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Abstract. This paper studies a class of delivery problems associated with the Chinese postman problem and a corresponding class of delivery games. A delivery problem in this class is determined by a connected graph, a cost function defined on its edges and a special chosen vertex in that graph which will be referred to as the post office. It is assumed that the edges in the graph are owned by different individuals and the delivery game is concerned with the allocation of the traveling costs incurred by the server, who starts at the post office and is expected to traverse all edges in the graph before returning to the post office. A graph G is called Chinese postman-submodular, or, for short, CP-submodular (CP-totally balanced, CP-balanced, respectively) if for each delivery problem in which G is the underlying graph the associated delivery game is submodular (totally balanced, balanced, respectively).

For undirected graphs we prove that CP-submodular graphs and CP-totally balanced graphs are weakly cyclic graphs and conversely. An undirected graph is shown to be CP-balanced if and only if it is a weakly Euler graph. For directed graphs, CP-submodular graphs can be characterized by directed weakly cyclic graphs. Further, it is proven that any strongly connected directed graph is CP-balanced. For mixed graphs it is shown that a graph is CP-submodular if and only if it is a mixed weakly cyclic graph.

Finally, we note that undirected, directed and mixed weakly cyclic graphs can be recognized in linear time.

Key words. cooperative games – Chinese postman problem

1. Introduction

A class of delivery games was introduced by Hamers et al. [7] to analyze a cost allocation problem which arises in some delivery problems on graphs. These delivery problems are associated with the Chinese postman problem [14, 2] and can be described as follows. A server is located at some fixed vertex of a graph G, to be referred to as the post office, and each edge of G belongs to a different player. The players need some service, e.g. mail delivery, and the nature of this service requires the server to travel from the post office and visits all edges (players) before returning to the post office. The cost allocation problem associated with this delivery problem is concerned with a fair allocation of the cost of a cheapest Chinese postman tour in the graph. That is, the cost of a cheapest tour, which starts at the post office, visits each edge of G at least once and returns to the post office. Following what is by now an established line of research, Hamers et al. [7] formulated this cost allocation problem as a cooperative game (N, c), referred to as a delivery game, where N is the set of players (edges) in the graph, and $c : 2^N \to \mathbb{R}$ is

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the characteristic function. For each $S \subseteq N$, $c(S)$ is the cost of a minimal (i.e. cheapest) S-tour, which starts at the post office, visits each edge in S at least once and returns to the post office. Solution concepts in cooperative game theory were then evaluated as possible cost allocation schemes for the above delivery problem.

One of the most prominent solution concepts in cooperative game theory is the core of a game. It consists of all vectors which distribute the cost incurred to N, $c(N)$, among the players in such way that no subset of players can be better off by seceding from the rest of the players and act on their own behalf. That is, a vector x is in the core of a game (N, c) if $\sum_{j \in N} x_j = c(N)$ and $\sum_{j \in S} x_j \leq c(S)$, for all $S \subset N$. A cooperative game whose core is not empty is said to be balanced, and if the core of any subgame of it is nonempty, it is said to be totally balanced.

In general, a delivery game associated with an undirected graph could have an empty core. However, Hamers et al. [7] have shown that a delivery game induced by a connected weakly Euler graph is balanced. Here, a graph G is called a weakly Euler graph if after the removal of the bridges in G, the remaining components are all Euler graphs or singletons. Alternatively, we can say that G is a weakly Euler graph if each biconnected component in G is eulerian. Further, Hamers [6] has shown that if a connected undirected graph is weakly cyclic, that is, every edge therein is contained in at most one circuit, then the associated delivery game is submodular. That is, the characteristic function c is submodular.

In this paper we study the class of delivery games induced by undirected, directed and mixed graphs. We define a graph to be Chinese Postman-submodular, Chinese Postman-totally balanced or Chinese Postman-balanced (or, for short, CP-submodular, CP-totally balanced and CP-balanced), if the corresponding delivery game is submodular, totally balanced, or balanced, respectively, for all edge costs and all locations of the post office. We prove that an undirected graph is CP-submodular if and only if it is CP-totally balanced, which holds if and only if it is weakly cyclic. An undirected graph is shown to be CP-balanced if and only if it is a weakly Euler graph. In contrast with the undirected case, we prove that any connected directed graph is CP-balanced. Further, we prove that a delivery game induced by a directed graph is submodular if and only if the directed graph is weakly cyclic. In a directed weakly cyclic graph each arc is contained in exactly one circuit. For a connected mixed graph, G is CP-submodular if and only if G is a mixed weakly cyclic graph. That is, each arc or edge therein is contained in at most one mixed circuit. Finally, we observe that undirected, directed and mixed weakly cyclic graphs can be recognized in linear time.

Our ability to characterize submodular delivery games is significant because submodular games are known to have nice properties, in the sense that some solution concepts for these games coincide and others have intuitive description. For example, for submodular games the Shapley value is the barycentre of the core [19], the bargaining set and the core coincide, the kernel coincide with the nucleolus [12], the τ-value [22] can be easily calculated and there is a polynomial algorithm for computing the nucleolus [10].

Some examples of submodular games which were studied in the literature include airport games [11], tree games [13,5], sequencing games [1,8] and certain communication games [15].

Finally, we note that results obtained in this paper are in similar vein to those derived in [9] and [4]. In particular, in [4], delivery games associated with the traveling salesman problem are investigated, and directed graphs which give rise to submodular delivery games are characterized.

The paper is organized as follows. Section 2 introduces the delivery problem and the associated delivery game. Section 3 investigates the delivery game when G is undirected, and Sect. 4 is devoted to delivery games defined on directed and mixed graphs.

2. Delivery problems and delivery games

We present in this section a class of delivery problems associated with the Chinese postman problem and a corresponding class of delivery games. However, before a formal description of the models is presented, we need to provide some background in cooperative game theory and recall some elementary graph theoretical definitions.

A *cooperative (cost) game* is a pair (N, c), where N is a finite set of players, c is a mapping, $c : 2^N \to \mathbb{R}$, with $c(\emptyset) = 0$, and 2^N is the collection of all subsets of N. A subset of N will be sometimes referred to as a *coalition*. A function $h : 2^N \to \mathbb{R}$ is said to be *subadditive* if $h(S) + h(T) \geq h(S \cup T)$ whenever $S \cap T = \emptyset$ and it is said to be *submodular* if

$$h(T \cup \{j\}) - h(T) \leq h(S \cup \{j\}) - h(S) \tag{1}$$

for all $j \in N$ with $S \subset T \subseteq N\backslash\{j\}$. Equivalently, h is submodular if

$$h(S \cup T) + h(S \cap T) \leq h(S) + h(T) \tag{2}$$

for all coalitions $S, T \in 2^N$. A game (N, c) is submodular or concave if and only if the map $c : 2^N \to \mathbb{R}$ is submodular.

An allocation $x = (x_i)_{i \in N} \in \mathbb{R}^N$ is a *core-element* of (N, c) if $\sum_{i \in N} x_i = c(N)$ and $\sum_{i \in S} x_i \leq c(S)$ for all $S \in 2^N$. The *core* of a game (N, c) consists of all core elements. A game is called *balanced* if its core is non-empty and it is *totally balanced* if for each $S \subset N$, (S, c_S) is balanced, where c_S is the restriction of c to the family of subsets of S. It follows from Shapley [19] that concave games are totally balanced.

Let $G = (V(G), E(G))$ be an undirected (directed) graph where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges (arcs) of G, respectively. An edge, $\{u, v\}$, in an undirected graph joins vertices u and v therein. If (u, v) is an arc from u to v in a directed graph (digraph), we will refer to u and v as the tail and head of arc (u, v), respectively. A *(directed) walk* in $G = (V(G), E(G))$ is a finite sequence of vertices and edges (arcs) of the form $v_1, e_1, v_2, ..., e_k, v_{k+1}$ with $k \geq 0, v_1, ..., v_{k+1} \in V(G), e_1, ..., e_k \in E(G)$ such that $e_j = \{v_j, v_{j+1}\}$ $(e_j = (v_j, v_{j+1}))$ for all $j \in \{1, ..., k\}$. Such a walk is said to be *closed* if $v_1 = v_{k+1}$. A *(directed) path* in G is a (directed) walk in which all vertices (except, possibly v_1 and v_{k+1}) and edges (arcs) are distinct. A closed (directed) path, i.e., a path in which $v_1 = v_{k+1}$, containing at least one edge (arc) is called a *(directed) circuit*. An undirected (directed) graph G is (*strongly*) *connected* if there is a (directed) path (from) between any vertex to any other vertex in G. An edge $b \in E(G)$ is called a *bridge* in a connected graph $G = (V(G), E(G))$

if the graph $(V(G), E(G)\backslash\{b\})$ is not connected. The set of bridges in G is denoted by $B(G)$.

Let $G = (V(G), E(G))$ be a (strongly) connected undirected (directed) graph, and let $v_0 \in V$ be an arbitrary vertex in $V(G)$, which will sometimes be referred to as the *post office* of G. An S-tour w.r.t. v_0 associated with $S \subseteq E(G)$ is a closed walk that starts at the post office v_0, visits each edge (arc) in S at least once and returns to v_0. Formally, we have:

Definition 1. *Let $G = (V(G), E(G), v_0)$ be a (strongly) connected undirected (directed) graph in which $v_0 \in V(G)$ is the post office. An S-tour w.r.t. v_0 in G is a closed (directed) walk $v_0, e_1, v_1, ..., v_{k-1}, e_k, v_0$ such that $S \subseteq \{e_j \mid j \in \{1, ..., k\}\}$.*

The set of S-tours associated with $S \subseteq E(G)$ is denoted by $D(S)$.

Alternatively, we can associate with an S-tour a eulerian multigraph H_S, that contains k copies of edges (arcs) which are traversed k times by the S-tour, $k \in \{0, 1, 2, ..., \mid E(G) \mid\}$. Let $t : E(G) \to [0, \infty)$ be a travel cost function associated with edges (arcs) of G. The travel cost of an S-tour $v_0, e_1, v_1, ..., v_{k-1}, e_k, v_0$ is naturally equal to $\sum_{j=1}^{k} t(e_j)$.

The class of delivery problems we analyze in this paper and the corresponding class of cost allocations problems arise in G when it is assumed that edges (arcs) therein belong to different players. Explicitly, assume that each edge (arc) in G belongs to a different player and that a server, located at v_0, is providing some service to players in G. The nature of this service requires that the server will travel along the edges (arcs) of G and return to v_0. The corresponding cost allocation problem is concerned with the allocation of the cost of providing the service to the players.

Formally, let $\Gamma = (E(G), (V(G), E(G), v_0), t)$ denote a *delivery problem*, where $E(G)$ is the set of players (edges, arcs), $(V(G), E(G), v_0)$ is a (strongly) connected undirected (directed) graph in which v_0 represents the post office and $t : E(G) \to [0, \infty)$ assigns travel costs to the edges (arcs).

Definition 2. *The delivery game $(E(G), c)$ corresponding to the delivery problem $\Gamma = (E(G), (V(G), E(G), v_0), t)$ is defined for all $S \subseteq E(G)$ by*

$$c(S) = \min_{v_0, e_1, ... e_k, v_0 \in D(S)} \sum_{j=1}^{k} t(e_j). \tag{3}$$

Obviously, if $\mathcal{E}(S)$ is the set of eulerian multigraphs that correspond to the set of delivery tours $D(S)$, we have that

$$c(S) = \min_{H_S \in \mathcal{E}(S)} t(H_S). \tag{4}$$

Note, if G is a connected undirected graph then a multigraph that optimizes (4) contains 0,1 or 2 copies of every edge of $E(G)$.

Clearly, c is subadditive. Moreover, since the travel cost function t is non-negative, delivery games are also monotonic, i.e. $c(S) \leq c(T)$ for all $S \subset T \subseteq E(G)$.

3. Submodular, totally balanced and balanced undirected graphs

We characterize in this section CP-submodular graphs, CP-totally balanced graphs and CP-balanced graphs, when the underlying graph G in the delivery problem is undirected. Explicitly, we prove that both CP-submodular graphs and CP-totally balanced graphs are weakly cyclic graphs, where an undirected graph is said to be *weakly cyclic* if it is connected and every edge therein is contained in at most one circuit. Or, alternatively, each biconnected component[1] is a circuit in this graph. Further, we prove in this section that an undirected graph is CP-balanced if and only if it is a weakly Euler graph.

Theorem 1. *For a connected undirected graph G, the following statements are equivalent:*

(i) G is weakly cyclic.
(ii) G is CP-submodular.
(iii) G is CP-totally balanced.

Proof. *(i)* → *(ii)*: Let $\Gamma = (E(G), (V(G), E(G), v_0), t)$ be a delivery problem and let $(E(G), c)$ be the corresponding delivery game. Let $S, T \subseteq E(G)$, then we have to show that

$$c(S \cup T) + c(S \cap T) \leq c(S) + c(T). \tag{5}$$

We prove (5) by induction on the number of edges $\mid E(G) \mid$. Obviously, if $\mid E(G) \mid = 1$, then G is CP-submodular. Assume that a weakly cyclic graph G, with $\mid E(G) \mid < n$, is CP-submodular, and let G be a weakly cyclic graph with $\mid E(G) \mid = n$. Let H_S and H_T be eulerian multigraphs that optimize (4) for coalitions S and T, respectively. Suppose first that some edge e is used by neither H_S nor H_T. Then, $G' = G \backslash \{e\}$ contains a weakly cyclic graph G^*, with $\mid E(G^*) \mid < n$, which contains the tours corresponding to H_S and H_T. By the induction hypothesis, G^* is CP-submodular. Consider the delivery problem $(E(G^*), (V(G^*), E(G^*), v_0), t_{|E(G^*)})$, which is the restriction of Γ to G^*, and its corresponding delivery game $(E(G^*), c^*)$. Then

$$\begin{aligned} c(S) + c(T) &= c^*(S) + c^*(T) \\ &\geq c^*(S \cup T) + c^*(S \cap T) \\ &\geq c(S \cup T) + c(S \cap T), \end{aligned}$$

where the first equality holds since H_S and H_T do not contain e, the first inequality follows from the induction hypothesis applied to G^* and the last inequality holds since the removal of e can only increase the cost of an $S \cup T$-tour or $S \cap T$-tour in G. Hence, to conclude the proof, we may assume that $H_S + H_T$ contains every edge of G with multiplicity at least 1, and every bridge therein with multiplicity at least 2. Let H_N be an optimal multigraph induced by an N-tour in G. Then H_N contains one copy of every edge belonging to a circuit in G and two copies of each bridge in G. Now, one can easily verify that for each vertex v incident to an edge $e \in S \cap T$, $H_S + H_T$ contains 4 edge-disjoint paths between v_0 to v. Hence, there are at least two edge-disjoint paths between

[1] A biconnected component of a graph G is a maximal subgraph of G in which for each triple of distinct vertices v, w, z there exists a path between v and w not containing z.

v and v_0 in the multigraph $H_S + H_T - H_N$. We conclude that there exist $S \cap T$-tours in G for which the corresponding multigraph $H_{S\cap T}$ is contained in $H_S + H_T - H_N$, implying that $c(S \cap T) \leq t(H_S + H_T - H_N) = t(H_S) + t(H_T) - t(H_N)$. Further, since delivery games are monotonic we have that $c(S \cup T) \leq c(N) = t(H_N)$. Hence,

$$\begin{aligned} c(S \cup T) + c(S \cap T) &\leq t(H_N) + (t(H_S) + t(H_T) - t(H_N)) \\ &= t(H_S) + t(H_T) \\ &= c(S) + c(T). \end{aligned}$$

(ii) → *(iii)*: Follows immediately, since each submodular game is totally balanced.
(iii) → *(i)*: Let $(E(G), (G, v_0), t)$ be a delivery problem and let $(E(G), c)$ be the corresponding delivery game. Suppose G is not weakly cyclic. Then, G contains a connected subgraph G^* of the form shown in Fig. 1.

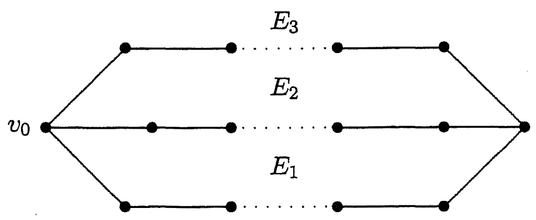

Fig. 1.

Let $E(G^*) = E_1 \cup E_2 \cup E_3$ in which E_1, E_2 and E_3 are the edge sets as depicted in Fig. 1. Let v_0, as indicated in Fig. 1, be the post office, let t be a travel cost function satisfying $\sum_{e\in E_j} t(e) > 0$ for $j = 1, 2, 3$ and $t(e)$ is arbitrarily large for $e \notin E_1 \cup E_2 \cup E_3$, and let $(E(G^*), c_{E(G^*)})$ be the subgame of $(E(G), c)$ corresponding to $E(G^*)$. We claim that with the above choice of v_0 and the cost function t, the core of $(E(G^*), c_{E(G^*)})$ is empty. Indeed, if the core is not empty, then there exists a vector x, $x \in I\!R^{E(G^*)}$, such that[2]

$$\begin{aligned} &x(E(G^*)) = c_{E(G^*)}(E(G^*)), \\ &x(E_1 \cup E_2) \leq t(E_1) + t(E_2), \\ &x(E_1 \cup E_3) \leq t(E_1) + t(E_3) \quad \text{and} \\ &x(E_2 \cup E_3) \leq t(E_2) + t(E_3). \end{aligned} \tag{6}$$

Summing the inequalities in (6) we obtain that

$$x(E(G^*)) \leq t(E_1) + t(E_2) + t(E_3) < c_{E(G^*)}(E(G^*)),$$

where the last strict inequality follows since $t(E_j) > 0$ for $j = 1, 2, 3$. We have obtained a contradiction, since it was assumed that $x(E(G^*)) = c_{E(G^*)}(E(G^*))$, and we conclude

2 For a vector $y \in I\!R^N$ and $S \subseteq N$ we let $y(S) = \sum_{j\in S} y_j$.

that $(E(G^*), c_{E(G^*)})$ is not balanced. Consequently, G is not CP-totally balanced.

□

Let us briefly consider the recognition problem of a weakly cyclic graph. The connectedness of any graph can be checked in linear time. Tarjan [21] showed that the biconnected components of a graph can be found in linear time with respect to the number of vertices and edges. In a weakly cyclic graph, the biconnected components are circuits. Since it can be checked in linear time whether a biconnected component is a circuit, we have proved the following proposition.

Proposition 1. *The computational complexity of determining whether a graph G is weakly cyclic is* $\mathcal{O}(\mid E(G) \mid, \mid V(G) \mid)$.

We conclude this section with a characterization of undirected CP-balanced graphs. Explicitly Hamers et al. [7] proved that a connected undirected weakly Euler graph is CP-balanced, and we prove below that an undirected CP-balanced graph must, in fact, be a weakly Euler graph.

Theorem 2. *A connected undirected graph G is a weakly Euler graph if and only if G is CP-balanced.*

Proof. If G is a weakly Euler graph, then $x \in \mathbb{R}^N$, defined by

$$x(e) = 2t(e) \text{ if } e \in B(G),$$
$$x(e) = t(e) \quad \text{otherwise},$$

is in the core of any delivery game that arises from G.

Suppose G is CP-balanced and assume, on the contrary, that G is not a weakly Euler graph. Let t be defined by $t(e) = 1$ for all $e \in E(G)$ and let H_N be an eulerian multigraph corresponding to an optimal CP-tour in $E(G)$, independent of v_0. Let D be the set of edges with multiplicity 2 in H_N. Since G is not a weakly Euler graph, there exists a non-bridge $e_0 \in D$. Let $(E(G), (G, v_0), t)$ be the delivery problem in which $e_0 = \{v_0, v_1\}$, and let $(E(G), c)$ be the corresponding delivery game. Suppose that $x \in Core((E(G), c))$. For any $e \in E(G)$, $x(e) = x(E(G)) - x(E(G)\backslash e) \geq c(E(G)) - c(E(G)\backslash e) \geq 0$, since $(E(G), c)$ is monotonic. Thus, $x(e) \geq 0$ for all $e \in E(G)$. Now, let $T := H_N \backslash 2e_0$ denote the multigraph derived from H_N by removing two copies of the edge e_0 therefrom. Since e_0 is a non-bridge, we can split T into two multigraphs T_0 and T_1, corresponding to the two walks between v_0 to v_1. Then

$$x(E(T_i) \cup e_0) \leq c(E(T_i) \cup e_0) = t(E(T_i)) + t(e_0) \quad (i = 0, 1),$$

where $E(T_i) \subseteq E(G)$ is the set of edges used by T_i. Adding these two inequalities yields:

$$x(E(G)) + x(D) \leq t(H_N) = x(E(G)).$$

Hence $x(D) \leq 0$. Since $x(e) \geq 0$, we have that $x(e) = 0$ for all $e \in D$. In particular, we have that $x(e_0) = 0$. Now,

$$x(E(G)\backslash e_0) \leq t(E(T)) = t(H_N) - 2t(e_0) = x(E(G)) - 2,$$

which implies that $x(e_0) \geq 2$, contradicting our earlier conclusion that $x(e_0) = 0$.

□

4. Submodular and balanced directed graphs

A strongly connected digraph is said to be weakly cyclic if each arc therein is contained in precisely one directed circuit. We prove in this section that CP-submodular digraphs are weakly cyclic and conversely, and we further show that any strongly connected digraph is CP-balanced.

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two connected graphs with $V(G_1) \cap V(G_2) = \emptyset$. A *1-sum* of G_1 and G_2 is obtained by coalescing one vertex in G_1 with another in G_2.

Proposition 2. *A weakly cyclic directed graph is a 1-sum of directed circuits.*

Proof. Clearly, a 1-sum of directed circuits is a weakly cyclic directed graph. To prove the other direction, let G be a weakly cyclic directed graph and assume, on the contrary, that it is not a 1-sum of directed circuits. Consider all subgraphs of G which do not contain one node cutsets. That is, subgraphs not containing a single node whose removal will disconnect the subgraph. Observe that all these subgraphs are strongly connected. Since G is assumed not to be a 1-sum of directed circuits, there exists one such a subgraph, A, of G which contains a directed circuit, C, and a node v_1 on C whose degree is strictly larger than two. We claim that there exists another node v_2 on C and a directed path, P, in A which traverses v_1 and v_2 and such that $E(P) \cap E(C) = \emptyset$, where $E(P)$ (respectively $E(C)$) is the arc set of P (respectively C). Indeed, assume that such a directed path does not exist, and let $e = (v_1, v_3)$ be an arc incident to v_1 which is not on C. Then, any directed path from v_3 to v_1 can only have node v_1 in common with C. Similarly, if (v_3, v_1) is an arc incident to v_1, $v_3 \notin C$, then any directed path from v_1 to v_3 will only have node v_1 in common with C. But this implies that v_1 is a one node cutset in A, which is a contradiction. Thus, A and therefore G contains a directed circuit C and a directed path P traverses v_1 and v_2, where v_1 and v_2 are nodes on C, such that $E(P) \cap E(C) = \emptyset$. This implies the existence of an arc in G which belongs to two directed circuits, contradicting our assumption that G is weakly cyclic.

□

The following Proposition demonstrates that a CP-submodular graph is weakly cyclic.

Proposition 3. *A CP-submodular digraph is weakly cyclic.*

Proof. Let $(E(G), (G, v_0), t)$ be a delivery problem and let $(E(G), c)$ be the corresponding delivery game. Suppose G is not weakly cyclic. Then, by Proposition 2, G is not a 1-sum of directed circuits. Hence, as it was demonstrated in the proof of Proposition 2, there exist a directed circuit C in G with nodes w_1, w_2 therein and a directed path $P : w_1 \rightarrow w_2$ with $E(P) \cap E(C) = \{w_1, w_2\}$. Suppose C is oriented clockwise. Then, there exist three internally vertex-disjoint directed paths $P_1 : w_1 \rightarrow w_2$, $P_2 : w_1 \rightarrow w_2$ and $P_3 : w_2 \rightarrow w_1$ in G. Let E_1, E_2 and E_3 be the sets of arcs contained in P_1, P_2, and P_3, respectively. Let w_1 be the post office, let $t(e) = 1$ for all arcs contained in P_1, P_2 and P_3, and let $t(e) = \max\{| P_1 |, | P_2 |, | P_3 |\} + 1$ for all other arcs e, where

$| P_j |$, $j = 1, 2, 3$ denotes the number of arcs in P_j. Then

$$\begin{aligned} c(E_1 \cup E_2 \cup E_3) + c(E_3) &= (| P_1 | + | P_2 | + 2 | P_3 |) \\ &\quad + (| P_3 | + \min\{| P_1 |, | P_2 |\}) \\ &> (| P_1 | + | P_3 |) + (| P_2 | + | P_3 |) \\ &= c(E_1 \cup E_3) + c(E_2 \cup E_3), \end{aligned}$$

implying that c is not a submodular function, contradicting our assumption that G is CP-submodular.

□

Let G be a weakly cyclic digraph and let v_0 be an arbitrary vertex therein. By Proposition 2, G is a 1-sum of directed circuits. Therefore, we can associate a directed tree $T(G, v_0)$ with (G, v_0) as follows. All arcs in the tree $T(G, v_0)$ are directed towards v_0, the root of the tree. A directed circuit in G, consisting of the arc set S, corresponds to an arc a_S in $T(G, v_0)$, and vertex v_S in $T(G, v_0)$ is the tail of arc a_S therein. Further, if two directed circuits, C_1 and C_2, consisting of arc sets S_1 and S_2 in G have a common vertex and the directed path from any node in C_1 to v_0 in G contains some arcs in C_2, then v_{S_2} is the head of arc a_{S_1} in $T(G, v_0)$. The directed circuit, C, that contains v_0 corresponds to arc a_s in $T(G, v_0)$ whose head therein is v_0.

Let $\Gamma = (E(G), (G, v_0), t)$ be a delivery problem associated with G. Its corresponding directed tree problem is defined to be $\mathcal{T}, \mathcal{T} = \{E(G), T(G, v_0), t^*\}$, where $T(G, v_0)$ is the directed tree associated with (G, v_0) and t^* is the cost function in $T(G, v_0)$ satisfying $t^*(a_S) = \sum_{e \in S} t(e)$, for every directed circuit consisting of arcs S in G. The players $E(G)$ are assigned to vertices in $T(G, v_0)$. Explicitly, if S is the set of players (arcs) in a circuit of G, its corresponding vertex, v_S, in $T(G, v_0)$ contains the set of players S.

Let $(E(G), c)$ be the delivery game corresponding to $\Gamma = (E(G), (G, v_0), t)$ and let $(E(G), c^*)$ be the game corresponding to $\mathcal{T} = (E(G), T(G, v_0), t^*)$, where, for each $S \subseteq E(G)$, $c^*(S)$ is the total cost of all arcs in the minimal subtree of $T(G, v_0)$ that is rooted at v_0 and contains all vertices which contain players in S. By construction of the tree graph $T(G, v_0)$, there is a one-to-one correspondence between arcs in the tree and circuits in G. From this observation and the location of the players at vertices in the tree it follows that

$$c(S) = c^*(S) \text{ for all } S \subseteq E(G). \tag{7}$$

Display (7) implies that delivery games which arise from weakly cyclic digraphs are equivalent to the class of tree games, introduced by Megiddo [13].[3] Granot et al. [5] observed that tree games are submodular, which, in combination with Proposition 3, results in the following Theorem.

Theorem 3. *A connected digraph G is weakly cyclic if and only if G is CP-submodular.*

[3] Megiddo's [13] (standard) tree games were defined for situations $\mathcal{T} = (E(G), T(G, v_0), t^*)$ in which $T(G, v_0)$ is a directed tree with v_0 as the unique root and t^* is the arc weight function. Players reside in vertices of the tree, except for the root vertex, and the cost function $c(.)$ in the tree game assigns to each set of players, S, the cost of a minimal subtree rooted at v_0, which contains all vertices which contain players in S.

Megiddo [13] proved that for tree games Shapley value can be computed in $\mathcal{O}(n)$ and the nucleolus can be computed in $\mathcal{O}(n^3)$, where n is the number of vertices in the tree. Galil [3] improved Meggido's algorithm and demonstrated that the nucleolus of a tree game can be computed in $\mathcal{O}(nlogn)$. In [5] and [18] other algorithms are developed for computing the nucleolus of a tree game. Obviously, all these algorithms can be used to compute the nucleolus of delivery games that arise from CP-submodular digraphs.

By contrast with the undirected case, the class of CP-totally balanced digraphs properly contains the class of CP-submodular digraphs. Indeed, in the following example we present a digraph for which the corresponding delivery game is totally balanced but not submodular.

Example 1. Consider the graph G with $V(G) = \{v_0, v_1, v_2\}$ and $E(G) = \{(v_0, v_1), (v_1, v_2), (v_2, v_0), (v_1, v_0)\}$. Let $(E(G), (G, v_0), t)$ be the delivery problem in which $t(e) = 1$ for all $e \in E(G)$ and arcs (v_0, v_1), (v_1, v_2), (v_2, v_0) and (v_1, v_0) are identified with players 1, 2, 3 and 4, respectively. Then, it is easy to verify that the corresponding delivery game $(E(G), c)$ is totally balanced. However, it is not CP-submodular, since

$$c(\{1,2,4\}) - c(\{1,4\}) = 5 - 2 = 3 > 1 = 3 - 2 = c(\{1,2\} - c(\{1\}).$$

Similarly, the following theorem demonstrates that, by contrast with the undirected case, a connected digraph is always CP-balanced.

Theorem 4. *A connected directed graph is CP-balanced.*

Proof. Let G be a connected digraph, with an associated delivery problem $\Gamma = (E(G), (G, v_0), t)$ and a corresponding delivery game $(E(G), c)$. We have to show that $(E(G), c)$ is balanced.

For $S \subseteq E(G)$, consider the following linear programming (LP) problem:

$$\begin{aligned} & c^*(S) = \min \sum_{i,j \in E(G)} t_{ij} x_{ij} \\ & \text{subject to} \\ & \sum_{j \in E(G)} x_{ji} - \sum_{j \in E(G)} x_{ij} = 0 \text{ for all } i \in E(G) \\ & x_{ij} \geq 1 \text{ for all arcs } (v_i, v_j) \in S, \\ & x_{ij} \geq 0 \text{ for all arcs } (v_i, v_j) \notin S, \end{aligned} \tag{8}$$

where t_{ij} denotes the cost of arc (v_i, v_j) and x_{ij} denotes the flow in arc (v_i, v_j). For $S = E(G)$ an optimal solution for (8) is a minimum cost circulation in G such that the flow in each arc is at least one. In fact, the optimal value of (8) for $S = E(G)$ is the cost of an optimal Chinese postman tour in G with cost function t (cf. [16]). Therefore, we conclude that $c^*(E(G)) = c(E(G))$. For $S \neq E(G)$ an optimal solution to (8) will consist of minimum cost circulations on G which may be disconnected. In fact, $c^*(S)$ is equal to the total cost of minimum cost (sub)tours that visit each arc of S at least once. In a minimal delivery tour of coalition S, each arc of S is also visited at least once. However, this tour has to be connected and must contain v_0. We conclude therefore that $c^*(S) \leq c(S)$ for all $S \subset E(G)$.

For a set of players (arcs) $S \subseteq E(G)$, let b^S denote the right hand side vector in (8). Then, one can easily verify that $b^S = \sum_{(i,j)\in S} b^{(i,j)}$, where $b^{(i,j)} = 1$ if $(v_i, v_j) \in S$ and $b^{(i,j)} = 0$ otherwise. Thus, (8) presents a linear production game formulation of $(E(G), c^*)$, and by Owen [17] it follows that $(E(G), c^*)$ is totally balanced. Since $c^*(E(G)) = c(E(G))$ and $c^*(S) \leq c(S)$ for each $S \subset E(G)$, it follows that $(E(G), c)$ is balanced.

□

We note that it follows from Owen that if u_{ij} is an optimal dual variable associated with the lower bound constraint in the LP problem (8) associated with $S = E(G)$, then $u = ((u_{ij}) : (v_i, v_j) \in E(G))$ is in the core of the delivery game $(E(G), c)$. Therefore, it follows from Tardos [20] that a core point in a delivery game associated with an arbitrary digraph can be found in strongly polynomial time.

Finally, we note that the recognition problem of a weakly cyclic digraph G can be solved by considering the undirected underlying graph associated with G. Then essentially the same procedure for the recognition problem in the undirected case can be applied to the directed case. The only difference lies in the last step where one has to verify if each biconnected component is a directed circuit. However, this step can also be done in linear time. Hence, we conclude that the recognition of a weakly cyclic digraph can be done in linear time.

We conclude this section by considering briefly the case where the underlying graph $G = (V(G), E(G))$ is mixed. That is, an element in $E(G)$, which will be referred to as a connection, is either an arc or an edge. P is said to be a mixed path from v_1 to v_2 in G if the underlying undirected graph associated with P is a path between v_1 and v_2, and all arcs in P are directed from v_1 to v_2. A mixed circuit in G is defined similarly.

A strongly connected mixed graph G is said to be weakly cyclic if each connection therein is contained in at most one mixed circuit. Then, using essentially the same proof technique as that used in Proposition 2, one can prove that if G is a strongly connected weakly cyclic mixed graph, then all subgraphs of G which do not contain one node cutset are mixed circuits. Further, we can show:

Theorem 5. *A strongly connected mixed graph G is weakly cyclic if and only if G is a CP-submodular graph.*

The proof of the only if part of Theorem 5 is similar to the proof that (*i*) implies (*ii*) in Theorem 1. The proof of the if part of Theorem 5 can be obtained using essentially Proposition 3 and the extension of Proposition 2 discussed in the previous paragraph.

Finally, we note that as in the undirected and directed cases, mixed weakly cyclic graphs can be recognized in linear time.

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