to buy less than half of the votes). Since players get more as proposers than as responders, expected payoffs conditional on a given coalition depend on whether one is usually a proposer or a responder for that coalition.

Remark 6.6 *Expected payoffs conditional on a concrete coalition being formed are* $[\frac{1}{n-1}, \ldots, \frac{1}{n-1}]$ *for a coalition containing all minor players and depend on the protocol for a coalition containing the apex player and a minor player.*

The first part of the remark follows from the fact that expected payoffs are the same for all minor players and the equilibrium is symmetric. As for the second part, we have argued above that the apex player will be in a coalition less often under the proportional protocol. Since his ex ante expected payoff remains unchanged, this implies that *conditional on being in a coalition* his payoff must be higher. We can calculate the exact (limit) expected payoff division using the equilibrium values of λ found in propositions 6.2 and 6.3 and the coalition probabilities computed above.

For the egalitarian protocol, a coalition of the apex player with a given minor player forms with probability $\frac{1}{n}$. It is proposed by the apex player with probability $\frac{1}{n}\frac{1}{n-1}$, and by the minor player with probability $\frac{1}{n}\frac{n-2}{n-1}$. As a proposer, the apex player gets $1 - \frac{1}{2n-3}$; as a responder he gets $\frac{n-2}{2n-3}$. His expected payoff conditional on being in the coalition is then

$$\frac{\frac{1}{n}\frac{1}{n-1}[1 - \frac{1}{2n-3}] + \frac{1}{n}\frac{n-2}{n-1}\frac{n-2}{2n-3}}{\frac{1}{n}} = \frac{n(n-2)}{(n-1)(2n-3)}.$$

This value is close to $\frac{1}{2}$.

For the proportional protocol, a coalition of the apex player and a given minor player forms with probability $\frac{1}{2n-3}$. It is proposed by the apex player with probability $\frac{n-2}{2n-3}\frac{1}{n-1}$ and by the minor player with probability $\frac{1}{2n-3}\frac{1}{n-1}$. As for the egalitarian protocol, the apex player gets $1 - \frac{1}{2n-3}$ as a proposer and $\frac{n-2}{2n-3}$ as a responder. His expected payoff conditional on being in the coalition is then

$$\frac{\frac{n-2}{2n-3}\frac{1}{n-1}\left[1 - \frac{1}{2n-3}\right] + \frac{1}{2n-3}\frac{1}{n-1}\frac{n-2}{2n-3}}{\frac{1}{2n-3}} = \frac{n-2}{n-1}.$$

That is, the expected division of payoffs conditional on a coalition of the apex player and a minor player is proportional to the number of votes each of the two players has.

6.2 Related literature

This section reviews some of the theoretical and experimental literature on apex games. The theoretical literature is divided in four groups: the stable demands literature, the two-stage literature, the imperfect competition literature and the bargaining set literature.

6.2.1 Stable demands

The stable demands literature (Albers (1974), Bennett (1983), Cross (1967), Morelli (1998), see also chapter 8) predicts that any minimal winning coalition may form. Given that a coalition forms, payoff division will be proportional.[9] Since probabilities are not assigned to each minimal winning coalition, there is no prediction of *ex ante* expected payoffs.

If the situation is modeled as an extensive form game (Selten (1981), Bennett and van Damme (1991), Morelli (1998)) the equilibrium strategies are not unique: the apex player can propose to any minor player and each minor player can propose the minor players coalition or a coalition with the apex player. "Natural" assumptions on the selection of the first proposer (egalitarian or proportional protocols) and on the mixed strategies (each player plays all strategies that yield the same payoff with equal probability) do not lead to expected *ex ante* payoffs proportional to the number of votes.[10] *Ex post* payoffs are proportional to the number of votes; *ex ante* payoffs may be proportional to the number of votes (if initial probabilities and mixed strategies are chosen in an appropriate way) but need not be so.

On the other hand, given that a coalition forms the payoff division is always proportional and does not depend on the protocol or on who was the proposer. Thus, the stable demands approach makes robust predictions *ex post*, whereas the current approach makes robust predictions *ex ante*.

[9]The *rationale* for this division varies across the stable demands literature. A simple justification is the partnership condition: given two players, either each of them needs the other to get his payoff demands, or neither needs the other.

[10]Neither do they lead to the Shapley or Banzhaf values.

6.2.2 Two-stage models

By two-stage models we mean models that keep coalition formation and payoff division separated. First, the coalition structure is determined (the procedure depends on the concrete model under consideration). Once the coalition structure is determined, payoffs are bargained upon. This stage is not modelled explicitly but in "reduced form": payoffs are assumed to correspond to some axiomatic solution concept, like the Owen value (used by Hart and Kurz (1983)) or the Myerson value[11] (used by Aumann and Myerson (1988)).

The two-stage approach provides possible justifications for the $\frac{1}{2}:\frac{1}{2}$ split in the two-person minimal winning coalition. This type of coalition, however, never arises in equilibrium and the apex player always gets a zero payoff.

A reason why the kernel assigns the unintuitive $\frac{1}{2}:\frac{1}{2}$ split to a coalition of the apex player and a minor player is the assumption that, when players consider alternative coalitions, they assign to the other players the payoff they get in the *current* coalition structure. Hart and Kurz (1983) instead assume that payoffs are given by the Owen value, and that if a group of players deviates the new payoffs are given by the Owen value of the *new* coalition structure. The new coalition structure depends on the reaction of the other players. Depending on the reaction of the other players, several notions of stability can be obtained. The concept of γ-stability assumes that coalitions that are left by even one member break apart into singletons; the concept of δ-stability assumes they stick together. A coalition structure is considered to be stable if no group of players can reorganize themselves in such a way that all its members are strictly better-off.

The payoff division corresponding to a coalition of the apex player with a minor player is $(\frac{1}{2},\frac{1}{2})$ if the other minor players are together in a coalition and $(\frac{n-2}{n-1},\frac{1}{n-1})$ if they remain singletons (the kernel makes the same prediction for both structures). This coalition structure is not stable because the apex player can form a coalition (larger than minimal winning) with other minor players. The coalition of all small players is γ-stable if $n \geq 5$.

Aumann and Myerson (1988) consider a link formation game in which players are offered the opportunity to form links and payoffs are determined by the Myerson value of the resulting graph. The $\frac{1}{2}:\frac{1}{2}$ split in a two-player coalition is then justified since communication is not possible among players in different coalitions, so that the apex and the minor player are in a symmetric situation. Because of this, the big player prefers to form a coalition larger than the "minimal winning".

[11] The Myerson value is the Shapley value of a graph-restricted game: see Myerson (1977).

In equilibrium, all minor players form a coalition. The reason is that, if any of them links with the apex player, the apex player will then link with his "optimal" number of minor players, and each minor player would get less than what he would get if he linked with all the other minor players.

6.2.3 Imperfect competition

Chatterjee et al. (1993) consider a proposal-making model in which a rule of order selects the first proposer and the order in which players respond to a proposal, and the first player to reject becomes the next proposer. They predict that a coalition of the apex player and a minor player will form and split the payoff equally regardless of the number of players. Expected payoffs depend on the rule of order and the strategy of the apex player. If each player has the same probability of becoming a proposer and the apex player proposes to each of the minor players with equal probability, expected payoffs are $(\frac{1}{2}, \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)})$.

The reason for the equal-split prediction is that the game fails to reflect competition between the minor players.

Suppose the minor players propose to the apex player. A minor player who rejects an offer will propose to the apex player in the next period and get a continuation value $z_m = \delta(1 - \delta z_a)$; this payoff is the same for all minor players, and it is independent of the proposing strategy of the apex player. If the apex player rejects a proposal, he gets $z_a = \delta(1 - \delta z_m)$ no matter to which minor player he proposes. These two equations determine z_a and z_m *independently of the number of minor players.* In the present chapter, the payoff of a player who rejects a proposal depends on how often other players propose to him, so that competitive pressures are reflected in the expected payoffs of the players.[12]

6.2.4 The bargaining set literature

The Aumann-Maschler (1964) bargaining set does not predict coalition formation, but it predicts the payoff division given that a coalition forms. For the coalition of the apex player with a minor player, it predicts any payoff division between the egalitarian and the

[12]Recall that in the game considered by Chatterjee et al. (1993) expected payoffs and continuation payoffs may be very different. Consider a protocol that always appoints a given minor player i to be the proposer. This implies that the expected payoff for a minor player $j \neq i$ (w_j in the notation of this paper) equals 0, whereas his continuation payoff z_i is approximately $\frac{1}{2}$. In the game with random proposers there is a close relation between expected payoffs and continuation payoffs. Since nature selects a new proposer once a proposal is rejected, we have $z_i = \delta w_i$.

proportional division. The kernel (Davis and Maschler, 1965) of this coalition structure predicts one of the extremes (equal payoff division). The other extreme is predicted by the competitive bargaining set (Horowitz, 1973).

6.2.5 Experimental evidence

Some of the experiments on apex games are characterized by unstructured, face-to-face communication (examples are Selten and Schuster (1968) and Albers (1978)); others are based on computer-controlled, highly structured anonymous interaction (examples are Horowitz and Rapoport (1974), and the experiments reported in Rapoport et al. (1979)).[13] The qualitative results of these experiments are fairly robust. Minimal winning coalitions form almost all the time (in some of the experiments these are the only allowed coalitions); a vast majority of those are coalitions of the apex player with a minor player, while the base coalition is very rare. The payoff division in the two-player coalition supports the predictions of the (Aumann-Maschler) bargaining set. In some of the studies (Horowitz and Rapoport (1974)) it is close to the proportional division, whereas other studies find payoffs uniformly distributed in the range of the bargaining set. The exact proportional division is not usually found since the base players prefer to form the base coalition other things equal, so that the apex player has to offer its partner a premium over the proportional division. The payoff division in the base player coalition is usually egalitarian, though Selten and Schuster (1968) and Albers (1978) found some cases of unequal divisions; in Albers (1978) this is related to the competition between the apex player and a block of minor players over a distinguished minor player.[14] Payoff division also depends on the *nominal* weights of the players (see, for example, Murnighan et al. (1977), were players play three versions of the four-player apex game, differing only in the nominal weights). The results also suggest that a higher number of base players leads to more frequent two-player coalitions and more competitive payoff division. There is also

[13] One may wonder why the most recent experimental papers concerning apex games are twenty years old. A possible reason is that, even though the computer controlled experiments tried to test cooperative cooperative solution concepts like the kernel and the bargaining set, it is obvious from the experimental setup that players are really involved in a noncooperative extensive form game (Selten, 1981). A need was felt to know the Nash equilibria of the games played in the laboratory, and, since the equilibria of the noncooperative games that were played were difficult to calculate, experimenters later looked at simpler bargaining procedures, and also at simpler characteristic function games (usually no more than three players, and three-player apex games are not very interesting!).

[14] Note that minor players do not profit from forming bargaining blocks in our noncooperative game; neither they do according to *ex ante* solution concepts like the Shapley value or the kernel.

an effect of the communication structure: in face-to-face communication, the base players are more successful in cooperating with each other and the apex player gets a lower share when a two-player coalition forms.

There are many reasons why the experimental results differ from the theoretical predictions in this chapter. The extensive form games being played are different, human subjects are boundedly rational and moreover there are several empirically relevant factors that the model does not capture, like the substantial transaction costs involved in forming larger coalitions, the solidarity between base players[15] and the saliency of payoff division norms based on the nominal weights of the players.

6.3 Possible extensions?

The main result of this chapter, namely the fact that expected payoffs are proportional to the number of votes, easily extends to all simple games with one large player and $n-1$ identical small players. In this sort of games, the large player has $n-2m$ votes ($0 < 2m < n$), each of the small players has 1 vote (thus the total number of votes is $2(n-m)-1$), and $n-m$ votes are needed to win. The large player needs m minor players to form a minimal winning coalition whereas a small player needs $n-m-1$ other small players. A smaller m represents a more favorable position of the large player: $m=1$ corresponds to apex games.

It can be checked that expected equilibrium payoffs are proportional to the number of votes of the players, both for the egalitarian and the proportional protocol.

This main result does not extend to all weighted majority games. This section includes two counterexamples, one for the egalitarian protocol and one for the proportional protocol.

6.3.1 Counterexample 1: egalitarian protocol

Consider a weighted majority game in which two large players have $k \geq 2$ votes each, $2k-1$ players have one vote each and $2k$ votes are needed to obtain a majority. The number of players is then $n = 2k+1$, and the total number of votes is $4k-1$. There are two types of minimal winning coalitions: the two large players, and a large player together with k small players. Consider the egalitarian protocol and suppose expected equilibrium

[15] An attempt to model reciprocal loyalty is made in Albers (1988).

payoffs are proportional to the number of votes of the players; this implies that players will propose only minimal winning coalitions. A small player will then include one and only one of the large players in the coalition he proposes; a large player is indifferent between proposing to the other large player or to k minor players.

One may suspect that a large player cannot get a payoff proportional to his number of votes for n large enough. As n grows, a large player becomes a proposer less often, thus he must be a responder more often to keep his expected payoff equal to his proportion of votes $\frac{k}{4k-1}$. Since the proportion of the total votes a large player has is roughly constant regardless of the number of players, his probability of being a responder must be close to 1 when the number of players is large. However, there is an upper bound to his probability of being the responder since a small player proposes to only one of the large players (together with $k-1$ small players).

The expected payoff for a large player w_l must satisfy the following equation, where λ is the probability with which a large player proposes to the other large player[16]

$$w_l = \tfrac{1}{n}[1 - \delta w_l] + \tfrac{n-2}{n}\tfrac{1}{2}\delta w_l + \tfrac{1}{n}\lambda \delta w_l.$$

The maximum possible value of w_l (corresponding to $\lambda = 1$) is smaller than $\frac{k}{4k-1}$ for any $k \geq 2$!

One can prove that in equilibrium a small player will propose a coalition of himself and the two large players with positive probability and a large player receives $k-1$ times what a small player receives instead of k times. Thus, large players are underpaid and coalitions larger than minimal form.

6.3.2 Counterexample 2: proportional protocol

Consider a weighted majority game with four players: a large player with 3 votes, two medium players with 2 votes and a small player with 1 vote. 5 votes are needed to obtain a majority, and thus there are two types of minimal winning coalitions: a player with 3 votes together with one of the players with 2 votes, or the two players with 2 votes together with the player with 1 vote. Thus, the largest and the smallest player are never together in a minimal winning coalition.

Suppose the protocol is proportional and expected equilibrium payoffs are also proportional. Then the equilibrium must be such that only minimal winning coalitions form,

[16] We have chosen symmetric strategies without loss of generality.

thus the player with 3 votes must propose to one of the players with 2 votes, and the player with 1 vote to both of them. A player with 2 votes has a choice between two minimal winning coalitions: one including the largest player and the other including the smallest player. If μ is the probability that a player with 2 votes proposes to the largest player, the following equations must be satisfied:

- $\frac{3}{8} = \frac{3}{8}(1 - \delta\frac{2}{8}) + \frac{4}{8}\mu\delta\frac{3}{8}$, describing the expected payoff for the largest player.
- $\frac{1}{8} = \frac{1}{8}(1 - 2\delta\frac{2}{8}) + \frac{4}{8}(1 - \mu)\delta\frac{1}{8}$, describing the expected payoff for the smallest player.

For δ close to 1, μ needs to be close to $\frac{1}{2}$ for the largest player to receive a proportional payoff, and close to 0 for the smallest player to receive a proportional payoff, a contradiction.

The game we have just examined is not *constant-sum*, that is, it may be the case that neither a coalition nor its complement are winning. For this sort of games, the proportional payoff division may not be in the kernel; indeed, this is the case in the example. Whether expected equilibrium payoffs will be proportional with a proportional protocol for all constant-sum weighted majority games remains an open question.

6.4 Concluding remarks

This chapter considers an application of the random proposers bargaining model to apex games. Expected *ex ante* payoffs are found to be proportional to the number of votes of the players, and this result is robust to (not too extreme) changes in the protocol. The probability of a coalition being formed as well as the expected division of payoffs given that a coalition is formed depend on the protocol.

The proposer has two sources of advantage in this game: the impatience of the players and the majority (rather than unanimity) rule. In fact, a proposer always gets more than half of the total payoff regardless of the number of players. The reason is that each responder receives a payoff proportional to his share of the total votes, and, since only minimal winning coalitions form, the sum of these shares is always smaller than $\frac{1}{2}$. This is striking if we think of large apex games with a minor player selected as a proposer.

Expected payoffs proportional to the number of votes may be desirable for fairness reasons. However, the fairness argument only applies for homogeneous majority games (games in which each minimal winning coalition has the same number of votes). This chapter concerns homogeneous apex games only. If the game is not homogeneous, the result that all minor players have the same expected payoff is not so obviously attractive. Moreover, the assumption that the protocol treats all the minor players equally and the requirement that the minor players follow similar strategies are less reasonable if the game is not homogeneous.

Part II

Cooperative Solution Concepts

Chapter 7

Bargaining Sets and Coalition Formation

Aumann and Maschler (1964) introduced the first bargaining set as "an attempt to translate into mathematical formulas what people may argue when faced with a cooperative n-person game described by a characteristic function". The bargaining set answers the following question: *given that a coalition structure forms*, what will be the payoff division? The bargaining set selects the outcomes that are stable in the sense that all "threats" players have (called *objections*) can be countered (by a so-called *counterobjection*). Peleg (1967) has shown that one can find payoffs that are stable in the sense of the Aumann-Maschler bargaining set for *any* coalition structure. Thus, the bargaining set does not answer the question of which coalitions will eventually form, and can be considered a contribution to "partial equilibrium analysis" (Aumann and Drèze (1974)).[1]

After the work by Aumann and Maschler, other bargaining sets have been constructed that differ in the concepts of objection and counterobjection. These other bargaining sets may be used to predict coalition formation: since they may be empty for some coalition structures, they predict that such coalition structures will not form. Horowitz (1973) introduces a bargaining set that selects the most competitive outcomes of the Aumann-Maschler bargaining set. Bennett (1983, 1984) introduces a bargaining set defined in a different payoff space (aspirations instead of imputations). Mas-Colell (1989) uses a bargaining set in which objections and counterobjections are made by coalitions instead of by individual players. The objective of his work was not to predict coalition forma-

[1]See however Shenoy (1979) for an attempt to use the bargaining set to predict coalition structures.

tion but to prove the equivalence between the bargaining set and the set of Walrasian allocations for large economies; for this, a large bargaining set is not a drawback since it only makes the equivalence result more remarkable. Zhou (1994) introduces a modification of the Mas-Colell bargaining set that aims at predicting coalition formation and at giving the concepts of objection and counterobjection an intuitive interpretation. The Zhou bargaining set greatly improves over the Mas-Colell bargaining set in terms of the intuitive content of its definition. However, it retains some unintuitive features. In this chapter, we discuss the difficulties with the Zhou bargaining set and compare it with the other bargaining sets. We also discuss some other solution concepts, namely the kernel, the aspiration kernel, the equal gains aspirations set and the aspiration core. This chapter serves as an introduction to the next chapter, where we will discuss a new bargaining set that improves on some dimensions over the existing ones.

7.1 The Aumann-Maschler bargaining set

Aumann and Maschler (1964) discussed several possible variants of the bargaining set. We refer here to the version in Davis and Maschler (1963, 1967), called the bargaining set $\mathcal{M}_1^{(i)}$.

The bargaining set $\mathcal{M}_1^{(i)}$ is defined on the space of individually rational payoff configurations.

Definition 7.1 Let (N, v) be a game in characteristic function form. An **individually rational payoff configuration** (i.r.p.c.) is a pair (x, π) where $x := (x_i)_{i\in N}$ is a payoff vector and $\pi := \{B_1, ..., B_m\}$ a coalition structure satisfying

$$x(B_j) = v(B_j); \quad j = 1, ..., m \text{ (group rationality)} \tag{7.1}$$

$$x_i \geq v(i); \quad i = 1, ..., n \text{ (individual rationality)} \tag{7.2}$$

For the special case of $\pi = \{N\}$, individually rational payoff configurations are also called *imputations*.

The concept of payoff configuration assumes that each coalition distributes its value among its members (no payoffs are wasted) and that no individual will accept a lower payoff than what he can secure by himself.

Definition 7.2 Let (x, π) be an i.r.p.c. Let i, j be two distinct players in a coalition B of π. An **objection** of i against j in (x, π) is a pair (S, y) satisfying

$$\begin{aligned} &i \in S,\ j \neq S \\ &y(S) = v(S) \\ &y_k > x_k\ \forall\ k \in S. \end{aligned}$$

The idea behind the concept of objection is that, if i can form a coalition without j that makes all members of the coalition better-off, i can demand that j gives up part of his share of $v(B)$ in favor of i, and use the possibility of forming S as a threat.

Definition 7.3 Let (S, y) be an objection of i against j in (x, π). A **counterobjection** to this objection is a pair (T, z) satisfying

$$\begin{aligned} &j \in T, i \notin T \\ &z(T) = v(T) \\ &z_k \geq y_k\ \forall k \in S \cap T \\ &z_k \geq x_k\ \forall k \in T \backslash S. \end{aligned}$$

If player j has a counterobjection, it means that he can protect his share of $v(B)$ without i. He can form coalition T, give everybody in T at least what they got in x and, if some members of T were offered a higher payoff by i, he can match the offer.

Definition 7.4 The **bargaining set** $\mathcal{M}_1^{(i)}$ is the set of individually rational payoff configurations s.t. every objection can be countered.

Peleg (1967) shows that, given any coalition structure π, one can find a payoff vector x s.t. $(x, \pi) \in \mathcal{M}_1^{(i)}$.

Notice that the definition of counterobjection does not exclude cases in which i and j can *both* make a profit by leaving B; this would be the case if $S \cap T = \varnothing$ and $v(T) > x(T)$. Furthermore, it is possible that everybody in the game can make a profit by moving from a coalition structure that is in the bargaining set to some other coalition structure. Indeed, the coalition structure of all singletons, each receiving $v(i)$, is always in the bargaining set. No player can object to this allocation, since objections have to be directed against a player in the same coalition. These features are justified by the fact that the bargaining set is defined for a given coalition structure.

Aumann and Drèze (1974) provide the following interpretation for the bargaining set (and in general for all solution concepts that are defined for a given coalition structure): players partition themselves according to the coalition structure π, and each coalition negotiates the division of the payoff on the assumption that π will be formed. This division may be based on the opportunities players have in other coalitions. The negotiations may break down, and at no time is asserted that they will succeed. What is being asserted is that if negotiations succeed and π forms, then the payoff division should be according to the bargaining set.

Maschler (1992) emphasizes that the fact that two payoff configurations are in the bargaining set does not mean that they are equally reasonable; all what is claimed is that payoff configurations outside the bargaining set are unstable. The question arises of how to select the "more reasonable" payoff configurations, and especially of how to select the more reasonable coalition structures.

Shenoy (1979) attempts to use the bargaining set $\mathcal{M}_1^{(i)}$ as well of other concepts to predict coalition formation. He models coalition formation as an *abstract game*. An abstract game is a pair (X, dom) where X is an arbitrary set whose elements are called *outcomes* and *dom* is an arbitrary binary relation defined on X and called *domination*. Shenoy considers two abstract games; we will discuss one of them in detail. In this game, the set X is the set of coalition structures, and domination is defined as follows.

Definition 7.5 Consider two coalition structures, π_1 and π_2. Call $S(\pi) \subseteq R_+^n$ the set of possible payoffs assigned by the bargaining set (or other solution concept) to coalition structure π. We say that π_1 **dominates** π_2 **via coalition** R (denoted $\pi_1\ dom_R\ \pi_2$) if

i) $R \in \pi_1$

and

ii) $\forall y \in S(\pi_2), \exists x \in S(\pi_1)$ such that $x_i > y_i\ \forall i \in R$.

Definition 7.6 We say that π_1 **dominates** π_2, denoted π_1 dom π_2, if there exists a nonempty coalition R such that $\pi_1\ dom_R\ \pi_2$.

Shenoy considers two solution concepts for abstract games: the core and the d-solution.[2]

[2] The core is the set of undominated outcomes.

The definition of domination above is controversial. When evaluating a move from π_2 to π_1, only the incentives of coalition R are examined. It may be that other players that must reorganize themselves in order to achieve π_1 have no incentive to do so. Moreover, this reorganization of players outside R may be essential in order to make the move from π_1 to π_2 profitable for all players in R. This is due to the fact that the stable payoffs for a coalition in the sense of the bargaining set may depend on how the remaining players are organized.

The following example shows that domination may rest upon the "wrong" conjectures of players in R.

Example 7.1 $N = \{1, 2, 3, 4\}$, $v(N) = 40$, $v(S) = 50$ for all S s.t. $|S| = 2$. All other values are zero.

According to the concept of dominance above, coalition structure $(12, 3, 4)$ dominates coalition structure (1234). The bargaining set for the first coalition structure assigns a payoff of zero to players 3 and 4; this implies that any division of $v(1, 2)$ between 1 and 2 is in the bargaining set. For the grand coalition, the bargaining set contains all imputations. It is clear that, whatever the imputation one starts from, players 1 and 2 cannot get more than 40, so that they can form coalition $\{1, 2\}$ and divide 50 between themselves in such a way that they are both better-off.

The reasoning above, however, rests on the fact that 3 and 4 remain singletons, so that each of players 1 and 2 can protect any share of 50 by forming a coalition with one of players 3 and 4 and offering him zero. If 3 and 4 would also form a coalition (as it seems reasonable), the only payoff vector in the bargaining set is $(25, 25, 25, 25)$. Coalition structure $(12, 34)$ does not dominate the grand coalition[3] via $\{1, 2\}$, since there are imputations for the grand coalition that give player 1 more than 25.

The d-solution is based on the concept of accessibility. We say that $x \in X$ is accessible from $y \in X$, denoted by $x \leftarrow y$, if there exist a chain of outcomes $z_1, ..., z_m$ such that

x *dom* z_1 *dom* z_2 ... *dom* z_m *dom* y.

An *elementary dynamic solution* is then defined as a set $S \subseteq X$ such that

if $x \in S$, $y \in X \backslash S$ then $y \not\leftarrow x$

and

if x, $y \in S$, then $x \leftarrow y$ and $y \leftarrow x$.

The *d-solution* is the union of all the elementary dynamic solutions. It is never empty for a finite X.

[3]Notice, however, that coalition structure $(12, 34)$ is accessible from the grand coalition.

7.2 The kernel

The kernel was introduced by Davis and Maschler (1965) as a subset of the bargaining set that is relatively easy to compute. Since then, it has become a solution concept on its own right.

Consider a cooperative game (N, v). Assume $v(S) \geq 0 \;\; \forall S \subseteq N$ and $v(i) = 0 \; \forall i \in N$. The kernel is defined on the space of individually rational payoff configurations.

Definition 7.7 Let (x, π) be an i.r.p.c. in a game (N, v) for an arbitrary coalition structure. The **excess** of a coalition S at x is $e(S, x) := v(S) - \sum_{i \in S} x_i$.

Definition 7.8 Let (x, π) be an i.r.p.c. for a cooperative game, and let k and l be two distinct players in a coalition B_j of π. The **surplus** of k over l at x is

$$s_{k,l}(x) := \max_{\substack{k \in S, \\ l \notin S}} e(S, x).$$

Definition 7.9 Let (N, v) be a cooperative game, π a coalition structure and $X(\pi)$ the set of payoff vectors x such that (x, π) is a i.r.p.c. The **kernel** $K(\pi)$ for π is

$$\mathcal{K}(\pi) := \{x \in X(\pi) : s_{k,l}(x) > s_{l,k}(x) \Longrightarrow x_l = 0, \text{ for all } k, l \in B \in \pi, k \neq l\}. \quad (7.3)$$

The idea of the kernel is that players demand equal gains (or losses) from cooperating with each other. If player k gains more (or loses less) from stopping cooperation with l than l would gain (or lose) from stopping cooperation with k, k will demand that l gives up some payoff in his favor, unless l is already receiving the amount he can guarantee by forming a singleton (0, given the normalization above).

The kernel is nonempty for all coalition structures (Davis and Maschler (1965)).

Davis and Maschler (1965) used the apex game to illustrate the predictions of the kernel (cf chapter 6).

Example 7.2 *(Apex game).* $N = \{1, 2, ..., n\}$, $v(S) = 1$ if $1 \in S$ and $N \backslash \{1\} \neq \varnothing$ or $S = N \backslash \{1\}$, $v(S) = 0$ otherwise.

Consider the coalition structure $\{\{1,2\},\{3\},\{4\},...,\{n\}\}$. The kernel for this coalition structure predicts that 1 and 2 receive $\frac{1}{2}$ each. Since the payoff for a coalition must equal its value, all singletons must receive zero. The surplus of 1 against 2 then equals $1 - x_1$, whereas the surplus of 2 against 1 equals $1 - x_2$. Thus $x_1 = x_2$. Player 1 needs only one of the other players, whereas player 2 needs all of them; however, since all those players receive zero, this makes no difference for payoffs. This is somehow disappointing, because intuitively the apex player is stronger and it seems that he should get more than half.

For the grand coalition, the kernel predicts $(\frac{n-2}{2n-3}, \frac{1}{2n-3}, ..., \frac{1}{2n-3})$. To see this, notice that all the minor players must get the same payoff; call this payoff x_2. The surplus of the apex player against a minor player then equals $1-x_2-x_1$, whereas the surplus of a minor player against the apex player equals $1-(n-1)x_2$. The equality $1-x_2-x_1 = 1-(n-1)x_2$ together with $x_1 + (n-1)x_2 = 1$ (that is, x must be an imputation) yields the result. If we interpret apex games as weighted majority games, the kernel predicts payoffs that are proportional to the number of votes of the players.[4]

7.3 The competitive bargaining set

Horowitz (1973) introduces a new bargaining set that aims at predicting payoff division in a highly competitive environment. The starting point of this work is the definition of counterobjection for the bargaining set $\mathcal{M}_1^{(i)}$. He argues that condition $z_k \geq y_k$ for all k in $S \cap T$ is reasonable, but condition $z_k \geq x_k$ for all k in $T \backslash S$ is open to criticism.

> The bargaining set theory assumes that player k is willing to join the counterobjection for at least his previous payoff x_k, regardless of his relative power defined by the characteristic function of the game. Why should player k help player i for any return z_k, $z_k \geq x_k$, ignoring potential coalitions with player j in which he may obtain a higher payoff?

He uses the apex game to illustrate this criticism. Consider the following payoff configuration for the five-person apex game: $\{(55,45,0,0,0),[12,3,4,5]\}$. Player 1 (the apex player) has an objection against player 2 through coalition $\{1,3\}$: he can offer 40 to player 3 and keep 60 for himself. Player 2 then has a counterobjection: he could form $\{2,3,4,5\}$ with payoff division (45, 40, 7, 8). Players 4 and 5 agree to 7 and 8 units

[4]While proportional payoffs may seem only too obvious, one must take into account that neither the Shapley value nor the Banzhaf (1965) value assign proportional payoffs to the apex game.

since they got zero in the original payoff configuration. Horowitz suggests that they may instead compete against player 2 and offer 60 or more to player 1. If players 4 and 5 make an offer to player 1, these offers can be considered as threats of 1 against 2.

In order to formalize these insights, Horowitz introduces the concept of multi-objection.

Definition 7.10 Suppose i has an objection against j in (x, π). Let $S_1, ..., S_t$ be t distinct coalitions, $t \geq 1$, through which player i may object against player j, and let $y^1, ..., y^t$ be the associated payoff vectors. Then the set of objections

$$\{(S^1, y^1), ..., (S^t, y^t)\}$$

is called a **multi-objection** of player i against player j in (x, π).

By the definition of objection, a multi-objection implies that for each $g = 1, 2, ..., t$,

$$\begin{aligned}
&i \in S^g, j \notin S^g, i, j \in B \in \pi \\
&\textstyle\sum_{k \in S^g} y_k^g = v(S^g) \\
&y_k^g > x_k \text{ for all } k \in S^g.
\end{aligned}$$

In the example of the apex game above, a multi-objection could be $(\{1, k\}, (60, 40))_{k=3}^5$ The interpretation would be that player 1 is ready to offer 40 units to the first player among $\{3, 4, 5\}$ who is willing to join him.

Definition 7.11 Suppose i has a multi-objection against j in (x, π). The pair (T, z) is called a **counter-multiobjection** of player j against i if

$$\begin{aligned}
&j \in T, i \notin T \\
&\textstyle\sum_{k \in T} z_k = v(T) \\
&z_k \geq x_k \text{ for all } k \in T \\
&z_k \geq y_k^g \text{ for all } k \in (T \cap S^g).
\end{aligned}$$

In the example of the apex game, player 2 has no counter-multiobjection, since he would have to give a payoff of 40 to each of players 3, 4 and 5.

Definition 7.12 The **competitive bargaining set** is the set of individually rational payoff configurations such that every multi-objection can be countered.

The competitive bargaining set is included in the bargaining set $\mathcal{M}_1^{(i)}$. For apex games, it gives a unique stable payoff division for each coalition structure (see Horowitz (1973)).

Example 7.3 The competitive bargaining set in the apex game for a coalition structure including $\{1, i\}$.

Call the payoff division $(1-x, x)$. The "most damaging" multi-objection by player 1 is that in which he offers each other minor player almost x. Thus, in order for player i to have a counter-multiobjection, he must be able to offer each other minor player x and still keep x for himself, thus $(n-1)x \leq 1$. If $(n-1)x \leq 1$, player i has a multiobjection against the apex player in which he gives every other minor player $\frac{1-x}{n-2} \geq \frac{1}{n-1}$. In order for the apex player to have a counter-multiobjection, $\frac{1-x}{n-2} \leq x$. Thus, the only stable payoff division is $(\frac{n-2}{n-1}, \frac{1}{n-1})$, proportional to the weights of the players in the homogeneous representation.

For the five-player case, the payoff division is $(75, 25)$. If the minor player would receive more, like in $(74, 26)$, the apex player could offer each other minor player 25, and the minor player cannot counterobject. If the minor player receives less, as in $(76, 24)$, he can offer each other minor player 25, and the apex player cannot match this offer.

The competitive bargaining set may be empty for all coalition structures in some games (Horowitz (1973)).

7.4 The aspiration bargaining set

Bennett (1983, 1984) introduced a bargaining set that can predict coalition formation. This bargaining set differs from the rest in that it is not defined in the space or imputations, but in the space of *aspirations.*

Definition 7.13 A payoff vector $x \in \mathbb{R}^n$ is an **aspiration** if it satisfies the following properties

1. $x(S) \geq v(S) \ \ \forall S \subseteq N$ *(maximality).*

2. $\forall i \in N \ \exists S$ containing i such that $x(S) = v(S)$ *(feasibility).*

Unlike the space of imputations, the space of aspirations does not presuppose the formation of any coalition structure.

Aspirations are interpreted as prices players set for their participation in a coalition. Feasibility captures the idea that an agent will not set a price so large that no coalition can afford to pay it given the prices set by the other players. The idea behind the requirement of maximality is that a player will increase his price if he can do it without completely pricing himself out of the market. While feasibility is rather compelling, maximality is more debatable: by raising his price, a player may not price himself out of *all* possible coalitions, but he may price himself out of a considerable number of them. Since eventual participation in a coalition is not guaranteed, it may be better not to raise one's price (unless one is an optimist!).

Let $x \in \mathbb{R}^n$ be an aspiration. The set $GC(x) := \{S \subseteq N \mid x(S) \leq v(S),\ S \neq \varnothing\}$ is called the *generating collection* of x. This is the set of coalitions that are feasible given x. Usually there is no partition $\pi \subseteq GC(x)$, and thus not all players can simultaneously get their payoff demands.

Aspiration solution concepts select a subset of the space of aspirations. An aspiration solution concept of interest in this chapter, originally introduced by Albers (1974), is the *partnering condition.*

Definition 7.14 Let x be an aspiration vector. Let $S_i(x)$ be the set of coalitions including i that are feasible given x. We say that j and k are **partners** if

$$\begin{aligned} &\text{either } S_j(x) = S_k(x) \\ &\text{or } S_j(x) \backslash S_k(x) \neq \varnothing \text{ and } S_k(x) \backslash S_j(x) \neq \varnothing. \end{aligned}$$

Definition 7.15 An aspiration x such that any two players are partners at x is called a **partnered aspiration**.

Players j and k are partners if either they need each other's cooperation in order to obtain their payoff demands or neither of them needs the other in order to obtain their payoff demand. The spirit of this concept is close to spirit of the bargaining set $\mathcal{M}_1^{(i)}$: if a payoff vector is in $\mathcal{M}_1^{(i)}$ for some coalition structure, given two players in the same coalition, either they need each other to keep their payoffs (there are no objections) or neither of them needs the other to keep his payoff (objections can be counterobjected). This is why the set of aspirations satisfying the partnering condition was called the *aspiration bargaining set* by Bennett.[5] Formally, objections and counterobjections are defined as follows:

Definition 7.16 Player i has an **aspiration objection** against j for the aspiration x if there exists a coalition S containing both i and j with $x(S) = v(S)$ and there exist a coalition T containing i but not j with $x(T) = v(T)$.

[5] Albers (1979) calls it the d-superkernel.

Definition 7.17 Player j has an **aspiration counterobjection** against i if there exists a coalition Z containing j but not i with $x(Z) = v(Z)$.

Definition 7.18 An aspiration x is in the **aspiration bargaining set** if there are no aspiration objections without counterobjections, or, analogously, if any two players are partners.

Bennett (1983) shows that the set of partnered aspirations is nonempty.

Selten (1981) studies a noncooperative model of characteristic function bargaining. He shows that every stationary perfect equilibrium is connected to an aspiration, and that refinements of the equilibrium concept can be introduced in order to reduce the equilibrium payoffs to those satisfying the partnership condition[6] (called *stable demand vectors* by Selten).

7.5 Equal gains aspirations

The equal gains property, introduced by Bennett (1983) is based on the idea that agents may demand equal gains from forming their "best" coalition rather than their "next best" coalition.

Consider an aspiration x and a coalition S in the generating collection of x. Suppose that an agent $i \in S$ can no longer form S, and that all other agents can still form coalitions in $GC(x)$. If agent i wants to attract agents to other coalition S', he will still have to pay them their components of x. The aspiration x has the equal gains property if all agents make the same loss from forming their next best coalition.

Definition 7.19 The **gain** to agent i from forming S in $GC(x)$ over his next best alternative, $g^i(S)$, is given by

$$g^i(S) = \max\{x(S') - v(S') \mid i \in S',\, S' \neq S\}.$$

[6]The basic idea of this refinement is as follows: suppose the equilibrium is such that all coalitions that are optimal for player i to propose contain player j, but player j has an optimal coalition not including player i. Then j can "provoke" i by proposing a higher share for himself and a lower share for player i, knowing that he faces no risk of being excluded from a coalition in doing so. Player i, one the other hand, cannot do the same to player j, since j can form a coalition that excludes i. Such situations are excluded by the refinement.

Definition 7.20 The aspiration x is called an **equal gains aspiration** if for each coalition S in $GC(x)$ and for each pair of agents i and j in S, $g^i(S) = g^j(S)$.

Notice that, when calculating $g^i(S)$, player i imagines that he cannot form S, whereas the remaining players in S can still form S.

Example 7.4 $N = \{1, 2, 3, 4\}$, $v(1, 2) = v(3, 4) = 50$, $v(1, 2, 3) = 60$, $v(1, 4) = 35$.

The aspiration vector $(25, 25, 25, 25)$ is equal gains. If player 1 cannot form coalition $\{1, 2\}$, his next best coalition is $\{1, 2, 3\}$. In this coalition he has to pay 25 to both players 2 and 3. This is reasonable as far as it concerns player 3, who can get 25 by forming a coalition with 4. Player 2 is also assumed to keep his demand unchanged, but in order for 2 to keep his demand, he needs to be able to form coalition $\{1, 2\}$.

An aspiration vector can be equal gains without being partnered. This will be the case if player i needs player j in his second best coalition, while player j does not need player i.

Example 7.5 $N = \{1, 2, 3\}$, $v(1, 2) = v(1, 3) = 10$, $v(N) = 15$. This is example 3 in Bennett (1983).

The aspiration vector $(5, 5, 5)$ is equal gains but not partnered.

Bennett (1983) shows that the set of equal gains aspirations is nonempty for any game.

7.6 The aspiration kernel

The aspiration kernel (Bennett (1985)) is an extension of the kernel to the space of aspirations.

Consider an aspiration x. Suppose that an agent i can no longer form coalitions including player j, and j cannot form coalitions including i. If agent i or j wants to attract agents to other coalition S', he will still have to pay them their components of x. The aspiration x is in the aspiration kernel if i and j make the same loss by having to form a coalition that excludes the other player.

Definition 7.21 Let x be an aspiration vector and i and j be two distinct players. The **surplus** of i over j at x, $s_{i,j}(x)$, is given by

$$s_{i,j}(x) = \max\{v(S') - x(S') \mid i \in S', \, j \notin S'\}.$$

Definition 7.22 The aspiration x is in the **aspiration kernel** if for any two players i, j in N, $s_{i,j}(x) = s_{j,i}(x)$.

Remark 7.1 *The aspiration kernel is a subset of the aspiration bargaining set.*

The aspiration bargaining set requires that $s_{i,j}(x)$ and $s_{j,i}(x)$ are both zero (if no player needs the other to keep his demand) or both negative (if each player needs the other to keep his demand) but not necessarily equal.

In computing $s_{i,j}$, player i assumes that the demands of all other players remain unchanged, even if players other than i and j need to be able to form coalitions including both i and j to keep their demands.

Example 7.6 $N = \{1, 2, 3\}$, $v(2) = 1$, $v(1, 3) = 6$, $v(N) = 15$.

Consider the aspiration vector $(5, 5, 5)$. Coalition $\{1, 2, 3\}$ is the only coalition in the generating collection of this aspiration vector. If we compute $s_{1,2}$ and $s_{2,1}$, both values equal -4. Player 2 can get 1 by himself, whereas player 1 can get 1 by forming a coalition with player 3 and giving him 5. Since player 3 needs both players 1 and 2 in order to keep his demand, there is no clear reason to assume that, in case players 1 and 2 do not cooperate with each other, 3 can still keep his demand.

7.7 The aspiration core

The aspiration core, also called the set of balanced aspirations, was introduced by Cross (1967). The idea behind this concept is that, while players will try to get as much as they possibly can (thus leading to the space of aspirations) competition among coalitions for potential members will drive up the payoffs of scarce players and drive down the payoffs of the others, resulting in the selection of those aspirations which have least sum.

Example 7.7 Consider the five-player apex game with $v(1, 2) = v(1, 3) = v(1, 4) = v(1, 5) = v(2, 3, 4, 5) = v(N) = 100$.

Consider the aspiration vector $(50, 50, 50, 50, 50)$. Each player is setting his demand as high as he possibly can given the demands of the other players. However, if we assume that the environment is highly competitive, these demands are not stable. Players 2, 3, 4 and 5 would all want to form a coalition with player 1. It is reasonable to assume that competition between players 2, 3, 4 and 5 will drive the "price" of player 1 up and the prices of players 2, 3, 4 and 5 down. Consider now the aspiration vector (60, 40, 40, 40, 40). This aspiration vector has a lower sum than the original one. Nevertheless, player 1 is still in high demand, and his price will go up. This process has a stable point, namely $(75, 25, 25, 25, 25)$. Starting from this point, deviations are still profitable. However, further deviations will bring the prices of the players back towards this point. Suppose that the coalition of all small players in under consideration. Player 1 may then offer player 2 a higher payoff (say, 26) but then player 1 only demands 74 and competition between minor players will drive his payoff back to 75.

Definition 7.23 The aspiration x is a **balanced aspiration** (is in the **aspiration core**) if $x'(N) \geq x(N)$ for every aspiration x'.

Thus, the set of balanced aspirations is the solution to the following linear programming problem

$$\min \sum_{i \in N} x_i$$
$$s.t. \sum_{i \in S} x_i \geq v(S).$$

The name of balanced aspirations is due to the following result.

Theorem 7.1 *(Bennett, 1983) The generating collection of the aspiration x is a balanced collection*[7] *if and only if the aspiration x is a balanced aspiration.*

Three of the four solution concepts we have seen for aspirations (excluding the aspiration kernel, that is included in the aspiration bargaining set) cannot be ranked in terms of inclusion. One can always find aspiration vectors that satisfy two of the three stability requirements but not the third (see Bennett (1983)).

[7]The collection of coalitions G is a balanced collection if there exists a set of weights $d_S \geq 0$ for each $S \in G$, such that for every agent,

$\sum_{S \in G} \delta_i(S) d_S = 1$

where $\delta_i(S) = \begin{cases} 1 \text{ if } i \in S \\ 0 \text{ if } i \notin S. \end{cases}$

For the example of apex games, these weights are $\frac{3}{4}$ for the minor player coalition and $\frac{1}{4}$ for each apex coalition.

7.8 The Mas-Colell bargaining set

Mas-Colell introduced a new bargaining set in 1989. The objective of his work was not to predict coalition formation, but to prove the equivalence of his bargaining set and Walrasian equilibrium allocations in an atomless exchange economy. This is perhaps the reason why the definition of counterobjection for the Mas-Colell bargaining set is difficult to interpret.

Mas-Colell (1989) defined the bargaining set on the space of imputations. An imputation in this context is a payoff vector $x \in \mathbb{R}^n$ such that $\sum_{i \in N} x_i = v(N)$. Thus, Mas-Colell does not require imputations to be individually rational. One can easily extend the definition to payoff configurations (that is, payoff vectors satisfying (7.1)) to allow for coalition structures not payoff equivalent to the grand coalition. We take this approach in the definition below.

Definition 7.24 An **objection** from a coalition S against a payoff configuration (x, π) is a pair (S, y) in which $\sum_{i \in S} y_i \leq v(S)$ and $y_i > x_i$ for all $i \in S$.

Definition 7.25 Let (S, y) be an objection against a payoff configuration (x, π). A **counterobjection** from a coalition T against (S, y) is a pair (T, z) in which $\sum_{j \in T} z_j \leq v(T)$ and

$$
\begin{aligned}
z_k &> y_k \text{ for all } k \in T \cap S && (7.4)\\
z_k &> x_k \text{ for all } k \in T \backslash S. && (7.5)
\end{aligned}
$$

Definition 7.26 An objection (S, y) against (x, π) is **justified** if there is no counterobjection from any other coalition T against (S, y).

Definition 7.27 The **Mas-Colell bargaining set** is the set of imputations for which there are no justified objections.

Vohra (1991) shows that the Mas-Colell bargaining set for the grand coalition is nonempty for weakly superadditive games[8] even after adding the individual rationality requirement.

[8]A game (N, v) is weakly superadditive if $v(S \cup \{i\}) \geq v(S) + v(i)$ for all S s.t. $i \notin S$.

7.9 The Zhou bargaining set

The Zhou (1994) bargaining set is defined in the space of payoff configurations. The distinctive feature of this bargaining set is that it imposes some additional requirements in the definition of a counterobjection.

Definition 7.28 An **objection** from a coalition S against a payoff configuration (x, π) is a pair (S, y) in which $\sum_{i\in S} y_i \leq v(S)$ and $y_i > x_i$ for all $i \in S$.

Definition 7.29 Let (S, y) be an objection against a payoff configuration (x, π). A **counterobjection** from a coalition T against (S, y) is a pair (T, z) in which $\sum_{j\in T} z_j \leq v(T)$ and

$$T\backslash S \neq \varnothing,\ S\backslash T \neq \varnothing \text{ and } T \cap S \neq \varnothing. \tag{7.6}$$

$$z_k \geq x_k \text{ for all } k \in T\backslash S, \text{ and } z_l \geq y_l \text{ for all } l \in S \cap T. \tag{7.7}$$

Definition 7.30 An objection (S, y) against (x, π) is **justified** if there is no counterobjection from any other coalition T against (S, y).

Definition 7.31 The **Zhou bargaining set** of a game (N, v) is the set of payoff configurations against which no coalitions have justified objections.

The conditions in (7.6) are imposed in order for a counterobjection to have a satisfactory interpretation. It must be launched by some player not in S (thus $T\backslash S \neq \varnothing$), it should not reinforce the original objection ($S\backslash T \neq \varnothing$), and it must nullify the objection ($S \cap T \neq \varnothing$).

Notice that (7.6) implies that there cannot be any counterobjections to an objection made by the grand coalition. This implies that, for a superadditive game, only efficient coalition structures are stable in the sense of the Zhou bargaining set.

Zhou (1994) describes his bargaining set as being "almost a subset" of the Mas-Colell bargaining set. The reason why it is not a subset is that it requires weak inequalities in (7.7). Zhou points out that requiring a strong inequality would make the bargaining set empty for some games.

Example 7.8 $N = \{1, 2, 3, 4\}$, $v(S) = 4$ if $S = \{2, 3\}$, $v(S) = 3$ if $S = \{1, 2\}$ or $\{3, 4\}$, $v(S) = |S|$ otherwise.

The coalition structure $\{1, 2, 3, 4\}$ together with $(1, 1, 1, 1)$ is in the Zhou bargaining set. An objection by $\{2, 3\}$ with $(2, 2)$ can be counterobjected by $\{3, 4\}$ with $(2, 1)$. If we would require a strict inequality, this allocation would not be in the bargaining set. Thus, it is not in the Mas-Colell Bargaining set.

The Zhou bargaining set is nonempty for any TU game (see Zhou (1994)).

7.10 Intuitive and unintuitive features of the Zhou bargaining set

Despite of the intuitive content of definition 7.29, the Zhou bargaining set contains some payoff configurations that rest on unintuitive counterobjections. The next subsections provide some examples.

7.10.1 A player may be "forced" to counterobject

The Zhou bargaining set allows for counterobjections that are not in the interest of the players. Consider the following example:

Example 7.9 $N = \{1, 2, 3, 4\}$, $v(S) = 4$ if $|S| = 2$ and $v(S) = |S|$ otherwise. This game corresponds to example 2.6 in Zhou (1994). The payoff configuration $\{(1, 1, 1, 1), [1234]\}$ is in the bargaining set.

The reason why this payoff configuration is in the bargaining set is that an objection like $\{\{1, 2\}, (2, 2)\}$ can be counterobjected by a coalition like $\{2, 3\}$ with a payoff division like $(2.1, 1.9)$.

There seems to be something counterintuitive in this reasoning. While players 3 or 4 can launch a counterobjection that nullifies the objection above, they can also improve their payoffs by forming coalition $\{3, 4\}$, without nullifying the objection. Furthermore, the counterobjection implies that all players will receive 1.

It seems that this sort of situation should be excluded. Intuitively, if players 3 and 4 would have a choice between counterobjecting and not counterobjecting, they would prefer not to counterobject. Then the objection would succeed and players 3 and 4 would have the opportunity to form a two-player coalition and earn each of them more than 1. However, while this is clear in this game, in more complicated games the decision between counterobjecting or not would not be straightforward. For example, consider the following game with five players

Example 7.10 $N = \{1, 2, 3, 4, 5\}$, $v(S) = 4$ if $|S| = 2$, $v(S) = |S|$ if $|S| \geq 3$ and $v(S) = 0$ if $|S| = 1$.

Now the success of an objection of players $\{1, 2\}$ would presumably lead to the formation of another coalition of size 2, and player 3 cannot be sure of getting into that coalition.

Example 7.10 shows the difficulties of trying to let players decide whether they want to counterobject or not. Still, when the success of an objection is compatible with a Pareto improvement (as in example 7.9), it seems reasonable to assume that the objection will be successful.

An alternative way of addressing this question would be to allow the objecting set to partition itself into several subcoalitions. This would exclude situations like example 7.9 (the grand coalition would object and (7.6) implies that there are no possible counterobjections). Allowing an objecting coalition to partition itself into several subcoalitions together with requirement (7.6) would solve the problem while keeping the nonemptiness result (see Zhou (1994)). Zhou (1994) argues that it may be difficult to coordinate this type of moves.

Since the Aumann-Maschler bargaining set is never empty, it also allows for the grand coalition in example 7.9. The same can be said about the kernel. The competitive bargaining set allows for the grand coalition as well: a multiobjection of, say, 1 against 2 can always be counterobjected: whatever the payoff player 1 offers to 3 and 4, player 2 can match player 1's offer. Both the Aumann-Maschler bargaining set and the competitive bargaining set look at whether the player affected by the objection can keep his payoff without the objecting player; this is clearly the case in example 7.9. The aspiration bargaining set and the other aspiration solution concepts, on the other hand, do not allow for the grand coalition in example 7.9 (or in general, for outcomes in which a coalition could form and pay its members more than their demands), since they are restricted to the space of aspirations in the first place.

7.10.2 "Strangers" can counterobject

The Aumann-Maschler bargaining set requires counterobjections to be launched by a player who was affected by the original objection. This requirement is in agreement with the original interpretation of a counterobjection as an argument that seeks to deter the objection. The Zhou bargaining set allows counterobjections by players that are not affected by original objection. Consider the following example.

Example 7.11 *(The modified glove game).* $N = \{1,2,3,4,5,6\}, v(S) = 10$ if $S = \{i,j\}$, $i \in \{5,6\}$, $j \in \{1,2,3,4\}$, $v(S) = 8$ if $|S| = 2$ and either $S = \{5,6\}$ or $S \subset \{1,2,3,4\}$. In all other cases, $v(S) = 0$.

Only pairs of players are valuable. Moreover, pairs formed by players of different type are more valuable than pairs of players of the same type.

Payoff configurations $\{(0,0,5,5,5,5),[1,2,35,46]\}$ and $\{(0,0,4,4,6,6),[1,2,35,46]\}$ are in the Zhou bargaining set. The reason is that, for any objection by coalition $\{1,2\}$, we can find a counterobjection by $\{1,5\}$ or $\{2,5\}$. We can think of those counterobjections as being launched by player 5. However, player 5 (or any other player for that matter) was not affected by the objection.

The Aumann-Maschler bargaining set, the kernel, the competitive bargaining set, the aspiration bargaining set and the aspiration kernel also allow for the outcomes $\{(0,0,5,5,5,5),[1,2,35,46]\}$ and $\{(0,0,4,4,6,6),[1,2,35,46]\}$ (with payoff demands $(5,5,5,5,5,5)$ and $(4,4,4,4,6,6)$ in the case of the aspiration bargaining set).

The Aumann-Maschler bargaining set and the competitive bargaining set do not even allow an objection by 1 or 2 in example 7.11. The reason is that objections always have to be against some player. Analogously, the kernel only considers pair of players that are together in a coalition.

Both demand vectors $(5,5,5,5,5,5)$ and $(4,4,4,4,6,6)$ are stable in the sense of the aspiration bargaining set and the aspiration kernel. The aspiration solution concepts offer only payoff demands, not coalition structures. As for coalition structures, it is predicted that coalitions in the generating collection of the stable aspirations can form, and other cannot. Given the second vector, coalition $\{1,2\}$ can form. Given the first vector, it can not. Thus, the aspiration bargaining set allows for outcomes that could be Pareto improved. Notice that the aspiration bargaining set is designed to judge whether the demand vector is stable, and this is done by checking whether each player can keep his demand without counting on other players. It passes no judgements on the coalition structure that is formed as such. Each player demanding 5 would still be in the aspiration bargaining set (though not in the aspiration core) for arbitrarily many players of one type and only two of the other type. Such examples illustrate the fact that the concept of aspiration is a very optimistic one, since it is only required that each demand is feasible in some coalition.

7.10.3 "Scarce" players are punished

Consider the following example.

Example 7.12 The game is as in example 7.9. Payoff configuration $\{(1,3,2,2),[12,34]\}$ is in the Zhou bargaining set.

The reason is that, if a coalition like $\{1,3\}$ would object with a payoff division like $(1.5,2.5)$, then coalition $\{1,4\}$ would counterobject with a payoff division like $(1.6,2.4)$. Thus, player 1 cannot benefit from the fact that he is "scarce" (if $\{1,4\}$ was not feasible, the objection would have succeeded) .

In the language of the Aumann-Maschler bargaining set, the objection should be interpreted as made by 1 against 2; then the counterobjecting coalition should include 2 and not 1.

Intuitively, it seems that the fact that a player is "scarce" should not hurt this player.

The Aumann-Maschler bargaining set does not have this problem in example 7.12. Player 1 can object against player 3 and player 3 cannot keep his payoff. Notice however that the scarce player may be a singleton and as such he is not allowed to object.

Example 7.13 $N=\{1,2,3\}$, $v(1,2)=v(1,3)=10$, $v(2,3)=8$.

Consider the coalition structure $\pi=(1,23)$ with payoff division $(0,4,4)$. Any objection of 2 against 3 through $\{1,2\}$ can be counterobjected by 3 trough $\{1,3\}$. Player 1 as such cannot object. Thus, this payoff configuration is the Aumann-Maschler bargaining set and also in the competitive bargaining set.

We have not defined what we understand by a "scarce" player. We would like to distinguish between cases like example 7.12, in which a player is obviously "exploited" and cases like the "musical chairs" game.

Example 7.14 *(musical chairs)* $N=\{1,2,3\}$, $v(S)=1$ *if* $|S|=2$, $v(S)=0$ *otherwise.*

A reasonable outcome of this game is $\{(\frac{1}{2}, \frac{1}{2}, 0), [12, 3]\}$. Player 3 would be better-off if only coalition $\{1, 3\}$ was feasible. Nevertheless, we do not want to call player 3 a scarce player.

The aspiration approach may clarify the question of scarcity of players. Defining a scarce player as a player such that other players would be prepared to give him more than his demand, maximality of the demand vectors precludes any player being scarce. Thus, in example 7.12, the payoff vector is not an aspiration because coalitions $\{1, 3\}$ and $\{1, 4\}$ could pay the players more than their demands. Coalition $\{2, 3\}$ in example 7.13 would be justified if player 1 demands 6, which is quite reasonable.

The Zhou bargaining set inherits some problems of the Aumann-Maschler bargaining set (players may be "forced" to counterobject and "scarce" players may be punished) and it adds a new problem: "strangers" (players that are not hurt by the objection) may launch a counterobjection. On the other hand, the Zhou bargaining set allows any player that is affected by an objection to counterobject, and this is arguably a desirable property. The following example illustrates this.

7.10.4 Should any affected player be allowed to counterobject?

Consider the following example.

Example 7.15 $N = \{1, 2, 3, 4\}$, $v(123) = 3$, $v(24) = v(34) = 2$, $v(S) = 0$ in all other cases.

Consider a payoff configuration like $\{(1, 1, 1, 0), [123, 4]\}$. This payoff configuration is in the Zhou bargaining set, since an objection by $\{3, 4\}$ can be counterobjected by $\{2, 4\}$. However, it is not in the Aumann-Maschler bargaining set because the objection by $\{3, 4\}$ can be interpreted as an objection of 3 against 1. Player 1 does not have any counterobjection. On the other hand, player 2 is also affected by the objection.

Shouldn't any affected player be allowed to counterobject? It is no use for player 2 to think that the objection is not really against him but against player 1; in fact, he would prefer that 3 objects against him since in this way he can defend himself.

The aspiration vector $(1, 1, 1, 1)$ is not partnered, balanced or equal gains. However, the outcome described above is perfectly reasonable: players 2 and 3 have decided not to fight with each other for the favors of player 4. The aspiration $(0, 1.5, 1.5, 0.5)$ may be more favorable for players 2 and 3 given that they get into a coalition, but there is nothing wrong with preferring a certain payoff of 1 to an uncertain payoff of 1.5.

In the bargaining sets we have discussed, objections and counterobjections are either personal ($\mathcal{M}_1^{(i)}$, aspiration bargaining set) or through coalitions (Mas-Colell, Zhou). One could define a hybrid bargaining set: any coalition can object (thus, objections do not need to be against any particular player) and any affected player (and only those) could counterobject. While this seems desirable in view of the examples above, one can also produce examples in which personalized objections are much more reasonable, as we will see in example 8.6 in the next chapter.

7.11 Conclusion

The Aumann-Maschler bargaining set answers the question of what will be the payoff division given a certain coalition structure. After this bargaining set, other solution concepts have appeared that attempt to determine a coalition structure endogenously. Clearly, a lot of progress has been done. It is also true that more work is needed. A completely satisfactory solution concept is not likely to arise. Trying the solution concepts on concrete examples that show the weaknesses of those solution concepts contributes at the very least to a deeper understanding of those concepts.

Chapter 8

The Stable Demand Set

The aspiration approach makes more reasonable predictions than other solution concepts, at least for some games. This approach, however, has the obvious drawback that it is restricted to the space of aspirations. It would be convenient to have a solution concept that, without being restricted to the space of aspirations, would make similar predictions to the aspiration approach. The stable demand set, introduced by Morelli (1998), is such a concept. This chapter, based on Morelli and Montero (2000) studies some of properties.

The stable demand set is a bargaining set. It can be described as a "hybrid" bargaining set, since it has some elements from the traditional bargaining sets, and some elements from the aspiration bargaining set. Objections in the stable demand set are defined as usual (an objection must make all objecting players strictly better-off), whereas counterobjections are restricted to use the original demand vector.

The stable demand set contains the core and is contained in the Mas-Colell bargaining set and in the Zhou bargaining set. It may be empty for some games, but it makes intuitive predictions for others. In this chapter, we will concentrate on constant-sum homogeneous weighted majority games. These games have received a lot of attention both in theory (starting with the book of von Neumann and Morgenstern) and in experiments. Moreover, they are very relevant for applications.

The existing cooperative solution concepts defined on the space of imputations (or payoff configurations) fail to provide an adequate prediction of the outcomes of weighted majority games. The core of any constant-sum essential game is empty. Value concepts do not yield predictions of what coalition will eventually form and how payoffs will be divided. The bargaining set and the kernel answer the question of payoff distribution conditional on a coalition structure, but they do not predict what coalition will form and the set of possible payoffs is often large. On the other hand, the cooperative solution concepts defined on the space of aspirations seem more appropriate to predict coalition formation, but they are defined on the restricted domain of aspirations. The stable demand set shares the spirit and some of the predictions of those solution concepts without being restricted to the space of aspirations.

The stable demand set is never empty for constant-sum homogeneous weighted majority games. It predicts that only minimal winning coalitions will form and that each player will receive a payoff proportional to the number of votes he brings to the coalition. While this seems reasonable, no other bargaining set makes the same prediction.

The remainder of this chapter is organized as follows. Section 8.1 is devoted to the definition of the stable demand set and the discussion of its relation to other solution concepts. Section 8.2 is devoted to the characterization of the stable demand set for constant-sum weighted majority games. Section 8.3 includes two illustrative examples in which the stable demand set selects a reasonable outcome where other bargaining sets are too large. Section 8.4 compares the stable demand set and the Zhou bargaining set in the light of the problems discussed in the previous chapter. Section 8.5 concludes.

8.1 The stable demand set: definition and properties

8.1.1 Definition

Let (N, v) be a zero-normalized characteristic function game. The stable demand set (henceforth SDS) is defined on the space

$$\mathfrak{X} := \{(\alpha, \pi) \in (\mathbb{R}^n_+ \times \Pi(N)) : \sum_{j \in S} \alpha_j \leq v(S) \text{ for all } S \in \pi, |S| > 1\}.$$

Thus, a candidate element for the SDS is a pair (α, π) where $\alpha \in \mathbb{R}^n_+$ is a *demand vector* specifying what each player charges for his cooperation with other players in a coalition and π is a *coalition structure* compatible with α in the sense that coalitions of more than one player must be able to afford the demands of their members; no such restriction is imposed on singletons.

For any given pair (α, π), actual payoffs are obtained using the following *payoff assignment rule*:

$$\alpha_i^\pi = \begin{cases} \alpha_i & \text{if } i \in S \in \pi : \sum_{j \in S} \alpha_j \leq v(S) \\ 0 & \text{otherwise.} \end{cases}$$

Players receive their demands provided that they are feasible for the coalition to which they belong, otherwise they receive zero. Notice that this implies that all singletons receive zero.

Like all bargaining sets, the SDS rests on the concepts of objection and counterobjection.

Definition 8.1 An **objection** of a coalition T against (α, π) is a pair (T, y) such that

$$\begin{aligned} &\textstyle\sum_{i \in T} y_i \leq v(T) \\ &y_i > \alpha_i^\pi \text{ for all } i \in T. \end{aligned}$$

Notice that objections are against actual payoffs, not against demands; thus, an objecting player may receive less than his demand in the objection.

Definition 8.2 Let (T, y) be an objection against (α, π). A coalition Z has a **counterobjection** against (T, y) if

$$\begin{aligned} &T \cap Z \neq \varnothing \\ &\textstyle\sum_{i \in Z} \alpha_i \leq v(Z) \\ &\alpha_i > y_i \quad \forall i \in T \cap Z. \end{aligned}$$

Thus, counterobjections in the SDS must use the *original* demand vector. We also require the inequality within $T \cap Z$ to be strict.

Definition 8.3 An objection against (α, π) is **acceptable** if it cannot be counterobjected.

Definition 8.4 A pair (α, π) belongs to the **SDS** if there is no acceptable objection to it.

A counterobjection in the SDS is restricted to use the original demand vector, while at the same time having a nonempty intersection with the objecting coalition and making players in the intersection better-off. Therefore, in order for a counterobjection to exist, it must be the case that at least one player in the objecting coalition S was receiving less than his demand. Thus

Remark 8.1 *Objections that use the demand vector α cannot be countered.*

8.1.2 Relation with other solution concepts

Aspiration solution concepts

The SDS is obviously related to the aspiration approach to coalition formation. However, the payoff demands in the SDS are *not* always aspirations: they satisfy maximality but they do not necessarily satisfy feasibility.

Lemma 8.1 *If (α, π) is in the SDS, then α is maximal but not necessarily feasible.*

Proof. If there was a pair (α, π) in the SDS and $S \subseteq N$ such that $\sum_{i \in S} \alpha_i < v(S)$, S itself would have an objection in which each player would receive more than his demand.

An example that shows that demands do not have to be feasible is a game with three players, where $v(1,2) = 6$, $v(i) = 0$, $v(S) = 1$ for every other S: in this case the vector $(3, 3, x)$ (together with the coalition structure $(12, 3)$) is in the SDS, for every x. ■

Even in games where the SDS contains only aspirations, it may contain aspirations that are not stable according to the aspiration solution concepts. Consider the following examples:

Example 8.1 $N = \{1, 2, 3\}$, $v(1,2) = v(1,3) = 10$, $v(1,2,3) = 15$. All other values are zero.

The demand vector $(5, 5, 5)$ together with the grand coalition is in the SDS. This demand vector is an aspiration, but it does not satisfy the partnership condition (player 2 needs player 1 in order to obtain his demand, but player 1 does not need player 2) and it is not equal gains or in the aspiration kernel (player 1 can maintain his demand without coalition N or without player 2, but player 2 can not).

Example 8.2 Consider the set of players $N = \{1, 2, 3, 4, 5, 6\}$ with

$$v(i,j) = 10 \text{ if } i \in \{5,6\} \text{ and } j \notin \{5,6\},\ v(i,j) = 8 \text{ otherwise},\ v(S) = 0 \text{ for } |S| \neq 2.$$

This is the modified glove game from the previous chapter. The only balanced aspiration vector is $(4, 4, 4, 4, 6, 6)$, but the SDS also contains $(5, 5, 5, 5, 5, 5)$ together with a coalition structure like $[1, 2, 35, 46]$.

It is easy to see that the reverse is also true: there are stable aspirations that are not stable demand vectors in the sense of the SDS. Consider the following game, proposed by Maschler (see Aumann and Drèze (1974)):

Example 8.3 $N = \{1, 2, 3\}$, $v(i) = 0$, $v(S) = 60$ if $|S| = 2$, $v(N) = 72$.

The aspiration vector $(30, 30, 30)$ is partnered, equal gains, balanced and it is in the aspiration kernel. It is not in the SDS because an objection by the grand coalition with a payoff division like $(31, 31, 20)$ cannot be countered. The SDS is empty in this game.

Maximality of the demand vectors together with the fact that the game is zero normalized implies that the SDS lies in the space of payoff configurations in terms of *realized outcomes.*

Remark 8.2 *If (α, π) is in the SDS, then (α^π, π) is a payoff configuration.*

The core

The fact that the SDS lies in the space of payoff configurations in terms of realized outcomes makes it comparable with other solution concepts that lie in the same space, like the core or most bargaining sets. Like all bargaining sets, the SDS contains the (coalition structure) core. This follows because the core is the set of allocations to which there are no objections.

Remark 8.3 *If (α, π) is in the SDS and $\alpha_i^\pi = \alpha_i$ for all i, then α is in the (coalition structure) core. In particular, if (α, N) is in the SDS, α is in the core.*

It follows from remark 8.3 that the grand coalition can never be stable in the sense of the SDS if the core of (N, v) is empty.

The SDS is a strict superset of the core, and as such contains some dominated payoff vectors. However, if (α, π) is in the SDS, then the induced payoff vector α^π cannot be dominated via a subset of a coalition in π or (if the game is cohesive) via the union of several elements of π. The first property is called *coalitional rationality* (see Aumann and Maschler (1964)); the second property can be called *union rationality.*

Proposition 8.1 *If (α, π) belongs to the SDS, then*

$$\sum_{i \in T} \alpha_i^\pi \geq v(T) \textit{ for all } T \subseteq S, \ S \in \pi. \qquad \text{(coalitional rationality)}$$

Moreover, if the game is cohesive

$$\nexists T : v(T) > \sum_{i \in T} \alpha_i^\pi \textit{ and } T \textit{ is the union of elements of } \pi. \qquad \text{(union rationality)}$$

Proof. The first part of the proposition follows trivially if T is a singleton; for larger sets, it follows because $\alpha_i^\pi = \alpha_i$ for all i in T, and we know that α satisfies maximality.

As for the second part, suppose $\exists T : v(T) > \sum_{i \in T} \alpha_i^\pi$ and T is the union of elements of π. If (α, π) is such that all players in T are receiving their demands, T itself has an acceptable objection. If (α, π) is such that not all players in T are receiving their demands, call Z the set of players who are receiving their demands. It follows immediately from coalitional rationality that $v(T) > \sum_{i \in T \cap Z} \alpha_i$. Furthermore, since the game is cohesive and T is the union of elements of π, $v(N) > \sum_{i \in Z} \alpha_i$. Consider an objection by the grand coalition that gives all players in Z more than their demands. A counterobjecting coalition C cannot include any players from Z. Thus, if a counterobjecting coalition (perhaps T itself) exists, this counterobjecting coalition itself has an acceptable objection. ■

Indeed, a reason why the SDS may be empty is that there are games in which coalitional rationality and union rationality cannot be achieved simultaneously. Consider example 8.3: coalitional rationality requires a two-player coalition to form, whereas union rationality requires the grand coalition to form, and thus the SDS is empty.

Example 8.2 shows that coalition structures that are not union rational can occur if the game is not cohesive and not all players receive their demands: the resulting payoff vector is dominated by coalition $\{1, 2\}$, and this coalition is the union of two elements of the coalition structure π.

Corollary 8.1 *If (N, v) is such that the grand coalition is the only efficient coalition structure, only the grand coalition can be stable in the sense of the SDS.*

Corollary 8.2 *If (N, v) is such that the grand coalition is the only efficient coalition structure, the SDS will be empty if the core is empty.*

Thus, the SDS does not improve over the core in games where the grand coalition is the only efficient coalition structure. The SDS is a useful solution concept for games that do not have this property, like quota games or simple games.

The Mas-Colell bargaining set

The SDS is a subset of the Mas-Colell bargaining set.

Proposition 8.2 *If (α, π) is in the SDS, then (α^π, π) is in the Mas-Colell bargaining set.*

Proof. The definition of objection in the SDS is equivalent to that in Mas-Colell (1989). The definition of counterobjection in the SDS is more demanding, since it requires objecting and counterobjecting coalition to have a nonempty intersection and the counterobjection must use the original demand vector. On the other hand, the Mas-Colell bargaining set requires strict inequality for all members of the counterobjecting coalition, whereas the SDS only requires strict inequality for the members in the intersection. Since payoff is transferable, this makes no practical difference. ■

The Zhou bargaining set

We now turn to the Zhou bargaining set. The SDS and the Zhou bargaining set share the same definition of objection. As for the definition of counterobjection, they cannot be directly ranked. The SDS makes counterobjecting more difficult since it requires counterobjections to use the original demand vector and some of the counterobjecting players to be better-off; on the other hand, it seems to make counterobjecting easier since it does not impose the requirements $Z \backslash T \neq \varnothing$ and $T \backslash Z \neq \varnothing$. However, we will show that the SDS is included in the Zhou bargaining set.

Proposition 8.3 *If (α, π) is in the SDS, then (α^π, π) is in the Zhou bargaining set.*

Proof. Suppose (α, π) is in the SDS but not in the Zhou bargaining set. Then, there must be a coalition T that has an objection y such that any counterobjecting coalition Z satisfies either $Z \subset T$ or $T \subset Z$.

Consider first the case in which T has an objection and $Z \subset T$ has a counterobjection. Then it must be the case that $\alpha_i > y_i > \alpha_i^\pi$ for all $i \in Z$, thus Z itself has an acceptable objection and (α, π) cannot be in the SDS.

Consider now the case in which T has an objection that can *only* be countered by supersets of T. Let Z be one of those supersets. Since Z is a superset of T, it follows that $\alpha_i > y_i$ for all i in T. Maximality together with the fact that the demand vector α is feasible for Z implies $\sum_{i \in Z} \alpha_i = v(Z)$. Z itself has an objection in which one of the players in T (say, j) receives less than his demand and each other player in Z receives more than his demand. In order for (α, π) to be in the SDS, there must be a counterobjection to this objection. Let Z' be one of the coalitions that have a counterobjection to this objection. Z' must contain player j and cannot contain any other player in T. But then Z' itself has a counterobjection to the original objection by coalition T, contradicting the assumption that this objection can only be countered by supersets[1] of T. ■

8.2 Characterization for constant-sum homogeneous weighted majority games

The predictions of the stable demand set for constant-sum homogeneous weighted majority games greatly improve over those of other solution concepts in terms of sharpness and intuitive content. The stable demand set predicts that only minimal winning coalitions will form and that the payoff division inside a coalition will be proportional to the vector of homogeneous weights. This section contains the proportionality results and some examples.

8.2.1 Definitions

Definition 8.5 A characteristic function game (N, v) is a **simple game** if

i) $v(\varnothing) = 0, \; v(N) = 1, \; v(S) = 0$ or 1.

ii) $v(S) = 1$ whenever $v(T) = 1$ for some $T \subseteq S$.

Denote by $\Omega \equiv \{S : v(S) = 1\}$ the set of *winning coalitions* (WC) and by $\Omega^m \equiv \{S : v(S) = 1, v(T) = 0 \; \forall T \subset S\}$ the set of *minimal winning coalitions* (MWC).

Definition 8.6 A simple game is called **proper** if $S \in \Omega$ implies $N \backslash S \notin \Omega$.

Definition 8.7 A simple game is called **strong** if $S \notin \Omega$ implies $N \backslash S \in \Omega$.

Definition 8.8 A simple game is called **constant-sum**[2] if it is proper and strong.

Definition 8.9 A simple game is a **weighted majority game** if there exists a vector of non-negative weights w and a number q called **quota** $(0 < q \leq \sum_{i=1}^{n} w_i)$ such that

$$S \in \Omega \iff \sum_{i \in S} w_i \geq q.$$

[1] Z' can only be a superset of T if T is a singleton. Notice that since the SDS only contains individually rational payoff configurations, any objecting set T must have at least two players.

[2] The reason for this terminology is that $v(S) + v(N \backslash S) = 1$ for all S.

A convenient notation for a weighted majority game is $[q; w_1, ..., w_n]$.

Definition 8.10 A weighted majority game admits a **homogeneous representation** if there exists a vector $w \geq 0$ (and an induced quota q) such that

$$\sum_{i \in S} w_i = q \; \forall S \in \Omega^m. \tag{8.1}$$

A weighted majority game in homogeneous representation is called a *homogeneous weighted majority game.*

Example 8.4 *(apex game)* $N = \{1,2,3,4\}$, $\Omega^m = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$.

Suppose player 1 holds 3 votes, player 2 holds 2, and players 3, 4 hold 1; this is a weighted majority representation of the simple game above. This representation is not homogeneous: there is one MWC with 5 votes, and three MWCs with 4 votes. An equivalent homogeneous representation of the same characteristic function game is one where player 1 has 2 votes, and players 2, 3 and 4 have 1 vote each. Each MWC has now 3 votes.

8.2.2 The proportionality result

For a given game (N, v), denote by M^i the set of MWCs containing player i, and the number of elements in M^i by m^i.

Lemma 8.2 *Consider a constant-sum homogeneous weighted majority game. For every player i, either $m^i = 0$ or $m^i \geq 2$.*

Proof. Suppose that for some i there exists a MWC $S \in M^i$. Since the game is strong, the coalition $T \equiv \{N \setminus S\} \cup \{i\}$ must be a winning coalition. Either T itself is a MWC, or there must exist $Z \subset T$ such that $Z \in \Omega^m$, and such a coalition Z must contain i, otherwise S would not have been winning in the first place; in either case, $m^i \geq 2$. ∎

Corollary 8.3 *For any MWC S containing player i, we can find another winning coalition S' such that $S \cap S' = \{i\}$.*

The results in the previous section have the following implication for the selection of coalition structures in simple games:

Lemma 8.3 *The only candidate pairs (α, π) for the SDS of any simple game are those where*

(i) π always includes a WC $S \in \Omega$.

(ii) For any $T \in \Omega$, $\sum_{i \in T} \alpha_i \geq v(T)$.

Proof. Part (1) follows because of union rationality for cohesive games (proposition 8.1); Part (2) follows because of maximality (lemma 8.1). ■

Corollary 8.4 *Let (α, π) be in the SDS of a simple game (N, v), and let i be a dummy player. Then $\alpha_i^\pi = 0$.*

We can now characterize the SDS of constant-sum homogeneous weighted majority games.

Theorem 8.1 *Consider a constant-sum homogeneous weighted majority game (N, v).*

(i) If there are no dummy players, the SDS is non-empty *and* only *contains pairs (α^*, π) where the* unique *stable demand vector α^* has*

$$\alpha_i^* = \frac{w_i}{q}, \quad \forall i \in N$$

and π always contains a coalition $S \in \Omega^m$.

(ii) If there are dummy players, the SDS is nonempty and only contains pairs (α^*, π) where $\alpha_i^* = \frac{w_i}{q}$ for all nondummy players and π contains a winning coalition $S \in \Omega$.

Proof. We first provide the proof for the case in which there are no dummy players.

Consider a pair where there is a MWC $S \in \pi$ and the demand vector is α^*. We first show that (α^*, π) is in the SDS of the game.

Since (N, v) is proper, any blocking coalition T must contain some agents in common with S (i.e., $T \cap S \neq \varnothing$). T can make an objection to (α^*, π) only if there exists a payoff vector y feasible for T such that $y_i > \frac{w_i}{q}$ $\forall i \in T \cap S$.

Define $Z := \{i \in T : y_i \geq \frac{w_i}{q}\}$. Because $Z \supset T \cap S$ and all agents in $T \cap S$ must receive more than $\frac{w_i}{q}$, it follows that Z is a losing coalition (Z winning would be unfeasible). Thus, there must be a WC $C \subseteq N \backslash Z$, and a MWC $C' \subseteq C$. Since the game is homogeneous, α^* is feasible for any MWC, thus coalition C' has a counterobjection: since the game is proper, $C' \cap T \neq \varnothing$. Moreover, the fact that C' is included in $N \backslash Z$ ensures that all players in $C' \cap T$ are strictly better-off.

We have shown that no objection to (α^*, π) can be acceptable, because every objection would lead to a counterobjection using α^* once again. In order to complete the proof, we now have to show that any pair (α, π) with $\alpha \neq \alpha^*$ (and $\alpha_i \neq \alpha_i^*$ for at least one nondummy player i) is vulnerable to acceptable objections.

Suppose (α, π) is in the SDS but $\alpha \neq \alpha^*$. We define the following sets:
$U := \{i \in N : \alpha_i < \alpha_i^*\}$, the set of "underdemanding" players.
$F := \{i \in N : \alpha_i = \alpha_i^*\}$, the set of players demanding exactly α^*.
$O := \{i \in N : \alpha_i > \alpha_i^*\}$, the set of "overdemanding" players.
There are two possibilities:
Case 1. $U = \varnothing$.
Case 2. $U \neq \varnothing$.

1. Since $U = \varnothing$ and $\alpha \neq \alpha^*$, it must be the case that $\alpha_i \geq \alpha_i^*$ for all i and $\alpha_j > \alpha_j^*$ for some j. We know from lemma 8.3 that π contains a winning coalition S such that $\sum_{i \in S} \alpha_i = v(S)$. Since $U = \varnothing$, coalitions containing overdemanding players are unfeasible, thus S must be such that $\alpha_i = \alpha_i^*$ for all $i \in S$, therefore $j \notin S$. Any MWC $T \ni j$ has an acceptable objection. In this objection, any player $i \in T \cap F$ receives $\alpha_i^* + \epsilon$ and any player i in $T \cap O$ receives some positive payoff. Notice that $j \in T$ and T being a MWC ensures that players in $T \cap F$ have less than q votes, so that this payoff distribution is feasible. Since $S \cap O = \varnothing$, any positive payoff makes the players in $T \cap O$ better-off, thus we indeed have an objection. There is no possible counterobjection, since it would have to include some players in $T \cap O$, and any coalition containing overdemanding players is unfeasible.

2. $U \neq \varnothing$ implies $O \neq \varnothing$ (otherwise we would contradict maximality). Again, we know from lemma 8.3 that π contains a winning coalition S such that $\sum_{i \in S} \alpha_i = v(S)$. We will distinguish two cases:

   Case 2(a). $S \subseteq F$.

   Case 2(b). $S \nsubseteq F$.

   (a) $S \subseteq F$. Take any player $j \in U$, and any MWC $T \ni j$. Denote $T \cup U$ by Z. Coalition Z has an acceptable objection in which every member of $Z \backslash O$ gets $\alpha_i^* + \epsilon$ and every member of $Z \cap O$ gets a positive payoff. To see this, notice that this payoff division is feasible for Z if ϵ is small enough.[3] $S \subseteq F$ implies that all players in Z are made better-off by the objection. Finally, since a counterobjection must include at least one player from $Z \cap O$, and it can include no players from U, it follows that no counterobjecting coalition is feasible given α.

(b) $S \nsubseteq F$. Lemma 8.3 implies $S \cap O \neq \varnothing$ and $S \cap U \neq \varnothing$. Moreover, there is a MWC $S' \subseteq S$ such that $S' \cap O \neq \varnothing$, $S' \cap U \neq \varnothing$ and $S \cap O \subseteq S'$. Take a player $i \in S' \cap U$. Corollary 8.3 implies that there exists another MWC $T \ni i$ such that $T \cap S' = \{i\}$. We can then repeat the argument above: coalition $Z := T \cup U$ has an objection with the payoff distribution we have described. The only reason why case 2b) differs from case 2a) is that, in constructing Z, we had to make sure that $Z \cap (S \cap O) = \varnothing$.

If there are dummy players, the reasoning we made above for games without dummy players can be replicated to conclude that any stable demand vector must assign a demand $\frac{w_i}{q}$ for all i with $m^i \geq 2$. As for the dummy players, their payoffs are always zero (corollary 8.4) but their demands are not restricted: a pair (α, π) in the SDS may well be such that $\alpha_i > 0$ for a dummy player i. We cannot construct an acceptable objection the way we did in case 1), because no winning coalition containing i can pay each of the other players $\frac{w_i}{q} + \epsilon$. A dummy player demanding zero may be part of a winning coalition S in π. ■

Remark 8.4 *Notice that the definition of the SDS depends only on the characteristic function, and hence it is invariant to the particular representation chosen.*

Thus, if a game admits a homogeneous representation, theorem 8.1 applies (with the understanding that $(w_i)_{i \in N}$ denotes the weights corresponding to an equivalent homogeneous representation[4]) regardless of whether the game is actually in homogeneous representation or not.

8.2.3 An example: the apex game

Consider the five-player apex game, represented by $[4; 3, 1, 1, 1, 1]$. Notice that there are two sorts of MWCs, namely:

[3]This follows because $Z \backslash O$ is a losing coalition. If it were winning, one could find a subset of $Z \backslash O$ having exactly q votes, and containing at least one player from U (because of the way Z has been constructed, $Z \backslash U$ is losing), contradicting maximality.

[4]Homogeneous representations are unique up to multiplication by a constant (see von Neumann and Morgenstern (1944)); thus, the vector α^* is unique for nondummy players.

- Apex coalitions: $\{1, i\}$.
- Minor player coalition: $\{2, 3, 4, 5\}$.

The SDS predicts that if an apex coalition forms then player 1 receives $\frac{3}{4}$, and the other player gets $\frac{1}{4}$ (proportional payoffs); if the minor player coalition forms they share equally ($\frac{1}{4}$ each).

The proof that the only stable demand vector (for this example) is the proportional one can be summarized as follows.

Consider first the proposal where the demand vector is $\alpha^* = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and the coalition structure includes an apex coalition. One possible type of objection to this proposal is another apex coalition with payoff division $(\frac{3}{4} + \epsilon, \frac{1}{4} - \epsilon)$. These objections are not acceptable because the minor player coalition can counterobject with the vector α^*. The second (and last) kind of objection would be one with the four small players together, where the "blocker" receives $\frac{1}{4} + \epsilon$ and the others share the rest; but then at least one of these other three small players must be receiving less than $\frac{1}{4}$, and can therefore counter by offering again α^* to the apex player.

Consider now the demand vector α^* together with $\pi = [1, 2345]$. The objection to be considered here is one by an apex coalition and payoffs $(\frac{3}{4} - \epsilon, \frac{1}{4} + \epsilon)$; to this, however, there exists a counterobjection by another apex coalition with $(\frac{3}{4}, \frac{1}{4})$. It is finally easy to see that no other pair can be in the SDS.

The core of the apex game is empty, as it would be for any other constant-sum essential game. The Shapley value only looks at the ex ante balance of power, which here implies $\frac{3}{5}$ for the big player and $\frac{1}{10}$ for each small player. Similarly, the nucleolus (that here coincides with the kernel) of the grand coalition has the allocation $(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$ (the homogeneous representation of the game itself)[5] and can again be interpreted as an ex ante evaluation. While this solution assigns each player the appropriate relative bargaining power, this property disappears when the nucleolus is computed with respect to other coalition structures: for example, the nucleolus of a coalition structure including an apex coalition gives each of the two players a payoff of $\frac{1}{2}$, in spite of the very different endowments.

The Aumann-Maschler bargaining set, on the other hand, not only allows for any coalition structure, but it also gives an uninformative prediction about the payoff distribution

[5] See Peleg (1968).

for some minimal winning coalitions. For the apex coalition, the share of the apex player can take any value between $\frac{3}{4}$ and $\frac{1}{2}$. The latter extreme ($x = (\frac{1}{2}, \frac{1}{2})$) is the kernel for any coalition structure including the apex coalition. The competitive bargaining set makes the same prediction as the SDS for the MWCs, but it allows for all coalition structures. The Zhou bargaining set also allows for the grand coalition to form and for the two-player coalition with equal division. The aspiration approach makes the same predictions as the SDS.

8.2.4 Comparison with other solution concepts for constant-sum weighted majority games

We conjecture that for strong homogeneous weighted majority games, all demand vectors that are stable in the sense of the SDS are balanced aspirations, and vice versa. Corollary 8.3 together with theorem 8.1 imply that all demand vectors that are stable in the sense of the SDS are also partnered, equal gains and in the aspiration kernel. The reverse is not true, as example 8.5 shows.

Remark 8.5 *The proportional payoff vector in theorem 8.1 is partnered and belongs to the aspiration kernel (since each nondummy player is at least in two MWCs, no player depends on any other player). It is also equal gains (for the same reason, the next best coalition is always as good as the best one).*

Remark 8.6 *One can find aspiration vectors that are partnered, equal gains and belong to the aspiration kernel without being proportional.*

Example 8.5 Consider the weighted majority game $[6; 4, 2, 2, 1, 1, 1]$. The demand vector $(\frac{14}{18}, \frac{7}{18}, \frac{7}{18}, \frac{2}{18}, \frac{2}{18}, \frac{2}{18})$ is partnered, equal gains and in the aspiration kernel.

Notice that each player can keep his demand without any of the other players, except for players 2 and 3, who need each other to obtain their demands. Thus, the demand vector is partnered. It is also equal gains (each player can keep his payoff after deleting his first-best coalition, except for players 2 and 3, who would make the same loss) and in the aspiration kernel (each player can keep his payoff without any of the others, except for players 2 and 3, who would lose the same if they do not cooperate). However, it is not proportional, since players 1, 2 and 3 make more than proportional demands and players 4, 5 and 6 make less than proportional demands.

As we know from theorem 8.1, only proportional payoffs can be in the SDS. We will nevertheless check that the demand vector $(\frac{14}{18}, \frac{7}{18}, \frac{7}{18}, \frac{2}{18}, \frac{2}{18}, \frac{2}{18})$ is not in the SDS for the purpose of illustration. There are two sorts of coalition structure that can accompany this demand vector: a coalition of player 1 with two of players 4, 5 and 6, and a coalition of players 2 and 3 with two of players 4, 5 and 6. In the first case, coalition $\{2,3,4,5\}$ has an acceptable objection with the proportional payoff division $(\frac{6}{18}, \frac{6}{18}, \frac{3}{18}, \frac{3}{18})$; in the second case, coalition $\{1,4,5\}$ has an acceptable objection with the proportional payoff division $(\frac{12}{18}, \frac{3}{18}, \frac{3}{18})$. These objections are acceptable because, since players 4 and 5 are receiving more than their demands, they cannot be included in any counterobjecting coalition; no winning coalition included in $\{1,2,3,6\}$ can afford the original demand vector.

8.3 Some other applications

8.3.1 Positive m-quota games

Quota games were introduced by Shapley (1953b). We will take a slightly different definition, due to Peleg (1965).

Definition 8.11 An n-person game (N, v) is an **m-quota game**, $1 < m < n$, if there exist n real numbers $q_1, ..., q_n$ such that $v(S) = \sum_S q_i$ when $|S| = m$, and $v(S) = 0$ otherwise.

We will consider the case of $q_i > 0$ for all i (positive quota games). A reasonable solution for positive quota games is that each player demands exactly his quota. This is the prediction of all the aspiration solution concepts (except the equal gains solution concept, that allows for other vectors as well) and also of the SDS. Quota games were used by Bennett (1985) to illustrate the advantages of the aspiration approach.

Proposition 8.4 *For positive m-quota games, the only demand vector in the SDS is q.*

Proof. Suppose at least one player demands less than his quota. Since the SDS only contains maximal demand vectors, we know that there are less than m underdemanding players. Proposition 8.1 implies that there is at least one feasible coalition containing m players. Any such coalition S must include *all* underdemanding players (otherwise the coalition resulting from deleting one player not in U and adding a player from U would have an acceptable objection). This implies that only one feasible coalition can form at a time. We can construct an objection including all players in U in an m-player coalition $T \neq S$ and giving them more than their demands (this is feasible since $m < n$ implies that at least one of the players is receiving zero; the objecting coalition T can be constructed by replacing one of the players in $S \backslash U$ by this player). This objection cannot be countered.

If no player demands less than his quota and at least one player j demands more, one can easily construct an acceptable objection taking an m-player objecting coalition S containing j that gives all players (if any) in $S \backslash O$ more than their demands. ∎

Proposition 8.5 *For positive m-quota games, the only aspiration vector that is partnered, balanced and in the aspiration kernel is q.*

Proof. The requirement of feasibility of aspirations exclude cases in which all players demand at least their quotas and at least one player demands strictly more.

Suppose at least one player demands less than his quota. By the same reasoning as above, if the demand vector is an aspiration any feasible coalition must contain all underdemanding players. Because of feasibility, each player in $N \backslash U$ belongs to at least one of such coalitions. Thus, each player in $N \backslash U$ depends on each player in U to get this demand. Since there are more than m players, no player in U depends on any player outside. This means that the aspiration vector is not partnered or in the aspiration kernel. The sum of all payoff demands can be decomposed in two terms, $\sum_S \alpha_i$ and $\sum_{N \backslash S} \alpha_i$, were S is some feasible coalition. The first of this terms equals $\sum_S q_i$ and the second is strictly larger, thus the demand vector would not be balanced. ∎

The aspiration approach and the SDS predict that an m-player coalition will form and divide payoffs according to the quotas. This prediction may look far too obvious. However, it does not follow from all solution concepts. The core is empty unless n is a multiple of m. The Shapley value does not assign players their quotas, even if n is a multiple of m. The Aumann-Maschler bargaining set allows for coalition structures where all players receive zero; the same can be said of the kernel. The Zhou bargaining set and the competitive bargaining set also have this problem.

Example 8.6 $N = \{1, 2, 3\}$, $q_1 = 50$, $q_2 = 30$, $q_3 = 10, m = 2$. This is the game described in the introduction of this thesis.

The core of this game is empty. The grand coalition with payoffs $(0, 0, 0)$ is clearly in the Aumann-Maschler bargaining set and in the competitive bargaining set: given any objection, the affected player can always protect his payoff by forming a singleton.

The Zhou bargaining set requires that the objecting coalition and the counterobjecting coalition have some element in common. However, since objections are not personalized, it is still possible to find a counterobjection for any objection. Consider an objection by $\{1,2\}$. Whatever the payoff division, at least one of the players (say, i) is not getting more than his quota. Then $\{i,3\}$ can counterobject. Notice that if instead objections were personalized (holding other features of the Zhou bargaining set constant), player 1 could object against player 3 via coalition $\{1,2\}$ with payoff division $(1,79)$. Since objecting and counterobjecting coalitions cannot be disjoint, player 3 could only counterobject via coalition $\{2,3\}$. It would then be impossible for player 3 to keep 0 for himself while still offering 79 to player 2.

Example 8.7 $N = \{1,2,3,4\}$, $q_1 = 50$, $q_2 = 30$, $q_3 = 20$, $q_4 = 10$, $m = 2$.

The core makes the obvious prediction for this game. However, the Shapley value equals $(35, 28\frac{1}{3}, 25, 21\frac{2}{3})$, and thus assigns values to the players that are less extreme than the quotas. The bargaining sets still allow for the grand coalition with payoff division $(0,0,0,0)$.

In the introduction we argued that a noncompetitive payoff division (in this context, a payoff division diverging from the quotas) can be rational because the players who are getting less than their quotas have higher chances of getting into a coalition. In example 8.7 all players can still get into a coalition when they demand their quotas, so that the quota solution is specially compelling.[6]

8.3.2 A modified glove game

A glove game has two types of players, and only pairs of players of different type can profit from cooperation. A modified glove game allows for gains from cooperation by pairs of players of the same type, but coalitions of players of the same type are less profitable (players of different types own a right-left pair of gloves, players of the same type own two identical gloves that they cannot use but they can still sell for their leather).

[6]This behavior is however not observed in experiments. The reason is that players usually form coalitions sequentially, whereas the aspiration solution concepts and the SDS are designed for simultaneous determination of the coalition structure and the payoff division. See Shapley (1953b) for a discussion of sequential coalition formation in quota games in connection with the concept of stable set.

Example 8.2 is a modified glove game. It is not a quota game, since players contribute more to coalitions of different type than to coalitions of the same type.[7]

We now examine some candidate outcomes for example 8.2.

- Each player joins a player of the same type and they split equally. An example is $\{[12, 34, 56], (4, 4, 4, 4, 4, 4)\}$.

This allocation does not include maximal payoffs (a coalition like $\{1, 5\}$ can pay its players more than their demands). Thus, it is not stable in the sense of the SDS or of the aspiration solution concepts (the demand vector is not even an aspiration).

This outcome is in the bargaining set. The only possible objection is through a coalition of players of different type, say, $\{1, 5\}$. This could be an objection of 1 against 2 or of 5 against 6. In either case, it can be countered through coalition $\{2, 6\}$. A similar reasoning applies to the Zhou bargaining set (an objection by $\{1, 5\}$ could be countered by either $\{2, 5\}$ or $\{1, 6\}$).

The competitive bargaining set also allows for this outcome. Any multi-objection, like $\{1, 5\}$, $\{2, 5\}$, $\{3, 5\}$, $\{4, 5\}$ with payoff division $(5, 5)$ could be counterobjected, by, say, $\{1, 6\}$. Whatever offers player 5 can make to the players of the other type can be matched by player 6.

- Players form pairs in such a way that there are as many pairs of players of different type as possible, and all pairs split the payoff equally. An example would be the payoff configuration $[12, 35, 46], (4, 4, 5, 5, 5, 5)\}$.

Again, these demands are not maximal. This outcome is nevertheless in the Aumann-Maschler bargaining set, in the competitive bargaining set and in the Zhou bargaining set.

- Players form as many pairs of players of different type as possible, and they split equally. The players who are left out remain singletons and (for the SDS and the aspiration approach) insist in getting 5 as well. An example would be the payoff configuration $\{[1, 2, 35, 46], (0, 0, 5, 5, 5, 5)\}$ associated to the demand vector $(5, 5, 5, 5, 5, 5)$.

[7] To be sure, we could use $(4, 4, 4, 4, 6, 6)$ as quota vector, but this "quota" would not be invariant to the number of players of each type in the game.

The demand vector is partnered, equal gains and in the aspiration kernel, but not balanced (the demand vector $(4,4,4,4,6,6)$ has a lower total sum). The pair $\{[1,2,35,46],$ $(5,5,5,5,5,5)\}$ is in the SDS.

This payoff configuration is in the Aumann-Maschler bargaining set (players 1 and 2 cannot even object), in the competitive bargaining set and in the Zhou bargaining set.

- Players form pairs in such a way that there are as many pairs of players of different type as possible, and split the payoffs so that each player of the first type receives 4 and each player of the second type receives 6. An example would be the payoff configuration $\{[12,35,46],(4,4,4,4,6,6)\}$.

This is probably the common sense prediction for the modified glove game. Indeed, the demand vector is partnered, balanced, equal gains and in the aspiration kernel; it is also stable in the sense of the SDS. The payoff configuration $\{[12,35,46],(4,4,4,4,6,6)\}$ is in all three bargaining sets, and in the core.

8.4 Intuitive and unintuitive features of the SDS

In the previous chapter, we discussed some pros and cons of the Zhou bargaining set, comparing it to other solution concepts. The SDS improves over the Zhou bargaining set in most of these dimensions.

The first problem of the Zhou bargaining set was that players might counterobject, therefore keeping the original payoff configuration in force, while at the same time having the possibility to form a coalition compatible with the objection and more profitable than the original payoff configuration. Since counterobjections use the original demand vector and demand vectors are always maximal, this cannot happen in the SDS.

The second problem was that "strangers" can counterobject. This problem is shared by the Zhou bargaining set and by the SDS. Adding the requirement that counterobjecting coalitions must include some player who was affected by the objection (in the sense that he was in a coalition with one of the objecting players in the original payoff configuration) would not affect the results on majority games and quota games and would eliminate this problem.

The third problem was that "scarce" players may be hurt by their scarcity. It is not clear how to define in general the scarcity of a player. For the SDS, and similarly to the aspiration solution concepts, we can define a player to be scarce if other players would be willing to pay him more than his demand. Taking this definition, maximality of the demand vector excludes the existence of scarce players.

So far the SDS performs weakly better than the Zhou bargaining set and similarly to the aspiration concepts.

The Zhou bargaining set has some desirable features as well. Indeed, there was a point in which the Zhou bargaining set was more intuitive than the aspiration bargaining set: it allows any player who is hurt by an objection to launch a counterobjection. This is also the case of the SDS. In example 7.15, the demand vector $(1,1,1,1)$ together with coalition structure $[123,4]$ is in the SDS.

In example 8.6, the fact that objections in the Zhou bargaining set are not personalized led to unintuitive results. Objections in the SDS are not personalized either, but the fact that counterobjections are restricted to use the original demand vector excluded any unreasonable outcomes.

Finally, the SDS eliminates all outcomes that are not coalitionally rational or (for cohesive games) union rational. This is not the case of other bargaining sets. Consider the following simple example:

Example 8.8 $N = \{1,2,3\}$, $v(S) = 1$ for all S with $|S| > 1$, $v(i) = 0$ for all i.

Payoff configuration $\{[1,2,3],(0,0,0)\}$ is in the Aumann-Maschler bargaining set, in the competitive bargaining set and in the Zhou bargaining set. This payoff configuration is not union rational, and is excluded from the SDS. Payoff configuration $\{[123],(\frac{1}{3},\frac{1}{3},\frac{1}{3})]$ is in those three bargaining sets and in the Mas-Colell bargaining set. Since it is not coalitionally rational, it is not in the SDS. The only pairs in the SDS are of the form $\alpha = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $\pi = [ij,k]$.

8.5 Conclusion

We have studied the properties of a new solution concept, the stable demand set. This solution concept shares some elements with the traditional bargaining sets and with the aspiration approach. While at one extreme the traditional bargaining sets allow objecting and counterobjecting coalitions to divide their payoffs in *any* way, and at the other extreme the aspiration bargaining set forces objecting and counterobjecting coalitions to use the *original* demand vector, the SDS allows objecting coalitions to use any payoff division but forces counterobjecting coalitions to use the original demand vector.

The SDS strictly contains the core and, at least for some interesting games, it makes sharper predictions than the bargaining sets. For the very relevant case of constant-sum homogeneous weighted majority games, it predicts that only minimal winning coalitions will form, and that players will split the payoff in proportion to their number of votes.

The SDS may be empty for some games, including the important class of games where the grand coalition is the only efficient coalition structure but the core is empty. The possible emptiness of the SDS may cast doubts over whether it is a real improvement over other solution concepts. After all, the aspiration solution concepts are similar and they are never empty. However, all those concepts are nonempty because they are restricted to the space of aspirations. The possibility of the SDS being empty is a price we have to pay for the unrestricted domain. Moreover, whether the SDS is empty or not may be taken as a test of how solid is the prediction of the aspiration approach for the corresponding game. We would then claim that the predictions of the aspiration approach are more solid for the apex game than for the three-person game in example 8.3.

We agree with Maschler (1992) that emptiness of a solution concept for some games is not a reason to abandon the concept.

> They [solution concepts that may be empty] represent desires for greater stability which, however, cannot always be met. But in those cases where they can be met, I see no reason not to recommend their outcomes.

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Index

Notation

Concepts

Samenvatting

Het onderwerp van het proefschrift is de formatie van coalities. Dit is een vaak voorkomend verschijnsel. Voorbeelden zijn fusies, internationale verdragen, samenwerkende wetenschappers en kabinetten zoals Paars II. In al die situaties is samenwerking voordelig, maar er bestaat echter een conflict over welke partijen een coalitie moeten sluiten en hoe de uitbetalingen die de samenwerking levert verdeeld moeten worden. Het doel van dit proefschrift is de volgende vragen te beantwoorden:

1. Welke coalities zullen er worden besloten?

2. Hoe zullen de leden van elke coalitie de uitbetalingen van de samenwerking verdelen?

De factoren die een rol spelen in het beantwoorden van deze vragen kunnen het best worden geïllustreerd met behulp van een voorbeeld.

Beschouw drie wetenschappers die in een kleine universiteit werken en Ans, Bouke en Cees heten. Ze zijn deskundig in verschillende vakken: Ans in macroeconomie, Bouke in microeconomie, en Cees in informatica. Er zijn drie mogelijkheden voor samenwerking: in de eerste plaats kunnen Ans en Cees een boek schrijven over macroeconomie (Ans zou dan de theorie behandelen en Cees zou enkele simulaties programmeren), ten tweede kunnen Bouke en Cees een boek schrijven over microeconomie (Bouke zou de theorie behandelen terwijl Cees experimenten zou programmeren), en tenslotte kunnen Ans en Bouke een boek schrijven over de relatie tussen individuele keuzes (waarin Bouke op zijn best is) en het gedrag van het geaggregeerde economische systeem (daar weet Ans alles van). De drie boeken zijn verwacht opbrengsten van respectievelijk 60, 80 en 40 op te leveren. Elk individu heeft tijd om maar op een boek te werken.

Welke coalitie zal er gevormd worden? Er moet gekozen worden. Er bestaat een conflict over welke coalitie moet gevormd worden (elk individu wil in de coalitie zitten) en over het verdelen van de uitbetalingen (als twee individu's een coalitie vormen, elk individu zou zoveel mogelijk willen krijgen).

De oplossing schijnt makkelijk te zijn als we aannemen dat medeauteurs de uitbetalingen gelijk verdelen. Iedere auteur verdient 30 met het eerste boek, 40 met het tweede en 20 met het derde. Het is dan te verwachten dat de tweede boek wordt geschreven: Bouke en Cees verdienen het meest met de tweede boek, en geen van hun kan profiteren van een boek schrijven met Ans.

De aanneming van gelijkverdeling is echter problematisch. Uit de situatie blijkt dat Cees het meest waardevol individu is. Waarom zou hij bereid zijn om maar de helft van de uitbetalingen te krijgen?

Inderdaad, terwijl Bouke en Cees een coalitie met gelijk verdeling aan het beschouwen zijn, kan Ans Cees aanspreken en hem 41 aanbieden als hij het boek over microeconomie met haar schrijft (dan zou Ans nog 19 verdienen). Bouke neemt daar geen genoegen mee een biedt Cees 42 aan, enzovoort. Welk bedrag kunnen zulke aanbiedingen bereiken? Stel je voor dat Ans en Bouke bieden beiden 55 aan. Als Cees een coalitie met Bouke sluit, verdient Bouke 25. Als Cees een coalitie met Ans sluit, verdient Ans 5. Dan zullen Bouke en Ans liever een coalitie sluiten, en bijvoorbeeld 30 en 10 verdienen.

De lezer kan de volgende stelling verifiëren: onafhankelijk van welke coalitie en welke verdienstenverdeling we nemen, kunnen we altijd twee individu's vinden die van een andere coalitie kunnen profiteren. In de taal van speltheorie, de kern (*core*) van het spel is leeg.

Betekent deze stelling dat we helemaal niets kunnen voorspellen over het resultaat van de interactie tussen onze drie individu's? Is iedere verdienstenverdeling even labiel?

Stel je voor dat Ans 10 vraagt voor haar medewerking in welke coalitie dan ook, terwijl Bouke 30 vraagt en Cees 50 (ze zetten een 'prijs' voor hun medewerking). Deze prijzen hebben een interessante eigenschap: het kan elk individu niet schelen met wie hij een coalitie sluit. Als (bijvoorbeeld) Ans meer zou vragen, niemand zou een coalitie met haar willen. Als ze juist minder zou vragen, iedereen zou een coalitie met haar willen, en competitie tussen de andere individu's zal de prijs van haar medewerking terug naar 10 brengen. De prijzen 10, 30 en 50 hebben de eigenschap dat competitie tussen individu's de prijzen naar deze bedragen brengt. In de taal van speltheorie, deze bedragen zijn de individu's hun *quota*.

Als we de quota als redelijke prijzen accepteren, blijft de vraag van welke coalitie vormt ontbeantwoordt. De verschillende individu's zijn weliswaar niet even waardevol, maar wel even voordelig als coalitiepartners.

De aanneming van gelijkverdeling leidt dus tot de stelling dat Bouke en Cees een coalitie zullen sluiten, terwijl de aanneming van competitie wel tot de uitvinding van de competitieve verdeling (de *quota*), maar niet tot een duidelijk conclusie over coalities leidt.

Laten we maar op het begin van deze redenering terugkomen. Stel je voor dat de coalitie tussen Bouke en Cees met een gelijkverdeling is nu onder beschouwing. Zou Cees argumenteren dat hij waardevoller is dan Bouke en daarom meer dan 40 verdient, of zou hij 'gul' zijn en gelijkverdeling aanvaarden? Wat is voordeliger: een zekere winst 40 of een winst van 50 maar alleen als het men lukt in de coalitie te zitten? Bovendien is de huidige verdeling (40 voor Bouke, 40 voor Cees) bijzonder veilig voor Cees: Ans kan niet profiteren van een coalitie met Bouke. Een 'gul' gedrag gericht om mens deelneming waarschijnlijker te maken is niet uiteenlopend en komt ook vaak voor in experimenten.

Het doel van dit proefschrift is voorspellingen te geven in situaties zoals de bovenstaande met behulp van speltheorie. Speltheorie is een verzameling analytische gereedschappen ontworpen met het doel onze verstand van de interactie (waaronder coalitievorming) tussen besluitvormers te verbeteren. Speeltheorie neemt aan dat besluitvormers doelgerichte keuzen maken (ze zijn rationeel) en ook rekening mee houden met de rationaliteit van anderen als ze verwachtingen vormen over hun gedrag.

Speltheorie is verdeeld in twee takken, namelijk coöperatieve en niet-coöperatieve speltheorie. De coöperatieve speltheorie houdt zich bezig met de 'grote kwesties' zoals welke uitbetalingen mensen kunnen bereiken als ze samenwerken, en niet met 'praktische kleinigheden' zoals welke strategieën men moet volgen om zulke uitbetalingen te bereiken. De niet-coöperatieve speltheorie houdt zich wel bezig met alle praktische onderdelen van de situatie zoals welke acties de individu's (de spelers) kunnen nemen, wat precies ze weten, in welke volgorde verschillende spelers beslissingen moeten nemen, enzovoort. De namen 'coöperatieve' en 'niet-coöperatieve' wijzen naar de gebruikelijke veronderstellingen van deze theorieën: de coöperatieve theorie neemt aan dat de spelers verplichtende contracten kunnen sluiten, terwijl de niet-coöperatieve theorie aanneemt dat verplichtende contracten in de situatie onder beschouwing onmogelijk zijn. Zulke classificatie is niet zonder uitzonderingen: niet-coöperatieve modellen met verplichtende contracten staan in dit proefschrift centraal.

Beide takken van de speltheorie hebben sterke kanten en zwakke kanten. Een coöperatieve model is van algemene toepassing, terwijl een niet-coöperatieve model alleen in specifieke situaties van toepassing is. Aan de andere kant zijn er veel mogelijke oplossingsconcepten voor coöperatieve modellen: elk oplossingconcept probeert de argumenten die mensen in hun onderhandelingen hanteren weer te geven. Aangezien de vele mogelijkheden waarover mensen beschikken, zal de menigvuldigheid van oplossingsconcepten niemand verbazen. De niet-coöperatieve theorie beschouwt de argumenten die mensen kunnen hanteren als een onderdeel van het model (en dus niet van de oplossingsconcept). De mogelijke argumenten worden dan zeer precies omschreven en streng beperkt tot de belangrijkste kenmerken van de werkelijke situatie. Voor niet-coöperatieve modellen bestaat er een geaccepteerd oplossingsconcept: het Nash evenwichtspunt. Het idee van het Nash evenwichtspunt is dat de spelers juiste verwachtingen vormen over de beslissingen van andere spelers en ook hun best doen gegeven wat de andere spelers hebben besloten.

Beide benaderingen zijn waardevol en steunen elkaar: coöperatieve modellen vestigen de aandacht op het wezen; niet-coöperatieve modellen zijn explicieter in hun veronderstellingen en daardoor kunnen ze licht werpen op coöperatieve oplossingsconcepten.

Hoofdstukken 1 en 2 van dit proefschrift zijn inleidend. Hoofdstuk 1 geeft een tamelijk informele inleiding naar het probleem van coalitievorming. Hoofstuk 2 bevat enkele technische benodigdheden en een overzicht van de vakliteratuur.

Deel I behandelt niet-coöperatieve modellen van coalitievorming. Het bestaat uit hoofdstukken 3, 4, 5 en 6.

Hoofdstuk 3 bestudeert niet-coöperatieve modellen van coalitievorming in spelen met *externe effecten*, dat wil zeggen, spelen waarin de formatie van een coalitie de uitbetalingen van andere coalities beïnvloedt. Deze mogelijkheid geschiedt in de meeste praktische situaties: fusies zijn ongunstig voor concurrenten, internationale verdragen met het doel de luchtvervuiling te verminderen zijn gunstig voor andere landen, enzovoort. De *waarde* (uitbetalingen) van een coalitie is dan afhankelijk van welke andere coalities vormen. Of er een bepaalde coalitie wordt gesloten, hangt van de verwachtingen over de wijze waarop buitenstaanders zich groeperen. Er zijn vele mogelijke verwachtingen: geheel optimistisch, geheel pessimistisch... We veronderstellen dat spelers *juiste* verwachtingen vormen: wat de spelers verwachten is in overeenstemming met wat er in de werkelijkheid zou gebeuren. Het hoofdstuk bevat ook noodzakelijke voorwaarden voor het bestaan van een efficiënt evenwicht: het begrip 'evenwicht karakteristieke functie' staat daar centraal. De conclusie van dit hoofdstuk is dat efficiëntie lastig te bereiken valt, zelfs als er 'perfecte informatie' is (de spelers weten precies welke preferenties andere spelers hebben, en ze weten ook alles wat er tijdens de onderhandelingen gebeurt) en verplichtende contracten. Het laatste deel van het hoofdstuk is besteed aan het vergelijken van enkele niet-coöperatieve modellen.

Hoofdstuk 4 concentreert zich op een bepaalde soort situaties met externe effecten: de *constantsomspelen in partitionele functie vorm*, dat wil zeggen, spelen waarin de som van de uitbetalingen voor alle coalities (maar *niet* de verdeling tussen coalities) constant blijft ongeacht welke coalities worden gevormd. Het grootste deel van het hoofstuk behandelt de driepersoonsspel. We veronderstellen dat alle tweepersoonscoalities voordelig zijn; verder veronderstellen we niets over de waarde van de coalities. De verwachtte uitbetalingen voor de spelers zijn vrijwel onafhankelijk van de details van het niet-coöperative model: als de details van het model veranderen (in het voordeel van een bepaalde speler), veranderen ook de strategieën die de spelers hanteren (in het nadeel van die speler) zodat de verwachtte uitbetaling onveranderd blijft. Als de regels van het spel symmetrisch zijn (geen enkele speler heeft een voorsprong of een achterstand), zijn alle tweepersoonscoalities even mogelijk. De verwachtte uitbetalingen hebben ook de volgende eigenschap: het kan de spelers niet schelen met welke andere speler ze een coalitie vormen (al kunnen de spelers heel verschillend zijn!).

Hoofdstuk 5 is volledig gewijd aan een klassieke economische probleem: de driepersoonsmarkt. Een verkoper kan een rare stoel verkopen aan een van twee kopers. Alle spelers weten precies hoeveel elke koper bereid is maximaal te betalen. Een van de kopers (de sterke koper) heeft meer waardering voor de stoel, en die zal de stoel krijgen (hij is bereid meer dan de andere koper te bieden). De aanwezigheid van de tweede koper zorgt echter voor een hogere prijs. De zwakke koper kan dan ervan profiteren en betaald worden door de koper om buiten de markt te blijven of door de verkoper om juist in de markt te zijn. De moraal van dit hoofdstuk is dat mensen die geen extra waarde leveren aan een spel maar de verdeling van de waarde beïnvloeden, kunnen ervan profiteren.

Hoofdstuk 6 behandelt een andere klassieke situatie: het *apex* spel. Een apex spel bestaat uit een grote speler (de apex) en enkele (minstens drie) kleine spelers. De apex speler heeft alleen een kleine speler nodig om een voordelige coalitie te sluiten; de kleine spelers zijn machteloos tenzij alle kleine spelers zich in een coalitie verenigen. De verwachtte uitbetalingen van de spelers in het evenwichtspunt van het niet-coöperatieve model corresponderen met een coöperatieve oplossingsconcept: de kernel.

Deel II behandelt coöperatieve modellen van coalitievorming. Het bestaat uit hoofdstukken 7 en 8.

Hoofdstuk 7 bespreekt enkele coöperatieve oplossingconcepten bekend als *bargaining sets* ('onderhandelingsverzamelingen'). Deze oplossingsconcepten zijn gebaseerd op de begrippen *objectie* ('bezwaar') en *counterobjectie* ('tegenbezwaar'). Stel je voor dat er een bepaalde verdeling van de spelers in coalities (een *coalitiestructuur*) samen met een distributie van de uitbetalingen binnen de coalities onder beschouwing is. Als een van de spelers een coalitie kan vormen die voordeliger voor hun leden is dan de huidige coalitiestructuur, bestaat er een objectie. Het bestaan van een objectie betekent niet meteen dan de huidige situatie labiel is. Een objectie meent de aandacht te vestigen op het feit dat een andere speler nu 'te veel' krijgt. Als deze speler zijn uitbetalingen kan beschermen door het vormen van een nieuwe coalitie, is er nog niets aan de hand. Het hoofdstuk bespreekt verschillende definities van de begrippen objectie en counterobjectie. Wie mag een objectie hebben en tegen wie? Moeten objecties eigenlijk altijd tegen iemand gericht zijn? Mogen verschillende objecties tegelijk voorkomen? Wie mag er een counterobjectie hebben? Wat voor eisen zijn gesteld aan een geldige counterobjectie?

Hoofdstuk 8 bestudeert een nieuwe onderhandelingsverzameling, de *stable demand set* ('stabiele eisen verzameling'). Dit oplossingsconcept is gebaseerd op het idee dat de spelers prijzen stellen voor hun samenwerking. De spelers krijgen niet altijd deze prijzen: in het bovenstaande verhaal over Ans, Bouke en Cees zijn (20, 20, 50) mogelijke prijzen. Als Ans en Bouke een coalitie sluiten, krijgen ze wat ze geëist hebben, terwijl Cees niets krijgt. De prijzen gelden alleen als men binnen een coalitie zit. Als Ans en Bouke een coalitie sluiten met een gelijkverdeling (elk krijgt 20) hebben Bouke en Cees er bezwaar tegen: ze kunnen een coalitie vormen en krijgen 40 per persoon. Dit is voordelig voor Bouke en ook voor Cees, die, door een verlaging van zijn eisen, krijgt nu 40 in plaats van nul. Gegeven deze objectie, een counterobjectie moet bewijzen dat de *oorspronkelijke* eisen (20, 20 en 50) stabiel zijn. Ans zou dan een coalitie met Cees kunnen sluiten waarin beiden hun oorspronkelijke eisen (20 en 50) krijgen. Zoiets is niet mogelijk aangezien de coalitie van Ans en Cees is maar 60 waard. De enige stabiele eisen corresponderen precies met de quota (10, 30, 50).

Hoofdstuk 8 bespreekt de eigenschappen van de stable demand set en de betrekkingen tussen dit en andere oplossingsconcepten. De stable demand set wordt berekend voor enkele vaak voorkomende spelen, zoals constant-som gewogen meerderheidsspelen, quotumsspelen en handschoenenspelen.

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MARIA MONTERO graduated in Economics from the University of Salamanca in 1995. She joined the CentER for Economic Research as a Ph.D. student in 1996. She is working on game theory, particularly on cooperative and noncooperative games of coalition formation. She currently holds a Marie Curie Fellowship at Dortmund University.

Coalition formation is a frequent phenomenon in everyday life. Firms merge, states sign treaties and authors write papers together. In most of these situations, the payoff of a coalition depends on whether and which other coalitions form: the formation of a coalition imposes externalities on outsiders. The first part of this thesis analyzes coalition formation in situations with externalities using noncooperative game theory. The predicted organization of players in coalitions will generally not be efficient. Besides efficiency, the issue of distribution is studied: individuals who do not bring additional value to a situation but can affect the distribution of this value can exploit their position. The thesis also points out the pros and cons of several noncooperative models. The second part of the thesis discusses a family of cooperative solution concepts called bargaining sets, and in particular the stable demand set. The stable demand set is found to make very reasonable predictions for two classes of empirically relevant situations: constant-sum majority games and quota games.

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