Cooperative decision making in a stochastic environment

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STELLINGEN BEHORENDE BIJ HET PROEFSCHRIFT

Cooperative Decision Making in a Stochastic Environment

van

Jeroen Suijs

I

"Motorracing is dangerous by definition. Motorracing is exciting. Motorracing is a sport that brings people to the limit, man and machine, and it must be seen as that. And people that are on the limit and equipment that work on the limit are bound to go wrong from time to time, and that must be taken into consideration to be a part of the game."

Ayrton Senna (1960 - 1994)

II

In een sociaal keuze probleem waar de individuen hun preferenties over de te kiezen alternatieven baseren op quasi-lineaire nutsfuncties, is de klasse van zogenaamde Groves-mechanismen de enige klasse die voldoet aan de eigenschap 'incentive compatibility' dan en slechts dan als het domein van preferenties graaf-samenhangend is.

zie ook: SUIJS, J.: *'On Incentive Compatibility and Budget Balancedness in Public Decision Making ,'* Economic Design, **2** (1996), 193-209.

III

Voor de klasse van cooperatieve spelen gebaseerd op machinevolgorde problemen zoals beschreven in CURIEL et al. (1989) is de Split Core de grootste verzameling die voldoet aan efficientie, monotonie en de dummy eigenschap.

zie ook: HAMERS, H., J. SUIJS, S. TIJS en P. BORM: 'The Split Core for Sequencing Games,' Games and Economic Behavior, **15** (1996), 165-176 en CURIEL, I., G. PEDERZOLI en S. TIJS: *'Sequencing Games,'* European Journal of Operational Research, **40** (1989), 344-351.

IV

De EGS-regel en de Split Core, twee oplossingsconcepten die de kostenbesparingen voortvloeiend uit een machinevolgorde probleem verdelen over de aanwezige jobs, kunnen gekarakteriseerd worden met behulp van consistentie indien men de niet-geaggregeerde variant van deze oplossingsconcepten beschouwd.

zie ook: SUIJS, J., H. HAMERS en S. TIJS: 'On Consistency of Reward Allocation Rules in Sequencing Situations,' in *Ten Years LNMB*, ed. by W. Klein Haneveld, O. Vrieze en L. Kallenberg. Amsterdam: CWI Tract, 1997.

V

Cooperatieve spelen die gebaseerd zijn op machinevolgorde problemen met meerdere machines en identieke bewerkingstijden voor de jobs zijn gebalanceerd.

zie ook: HAMERS, H., F. KLIJN en J. SUIJS: 'Balancedness of m-Machine Sequencing Games,' *Working Paper Tilburg University.*

VI

De Formule 1 Grand Prix races zullen nog spannender worden wanneer het gebruik van voor- en achtervleugels tot een minimum beperkt wordt, en het gebruik van het zogenaamde 'ground-effect' weer wordt toegestaan in het ontwerp van Formule 1 racewagens.

VII

De voornaamste reden waarom zowel kinderen als volwassenen vallen wanneer zij voor het eerst leren fietsen, is niet zozeer het gebrek aan evenwicht door het ontbreken van een of meer extra wielen, maar wel het feit dat men niet bekend is met het stuurgedrag van een fiets: bij normale snelheden heeft een stuurbeweging naar links een bocht naar rechts tot gevolg en omgekeerd.

zie ook: CODE, K.: *A Twist of the Wrist II - The Basics of High-Performance Motorcycling Riding,* Motorbooks International, 1993 en ROBINSON, J.: *Motorcycle Tuning – Chassis*, Butterworth-Heinemann, 1994.

VIII

In tegenstelling tot hetgeen voetbalverslaggevers regelmatig beweren, is geluk niet afdwingbaar.

IX

Mensen die van zichzelf zeggen dat ze prettig gestoord zijn, zijn inderdaad gestoord.

X

Mensen hebben de neiging zich arroganter te gaan gedragen naarmate hun affiniteit met de voetbalclub Ajax toeneemt.

Cooperative Decision Making in a Stochastic Environment

Cooperative Decision Making in a Stochastic Environment

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Katholieke Universiteit Brabant, op gezag van de rector magnificus, prof. dr. L.F.W. de Klerk, in het openbaar te verdedigen ten overstaan van een door het college van decanen aangewezen commissie in de aula van de Universiteit op vrijdag 27 maart 1998 om 16.15 uur door

JERONIMUS PETRUS MARTINUS SUIJS

geboren op 24 februari 1971 te Dongen.

PROMOTOR: prof. dr. S.H. Tijs
COPROMOTOR: dr. P.E.M. Borm

Acknowledgements

About six years ago, spring 1992, I had it all figured out: graduate as soon as possible, find a job and make lots of money. Two years later, however, I shared a small office and was earning less than minimum wage; On the brighter side though, I played games all day long, had no commitments and no responsibilities, well, almost none. The only thing I had to do was to write a Ph.D. thesis ... in four years time. That is not bad at all if you ask me. So thanks ever so much to Stef Tijs and Peter Borm for showing me the pleasure one can take in scientific research and game theory in particular. If it was not for them and their excellent supervision and ever lasting enthusiasm, this thesis would definitely not have existed. But my supervisors are not the only persons I am indebted to. Many thanks go to Anja De Waegenaere, who participated in some major parts of the research. Her non-game-theoretical view of the matter really contributed something special. Furthermore, I would like to thank Carles Rafels, Hans Peters and Jos Potters for joining the thesis committee.

Naturally, thanks go out to my parents, family, and friends. I am grateful to Tilburg University, the Department of Econometrics, and CentER. Not only did they considerably increase my wage over the past four years, but they also created a pleasant and stimulating environment to work in. Of course, for this latter part also my colleagues are to blame. Thanks!

Finally, I want to thank everyone who, in one way or another, contributed to this thesis or only thinks he did.

Contents

Notations

Many of the notations we use are defined in the text at their first appearance. The following symbols and notations are used troughout this monograph.

The set $\{0, 1, 2, \ldots\}$ of natural numbers is denoted by $\mathbb{N}$ and the set of real numbers is denoted by $\mathbb{R}$. For any finite set N we denote the power set by 2^N, that is, $2^N = \{S | S \subset N\}$, and the number of elements by $\#N$. Furthermore, $\mathbb{R}^N$ denotes the set of all functions from N to $\mathbb{R}$. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x_i)_{i \in N}$.

The set of real valued stochastic variables with finite expectation is denoted by $L^1(\mathbb{R})$. For a brief survey on probability theory and stochastic variables we refer to Appendix B.

To improve the readability of this text, we try to keep to the following notations as much as possible. Given a finite set N, the elements of $\mathbb{R}^N$ are denoted by lower case letters and subsets of $\mathbb{R}^N$ are denoted by upper case letters. So, $x \in \mathbb{R}^N$ and $X \subset \mathbb{R}^N$. Similarly, random variables belonging to the set $L^1(\mathbb{R})$ are denoted by bold faced upper case letters while sets of random variables are denoted by calligraphic letters. Thus, $\boldsymbol{X} \in L^1(\mathbb{R})$ and $\mathcal{X} \subset L^1(\mathbb{R})$.

1 Introduction

Three women, who once made their way to the pinnacle of show business but are now down and out, live their lives drifting about on the streets reminiscing the good old days. One day, when going through their daily routine of searching other people's garbage for food, something strange happened. As they lifted the lid of yet another garbage can, deepest thunder shook the skies. Suddenly a Genie appeared - right before their very eyes! "I'm the Genie of the garbage can," said the Genie with a laugh. "I've been stuck inside this garbage can for over three weeks! You opened the can and set me free. As a sign of my gratitude, I will let you have a wish each and make them all come true." The women, let us call them Linda, Roos, and Jessica, are stunned. Not completely back to their senses, they happily accept the Genie's offer and, God knows why, wish for two spades and a Ph.D. in economics, respectively. "Oh dear," says the Genie, "I think you poor little girls have slightly overestimated my magic powers. The spades are no problem, I can do that, but a Ph.D. in economics, that is way beyond my powers . . . You know what, let me give you this old treasure map instead. It goes better with the spades anyway," after which he disappears into distance leaving the girls behind with two spades and a treasure map.

Dreaming of all the riches that will be theirs soon, Linda, Roos, and Jessica go on their way hunting the treasure. Once arrived on the spot, Linda and Roos immediately put their spades into action and, after a few hours of digging, they have finally found it: a huge fridge filled with 24 kilos of food. Though not quite the treasure they expected, the sight of all that delicious food makes their mouths water. Especially Linda and Roos, who worked up quite an appetite by digging up the fridge, can hardly wait to divide the food. But how are they going to divide the food?

Suppose Roos proposes an equal division of the food, that is, 8 kilos each. Then Jessica may claim a higher share by arguing that, without the use of her map, the other two would

never have found the treasure. She should, however, not forget that without a spade she would not have dug out the treasure either, making her claim for a larger share rather weak. Her claim for more food would be much stronger though, if she collaborates with Linda. Presently, Linda and Jessica would receive 8 kilos each, making a total of 16 kilos for the both of them. Since Linda and Jessica together possess the map as well as a spade, they could have found the treasure without any help from Roos. In that case, they would have 24 kilos for the two of them. Compared with the 16 they get at the moment, their claim for a larger share of the food is reasonable, if not obvious. Moreover, Roos cannot claim anything; she would never have found the treasure on her own. This means that Linda and Jessica can claim all of Roos's share, leaving Roos with 0 kilos and Linda and Jessica with 12 kilos each.

This distribution is, to say the least, not Roos's favorite. But what can she do? On her own, there is no way she can convince the others to give her a share of the food. So for any chance to get some food, she has to turn to either Linda or Jessica for support; and for this case, she should turn to Jessica. For if Roos and Jessica collaborate, they can claim Linda's share of 12 kilos by using a similar argument as Linda and Jessica used to claim Roos's share. Now Roos receives 6 kilos, Jessica receives 18, and Linda receives nothing. Linda for her part then unites again with Jessica, to claim Roos's 6 kilos of food. The resulting distribution gives 3 kilo to Linda, 21 to Jessica, and nothing to Roos. These claims will go back and forth until Jessica receives all 24 kilos of food and Linda and Roos receive nothing. Though this distribution seems unfair to both Linda and Roos, they cannot really object against it. Since they do not possess the map, they would never have found the treasure, making any claim for more food not very convincing.

The situation described in the example can be interpreted as a cooperative decision making problem: a situation involving several persons with possibly diverse interests, who can benefit from cooperating with each other. Linda, Roos, and Jessica, for instance, got some food by cooperating with each other. When dividing the money, however, mutual conflicts arose because each of them wanted her share to be as large as possible. For analyzing such situations we can turn to cooperative game theory. In fact, the allocation that gives everything to Jessica is a so-called core-allocation of the game played by Linda, Roos, and Jessica. But cooperative game theory, of course, has not been developed for analyzing fictitious examples like this one. Since its introduction in VON NEUMANN and MORGENSTERN (1944), cooperative game theory serves as a way to model the economic behavior of individuals. To illustrate, consider the following examples of cooperative decision making problems that have been subject to a game theoretical analysis in the past.

Example 1.1 In an exchange economy, each agent is endowed with a certain bundle of consumption goods. Now, each agent can just consume his own bundle, but it could well be

possible that an agent possesses some commodities he does not like. He would rather like to exchange these ones for goods he likes better. So, agents may benefit from exchanging their initial endowments. The question that remains is, which commodities will eventually change hands? These cooperative decision making problems, also referred to as market games, are analyzed in e.g. SCARF (1967) and SHAPLEY and SHUBIK (1969).

Example 1.2 The following class of so-called linear production games is introduced in OWEN (1975). These games are derived from the following type of situations. Consider a group of agents, each having a bundle of resources which can be used to produce a number of consumption goods. For the production of these goods, every agent has access to the same linear production technology. Once the goods are produced, they are sold for given prices. In this situation the agents could individually use their own resources to produce a bundle of consumption goods that maximizes the revenues. But they can probably do better if they cooperate with each other and combine their resources. For instance, there may be a consumption good that an agent cannot produce on his own, because he lacks some of the resources to produce it. Cooperating with agents who do possess the absent resources, then enables them to produce this good, and, subsequently, pocket the corresponding revenues. The problem that needs to be solved is, how to divide the revenues among the agents.

Example 1.3 Consider a large firm that consists of several divisions, which all have to make use of the same facility - for instance, a repair and maintenance facility - that can only serve one division at a time. When divisions ask for service of the repair and maintenance facility, each division incurs costs for the time it is waiting for service. Now, suppose that a fixed number of divisions has requested service from this facility and that initially they will be served on a first come first served basis. Since the service demanded by the several divisions need not be equally urgent, a division with relatively high urgency may significantly decrease its waiting costs if it could move further forward in the queue. So, by rearranging their positions in the queue, the divisions might decrease the total waiting costs. These situations are known in the literature as sequencing games and were introduced by CURIEL, PEDERZOLI, and TIJS (1989). They proposed a 'fair' division of the total benefits, that is, the difference in total waiting costs between the initial serving order and the optimal serving order.

Cooperative game theory is a mathematical tool to analyze decision making problems like the ones presented in Examples 1.1 - 1.3. A cooperative game is usually either of two types: a cooperative game with transferable utility, henceforth referred to as a TU-game, or a cooperative game with non-transferable utility, henceforth referred to as an NTU-game. The main difference between these two types of games is the way in which the benefits of cooperation are described. Roughly speaking, a TU-game describes for each subgroup the

benefits from cooperation by a single number, while an NTU-game describes these benefits by a set of utility vectors. In fact, the games referred to in Example 1.2 and Example 1.3 are both TU-games while the market games discussed in Example 1.1 are of the NTU-type.

These examples all have one common aspect though, namely, that all benefits from cooperation are deterministic. When several persons decide to cooperate with each other, they exactly know beforehand what the benefits will be. One can, however, think of situations where this is not the case. In our introductory example, for instance, Linda and Roos have to decide whether or not they make the effort to dig up the treasure before they know what this treasure will bring them. The situations described in Example 1.2 and Example 1.3 can also be formulated in an uncertain setting. To start with the linear production situations, when production takes a considerable amount of time, prices may have changed during production. So, at the time that agents decide upon their production plan, they are not exactly sure at what prices these goods can be sold. As a consequence, they may have to choose between a production plan that yields high revenues with only small probability and a production plan that yields only moderate revenues with high probability. A similar reasoning holds for the sequencing situations described in Example 1.3. The service time a division needs is only known with certainty once the service has ended. So, when divisions rearrange the serving order, the benefits they generate are uncertain due to the unknown serving times.

An example of a different order involves insurance problems. A health-insurance, for instance, can be interpreted as cooperation between an individual and an insurance company. An individual may find the costs resulting from any health problems too high to bear on his own. Therefore, he could turn to an insurance company, who, in exchange for an insurance premium, takes over part of these costs. So, an insurance is just a redistribution of yet unknown costs between an individual and an insurance company. Furthermore, cooperation yields no decrease in the total costs, both parties possibly benefit from redistributing these costs only.

These kind of cooperative decision making problems, where the benefits from cooperation are not known with certainty, are the subject of this thesis. The aim is to develop a theory for these so-called stochastic cooperative games by using traditional cooperative game theory as a guideline. Here, traditional alludes to both TU-games and NTU-games. For these types of games, attention has been and still is focused on two issues, namely, which individuals will eventually cooperate with each other, and how should the resulting benefits be divided? Although both questions may be considered equally important, most of the past research on cooperative game theory has focused on the latter question. In fact, the same thing can be said about this monograph; it does not focus on the problem of coalition formation. Instead, it presumes that one large coalition will be formed, and focuses on the allocation problem of the corresponding benefits.

This monograph is organized as follows. Chapter 2 introduces the reader into deterministic cooperative game theory. We formulate a mathematical framework to describe cooperative decision making problems in general. For analyzing these problems we then turn to cooperative game theory. Which type of cooperative games to use depends on the characteristics of the cooperative decision making problem that is under consideration. Examples of cooperative games are, of course, TU-games and NTU-games, but also chance-constrained games. In Chapter 2 we provide a brief survey of each of the three types of games.

We start with explaining when a situation can be modeled as an NTU- or TU-game, followed by a short survey of both areas. For TU-games we provide the definitions of some well-known solution concepts like the core, the Shapley value, and the nucleolus. Furthermore, we state the main results in this regard. With respect to the class of NTU-games, we show to what extent the concepts that we discussed for TU-games also apply to NTU-games. It should be noted, however, that both surveys are rather concise. They confine to providing only those concepts that are needed in the forthcoming chapters on stochastic cooperative games.

In a separate section we discuss chance-constrained games. These games are introduced in CHARNES and GRANOT (1973) to encompass situations where the benefits obtained by the agents are random variables. Their attention is also focused on dividing the benefits of the grand coalition. Although the benefits are random, the authors allocate a deterministic amount in two stages. In the first stage, before the realization of the benefits is known, payoffs are promised to the individuals. In the second stage, when the realization is known, the payoffs promised in the first stage are modified if needed. In several papers, Charnes and Granot introduce some allocation rules for the first stage like the prior core, the prior Shapley value, and the prior nucleolus. To modify these so-called prior allocations in the second stage they defined the two-stage nucleolus. We will discuss all these solution concepts and illustrate them with some examples.

Chapter 3 is partly based on SUIJS, BORM, DE WAEGENAERE, and TIJS (1995) and SUIJS and BORM (1996). It introduces stochastic cooperative games, a class of games that deals with the same kind of problems as the chance-constrained games do, albeit in a completely different way. A drawback of the model introduced by CHARNES and GRANOT (1973) is that it does not explicitly take into account the individual's behavior towards risk. The effects of risk averse behavior, for example, are difficult to trace in this model. The model we introduce includes the preferences of the individuals. Any kind of behavior towards risk, from risk loving behavior to risk averse behavior, can be expressed by these preferences. Another major difference is the way in which the benefits are allocated. As opposed to a two-stage allocation, which assigns a deterministic payoff to each agent, an allocation in a stochastic cooperative game assigns a random payoff to each agent. Furthermore, for a two-stage allocation the agents must come to an agreement twice. In the first stage, before the realization of the payoff is known, they have to agree on a prior allocation. In the second stage, when the realization is known, they have to agree on how the prior payoff is modified. For stochastic cooperative games on the other

hand, the agents decide on the allocation before the realization is known. As a result, random payoffs are allocated so that no further decisions have to be taken once the realization of the payoff is known.

Besides the definition of the model, Chapter 3 also provides some examples arising from linear production situations, sequencing situations and financial markets. A separate section is devoted to the preferences of the agents. We show which type of preferences allow for risk averse, risk neutral, or risk loving behavior. Furthermore, we define several conditions that will be imposed upon the preferences in future chapters. In particular, we introduce a special class of preferences such that the random benefits can be represented by a deterministic number, also known as the certainty equivalent. For this specific class we can describe a stochastic cooperative game by a TU-game.

In Chapter 4 we extend the notions like core, superadditivity, and convexity of TU- and NTU-games to the class of stochastic cooperative games. In a nutshell, the core consists of those allocations that please each coalition in the sense that this coalition receives at least as much as it can obtain on its own. For TU-games, there is the well-known result that a core-allocation exists if and only if the game is balanced. For NTU-games, a balancedness condition also exists, except that in this regard it is only a sufficient condition for nonemptiness of the core. For stochastic cooperative games, we formulate such a balancedness condition for a particular subclass.

Superadditivity and convexity tell us something about how the benefits increase with the coalition size. To be more precise, superadditivity implies that two disjoint coalitions are better off by forming one large coalition. Hence, disjoint coalitions have an incentive to merge. Convexity then implies that this incentive increases as the size of the coalitions increases. Thus, for a coalition it is better to merge with a large coalition than with a small one. For TU-games it is known that every convex game is superadditive. Furthermore, a TU-game is convex if and only if all marginal vectors are core-allocations. For explaining a marginal vector, suppose that the grand coalition is formed in a specific order. Thus we begin with a one-person coalition, then an other agents joins this coalition, followed by another one, and so on. The marginal vector with respect to this order then assigns to each agent his marginal contribution, that is, the amount with which the benefits change when he joins a coalition. With respect to NTU-games, again only weaker results hold. For an NTU-game, two definitions of convexity exist. Both imply superadditivity and a nonempty core. Neither of the two, however, implies that all marginal vectors belong to the core. For stochastic cooperative games, we extend the definitions of superadditivity and convexity for TU-games, along the lines of SUIJS and BORM (1996). Our extension of convexity is based on the interpretation that a coalition can improve its payoff more by joining a large coalition instead of a small one. We show that convex stochastic cooperative games are superadditive. Furthermore, we show that convexity not only implies that core-allocations exist, but also that all marginal allocations belong to the core of the game. The reverse of this statement, however, is shown to be false: a stochastic cooperative

game is not necessarily convex if all marginal vectors belong to the core. Finally, we show that this new notion of convexity also leads to a new notion of convexity for NTU-games. Since the same result can be derived for NTU-games, that is, all marginal vectors are core-allocations, this new type of convexity thus differs from the existing notions of convexity for NTU-games.

Chapter 5 discusses a nucleolus for stochastic cooperative games that is introduced in SUIJS (1996). The nucleolus, a solution concept for TU-games, originates from SCHMEIDLER (1969). This solution concept yields an allocation such that the excesses of the coalitions are the lexicographical minimum. The excess describes how dissatisfied a coalition is with the proposed allocation. The larger the excess of a particular allocation, the more a coalition is dissatisfied with this allocation. For Schmeidler's nucleolus the excess is defined as the difference between the payoff a coalition can obtain when cooperating on its own and the payoff received by the proposed allocation. So, when less is allocated to a coalition, the excess of this coalition increases and the other way around.

Since the nucleolus depends mainly on the definition of the excess, we only need to specify the excesses for a stochastic cooperative game in order to define a nucleolus. Unfortunately, this is not that simple. Defining excess functions for stochastic cooperative games appears to be not as straightforward as for TU-games. How should one quantify the difference between the random payoff a coalition can achieve on its own and the random payoff received by the proposed allocation when the behavior towards risk can differ between the members of this coalition? Moreover, the excess of one coalition should be comparable to the excess of another coalition.

For defining the excess for stochastic cooperative games we interpret the excess of Schmeidler's nucleolus in a slightly different way. Bearing the conditions of the core in mind, this excess can be interpreted as follows. Given an allocation of the grand coalition's payoff we distinguish two cases. In the first case, a coalition has an incentive to part company with the grand coalition. Then the excess equals the minimal amount of money a coalition needs on top of what they already get such that this coalition is willing to stay in the grand coalition. In the second case, a coalition has no incentive to leave the grand coalition. Then the excess equals minus the maximal amount of money that can be taken away from this coalition such that this coalition still has no incentive to leave the grand coalition. This interpretation is used to define the excess for stochastic cooperative games.

Once the excess is defined, the nucleolus is defined in a similar way as for TU-games. We show that the nucleolus is a well-defined solution concept for stochastic cooperative games. Furthermore, we show that the nucleolus is a subset of the core, whenever the core is nonempty.

Chapter 6, the final chapter, contains an application of stochastic cooperative games to insurance problems and is based on SUIJS, DE WAEGENAERE, and BORM (1996). The insurance of possible personal losses as well as the reinsurance of the portfolios of insurance companies is modeled by means of a stochastic cooperative game. For, by cooperating with insurance companies an individual is able to transfer (part of) his future random losses to the insurance

companies. Thus, in doing so, he concludes an insurance deal. Similarly, by cooperating with other insurers an insurance company can transfer (parts of) her insurance portfolio to the other insurers. So, the insurance company concludes a reinsurance deal.

In this chapter our attention is focused on Pareto optimal allocations of the risks, and on the question which premiums are fair to charge for these risk exchanges. A Pareto optimal allocation is such that there exists no other allocation which is better for all participants. We show that there is essentially a unique Pareto optimal allocation of risk. It will appear that this Pareto optimal allocation of the risk is independent of the insurance premiums that are paid for these risk exchanges. For determining fair premiums, we look at the core of the insurance game. We show that the core is nonempty and that the zero utility principle for calculating premiums results in a core allocation.

2 Cooperative Game Theory

This chapter provides the reader with a brief introduction into cooperative game theory, making the usual distinction between games with transferable utility (TU-games) and games with non-transferable utility (NTU-games). A cooperative game mathematically models a multi-person decision making problem, in which individuals can obtain certain benefits by cooperating with each other. The questions it tries to answer are which individuals will eventually cooperate, and how do they divide the resulting benefits. The first question concerning the formation of coalitions falls outside the scope of this text. Instead, it is assumed that all individuals cooperate, so that our attention is restricted to dividing the benefits only.

We start with defining a mathematical framework to describe cooperative decision making problems in general. Section 2.2 then shows under which conditions such a problem gives rise to an NTU-game and a TU-game, respectively. By means of an example we try to explain when utility is transferable and when not and what the consequences are for describing the benefits. Subsequently, we state the formal definition of transferable utility. Next, in Section 2.3, a brief survey of TU-games follows. We discuss properties like superadditivity and convexity and define the solution concepts the core, the Shapley value, and the nucleolus. Furthermore, we state the main results concerning these concepts. The succeeding section then shows to what extent these results cary over to the class of NTU-games. Note, however, that both surveys are far from complete. They only provide the concepts necessary for understanding the remainder of this monograph. The chapter ends with a summary of chance-constrained games, in which the benefits from cooperation are stochastic variables. This summary is based on CHARNES and GRANOT (1973),(1976),(1977), and GRANOT (1977) and states the definitions of the so-called prior core, prior Shapley value, prior nucleolus, and the two-stage nucleolus. Furthermore, we compare these definitions with the corresponding ones for TU-games.

2.1 Cooperative Decision Making Problems

A cooperative decision making problem describes a situation involving several persons who can obtain benefits by cooperating. The problems they face are who will cooperate with whom, and how will the corresponding benefits be divided. Obviously, in which coalition someone is going to take part, depends on what part of the benefits this coalition has to offer. So, a coalition is only likely to be formed, if all the members of this coalition agree on a specific distribution of the benefits. Finding such an agreement, however, could be troublesome when these members have mutually conflicting interests.

For analyzing such situations, let us formulate a general mathematical framework that captures various cooperative decision making problems. For starters, let $N = \{1, 2, \ldots, n\}$ denote the finite set of individuals. To specify the benefits from cooperation, let Y be some topological space representing the outcome space, whose interpretation depends on the decision making problem that is under consideration. Moreover, this outcome space Y is such that the benefits generated by each coalition $S \subset N$, can be represented by a subset $Y_S \subset \prod_{i \in S} Y$. An outcome $(y_i)_{i \in S} \in Y_S$ then yields the payoff y_i to agent i. In order to evaluate different outcomes, each individual $i \in N$ has a preference relation $\succsim_i$ over the outcome space Y. So, agent i is only interested in what he himself receives. He does not take into account the payoffs of the other agents. Then given any two outcomes $y, \hat{y} \in Y$, we write $y \succsim_i \hat{y}$ if individual i finds the outcome y as least as good as the outcome $\hat{y}$. If agent i finds the outcome y strictly better than the outcome $\hat{y}$ we write $y \succ_i \hat{y}$. The preference relation $\succ_i$ is called the asymmetric part of $\succsim_i$; it holds that $y \succ_i \hat{y}$ if $y \succsim_i \hat{y}$ but not $\hat{y} \succsim_i y$. If agent i is indifferent between the outcomes y and $\hat{y}$ we write $y \sim_i \hat{y}$. The preference relation $\sim_i$ is the symmetric part of $\succsim_i$; it holds that $y \sim_i \hat{y}$ if both $y \succsim_i \hat{y}$ and $\hat{y} \succsim_i y$. Summarizing, a cooperative decision making model can be described by a tuple $(N, \{Y_S\}_{S \subset N}, \{\succsim_i\}_{i \in N})$. Let us illustrate this notation with some examples.

Example 2.1 For the situation described in Example 1.1, the set $N = \{1, 2, \ldots, n\}$ represents the individuals participating in the exchange economy. Let $\omega^i \in \mathbb{R}^m_+$ denote the bundle of consumption goods owned by agent i. So, there are m types of consumption goods and ω^i_j is the amount that agent i possesses of good j. Since an outcome describes a redistribution of the available consumption goods yielding each agent a commodity bundle in $\mathbb{R}^m_+$, the outcome space Y equals the commodity space $\mathbb{R}^m_+$. Next, consider a coalition $S \subset N$ of agents. The total number of commodities available to coalition S equals $\sum_{i \in S} \omega^i$. Hence, the outcome space Y_S equals

$$Y_S = \{(c^i)_{i \in S} \in (\mathbb{R}^m_+)^S | \sum_{i \in S} c^i \leq \sum_{i \in S} \omega^i\},$$

with $c^i \in \mathbb{R}^m_+$ representing the bundle of consumption goods assigned to agent $i \in S$. Note that the $\leq$-sign allows the agent the dispose some of the goods if they want to. Furthermore,

since $c^i \geq 0$ for all $i \in S$ the set Y_S is compact in $(\mathbb{R}^m_+)^S$. Now, a coalition S can benefit from cooperation if there exists an outcome $(c^i)_{i \in S} \in Y_S$ such that $c^i \succ_i \omega^i$ for each $i \in S$, that is, each agent $i \in S$ strictly prefers the bundle c^i to his initial endowment ω^i.

Example 2.2 Consider the linear production situation presented in Example 1.2. This situation can be formulated as a cooperative decision making problem in the following way. Let $N = \{1, 2, \ldots, n\}$ denote the set of agents. To specify the benefits from production, let r be the number of different resources and let m be the number of different consumption goods. Furthermore, let p_j be the price at which good $j = 1, 2, \ldots, m$ can be sold. Since the benefits consist of a division of the revenues generated by the sale of consumption goods, an outcome specifies for each individual i his share in the revenues. Hence, the outcome space equals $Y = \mathbb{R}$. The resources that are needed to produce a consumption bundle $c \in \mathbb{R}^m_+$ equal Ac, where $A \in \mathbb{R}^{r \times m}$ represents the linear production technology, which, by the way, is available to each agent. Now, if $b^i \in \mathbb{R}^r_+$ denotes the resource bundle of agent i, then the consumption bundles c that agent i can produce are $C(\{i\}) = \{c \in \mathbb{R}^m_+ | \, Ac \leq b^i\}$. Hence, the corresponding revenues he can obtain equal

$$Y_{\{i\}} \;=\; \{x \in \mathbb{R} | \; \exists_{c \in C(\{i\})} : x_i \leq p^\top c\}.$$

Since a coalition $S \subset N$ can use the resources of all its members, the feasible consumption bundles are $C(S) = \{c \in \mathbb{R}^m_+ | \, Ac \leq \sum_{i \in S} b^i\}$. An outcome for coalition S then equals a distribution of the revenues this coalition can obtain, that is,

$$Y_S \;=\; \{(x_i)_{i \in S} \in \mathbb{R}^S | \; \exists_{c \in C(S)} : \sum_{i \in S} x_i \leq p^\top c\}.$$

Finally, it seems likely that the preferences of an agent are such that the more money he receives the better. So, for $x, \hat{x} \in \mathbb{R}$ we have that $x \succsim_i \hat{x}$ if $x \geq \hat{x}$.

Example 2.3 Let us return to the sequencing situation described in Example 1.3. When modeling this sequencing situation as a cooperative decision making problem, the set $N = \{1, 2, \ldots, n\}$ denotes the divisions of the firm. Next, let $\sigma : N \to \{1, 2, \ldots, n\}$ denote the order in which the divisions are served, with $\sigma(i) = j$ meaning that division i takes position j in the queue. Furthermore, denote the initial serving order based on the 'first come first served' principle by σ_0. Now, it would be straightforward to take the outcome space Y to be the set of all possible serving orders σ. For this situation, however, this outcome space is too restrictive. For if two divisions i and j trade places in the serving order, the waiting time increases for, say, division i, and the waiting time decreases for division j. Since an increase in waiting time for division i increases its costs, division i only agrees to trade places with division j, if division i is compensated for her increased waiting costs. To enable such

compensations we allow transfer payments of money between the divisions. The outcome space thus becomes $Y = \mathbb{R} \times \Pi_N$, where $\mathbb{R}$ represents the amount of money a division receives and $\Pi_N = \{\sigma : N \to \{1, 2, \ldots, n\} | \ \forall_{i,j \in N: i \neq j} : \sigma(i) \neq \sigma(j)\}$ denotes the set of all possible serving orders. To determine the outcome space of a coalition S of divisions, note that a restraint is put on the rearrangements this coalition can make with respect to the serving order. Two divisions i and j of a coalition are only allowed to change places if all divisions that are waiting in between these two divisions also participate in the coalition. The outcome space for coalition S thus equals

$$Y_S = \{(x_i, \sigma)_{i \in S} \in \prod_{i \in S} (\mathbb{R} \times \Pi_S) | \ \sum_{i \in S} x_i \leq 0\},$$

with $\Pi_S = \{\sigma \in \Pi_N | \ \forall_{i \in S} \forall_{j \notin S} : \sigma(j) < \sigma(i) \Leftrightarrow \sigma_0(j) < \sigma_0(i)\}$ the set of feasible rearrangements for coalition S. The vector $(x_i)_{i \in S}$ represents the transfer payments between the divisions in coalition S. Since the amounts of money received by some divisions have to be paid by the other divisions, we have that $\sum_{i \in S} x_i \leq 0$.

Over the years, several game theoretical models have been introduced to deal with cooperative decision making problems like the ones presented in Examples 2.1 - 2.3. From all of these models, TU-games and NTU-games certainly are the most commonly used. Other, less frequently appearing cooperative games are, amongst others, multi-criteria games, multi-choice games and chance-constrained games. Which of these models best suits a specific cooperative decision making problem depends on the characteristics of the problem under consideration.

The exchange economy presented in Example 1.1 and Example 2.1, for instance, can be modelled as an NTU-game if the preferences of each agent can be represented by a utility function. This means that for each agent $i \in N$ there exists a function $U_i : \mathbb{R}^m_+ \to \mathbb{R}$ such that for any $c^i, \hat{c}^i \in \mathbb{R}^m_+$ it holds that $c^i \succsim_i \hat{c}^i$ if and only if $U_i(c^i) \geq U_i(\hat{c}^i)$, i.e., a bundle c^i is preferred to $\hat{c}^i$ if and only if the utility agent i gets from c^i exceeds the utility he gets from $\hat{c}^i$. If, however, the preferences are such that agent i prefers the bundle c^i to $\hat{c}^i$ if and only if $c^i_j \geq \hat{c}^i_j$ for $j = 1, 2, \ldots, m$, then a multi-criteria game applies best. Multi-choice games on the other hand, apply when an agent not only has to decide which coalition to join, but also has to decide upon the activity level of his cooperation. Such situations occur, for example, in construction work, where several firms work together on the construction of a building. Now, if these firms get paid dependent on the completion date, the revenues not only depend on which firms participate in the construction, but also on the effort they put into it.

Multi-criteria games and multi-choice games though, are not further discussed in this monograph. The interested reader is referred to BERGSTRESSER and YU (1977) and ANAND, SHASHISHEKHAR, GHOSE and PRASAD (1995) for multi-criteria games and to HSIAO and RAGHAVAN (1993) and NOUWELAND, POTTERS, TIJS and ZARZUELO (1995) for multi-choice games. In the forthcoming sections we focus on TU-games, NTU-games, and chance-

constrained games only. The next section explains when a cooperative decision making problem can be modeled as an NTU-game and TU-game, respectively.

2.2 Transferable and Non-transferable Utility

The terms transferable and non-transferable utility refer to the agents' preferences in a cooperative decision making problem. First of all, they imply that the preferences can be represented by a utility function. Given a cooperative decision making problem $(N, \{Y_S\}_{S\subset N}, \{\succsim_i\}_{i\in N})$, this means that for each agent $i \in N$ there exists a utility function $U_i : Y \to \mathbb{R}$ such that for any $y, \hat{y} \in Y$ it holds that $y \succsim_i \hat{y}$ if and only if $U_i(y) \geq U_i(\hat{y})$. Not every preference relation, however, can be represented by such a utility function. For this to be the case, consider the following three properties.

(P1) A preference relation $\succsim_i$ is *complete* if for any two outcomes $y, \hat{y} \in Y$ it holds that $y \succsim_i \hat{y}$, or $\hat{y} \succsim_i y$, or both.

(P2) A preference relation $\succsim_i$ is *transitive* if for any $y, \hat{y}, \tilde{y} \in Y$ such that $y \succsim_i \hat{y}$ and $\hat{y} \succsim_i \tilde{y}$, it also holds true that $y \succsim_i \tilde{y}$.

(P3) A preference relation $\succsim_i$ is *continuous* if for every $y \in Y$ the sets $\{\hat{y} \in Y \mid \hat{y} \succ_i y\}$ and $\{\hat{y} \in Y \mid y \succ_i \hat{y}\}$ are open in Y.

Property (P1) states that an agent can rank any two outcomes y and $\hat{y}$. So the case that they are incomparable does not occur. Property (P2) states that if an agent prefers the outcome y to the outcome $\hat{y}$, and he prefers $\hat{y}$ to $\tilde{y}$, then he also prefers y to $\tilde{y}$. Property (P3) is a kind of smoothness condition. Roughly speaking, it states that if an agent strictly prefers the outcome $\hat{y}$ to the outcome y, then he also strictly prefers the outcome $\hat{y}$ to all outcomes $\tilde{y}$ that differ only slightly from the outcome y.

The following result, which can be found in DEBREU (1959), provides sufficient conditions on $\succsim_i$ so that it can be represented by a utility function.

Theorem 2.1 Let $\succsim_i$ be a preference relation on a connected topological space Y. If $\succsim_i$ satisfies conditions (P1) - (P3), then there exists a continuous utility function $U_i : Y \to \mathbb{R}$ such that for any $y, \hat{y} \in Y$ we have that $y \succsim_i \hat{y}$ if and only if $U_i(y) \geq U_i(\hat{y})$.

Note that this utility function U_i is not uniquely determined. For if U_i represents the preferences $\succsim_i$, then so does any monotonic transformation of U_i. This means that if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then the utility function $\hat{U}_i$ defined by $\hat{U}_i(t) = f(U_i(t))$ for all $t \in \mathbb{R}$ also represents the preference relation $\succsim_i$.

For the remainder of this section assume that the conditions of Theorem 2.1 are satisfied. So, given a coalition S and its outcome space Y_S, we can determine the utility levels the

agents in this coalition can obtain individually. Then a cooperative decision making model $(N, \{Y_S\}_{S \subset N}, \{\succsim_i\}_{i \in N})$ can be modelled as an NTU-game (N, V), where N is the set of agents and $V(S) \subset \mathbb{R}^S$ is the set of utility levels coalition S can obtain, that is,

$$V(S) = \{(z_i)_{i \in S} \in \mathbb{R}^S | \exists_{(y_i)_{i \in S} \in Y_S} \forall_{i \in S} : z_i \leq U_i(y_i)\}. \tag{2.1}$$

Note that for mathematical convenience the set $V(S)$ not only includes the utility levels $(U_i(y_i))_{i \in S}$ for all $(y_i)_{i \in S} \in Y_S$, but also the utility levels $z \in \mathbb{R}^S$ which are worse. Finally, note that the preferences $\succsim_i$ need not be included in the description (N, V) of an NTU-game, because for each agent it holds that the higher his utility the better.

NTU-games were introduced in AUMANN and PELEG (1960) and generalize TU-games, which were already introduced in VON NEUMANN and MORGENSTERN (1944). Whether or not an NTU-game can be reduced to a TU-game, depends on the agents' utility functions. In fact, it depends on whether or not adding the individual utilities makes any sense. If so, the benefits from cooperation may be determined by adding the individual utilities and maximizing the resulting sum over all available outcomes. A TU-game can thus be described by a pair (N, v), where N is the set of agents and

$$v(S) = \max_{(y_i)_{i \in S} \in Y_S} \sum_{i \in S} U_i(y_i) \tag{2.2}$$

are the benefits for coalition $S \subset N$. Note that, by the same argument as for NTU-games, we do not need to include the preference relations $\succsim_i$ for $i \in N$, in the description (N, v) of a TU-game.

To explain the difference between transferable and non-transferable utility, let us return to the introductory example involving Linda, Roos, and Jessica. Jessica feels a bit sorry for Linda and Roos that they are left empty handed. For consolation, she therefore gives them her ticket to a Marco Borsato concert. Now Linda and Roos have one ticket for the both of them. Obviously, only one of them can actually go and attend the concert. So, who will be the lucky person to get the ticket?

Let us begin with specifying the outcome space Y and $Y_{\{L,R\}}$, respectively. To describe a distribution of the benefits, let the vector $y = (y_L, y_R)$ denote a distribution of the ticket. Since the ticket is useless when divided into smaller parts, it can be given to either Linda, to Roos, or, in case of disposal, to neither of them. Hence, the only possible allocations are $(1, 0)$, $(0, 1)$, and $(0, 0)$. Now, if the ticket is given to, say, Linda then Roos receives nothing. Obviously, Roos disapproves of this allocation. Similarly, if Roos receives the ticket, then Linda gets nothing and she disapproves of the allocation. In order to make some progress in this bargaining process, let us introduce transfer payments. So if, for instance, Linda receives the ticket she can pay money to Roos to compensate her for not receiving the ticket. Let the vector (x_L, x_R) denote these transfer payments. So, an outcome for, say, Linda states the

amount of money she receives and who receives the ticket. This means that we can take the outcome space Y equal to

$$Y = \mathbb{R} \times \{(1,0),(0,1),(0,0)\}.$$

Now, let us determine the outcome space $Y_{\{L,R\}} \subset Y \times Y$. Since the amount of money received by one person has to be paid by the other one, we have that $x_L + x_R \leq 0$. The outcome space $Y_{\{L,R\}}$ thus equals

$$Y_{\{L,R\}} = \{((x_L,y),(x_R,y)) \in Y \times Y |\, x_L + x_R \leq 0 \text{ and } y \in \{(1,0),(0,1),(0,0)\}\}.$$

Next, let the preferences of Linda and Roos be described by the following the utility functions:

$$\begin{aligned} U_L(x,y) &= \begin{cases} x+200, & \text{if } y=(1,0), \\ x, & \text{if } y \neq (1,0) \end{cases} \\ U_R(x,y) &= \begin{cases} x+400, & \text{if } y=(0,1), \\ x, & \text{if } y \neq (0,1), \end{cases} \end{aligned}$$

This means that Linda prefers an allocation (x,y) to the allocation $(\hat{x},\hat{y})$ if $U_L(x,y) > U_L(\hat{x},\hat{y})$. Similarly, Roos prefers (x,y) to $(\hat{x},\hat{y})$ if $U_R(x,y) > U_R(\hat{x},\hat{y})$. For instance, if $(x_L,x_R) = (-100,100)$ and $y=(1,0)$ then Linda's utility equals $-100+200=100$ and Roos's utility equals 100. Furthermore, if the ticket is given to Roos instead of Linda, that is, $y=(0,1)$, then Linda's utility equals -100 and Roos's utility equals 500. Hence, Linda prefers the first allocation to the latter, while Roos prefers the latter to the first.

Now, let us start with describing the benefits in terms of NTU-games. For this purpose we determine the utility levels of all possible allocations. First, consider the allocations that assign the ticket to neither Linda nor Roos, that is, $y=(0,0)$. Then given an allocation (x_L,x_R), Linda's utility equals x_L and Roos's utility equals x_R. Since $x_L + x_R \leq 0$, the utility levels Linda and Roos can obtain are

$$\{(x_L,x_R)|\, x_L + x_R \leq 0\}.$$

Second, consider the allocations that assign the ticket to Linda, that is $y=(1,0)$. Then Linda's utility equals x_L+200 and Roos's utility equals x_R. The utility levels they can obtain in this way are

$$\{(x_L+200,x_R)|\, x_L + x_R \leq 0\}.$$

Finally, consider the allocations that assign the ticket to Roos, that is, $y=(0,1)$. In that case, Linda's utility equals x_L and Roos's utility equals x_R+400. Hence, the corresponding utility levels are

$$\{(x_L,x_R+400)|\, x_L + x_R \leq 0\}.$$

As can be seen in Figure 2.1, which pictures these three sets, the set $\{(x_L, x_R+400)|\ x_L+x_R \leq 0\}$ contains both $\{(x_L+200, x_R)|\ x_L+x_R \leq 0\}$ and $\{(x_L, x_R)|\ x_L+x_R \leq 0\}$. So, instead of stating that Linda and Roos's benefits are a ticket for a Marco Borsato concert, we can say that cooperation earns them

$$V(\{L,R\}) \ = \ \{(x_L, x_R+400)|\ x_L+x_R \leq 0\}.$$

Formulating the benefits in this way is typical for NTU-games.

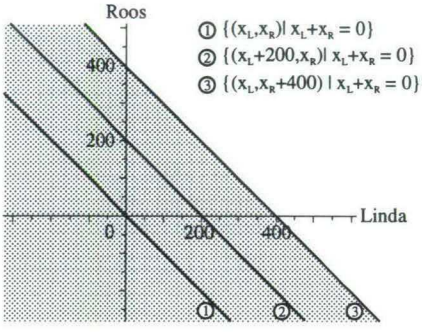


Figure 2.1

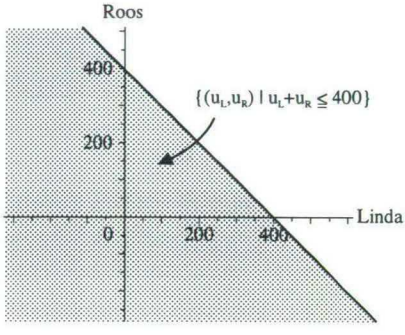


Figure 2.2

Next, let us describe the benefits in terms of TU-games. This means that we add the individual utilities and maximize this sum over all possible allocations. Solving

$$\begin{array}{ll} \text{maximize} & x_L + x_R + (200, 400)y^{\top} \\ \text{subject to:} & x_L + x_R \leq 0, \\ & y \in \{(1,0),(0,1),(0,0)\} \end{array}$$

yields a maximum of 400. So, instead of stating that Linda and Roos's benefits are a ticket for a Marco Borsato concert, we can say that cooperation earns them 400 units of utility. What is left to check is if adding the individual utilities is a justified operation, that is, does it make any difference whether we state the benefits as 400 or as $\{(x_L, x_R+400)|\ x_L+x_R \leq 0\}$. In answering this question, let us examine the utility levels Linda and Roos can obtain when they may allocate 400 units of utility. If u_L denotes the amount of utility that Linda receives and u_R the amount that Roos receives, then the utility levels they can realize are described by the set

$$\{(u_L, u_R)|\ u_L+u_R \leq 400\}.$$

This set is pictured in Figure 2.2. Comparing Figure 2.1 to Figure 2.2 reveals that the set $\{(u_L, u_R)|\ u_L+u_R \leq 400\}$ coincides with the set $\{(x_L, x_R+400)|\ x_L+x_R \leq 0\}$. Hence, each allocation $((x_L, y), (x_R, y))$ corresponds with an allocation $u = (u_L, u_R)$ of 400 units of utility, and the other way around; for each allocation $u = (u_L, u_R)$ there exists an

allocation $((x_L, y), (x_R, y)) \in Y_{\{L,R\}}$ such that the corresponding utilities coincide, that is, $U_L(x_L, y) = u_L$ and $U_R(x_R, y) = u_R$. This implies that it is indeed correct to state Linda and Roos's benefits to be 400.

Determining the benefits by maximizing the sum of the individual utilities does not work in general. Suppose, for instance, that Linda and Roos's utility functions are given by

$$\begin{aligned} W_L(x,y) &= \begin{cases} f(x) + 200, & \text{if } y = (1,0), \\ f(x), & \text{if } y \neq (1,0) \end{cases} \\ W_R(x,y) &= \begin{cases} f(x) + 400, & \text{if } y = (0,1), \\ f(x), & \text{if } y \neq (0,1), \end{cases} \end{aligned}$$

where

$$f(t) = \begin{cases} 2t, & \text{if } t \leq 0 \\ \frac{1}{2}t, & \text{if } t > 0. \end{cases}$$

So, Linda's utility for money depends on whether she receives money or has to pay it. Of course, a similar argument holds for Roos.

Again, let us start with describing the benefits in terms of NTU-games. To determine all utility levels Linda and Roos can realize, we distinguish three cases. In the first case, neither of them gets the ticket, that is, $y = (0,0)$. This means that Linda's utility equals $f(x_L)$ and that Roos's utility equals $f(x_R)$. Since $x_L + x_R \leq 0$, the utility levels they can obtain are

$$\{(f(x_L), f(x_R)) |\, x_L + x_R \leq 0\}.$$

In the second case, Linda gets the ticket, so that $y = (1,0)$. Then Linda's utility equals $f(x_L) + 200$ and Roos's utility equals $f(x_R)$. The utility levels they can achieve in this way thus are

$$\{(f(x_L) + 200, f(x_R)) |\, x_L + x_R \leq 0\}.$$

Finally, in the third case, Roos gets the ticket. Hence, $y = (0,1)$ so that Linda's utility equals $f(x_L)$ and Roos's utility equals $f(x_R) + 400$. The feasible utility levels then equal

$$\{(f(x_L), f(x_R) + 400) |\, x_L + x_R \leq 0\}.$$

Combining these three possibilities, yields the utility levels presented in Figure 2.3.

In contrast to the previous situation, describing the benefits by the maximal sum of the individual utilities is not possible. This can be seen as follows. The maximum equals 400 and is attained in the allocation $((x_L, y), (x_R, y)) = ((0,(0,1)), (0,(0,1))$, which corresponds to the point $(0, 400)$ in Figure 2.3. Now, if we state that the benefits equal 400 units of utility, then Linda and Roos may allocate this 400 between the two of them. This means that they can obtain the following utility levels

$$\{(w_L, w_R) |\, w_L + w_R \leq 400\},$$

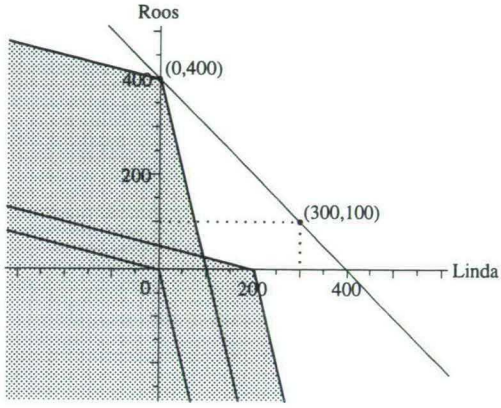


Figure 2.3

which, among others, also includes the allocation $(300, 100)$. But as Figure 2.3 shows, there exists no allocation $((x_L, y), (x_R, y)) \in Y_{\{L,R\}}$ that yields Linda and Roos a utility level of 300 and 100, respectively. Apparently, allocating 400 units of utility is not similar to allocating the ticket. So, what makes the difference then? Why does adding the utility functions U_L and U_R not raise any problems, while adding W_L and W_R does?

The answer to this question is the following property, which U_L and U_R possess, namely, that a transfer of money from one person to another, corresponds with a transfer of utility, hence the distinction between games with transferable utility and games with non-transferable utility. To explain this transferable utility property in more detail, consider an arbitrary allocation $((x_L, y), (x_R, y)) \in Y_{\{L,R\}}$ such that $y = (0, 1)$, that is, Roos receives the ticket. The utility levels $U_L(x_L, y)$ and $U_R(x_R, y)$ that correspond with this allocation thus are x_L for Linda and $x_R + 400$ for Roos. Now, suppose that an amount of t dollar is transferred from Roos to Linda, then Linda receives t dollar and Roos receives the ticket and $-t$ dollar. This transfer payment increases Linda's utility with t units to $x_L + t$, and it decreases Roos's utility with t units to $x_R - t + 400$. In other words, transferring t dollar from Roos to Linda corresponds with transferring t units of utility from Roos to Linda. So instead of transferring money they transfer units of utilities.

A consequence of this transferable utility property is that each individual's valuation for the ticket does not depend on the amount of money she receives. Indeed, Linda's utility equals $x_L + 200$ if she gets the ticket and x_L if she does not get it. The difference always amounts to 200, whatever the value of x_L. So Linda's value for the ticket is 200 units of utility. Similarly, it follows that Roos's value for the ticket is 400 units of utility.

Since utility is transferable, the more units of utility Linda and Roos can divide, the better it is for the both of them. This means that we need to focus on the allocations that maximize

the sum of the utilities. This sum equals 0 if no one receives the ticket, 200 if Linda receives the ticket, and 400 if Roos receives the ticket. Hence, the ticket should be given the Roos, so that the benefits equal 400 units of utility.

That the utility functions W_L and W_R do not satisfy the transferable utility property is easily seen as follows. Consider the allocation $((x_L, y), (x_R, y)) = ((0, (0, 1)), (0, (0, 1)))$, which corresponds to the point $(0, 400)$ in Figure 2.2. Now, if we transfer \$ 200 from Roos to Linda, so that Linda receives \$ 200 and Roos pays \$ 200, then Linda's utility only increases with 100 to $f(200) = 100$. Roos's utility, on the other hand, decreases with 400 to $f(-200) + 400 = 0$. Hence, transferring \$ 200 dollar from Roos to Linda does not transfer 200 units of utility. In fact, it decreases the total utility with 300. Although total utility has decreased, this does not mean that the allocation $((200, (0, 1)), (-200, (0, 1)))$ is worse than $((0, (0, 1), (0, (0, 1)))$; both allocations are optimal in the sense, that there exists no other allocation that yields a higher utility to both Linda and Roos (see also Figure 2.3).

The example above shows that not every utility function satisfies the transferable utility property. So, what utility functions do satisfy this transferable utility property, and, maybe even more important, which preferences can be represented by such utility functions? To start with the first question, recall that utility is transferable if a transfer of money between agents corresponds to a transfer of utility. For this to occur in the first place, the outcome space Y must at least allow for transfer payments between the agents. For if agents cannot transfer any money, a transfer of money can certainly not correspond to a transfer of utility. This means that Y can be written as $\mathbb{R} \times \tilde{Y}$, where $\mathbb{R}$ represents the amount of money an agent receives and $\tilde{Y}$ represents all other outcomes (see also Example 2.3 on sequencing situations). Then the outcome space Y_S for coalition S is a subset of $\prod_{i \in S}(\mathbb{R} \times \tilde{Y})$. The utility that agent $i \in S$ assigns to an outcome $((x_i, y_i))_{i \in S} \in Y_S$ equals $U_i(x_i, y_i)$. Moreover, it satisfies the transferable utility property if the utility function U_i is linearly separable in money, that is, if there exists a function $v_i : \tilde{Y} \to \mathbb{R}$ such that for each outcome $(x_i, y_i) \in \mathbb{R} \times \tilde{Y}$ it holds that

$$U_i(x_i, y_i) \;=\; x_i + v_i(y_i). \tag{2.3}$$

To see that utility is indeed transferable, consider two agents i and j such that $U_i(x_i, y_i) = x_i + v_i(y_i)$ and $U_j(x_j, y_j) = x_j + v_j(y_j)$. Let $i, j \in S$ and take an outcome $((x_k, y_k))_{k \in S} \in Y_S$. Next, let τ be a transfer payment of t dollars from agent i to agent j. Hence, agent i receives $x_i - t$ dollar and agent j receives $x_j + t$ dollar. Then $U_i(x_i - t, y_i) = x_i - t + v_i(y_i)$ and $U_j(x_j + t, y_j) = x_j + t + v_i(y_j)$. Since the transfer payment of t dollars from agent i to agent j decreases agent i's utility with t, and increases agent j's utility with t, it corresponds to a transfer of t units of utility. Furthermore, if all utility functions are of the form (2.3) then the benefits of coalition S can be described by the maximum sum of the individual utilities. The proof is straightforward. Since transfer payments between the agents are allowed, this means that for each outcome $((x_i, y_i))_{i \in S} \in Y_S$ it holds that $\sum_{i \in S} x_i \leq 0$. Then given the utility

functions $U_i(x_i, y_i) = x_i + v_i(y_i)$ for all $i \in S$ the benefits in terms of an NTU-game equals

$$V(S) = \{z \in \mathbb{R}^S | \exists_{((x_i,y_i))_{i\in S} \in Y_S} \forall_{i\in S} : z_i \leq x_i + v_i(y_i)\}.$$

Assuming that the maximum of the sum of the individual utility levels exists, it equals

$$\begin{aligned} v(S) &= \max\left\{\sum_{i\in S}(x_i + v_i(y_i)) | \, ((x_i, y_i))_{i\in S} \in Y_S\right\} \\ &= \max\left\{\sum_{i\in S} v_i(y_i) | \, ((x_i, y_i))_{i\in S} \in Y_S\right\}. \end{aligned}$$

When coalition S allocates $v(S)$, the utility levels the members of S can obtain equal $\{z \in \mathbb{R}^S | \sum_{i\in S} z_i \leq v(S)\}$. Hence, it is sufficient to show that $V(S) = \{z \in \mathbb{R}^S | \sum_{i\in S} z_i \leq v(S)\}$. That $V(S) \subset \{z \in \mathbb{R}^S | \sum_{i\in S} z_i \leq v(S)\}$ follows immediately from

$$v(S) = \max\{\sum_{i\in S}(x_i + v_i(y_i)) | \, ((x_i, y_i))_{i\in S} \in Y_S\} = \max_{z\in V(S)} \sum_{i\in S} z_i.$$

For the reverse inclusion, take $\zeta \in \{z \in \mathbb{R}^S | \sum_{i\in S} z_i \leq v(S)\}$ and let $((0, y_i))_{i\in S} \in Y_S$ be such that $\sum_{i\in S} v_i(y_i) = \max\{\sum_{i\in S} v_i(\tilde{y}_i) | \, ((\hat{x}_i, \tilde{y}_i))_{i\in S} \in Y_S\} = v(S)$. Next, define $x_i = \zeta_i - v_i(y_i)$ for all agents $i \in S$. Then

$$\sum_{i\in S} x_i = \sum_{i\in S} \zeta_i - v_i(y_i) \leq v(S) - \max\left\{\sum_{i\in S} v_i(\tilde{y}_i) | \, ((0, \tilde{y}_i))_{i\in S} \in Y_S\right\} = 0,$$

and, consequently, $((x_i, y_i))_{i\in S} \in Y_S$. Since $U_i(x_i, y_i) = x_i + v_i(y_i) = \zeta_i$ for all $i \in S$ it follows that $\{z \in \mathbb{R}^S | \sum_{i\in S} z_i \leq v(S)\} \subset V(S)$.

Summarizing, if the utility functions are linearly separable in money we can model a cooperative decision making problem as a TU-game.

Next, let us turn to the second question, that is, which preferences can be represented by a utility function that is linearly separable in money. For characterizing these preferences we need the properties (P1), (P2) and three additional properties. So, let $\succsim_i$ be a preference relation on an outcome space $Y = \mathbb{R} \times \tilde{Y}$.

(P4) The preferences $\succsim_i$ are *strictly monotonic in* x if for any $(x, y), (\tilde{x}, y) \in \mathbb{R} \times \tilde{Y}$ it holds that $(x, y) \succ_i (\tilde{x}, y)$ if and only if $x > \tilde{x}$.

(P5) For any outcomes $(x, y), (\tilde{x}, \tilde{y}) \in \mathbb{R} \times \tilde{Y}$ with $(x, y) \succsim_i (\tilde{x}, \tilde{y})$ there exists $t \in \mathbb{R}$ such that $(x, y) \sim_i (\tilde{x} + t, \tilde{y})$.

(P6) For any outcomes $(x, y), (\tilde{x}, \tilde{y}) \in \mathbb{R} \times \tilde{Y}$ with $(x, y) \sim_i (\tilde{x}, \tilde{y})$ and every $t \in \mathbb{R}$ it holds that $(x + t, y) \sim_i (\tilde{x} + t, \tilde{y})$.

Property (P4) means that agent i prefers more money to less. Property (P5) states that a change in the outcome from (x, y) to $(\tilde{x}, \tilde{y})$ can be compensated by an amount of money t. Finally, property (P6) implies that agent i's preferences over the outcome space $\tilde{Y}$ does not depend on the amount of money x_i he receives.

The following theorem is due to AUMANN (1960) and KANEKO (1976) and characterizes preference relations with transferable utility.

Theorem 2.2 Let $\succsim_i$ be a preference relation on the outcome space $Y = \mathbb{R} \times \tilde{Y}$. If $\succsim_i$ satisfies conditions (P1), (P2), and (P4) - (P6), then there exists a utility function $v_i : \tilde{Y} \to \mathbb{R}$ such that for any $(x, y), (\hat{x}, \hat{y}) \in Y$ it holds that $(x, y) \succsim_i (\hat{x}, \hat{y})$ if and only if $x + v_i(y) \geq \hat{x} + v_i(\hat{y})$.

Remark that if U_i represents the preferences $\succsim_i$, then so does any monotonic transformation of U_i. The property of linear separability in money, however, may be lost after such monotonic transformations. For instance, suppose $U_i(x, y) = x + v_i(y)$ represents agent i's preferences $\succsim_i$ on $\mathbb{R} \times \tilde{Y}$. The utility function $\hat{U}_i$ defined by $\hat{U}_i(x, y) = e^{U_i(x,y)}$ for all $(x, y) \in \mathbb{R} \times \tilde{Y}$ also represents $\succsim_i$. The utility function $\hat{U}_i$, however, is not linearly separable in money since $\hat{U}_i(x, y) = e^{x+v_i(y)}$ for all $(x, y) \in \mathbb{R} \times \tilde{Y}$.

2.3 Cooperative Games with Transferable Utility

A *cooperative game with transferable utility*, or *TU-game*, is described by a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of agents and $v : 2^N \to \mathbb{R}$ is the *characteristic function* assigning to each coalition $S \subset N$ a value $v(S)$, representing the benefits from cooperation. In particular we have that $v(\emptyset) = 0$.

Example 2.4 Our introductory example involving Linda, Roos, and Jessica was considered to be a three-person NTU-game with $N = \{L, R, J\}$ and $v(S) = 24$ if $S \in \{\{L, J\}, \{R, J\}, \{L, R, J\}\}$ and $v(S) = 0$ otherwise. Thus the value of a coalition S is worth 24 if coalition S can find the treasure on its own, and its value is zero if this is not possible.

Example 2.5 For the linear production situation formulated in Example 2.2 we have the outcome space $Y = \mathbb{R}$ and for each $S \subset N$

$$Y_S = \{(x_i)_{i \in S} \in \mathbb{R}^S | \, \exists_{c \in C(S)} : \sum_{i \in S} x_i \leq p^\top c\}$$

with $C(S) = \{c \in \mathbb{R}^m_+ | \, Ac \leq \sum_{i \in S} b^i\}$ the set of feasible production plans. Since the outcome space Y represents allocations of money only, transferable utility implies that $U_i(x) = x$ for all $x \in \mathbb{R}$ and all $i \in N$. A linear production game (N, v) is then defined by

$$v(S) = \max\{\sum_{i \in S} U_i(x_i) | \, (x_i)_{i \in S} \in Y_S\} = \max\left\{p^\top c | \, Ac \leq \sum_{i \in S} b^i\right\}$$

for all $S \subset N$.

Example 2.6 Recall that for the sequencing situation described in Example 2.3 we have $N = \{1, 2, \ldots, n\}$, the set of divisions waiting for service,

$$Y = \mathbf{R} \times \Pi_N,$$

the outcome space, and

$$Y_S = \{((x_i, \sigma))_{i \in S} \in \prod_{i \in S} Y | \sum_{i \in S} x_i \leq 0 \text{ and } \sigma \in \Pi_S\}$$

the outcome space for coalition S, where Π_S denotes the set of admissible rearrangements with respect to the initial serving order σ_0. To comply with CURIEL ET AL. (1989), let the waiting costs for division $i \in N$ be given by $k_i(\sigma) = \sum_{j \in N: \sigma(j) \leq \sigma(i)} \alpha_i p_j$ with p_j the service time of division j and $\alpha_j > 0$. Thus, the waiting costs increase linearly with the waiting time. Then given a serving order σ the cost savings for division i equal $k_i(\sigma_0) - k_i(\sigma)$. Next, let $U_i(x, \sigma) = x + k_i(\sigma_0) - k_i(\sigma)$ for all $(x, \sigma) \in Y_S$. So, given an outcome (x, σ) division i's utility equals the amount of money x it receives plus the cost savings that correspond with the serving order σ. A sequencing game (N, v) is then defined by

$$\begin{aligned} v(S) &= \max\left\{\sum_{i \in S} U_i(x_i, \sigma) | \, ((x_i, \sigma))_{i \in S} \in Y_S\right\} \\ &= \max\left\{\sum_{i \in S} (k_i(\sigma_0) - k_i(\sigma)) | \, \sigma \in \Pi_S\right\}, \end{aligned}$$

for all $S \subset N$.

A TU-game (N, v) is called *superadditive* if for all disjoint $S, T \subset N$ it holds that

$$v(S) + v(T) \leq v(S \cup T). \tag{2.4}$$

This means that two disjoint coalitions do not suffer from forming one large coalition. Many economic situations will lead to superadditive games, for if two disjoint coalitions S and T join forces, they can often guarantee themselves the payoff $v(S) + v(T)$ by operating separately.

A TU-game (N, v) is *convex* if for all $i \in N$ and all $S \subset T \subset N \backslash \{i\}$ it holds that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T). \tag{2.5}$$

Convexity thus implies that an agent contributes more to the benefits of a coalition when this coalition becomes larger. Convexity first appeared in SHAPLEY (1971), and besides the definition given by (2.5), several other, equivalent definitions exist. Two of those are the following. The first one is quite similar to (2.5) and reads as follows: for each $U \subset N$ and each $S \subset T \subset N \backslash U$ it holds that

$$v(S \cup U) - v(S) \leq v(T \cup U) - v(T). \tag{2.6}$$

So, also if more than one agent joins a coalition, the change in the benefits increases when this coalition increases. The second alternative definition, on the other hand, differs significantly from expression (2.5), namely, for every $S, T \subset N$ it holds that

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T). \tag{2.7}$$

One can show that expression (2.7) implies that $v(S \cup T) - v(S) - v(T)$ is increasing in both S and T. Since $v(S \cup T) - v(S) - v(T)$ can be interpreted as the incentive for coalitions S and T to merge, convexity thus means that larger coalitions have a bigger incentive to join forces. Hence, convexity is a much stronger condition than superadditivity, which only states that two disjoint coalitions have an incentive to merge; superadditivity does not say anything about the magnitude of this incentive. That convex games are superadditive also follows immediately from (2.7) by taking S and T disjoint. Convexity, however, is not as generally satisfied as superadditivity. The linear production games described in Example 2.5, for instance, are superadditive but not necessarily convex.

What we are interested in are 'fair' allocations of the benefits $v(N)$. So, assuming that all agents are willing to cooperate with each other, how should the benefits $v(N)$ be divided so that each agent is satisfied. The game theoretical literature proposes many allocation rules as a solution to this problem. As we will show, some of these rules are generally applicable while others are designed to suit only very specific TU-games. Each of these rules, of course, has it own pros and cons and deciding which allocation rule applies best is a problem in itself.

An *allocation* of $v(N)$ is described by a vector $x \in \mathbb{R}^n$ such that $\sum_{i \in N} x_i \leq v(N)$. Moreover, an allocation x is called *efficient* or *Pareto optimal* if there exists no allocation $\hat{x}$ that yields every agent a higher payoff, i.e., $\hat{x}_i > x_i$ for all $i \in N$. Obviously, each agent i is only willing to participate in the grand coalition if it pays him at least as much as he can obtain on his own. This means that an allocation x must be such that $x_i \geq v(\{i\})$ for all $i \in N$. These allocations are called *individually rational*. The set of all individually rational and Pareto optimal allocations is called the *imputation set* and is defined by

$$I(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N), \forall_{i \in N} : x_i \geq v(\{i\})\}. \tag{2.8}$$

Note that the imputation set is nonempty if the game is superadditive.

The argument that an agent does not participate in the grand coalition N if he does not receive at least $v(\{i\})$ is reasonable. Moreover, it does not only apply to individuals, but also to coalitions. A coalition S has incentives to part company with the grand coalition if it can improve the payoff of each member. Mathematically, this means that given an allocation x of $v(N)$, coalition S has incentives to leave coalition N if there exists an allocation y of $v(S)$ such that $y_i > x_i$ for all $i \in S$. It is a straightforward exercise to check that such an allocation y does not exist if and only if $\sum_{i \in S} x_i \geq v(S)$. Hence, no coalition has incentives to separate

from the grand coalition if the allocation x is such that $\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$. Such an allocation x is called a *core-allocation*. The *core* of a TU-game (N, v) is thus defined by

$$C(v) = \{x \in I(v) | \forall_{S \subset N} : \sum_{i \in S} x_i \geq v(S)\}. \tag{2.9}$$

Although core-allocations provide the agents with an incentive to maintain the grand coalition, they need not always exist, as the following example shows.

Example 2.7 Consider a three-person TU-game (N, v) with $v(\{i\}) = 0$ for all $i \in N$ and $v(S) = 1$ for all $S \subset N$ with $|S| \geq 2$. Note that this game is superadditive but that its core is empty. To see that no core allocation exists, take $x \in I(v)$. So, $\sum_{i \in N} x_i = 1$ and $x_i \geq 0$ for $i = 1, 2, 3$. If x would be a core-allocation it must satisfy $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, and $x_2 + x_3 \geq 1$. Adding these three inequalities yields $2(x_1 + x_2 + x_3) \geq 3$. Since $x_1 + x_2 + x_3 = 1$ this inequality cannot be satisfied. Hence, the core is empty.

A necessary and sufficient condition for nonemptiness of the core is given in BONDAREVA (1963) and SHAPLEY (1967). In order to formulate this condition we use the notion of balanced maps. Therefore, define for each $S \subset N$ the vector $e^S \in \mathbb{R}^N$ by $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. A map $\lambda : 2^N \to [0, 1]$ is a *balanced map* if for all $i \in N$ it holds that $\sum_{S \subset N} \lambda(S) e^S = e^N$.

Theorem 2.3 Let (N, v) be a TU-game. Then $C(v) \neq \emptyset$ if and only if for each balanced map λ it holds that

$$\sum_{S \subset N} \lambda(S) v(S) \leq v(N). \tag{2.10}$$

TU-games that have a nonempty core are called *balanced games*. Furthermore, if also every subgame $(S, v_{|S})$ of the TU-game (N, v) has a nonempty core, the game is called *totally balanced*. Here, the characteristic function of a subgame $(S, v_{|S})$ is defined as $v_{|S}(T) = v(T)$ for all $T \subset S$. The linear production games and sequencing games presented in Example 2.5 and Example 2.6, respectively, are examples of totally balanced games.

A sufficient but not necessary condition for nonemptiness of the core is provided in SHAPLEY (1971) and reads as follows:

Theorem 2.4 Let (N, v) be a TU-game. If (N, v) is convex, then $C(v) \neq \emptyset$.

When dividing the benefits of the grand coalition N, it is likely that the agents focus on core-allocations only. For if an allocation outside the core is proposed, at least one coalition can threaten to leave the grand coalition if it does not receive a larger share of the benefits. The core, however, need not give a decisive answer to what allocation the agents should agree upon. Although the core can consist of only one allocation, most of the time it turns out to be a fairly large set of allocations. For illustration, consider the following two examples.

Example 2.8 Consider the game (N, v) of Example 2.4 with $N = \{L, R, J\}$ and $v(S) = 24$ if $S \in \{\{L, J\}, \{R, J\}, \{L, R, J\}\}$ and $v(S) = 0$ otherwise. So, $(x_L, x_R, x_J) \in \mathbb{R}^3_+$ is a core-allocation if $x_L + x_J \geq 24$, $x_R + x_J \geq 24$ and $x_L + x_R + x_J = 24$. Adding the two inequalities yields $x_L + x_R + x_J + x_J \geq 48$. Substituting $x_L + x_R + x_J = 24$ gives that $x_J \geq 24$. Together with $x_L \geq 0$ and $x_R \geq 0$ this implies that $(x_L, x_R, x_J) = (0, 0, 24)$ is the only solution. Hence, $C(v) = \{(0, 0, 24)\}$.

Example 2.9 Consider a three-person TU-game (N, v) with $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = 4$, $v(\{1, 3\} = 3$, $v(\{2, 3\} = 2$, and $v(\{1, 2, 3\}) = 8$. The core $C(v)$ of this game is depicted in Figure 2.4 and contains infinitely many allocations.

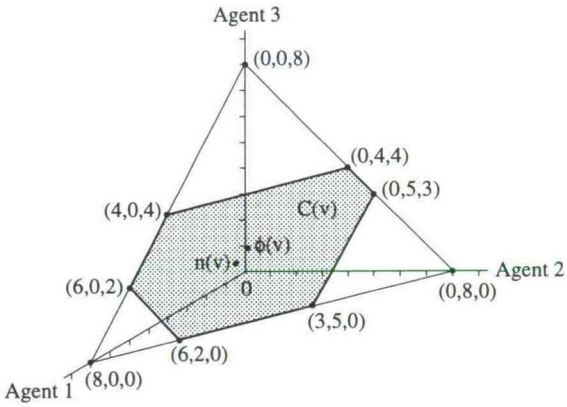


Figure 2.4

In contrast with the core, which can be a set of allocations, an allocation rule assigns to each TU-game (N, v) exactly one allocation. A well known allocation rule is the Shapley value, introduced in SHAPLEY (1953). This allocation rule is based on the marginal vectors of a TU-game (N, v). To explain the definition of a marginal vector, let σ be an ordering of the agents. So, $\sigma : N \to \{1, 2, \ldots, n\}$ is a bijection with $\sigma(i) = j$ meaning that agent i is in position j. Next, let the agents enter a room one at a time according to the order σ. Thus, agent $\sigma^{-1}(1)$ enters first, then agent $\sigma^{-1}(2)$, and so on. When an agent enters the room, he joins the coalition that is already there. Furthermore, the payoff he receives equals the amount with which the benefits of the coalition changes. The resulting allocation is called the *marginal vector* with respect to the order σ and equals

$$m_i^\sigma(v) = v(\{j \in N | \sigma(j) \leq \sigma(i)\}) - v(\{j \in N | \sigma(j) < \sigma(i)\}), \tag{2.11}$$

for all $i \in N$. Note that $m^\sigma(v)$ is indeed an allocation since $\sum_{i \in N} m_i^\sigma(v) = v(N)$. Presuming that each of the $n!$ different orders occurs equally likely, the *Shapley value* $\phi(v)$ assigns to each agent i his expected marginal contribution, that is,

$$\begin{aligned} \phi_i(v) &= \frac{1}{n!} \sum_{\sigma \in \Pi_N} m_i^\sigma(v) \\ &= \sum_{S \subset N \setminus \{i\}} \frac{\#S!(n-1-\#S)!}{n!}(v(S \cup \{i\}) - v(S)), \end{aligned} \tag{2.12}$$

with $\Pi_N = \{\sigma : N \to \{1, 2, \ldots, n\} | \forall_{i,j \in N: i \neq j} : \sigma(i) \neq \sigma(j)\}$ the set of all orderings of N. Let us calculate the Shapley value for the game given in Example 2.9.

Example 2.10 Consider the game defined in Example 2.9. Let σ_i, $i = 1, 2, \ldots, 6$ denote the orders 123, 132, 312, 321, 231, and 213, respectively. Then $m_1^{\sigma_1}(v) = v(\{1\}) - v(\emptyset) = 0$, $m_2^{\sigma_1}(v) = v(\{1,2\}) - v(\{1\}) = 4$, and $m_3^{\sigma_1}(v) = v(\{1,2,3\}) - v(\{2,3\}) = 4$. So, $m^{\sigma_1}(v) = (0,4,4)$. Similarly, one calculates $m^{\sigma_2}(v) = (0,5,3)$, $m^{\sigma_3}(v) = (3,5,0)$, $m^{\sigma_4}(v) = (6,2,0)$, $m^{\sigma_5}(v) = (6,0,2)$, and $m^{\sigma_6}(v) = (4,0,4)$. The Shapley value thus equals $\phi(v) = \frac{1}{6}\sum_{i=1}^{6} m^{\sigma_i}(v) = \frac{1}{6}(19, 16, 13)$ (see also Figure 2.4).

As can be seen in Figure 2.4, the Shapley value is a core-allocation for this particular game. In general, however, this is not the case as the following example shows.

Example 2.11 Consider again the game presented in Example 2.4. To calculate the Shapley value, let σ_i, $i = 1, 2, \ldots, 6$ denote the orders LRJ, LJR, JLR, JRL, RJL, and RLJ, respectively. Then $m_L^{\sigma_1}(v) = v(\{L\}) - v(\emptyset) = 0$, $m_R^{\sigma_1}(v) = v(\{L,R\}) - v(\{L\}) = 0$, and $m_J^{\sigma_1}(v) = v(\{L,R,J\}) - v(\{L,R\}) = 24$. So, $m^{\sigma_1}(v) = (0,0,24)$. Similarly, one calculates $m^{\sigma_2}(v) = (0,0,24)$, $m^{\sigma_3}(v) = (24,0,0)$, $m^{\sigma_4}(v) = (0,24,0)$, $m^{\sigma_5}(v) = (0,0,24)$, and $m^{\sigma_6}(v) = (0,0,24)$. The Shapley value then equals $\phi(v) = \frac{1}{6}\sum_{i=1}^{6} m^{\sigma_i}(v) = (4,4,16)$. Note that $\phi(v)$ is not a core-allocation since $\phi_L(v) + \phi_J(v) = 20 < 24 = v(\{L,J\})$.

So, the Shapley value need not result in a core-allocation. SHAPLEY (1971) provides a sufficient condition for $\phi(v)$ to be an element of the core.

Theorem 2.5 Let (N, v) be a TU-game. If (N, v) is convex, then $\phi(v) \in C(v)$.

The proof of this proposition follows from a result also due to SHAPLEY (1971), namely, that for convex games all marginal vectors belong to the core. The converse of this statement is given in ICHIISHI (1981). Hence, a TU-game (N, v) is convex if and only if all marginal vectors are elements of the core $C(v)$.

As opposed to the Shapley value, the nucleolus is an allocation rule that results in a core-allocation whenever the core is nonempty. The nucleolus for TU-games is introduced in SCHMEIDLER (1969) and minimizes the maximal complaint of the coalitions. Given an allocation $x \in I(v)$, the complaint of a coalition S expresses how dissatisfied this coalition is with the proposed allocation x. The complaint of coalition S is defined as the difference between what this coalition can obtain on its own and what the allocation x assigns to the members of S. So, the less x assigns to the members of S, the higher the complaint of this coalition will be. Then, roughly speaking, the nucleolus is that allocation of the imputation set $I(v)$ that minimizes the maximal complaint of all coalitions.

For a formal definition of the nucleolus, let (N, v) be a TU-game such that the imputation set $I(v)$ is nonempty. Given an allocation x, the complaint, or excess, of coalition S is defined by $E(S, x) = v(S) - \sum_{i \in S} x_i$. Note that the excess $E(S, x)$ increases if $\sum_{i \in S} x_i$ decreases. Next, let $E(x) = (E(S, x))_{S \subset N}$ be the vector of excesses and let $\theta \circ E(x)$ denote the vector of excesses with its elements arranged in decreasing order. The *nucleolus* is then defined by

$$n(v) \;=\; \{x \in I(v) | \, \forall_{y \in I(v)} : \theta \circ E(x) \leq_{lex} \theta \circ E(y)\}. \tag{2.13}$$

Here, $\leq_{lex}$ denotes the lexicographical ordering on $\mathbb{R}^{2^N}$. Given two vectors $x, y \in \mathbb{R}^{2^N}$ we have $x <_{lex} y$ if there exists $k \in \mathbb{N}$ such that $x_i = y_i$ for $i = 1, 2, \ldots, k$ and $x_{k+1} < y_{k+1}$. So, when comparing two vectors, the lexicographical ordering first compares the first element of each vector, if they are equal it compares the second element of each vector, and so on. The following result is due to SCHMEIDLER (1969).

Theorem 2.6 Let (N, v) be a TU-game. If $I(v) \neq \emptyset$, then the nucleolus $n(v)$ is a singleton. Moreover, $n(v) \subset C(v)$ whenever $C(v) \neq \emptyset$.

Example 2.12 Consider again the game presented in Example 2.9. The nucleolus for this game equals $n(v) = (3\frac{1}{2}, 2\frac{1}{2}, 2)$ with the maximal complaint being $E(\{3\}, n(v)) = E(\{1, 2\}, n(v)) = -2$. Figure 2.4 shows that the nucleolus $n(v)$ is indeed a core-allocation.

Although the nucleolus has the nice property that it always results in a core-allocation, provided one exists, it is not that easy to calculate. Even for a three person example the calculations are extensive, hence the absence of any calculations in Example 2.12. There are special classes of TU-games though, for which the nucleolus is determined more easily. Examples of such TU-games are big boss games (see MUTO, NAKAYAMA, POTTERS, and TIJS (1988)) and airport games (see LITTLECHILD (1974)).

The two allocation rules discussed thus far are applicable to every (superadditive) TU-game. When considering a particular class of TU-games though, a more intuitive allocation rule may exist that applies to this particular setting only. Besides having a more appealing

interpretation, these rules are often easier to determine than both the Shapley value and the nucleolus. This, for instance, is the case for linear production games and sequencing games. We therefore conclude this section on TU-games with two examples of such specific allocation rules, one applying to linear production games and the other applying to sequencing games.

Example 2.13 Recall that a linear production game (N, v) is defined by

$$v(S) = \max\left\{p^\top c |\ Ac \leq \sum_{i \in S} b^i, c \geq 0\right\}$$

for all $S \subset N$. One can, of course, calculate the Shapley value and nucleolus for these games. Since linear production games are totally balanced, we even know that the nucleolus results in a core-allocation. The following, relatively easy procedure, however, also yields a core-allocation for every linear production game (N, v).

Consider the following linear program to determine the value $v(N)$ of the grand coalition

$$v(N) = \max\left\{p^\top c |\ Ac \leq \sum_{i \in N} b^i, c \geq 0\right\}.$$

The dual of this linear program is

$$\min\left\{y^\top \sum_{i \in N} b^i |\ y^\top A \geq p, y \geq 0\right\}.$$

Now, if $\hat{y} \in \mathbb{R}^r_+$ denotes an optimal solution of the dual program, OWEN (1975) shows that the allocation x defined by $x_i = \hat{y}^\top b^i$ is a core-allocation of the linear production game (N, v). For interpreting this allocation, note that the optimal solution $\hat{y}$ represents the shadow prices of the resources. This means that if, say, the amount of resource j increases with 1 unit, then the maximal revenues increase with approximately $\hat{y}_j$. In other words, $\hat{y}_j \cdot \sum_{i \in N} b^i_j$ denotes the contribution of resource j in the maximal revenues. The allocation $(\hat{y}^\top b^i)_{i \in N}$ then gives each agent the value of his resource bundle.

Example 2.14 For the sequencing games described in Example 2.6, CURIEL ET AL. (1989) introduced the Equal Gain Splitting rule. For understanding how this rule works, let us go into more detail on the optimal serving order. For this purpose, consider two divisions i and j such that division j is the immediate follower of division i. Recall that the waiting costs for divisions i and j are given by $k_i(\sigma) = \sum_{k \in N: \sigma(k) \leq \sigma(i)} \alpha_i p_k$ and $k_j(\sigma) = \sum_{k \in N: \sigma(k) \leq \sigma(j)} \alpha_j p_k$, respectively. Now, if division i and j change their positions in the serving order, then division i's waiting time increases with p_j. Hence, the waiting costs for division i increase with $\alpha_i p_j$. Similarly, the waiting time for division j decreases with p_i, so that its waiting costs decrease with $\alpha_j p_i$. The benefits from this change in position thus equal $\alpha_j p_i - \alpha_i p_j$. This implies that division i and j trade places if and only if the benefits are positive, that is, $\alpha_j p_i - \alpha_i p_j > 0$.

It can be shown that an optimal order is reached by consecutively switching the places of two divisions, one succeeding the other, whenever such a change is beneficial. Moreover, all beneficial switches occur in this procedure. The *Equal Gain Splitting* rule, or EGS-rule, then divides the benefits of each switch equally among the two divisions that are involved in this switch. Hence, division i receives

$$EGS_i(v) = \sum_{\substack{j\in N:\ \sigma_0(j) > \sigma_0(i) \\ \alpha_j p_i - \alpha_i p_j > 0}} \tfrac{1}{2}(\alpha_j p_i - \alpha_i p_j) + \sum_{\substack{k\in N:\ \sigma_0(k) < \sigma_0(i) \\ \alpha_i p_k - \alpha_k p_i > 0}} \tfrac{1}{2}(\alpha_i p_k - \alpha_k p_i).$$

CURIEL ET AL. (1989) shows that the EGS-rule always results in a core-allocation of the corresponding sequencing game. Other allocation rules arise when the benefits $\alpha_j p_i - \alpha_i p_j$ are not divided equally among the divisions i and j, but according to some distribution code $\gamma = \{\gamma_{ij}\}_{i,j\in N: i<j}$. This means that if divisions i and j change their positions, division i receives the fraction $\gamma_{ij} \in [0,1]$ of the benefits $\alpha_j p_i - \alpha_i p_j$ and division j receives the fraction $1 - \gamma_{ij} \in [0,1]$. These allocation rules are considered in HAMERS, SUIJS, TIJS and BORM (1996). For an extensive study on (generalized) sequencing situations we refer to HAMERS (1996).

2.4 Cooperative Games with Non-transferable Utility

A *cooperative game with non-transferable utility*, or *NTU-game*, is described by a pair (N, V), where $N = \{1, 2, \ldots, n\}$ denotes the set of agents and V is a map assigning to each coalition S a nonempty and closed subset $V(S)$ of $\mathbb{R}^S$ such that

$$V(\{i\}) = (-\infty, v(\{i\})],\ v(\{i\}) \in \mathbb{R} \tag{2.14}$$

for all $i \in N$,

$$\text{if } x \in V(S) \text{ and } \hat{x} \leq x \text{ then also } \hat{x} \in V(S), \tag{2.15}$$

for all $S \subset N$, and

$$V(S) \cap \{x \in \mathbb{R}^S \mid \forall_{i\in S} : x_i \geq v(\{i\})\} \tag{2.16}$$

is bounded for all $S \subset N$. The set $V(S)$ represents the utility vectors the members of coalition S can obtain when cooperating (see also (2.1)). For mathematical reasons it is sometimes useful to extend the utility vectors in $V(S)$ with zeros for the agents that do not belong to coalition S. So, let us also define

$$\tilde{V}(S) = \{x \in \mathbb{R}^N \mid (x_i)_{i\in S} \in V(S), \forall_{i\notin S} : x_i = 0\} \tag{2.17}$$

for all $S \subset N$.

Example 2.15 Every TU-game (N, v) can be modelled as an NTU-game (N, V) by defining

$$V(S) = \{x \in \mathbb{R}^S | \sum_{i \in S} x_i \leq v(S)\}$$

for all $S \subset N$.

Example 2.16 Consider the model of an exchange economy presented in Example 2.1. Assuming that the agents' preferences can be represented by a continuous utility function U_i, this situation gives rise to the following NTU-game (N, V) with

$$V(S) = \{z \in \mathbb{R}^S | \exists_{(c^i)_{i \in S} \in Y_S} \forall_{i \in S} : z_i \leq U_i(c^i)\},$$

for all $i \in S$. Note that since Y_S is compact and U_i is continuous for all $i \in N$, condition (2.16) is satisfied.

An NTU-game (N, V) is called *superadditive* if for all disjoint $S, T \subset N$ we have that

$$V(S) \times V(T) \subset V(S \cup T). \tag{2.18}$$

Similar as for TU-games, superadditivity implies that two disjoint coalitions do not suffer from merging into one large coalition. Usually, superadditivity is satisfied. For two disjoint coalitions S and T can at least generate the benefits $V(S) \times V(T)$ by pretending a merger and, in the meantime, operating separately. An equivalent statement for superadditivity is that $\tilde{V}(S) + \tilde{V}(T) \subset \tilde{V}(S \cup T)$ for all disjoint $S, T \subset N$.

An NTU-game (N, V) is called *ordinal convex* if for all $S, T \subset N$ and all $x \in \mathbb{R}^N$ the following statement is true: if $(x_i)_{i \in S} \in V(S)$ and $(x_i)_{i \in T} \in V(T)$ then either $(x_i)_{i \in S \cap T} \in V(S \cap T)$ or $(x_i)_{i \in S \cup T} \in V(S \cup T)$. Ordinal convexity is introduced in VILKOV (1977) and extends the definition of convexity as given in (2.7) to NTU-games. Another such an extension is cardinal convexity, introduced in SHARKEY (1981). For defining cardinal convexity, let (N, V) be an NTU-game such that $V(\{i\}) = (-\infty, 0]$ for all $i \in N$. Then (N, V) is *cardinal convex* if for all $S, T \subset N$ it holds that

$$\tilde{V}(S) + \tilde{V}(T) \subset \tilde{V}(S \cup T) + \tilde{V}(S \cap T). \tag{2.19}$$

Note that $V(\{i\}) = (-\infty, 0]$ is not really a restriction because of the monotonic transformations we may take from the utility functions. Ordinal and cardinal convexity are not equivalent though, SHARKEY (1981) provides two examples of NTU-games, one being ordinal but not cardinal convex and the other being cardinal but not ordinal convex.

As was the case for TU-games, we assume that the grand coalition N is formed and focus on 'fair' allocations, that is, utility vectors $x \in V(N)$ at which all agents are satisfied. In this context, an *allocation* for coalition S is a utility vector $x \in V(S)$. An allocation $x \in V(S)$

is called *efficient* or *Pareto optimal* if there exists no other allocation $\hat{x} \in V(S)$ that yields all agents in S a higher utility, i.e., $\hat{x}_i > x_i$ for all $i \in S$. Furthermore, an agent will only participate in the grand coalition if this yields him as least as much as he can obtain on his own. This means that only those allocations $x \in V(N)$ are eligible such that for each agent $i \in N$ it holds that $x_i \geq v(\{i\})$. These allocations are called *individually rational* . Furthermore, the set of all efficient and individually rational allocations is called the *imputation set* and is given by

$$I(V) = \{x \in V(N) | \nexists_{\hat{x} \in V(N)} \forall_{i \in N} : \hat{x}_i > x_i, \forall_{i \in N} : x_i \geq v(\{i\})\}. \tag{2.20}$$

Analogous to TU-games one can define the core of an NTU-game. A core-allocation is such that no coalition S can improve the payoff of each member by separating from the grand coalition N and operating on its own. A coalition S can improve upon the allocation $x \in V(N)$ if there exists an allocation $\hat{x} \in V(S)$ such that $\hat{x}_i > x_i$ for all $i \in S$. The *core* of an NTU-game is thus given by

$$C(V) = \{x \in V(N) | \forall_{S \subset N} \nexists_{\hat{x} \in V(S)} \forall_{i \in S} : \hat{x}_i > x_i\}. \tag{2.21}$$

Example 2.17 Consider the following two person market game with two consumption goods. Let $\omega^1 = (2,3)$ and $\omega^2 = (4,1)$ be the endowments of agents 1 and 2, respectively. Next, let $c^i = (c_1^i, c_2^i)$ denote the consumption bundle for agent i. The preferences for agent 1 are described by the utility function $U_1(c_1^1, c_2^1) = 2\sqrt{c_1^1} + \sqrt{c_2^1}$ and the preferences for agent 2 are described by $U_2(c_1^2, c_2^2) = \sqrt{c_1^2} + 2\sqrt{c_2^2}$. The corresponding NTU-game equals

$$\begin{aligned} V(\{1\}) &= \{x \in \mathbf{R} | \, x \leq U_1(2,3) = 2\sqrt{2} + \sqrt{3}\} \\ V(\{2\}) &= \{x \in \mathbf{R} | \, x \leq U_2(4,1) = 4\} \\ V(\{1,2\}) &= \{x \in \mathbf{R}^2 | \, \exists_{c^1, c^2 \in \mathbf{R}^2 : (c^1, c^2) \leq (6,4)} : x_i \leq U_i(c^i), i = 1,2\} \end{aligned}$$

and is depicted in Figure 2.5. As one can see, the core $C(V)$ of this game is nonempty and equals $I(V)$.

From Example 2.15 we know that every TU-game can be modelled as an NTU-game. Since the core of a TU-game consists of the same utility vectors as the core of the corresponding NTU-game, it follows from Example 2.7 that also for NTU-games the core can be empty. Furthermore, a necessary and sufficient condition like the balancedness condition for TU-games does not exist. Although a notion of balancedness exists, it only provides a sufficient condition for nonemptiness of the core. The following result is a consequence of the main theorem in SCARF (1967).

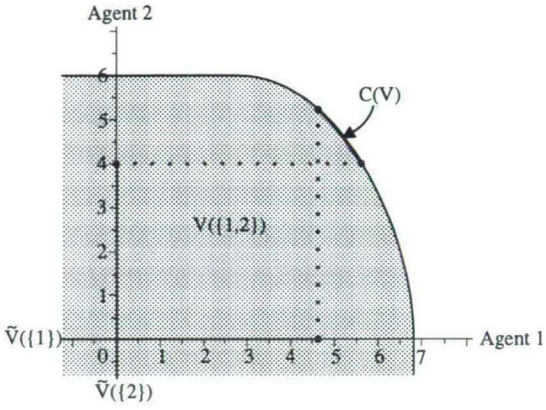


Figure 2.5

Theorem 2.7 Let (N, V) be an NTU-game. If for all balanced maps λ (see page 28) it holds that $\sum_{S \subset N} \lambda(S)\tilde{V}(S) \subset \tilde{V}(N)$, then $C(V) \neq \emptyset$.

As was the case for TU-games convexity is also a sufficient condition for a nonempty core. The following two results can be found in VILKOV (1977) and SHARKEY (1981), respectively.

Theorem 2.8 Let (N, V) be an NTU-game. If (N, V) is ordinally convex, then $C(V) \neq \emptyset$.

Theorem 2.9 Let (N, V) be an NTU-game. If (N, V) is cardinally convex, then $C(V) \neq \emptyset$.

An NTU-game with a nonempty core is called *balanced*. Furthermore, if for every subgame $(S, V_{|S})$ the core is nonempty, then the game is called *totally balanced*. The subgame $(S, V_{|S})$ is defined by $V_{|S}(T) = V(T)$ for all $T \subset S$. Market games, for example, are totally balanced NTU-games.

As shown above, properties like superadditivity, convexity, and balancedness extend reasonably well to the class of NTU-games. The same can be said for the Shapley value, as opposed to the nucleolus which has yet to be defined for NTU-games. For the Shapley value several extensions have been introduced. For example, the Harsanyi-value (HARSANYI (1963)), the Shapley NTU-value (SHAPLEY (1969)), the egalitarian solution (KALAI and SAMET (1985)), the consistent Shapley value (MASCHLER and OWEN (1992)), and the marginal based compromise value (OTTEN, BORM, PELEG and TIJS (1994)). In the remainder of this section we confine ourselves to a brief discussion of the Shapley NTU-value and the marginal based compromise value only.

Since the Shapley value was introduced in the previous section as the average of all marginal vectors, a straightforward extension to the class of NTU-games would thus be averaging the marginal vectors. Unfortunately, this procedure does not work. Although marginal vectors can be defined, the average of all these allocations can fail Pareto optimality. So, when extending the Shapley value to NTU-games, one has to employ a different approach.

Let us start with explaining Shapley's approach that led to the Shapley NTU-value for an NTU-game (N, V). For this purpose, assume that for each allocation $x \in V(S)$ the total utility for coalition S can be represented by a weighted sum of the individual utility levels. So, given a weight vector $w \in \mathbb{R}^N_+$ with $\sum_{i \in N} w_i = 1$, the total weighted utility that coalition S assigns to the allocation x equals $\sum_{i \in S} w_i x_i$. If we also assume that these weighted utilities are transferable, each weight vector w gives rise to a TU-game (N, v_w) with

$$v_w(S) \;=\; \sup\{\sum_{i \in S} w_i x_i |\; x \in V(S)\}, \tag{2.22}$$

for all $S \subset N$. For these games we can determine the Shapley value $\phi(v_w)$. Since the Shapley value $\phi(v_w)$ represents an allocation of the weighted sum of utilities, the resulting allocation need not be attainable for coalition N. So, what we are interested in are the attainable allocations $x \in V(N)$ for which the vector of weighted utilities $(w_i x_i)_{i \in N}$ coincides with $\phi(v_w)$. The *Shapley NTU-value* is thus defined by

$$\Phi(V) \;=\; \{x \in V(N) |\; \exists_{w \in \Delta^N} \forall_{i \in N} : \phi_i(v_w) = w_i x_i\}, \tag{2.23}$$

where $\Delta^N = \{w \in \mathbb{R}^N_+ |\, \sum_{i \in N} w_i = 1\}$.

Example 2.18 Consider the following two-person NTU-game (N, V) with $V(\{i\}) = (-\infty, 0]$ for $i = 1, 2$ and $V(\{1, 2\})$ as depicted in Figure 2.6. Let $w = (w_1, w_2)$ be a weight vector. Then for all $w \in \Delta^N$ it holds that $v_w(\{1\}) = v_w(\{2\}) = 0$ and

$$v_w(\{1,2\}) \;=\; \begin{cases} 4 - 2w_1, & \text{if } w_1 \leq \frac{2}{5} \\ 2 + 3w_1, & \text{if } w_1 > \frac{2}{5}. \end{cases}$$

This yields

$$\phi(v_w) \;=\; \begin{cases} (2 - w_1, 2 - w_1), & \text{if } w_1 \leq \frac{2}{5} \\ (1 + \frac{3}{2}w_1, 1 + \frac{3}{2}w_1), & \text{if } w_1 > \frac{2}{5}. \end{cases}$$

Next, let $w = (\frac{2}{5}, \frac{3}{5})$ and consider the allocation $(4, \frac{8}{3}) \in V(\{1, 2\})$. The corresponding weighted utilities equal $\frac{2}{5} \cdot 4 = \frac{8}{5}$ for agent 1 and $\frac{3}{5} \cdot \frac{8}{3} = \frac{8}{5}$ for agent 2. Since this coincides with $\phi(v_w)$ it holds that $(\frac{8}{5}, \frac{8}{5}) \in \Phi(V)$.

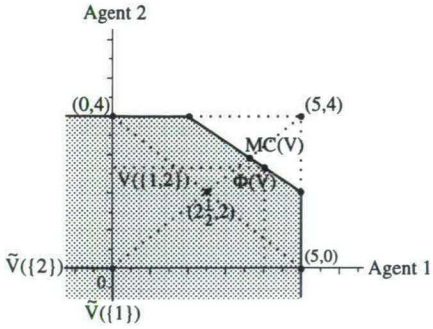


Figure 2.6

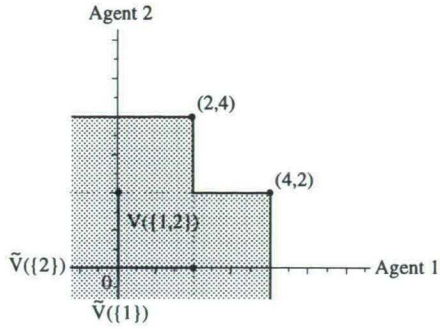


Figure 2.7

Example 2.19 Consider the two-person NTU-game (N, V) depicted in Figure 2.7. So, $V(\{i\}) = (-\infty, 2]$ for $i = 1, 2$ and

$$V(\{1,2\}) = \{(x_1, x_2) \in \mathbb{R}^2 | \, (x_1, x_2) \leq (2,4)\} \cup \{(x_1, x_2) \in \mathbb{R}^2 | \, (x_1, x_2) \leq (4,2)\}.$$

For this game it holds that $\Phi(V) = \emptyset$. To see this, let $w = (w_1, w_2)$ be a weightvector. Then $v_w(\{1\}) = 2w_1$, $v_w(\{2\}) = 2w_2 = 2(1 - w_1)$, and

$$v_w(\{1,2\}) = \begin{cases} 4 - 2w_1, & \text{if } w_1 \leq \frac{1}{2} \\ 2 + 2w_1, & \text{if } w_1 > \frac{1}{2} \end{cases}$$

Hence, the Shapley value $\phi(v_w)$ equals

$$\phi(v_w) = \begin{cases} (1 + w_1, 3 - 3w_1), & \text{if } w_1 \leq \frac{1}{2} \\ (3w_1, 2 - w_1), & \text{if } w_1 > \frac{1}{2} \end{cases}$$

Now, if $w_1 = 0$ then $\phi(v_w) = (1, 3)$. Since $w_1 = 0$ there exists no $x \in V(\{1,2\})$ such that $1 = w_1 x_1$ and $3 = w_2 x_2$. If $w_1 \in (0, \frac{1}{2})$, the only candidate to belong to $\Phi(V)$ is the allocation $(2, 4)$. But if $(2, 4) \in \Phi(V)$ it must hold that $\phi_1(v_w) = 2w_1$ and $\phi_2(v_w) = 4w_2 = 4(1 - w_1)$. Since both equations imply that $w_1 = 1$, we have a contradiction. Similar contradictions follow if $w_1 \in [\frac{1}{2}, 1]$, hence $\Phi(V) = \emptyset$.

As Example 2.19 shows, the Shapley NTU-value need not always exist. SHAPLEY (1969) provides sufficient conditions for nonemptiness. It does, however, not guarantee unicity of the Shapley NTU-value.

Theorem 2.10 Let (N, V) be a superadditive NTU-game. If $V(N)$ is a convex subset of $\mathbb{R}^N$ then $\Phi(V) \neq \emptyset$.

As follows from the definition of the Shapley NTU-value, the marginal vectors of the NTU-game itself are not taken into account. The marginal based compromise value, as its name implies, does make use of these marginal vectors. For defining a marginal vector of an NTU-game (N, V), let $\sigma \in \Pi_N$ be an ordering of the agents. Similar to TU-games, we assign to each agent i his marginal contribution to the benefits of the coalition that is formed by the agents preceding agent i in the order σ. The *marginal vector* $m^\sigma(V)$ is thus defined recursively as follows. Let i_1 be the first agent in the order σ, that is $\sigma(i_1) = 1$. He receives

$$m^\sigma_{i_1}(V) \;=\; \sup\{x_{i_1} \,|\, x \in V(\{i_1\})\}.$$

The second agent i_2 then receives

$$m^\sigma_{i_2}(V) \;=\; \sup\{x_{i_2} \,|\, x \in V(\{i_1, i_2\}) : x_{i_1} = m^\sigma_{i_1}(V)\}.$$

Continuing this procedure yields agent i_k a payoff equal to

$$m^\sigma_{i_k}(V) \;=\; \sup\{x_{i_k} \,|\, x \in V(\{i_1, i_2, \ldots, i_k\}) : \forall_{j=1,2,\ldots,k-1} : x_{i_j} = m^\sigma_{i_j}(V)\}. \tag{2.24}$$

Note that a marginal vector $m^\sigma(V)$ is well defined if the game is superadditive. Furthermore, note that by definition $m^\sigma(V) \in V(N)$ and that $m^\sigma(V)$ satisfies Pareto optimality. Moreover, if this procedure is applied to TU-games the original marginal vectors result.

The marginal based compromise value is defined for NTU-games (N, V) with $V(\{i\}) = (-\infty, 0]$ for all $i \in N$. Let $\overline{m}(V) = \sum_{\sigma \in \Pi_N} m^\sigma(V)$. Then the *marginal based compromise value*, or *MC-value*, is defined by

$$MC(V) \;=\; \alpha_V \overline{m}(V), \tag{2.25}$$

where a_V is such that $\alpha_V = \sup\{\alpha \in \mathbb{R}_+ | \alpha\overline{m}(V) \in V(N)\}$. So, the MC-value is the largest multiple of the vector $\overline{m}(V)$ that still belongs to $V(N)$. Hence, it is Pareto optimal. Furthermore, the MC-value coincides with the Shapley value on the class of TU-games and with the consistent Shapley value on the class of so-called hyperplane games (see MASCHLER and OWEN (1989)).

Example 2.20 Consider the game illustrated in Figure 2.6. Since it is a two- person game, there are only two different marginal vectors. Let σ_1 denote the order 12 and σ_2 the order 21. Then $m_1^{\sigma_1}(V) = \sup\{x | x \in V(\{1\})\} = 0$ and $m_2^{\sigma_1}(V) = \sup\{x_2 | (x_1, x_2) \in V(\{1,2\}) : x_1 = 0\} = 4$. Similarly, $m_2^{\sigma_2}(V) = \sup\{x | x \in V(\{2\})\} = 0$ and $m_1^{\sigma_2}(V) = \sup\{x_1 | (x_1, x_2) \in V(\{1,2\}) : x_2 = 0\} = 5$. This implies that $\overline{m}(V) = (0,4) + (5,0) = (5,4)$ and that $MC(V) = (3\frac{7}{11}, 2\frac{10}{11})$. Finally, note that the average of the marginal allocations equals $(2\frac{1}{2}, 2)$ and that it violates Pareto optimality since it is in the interior of $V(N)$.

2.5 Chance-Constrained Games

Chance-constrained games as introduced by CHARNES and GRANOT (1973) extend the theory of cooperative games in characteristic function form to situations where the benefits from cooperation are random variables. So, when several agents decide to cooperate, they do not exactly know the benefits this cooperation generates. What they do know is the probability distribution function of these benefits. Let $\boldsymbol{V}(S)$ denote the random variable describing the benefits of coalition S. Furthermore, denote its probability distribution function by $F_{\boldsymbol{V}(S)}$. Thus,

$$F_{\boldsymbol{V}(S)}(t) \;=\; \mathbf{P}\{\boldsymbol{V}(S) \leq t\}, \tag{2.26}$$

for all $t \in \mathbb{R}$. Then a *chance-constrained game* is defined by the pair $(N, \boldsymbol{V})$, with $\boldsymbol{V}$ the characteristic function assigning to each coalition S the nonnegative random benefits $\boldsymbol{V}(S)$. Note that chance-constrained games are based on the formulation (N, v) of TU-games with the deterministic benefits $v(S)$ replaced by stochastic benefits $\boldsymbol{V}(S)$. This, however, does not imply that the preferences of the agents are also linearly separable in money. In fact, the individual preferences are of no account in this model.

For dividing the benefits of the grand coalition, the authors propose two-stage allocations. In the first stage, when the realization of the benefits is still unknown, each agent is promised a certain payoff. These so-called prior payoffs are such that there is a fair chance that they are realized. Once the benefits are known, the total payoff allocated in the prior payoff can differ from what is actually available. In that case, we come to the second stage and modify the prior payoff in accordance with the realized benefits.

Let us start with discussing the prior allocations. A prior payoff is denoted by a vector $x \in \mathbb{R}^N$, with the interpretation that agent $i \in N$ receives the amount x_i. To comply with the condition that there is a reasonable probability that the promised payoffs can be kept, the prior payoff x must be such that

$$\underline{\alpha}(N) \;\leq\; \mathbf{P}(\{\boldsymbol{V}(N) \leq \sum_{i \in N} x_i\}) \;=\; F_{\boldsymbol{V}(N)}(\sum_{i \in N} x_i) \;\leq\; \overline{\alpha}(N), \tag{2.27}$$

with $0 < \underline{\alpha}(N) \leq \overline{\alpha}(N) < 1$. This condition assures that the total amount $\sum_{i \in N} x_i$ that is allocated is not too low or too high. Note that expression (2.27) can also be written as

$$\xi_{\underline{\alpha}(N)}(\boldsymbol{V}(N)) \;\leq\; \sum_{i \in N} x_i \;\leq\; \xi_{\overline{\alpha}(N)}(\boldsymbol{V}(N)).$$

To come to a prior core for chance-constrained games, one needs to specify when a coalition S is satisfied with the amount $\sum_{i \in S} x_i$ it receives, so that it does not threaten to leave the grand coalition N. CHARNES and GRANOT (1973) assume that a coalition S is satisfied with what it gets, if the probability that they can obtain more on their own is small enough. This

means that for each coalition $S \neq N$ there exists a number $\alpha(S) \in (0,1)$ such that coalition S is willing to participate in the coalition N whenever $\mathbf{P}(\{\boldsymbol{V}(S) \leq \sum_{i\in S} x_i\}) \geq \alpha(S)$. The number $\alpha(S)$ is a measure of assurance for coalition S. Note that the measure of assurance may vary over the coalitions. Furthermore, they reflect the coalition's attitude towards risk, the willingness to bargain with other coalitions, and so on. The *prior core* of a chance-constrained game $(N, \boldsymbol{V})$ is then defined by

$$C(\boldsymbol{V}) = \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} \forall_{S\subset N: S\neq N} : F_{\boldsymbol{V}(S)}(\sum_{i\in S} x_i) \geq \alpha(S), \\ \underline{\alpha}(N) \leq F_{\boldsymbol{V}(N)}(\sum_{i\in N} x_i) \leq \overline{\alpha}(N) \end{array} \right\}. \tag{2.28}$$

Note that the prior-core is an extension of the core for TU-games. To see this, consider the deterministic benefits $v(S)$ of coalition S. Since $\alpha(S) > 0$ the condition $\mathbf{P}(\{v(S) \leq \sum_{i\in S} x_i\}) \geq \alpha(S)$ is satisfied if and only if $\sum_{i\in S} x_i \geq v(S)$.

Example 2.21 Consider the following three-person chance-constrained game $(N, \boldsymbol{V})$ with $\boldsymbol{V}(\{i\}) = 0, i = 1,2,3$, $\boldsymbol{V}(S) \sim U(0,2)$ if $\#S = 2$, and $\boldsymbol{V}(N) \sim U(1,2)$. Furthermore, let $\alpha(S) = \frac{1}{2}$ for all $S \neq N$, $\underline{\alpha}(N) = \frac{2}{5}$, and $\overline{\alpha}(N) = \frac{3}{5}$. Next, let $x \in \mathbb{R}^3$ be a prior allocation. Since

$$F_{\boldsymbol{V}(N)}(t)) = \begin{cases} 0, & \text{if } t \leq 1, \\ t-1, & \text{if } 1 < t < 2, \\ 1, & \text{if } 2 \leq t, \end{cases}$$

condition (2.27) implies that $1\frac{2}{5} \leq \sum_{i\in N} x_i \leq 1\frac{3}{5}$. For the one-person coalitions we have that $F_{\boldsymbol{V}(\{i\})}(x_i) \geq \frac{1}{2}$ if $x_i \geq 0$. Furthermore, since for all two-person coalitions S it holds that

$$F_{\boldsymbol{V}(S)}(t)) = \begin{cases} 0, & \text{if } t \leq 0, \\ \frac{1}{2}t, & \text{if } 0 < t < 2, \\ 1, & \text{if } 2 \leq t, \end{cases}$$

it follows that $\sum_{i\in S} x_i \geq 1$ if $F_{\boldsymbol{V}(S)}(\sum_{i\in S} x_i) \geq \frac{1}{2}$. Hence, the prior core of this game is given by

$$C(\boldsymbol{V}) = \{x \in \mathbb{R}^3_+ \mid \forall_{S\subset N:\#S=2} : \sum_{i\in S} x_i \geq 1, 1\tfrac{2}{5} \leq \sum_{i\in N} x_i \leq 1\tfrac{3}{5}\}.$$

A necessary and sufficient condition for nonemptiness of the core is given by the following theorem, which can be found in CHARNES and GRANOT (1973).

Theorem 2.11 Let $(N, \boldsymbol{V})$ be a chance-constrained game. Then $C(\boldsymbol{V}) = \emptyset$ if and only if $\mu > \overline{\alpha}(N)$ with

$$\begin{array}{rll} \mu = & \min & F_{\boldsymbol{V}(N))}(\sum_{i\in N} x_i) \\ & \text{s.t.:} & F_{\boldsymbol{V}(S)}(\sum_{i\in S} x_i) \geq \alpha(S) \\ & & F_{\boldsymbol{V}(N)}(\sum_{i\in N} x_i) \geq \overline{\alpha}(N). \end{array}$$

A prior Shapley value is defined in the same way as for TU-games. The agents enter a room one at a time and, subsequently, receive their contribution to the benefits. The prior Shapley value then assigns to each agent the expected payoff of this procedure. Thus, given a chance-constrained game $(N, \boldsymbol{V})$, the *prior Shapley value* $\phi(\boldsymbol{V})$ is defined by

$$\phi_i(\boldsymbol{V}) = \sum_{S \subset N \setminus \{i\}} \frac{\#S!(n-1-\#S)!}{n!} \left(E(\boldsymbol{V}(S \cup \{i\})) - E(\boldsymbol{V}(S))\right), \tag{2.29}$$

for all $i \in N$.

Note that the prior-Shapley value extends the Shapley value for TU-games. This follows immediately from the fact that $E(v(S)) = v(S)$ in the deterministic case.

Example 2.22 Consider the game defined in Example 2.21. We have that $E(\boldsymbol{V}(\{i\})) = 0$ for $i = 1, 2, 3$, $E(\boldsymbol{V}(S)) = \frac{1}{2}$ if $\#S = 2$, and $E(\boldsymbol{V}(N)) = 1\frac{1}{2}$. This implies that

$$\begin{aligned}
\phi_1(\boldsymbol{V}) &= \tfrac{1}{6}\left(2(E(\boldsymbol{V}(\{1\})) - E(\boldsymbol{V}(\emptyset))) + E(\boldsymbol{V}(\{1,2\})) - E(\boldsymbol{V}(\{2\}))+\right. \\
&\quad \left. E(\boldsymbol{V}(\{1,3\})) - E(\boldsymbol{V}(\{3\})) + 2(E(\boldsymbol{V}(\{1,2,3\})) - E(\boldsymbol{V}(\{2,3\})))\right) \\
&= \tfrac{1}{6}(2 \cdot (0-0) + 2 \cdot (\tfrac{1}{2} - 0) + 2 \cdot (1\tfrac{1}{2} - \tfrac{1}{2})) \\
&= \tfrac{1}{2}.
\end{aligned}$$

Similarly, it follows that $\phi_2(\boldsymbol{V}) = \frac{1}{2}$ and $\phi_3(\boldsymbol{V}) = \frac{1}{2}$. Note that $\phi(\boldsymbol{V}) \in C(\boldsymbol{V})$.

For defining a prior nucleolus, one needs to specify the excess of a coalition at a given allocation x. Applying the definition used for TU-games results in the excess $E(S, x) = \boldsymbol{V}(S) - \sum_{i \in S} x_i$. Since $\boldsymbol{V}(S)$ is a stochastic variable, the excess is also a stochastic variable. But this raises a problem, because for determining the nucleolus we need to arrange the excesses in a decreasing order. For stochastic variables, however, there is no straightforward criterium that states when one stochastic variable is larger than another one. So, for chance-constrained games another definition of the excess is needed. CHARNES and GRANOT (1976) express the excess of coalition S at an allocation x by $1 - F_{\boldsymbol{V}(S)}(\sum_{i \in S} x_i)$, the probability that $\boldsymbol{V}(S)$ exceeds $\sum_{i \in S} x_i$. This implies that the excess decreases with $\sum_{i \in S} x_i$. With this definition the prior nucleolus is defined similarly to the nucleolus for TU-games. This means that for each feasible allocation $x \in Y$, where

$$Y = \{x \in \mathbf{R}^N \mid \forall_{i \in N} : x_i \geq 0, \underline{\alpha}(N) \leq F_{\boldsymbol{V}(N)}(\sum_{i \in N} x_i) \leq \overline{\alpha}(N)\},$$

we have $E(S, x) = 1 - F_{\boldsymbol{V}(S)}(\sum_{i \in S} x_i)$ for all $S \subset N$. Furthermore, $E(x) = (E(S, x))_{S \subset N}$ is the vector of excesses and $\theta \circ E(x)$ denotes the vector of excesses with its elements arranged in decreasing order. The *prior nucleolus* of a chance-constrained game is then defined by

$$n(\boldsymbol{V}) = \{x \in Y \mid \forall_{y \in Y} : \theta \circ E(x) \leq_{lex} \theta \circ E(y)\} \tag{2.30}$$

The prior nucleolus is not an extension of the nucleolus for TU-games. For if the benefits are deterministic, the excess $E(S,x) = 1 - F_{\boldsymbol{V}(S)}(\sum_{i\in S} x_i)$ equals either zero or one, depending on whether $\sum_{i\in S} x_i \geq v(S)$ or $\sum_{i\in S} x_i < v(S)$.

Example 2.23 Consider again the game presented in Example 2.21. Then for each feasible allocation x we have that $E(\{i\},x) = 0$ for $i = 1,2,3$, $E(S,x) = 1 - \frac{1}{2}\sum_{i\in S} x_i$ if $\#S = 2$ and $E(N,x) = 2 - \sum_{i\in N} x_i$. The prior nucleolus of this game equals $n(\boldsymbol{V}) = \frac{1}{15}(8,8,8)$. Note that $n(\boldsymbol{V}) \in C(\boldsymbol{V})$ and that $n_i(\boldsymbol{V}) > \phi_i(\boldsymbol{V})$ for $i = 1,2,3$.

CHARNES and GRANOT (1976) show that the prior nucleolus is a well defined allocation. The prior nucleolus, however, need not be in the prior core, as the following example shows.

Example 2.24 Consider the following three-person chance-constrained game $(N,\boldsymbol{V})$ with $\boldsymbol{V}(\{i\}) = 0$, $i = 1,2,3$, $\boldsymbol{V}(S) \sim U(0,2)$ if $|S| \geq 2$. Furthermore, let $\alpha(\{i\}) = \frac{1}{2}$ for $i = 1,2,3$, $\alpha(\{1,2\}) = \frac{1}{8}$, $\alpha(\{1,3\}) = \frac{2}{8}$, $\alpha(\{2,3\}) = \frac{3}{8}$, and $\underline{\alpha}(N) = \overline{\alpha}(N) = \frac{3}{8}$. Then $x \in \mathbb{R}^3_+$ is an element of the prior core if

$$\begin{array}{llll} x_1+ & x_2+ & x_3 & = \frac{3}{4} \\ x_1+ & x_2 & & \geq \frac{1}{4} \\ x_1+ & & x_3 & \geq \frac{1}{2} \\ & x_2+ & x_3 & \geq \frac{3}{4}. \end{array}$$

The only solution of this system of inequalities is $x = (0, \frac{1}{4}, \frac{1}{2})$. Hence, $C(\boldsymbol{V}) = \{(0, \frac{1}{4}, \frac{1}{2})\}$. For determining the nucleolus, note that for any feasible allocation x it holds that $E(\{i\},x) = 0$ for $i = 1,2,3$ and that $E(S,x) = 1 - \frac{1}{2}x(S)$ for $\#S = 2$. One can check that the prior nucleolus then equals $n(\boldsymbol{V}) = \frac{1}{4}(1,1,1)$, which is not contained in the prior core.

Thus far, we only discussed first stage allocations. As mentioned before, once the realization of the benefits is known, the first stage allocation might need some modifications. For these modifications, CHARNES and GRANOT (1977) introduce the two-stage nucleolus. Given the prior allocation x, for instance, the prior Shapley value, and the realization $\hat{v}(N)$ of $\boldsymbol{V}(N)$, the two-stage nucleolus allocates $\hat{v}(N)$ in such a way that the maximal complaint of the coalitions is minimized. An allocation of $\hat{v}(N)$ is a vector $y \in \hat{Y} = \{\hat{y} \in \mathbb{R}^n_+ | \sum_{i\in N} \hat{y}_i = \hat{v}(N)\}$. Given an allocation $y \in \hat{Y}$ the complaint of coalition S with respect to this modification is defined by

$$\begin{aligned} E(S,y;x) &= \mathbf{P}(\{\boldsymbol{V}(S) > \textstyle\sum_{i\in S} y_i\}) - \mathbf{P}(\{\boldsymbol{V}(S) > \textstyle\sum_{i\in S} x_i\}) \\ &= F_{\boldsymbol{V}(S)}(\textstyle\sum_{i\in S} x_i) - F_{\boldsymbol{V}(S)}(\textstyle\sum_{i\in S} y_i). \end{aligned}$$

So, the complaint increases if $\sum_{i\in S} y_i$ decreases and the other way around. Now, if $E(y;x) = (E(S,y;x))_{S\subset N}$ denotes the vector of excesses and $\theta \circ E(y;x)$ the vector of excesses with

the elements ordered in a decreasing way, the *two-stage nucleolus* with respect to the prior allocation x is given by

$$\hat{n}(\boldsymbol{V};x) \;=\; \{y \in \hat{Y} \,|\, \forall_{\hat{y}\in\hat{Y}} : \theta \circ E(y;x) \leq_{lex} \theta \circ E(\hat{y};x)\}. \tag{2.31}$$

Example 2.25 Consider the game defined in Example 2.21. Recall that the prior Shapley value equals $\phi(\boldsymbol{V}) = \frac{1}{2}(1,1,1)$. Now, suppose that the realization of $\boldsymbol{V}(N)$ is 1, then the prior-allocation needs to be modified. Let $\hat{Y} = \{\hat{y} \in \mathbb{R}^3_+ | \sum_{i\in N} \hat{y}_i = 1\}$. Furthermore, for $y \in \hat{Y}$ we have that $E(\{i\}, y; \phi(\boldsymbol{V})) = 0$ for $i = 1,2,3$, $E(S, y; \phi(\boldsymbol{V})) = \frac{1}{2}(\sum_{i\in S} \phi_i(\boldsymbol{V}) - \sum_{i\in S} y_i)$ if $\#S = 2$, and $E(N, y; \phi(\boldsymbol{V})) = 1\frac{1}{2} - \sum_{i\in N} y_i$. This implies that $\hat{n}(\boldsymbol{V}; \phi(\boldsymbol{V})) = \frac{1}{3}(1,1,1)$.

3 Stochastic Cooperative Games

For TU-games, the payoff of a coalition is assumed to be known with certainty. In many cases though, the payoffs to coalitions can be uncertain. This would not raise a problem if the agents can await the realizations of the payoffs before deciding which coalitions to form and which allocations to settle on. But if the formation of coalitions and allocations has to take place before the payoffs are realized, the theory of TU-games does no longer apply.

Section 2.5 of the previous chapter discussed chance-constrained games that are introduced in CHARNES and GRANOT (1973). For these games the value of a coalition S is allowed to be a stochastic variable. They suggested to allocate the stochastic payoff of the grand coalition in two stages. In the first stage, so-called prior payoffs are promised to the agents. These prior payoffs are such that there is a good chance that these promises can be realized. In the second stage the realization of the stochastic payoff is awaited and, subsequently, a possibly nonfeasible prior payoff vector has to be adjusted to this realization in some way.

In this chapter we will not follow the route set out by CHARNES and GRANOT(1973). Instead we will introduce a different, more extensive model. The main reason for this is that the model used by CHARNES and GRANOT (1976) leaves the preferences of the agents unspecified. The model we introduce includes the agents' preferences so that we can take into account each individual's behavior towards risk. Furthermore, it allocates random payoffs to each agent instead of the two-stage deterministic allocations in chance-constrained games.

3.1 The Model

For a better understanding of our definition of a stochastic cooperative game, let us start with describing some examples of cooperative decision making problems in a stochastic environment. So, let us return to the linear production and sequencing situations and formulate them

in a stochastic setting.

Example 3.1 Recall from Example 2.2 that for the deterministic case the outcome space Y in a linear production situation equals $\mathbb{R}$ and that for each coalition S it holds that

$$Y_S = \{(x_i)_{i\in S} \in \mathbb{R}^S | \exists_{c\in C(S)} : \sum_{i\in S} x_i = p^\top c\},$$

with $C(S) = \{c \in \mathbb{R}^m_+ | Ac \leq \sum_{i\in S} b^i\}$ the set of feasible production plans. As already mentioned in the introduction, we can formulate this problem in a stochastic setting. Presuming that production takes a considerable amount of time, prices can change between the moment that a decision is taken on the production plan and the moment that this production plan is realized. So, the individuals do not exactly know the prices when they have to decide on their production plan. Let the stochastic variable $\boldsymbol{P}_j \in L^1(\mathbb{R})$ describe the price of commodity $j = 1, 2, \ldots, m$. Then given a feasible production plan $c \in C(S)$ for coalition S, the revenues equal $\boldsymbol{P}^\top c$ with $\boldsymbol{P} = (\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_m)$. In our model the revenues are stochastic, each individual receives a stochastic payoff and we choose the outcome space $\mathcal{Y}$ to be $L^1(\mathbb{R})$. Furthermore, an allocation of the revenues $\boldsymbol{P}^\top c$ to the members of coalition S is a vector $\boldsymbol{Y} \in \prod_{i\in S} \boldsymbol{Y} = L^1(\mathbb{R})^S$ such that $\sum_{i\in S} \boldsymbol{Y}_i \leq \boldsymbol{P}^\top c$. The outcome space $\mathcal{Y}_S$ thus equals

$$\mathcal{Y}_S = \{(\boldsymbol{Y}_i)_{i\in S} \in L^1(\mathbb{R})^S | \exists_{c\in C(S)} : \sum_{i\in S} \boldsymbol{Y}_i \leq \boldsymbol{P}^\top c\}. \tag{3.1}$$

If $\succsim_i$ describes the preferences of agent $i \in N$ over the set of random payoffs $L^1(\mathbb{R})$, then the triple $(N, \{\mathcal{Y}_S\}_{S\subset N}, \{\succsim_i\}_{i\in N})$ describes a linear production situation with stochastic prices.

Example 3.2 Consider the sequencing situation described in Example 2.3. In the more realistic case, the service times of the divisions are only known with certainty once the service has ended. As long as they are waiting for service, the divisions only know their serving times by approximation. So, let the stochastic variable $\boldsymbol{P}_i \in L^1(\mathbb{R})$ describe the service time of division i. Furthermore, let $k_i : \mathbb{R} \to \mathbb{R}_+$ be the cost function of division i. Thus, division i incurs costs $k_i(t)$ if the waiting time equals t. Then, given a serving order $\sigma \in \Pi_N$, the total cost savings equal

$$\sum_{i\in N} k_i \left(\sum_{j\in N:\sigma_0(j)\leq\sigma_0(i)} \boldsymbol{P}_j \right) - \sum_{i\in N} k_i \left(\sum_{j\in N:\sigma(j)\leq\sigma(i)} \boldsymbol{P}_j \right).$$

Since these cost savings are stochastic the outcome space can be chosen $\mathcal{Y} = L^1(\mathbb{R})$. Similarly as in Example 3.1 one defines

$$\mathcal{Y}_S = \{(\boldsymbol{Y}_i)_{i\in S} \in L^1(\mathbb{R})^S | \exists_{\sigma\in\Pi_S} : \sum_{i\in S} \boldsymbol{Y}_i \leq \boldsymbol{K}_S(\sigma)\}, \tag{3.2}$$

with $\boldsymbol{K}_S(\sigma) = \sum_{i\in S} k_i \left(\sum_{j\in N:\sigma_0(j)\leq\sigma_0(i)} \boldsymbol{P}_j\right) - \sum_{i\in S} k_i \left(\sum_{j\in N:\sigma(j)\leq\sigma(i)} \boldsymbol{P}_j\right)$ the random cost savings for coalition S. Note that the serving order is no longer explicitly contained in the outcome space, as is the case in Example 2.3. A sequencing situation with stochastic processing times can thus be described by $(N, \{\mathcal{Y}_S\}_{S\subset N}, \{\succsim_i\}_{i\in N})$, where $\succsim_i$ are agent i's preferences over the set $L^1(\mathbb{R})$ of random payoffs.

The third and final example concerns financial markets. For a general equilibrium model on financial markets the reader is referred to MAGILL and SHAFER (1991). The example we provide will show some substantial differences with the model considered by MAGILL and SHAFER (1991). First, our example focuses on cooperation between the agents, and second, the assets we consider are indivisible goods.

Example 3.3 Let N be a set of agents, each having an initial endowment m^i of money. Furthermore, we have a set F of assets, where each asset $f \in F$ has a price π_f and stochastic revenues $\boldsymbol{R}_f \in L^1(\mathbb{R})$. Each agent can invest his money in a portfolio of assets and obtain stochastic revenues. We allow the set F to contain identical assets, so that we do not need to specify the amount that each agent buys of a specific asset. Instead of buying portfolios individually, agents can also cooperate with each other, combine their endowments of money, and invest in a more diversified portfolio of assets. Although cooperation enables them to make their investment less risky, they also have to agree on a division of the joint revenues. To model this as a cooperative decision making model, note that the agents only need to divide the stochastic revenues of their portfolio, so that we can again choose the outcome space $\mathcal{Y}$ to be $L^1(\mathbb{R})$. Next, let $A \subset F$ be a portfolio of assets. Then a portfolio A is affordable for coalition S if they can pay for all the assets in the portfolio, that is, $\sum_{f\in A} \pi_f \leq \sum_{i\in S} m^i$. Let $A(S) = \{A \subset F | \sum_{f\in A} \pi_f \leq \sum_{i\in S} m^i\}$ denote the set of affordable portfolios for coalition S. The outcome space $\mathcal{Y}_S$ then equals

$$\mathcal{Y}_S = \{(\boldsymbol{Y}_i)_{i\in S} \in L^1(\mathbb{R})^S | \exists_{A\in A(S)} : \sum_{i\in S} \boldsymbol{Y}_i \leq \sum_{f\in A} \boldsymbol{R}_f\}. \tag{3.3}$$

In the above mentioned examples coalitions have to choose between several actions, each action possibly yielding different stochastic revenues. In other words, each coalition can choose their benefits from a collection $\mathcal{V}(S) \subset L^1(\mathbb{R})$ of random variables. This observation leads to the following definition of a stochastic cooperative game.

A *stochastic cooperative game* is described by a tuple $(N, \mathcal{V}, \{\succsim_i\}_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is the set of agents, $\mathcal{V}$ a map assigning to each coalition S a collection of stochastic payoffs $\mathcal{V}(S)$, and $\succsim_i$ the preference relation of agent i over the set of random payoffs $L^1(\mathbb{R})$. So, in particular, it is assumed that the random payoffs are expressed in some infinitely divisible commodity like money. Benefits consisting of several different or indivisible commodities are excluded.

Besides the preferences, note that the collection of random payoffs $\mathcal{V}(S)$ is another difference with a chance-constrained game, which only assigns one random payoff $\boldsymbol{V}(S)$ to each coalition. This would not be a restriction though, if it is possible to pick a 'best' stochastic payoff $\boldsymbol{V}(S) \in \mathcal{V}(S)$. For the linear production situations with stochastic prices, however, it is usually not possible to determine unambiguously the production plan that yields the best revenues. Hence, such situations cannot be modelled as chance-constrained games. The same holds for the other two examples we discussed.

Example 3.4 Consider the linear production situation with stochastic prices presented in Example 3.1. This situation is described as a stochastic cooperative game $(N, \mathcal{V}, \{\succsim_i\}_{i\in N})$ with

$$\mathcal{V}(S) = \{\boldsymbol{X} \in L^1(\mathbb{R}) | \exists_{c\in C(S)} : \boldsymbol{X} = \boldsymbol{P}^\top c\},$$

for each $S \subset N$.

Example 3.5 The sequencing situation with stochastic serving times as described in Example 3.2 is written as a stochastic cooperative game $(N, \mathcal{V}, \{\succsim_i\}_{i\in N})$, where

$$\mathcal{V}(S) = \{\boldsymbol{X} \in L^1(\mathbb{R}) | \exists_{\sigma\in\Pi_S} : \boldsymbol{X} = \boldsymbol{K}_S(\sigma)\},$$

for all $S \subset N$.

Example 3.6 The financial markets presented in Example 3.3 are described by the following stochastic cooperative game $(N, \mathcal{V}, \{\succsim_i\}_{i\in N})$ with

$$\mathcal{V}(S) = \{\boldsymbol{X} \in L^1(\mathbb{R}) | \exists_{A\in A(S)} : \boldsymbol{X} = \sum_{f\in A} \boldsymbol{R}_f\},$$

for all $S \subset N$.

As is the case for TU- and NTU-games, the main issue for stochastic cooperative games is to find an appropriate allocation of the stochastic payoff of the grand coalition. In Examples 3.1 - 3.3 an *allocation* of a random payoff $\boldsymbol{X}$ for a coalition S is a vector $(\boldsymbol{Y}_i)_{i\in S} \in L^1(\mathbb{R})^S$ such that $\sum_{i\in S} \boldsymbol{Y}_i \leq \boldsymbol{X}$. The interpretation is that agent $i \in S$ receives the random payoff $\boldsymbol{Y}_i$. This definition induces a very large class of allocations, which, on the one hand, is nice, but, on the other hand, will give computational difficulties. Therefore we restrict our attention to a specific class of allocations.

Let $S \subset N$ and let $\boldsymbol{X} \in \mathcal{V}(S)$ be a stochastic payoff for coalition S. An *allocation* of $\boldsymbol{X}$ can be represented by a pair $(d, r) \in \mathbb{R}^S \times \mathbb{R}^S$ with $\sum_{i\in S} d_i \leq 0$, $\sum_{i\in S} r_i = 1$, and $r_i \geq 0$ for all $i \in S$. Given a pair (d, r), agent $i \in S$ then receives the random payoff $d_i + r_i\boldsymbol{X}$. So,

an allocation consists of two parts. The first part represents deterministic transfer payments between the agents in S. Note that the $\leq$-sign allows the agents to discard some of the money. The second part then allocates a fraction of the random payoff to each agent in S. Note that we can indeed allocate fractions of the stochastic revenues $\boldsymbol{X}$ because of our assumption that $\boldsymbol{X}$ represents money or any other comparable commodity. The *class of stochastic cooperative games* with agent set N adopting this restricted definition of an allocation is denoted by $SG(N)$ and its elements are denoted by Γ. Furthermore, let $\mathcal{Z}_\Gamma(S)$ denote the set of allocations coalition S can obtain. Hence,

$$\mathcal{Z}_\Gamma(S) \;=\; \{(d_i + r_i\boldsymbol{X})_{i\in S} |\; \boldsymbol{X} \in \mathcal{V}(S), (d,r) \in H^S \times \Delta^S\}, \tag{3.4}$$

where $H^S = \{d \in \mathbb{R}^S | \sum_{i\in S} d_i \leq 0\}$ and $\Delta^S = \{r \in \mathbb{R}^S_+ | \sum_{i\in S} r_i = 1\}$.

3.2 Preferences on Stochastic Payoffs

One of the main differences between stochastic cooperative games and chance-constrained games is that the former explicitly takes into account the preferences of the individuals. Reason for this is that the behavior towards risk may vary between different individuals. In general, three different kinds of behavior are distinguished, i.e., risk averse, risk neutral, and risk loving behavior. To formalize these three types, consider the space $L^1(\mathbb{R})$ of stochastic variables and let $\succsim_i$ describe the preferences of individual i over $L^1(\mathbb{R})$. Then individual i is said to behave *risk averse* if for all $\boldsymbol{Y} \in L^1(\mathbb{R})$ it holds that $E(\boldsymbol{Y}) \succsim_i \boldsymbol{Y}$. So, he rather receives the expected payoff with certainty than the random payoff itself. Similarly, individual i is *risk loving* if the reverse holds, that is, for all $\boldsymbol{Y} \in L^1(\mathbb{R})$ it holds that $\boldsymbol{Y} \succsim_i E(\boldsymbol{Y})$. Thus he prefers receiving the random payoff to receiving its expected value. Finally, individual i is called *risk neutral* if $\boldsymbol{Y} \sim_i E(\boldsymbol{Y})$ for all $\boldsymbol{Y} \in L^1(\mathbb{R})$. This means that he is indifferent between receiving the random payoff and receiving the certain payoff $E(\boldsymbol{Y})$. So a risk neutral person is only interested in the expected payoff he receives. He does not care about the difference between the possible realizations. Note, however, that these conditions only specify whether an agent is risk averse, risk neutral, or risk loving. It does not say anything about, say, the degree of risk aversion.

Next, let us consider some examples of preferences on stochastic variables and see how they express different kinds of risk behavior.

A natural way of ordering stochastic payoffs is by means of stochastic dominance. Let $\boldsymbol{X}, \boldsymbol{Y} \in L^1(\mathbb{R})$ be stochastic variables and denote by $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Y}}$ the probability distribution functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Then $\boldsymbol{X}$ *stochastically dominates* $\boldsymbol{Y}$, in notation $\boldsymbol{X} \succsim_F \boldsymbol{Y}$, if and only if for all $t \in \mathbb{R}$ it holds that $F_{\boldsymbol{X}}(t) \leq F_{\boldsymbol{Y}}(t)$. Moreover, we have $\boldsymbol{X} \succ_F \boldsymbol{Y}$ if and only if for all $t \in \mathbb{R}$ it holds that $F_{\boldsymbol{X}}(t) \leq F_{\boldsymbol{Y}}(t)$ with strict inequality for at least one $t \in \mathbb{R}$. Hence, $\boldsymbol{X}$ stochastically dominates $\boldsymbol{Y}$ if for any $t \in \mathbb{R}$ the random variable $\boldsymbol{X}$ yields

at most the value t with a lower probability than $\boldsymbol{Y}$ does. Note that this preference relation is incomplete. Many stochastic variables will be incomparable with respect to $\succsim_F$. Unless $\boldsymbol{Y}$ is degenerate, the random payoff $\boldsymbol{Y}$ and $E(\boldsymbol{Y})$, for instance, are incomparable. Consequently, $\succsim_F$ does not imply risk averse, risk neutral, or risk loving behavior. Intuitively though, one may expect that every rationally behaving agent, whether he is risk averse, risk neutral or risk loving, will prefer a stochastic payoff $\boldsymbol{X}$ to $\boldsymbol{Y}$ if $\boldsymbol{X} \succsim_F \boldsymbol{Y}$. So for any other preference relation $\succsim_i$ one may at least expect that $\boldsymbol{X} \succsim_i \boldsymbol{Y}$ whenever $\boldsymbol{X} \succsim_F \boldsymbol{Y}$.

In Section 2.2 we saw that under certain conditions, the preferences of an agent can be represented by a utility function. Since in that case the outcome space Y only needs to be connected, Theorem 2.1 also holds if $Y = L^1(\mathbb{R})$. So, if an agent's preferences $\succsim_i$ are complete, transitive, and continuous on $L^1(\mathbb{R})$, then there exists a continuous utility function $W_i : L^1(\mathbb{R}) \to \mathbb{R}$ such that for any $\boldsymbol{X}, \boldsymbol{Y} \in L^1(\mathbb{R})$ the following statement is true: $\boldsymbol{X} \succsim_i \boldsymbol{Y}$ if and only if $W_i(\boldsymbol{X}) \geq W_i(\boldsymbol{Y})$. Note that if the preferences in a stochastic cooperative game $\Gamma = (N, \mathcal{V}, \{\succsim_i\}_{i \in N})$ are represented by a utility function $W_i : L^1(\mathbb{R}) \to \mathbb{R}$, then we can also define an NTU-game (N, V) by

$$V(S) \;=\; \{x \in \mathbb{R}^S \mid \exists_{(d_i + r_i \boldsymbol{X})_{i \in S} \in \mathcal{Z}_\Gamma(S)} \forall_{i \in S} : x_i \leq W_i(d_i + r_i \boldsymbol{X})\},$$

for all $S \subset N$.

A special subclass of utility functions are the ones that correspond to expected utility. This means that there exists another utility function $U_i : \mathbb{R} \to \mathbb{R}$ such that $W_i(\boldsymbol{X}) = E(U_i(\boldsymbol{X}))$ for all $\boldsymbol{X} \in L^1(\mathbb{R})$. The utility function U_i is also called a *von Neumann-Morgenstern utility function*. It implies that individual i prefers one random payoff to another one if and only if the expected utility of the first exceeds the expected utility of the latter, that is, for any $\boldsymbol{X}, \boldsymbol{Y} \in L^1(\mathbb{R})$ it holds that $\boldsymbol{X} \succsim_i \boldsymbol{Y}$ if and only if $E(U_i(\boldsymbol{X})) \geq E(U_i(\boldsymbol{Y}))$. Agents having such preferences are called expected utility maximizers.

Note that if U_i is a von Neumann-Morgenstern utility function representing the preferences $\succsim_i$, then so does the utility function $\hat{U}_i(x) = aU_i(x) + b$ $(x \in \mathbb{R})$ with $a > 0$ and $b \in \mathbb{R}$. Note that only positive linear transformations are allowed instead of arbitrary monotonic transformations. Furthermore, note that $E(U_i(\boldsymbol{X})) \geq E(U_i(\boldsymbol{Y}))$ whenever $\boldsymbol{X}$ stochastically dominates $\boldsymbol{Y}$.

Von Neumann-Morgenstern utility functions are commonly used to model the preferences of an individual. In particular, they can express different kinds of behavior towards risk. So does risk averse behavior correspond to a concave utility function, risk neutral behavior to a linear utility function, and risk loving behavior to a convex utility function. Furthermore, if the utility function is twice differentiable, one can determine other measures of risk aversion like absolute risk aversion, relative risk aversion, and partial risk aversion, which all express a degree of risk aversion. We will not explain these terms in detail though, the interested reader is referred to EECKHOUDT and GOLLIER (1995).

Besides maximizing expected utility, one can think of other ways to describe a preference relation over stochastic payoffs. One such a way considers a particular quantile of the stochastic payoff. This means that for a given value of $\alpha \in (0,1)$ it holds that $\boldsymbol{X} \succsim_\alpha \boldsymbol{Y}$ if and only if $\xi_\alpha(\boldsymbol{X}) \geq \xi_\alpha(\boldsymbol{Y})$. Furthermore, note that $\boldsymbol{X} \succsim_\alpha \boldsymbol{Y}$ if $\boldsymbol{X} \succsim_F \boldsymbol{Y}$ and that $\boldsymbol{X} \succsim_F \boldsymbol{Y}$ if for all $\alpha \in (0,1)$ it holds that $\boldsymbol{X} \succsim_\alpha \boldsymbol{Y}$. These type of preferences occur, for instance, in insurance problems when the insurance premium is based on the percentile-principle. As Example 3.7 will show, these preferences do not allow for risk averse, risk neutral, or risk loving behavior in the sense as defined on page 49. Nevertheless, a low value of α can be associated with some kind of 'risk averse' behavior. For a low value of α implies that attention is focused on the worse outcomes, which is more likely behavior for people who do not like to take risks than for people who do like it. Using a similar argument, a high value of α shows some kind of 'risk loving' behavior.

Example 3.7 Consider $\succsim_\alpha$-preferences with $\alpha \in (0,1)$ and define

$$\boldsymbol{X} = \begin{cases} 0, & \text{with probability } p \\ 1, & \text{with probability } 1-p, \end{cases}$$

and

$$\boldsymbol{Y} = \begin{cases} 0, & \text{with probability } q \\ 1, & \text{with probability } 1-q, \end{cases}$$

where $0 < q < \alpha < p < 1$. Then $\xi_\alpha(\boldsymbol{X}) = 0$ and $\xi_\alpha(\boldsymbol{Y}) = 1$ while $\xi_\alpha(E(\boldsymbol{X})) = E(\boldsymbol{X}) = 1-p$ and $\xi_\alpha(E(\boldsymbol{Y})) = E(\boldsymbol{Y}) = 1-q$. Hence, $E(\boldsymbol{X}) \succ_\alpha \boldsymbol{X}$ and $\boldsymbol{Y} \succ_\alpha E(\boldsymbol{Y})$, so that $\succsim_\alpha$ implies neither risk averse nor risk neutral nor risk loving behavior.

The third type of preferences we consider appear in portfolio decision theory. An agent's preferences over different portfolios often depends on the expected returns of this portfolio and - provided that it exists - the variance of the returns, which is interpreted as a measure for the risk of a portfolio. Formally, this means that agent i has a utility function $U_i : \mathbb{R}^2 \to \mathbb{R}$ such that he weakly prefers the portfolio with returns $\boldsymbol{X} \in L^1(\mathbb{R})$ to a portfolio $\boldsymbol{Y}$ if and only if $U_i(E(\boldsymbol{X}), V(\boldsymbol{X})) \geq U_i(E(\boldsymbol{Y}), V(\boldsymbol{Y}))$. A simple example of such a utility function is $U_i(E(\boldsymbol{X}), V(\boldsymbol{X})) = E(\boldsymbol{X}) + b\sqrt{V(\boldsymbol{X})}$, where $b \in \mathbb{R}$. Then $b < 0$ implies risk averse behavior, $b = 0$ implies risk neutral behavior, and $b > 0$ implies risk loving behavior. This utility function, however, violates the condition we posed at the beginning of this section, namely, that $\boldsymbol{X}$ is preferred to $\boldsymbol{Y}$ if $\boldsymbol{X}$ stochastically dominates $\boldsymbol{Y}$.

Example 3.8 Given $\boldsymbol{X}, \boldsymbol{Y} \in L^1(\mathbb{R})$ let $\boldsymbol{X} \succsim \boldsymbol{Y}$ if $E(\boldsymbol{X}) - 2\sqrt{V(\boldsymbol{X})} \geq E(\boldsymbol{Y}) - 2\sqrt{V(\boldsymbol{Y})}$. Take $\boldsymbol{X} \sim U(0,2)$ and $\boldsymbol{Y} \sim U(0,1)$ so that $\boldsymbol{X}$ is stochastically dominant to $\boldsymbol{Y}$. Since $E(\boldsymbol{X}) = 1$, $V(\boldsymbol{X}) = \frac{1}{3}$, $E(\boldsymbol{Y}) = \frac{1}{2}$, and $V(\boldsymbol{Y}) = \frac{1}{12}$ it follows that $E(\boldsymbol{X}) - 2\sqrt{V(\boldsymbol{X})} = 1 - \frac{2}{3}\sqrt{3} < \frac{1}{2} - \frac{1}{3}\sqrt{3} = E(\boldsymbol{Y}) - 2\sqrt{V(\boldsymbol{Y})}$. Hence, $\boldsymbol{X} \prec \boldsymbol{Y}$.

In the forthcoming analysis of stochastic cooperative games, our attention is focused on two different subclasses of games that are characterized by the conditions we impose on the preferences of the individuals. The first class is mainly determined by a continuity property. As we will show, the standard continuity condition is too restrictive for our model. Therefore, we introduce a weaker form of continuity. For the second class, we consider preferences for which a random payoff can be represented by its so-called certainty equivalent.

3.2.1 Weakly Continuous Preferences

Consider the following three properties for the preference relation $\succsim_i$ of agent i:

(C1) $\succsim_i$ is complete and transitive on $L^1(\mathbb{R})$;

(C2) for all $\boldsymbol{X} \in L^1(\mathbb{R})$ and all $d > 0$ we have that $\boldsymbol{X} + d \succ_i \boldsymbol{X}$;

(C3) for any $\boldsymbol{X}, \boldsymbol{Y} \in L^1(\mathbb{R})$ there exist $\bar{d}, \underline{d} \in \mathbb{R}$ such that $\boldsymbol{X} + \underline{d} \prec_i \boldsymbol{Y} \prec_i \boldsymbol{X} + \bar{d}$;

(C4) $\succsim_i$ is *continuous*, i.e., the sets $\{F_{\boldsymbol{X}} \in \mathcal{F} | \boldsymbol{X} \succ_i \boldsymbol{Y}\}$ and $\{F_{\boldsymbol{X}} \in \mathcal{F} | \boldsymbol{X} \prec_i \boldsymbol{Y}\}$ are open in the metric space $(\mathcal{F}, \rho)$ (see (B.6)) for all $\boldsymbol{Y} \in L^1(\mathbb{R})$.[1]

Condition (C2) states that the preferences are strictly increasing in the deterministic amount of money one receives. Condition (C3) then states that by adding the appropriate deterministic amount of money a stochastic payoff $\boldsymbol{X}$ can be made strictly worse than the stochastic payoff $\boldsymbol{Y}$ and the other way around. Finally, the continuity condition (C4) can be interpreted as follows. Given that a stochastic payoff $\boldsymbol{Y}$ is strictly preferred to the stochastic payoff $\boldsymbol{X}$, then $\boldsymbol{Y}$ is also strictly preferred to all stochastic payoffs that differ only slightly from $\boldsymbol{X}$. Note that the conditions (C1) - (C4) imply that for any $\boldsymbol{X}, \boldsymbol{Y} \in L^1(\mathbb{R})$ there exists $d \in \mathbb{R}$ such that $d + \boldsymbol{X} \sim_i \boldsymbol{Y}$. Condition (C4), however, turns out to be rather strong. As the following examples show, neither $\succsim_\alpha$-preferences nor preferences based on von Neumann-Morgenstern utility functions satisfy continuity.

Example 3.9 Consider $\succsim_\alpha$-preferences with $\alpha = \frac{1}{2}$. Take $\boldsymbol{Y} = \frac{3}{4}$ and $\boldsymbol{X}^k$ such that

$$F_{\boldsymbol{X}^k}(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{1}{2} - \frac{1}{k} + \frac{2}{k}t, & \text{if } 0 \leq t < 1 \\ 1, & \text{if } 1 \leq t. \end{cases}$$

for all $t \in \mathbb{R}$ and $k \geq 2$. The random variable $\boldsymbol{X}^k$ attains the values 0 and 1 with probability $\frac{1}{2} - \frac{1}{k}$ each, while the remaining $\frac{2}{k}$ is equally distributed over the open interval $(0, 1)$. Then $\xi_\alpha(\boldsymbol{X}^k) = \frac{1}{2} < \frac{3}{4} = \xi_\alpha(\boldsymbol{Y})$ so that $\boldsymbol{X}^k \in \{F_{\boldsymbol{Z}} \in \mathcal{F} | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$ for all $k \geq 2$. Since the sequence $\boldsymbol{X}^k$ weakly converges to $\boldsymbol{X}$ with $\mathbf{P}(\{\boldsymbol{X} = 0\}) = \mathbf{P}(\{\boldsymbol{X} = 1\}) = \frac{1}{2}$ it follows that $\xi_\alpha(\boldsymbol{X}) = 1$. Hence, $\boldsymbol{X} \notin \{F_{\boldsymbol{Z}} \in \mathcal{F} | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$ so that $\succsim_\alpha$ is not continuous.

[1] If the preferences are complete, an equivalent statement is that $\{F_{\boldsymbol{X}} \in \mathcal{F} | \boldsymbol{X} \succsim_i \boldsymbol{Y}\}$ and $\{F_{\boldsymbol{X}} \in \mathcal{F} | \boldsymbol{X} \precsim_i \boldsymbol{Y}\}$ are closed sets in $(\mathcal{F}, \rho)$ for all $\boldsymbol{Y} \in L^1(\mathbb{R})$.

Example 3.10 Let $U_i : \mathbb{R} \to \mathbb{R}$ be a utility function defined by $U_i(t) = -e^{-t}$ for all $t \in \mathbb{R}$. Take $\boldsymbol{Y} = -\log 2$ and

$$\boldsymbol{X}^k = \begin{cases} 0, & \text{with probability } 1 - \frac{1}{k} \\ -\log(2k), & \text{with probability } \frac{1}{k} \end{cases}$$

for $k \geq 2$. Then $E(U_i(\boldsymbol{X}^k)) = (1 - \frac{1}{k})(-1) + \frac{1}{k}(-2k) = \frac{1}{k} - 3 < -2 = E(U_i(\boldsymbol{Y}))$ so that $\boldsymbol{X}^k \in \{F_{\boldsymbol{Z}} \in \mathcal{F} | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$ for all $k \geq 2$. Next, let $\boldsymbol{X} \in L^1(\mathbb{R})$ be such that $\mathbb{P}(\boldsymbol{X} = 0) = 1$. Since $E(U_i(\boldsymbol{X})) = -1$ we have that $\boldsymbol{X} \notin \{F_{\boldsymbol{Z}} \in \mathcal{F} | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$. From the fact that the sequence $\boldsymbol{X}^k$ weakly converges to $\boldsymbol{X}$ it then follows that $\{F_{\boldsymbol{Z}} \in \mathcal{F} | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$ is not closed. Since the preferences based on U_i are complete, this implies that they are not continuous.

Note that the utility function in Example 3.10 is unbounded. For bounded utility functions, expected utility does lead to a continuous preference relation. This follows directly from Helly's Theorem B.1 that is stated in Appendix B.

As the two examples show, continuity is too strong a condition. So, if we want something like continuity to be satisfied, we need some further assumptions. Actually, only one additional assumption and a weaker form of continuity is what we need. Let $\Gamma \in SG(N)$ be a stochastic cooperative game such that $\mathcal{V}(S)$ is finite for every $S \subset N$. So, each coalition only has a finite number of random benefits to choose from. Note that this is the case for the sequencing games and financial markets presented in Example 3.5 and Example 3.6, but not for the linear production games described in Example 3.4.

For our weaker form of continuity, we make use of the special structure of the allocations. This means that we only need continuity on specific subsets of random variables only. Therefore, let $\Gamma = (N, \mathcal{V}, \{\succsim_i\}_{i \in N}) \in SG(N)$ and suppose that coalition $S \subset N$ has formed. Then the set of payoffs agent $i \in S$ possibly obtains equals

$$\{d_i + r_i \boldsymbol{X} | \ \boldsymbol{X} \in \mathcal{V}(S), d_i \in \mathbb{R}, r_i \in [0,1]\}.$$

Consequently, the set of payoffs agent i possibly obtains in the game Γ equals

$$\{d_i + r_i \boldsymbol{X} | \ \exists_{S \subset N : i \in S, \# S \geq 2} : \boldsymbol{X} \in \mathcal{V}(S), d_i \in \mathbb{R}, r_i \in [0,1]\} \ \cup \ \mathcal{V}(\{i\}). \tag{3.5}$$

So, what we actually need is that the preferences $\succsim_i$ are continuous on the set defined in (3.5). Formulating continuity in this way though, implies that continuity depends on the game Γ. Since that is not desirable, we formulate it in the following, more generalized way. For this purpose, let $\mathcal{X} \subset L^1(\mathbb{R})$ be a finite subset of random variables. Define

$$\mathcal{L}(\mathcal{X}) \ = \ \{d + r\boldsymbol{X} | \ \boldsymbol{X} \in \mathcal{X}, d \in \mathbb{R}, r \in [0,1]\}, \tag{3.6}$$

and

$$\mathcal{F}(\mathcal{X}) = \{F_{\boldsymbol{Z}} | \, \boldsymbol{Z} \in \mathcal{L}(\mathcal{X})\}, \tag{3.7}$$

as the set of probability distribution functions corresponding to the random payoffs of the form $d + r\boldsymbol{X}$ that can be constructed out of the set $\mathcal{X}$. Note that $(\mathcal{F}(\mathcal{X}), \rho)$ constitutes a metric space (see also B.6). The modified continuity condition then reads as follows

(C5) $\succsim_i$ satisfies *weak continuity* if for all finite subsets $\mathcal{X} \subset L^1(\mathbb{R})$ it holds that the sets $\{F_{\boldsymbol{Z}} \in \mathcal{F}(\mathcal{X}) | \boldsymbol{Z} \succ_i \boldsymbol{Y}\}$ and $\{F_{\boldsymbol{Z}} \in \mathcal{F}(\mathcal{X}) | \boldsymbol{Z} \prec_i \boldsymbol{Y}\}$ are open in the metric space $(\mathcal{F}(\mathcal{X}), \rho)$ for all $\boldsymbol{Y} \in \mathcal{L}(\mathcal{X})$. [2] [3]

Note that since $\mathcal{V}(S)$ is finite by assumption, weak continuity implies that the preferences are also continuous on the set formulated in (3.5). Furthermore, note that weak continuity does not depend on a stochastic cooperative game $\Gamma \in SG(N)$.

Let $CG(N) \subset SG(N)$ denote the class of stochastic cooperative games for which $\mathcal{V}(S)$ is finite for each $S \subset N$ and the preferences $\succsim_i$ of each individual $i \in N$ satisfy the conditions (C1), (C2), (C3), and (C5).

3.2.2 Certainty Equivalents

In this section we focus on a special class of stochastic cooperative games to which one can associate a TU-game. For the games in this subclass the preferences $\{\succsim_i\}_{i \in N}$ are such that for each $i \in N$ there exists a function $m_i : L^1(\mathbb{R}) \to \mathbb{R}$ satisfying

(M1) for all $\boldsymbol{X}, \boldsymbol{Y} \in L^1(\mathbb{R}) : \boldsymbol{X} \succsim_i \boldsymbol{Y}$ if and only if $m_i(\boldsymbol{X}) \geq m_i(\boldsymbol{Y})$;

(M2) for all $d \in \mathbb{R} : m_i(d) = d$;

(M3) for all $\boldsymbol{X} \in L^1(\mathbb{R}) : m_i(\boldsymbol{X} - m_i(\boldsymbol{X})) = 0$,

(M4) for all $\boldsymbol{X} \in L^1(\mathbb{R})$ and all $d, d' \in \mathbb{R}$ with $d < d' : m_i(d + \boldsymbol{X}) < m_i(d' + \boldsymbol{X})$.

The interpretation is that $m_i(\boldsymbol{X})$ equals the amount of money m for which agent i is indifferent between receiving the amount $m_i(\boldsymbol{X})$ with certainty and receiving the stochastic payoff $\boldsymbol{X}$. The amount $m_i(\boldsymbol{X})$ is called the *certainty equivalent* of $\boldsymbol{X}$. Condition (M1) states that agent i weakly prefers one stochastic payoff to another one if and only if the certainty equivalent of the former is greater than or equal to the certainty equivalent of the latter. Condition (M2) states that the certainty equivalent of a deterministic payoff d equals d itself. From the conditions (M1) and (M2) it then follows that $\boldsymbol{X} \sim_i m_i(\boldsymbol{X})$ for all $\boldsymbol{X} \in L^1(\mathbb{R})$. Condition (M3) states that an agent is indifferent between receiving the stochastic payoff $\boldsymbol{X} - m_i(\boldsymbol{X})$ and receiving the

[2] If the preferences are complete, an equivalent statement is that $\{F_{\boldsymbol{Z}} \in \mathcal{F}(\mathcal{X}) | \boldsymbol{Z} \succsim_i \boldsymbol{Y}\}$ and $\{F_{\boldsymbol{Z}} \in \mathcal{F}(\mathcal{X}) | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$ are closed sets in $(\mathcal{F}(\mathcal{X}), \rho)$ for all $\boldsymbol{Y} \in \mathcal{L}(\mathcal{X})$.

[3] For ease of notation, the sets $\{F_{\boldsymbol{Z}} \in \mathcal{F}(\mathcal{X}) | \boldsymbol{Z} \succsim_i \boldsymbol{Y}\}$ and $\{F_{\boldsymbol{Z}} \in \mathcal{F}(\mathcal{X}) | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$ are often denoted by $\{\boldsymbol{Z} \in \mathcal{L}(\mathcal{X}) | \boldsymbol{Z} \succsim_i \boldsymbol{Y}\}$ and $\{\boldsymbol{Z} \in \mathcal{L}(\mathcal{X}) | \boldsymbol{Z} \precsim_i \boldsymbol{Y}\}$, respectively.

payoff zero. Finally, condition (M4) is equivalent to (C2); it implies that the preferences over stochastic payoffs of the form $d + \boldsymbol{X}$ are monotonically increasing in d. Note that condition (M3) is not implied by the other three conditions as the following example shows.

Example 3.11 Take the utility function equal to $U_i(x) = \sqrt{x+1}$, $x \geq -1$, so that $\boldsymbol{X} \succsim_i \boldsymbol{Y}$ if and only if $E(U_i(\boldsymbol{X})) \geq E(U_i(\boldsymbol{Y}))$. Take $\boldsymbol{X}$ such that $\mathbb{P}(\{\boldsymbol{X} = 0\}) = \mathbb{P}(\{\boldsymbol{X} = 1\}) = \frac{1}{2}$. Then $E(U_i(\boldsymbol{X})) = \frac{1}{2}(\sqrt{2}+1)$ and $m_i(\boldsymbol{X}) = U_i^{-1}(E(U_i(\boldsymbol{X}))) = \frac{1}{2}(\sqrt{2} - \frac{1}{2})$. Note that m_i satisfies (M1), (M2) and (M4) but that

$$m_i(\boldsymbol{X} - m_i(\boldsymbol{X})) = \left(\tfrac{1}{2}(\sqrt{\tfrac{5}{4} - \tfrac{1}{2}\sqrt{2}}) + \tfrac{1}{2}(\sqrt{\tfrac{9}{4} - \tfrac{1}{2}\sqrt{2}})\right)^2 - 1 = -0.1714 \neq 0.$$

Conditions (M3) and (M4) are equivalent to condition (M5) below:

(M5) for all $\boldsymbol{X} \in L^1(\mathbb{R})$ and all $d \in \mathbb{R}$: $m_i(d + \boldsymbol{X}) = d + m_i(\boldsymbol{X})$.

Obviously, condition (M5) implies conditions (M3) and (M4). For the converse, suppose that $m_i(d + \boldsymbol{X}) > d + m_i(\boldsymbol{X})$ for some $d \in \mathbb{R}$ and some $\boldsymbol{X} \in L^1(\mathbb{R})$. Then we get the following contradiction,

$$0 = m_i(d + \boldsymbol{X} - (m_i(d + \boldsymbol{X}))) < m_i(d + \boldsymbol{X} - (d + m_i(\boldsymbol{X}))) = m_i(\boldsymbol{X} - m_i(\boldsymbol{X})) = 0.$$

Here the first and the last equality follow from condition (M3) and the inequality follows from condition (M4). Of course, a similar argument holds if one would suppose that $m_i(d + \boldsymbol{X}) < d + m_i(\boldsymbol{X})$.

Example 3.12 Consider the preferences based on a utility function of the form $U(t) = \beta.e^{-\alpha t}$, $(t \in \mathbb{R})$, where $\beta < 0$ and $\alpha > 0$. The certainty equivalent of $\boldsymbol{X} \in L^1(\mathbb{R})$ can be defined by $m(\boldsymbol{X}) = U^{-1}(E(U(\boldsymbol{X})))$. It is easy to check that m satisfies conditions (M1), (M2) and (M4). For condition (M3), let $\boldsymbol{X} \in L^1(\mathbb{R})$. Then $U^{-1}(t) = -\frac{1}{\alpha}\log\left(\frac{t}{\beta}\right)$ and

$$\begin{aligned}
m(\boldsymbol{X} - m(\boldsymbol{X})) &= U^{-1}(E(U(\boldsymbol{X} - m(\boldsymbol{X})))) \\
&= -\tfrac{1}{\alpha}\log\left(\tfrac{1}{\beta}\int \beta \cdot e^{-\alpha(t - m(\boldsymbol{X}))} dF_{\boldsymbol{X}}(t)\right) \\
&= -\tfrac{1}{\alpha}\log\left(e^{\alpha m(\boldsymbol{X})}\tfrac{1}{\beta}\int \beta \cdot e^{-\alpha t} dF_{\boldsymbol{X}}(t)\right) \\
&= -m(\boldsymbol{X}) - \tfrac{1}{\alpha}\log\left(\tfrac{1}{\beta}(\int \beta e^{-\alpha t} dF_{\boldsymbol{X}}(t))\right) \\
&= -m(\boldsymbol{X}) + m(\boldsymbol{X}) = 0.
\end{aligned}$$

Finally, note that U is a monotonically increasing and concave function and thus implies risk averse behavior.

Example 3.13 Let the preferences $\succsim_\alpha$ be such that for $X, Y \in L^1(\mathbb{R})$ it holds that $\boldsymbol{X} \succsim_\alpha \boldsymbol{Y}$ if $\xi_\alpha(\boldsymbol{X}) \geq \xi_\alpha(\boldsymbol{Y})$, where $\alpha \in (0,1)$. With the certainty equivalent of $\boldsymbol{X} \in L^1(\mathbb{R})$ given by $m(\boldsymbol{X}) = \xi_\alpha(\boldsymbol{X})$, the conditions (M1), (M2), and (M5) are satisfied. That (M1) and (M2) are fulfilled is straightforward. For (M5) note that

$$\begin{aligned}
\xi_\alpha(d + \boldsymbol{X}) &= \sup\{t \in \mathbb{R} |\ \mathbb{P}(\{d + \boldsymbol{X} < t\}) \leq \alpha\} \\
&= \sup\{t \in \mathbb{R} |\ \mathbb{P}(\{\boldsymbol{X} < t - d\}) \leq \alpha\} \\
&= \sup\{t + d \in \mathbb{R} |\ \mathbb{P}(\{\boldsymbol{X} < t\}) \leq \alpha\} \\
&= d + \sup\{t \in \mathbb{R} |\ \mathbb{P}(\{\boldsymbol{X} < t\}) \leq \alpha\} \\
&= d + \xi_\alpha(\boldsymbol{X}),
\end{aligned}$$

for all $d \in \mathbb{R}$ and all $\boldsymbol{X} \in L^1(\mathbb{R})$. Hence, $m(d + \boldsymbol{X}) = d + m(\boldsymbol{X})$.

Note that condition (M5) shows some resemblance with the transferable utility property defined in 2.3. Transferable utility implies that utility is linearly separable in the amount of money one receives while condition (M5) states that the certainty equivalent is linearly separable in the deterministic amount of money one receives. The similarities between (M5) and transferable utility go even further. As was the case for transferable utility, we can represent the stochastic benefits of each coalition by a single number if the preferences of each agent can be described by certainty equivalents.

Let $\Gamma \in SG(N)$ be a stochastic cooperative game satisfying conditions (M1) - (M4). Take $S \subset N$. An allocation $(d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ is *Pareto optimal* for coalition S if there exists no allocation $(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ such that $\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}} \succ_i d_i + r_i\boldsymbol{X}$ for all $i \in S$. Pareto optimal allocations are characterized by the following proposition.

Proposition 3.1 Let $\Gamma \in SG(N)$ satisfy conditions (M1) - (M4). Then $(d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ is Pareto optimal if and only if

$$\sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}) = \max\left\{\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) |\ (\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)\right\}. \tag{3.8}$$

PROOF: Let $(d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ be such that it is not Pareto optimal. Then there exists an allocation $(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ such that $\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}} \succ_i d_i + r_i\boldsymbol{X}$ for all $i \in S$. Since Γ satisfies (M1) -(M4) this is equivalent to $m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) > m_i(d_i + r_i\boldsymbol{X})$ for all $i \in S$. Hence,

$$\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) > \sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}),$$

so that expression (3.8) does not hold.

Next, let $(d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ be such that

$$\sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}) < \max\left\{\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) |\ (\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)\right\}.$$

Then there exists $(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ such that

$$\sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}) \;<\; \sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}). \tag{3.9}$$

Define for each $i \in S$

$$\begin{aligned}\delta_i \;=\;& \hat{d}_i - m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) + m_i(d_i + r_i\boldsymbol{X}) \\ &+ \frac{1}{\#S}\left(\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) - \sum_{i\in S} m_i(d_i + r_i\boldsymbol{X})\right).\end{aligned}$$

Then $(\delta_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ because $\sum_{i\in S}\delta_i = \sum_{i\in S}\hat{d}_i \leq 0$. Furthermore, for each $i \in S$ it holds that

$$\begin{aligned}m_i(\delta_i + \hat{r}_i\hat{\boldsymbol{X}}) \;&=\; \delta_i - \hat{d}_i + m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) \\ &=\; m_i(d_i + r_i\boldsymbol{X}) + \frac{1}{\#S}\left(\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) - \sum_{i\in S} m_i(d_i + r_i\boldsymbol{X})\right) \\ &>\; m_i(d_i + r_i\boldsymbol{X}),\end{aligned}$$

where the first equality follows from (M5) and the inequality follows from (3.9). Since this implies that $\delta_i + \hat{r}_i\hat{\boldsymbol{X}} \succ_i d_i + r_i\boldsymbol{X}$ for all $i \in S$, the allocation $(d_i + r_i\boldsymbol{X})_{i\in S}$ violates Pareto optimality. ∎

For interpreting condition (3.8), consider a particular allocation for coalition S. Now suppose that each member pays the certainty equivalent of the random payoff he receives. Then we know from condition (M3) that the initial wealth of each member has not changed. Furthermore, this coalition still has to divide the certainty equivalents that have been paid by its members. Since the preferences are strictly increasing in the deterministic amount of money one receives, the more money a coalition can divide, the better it is for all its members. So, the best way to allocate the stochastic benefits, is the one that maximizes the sum of the certainty equivalents. Furthermore, we can describe the stochastic benefits of each coalition by the maximum sum of the certainty equivalents they can obtain, provided that this maximum exists, of course. This follows from the fact that for each $S \subset N$ it holds that

$$\{(m_i(d_i + r_i\boldsymbol{X}))_{i\in S} \mid (d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)\} \;=\; \{x \in \mathbf{R}^S \mid \sum_{i\in S} x_i \leq v_\Gamma(S)\}, \tag{3.10}$$

where

$$v_\Gamma(S) \;=\; \max\left\{\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) \mid (\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)\right\}. \tag{3.11}$$

Expression (3.10) means that it does not matter for coalition S whether they allocate a random payoff $\boldsymbol{X} \in \mathcal{V}(S)$ or the deterministic amount $v_\Gamma(S)$. To see that this equality does indeed

hold, note that the inclusion '$\subset$' follows immediately from the definition of $v_\Gamma(S)$. For the reverse inclusion '$\supset$', let $y \in \{x \in \mathbf{R}^S | \sum_{i\in S} x_i \leq v_\Gamma(S)\}$. Next, let $(d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ be such that $\sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}) = v_\Gamma(S)$ and define $\delta_i = y_i + d_i - m_i(d_i + r_i\boldsymbol{X})$ for each $i \in S$. Since $\sum_{i\in S} \delta_i \leq 0$ and

$$\begin{aligned} m_i(\delta_i + r_i\boldsymbol{X}) &= m_i(\delta_i - d_i + d_i + r_i\boldsymbol{X}) \\ &= \delta_i - d_i + m_i(d_i + r_i\boldsymbol{X}) \\ &= y_i + d_i - m_i(d_i + r_i\boldsymbol{X}) - d_i + m_i(d_i + r_i\boldsymbol{X}) \;=\; y_i \end{aligned}$$

for all $i \in S$ it holds that $y \in \{(m_i(d_i + r_i\boldsymbol{X}))_{i\in S} |\ (d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)\}$. Hence, the benefits of coalition S can be represented by $v_\Gamma(S)$. So, if for a stochastic cooperative game $\Gamma \in SG(N)$ the value $v_\Gamma(S)$ is well defined for each coalition $S \subset N$ we can also describe the game Γ by a TU-game (N, v_Γ) with $v_\Gamma(S)$ as in (3.11). Let $MG(N) \subset SG(N)$ denote the class of stochastic cooperative games for which the conditions (M1) - (M4) are satisfied and the game (N, v_Γ) is well defined.

4 The Core, Superadditivity, and Convexity

In this chapter we consider the core for stochastic cooperative games. Given its interpretation for both TU- and NTU-games, the core extends fairly straightforward to the class of stochastic cooperative games. A balancedness condition like the one for TU-games, however, does not yet exist. We can only provide such a condition for the subclass MG(N) of stochastic cooperative games for which certainty equivalents are well defined. Besides the core, we also extend the definitions of superadditivity and convexity to the class of stochastic cooperative games. We show that a convex stochastic cooperative game is superadditive and has a nonempty core. Furthermore, we define marginal vectors and show that for each convex stochastic cooperative game all marginal vectors belong to the core.

4.1 The Core of a Stochastic Cooperative Game

Section 2.3 and Section 2.4 considered the core for TU-games and NTU-games, respectively. For these games, an allocation is a core-allocation if no coalition has an incentive to part company with the grand coalition. This same principle is used to define a core for stochastic cooperative games. Therefore, we need to specify under which conditions a coalition has an incentive to leave the grand coalition. So, let $\Gamma \in SG(N)$ be a stochastic cooperative game and let $(d_i + r_i \boldsymbol{X})_{i \in N} \in \mathcal{Z}_\Gamma(N)$ be an allocation for N. Coalition S has an incentive to leave coalition N and start cooperating on its own if it can improve on the payoff of each of its members. This means that there exists an allocation $(\hat{d}_i + \hat{r}_i \hat{\boldsymbol{X}})_{i \in S} \in \mathcal{Z}_\Gamma(S)$ such that $\hat{d}_i + \hat{r}_i \hat{\boldsymbol{X}} \succ_i d_i + r_i \boldsymbol{X}$ for all $i \in S$. The *core* of a stochastic cooperative game $\Gamma \in SG(N)$ is thus defined by

$$\mathcal{C}(\Gamma) = \left\{ (d_i + r_i \boldsymbol{X})_{i \in N} \in \mathcal{Z}_\Gamma(N) \;\middle|\; \begin{array}{l} \forall_{S \subset N} \; \nexists_{(\hat{d}_i + \hat{r}_i \hat{\boldsymbol{X}})_{i \in S} \in \mathcal{Z}_\Gamma(S)} \forall_{i \in S} : \\ \qquad \hat{d}_i + \hat{r}_i \hat{\boldsymbol{X}} \succ_i d_i + r_i \boldsymbol{X} \end{array} \right\}. \tag{4.1}$$

A stochastic cooperative game $\Gamma \in SG(N)$ with a nonempty core is called *balanced*. Furthermore, if the core of every subgame $\Gamma_{|S}$ is nonempty, then Γ is called *totally balanced*. The subgame $\Gamma_{|S}$ is given by $(S, \mathcal{V}_{|S}, \{\succsim_i\}_{i \in S})$ where $\mathcal{V}_{|S}(T) = \mathcal{V}(T)$ for all $T \subset S$.

Proposition 4.1 The linear production games with stochastic prices defined in Example 3.4 are totally balanced if the agents are expected utility maximizers and the utility function U_i is concave for each agent $i \in N$.

PROOF: Let $\Gamma \in SG(N)$ be such a linear production game and define the corresponding NTU-game (N, V) by

$$V(S) = \{x \in \mathbb{R}^S \mid \exists_{(d_i + r_i \boldsymbol{X})_{i \in S} \in \mathcal{Z}_\Gamma(S)} \forall_{i \in S} : x_i \leq E(U_i(d_i + r_i \boldsymbol{X}))\},$$

for all $S \subset N$. We first show that $\mathcal{C}(\Gamma) \neq \emptyset$ if $C(V) \neq \emptyset$. For this, suppose that $\mathcal{C}(\Gamma) = \emptyset$ and let $x \in V(N)$. From the definition of $V(N)$ it follows that there exists an allocation $(d_i + r_i \boldsymbol{X})_{i \in N} \in \mathcal{Z}_\Gamma(N)$ such that $x_i \leq E(U_i(d_i + r_i \boldsymbol{X}))$ for all $i \in N$. Since $\mathcal{C}(\Gamma) = \emptyset$ there exists a coalition S and an allocation $(\hat{d}_i + \hat{r}_i \hat{\boldsymbol{X}})_{i \in S} \in \mathcal{Z}_\Gamma(S)$ such that $E(U_i(\hat{d}_i + \hat{r}_i \hat{\boldsymbol{X}})) > E(U_i(d_i + r_i \boldsymbol{X}))$ for all $i \in S$. Define $y \in \mathbb{R}^S$ with $y_i = E(U_i(\hat{d}_i + \hat{r}_i \hat{\boldsymbol{X}}))$ for all $i \in S$. Since $y \in V(S)$ and $y_i > x_i$ for each $i \in S$ it follows that $x \notin C(V)$. Hence, $C(V) = \emptyset$. So, for $\mathcal{C}(\Gamma)$ to be nonempty it suffices to show that $C(V)$ is nonempty.

In order to show that $C(V)$ is nonempty we apply Theorem 2.7. Thus, for each balanced map λ we have to show that $\sum_{S \subset N} \lambda(S) \tilde{V}(S) \subset \tilde{V}(N)$.

Let $x^S \in \tilde{V}(S)$ for each $S \subset N$, let λ be a balanced map, and define $x = \sum_{S \subset N} \lambda(S) x^S$. Note that $x_i^S = 0$ if $i \notin S$. Hence, $x_i = \sum_{S \subset N} \lambda(S) x_i^S = \sum_{S \subset N : i \in S} \lambda(S) x_i^S$ for all $i \in N$. Take $\boldsymbol{Y}^S \in \mathcal{Z}_\Gamma(S)$ such that $E(U_i(\boldsymbol{Y}_i^S)) \geq x_i^S$ for all $i \in S$. Furthermore, let $y^S \in C(S)$ be a production plan such that $\sum_{i \in S} \boldsymbol{Y}_i^S = \boldsymbol{P}^\top y^S$. Recall that $y^S \in C(S)$ implies that $Ay^S \leq \sum_{i \in S} b(i)$. Next, define $z = \sum_{S \subset N} \lambda(S) y^S$ and $\boldsymbol{Z}_i = \sum_{S \subset N : i \in S} \lambda(S) \boldsymbol{Y}_i^S$ for $i \in N$. Since

$$\begin{aligned} Az &= A \sum_{S \subset N} \lambda(S) y^S = \sum_{S \subset N} \lambda(S) A y^S \leq \sum_{S \subset N} \lambda(S) \sum_{i \in S} b(i) \\ &= \sum_{i \in N} b(i) \sum_{S \subset N : i \in S} \lambda(S) = \sum_{i \in N} b(i), \end{aligned}$$

it follows that $z \in C(N)$. Furthermore, we have that

$$\begin{aligned} \boldsymbol{P}^\top z &= \boldsymbol{P}^\top \sum_{S \subset N} \lambda(S) y^S = \sum_{S \subset N} \lambda(S) \boldsymbol{P}^\top y^S \\ &= \sum_{S \subset N} \lambda(S) \sum_{i \in S} \boldsymbol{Y}_i^S = \sum_{i \in N} \sum_{S \subset N : i \in S} \lambda(S) \boldsymbol{Y}_i^S = \sum_{i \in N} \boldsymbol{Z}_i, \end{aligned}$$

so that $\boldsymbol{Z} \in \mathcal{Z}_\Gamma(N)$ and, consequently, that $(E(U_i(\boldsymbol{Z}_i)))_{i\in N} \in \tilde{V}(N)$. From

$$\begin{aligned} E(U_i(\boldsymbol{Z}_i)) &= E(U_i(\sum_{S\subset N: i\in S} \lambda(S)\boldsymbol{Y}_i^S)) \geq E(\sum_{S\subset N: i\in S} \lambda(S)U_i(\boldsymbol{Y}_i^S)) \\ &= \sum_{S\subset N: i\in S} \lambda(S)E(U_i(\boldsymbol{Y}_i^S)) \geq \sum_{S\subset N: i\in S} \lambda(S)x_i = x_i, \end{aligned}$$

it then follows that $x \in \tilde{V}(N)$. Hence, $C(V) \neq \emptyset$ and, consequently, $\mathcal{C}(\Gamma) \neq \emptyset$. Since each subgame $(N, \mathcal{V}_{|S}, \{\succsim_i\}_{i\in S})$ is also a linear production game with stochastic prices, we know that $\mathcal{C}(\Gamma_{|S}) \neq \emptyset$. This implies that these games are totally balanced. ∎

For the class of TU- and NTU-games we saw that the core can be an empty set. The same holds for the class of stochastic cooperative games. We will provide an example of a game with an empty core later in this section, but first we take a closer look at the class $MG(N)$. Recall that for each $\Gamma \in MG(N)$ there corresponds a TU-game (N, v_Γ) with

$$v_\Gamma(S) = \max\{\sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}) \mid (d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)\},$$

for all $S \subset N$. Concerning the nonemptiness of the core of a game $\Gamma \in MG(N)$ we have the following result.

Theorem 4.2 Let $\Gamma \in MG(N)$. Then $\mathcal{C}(\Gamma) = \emptyset$ if and only if $C(v_\Gamma) = \emptyset$.

PROOF: Let $\Gamma \in MG(N)$ be such that $C(v_\Gamma) = \emptyset$. Take $(d_i + r_i\boldsymbol{X})_{i\in N} \in \mathcal{Z}_\Gamma(N)$. From $C(v_\Gamma) = \emptyset$ it then follows that $\sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}) < v_\Gamma(S)$ for some $S \subset N$. Take $(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ such that $\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) = v_\Gamma(S)$. Define

$$\begin{aligned} \delta_i &= m_i(d_i + r_i\boldsymbol{X}) - m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) + \hat{d}_i + \\ &\quad \frac{1}{\#S}\left(\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) - \sum_{i\in S} m_i(d_i + r_i\boldsymbol{X})\right) \end{aligned}$$

for all $i \in S$, so that

$$\begin{aligned} m_i(\delta_i + \hat{r}_i\hat{\boldsymbol{X}}) &= m_i(\delta_i - \hat{d}_i + \hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) \\ &= \delta_i - \hat{d}_i + m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) \\ &= m_i(d_i + r_i\boldsymbol{X}) + \frac{1}{\#S}\left(\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) - \sum_{i\in S} m_i(d_i + r_i\boldsymbol{X})\right) \\ &> m_i(d_i + r_i\boldsymbol{X}) \end{aligned}$$

for all $i \in S$. Moreover, since $\sum_{i\in S} \delta_i = \sum_{i\in S} \hat{d}_i \leq 0$ it holds that $(\delta_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)$. From $\delta_i + \hat{r}_i\hat{\boldsymbol{X}} \succ_i d_i + r_i\boldsymbol{X}$ for all $i \in S$ it then follows that $(d_i + r_i\boldsymbol{X})_{i\in N} \notin \mathcal{C}(\Gamma)$. Consequently, we must have that $\mathcal{C}(\Gamma) = \emptyset$.

Next, let $\mathcal{C}(\Gamma) = \emptyset$ and let x be an allocation of $v_\Gamma(N)$, i.e., $\sum_{i\in N} x_i \leq v_\Gamma(N)$. Take $(d_i + r_i\boldsymbol{X})_{i\in N} \in \mathcal{Z}_\Gamma(N)$ such that $\sum_{i\in N} m_i(d_i + r_i\boldsymbol{X}) = v_\Gamma(N)$. Define $\delta_i = x_i - m_i(d_i + r_i\boldsymbol{X}) + d_i$ for all $i \in N$, so that

$$\begin{aligned} m_i(\delta_i + r_i\boldsymbol{X}) &= m_i(\delta_i - d_i + d_i + r_i\boldsymbol{X}) \\ &= \delta_i - d_i + m_i(d_i + r_i\boldsymbol{X}) \\ &= x_i - m_i(d_i + r_i\boldsymbol{X}) + d_i - d_i + m_i(d_i + r_i\boldsymbol{X}) \;=\; x_i, \end{aligned}$$

for all $i \in N$. Furthermore, from $\sum_{i\in N} \delta_i \leq 0$ it follows that $(\delta_i + r_i\boldsymbol{X})_{i\in N} \in \mathcal{Z}_\Gamma(N)$. Since $\mathcal{C}(\Gamma) = \emptyset$ there exists a coalition $S \subset N$ and an allocation $(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ such that $\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}} \succ_i \delta_i + r_i\boldsymbol{X}$ holds for all $i \in S$. This implies that $m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) > m_i(\delta_i + r_i\boldsymbol{X}) = x_i$ for all $i \in S$ so that

$$\sum_{i\in S} x_i \;<\; \sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{\boldsymbol{X}}) \;\leq\; v_\Gamma(S),$$

Hence, $x \notin C(v_\Gamma)$ so that $C(v_\Gamma) = \emptyset$. ∎

Theorem 4.2 can also be stated in terms of allocations: if $(d_i + r_i\boldsymbol{X})_{i\in N} \in \mathcal{Z}_\Gamma(N)$ and $x \in \mathbb{R}^N$ are such that $m_i(d_i + r_i\boldsymbol{X}) = x_i$ for all $i \in N$ then

$$(d_i + r_i\boldsymbol{X})_{i\in N} \in \mathcal{C}(\Gamma) \text{ if and only if } x \in C(v_\Gamma).$$

Furthermore, we know that the core of the game (N, v_Γ) is nonempty if and only if the balancedness condition formulated in (2.10) is satisfied. Theorem 4.2 then provides an easy way to construct a stochastic cooperative game with an empty core.

Example 4.1 Consider a three-person stochastic cooperative game $\Gamma \in MG(N)$ with $\mathcal{V}(\{i\}) = \{0\}$, $i = 1, 2, 3$ and $\mathcal{V}(S) = \{\boldsymbol{X}_S\}$ with $\boldsymbol{X}_S \sim U(0, 2)$ if $\#S \geq 2$. Furthermore, let all agents be risk neutral. This means that agent i's certainty equivalent of a random payoff $\boldsymbol{X} \in L^1(\mathbb{R})$ equals $m_i(\boldsymbol{X}) = E(\boldsymbol{X})$. Consequently, we have for $(d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ it holds that

$$\begin{aligned} \sum_{i\in S} m_i(d_i + r_i\boldsymbol{X}) &= \sum_{i\in S} E(d_i + r_i\boldsymbol{X}) \;=\; \sum_{i\in S} d_i + E\left(\sum_{i\in S} r_i\boldsymbol{X}\right) \\ &= \sum_{i\in S} d_i \;+\; E(\boldsymbol{X}). \end{aligned}$$

The corresponding TU-game (N, v_Γ) then equals $v_\Gamma(S) = 0$ if $\#S = 1$ and $v_\Gamma(S) = 1$ if $\#S \geq 2$. For the game (N, v_Γ) it holds that $C(v_\Gamma) = \emptyset$. Hence, by Theorem 4.2, thus also $\mathcal{C}(\Gamma) = \emptyset$.

So, for the class $MG(N)$ of stochastic cooperative games we can relatively easy check whether the core is empty or not. For the more general class $SG(N)$, however, necessary and sufficient conditions for nonemptiness of the core do not yet exist.

4.2 Superadditive Games

For introducing superadditivity for stochastic cooperative games recall that for both TU- and NTU-games the underlying idea of superadditivity is that two disjoint coalitions can do (weakly) better by forming one coalition. Therefore, we propose the following definition of superadditivity, which conceptually is not only applicable to stochastic cooperative games, but also to TU- and NTU-games. Let $\Gamma \in SG(N)$. Then Γ is called *superadditive* if for any disjoint $S, T \subset N$ it holds that for every $(d_i^S + r_i^S \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$ and every $(d_i^T + r_i^T \boldsymbol{X}^T)_{i \in T} \in \mathcal{Z}_\Gamma(T)$ there exists an allocation $(d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T})_{i \in S \cup T} \in \mathcal{Z}_\Gamma(S \cup T)$ such that

$$\begin{array}{ll} d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T} \succsim_i d_i^S + r_i^S \boldsymbol{X}^S & \text{for all } i \in S, \\ d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T} \succsim_i d_i^T + r_i^T \boldsymbol{X}^T & \text{for all } i \in T. \end{array} \tag{4.2}$$

So whatever allocation the coalitions S and T agree on separately, they can always (weakly) improve their payoffs by forming one large coalition. Formulated in the context of NTU-games this definition reads as follows. For all disjoint $S, T \subset N$ it holds that for each allocation $x^S \in V(S)$ and each allocation $x^T \in V(T)$ there exists an allocation $x^{S \cup T} \in V(S \cup T)$ such that $x_i^{S \cup T} \geq x_i^S$ for all $i \in S$ and $x_i^{S \cup T} \geq x_i^T$ for all $i \in T$. Then it is not difficult to check that this definition is equivalent to the definition of superadditivity for NTU-games given in (2.18), i.e., for all disjoint $S, T \subset N$ it holds that $V(S) \times V(T) \subset V(S \cup T)$.

With respect to superadditivity we can derive a similar result as Theorem 4.2 for the class $MG(N)$.

Theorem 4.3 Let $\Gamma \in MG(N)$. Then Γ is superadditive if and only if (N, v_Γ) is superadditive.

PROOF: Let $\Gamma \in MG(N)$ be superadditive. Take $S, T \subset N$ disjoint and let $(d_i^S + r_i^S \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$ and $(d_i^T + r_i^T \boldsymbol{X})_{i \in T} \in \mathcal{Z}_\Gamma(T)$ be Pareto optimal, that is, $\sum_{i \in S} m_i(d_i^S + r_i^S \boldsymbol{X}^S) = v_\Gamma(S)$ and $\sum_{i \in T} m_i(d_i^T + r_i^T \boldsymbol{X}^T) = v_\Gamma(T)$. Then superadditivity implies that there exists $(d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T})_{i \in S \cup T} \in \mathcal{Z}_\Gamma(S \cup T)$ such that $d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T} \succsim_i d_i^S + r_i^S \boldsymbol{X}^S$ for all $i \in S$ and $d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T} \succsim_i d_i^T + r_i^T \boldsymbol{X}^T$ for all $i \in T$. Hence,

$$\begin{aligned} v_\Gamma(S \cup T) &\geq \sum_{i \in S \cup T} m_i(d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T}) \\ &\geq \sum_{i \in S} m_i(d_i^S + r_i^S \boldsymbol{X}^S) + \sum_{i \in T} m_i(d_i^T + r_i^T \boldsymbol{X}^T) = v_\Gamma(S) + v_\Gamma(T) \end{aligned}$$

so that (N, v_Γ) is superadditive.

Let (N, v_Γ) be superadditive. Take $S, T \subset N$ disjoint and let $(d_i^S + r_i^S \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$ and $(d_i^T + r_i^T \boldsymbol{X}^T)_{i \in T} \in \mathcal{Z}_\Gamma(T)$. So, $\sum_{i \in S} m_i(d_i^S + r_i^S \boldsymbol{X}^S) \leq v_\Gamma(S)$ and $\sum_{i \in T} m_i(d_i^T + r_i^T \boldsymbol{X}^T) \leq v_\Gamma(T)$. Now, let $(d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T})_{i \in S \cup T} \in \mathcal{Z}_\Gamma(S \cup T)$ be Pareto optimal and define

$$\delta_i = \begin{cases} m_i(d_i^S + r_i^S \boldsymbol{X}^S) + d_i^{S \cup T} - m_i(d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T}) & \text{if } i \in S, \\ m_i(d_i^T + r_i^T \boldsymbol{X}^T) + d_i^{S \cup T} - m_i(d_i^{S \cup T} + r_i^{S \cup T} \boldsymbol{X}^{S \cup T}) & \text{if } i \in T. \end{cases}$$

Then applying condition (M5) yields that $m_i(\delta_i + r_i^{S\cup T}\boldsymbol{X}^{S\cup T}) = m_i(d_i^S + r_i^S\boldsymbol{X}^S)$ if $i \in S$ and $m_i(\delta_i + r_i^{S\cup T}\boldsymbol{X}^{S\cup T}) = m_i(d_i^T + r_i^T\boldsymbol{X}^T)$ if $i \in T$. Hence, $\delta_i + r_i^{S\cup T}\boldsymbol{X}^{S\cup T} \succsim_i d_i^S + r_i^S\boldsymbol{X}^S$ for all $i \in S$ and $\delta_i + r_i^{S\cup T}\boldsymbol{X}^{S\cup T} \succsim_i d_i^T + r_i^T\boldsymbol{X}^T$ for all $i \in T$. So, for Γ to be superadditive we only need to prove that $(\delta_i + r_i^{S\cup T}\boldsymbol{X}^{S\cup T})_{i\in S\cup T} \in \mathcal{Z}_\Gamma(S\cup T)$. But this follows from

$$\begin{aligned}\sum_{i\in S\cup T}\delta_i &= \sum_{i\in S} m_i(d_i^S + r_i^S\boldsymbol{X}^S) + \sum_{i\in T} m_i(d_i^T + r_i^T\boldsymbol{X}^T)\\ &\quad + \sum_{i\in S\cup T} d_i^{S\cup T} - \sum_{i\in S\cup T} m_i(d_i^{S\cup T} + r_i^{S\cup T}\boldsymbol{X}^{S\cup T})\\ &\le v_\Gamma(S) + v_\Gamma(T) - v_\Gamma(S\cup T) \le 0,\end{aligned}$$

where the first inequality follows from $\sum_{i\in S\cup T} d_i^{S\cup T} \le 0$ and the second inequality follows from the superadditivity of (N, v_Γ). ∎

Note that the game in Example 4.1 is superadditive. So, like for TU- and NTU-games we need a stronger condition for nonemptiness of the core than superadditivity. One such a condition is convexity.

4.3 Convex Games

For our definition of convexity, we take the convexity for TU-games as formulated in (2.6) as a starting point. So, a TU-game (N, v) is called convex if for each $U \subset N$ and each $S \subset T \subset N\backslash U$ it holds that

$$v(S\cup U) - v(S) \le v(T\cup U) - v(T). \tag{4.3}$$

This means that for a coalition it is more profitable to join a larger coalition. Now, we apply this idea to stochastic cooperative games. Let $\Gamma \in SG(N)$ and define for each $S \subset N$ the set of individually rational allocations by

$$\mathcal{IR}_\Gamma(S) = \{(d_i + r_i\boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S) \mid \forall_{i\in S}\ \nexists_{\boldsymbol{Y}\in\mathcal{V}(\{i\})} : \boldsymbol{Y} \succ_i d_i + r_i\boldsymbol{X}\}.$$

Then Γ is called *convex* if for each $U \subset N$ and each $S \subset T \subset N\backslash U$ the following statement is true: for all $(d_i^S + r_i^S\boldsymbol{X}^S)_{i\in S} \in \mathcal{IR}_\Gamma(S)$, all $(d_i^T + r_i^T\boldsymbol{X}^T)_{i\in T} \in \mathcal{IR}_\Gamma(T)$ and all $(d_i^{S\cup U} + r_i^{S\cup U}\boldsymbol{X}^{S\cup U})_{i\in S\cup U} \in \mathcal{Z}_\Gamma(S\cup U)$ satisfying

$$d_i^{S\cup U} + r_i^{S\cup U}\boldsymbol{X}^{S\cup U} \succsim_i d_i^S + r_i^S\boldsymbol{X}^S,$$

for all $i \in S$, there exists an allocation $(d_i^{T\cup U} + r_i^{T\cup U}\boldsymbol{X}^{T\cup U})_{i\in T\cup U} \in \mathcal{Z}_\Gamma(T\cup U)$ such that

$$\begin{array}{ll} d_i^{T\cup U} + r_i^{T\cup U}\boldsymbol{X}^{T\cup U} \succsim_i d_i^T + r_i^T\boldsymbol{X}^T & \text{for all } i \in T, \text{ and}\\ d_i^{T\cup U} + r_i^{T\cup U}\boldsymbol{X}^{T\cup U} \succsim_i d_i^{S\cup U} + r_i^{S\cup U}\boldsymbol{X}^{S\cup U} & \text{for all } i \in U.\end{array} \tag{4.4}$$

So whatever individually rational allocation the coalitions S and T agree on separately, then given an allocation for coalition $S \cup U$ such that coalition S is willing to let coalition U join, the members of coalition U can obtain (weakly) better payoffs by joining the larger coalition T. By taking $S = \emptyset$ in the definition of convexity it follows immediately that convex games are superadditive.

For the next theorem, we need to define marginal vectors for stochastic cooperative games. Recall that a marginal vector with respect to an order $\sigma \in \Pi_N$ gives each agent his marginal contribution to the benefits of the coalition that is formed by the preceding agents. Keeping this in mind we can define a marginal vector. For this purpose, take $\Gamma \in CG(N)$ and let us abbreviate the notation $(d_i + r_i \boldsymbol{X})_{i \in S}$ of an allocation to $\boldsymbol{Y}$. Take $i \in N$, $S \subset N \backslash \{i\}$ with $S \neq \emptyset$ and $\boldsymbol{Y} \in \mathcal{Z}_\Gamma(S)$. Define

$$\mathcal{B}(S, \boldsymbol{Y}, i) = \{\hat{\boldsymbol{Y}} \in \mathcal{Z}_\Gamma(S \cup \{i\}) | \forall_{j \in S} : \hat{\boldsymbol{Y}}_j \succsim_j \boldsymbol{Y}_j\},$$

as the set of all allocations for coalition $S \cup \{i\}$ which all members of S weakly prefer to the allocation $\boldsymbol{Y} \in \mathcal{Z}_\Gamma(S)$. Note that there is no lower bound on agent i's payoff, the set $\mathcal{B}(S, \boldsymbol{Y}, i)$ is nonempty. Furthermore, let $\boldsymbol{B}(S, \boldsymbol{Y}, i) \in \mathcal{B}(S, \boldsymbol{Y}, i)$ be the most preferred allocation for agent i in this set, that is, $\boldsymbol{B}_i(S, \boldsymbol{Y}, i) \succsim_i \hat{\boldsymbol{Y}}_i$ for all $\hat{\boldsymbol{Y}} \in \mathcal{B}(S, \boldsymbol{Y}, i)$. Since the preferences are assumed to be complete, transitive and weakly continuous such a best allocation exists. A marginal vector $\boldsymbol{M}^\sigma(\mathcal{V}) \in \mathcal{Z}_\Gamma(N)$ with respect to order $\sigma \in \Pi_N$ is now constructed as follows. Let $i_k \in N$ be such that $\sigma(i_k) = k$ for $k = 1, 2, \ldots, n$. Let $\boldsymbol{Y}^1 \in \mathcal{Z}_\Gamma(\{i_1\})$ be individually rational. Thus $\boldsymbol{Y}^1 \in \mathcal{IR}_\Gamma(\{i_1\})$. Take $\boldsymbol{Y}^2 \in \mathcal{Z}_\Gamma(\{i_1, i_2\})$ such that $\boldsymbol{Y}^2 = \boldsymbol{B}(\{i_1\}, \boldsymbol{Y}^1, i_2)$. Next, take $\boldsymbol{Y}^3 \in \mathcal{Z}_\Gamma(\{i_1, i_2, i_3\})$ such that $\boldsymbol{Y}^3 = \boldsymbol{B}(\{i_1, i_2\}, \boldsymbol{Y}^2, i_3)$. Continuing this procedure yields an allocation $\boldsymbol{Y}^n \in \mathcal{Z}_\Gamma(N)$. Finally, define a *marginal vector* with respect to order σ by $\boldsymbol{M}^\sigma(\mathcal{V}) = \boldsymbol{Y}^n$.

Theorem 4.4 Let $\Gamma \in CG(N)$ be a stochastic cooperative game. If Γ is convex, then all marginal vectors $\boldsymbol{M}^\sigma(\mathcal{V})$, $\sigma \in \Pi_N$ belong to the core. Hence, $\mathcal{C}(\Gamma) \neq \emptyset$.

PROOF: Let $\Gamma \in CG(N)$ be convex and take $\sigma \in \Pi_N$. For ease of notation let us assume that $\sigma(i) = i$ for all $i \in N$. Furthermore, let $\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots, \boldsymbol{Y}^n$ be the n allocations that lead to a marginal vector $\boldsymbol{M}^\sigma(\mathcal{V})$.

In order to prove that $\boldsymbol{M}^\sigma(\mathcal{V})$ is a core allocation, we prove for $k = 1, 2, \ldots, n$ that $\boldsymbol{Y}^k$ is a core allocation for the corresponding subgame Γ^k with agent set $\{1, 2, \ldots, k\}$. The proof of the latter statement goes by induction on k.

For $k = 1$ it is obvious that $\boldsymbol{Y}^1$ belongs to the core of Γ^1. Now suppose that $\boldsymbol{Y}^k \in \mathcal{C}(\Gamma^k)$ for $k = 1, 2, \ldots, m-1$ and recall that $\boldsymbol{Y}^m = \boldsymbol{B}(\{1, 2, \ldots, m-1\}, \boldsymbol{Y}^{m-1}, m)$. To prove that $\boldsymbol{Y}^m \in \mathcal{C}(\Gamma^m)$ consider a coalition $S \subset \{1, 2, \ldots, m-1\}$.

Since $\boldsymbol{Y}^{m-1} \in \mathcal{C}(\Gamma^{m-1})$ and $\boldsymbol{Y}^m_j \succsim_j \boldsymbol{Y}^{m-1}_j$ for all $j \in S$ it follows that coalition S has no incentive to leave the coalition $\{1, 2, \ldots, m\}$.

Next, we show that also the coalition $S \cup \{m\}$ has no incentive to leave the coalition $\{1, 2, \ldots, m\}$ if $\boldsymbol{Y}^m$ is allocated. For this, let $\hat{\boldsymbol{Y}}^S \in \mathcal{IR}_\Gamma(S)$ be such that for each $j \in S$ the following statement is true: $\hat{\boldsymbol{Y}}_j^S \sim_j \boldsymbol{X}$ with $\boldsymbol{X} \in \mathcal{IR}_\Gamma(\{j\})$. So, agent j is indifferent between receiving the payoff $\hat{\boldsymbol{Y}}_j^S$ and the best payoff he can obtain on his own. Note that $\hat{\boldsymbol{Y}}^S$ exists by the weak continuity of $\{\succsim_i\}_{i \in S}$. Furthermore, let $\hat{\boldsymbol{Y}}^{S \cup \{m\}} = \boldsymbol{B}(S, \hat{\boldsymbol{Y}}^S, m)$. So $\hat{\boldsymbol{Y}}_m^{S \cup \{m\}}$ is the best payoff agent m can obtain when cooperating with coalition S. Next, let $T = \{1, 2, \ldots, m-1\}$ and $U = \{m\}$. Since $\boldsymbol{Y}^{m-1} \in \mathcal{IR}_\Gamma(\{1, 2, \ldots, m-1\})$ it follows from convexity that there exists an allocation $\boldsymbol{Z} \in \mathcal{Z}_\Gamma(\{1, 2, \ldots, m\})$ such that $\boldsymbol{Z}_j \succsim_j \boldsymbol{Y}_j^{m-1}$ for all $j \in \{1, 2, \ldots, m-1\}$ and $\boldsymbol{Z}_m \succsim_m \hat{\boldsymbol{Y}}_m^{S \cup \{m\}}$. Since $\boldsymbol{Z} \in \mathcal{B}(\{1, 2, \ldots, m-1\}, \boldsymbol{Y}^{m-1}, m)$ and $\boldsymbol{Y}^m = \boldsymbol{B}(\{1, 2, \ldots, m-1\}, \boldsymbol{Y}^{m-1}, m)$ we have that $\boldsymbol{Y}_m^m \succsim_m \boldsymbol{Z}_m \succsim_m \hat{\boldsymbol{Y}}_m^{S \cup \{m\}}$. From the fact that $\hat{\boldsymbol{Y}}_m^{S \cup \{m\}}$ is the best payoff agent m can obtain when cooperating with coalition S, there exists no individually rational allocation for coalition S that yields agent m a strictly better payoff than $\boldsymbol{Y}_m^m$. Hence, coalition S has no incentive to part company with the coalition $\{1, 2, \ldots, m\}$ if $\boldsymbol{Y}^m$ is allocated. Consequently, we have that $\boldsymbol{Y}^m \in \mathcal{C}(\Gamma^m)$. Taking $m = n$ then gives that $\boldsymbol{Y}^n \in \mathcal{C}(\Gamma^n) = \mathcal{C}(\Gamma)$. Thus, $\boldsymbol{M}^\sigma(\mathcal{V}) = \boldsymbol{Y}^n \in \mathcal{C}(\Gamma)$. ∎

The reverse of this theorem, however, is not true. The following example shows that if all marginal vectors belong to the core, then the stochastic cooperative game need not be convex.

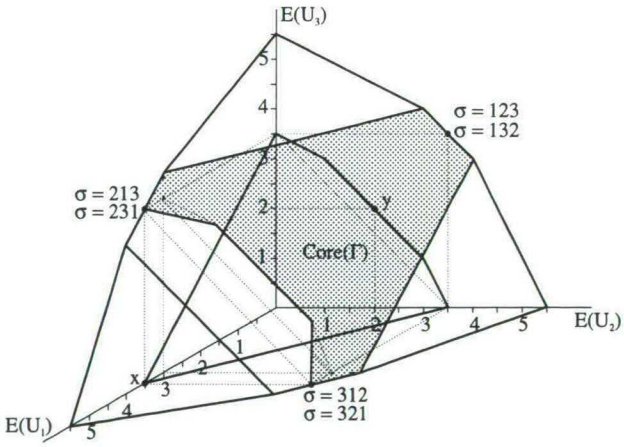


Figure 4.1

Example 4.2 Let Γ be a three-person stochastic cooperative game such that $\mathcal{V}(S) = \{\boldsymbol{X}_S\}$ for all $S \subset N$. Let $\boldsymbol{X}_{\{i\}} = 0$ for all $i \in N$, $\boldsymbol{X}_S = \frac{7}{4}$ for $S = \{1,2\},\{1,3\}$, $\boldsymbol{X}_N = 3\frac{1}{2}$ and $\boldsymbol{X}_{\{2,3\}} = \boldsymbol{Y}$, where $\boldsymbol{Y}$ is a random variable attaining the values 0 and 4 with probability $\frac{1}{2}$ each. Furthermore, each agent maximizes his expected utility according to the following utility function

$$U_i(t) = \begin{cases} 2t & \text{, if } t \leq 2, \\ t+2 & \text{, if } t > 2 \end{cases}$$

The expected utilities the agents can obtain in the various coalitions are depicted in Figure 4.1. Note that this game is superadditive. The marginal vectors of this game yield expected utilities of $(3\frac{1}{2}, 3\frac{1}{2}, 0)$, $(3\frac{1}{2}, 0, 3\frac{1}{2})$ and $(0, 3\frac{1}{2}, 3\frac{1}{2})$, respectively. Moreover, the marginal vectors belong to the core of the game. This game, however, is not convex. To see this, let $S = \{3\}, T = \{2,3\}$ and $U = \{1\}$. Take $(d_i^S + r_i^S \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$ arbitrary and take $(d_i^{S\cup U} + r_i^{S\cup U} \boldsymbol{X}^{S\cup U})_{i \in S\cup U} \in \mathcal{Z}_\Gamma(S \cup U)$ such that $d = (0,0)$ and $r = (1,0)$, i.e., agent 1 receives $\frac{7}{4}$ and agent 2 receives 0. The expected utilities of this allocation equal $3\frac{1}{2}$ for agent 1 and 0 for agent 2 (see also the point x in Figure 4.1). Next, let $(d_i^T + r_i^T \boldsymbol{X}^T)_{i \in T} \in \mathcal{Z}_\Gamma(T)$ be such that $d = (0,0)$ and $r = (\frac{1}{2}, \frac{1}{2})$. So, both agents 2 and 3 receive $\frac{1}{2}\boldsymbol{Y}$. The expected utilities then equal 2 for both players (see also the point y in Figure 4.1). Now if the game is convex there must exist an allocation $(d_i^{T\cup U} + r_i^{T\cup U} \boldsymbol{X}^{T\cup U})_{i \in T\cup U} \in \mathcal{Z}_\Gamma(T \cup U)$ such that the expected utilities are at least 2 for agents 2 and 3 and at least $3\frac{1}{2}$ for agent 1. Since $T \cup U = N$ this means that we have to allocate $\boldsymbol{X}_N = 3\frac{1}{2}$. If agents 2 and 3 must receive an expected utility of at least 2, this implies that each of them receives at least 1. Since only $3\frac{1}{2}$ can be allocated this implies that agent 1 can receive at most $1\frac{1}{2}$, yielding a utility of at most 3. Hence, the game is not convex.

The above mentioned results also hold for NTU-games if we formulate convexity based on (4.4) in the following way. An NTU-game (N, V) satisfies convexity if for every $U \subset N$ and every $S \subset T \subset N \backslash U$ the following statement holds. For every individually rational $x^S \in V(S)$, every individually rational $x^T \in V(T)$ and every $x^{S\cup U} \in V(S \cup U)$ such that $x_i^{S\cup U} \geq x_i^S$ for all $i \in S$, there exists $x^{T\cup U} \in V(T \cup U)$ satisfying

$$\begin{array}{lll} x_i^{T\cup U} & \geq & x_i^T \quad \text{for all } i \in T, \text{ and} \\ x_i^{T\cup U} & \geq & x_i^{S\cup U} \quad \text{for all } i \in U. \end{array}$$

In this way, convex NTU-games are totally balanced and every marginal vector belongs to the core. This also implies that our definition of convexity is not equivalent to ordinal or cardinal convexity - see VILKOV (1977) and SHARKEY (1981) for their respective definitions - since for both notions of convexity not all marginal vectors belong to the core.

Finally, let us focus once more on the class $MG(N)$ of stochastic cooperative games.

Theorem 4.5 Let $\Gamma \in MG(N)$. Then Γ is convex if and only if (N, v_Γ) is convex.

PROOF: Let $\Gamma \in MG(N)$ be a convex game. Take $U \subset N$ and $S \subset T \subset N \backslash U$. Next, let $(d_i^S + r_i^S \boldsymbol{X}^S)_{i \in S} \in \mathcal{IR}_\Gamma(S)$ and $(d_i^T + r_i^T \boldsymbol{X}^T)_{i \in T} \in \mathcal{IR}_\Gamma(T)$ be Pareto optimal for S and T, respectively. Note that such allocations exists since $\Gamma \in MG(N)$. Now, take $(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U})_{i \in S \cup U} \in \mathcal{Z}_\Gamma(S \cup U)$ such that it is Pareto optimal and $d_i^S + r_i^S \boldsymbol{X}^S \sim_i d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}$ for all $i \in S$. By the convexity of Γ there exists a Pareto optimal allocation $(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) \in \mathcal{Z}_\Gamma(T \cup U)$ which (weakly) improves everyone's payoff. This implies that

$$\begin{aligned}
v_\Gamma(S \cup U) - v_\Gamma(S) &= \sum_{i \in S \cup U} m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) - \sum_{i \in S} m_i(d_i^S + r_i^S \boldsymbol{X}^S) \\
&= \sum_{i \in U} m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) \\
&\leq \sum_{i \in U} m_i(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) \\
&= v_\Gamma(T \cup U) - \sum_{i \in T} m_i(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) \\
&\leq v_\Gamma(T \cup U) - v_\Gamma(T),
\end{aligned}$$

where the last equality follows from the Pareto optimality of $(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U})_{i \in T \cup U}$. The last inequality follows from the fact that for all $i \in T$ the allocation $(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U})_{i \in T \cup U}$ is a weak improvement of the Pareto optimal allocation $(d_i^T + r_i^T \boldsymbol{X}^T)_{i \in T}$. Hence, (N, v_Γ) is convex.

Let (N, v_Γ) be convex. Since this implies that (N, v_Γ) is superadditive, it follows from Theorem 4.3 that Γ is superadditive. Next, let $U \subset N$ and $S \subset T \subset N \backslash U$. Take $(d_i^S + r_i^S \boldsymbol{X}^S)_{i \in S} \in \mathcal{IR}_\Gamma(S)$ and $(d_i^T + r_i^T \boldsymbol{X}^T)_{i \in T} \in \mathcal{IR}_\Gamma(T)$ such that each allocation is Pareto optimal. Note that if we can show that condition (4.4) holds for Pareto optimal allocations, then it also holds for all other allocations. Next, take $(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U})_{i \in S \cup U} \in \mathcal{Z}_\Gamma(S \cup U)$ such that $d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U} \succsim_i d_i^S + r_i^S \boldsymbol{X}^S$ for all $i \in S$. Note that such an allocation exists by the superadditivity of Γ. Moreover, superadditivity implies that

$$\sum_{i \in U} m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) \leq v_\Gamma(S \cup U) - v_\Gamma(S). \tag{4.5}$$

Next, take $(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U})_{i \in T \cup U} \in \mathcal{Z}_\Gamma(T \cup U)$ such that it satisfies Pareto optimality and such that $d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U} \sim_i d_i^T + r_i^T \boldsymbol{X}^T$ for all $i \in T$. That such an allocation indeed exists follows from the superadditivity of Γ and the fact that the certainty equivalents satisfy (M5). Define $\delta_i = d_i^{T \cup U}$ for all $i \in T$ and

$$\begin{aligned}
\delta_i &= m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) + d_i^{T \cup U} - m_i(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) + \\
&\quad \frac{1}{\#U} \left(v_\Gamma(T \cup U) - v_\Gamma(T) - \sum_{i \in U} m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) \right),
\end{aligned}$$

for all $i \in U$. From

$$\begin{aligned}
\sum_{i \in T \cup U} \delta_i &= v_\Gamma(T \cup U) - v_\Gamma(T) + \sum_{i \in T \cup U} d_i^{T \cup U} - \sum_{i \in U} m_i(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) \\
&\leq v_\Gamma(T \cup U) - v_\Gamma(T) - \sum_{i \in U} m_i(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) \\
&= v_\Gamma(T \cup U) - v_\Gamma(T) \\
&\quad - \sum_{i \in T \cup U} m_i(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) + \sum_{i \in T} m_i(d_i^{T \cup U} + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) \\
&= v_\Gamma(T \cup U) - v_\Gamma(T) - v_\Gamma(T \cup U) + v_\Gamma(T) \;=\; 0,
\end{aligned}$$

it follows that $(\delta_i + r_i^{T \cup U} \boldsymbol{X}^{T \cup U})_{i \in T \cup U} \in \mathcal{Z}_\Gamma(T \cup U)$. Then applying condition (M5) yields for all $i \in U$ that

$$\begin{aligned}
m_i(\delta_i + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) = \; & m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) + \\
& \frac{1}{\#U} \left(v_\Gamma(T \cup U) - v_\Gamma(T) - \sum_{i \in U} m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) \right).
\end{aligned}$$

Since $v_\Gamma(T \cup U) - v_\Gamma(T) - \sum_{i \in U} m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}) \geq 0$ by (4.5) and the convexity of (N, v_Γ), it follows that $m_i(\delta_i + r_i^{T \cup U} \boldsymbol{X}^{T \cup U}) \geq m_i(d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U})$ and, consequently, that $\delta_i + r_i^{T \cup U} \boldsymbol{X}^{T \cup U} \succsim_i d_i^{S \cup U} + r_i^{S \cup U} \boldsymbol{X}^{S \cup U}$ for all $i \in U$. Hence, Γ is convex. ∎

In a similar way as for Theorem 4.5 one can show that in the context of TU-games the definitions of convexity provided in (2.6) and the TU-formulation of (4.4) are equivalent. Hence, the convexity introduced in this section is an extension of the convexity definition for TU-games.

4.4 Remarks

When introducing convexity for stochastic cooperative games we also defined marginal vectors for these type of games. This naturally raises the question if we can also define a Shapley value for stochastic cooperative games. For TU-games, the Shapley value is defined as the average of all marginal vectors. For stochastic cooperative games though, taking the average of the marginal vectors does not give the desired result. As is the case for NTU-games, the average need no longer be Pareto optimal. Furthermore, for each order $\sigma \in \Pi_N$ the marginal vector is not necessarily uniquely determined. Although the expected utility levels are uniquely determined, this need not be the case for the random payoffs itself. Hence, for each order there can exist several, maybe even an infinite number of marginal vectors.

So, when extending the Shapley value to stochastic cooperative games one encounters two major problems. First, the average of the marginal vectors can violate Pareto optimality and, second, a marginal vector need not be unique.

5 A Nucleolus for Stochastic Cooperative Games

The nucleolus, a solution concept for TU-games, originates from SCHMEIDLER (1969). This solution concept yields an allocation that minimizes the excesses of the coalitions in a lexicographical way. The excess describes how dissatisfied a coalition is with the proposed allocation. The larger the excess of a particular allocation, the more a coalition is dissatisfied with this allocation. For Schmeidler's nucleolus the excess is defined as the difference between the payoff a coalition can obtain when cooperating on its own and the payoff received by the proposed allocation. So, when less is allocated to a coalition, the excess of this coalition increases and the other way round.

Since the nucleolus depends mainly on the definition of the excess, other nucleoli are found when different definitions of excesses are used. Such a general approach can be found in POTTERS and TIJS (1992) and MASCHLER, POTTERS, and TIJS (1992). They introduced the general nucleolus as the solution that minimizes the maximal excess of the coalitions, using generally defined excess functions.

A similar argument holds for stochastic cooperative games. If we can specify the excesses we can define a nucleolus for these games. Unfortunately, this is not that simple. Defining excess functions for stochastic cooperative games appears to be not as straightforward as for deterministic cooperative games. Indeed, how should one quantify the difference between the random payoff a coalition can achieve on its own and the random payoff received by the proposed allocation when the behavior towards risk can differ between the members of this coalition? Furthermore, the excess of one coalition should be comparable to the excess of another coalition.

CHARNES and GRANOT (1976) introduced a nucleolus for chance-constrained games. There, the excess was based on the probability that the payoff a coalition can obtain when cooperating on its own, exceeds the payoff they obtained in the proposed allocation. Indeed,

it is quite reasonable to assume that a coalition is less satisfied with the proposed allocation if this probability increases.

For the excess defined in this paper we interpret the excess of Schmeidler's nucleolus in a slightly different way. Bearing the conditions of the core in mind, this excess can be interpreted as follows. Given an allocation of the grand coalition's payoff we distinguish two cases. In the first case, a coalition wants to leave the grand coalition. Then the excess equals the minimal amount of money a coalition needs on top of what they already get such that this coalition is willing to stay in the grand coalition. In the second case, a coalition has no incentive to leave the grand coalition. Then the excess equals minus the maximal amount of money that can be taken away from this coalition such that this coalition still has no incentive to leave the grand coalition. This interpretation is used to define the excess for stochastic cooperative games.

5.1 Preliminary Definitions

In this section we go through some necessary preliminaries. Consider a stochastic cooperative game $\Gamma = (N, \mathcal{V}, \{\succsim_i\}_{i\in N}) \in CG(N)$. For ease of notation, assume that $\mathcal{V}(S) = \{\boldsymbol{X}_S\}$ for all $S \in N$. The results presented in this chapter also hold though if $\mathcal{V}(S)$ is finite for all $S \subset N$. Next, take $i \in N$ and define $\mathcal{X}_i = \{\boldsymbol{X}_S | i \in S, \#S \geq 2\}$ so that

$$\mathcal{L}(\mathcal{X}_i) = \{d + r\boldsymbol{X} | \boldsymbol{X} \in \mathcal{X}_i, d \in \mathbb{R}, r \in [0,1]\} \cup \{\boldsymbol{X}_{\{i\}}\}$$

is the set of allocations that agent i can possibly receive in the game Γ. Furthermore, define

$$\mathcal{F}(\mathcal{X}_i) = \{F_{\boldsymbol{Z}} | \boldsymbol{Z} \in \mathcal{L}(\mathcal{X}_i)\}.$$

Recall that $\Gamma \in CG(N)$ implies that $\succsim_i$ is continuous on $\mathcal{F}(\mathcal{X}_i)$.

In order to define a nucleolus one needs to specify for each coalition $S \subset N$ an excess function E_S. The excess function assigns to each allocation $(d_i + r_i\boldsymbol{X}_N)_{i\in N} \in \mathcal{Z}_\Gamma(N)$ of the grand coalition N a real number representing the complaint of coalition S. The larger the complaint of a coalition the more this coalition is dissatisfied with the proposed allocation. For the excess function introduced in this chapter we need the following notation. Define

$$I_\Gamma(S) = \{(d,r) \in \mathbb{R}^S \times \Delta^S | \forall_{i\in S} : d_i + r_i X_S \succsim_i X_{\{i\}}\},$$

as the set of possibly nonfeasible individually rational allocations for coalition S. Note that $I_\Gamma(S)$ is a subset of $\mathbb{R}^S \times \Delta^S$ and not a subset of $L^1(\mathbb{R})^S$. So it contains the pairs (d,r) that lead to an allocation $(d_i + r_i\boldsymbol{X}_S)_{i\in S}$ instead of the allocations itself. In the remainder of this chapter we refer to both (d,r) and $(d_i + r_i\boldsymbol{X}_S)_{i\in S}$ as an allocation. An allocation $(d,r) \in I_\Gamma(S)$ is called feasible if $\sum_{i\in S} d_i \leq 0$. So, define

$$IR_\Gamma(S) = \{(d,r) \in I_\Gamma(S) | \sum_{i\in S} d_i \leq 0\},$$

as the set of feasible individually rational allocations for coalition S. We assume that $IR_\Gamma(S) \neq \emptyset$ for all $S \subset N$. Note that this assumption is satisfied if Γ is superadditive. Moreover, it should be noted that a coalition S is unlikely to be formed when $IR_\Gamma(S) = \emptyset$. Since in that case for every allocation of $\boldsymbol{X}_S$ there is at least one member of S whose payoff is not individually rational. Hence, he would be better off by leaving coalition S and forming a coalition on his own. Furthermore, define

$$PO_\Gamma(S) = \{(d,r) \in IR_\Gamma(S) | \nexists_{(d',r') \in IR_\Gamma(S)} \forall_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \succ_i d_i + r_i \boldsymbol{X}_S\},$$

as the set of feasible Pareto optimal allocations for S. Note that assumption (C2) in Section 3.2.1 implies that $\sum_{i \in S} d_i = 0$ whenever $(d,r) \in PO_\Gamma(S)$.

For gaining a clearer insight into the situation and the (forthcoming) mathematics in particular, we make use of a simplified graphical representation of the problem. At the moment this might seem a bit overdone, but for the remainder of this paper these figures might turn out to be helpful. The notions introduced in the preceding paragraph are illustrated in Figure 5.1.

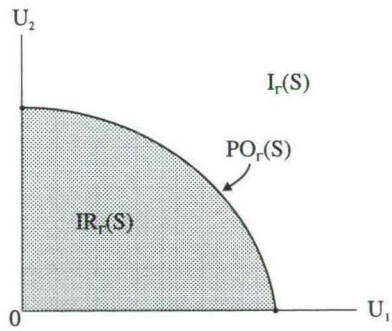


Figure 5.1

Figure 5.1 represents a stochastic cooperative game with two expected utility maximizing agents. The axes represent the utility levels of the agents. For simplicity, we have assumed that payoffs are individually rational if and only if the corresponding expected utility is greater than or equal to zero. So, the utility levels of both agents corresponding to the set $I_\Gamma(S)$ is represented by the positive orthant. Furthermore, the utility levels corresponding to the sets $IR_\Gamma(S)$ of individually rational allocations is depicted by the shaded area, and the utility levels corresponding to the set $PO_\Gamma(S)$ of Pareto optimal allocations is depicted by the bold printed curve. Finally, note that this and the forthcoming figures do not arise from a concrete example.

In Figure 5.1 the sets of utility vectors corresponding to $IR_\Gamma(S)$ and $PO_\Gamma(S)$, respectively are compact. The following propositions show that also the sets $IR_\Gamma(S)$ and $PO_\Gamma(S)$ are compact subsets for the class $CG(N)$ of stochastic cooperative games. The proofs of both propositions can be found in Appendix A.1.

Proposition 5.1 $IR_\Gamma(S)$ is a compact subset of $I_\Gamma(S)$ for each coalition $S \subset N$.

Proposition 5.2 The set of Pareto optimal allocations $PO_\Gamma(S)$ is a compact subset of $I_\Gamma(S)$ for each coalition $S \subset N$.

Furthermore, we need to consider the following sets. Define for each $S \subset N$

$$PD_\Gamma(S) = \{(d,r) \in I_\Gamma(S) | \exists_{(d',r') \in PO_\Gamma(S)} \forall_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \succsim_i d_i + r_i \boldsymbol{X}_S\}$$

as the set of (possibly nonfeasible) allocations that are (weakly) dominated by a Pareto optimal allocation, and

$$NPD_\Gamma(S) = \{(d,r) \in I_\Gamma(S) | \nexists_{(d',r') \in PO_\Gamma(S)} \forall_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \succsim_i d_i + r_i \boldsymbol{X}_S\}$$

as the set of (possibly nonfeasible) allocations that are not dominated by Pareto optimal allocations. Note that $IR_\Gamma(S) \subset PD_\Gamma(S)$. The reverse, however, need not be true, as the next example shows.

Example 5.1 Consider the following two person example. Let $\boldsymbol{X}_S$ be such that $-\boldsymbol{X}_S$ is exponentially distributed with expectation equal to 1 for all $S \subset N$. Furthermore let agents 1 and 2 be expected utility maximizers with utility functions $U_1(t) = -e^{-0.5t}$ and $U_2(t) = -e^{-0.25t}$, respectively. Then $E(U_1(d_1 + r_1 \boldsymbol{X}_{\{1,2\}})) = -e^{-d_1} \frac{1}{1-0.5r_1}$ and $E(U_2(d_2 - r_2 \boldsymbol{X}_{\{1,2\}})) = -e^{-d_2} \frac{1}{1-0.25r_2}$. An allocation $(d,r) \in I_\Gamma(\{1,2\})$ is individually rational if $E(U_1(d_1 + r_1 \boldsymbol{X}_{\{1,2\}})) \geq -2$ and $E(U_2(d_2 + r_2 \boldsymbol{X}_{\{1,2\}})) \geq -1.25$. Furthermore, (d,r^*) is Pareto optimal if and only if $r_1^* = \frac{1}{3}$ and $r_2^* = \frac{2}{3}$ (see WILSON (1968) or Proposition 6.1 for a similar result). Now, consider the allocation (d,r) with $d_1 = 0.1$, $d_2 = 0.1$, $r_1 = 1$ and $r_2 = 0$. Since $d_1 + d_2 > 0$ this allocation is nonfeasible. However, the Pareto optimal allocation (d^*, r^*) with $d_1^* = -0.9$, $d_2^* = 0.9$, $r_1^* = \frac{1}{3}$ and $r_2^* = \frac{2}{3}$ is feasible and preferred by both agents. Indeed,

$$E(U_1(d_1^* + r_1^* \boldsymbol{X}_{\{1,2\}})) = -1.8820 > -1.9025 = E(U_1(d_1 + r_1 \boldsymbol{X}_{\{1,2\}}))$$

and

$$E(U_2(d_2^* + r_2^* \boldsymbol{X}_{\{1,2\}})) = -0.9582 > -0.9753 = E(U_2(d_2 + r_2 \boldsymbol{X}_{\{1,2\}})).$$

So even nonfeasible allocations can be Pareto dominated.

The next proposition states a rather intuitive result. Namely that for every Pareto dominated allocation (d, r) and every non-Pareto dominated allocation (d', r'), which all members of S weakly prefer to the Pareto dominated allocation (d, r), there exists a Pareto optimal allocation such that for each agent the Pareto optimal allocation is weakly better than (d, r) but weakly worse than (d', r').

Proposition 5.3 Let $\Gamma \in CG(N)$. Take $(d, r) \in PD_\Gamma(S)$ and $(\tilde{d}, \tilde{r}) \in NPD_\Gamma(S)$ such that $d_i + r_i \boldsymbol{X}_S \precsim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$ for all $i \in S$. Then there exists $(\hat{d}, \hat{r}) \in PO_\Gamma(S)$ such that

$$d_i + r_i \boldsymbol{X}_S \precsim_i \hat{d}_i + \hat{r}_i \boldsymbol{X}_S \precsim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$$

for all $i \in S$.

PROOF: See Appendix A.1. ∎

A direct consequence of this proposition is that for each allocation $(d, r) \in IR_\Gamma(S)$ there exists a Pareto optimal allocation (d', r') such that $d'_i + r'_i \boldsymbol{X}_S \succsim_i d_i + r_i \boldsymbol{X}_S$ for all $i \in S$. Moreover, since $IR_\Gamma(S)$ is assumed to be nonempty we have that for each $(d, r) \in NPD_\Gamma(S)$ there exists $(d', r') \in PO_\Gamma(S)$ such that $d'_i + r'_i \boldsymbol{X}_S \precsim_i d_i + r_i \boldsymbol{X}_S$ for all $i \in S$.

Finally, we introduce three sets. Therefore, let $(d, r) \in IR_\Gamma(N)$ be an individually rational allocation for the grand coalition N. Take $S \subset N$ and define

$$W_S((d, r)) = \{(d', r') \in IR_\Gamma(S) | \forall_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \precsim_i d_i + r_i \boldsymbol{X}_N\}$$

as the set of individually rational allocations for coalition S which are weakly worse than the payoff $d_i + r_i \boldsymbol{X}_N$ for every member of S, and,

$$B_S((d, r)) = \{(d', r') \in IR_\Gamma(S) | \forall_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \succsim_i d_i + r_i \boldsymbol{X}_N\}$$

as the set of individually rational allocations for coalition S which are weakly better than the payoff $d_i + r_i \boldsymbol{X}_N$ for every member of S. Furthermore, define

$$PO^*_S((d, r)) = (W_S((d, r)) \cup B_S((d, r))) \cap PO_\Gamma(S),$$

as the set of Pareto optimal allocations for coalition S which are either weakly worse than $d_i + r_i \boldsymbol{X}_N$ for all members of S or weakly better than $d_i + r_i \boldsymbol{X}_N$ for all members of S. These three sets are illustrated in Figure 5.2. Note that $B_S((d, r))$ can be empty.

5.2 A Nucleolus for Stochastic Cooperative Games

With the definitions and notions introduced in the previous section we can now define an excess function and, consequently, a nucleolus for stochastic cooperative games. The excess function

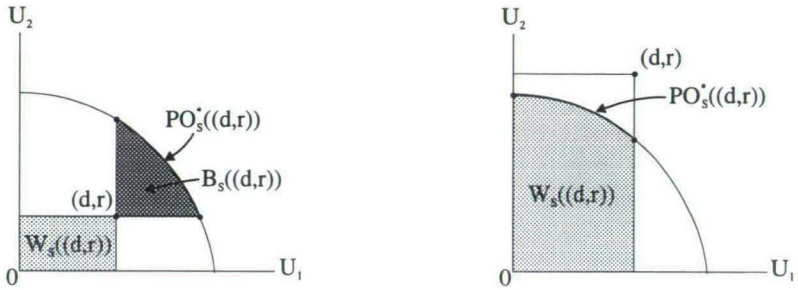


Figure 5.2

$E_S : IR_\Gamma(N) \to \mathbb{R}$ of coalition S is defined as follows. Take $(d, r) \in IR_\Gamma(N)$. Then the excess for coalition S is defined by

$$E_S((d,r)) = \min_{(d',r') \in PO^*_S((d,r))} \{\sum_{i \in S} \delta_i | \forall_{i \in S} : \delta_i \in \mathbb{R} \text{ and } d'_i + r'_i \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_N + \delta_i\}.$$

We will show later on that this minimum is well defined. For an interpretation of the excess, let us focus on the core conditions. So, given a proposed allocation (d, r) does a coalition S have an incentive to leave the grand coalition or not?

First, consider again the excess as used in SCHMEIDLER (1969). There, the excess can be interpreted as the minimum amount of money a coalition needs on top of what they already receive from the proposed allocation, such that they are indifferent between staying in the grand coalition and leaving the grand coalition. This interpretation is now applied to stochastic cooperative games. For this, note that given an allocation $(d, r) \in IR_\Gamma(N)$ a coalition S is indifferent between staying in the grand coalition N and leaving if there exists an allocation $(d', r') \in PO^*_S((d, r))$ such that each agent $i \in S$ is indifferent between receiving the payoff $d'_i + r'_i \boldsymbol{X}_S$ and the payoff $d_i + r_i \boldsymbol{X}_N$. So, coalition S cannot do strictly better by leaving the grand coalition but if they do split off they can allocate their payoff in such a way that no member is strictly worse off.

Now, suppose that a coalition S has an incentive to part company with the grand coalition N. So, there exists an allocation $(\tilde{d}, \tilde{r}) \in IR_\Gamma(S)$ such that each agent $i \in S$ strictly prefers the payoff $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$ to the payoff $d_i + r_i \boldsymbol{X}_N$. To keep this coalition in the grand coalition the payoff to the members of S must increase. This can be done by giving each member $i \in S$ a deterministic amount of money δ_i. Hence, their payoff becomes $d_i + \delta_i + r_i \boldsymbol{X}_N$. The excess of coalition S then equals the minimal amount of money they need so that they are just willing to stay in the grand coalition.

Next, suppose that a coalition S does not have an incentive to split off from the grand coalition. Hence, this coalition receives at least what they can achieve on their own. Consequently, one can decrease the payoff of each member $i \in S$ with a deterministic amount δ_i. Then the excess equals the maximal amount of money that can be taken away from this coalition such that they are still staying in the grand coalition.

Summarizing, a positive excess $E_S((d,r))$ represents the minimum amount of money that coalition S needs in order to be satisfied with the allocation (d,r). Moreover, if (d,r) and (d',r') are allocations of $\boldsymbol{X}_N$ such that each agent $i \in S$ prefers $d_i + r_i\boldsymbol{X}_N$ to $d'_i + r'_i\boldsymbol{X}_N$ then $E_S((d,r)) < E_S((d',r'))$. Hence, the excess decreases when each agent $i \in S$ improves his payoff. So, in a specific way the excess $E_S((d,r))$ describes how much coalition S is satisfied with the allocation (d,r). Finally, since all agents' preferences are monotonically increasing in the amount of money d they receive (see assumption (C2)) it is reasonable to say that one coalition is more satisfied with a particular allocation than another coalition if the first coalition needs less money to be satisfied than the latter one, or, in other words, if the excess of the first coalition is less than the excess of the latter. This last observation leads to the following definition of a nucleolus.

Let $\Gamma \in CG(N)$ be a cooperative game with stochastic payoffs and let

$$E_S((d,r)) = \min_{(d',r') \in PO^*_S((d,r))} \{\sum_{i \in S} \delta_i | \, \forall_{i \in S} : d'_i + r'_i\boldsymbol{X}_S \sim_i d_i + r_i\boldsymbol{X}_N + \delta_i\} \tag{5.1}$$

describe the excess of coalition S at allocation $(d,r) \in IR_\Gamma(N)$. Next, denote by $E((d,r))$ the vector of excesses at allocation (d,r) and let $\theta \circ E((d,r))$ denote the vector of excesses with its elements arranged in decreasing order. The *nucleolus* $\mathcal{N}(\Gamma)$ of the game $\Gamma \in CG(N)$ is then defined by

$$\mathcal{N}(\Gamma) = \{(d,r) \in IR_\Gamma(N) | \forall_{(d',r') \in IR_\Gamma(N)} : \theta \circ E((d,r)) \leq_{lex} \theta \circ E((d',r'))\}, \tag{5.2}$$

where $\leq_{lex}$ is the lexicographic ordering. Next, we show that the nucleolus is a well defined solution concept for the class $CG(N)$ of stochastic cooperative games.

Theorem 5.4 Let Γ be a stochastic cooperative game. If $\Gamma \in CG(N)$ and $IR_\Gamma(N) \neq \emptyset$ then $\mathcal{N}(\Gamma) \neq \emptyset$.

PROOF: In proving the nonemptiness of the nucleolus $\mathcal{N}(\Gamma)$ we make use of the results stated in MASCHLER, POTTERS and TIJS (1992). They introduced a nucleolus for a more general framework and showed that the nucleolus is nonempty if the domain is compact and the excess functions are continuous. Thus, we have to show that $IR_\Gamma(N)$ is compact and that $E_S((d,r))$ is continuous in (d,r) for each $(d,r) \in IR_\Gamma(N)$ and each $S \subset N$. The compactness of $IR_\Gamma(N)$ follows immediately from Proposition 5.1. The continuity proof is a bit more complicated and consists of the following parts.

First, it follows from Lemma A.1 in Appendix A.1 that $PO^*_S((d,r))$ is a nonempty compact subset of $PO_\Gamma(S)$. Next, let us introduce the following multifunction

$$\overline{E}_S((d,r)) = \{\sum_{i\in S} \delta_i | \exists_{(d',r')\in PO^*_S((d,r))} : d'_i + r'_i \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_N + \delta_i\}.$$

Hence, $E_S((d,r)) = \min \overline{E}_S((d,r))$. Lemma A.2 shows that $\overline{E}_S((d,r))$ is a compact subset of $\mathbb{R}$ for each allocation $(d,r) \in IR_\Gamma(N)$. This implies that the minimum in (5.1) exists. Subsequently, Lemma A.3 and Lemma A.4 show that this multifunction is upper and lower semi continuous, respectively. From Lemma A.5 it then follows that the excess function E_S is continuous. ∎

5.3 The Nucleolus, the Core and Certainty equivalents

For deterministic cooperative games it is known that the nucleolus as defined in SCHMEIDLER (1969) is a core allocation whenever the core is nonempty. A similar result can be derived for the nucleolus $\mathcal{N}(\Gamma)$ introduced in this chapter. For this, recall that an allocation $(d,r) \in IR_\Gamma(N)$ is a core allocation for the game Γ if for each coalition $S \subset N$ there exists no allocation $(\bar{d},\bar{r}) \in IR_\Gamma(S)$ such that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \succ_i d_i + r_i \boldsymbol{X}_N$ for all $i \in S$. The set of all core allocations is denoted by $\mathcal{C}(\Gamma)$.

Theorem 5.5 Let $\Gamma \in CG(N)$. If $\mathcal{C}(\Gamma) \neq \emptyset$ then $\mathcal{N}(\Gamma) \subset Core(\Gamma)$.

PROOF: Take $(d,r) \in IR_\Gamma(N)$ and $S \subset N$. Let $(\tilde{d}^S, \tilde{r}^S) \in I_\Gamma(S)$ be such that $\tilde{d}^S_i + \tilde{r}^S_i \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_N$ for all $i \in S$. Moreover, let $(\bar{d},\bar{r}) \in PO^*_S((d,r))$ and $\delta \in \mathbb{R}^S$ be such that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_N + \delta_i$ for all $i \in S$ and $\sum_{i\in S} \delta_i = E_S((d,r))$. Regarding the sign of the excess, we distinguish three cases.

First, suppose $(\tilde{d}^S, \tilde{r}^S) \in PD_\Gamma(S) \backslash PO_\Gamma(S)$. Then $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \succsim_i \tilde{d}^S_i + \tilde{r}^S_i \boldsymbol{X}_S$ for all $i \in S$. Hence, $\delta_i \geq 0$ for all $i \in S$. Since $(\tilde{d}^S, \tilde{r}^S)$ is not Pareto optimal there exists $j \in S$ such that $\bar{d}_j + \bar{r}_j \boldsymbol{X}_S \succ_j \tilde{d}^S_j + \tilde{r}^S_j \boldsymbol{X}_S \sim_j d_j + r_j \boldsymbol{X}_N$. Then $\delta_j > 0$ and, consequently, $E_S((d,r)) = \sum_{i\in S} \delta_i > 0$.

Second, suppose $(\tilde{d}^S, \tilde{r}^S) \in PO_\Gamma(S)$. This implies that $0 \in \overline{E}_S((d,r))$. Hence, $E_S((d,r)) \leq 0$.

Third, suppose $(\tilde{d}^S, \tilde{r}^S) \in NPD_\Gamma(S) \backslash PO_\Gamma(S)$. Then $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \precsim_i \tilde{d}^S_i + \tilde{r}^S_i \boldsymbol{X}_S$ for all $i \in S$. Hence, $\delta_i \leq 0$ for all $i \in S$. Moreover, since $(\tilde{d}^S, \tilde{r}^S)$ is not Pareto optimal there exists $j \in S$ such that $\bar{d}_j + \bar{r}_j \boldsymbol{X}_S \prec_j \tilde{d}^S_i + \tilde{r}^S_j \boldsymbol{X}_S \sim_j d_j + r_j \boldsymbol{X}_N$. So, $\delta_j < 0$ and, consequently, $E_S((d,r)) = \sum_{i\in S} \delta_i < 0$.

Now we show that the excess vector corresponding to a core allocation is lexicographically smaller then the excess vector corresponding to an allocation that does not belong to

the core. This implies that the latter allocation cannot belong to the nucleolus of the game whenever core allocations exist. Hence, the nucleolus must be a subset of the core.

Take $(d,r) \in \mathcal{C}(\Gamma)$ and $(d',r') \notin \mathcal{C}(\Gamma)$. Since $(d,r) \in \mathcal{C}(\Gamma)$ it follows from the core conditions that $(\tilde{d}^S, \tilde{r}^S) \in NPD_\Gamma(S)$ for all $S \subset N$. Hence, $E_S((d,r)) \leq 0$ for all $S \subset N$. Since $(d',r') \notin \mathcal{C}(\Gamma)$ there exists a coalition $S \subset N$ and an allocation $(\hat{d},\hat{r}) \in IR_\Gamma(S)$ for S such that $\hat{d}_i + \hat{r}_i \boldsymbol{X}_S \succ_i d'_i + r'_i \boldsymbol{X}_N$ for all $i \in S$. Hence, $(\tilde{d}'^S, \tilde{r}'^S) \in PD_\Gamma(S) \backslash PO_\Gamma(S)$ and, consequently, $E_S((d',r')) > 0$. This implies that $\theta \circ E_S((d,r)) <_{lex} \theta \circ E_S((d',r'))$. Thus $(d',r') \notin \mathcal{N}(\Gamma)$. ∎

Next, consider the class $MG(N)$ of stochastic cooperative games introduced in Section 3.2.2. Recall that each stochastic cooperative game $\Gamma \in MG(N)$ gives rise to a TU-game (N, v_Γ) with

$$v_\Gamma(S) \;=\; \max\{\sum_{i\in S} m_i(d_i + r_i \boldsymbol{X}) | (d_i + r_i \boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)\}$$

for all $S \subset N$. Furthermore, recall that an allocation $(d_i + r_i \boldsymbol{X})_{i\in S} \in \mathcal{Z}_\Gamma(S)$ is Pareto optimal if and only if

$$\sum_{i\in S} m_i(d_i + r_i \boldsymbol{X}) \;=\; v_\Gamma(S).$$

We will show that an allocation $(d,r) \in IR_\Gamma(N)$ belongs to the nucleolus $\mathcal{N}(\Gamma)$ if and only if the corresponding allocation $(m_i(d_i + r_i \boldsymbol{X}_N))_{i\in N}$ is the nucleolus of the corresponding TU-game (N, v_Γ).

Theorem 5.6 Let $\Gamma \in MG(N)$ such that $IR_\Gamma(S) \neq \emptyset$. Take $(d_i + r_i \boldsymbol{X}_N)_{i\in N} \in \mathcal{Z}_\Gamma(N)$ and let $y \in \mathbb{R}^N$ be such that $m_i(d_i + r_i \boldsymbol{X}_N) = y_i$ for all $i \in N$. Then $(d_i + r_i \boldsymbol{X}_N)_{i\in N} \in \mathcal{N}(\Gamma)$ if and only if $y = n(v_\Gamma)$.

PROOF: Let $\Gamma \in MG(N)$ and take $(d,r) \in IR_\Gamma(N)$. Let $y \in \mathbb{R}^N$ be such that $m_i(d_i + r_i \boldsymbol{X}_N) = y_i$ for all $i \in N$. The excess of coalition S then equals

$$\begin{aligned}
E_S((d,r)) &= \min_{(d',r')\in PO^*_S((d,r))} \{\sum_{i\in S} \delta_i | \forall_{i\in S} : d'_i + r'_i \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_N + \delta_i\} \\
&= \min_{(d',r')\in PO^*_S((d,r))} \{\sum_{i\in S} \delta_i | \forall_{i\in S} : m_i(d'_i + r'_i \boldsymbol{X}_S) = m_i(d_i + r_i \boldsymbol{X}_N + \delta_i)\} \\
&= \min_{(d',r')\in PO^*_S((d,r))} \{\sum_{i\in S} \delta_i | \forall_{i\in S} : m_i(d'_i + r'_i \boldsymbol{X}_S) = \delta_i + m_i(d_i + r_i \boldsymbol{X}_N)\} \\
&= \min_{(d',r')\in PO^*_S((d,r))} \sum_{i\in S} (m_i(d'_i + r'_i \boldsymbol{X}_S) - m_i(d_i + r_i \boldsymbol{X}_N)) \\
&= v_\Gamma(S) - \sum_{i\in S} y_i.
\end{aligned}$$

So, the excess of coalition S at allocation $(d_i + r_i \boldsymbol{X}_N)_{i\in N}$ equals the excess $E(S,y)$ introduced

by SCHMEIDLER (1969) of coalition S for the TU-game (N, v_Γ) (see page 31). Moreover, for each allocation $(d, r) \in PO_\Gamma(N)$ in Γ the vector $(m_i(d_i + r_i \boldsymbol{X}_N))_{i \in N}$ is an allocation of $v_\Gamma(N)$ in (N, v_Γ) and, vice versa, for each allocation x of $v_\Gamma(N)$ there exists an allocation $(d, r) \in PO_\Gamma(N)$ in Γ such that $m_i(d_i + r_i \boldsymbol{X}_N) = x_i$ for all $i \in N$. Hence, an allocation $(d_i + r_i \boldsymbol{X}_N)_{i \in N}$ belongs to the nucleolus $\mathcal{N}(\Gamma)$ of the game Γ if and only if the corresponding allocation $(m_i(d_i + r_i \boldsymbol{X}_N))_{i \in N} = y$ equals the nucleolus $n(v_\Gamma)$ of the corresponding TU-game (N, v_Γ). ∎

Example 5.2 Consider the following three person game Γ. Let $-\boldsymbol{X}_{\{i\}} \sim \text{Exp}(1)$ for $i = 1, 2, 3$ and let $\boldsymbol{X}_S = \sum_{i \in S} \boldsymbol{X}_{\{i\}}$ if $|S| \geq 2$. So, each agent individually faces a random cost which is exponentially distributed with expectation equal to 1. The cost of a coalition then equals the sum of the cost of the members of this coalition. Furthermore, all agents are expected utility maximizers with utility functions $U_1(t) = -e^{-0.5t}$, $U_2(t) = -e^{-0.33t}$ and $U_3(t) = -e^{-0.25t}$, respectively. For the certainty equivalent m_i it holds that $m_i(d_i + r_i \boldsymbol{X}_S) = U_i^{-1}(E(U_i(d_i + r_i \boldsymbol{X}_S)))$. For the corresponding TU-game (N, v_Γ) of Γ we then get $v_\Gamma(\{1\}) = -1.3863$, $v_\Gamma(\{2\}) = -1.2164$, $v_\Gamma(\{3\}) = -1.1507$, $v_\Gamma(\{1,2\}) = -2.2314$, $v_\Gamma(\{1,3\}) = -2.1878$, $v_\Gamma(\{2,3\}) = -2.1582$ and $v_\Gamma(\{1,2,3\}) = -3.1800$. The nucleolus $n(v_\Gamma)$ of this game is equal to $(-1.0933, -1.0633, -1.0234)$. To determine the nucleolus $\mathcal{N}(\Gamma)$ note that an allocation $(d_i + r_i \boldsymbol{X}_N)_{i \in N} \mathcal{Z}_\Gamma(N)$ is Pareto optimal if and only if $r = \frac{1}{9}(2, 3, 4)$. Then the only allocation $(d_i + r_i \boldsymbol{X}_N)_{i \in N}$ for which $(m_i(d_i + r_i \boldsymbol{X}_N))_{i \in N} = n(v_\Gamma)$ is the allocation $(d_i^* + r_i^* \boldsymbol{X}_N)_{i \in N}$ with $(d_1^*, d_2^*, d_3^*) = (-0.3865, -0.0034, 0.3899)$ and $(r_1^*, r_2^*, r_3^*) = \frac{1}{9}(2, 3, 4)$. Hence, $\mathcal{N}(\Gamma) = \{(d_i^* + r_i^* \boldsymbol{X}_N)_{i \in N}\}$.

6 Insurance Games

Classical actuarial theory has mainly focused on insurance problems from the insurer's point of view. Most of the attention is dedicated to the determination of an appropriate premium for the insured risk. Obviously, the nature of the risk is a substantial factor in this process. In this respect, there is an important difference whether the risk arises from the 'life' or the 'non life' sector. For the first, there is a profusion of statistical data on the expected remaining life available, which makes the calculation of an appropriate premium relatively easy. For the latter, however, things are a bit more complicated. In 'non life' insurance the risk is not always easy to capture in a statistical framework. Therefore, several premium calculation principles have been developed to serve this purpose, see for instance GOOVAERTS, DE VYLDER and HAEZENDONCK (1984).

These calculation principles, however, only take into account a part of the insurer's side of the deal. More precisely, they consider whether the premium is high enough to cover the risk. Competition arising from the presence of other insurers on the one hand, and the interests of the insured, on the other hand, are mostly ignored. It is, of course, better to consider all these aspects in an insurance deal, since the premium should not only be high enough to compensate the insurer for bearing the individual's risk, it should also be low enough so that an individual is willing to insure his risk (or a part of it) for this premium. The economic models for (re)insurance markets, which were developed from the 1960's on (cf. BORCH (1962A) and BÜHLMANN (1980), (1984)), consider indeed the interests of both the insurers and the insured. These models incorporate the possibility to study problems concerning fairness, Pareto optimality and market equilibrium. BÜHLMANN (1980), for example, shows that the Esscher calculation principle results in a Pareto optimal outcome.

More recently, also game theory is used to model the interests of all parties in an insurance problem. Especially when insurance companies incorporate subadditive premiums,

individuals can save on the premium if they decide to take a collective insurance instead of an individual one. This situation is discussed in ALEGRE and MERCÈ CLARAMUNT (1995). Other applications of cooperative game theory in insurance can be found in BORCH (1962B) and LEMAIRE (1991).

Cooperative game theory, however, still has to establish itself as an appropriate tool for exploring insurance problems. A reason for this is due to the inability of traditional cooperative game theory to incorporate the uncertainties, which play such an important role in insurance. Indeed, in classical theory the gains coalitions can obtain by cooperating are assumed to be known with certainty. In the preceding chapters, however, we have introduced and developed a model, which overcomes this problem.

This chapter shows how this new game theoretical model applies to problems in insurance. Our game theoretical approach allows for insurance of personal losses as well as reinsurance of the portfolios of insurance companies. By cooperating with insurance companies individual persons are able to transfer their future random losses to the cooperating insurance companies. Thus in doing so, individual persons conclude an insurance deal. Similarly, by cooperating with other insurers an insurance company can transfer (parts of) her insurance portfolio to the other insurers. So, the insurance company concludes a reinsurance deal.

In this model our attention is focused on Pareto optimal allocations of the risks, and on the question which premiums are fair to charge for these risk exchanges. A Pareto optimal allocation is such that there exists no other allocation which is better for all persons and insurers taking part in the game. We show that there is essentially a unique Pareto optimal allocation of risk. It will appear that this Pareto optimal allocation of the risk is independent of the insurance premiums that are paid for these risk exchanges. For determining fair premiums, we look at the core of the insurance game. A core allocation divides the gains of cooperation in such a way that no subcoalition has an incentive to split off. We show that the core is nonempty for insurance games. Moreover, we show that the zero utility principle for calculating premiums (see GOOVAERTS, DE VYLDER and HAEZENDONCK (1984)) results in a core allocation.

6.1 Insurance Games

For modeling insurance problems we use a slightly modified version of our model of a stochastic cooperative game as introduced in Chapter 3. We show that by cooperating, individuals and insurers can redistribute their risks and, consequently, improve their welfare. First, we need to specify the agents that participate in the game. An agent can be one of two types; an agent is either an individual person or an insurer. The set of individual persons is denoted by N_P and the set of insurers is denoted by N_I. Hence, the agents of the game are denoted by the set $N_I \cup N_P$.

Next, all agents are assumed to be risk averse expected utility maximizers. This means

that an agent prefers one risk to another if the expected utility of the first exceeds the expected utility of the latter. Note that insurers are also assumed to be risk averse. Furthermore, we assume that the utility function for each agent $i \in N_I \cup N_P$ can be described by $U_i(t) = \beta_i e^{-\alpha_i t}$, $(t \in \mathbb{R})$, with $\beta_i < 0$, $\alpha_i > 0$. Since $\beta_i < 0$ and $\alpha_i > 0$ imply concavity for the utility functions U_i, we have that each agent is risk averse. Recall that risk aversion implies that for each random loss $\boldsymbol{X}$ an agent prefers receiving the expected loss $E(\boldsymbol{X})$ with certainty to receiving the random loss $\boldsymbol{X}$. By changing the signs of the parameters β_i and α_i the utility function becomes convex, and, as a consequence, the agent will be risk loving. Regarding the situations where one or more risk neutral/loving insurers are involved we confine ourselves to a brief discussion later on. Finally, note that since the utility functions are exponential the expected utility of a random loss $\boldsymbol{X}$ need not always exist. In this chapter, however, we implicitly assume that the risks are such that the expected utility exists.

To describe the future random losses of an agent, we introduce the following notation. Let $\{\boldsymbol{Y}_k \sim \text{Exp}(\mu_k) | k \in K\}$ be a finite collection of independent exponentially distributed random variables. These variables can be interpreted as describing the random losses that could occur to individuals. They describe, for example, the monetary damages caused by cars, bikes, fires, or other people. The loss $\boldsymbol{X}_i$ for agent i then equals

$$\boldsymbol{X}_i = \sum_{k \in K} f_{ik} \boldsymbol{Y}_k, \tag{6.1}$$

where $0 \leq f_{ik} \leq 1$ for all $k \in K$. In particular we define $K_i = \{k \in K | f_{ik} \neq 0\}$ for all $i \in N_I \cup N_P$. So, if agent i is an insurer, then the loss $\boldsymbol{X}_i$ represents the loss of insurer i's portfolio. Moreover, the insurance portfolio $\boldsymbol{X}_i$ can be a combination of many random losses. In fact, they are the fractions f_{ik} of the losses that individuals have insured at this particular insurer. If agent i is an individual person then $\boldsymbol{X}_i$ represents the random loss this individual might want to insure. Note that the portfolios of different agents may be stochastically dependent, albeit in a very specific way; an individual can insure part of his loss at insurer i and another part of the same loss at insurer j.

Now, let us focus on the possibilities that occur when agents decide to cooperate. Therefore, consider a coalition S of agents. If the members of S decide to cooperate, the total loss $\boldsymbol{X}_S \in L^1(\mathbb{R})$ of coalition S equals the sum of the individual losses of the members of S, i.e., $\boldsymbol{X}_S = \sum_{i \in S} \boldsymbol{X}_i$. Subsequently, the loss $\boldsymbol{X}_S$ has to be allocated to the members of S.

In Chapter 3 we described an allocation of the random payoff $\boldsymbol{X}_S$ to the members of coalition S by means of a pair (d, r). Applying this definition to insurance games, however, raises a problem. Given an allocation (d, r) we have that agent $i \in S \cap N_P$ receives $d_i + r_i \boldsymbol{X}_S$. Thus $\boldsymbol{X}_S$ not only consists of the future random losses of agent i, but also of the future random losses of all other individuals $j \in S \cap N_P$. Hence, if agent i receives $d_i + r_i \boldsymbol{X}_S$ he receives (part of) the random losses of his fellow agents $j \in S \cap N_P$. Furthermore, this means that an agent $j \in S \cap N_P$ transfers (part of) his random losses to agent i, or, put in other words,

agent j insures (part of) his random losses at agent i. But this is rather unusual; agents only make insurance deals with insurance companies and not with other individuals. So, we need to modify our definition of an allocation so as to incorporate transfers of random losses from individuals to insurance companies only. The option we choose for is to replace the vector $r \in \Delta^S$ by a matrix $R \in \mathbb{R}^{S \times S}$. An element r_{ij} then represents the fraction of agent j's random loss that he transfers to agent i. Then by imposing the right conditions on R we can guarantee that individuals cannot transfer any risks among each other.

For explaining an allocation of the loss $\boldsymbol{X}_S$ in more detail, we distinguish between the following three cases. In the first case, coalition S consists of insurers only. So, $S \subset N_I$. Such a coalition is assumed to allocate the loss $\boldsymbol{X}_S$ in the following way. First, a coalition S allocates a fraction $r_{ij} \in [0,1]$ of the loss $\boldsymbol{X}_j$ of insurer $j \in S$ to insurer $i \in S$. So, insurer i bears a total loss of $\sum_{j \in S} r_{ij} \boldsymbol{X}_j$, where $r_{ij} \in [0,1]$ and $\sum_{k \in S} r_{kj} = 1$. This is called proportional (re)insurance. This part of the allocation of $\boldsymbol{X}_S$ for coalition S is described by a matrix $R \in \mathbb{R}_+^{S \times S}$, where r_{ij} represents the fraction insurer i bears of insurer j's loss $\boldsymbol{X}_j$. Second, the insurers are allowed to make deterministic transfer payments. This means that each insurance company $i \in S$ also receives an amount $d_i \in \mathbb{R}$ such that $\sum_{j \in S} d_j \leq 0$. These transfer payments can be interpreted as the aggregate premium insurers have to pay for the actual risk exchanges.

In the second case, coalition S consists of individual persons only. So, $S \subset N_P$. Then the gains of cooperation are assumed to be nil. That is, we do not allow any risk exchanges between the persons themselves. For, that is what the insurers are for in the first place. As a result, the only allocations (d, R) of $\boldsymbol{X}_S$ which are allowed are of the form $r_{ii} = 1$ for all $i \in S$ and $r_{ij} = 0$ for all $i, j \in S$ with $i \neq j$.

In the third and last case, coalition S consists of both insurers and individual persons. So, $S \subset N_I \cup N_P$. Now cooperation can take place in two different ways. First, insurers are allowed to exchange (parts of) their portfolios with other insurers, and, second, individual persons may transfer (parts of) their risks to insurers. Again, individual persons are not allowed to exchange risks with each other. Moreover, we assume that insurers cannot transfer (parts of) their portfolios to individuals.

Summarizing we can say that there are several restrictions on allocations. To be more precise, denote by S_I the set of insurers of coalition S, i.e., $S_I = S \cap N_I$, and by S_P the set of individuals of coalition S, i.e., $S_P = S \cap N_P$. Then an allocation $(d, R) \in \mathbb{R}^S \times \mathbb{R}_+^{S \times S}$ is feasible for the coalition S if for all $i \in S_P$ and all $j \in S$ with $i \neq j$ it holds that $r_{ij} = 0$ and $\sum_{i \in S} r_{ij} = 1$ for all $j \in S$. Furthermore, given an allocation (d, R) of the random loss $\boldsymbol{X}_S$, agent $i \in S$ receives $d_i + \sum_{j \in S} r_{ij} X_j = d_i + R_i \boldsymbol{X}^S$, where R_i denotes the i-th row of R and $\boldsymbol{X}^S = (\boldsymbol{X}_i)_{i \in S}$.

Example 6.1 Let $N_I = \{1, 2\}$, $N_P = \{4, 5\}$ and $K = \{a, b, c, d, e\}$. So, there are five independent exponentially distributed risks. Next, suppose that $\boldsymbol{X}_1 = \frac{1}{3}\boldsymbol{Y}_a + \boldsymbol{Y}_b$, $\boldsymbol{X}_2 =$

$\frac{1}{3}\boldsymbol{Y}_a + \boldsymbol{Y}_c$, $\boldsymbol{X}_4 = \boldsymbol{Y}_d$ and $\boldsymbol{X}_5 = \boldsymbol{Y}_e$. Consider the coalition $S = \{1,4,5\}$. Then $\boldsymbol{X}_S = \boldsymbol{X}_1 + \boldsymbol{X}_4 + \boldsymbol{X}_5 = \frac{1}{3}\boldsymbol{Y}_a + \boldsymbol{Y}_b + \boldsymbol{Y}_d + \boldsymbol{Y}_e$. A feasible allocation for S is the following. Let $d = (3,-2,-1)$ and $r_{11} = 1$, $r_{14} = \frac{1}{2}$, $r_{44} = \frac{1}{2}$, $r_{15} = \frac{1}{5}$ and $r_{55} = \frac{4}{5}$. Then insurer 1 receives

$$d_1 + R_1\boldsymbol{X}_S = 3 - (\boldsymbol{X}_1 + \tfrac{1}{2}\boldsymbol{X}_4 + \tfrac{1}{5}\boldsymbol{X}_5) = 3 - (\tfrac{1}{3}\boldsymbol{Y}_a + \boldsymbol{Y}_b + \tfrac{1}{2}\boldsymbol{Y}_d + \tfrac{4}{5}\boldsymbol{Y}_e),$$

individual 4 receives

$$d_4 + R_4\boldsymbol{X}_S = -2 - \tfrac{1}{2}\boldsymbol{X}_4 = -2 - \tfrac{1}{2}\boldsymbol{Y}_d,$$

and individual 5 receives

$$d_5 + R_5\boldsymbol{X}_S = -1 - \tfrac{4}{5}\boldsymbol{X}_4 = -1 - \tfrac{4}{5}\boldsymbol{Y}_e.$$

So, individuals 4 and 5 pay a premium of 2 and 1, respectively, to insurer 1 for the insurance of their losses.

In conclusion, an insurance game Γ with agent set $N_I \cup N_P$ is described by the tuple $(N_I \cup N_P, \mathcal{V}, (U_i)_{i\in N_I\cup N_P})$, where N_I is the set of insurers, N_P the set of individuals, $\mathcal{V}(S) = \{\sum_{i\in S}\boldsymbol{X}_i\}$ the random loss for coalition S, and U_i the utility function for agent $i \in N_I \cup N_P$. The class of all such insurance games with insurers N_I and individuals N_P is denoted by $IG(N_I, N_P)$.

6.1.1 Pareto Optimal Distributions of Risk

Since the preferences of both an individual and an insurer are described by means of a utility function we can look at the certainty equivalent of random payoffs for each of them. The certainty equivalent of a random payoff is the amount of money for which an agent is indifferent between receiving the random payoff and receiving this amount of money with certainty. For the utility functions considered in our model, we can define the certainty equivalent of a random payoff $\boldsymbol{X}$ by $m_i(\boldsymbol{X}) = U_i^{-1}(E(U_i(\boldsymbol{X})))$ provided that the expected utility exists. Then for all these random payoffs $\boldsymbol{X}$ it holds that $E(U_i(m_i(\boldsymbol{X})) = U_i(m_i(\boldsymbol{X})) = E(U_i(\boldsymbol{X}))$. Since the expected utilities equal each other, agent i is indifferent between receiving the random payoff $\boldsymbol{X}$ and the deterministic payoff $m_i(\boldsymbol{X})$. Moreover, for the insurance games introduced in the previous section the exponential utility functions are such that the results stated in Section 3.2.2 on certainty equivalents apply. One of these results concerns the Pareto optimality of an allocation. For insurance games this result reads as follows.

Proposition 6.1 Let $\Gamma \in IG(N_I, N_P)$ and $S \subset N_I \cup N_P$. An allocation $(d_i + R_i\boldsymbol{X}^S)_{i\in S} \in \mathcal{Z}_\Gamma(S)$ is Pareto optimal for coalition S if and only if

$$\sum_{i\in S} m_i(d_i + R_i\boldsymbol{X}^S) = \max\left\{\sum_{i\in S} m_i(\tilde{d}_i + \tilde{R}_i\boldsymbol{X}^S) \mid (\tilde{d}_i + \tilde{R}_i\boldsymbol{X}^S)_{i\in S} \in \mathcal{Z}_\Gamma(S)\right\}. \quad (6.2)$$

So, an allocation is Pareto optimal for coalition S if and only if this allocation maximizes the sum of the certainty equivalents. To determine these allocations, we first need to calculate the certainty equivalent of an allocation $(d_i + R_i \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$ for agent $i \in S$. Therefore, let $S \subset N_I \cup N_P$ and $(d_i + R_i \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$. The random loss coalition S has to allocate equals $\boldsymbol{X}_S = \sum_{i \in S} \boldsymbol{X}_i$. Given a feasible allocation $(d_i + R_i \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$, the random payoff to agent $i \in S$ equals

$$d_i + R_i \boldsymbol{X}^S = d_i - \sum_{j \in S} r_{ij} \boldsymbol{X}_j$$

if $i \in S_I$ and

$$d_i + R_i \boldsymbol{X}^S = d_i - r_{ii} \boldsymbol{X}_i$$

if $i \in S_P$. Consequently, we have that the certainty equivalent of $(d_i + R_i \boldsymbol{X}^S)_{i \in S}$ equals[1]

$$m_i(d_i + R_i \boldsymbol{X}^S) = \begin{cases} d_i + \sum\limits_{k \in K} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k} \alpha_i r_{ii} f_{ik}\right), & \text{if } i \in S_P, \\ d_i + \sum\limits_{j \in S} \sum\limits_{k \in K} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k} \alpha_i r_{ij} f_{jk}\right), & \text{if } i \in S_I. \end{cases} \tag{6.3}$$

The sum of the certainty equivalents then equals

$$\begin{aligned} \sum_{i \in S} m_i(d_i + R_i \boldsymbol{X}^S) &= \sum_{i \in S} d_i + \sum_{i \in S_P} \sum_{k \in K} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k} \alpha_i r_{ii} f_{ik}\right) \\ &\quad + \sum_{i \in S_I} \sum_{j \in S} \sum_{k \in K} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k} \alpha_i r_{ij} f_{jk}\right). \end{aligned} \tag{6.4}$$

Since $\sum_{i \in S} d_i = 0$ for Pareto optimal allocations, we have for these allocations that the sum of the certainty equivalents is independent of the vector of transfer payments d. Intuitively, this is quite clear. For since $\sum_{h \in S} d_h = 0$, an increase in d_i for agent i implies that d_j decreases for at least one other agent j. Consequently, Pareto optimality is solely determined by the choice of the allocation risk exchange matrix R of the random losses. In fact, the next theorem shows that there is a unique allocation risk exchange matrix R^* inducing Pareto optimality.

Theorem 6.2 Let $\Gamma \in IG(N_I, N_P)$ and $S \subset N_I \cup N_P$. An allocation $(d_i + R_i^* \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$ is Pareto optimal for S if and only if

$$r_{ij}^* = \begin{cases} \dfrac{\frac{1}{\alpha_i}}{\sum_{h \in S_I} \frac{1}{\alpha_h}} & \text{, if } i, j \in S_I, \\ \dfrac{\frac{1}{\alpha_i}}{\sum_{h \in S_I \cup \{j\}} \frac{1}{\alpha_h}} & \text{, if } i \in S_I \cup \{j\} \text{ and } j \in S_P, \\ 0 & \text{, otherwise.} \end{cases}$$

[1] The proof is stated in Appendix A.2.

PROOF: We have to show that R^* is the unique solution of

$$\begin{aligned}
\max \sum_{i\in S_P}\sum_{k\in K} \tfrac{1}{\alpha_i}\log\left(1-\tfrac{1}{\mu_k}\alpha_i r_{ii} f_{ik}\right) + \sum_{i\in S_I}\sum_{j\in S}\sum_{k\in K} \tfrac{1}{\alpha_i}\log\left(1-\tfrac{1}{\mu_k}\alpha_i r_{ij} f_{jk}\right)\\
\text{s.t.:}\quad r_{jj}+\textstyle\sum_{i\in S_I} r_{ij} &= 1, \quad \text{for all } j\in S_P,\\
\textstyle\sum_{i\in S_I} r_{ij} &= 1, \quad \text{for all } j\in S_I,\\
r_{ii} &\geq 0, \quad \text{if } i\in S_P,\\
r_{ij} &\geq 0, \quad \text{if } i\in S_I \text{ and } j\in S.
\end{aligned}$$

Since the objective function is strictly concave in r_{ij} for all relevant combinations of $i,j\in S$, it is sufficient to prove that R^* solves this maximization problem. The Karush-Kuhn-Tucker conditions[2] tell us that this is indeed the case if there exists $\lambda_j\in\mathbb{R}$ $(j\in S)$, $\nu_{jj}\geq 0$ $(j\in S_P)$ and $\nu_{ij}\geq 0$ $(i\in S_I,\ j\in S)$ such that

$$\begin{aligned}
\textstyle\sum_{k\in K_j}\frac{-1}{\frac{\mu_k}{f_{jk}}-\alpha_j r_{jj}} &= \lambda_j-\nu_{jj}, && \text{for all } j\in S_P,\\
\textstyle\sum_{k\in K_j}\frac{-1}{\frac{\mu_k}{f_{jk}}-\alpha_i r_{ij}} &= \lambda_j-\nu_{ij}, && \text{for all } i\in S_I \text{ and all } j\in S,\\
\nu_{ii}r_{ii} &= 0, && \text{for all } i\in S_P,\\
\nu_{ij}r_{ij} &= 0, && \text{for all } i\in S_I \text{ and all } j\in S.
\end{aligned}$$

Substituting r^*_{ij} gives $\nu_{ij}=0$ for all relevant combinations of $i,j\in S$ and

$$\begin{aligned}
\lambda_j &= -\textstyle\sum_{k\in K_j} f_{jk}\left(\mu_k-\frac{f_{jk}}{\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{-1}, && \text{for all } j\in S_P,\\
\lambda_j &= -\textstyle\sum_{k\in K_j} f_{jk}\left(\mu_k-\frac{f_{jk}}{\sum_{h\in S_I}\frac{1}{\alpha_h}}\right)^{-1}, && \text{for all } j\in S_I.
\end{aligned}$$

Consequently, R^* solves the maximization problem. ∎

So, for a Pareto optimal allocation of a loss $\boldsymbol{X}_j$ within S one has to distinguish between two cases. In the first case the index j refers to an insurer and in the second case j refers to an individual. When $\boldsymbol{X}_j$ is the loss of insurer $j\in S_I$, the loss is allocated proportionally to $\frac{1}{\alpha_i}$ among all insurers in coalition S. When $\boldsymbol{X}_j$ is the loss of individual $j\in S_P$, the loss is

[2] The Karush-Kuhn-Tucker conditions read as follows:

$$\begin{aligned}
\text{If } f(x) &= \max_y f(y)\\
&\text{s.t.}\quad g_k(y)\leq 0, \quad k\in K\\
&\qquad\ \ g_l(y)=0, \quad l\in L
\end{aligned}$$

then there exist $\nu_k\geq 0$ $(\forall k\in K)$ and $\lambda_l\in\mathbb{R}$ $(\forall l\in L)$ such that

$$\begin{aligned}
\nabla f(x) &= \textstyle\sum_{k\in K}\nu_k\cdot\nabla g_k(x) + \sum_{l\in L}\lambda_l\cdot\nabla g_l(x)\\
\nu_k\cdot g_k(x) &= 0, \text{ for all } k\in K.
\end{aligned}$$

Moreover, if f is strictly concave and g_k $(k\in K)$, g_l $(l\in L)$ are convex then the reverse of the statement also holds and the maximum is unique.

allocated proportionally to $\frac{1}{\alpha_i}$ among all insurers in coalition S and individual j himself. Note that by the feasibility constraints nothing is allocated to the other individuals. Furthermore, Pareto optimality does not depend on the parameters μ_k of the losses $\boldsymbol{Y}_k$ $(k \in K)$. Finally, remark that if only reinsurance of the insurance portfolios is considered, that is, $N_P = \emptyset$ then the Pareto optimal allocation coincides with the Pareto optimal allocation of (re)insurance markets discussed in BÜHLMANN (1980).

Example 6.2 In this example all monetary amounts can be assumed to be in thousands of dollars. Consider the following situation in automobile insurance with three insurance companies and two individual persons. So, $N_I = \{1, 2, 3\}$ and $N_P = \{4, 5\}$. The utility function of each agent can be described by $U_i(t) = e^{-\alpha_i t}$ with $\alpha_1 = 0.33$, $\alpha_2 = 0.1$, $\alpha_3 = 0.25$, $\alpha_4 = 0.4$ and $\alpha_5 = 0.25$, respectively. Each insurance company bears the risk of all the cars contained in its insurance portfolio. A car can be one of two types. The first type corresponds to an average saloon car which generates relatively low losses. The second type corresponds to an exclusive sportscar generating relatively high losses. Formally, the monetary loss generated by a car is described by the exponential probability distribution Exp(5) when it is of type 1 and by Exp(0.5) when it is of type 2. Thus the expected loss of a type 1 car and a type 2 car equal \$ 0.2 and \$ 2, respectively.

The insurance portfolio of insurer 1 consists of 1800 cars of type 1 and 10 cars of type 2. For insurer 2 the portfolio consists of 900 cars of type 1 and 25 cars of type 2. Finally, the portfolio of insurer 3 consists of 300 cars of type 1 and 90 cars of type 2. The expected loss for insurer 1 then equals $1800 \cdot 0.2 + 10 \cdot 2 =$ \$ 380. The expected losses for insurer 2 and 3 then equal \$ 230 and \$ 240, respectively. The two individual persons each possess one car. Player 4's car is of type 1 and player 5's car is of type 2. So, the expected losses are \$ 0.2 and \$ 2, respectively.

Next, let $\boldsymbol{X}_i$ denote the loss of agent i. If all agents cooperate, the Pareto optimal risk allocation matrix of the total random loss $\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4 + \boldsymbol{X}_5$ equals

$$R^* = \begin{bmatrix} \frac{3}{17} & \frac{3}{17} & \frac{3}{17} & \frac{6}{39} & \frac{3}{21} \\ \frac{10}{17} & \frac{10}{17} & \frac{10}{17} & \frac{20}{39} & \frac{10}{21} \\ \frac{4}{17} & \frac{4}{17} & \frac{4}{17} & \frac{8}{39} & \frac{4}{21} \\ 0 & 0 & 0 & \frac{5}{39} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{21} \end{bmatrix}.$$

Consequently, a Pareto optimal allocation $(d_i + R_i^* \boldsymbol{X}^N)_{i \in N} \in \mathcal{Z}_\Gamma(S)$ yields the payoffs

$$\begin{aligned} d_1 + R_1^* \boldsymbol{X}^N &= d_1 - \tfrac{3}{17}(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3) - \tfrac{6}{39}\boldsymbol{X}_4 - \tfrac{3}{21}\boldsymbol{X}_5, \\ d_2 + R_2^* \boldsymbol{X}^N &= d_2 - \tfrac{10}{17}(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3) - \tfrac{20}{39}\boldsymbol{X}_4 - \tfrac{10}{21}\boldsymbol{X}_5, \\ d_3 + R_3^* \boldsymbol{X}^N &= d_3 - \tfrac{4}{17}(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3) - \tfrac{8}{39}\boldsymbol{X}_4 - \tfrac{4}{21}\boldsymbol{X}_5, \end{aligned}$$

$$\begin{aligned} d_4 + R_4^* \boldsymbol{X}^N &= d_4 - \tfrac{5}{39}\boldsymbol{X}_4, \\ d_5 + R_5^* \boldsymbol{X}^N &= d_5 - \tfrac{4}{21}\boldsymbol{X}_5. \end{aligned}$$

The determination of the allocation risk exchange matrix is, of course, only one part of the allocation. We still have to determine the vector of transfer payments d, that is, the premiums that have to be paid. Although an allocation $(d_i + R_i^* \boldsymbol{X}^S)_{i \in S}$ may be Pareto optimal for any choice of d, not every d is satisfactory from a social point of view. An insurer will not agree with insuring the losses of other agents if he is not properly compensated, that is, if he does not receive a fair premium for the insurance. Similarly, insurance companies and individuals only agree to insure their losses if the premium they have to pay is reasonable. Consequently, there is a conflict of interests; both insurance companies and individuals want to pay a low premium for insuring their own losses, while insurance companies want to receive a high premium for bearing the losses of other agents. So the question remains which premiums are reasonable? This is the subject of the next subsection.

6.1.2 The Core of Insurance Games

In our quest for fair premiums we look at core allocations of insurance games. The core is one of the most important solution concepts in game theory. It is generally accepted by game theorists that if the core is a nonempty set of allocations, then the allocation on which the agents agree should be a core allocation. The core contains allocations that induce a form of stability for the coalition of all agents involved. In the context of insurance games, an allocation is a core allocation if there is no subcoalition that has an incentive to part company with the grand coalition $N = N_I \cup N_P$ because this subcoalition can achieve a better allocation on their own. Formally, this means that an allocation $(d_i + R_i \boldsymbol{X}^N)_{i \in N} \in \mathcal{Z}_\Gamma(N)$ is a core allocation if for each coalition $S \subset N$ there exists no allocation $(\tilde{d}_i + \tilde{R}_i \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$ for coalition S such that each agent $i \in S$ prefers the payoff $\tilde{d}_i + \tilde{R}_i \boldsymbol{X}^S$ to the payoff $d_i + R_i \boldsymbol{X}^N$, i.e., $E(U_i(\tilde{d}_i + \tilde{R}_i \boldsymbol{X}^S)) > E(U_i(d_i + R_i \boldsymbol{X}^N))$ for all $i \in S$. The set of all core allocations for a game $\Gamma \in IG(N_I, N_P)$ is denoted by $\mathcal{C}(\Gamma)$. Note that a core allocation is Pareto optimal for $N_I \cup N_P$. Hence, $(d_i + R_i \boldsymbol{X}^N)_{i \in N} \in \mathcal{C}(\Gamma)$ implies that the allocation risk exchange matrix R has the structure of the Pareto optimal allocation risk exchange matrix R^* as described in Theorem 6.2 with $S = N_I \cup N_P$.

We will show that insurance games are totally balanced, that is, the core of an insurance game is nonempty and the core of every subgame is nonempty. This means that there always exists an allocation of $N_I \cup N_P$ which is stable in the sense as described above. To prove this result, we make use of the results stated in Section 3.2.2 and Section 4.1.

First, we associate with each insurance game $\Gamma \in IG(N_I, N_P)$ a TU-game (N, v_Γ) with $N = N_I \cup N_P$. Let $S \subset N_I \cup N_P$. The value $v_\Gamma(S)$ of coalition S in the game (N, v_Γ) is defined by

$$v_\Gamma(S) = \max\left\{\sum_{i\in S} m_i(d_i + R_i\boldsymbol{X}^N) \mid (d_i + R_i\boldsymbol{X}^N)_{i\in N} \in \mathcal{Z}_\Gamma(N)\right\}. \tag{6.5}$$

Recall that the payoff $v_\Gamma(S)$ is based on Proposition 6.1, which states that an allocation is Pareto optimal for S if and only if the sum of the corresponding certainty equivalents equals $v_\Gamma(S)$. The following result is a consequence of Theorem 4.2.

Proposition 6.3 Let $\Gamma \in IG(N_I, N_P)$ be an insurance game and let (N, v_Γ) be the corresponding TU-game. Then $\mathcal{C}(\Gamma) \neq \emptyset$ if and only if $C(v_\Gamma) \neq \emptyset$. Moreover, let $(d_i + R_i\boldsymbol{X}^N)_{i\in N} \in \mathcal{Z}_\Gamma(N)$ and let $y \in \mathbb{R}^N$ be such that $m_i(d_i + R_i\boldsymbol{X}^N) = y_i$ for all $i \in N$. Then

$$(d_i + R_i\boldsymbol{X}^N)_{i\in N} \in \mathcal{C}(\Gamma) \text{ if and only if } y \in C(v_\Gamma).$$

So, to prove nonemptiness of the core of insurance games it is sufficient to prove that the core of the corresponding TU-game is nonempty. For this we can apply the Bondareva Shapley Theorem (see 2.3) to check nonemptiness of the core.

Theorem 6.4 Let $\Gamma \in IG(N_I, N_P)$. Then $\mathcal{C}(\Gamma) \neq \emptyset$.

PROOF: First, recall that $K_j = \{k \in K \mid f_{jk} \neq 0\}$ for all $j \in N = N_I \cup N_P$. Then for $S \subset N$ we have for all $d \in \mathbb{R}^S$ that

$$\begin{aligned}
v_\Gamma(S) &= \sum_{i\in S} m_i(d_i + R_i^*\boldsymbol{X}^S) \\
&= \sum_{i\in S_I}\sum_{j\in S_I}\sum_{k\in K_j} \tfrac{1}{\alpha_i}\log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}\right) + \\
&\quad \sum_{i\in S_I}\sum_{j\in S_P}\sum_{k\in K_j} \tfrac{1}{\alpha_i}\log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right) + \qquad (6.6)\\
&\quad \sum_{i\in S_P}\sum_{k\in K_i} \tfrac{1}{\alpha_i}\log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}}\sum_{h\in S_I\cup\{i\}}\frac{1}{\alpha_h}}\right) \\
&= \sum_{i\in S_I}\sum_{j\in S_I}\sum_{k\in K_j} \tfrac{1}{\alpha_i}\log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}\right) \\
&\quad + \sum_{j\in S_P}\sum_{i\in S_I\cup\{j\}}\sum_{k\in K_j} \tfrac{1}{\alpha_i}\log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right) \\
&= \sum_{j\in S_I}\sum_{k\in K_j} \tfrac{f_{jk}}{\mu_k}\log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}} \\
&\quad + \sum_{j\in S_P}\sum_{k\in K_j} \tfrac{f_{jk}}{\mu_k}\log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}, \qquad (6.7)
\end{aligned}$$

where the second equality follows from Theorem 6.2 and expression (6.4). Next, let $\lambda : 2^{N_I \cup N_P} \to \mathbb{R}_+$ be a balanced map. Then

$$\begin{aligned}
\sum_{S\subset N} \lambda(S)x_S &= \sum_{S\subset N}\sum_{j\in S_I}\sum_{k\in K_j} \lambda(S)\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}} \\
&\quad + \sum_{S\subset N}\sum_{j\in S_P}\sum_{k\in K_j} \lambda(S)\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}} \\
&\leq \sum_{S\subset N}\sum_{j\in S_I}\sum_{k\in K_j} \lambda(S)\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}} \\
&\quad + \sum_{S\subset N}\sum_{j\in S_P}\sum_{k\in K_j} \lambda(S)\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}} \\
&= \sum_{j\in N_I}\sum_{S\subset N:j\in S}\lambda(S)\sum_{k\in K_j}\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}} \\
&\quad + \sum_{j\in N_P}\sum_{S\subset N:j\in S}\lambda(S)\sum_{k\in K_j}\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}} \\
&= \sum_{j\in N_I}\sum_{k\in K_j}\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}} \\
&\quad + \sum_{j\in N_P}\sum_{k\in K_j}\tfrac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}} \\
&= v_\Gamma(N)
\end{aligned}$$

where the inequality follows from Lemma A.6 with $c = 0$ and the third equality follows from $\sum_{S\subset N:j\in S}\lambda(S) = 1$ for all $j \in N$. Applying Proposition 6.3 and Theorem 2.3 then completes the proof.

Example 6.3 Consider the situation described in Example 6.2. In order to calculate the certainty equivalent of this insurance game, note that since $f_{jk} = 1$ for all $k \in K_j$ and all $j \in N_I$ we have

$$\begin{aligned}
v_\Gamma(S) &= \sum_{j\in S_I}\sum_{k\in K_j}\left(\sum_{i\in S_I}\tfrac{1}{\alpha_i}\right)\log\left(1-\frac{1}{\mu_k\sum_{i\in S_I}\frac{1}{\alpha_i}}\right) + \\
&\quad \sum_{j\in S_P}\sum_{k\in K_j}\left(\sum_{i\in S_I\cup\{j\}}\tfrac{1}{\alpha_i}\right)\log\left(1-\frac{1}{\mu_k\sum_{i\in S_I\cup\{j\}}\frac{1}{\alpha_i}}\right)
\end{aligned}$$

for all $S \subset N_I \cup N_P$ (cf. expression (6.7)). Hence, we get

$$v_\Gamma(\{1\}) = 1800 \cdot 3 \log\left(1 - \tfrac{1}{5 \cdot 3}\right) + 10 \cdot 3 \log\left(1 - \tfrac{1}{0.5 \cdot 3}\right) = -405.52.$$

Similarly, one can calculate the value $v_\Gamma(S)$ for each coalition S. These values are presented in Table 6.1.

S	$v_\Gamma(S)$	S	$v_\Gamma(S)$	S	$v_\Gamma(S)$
$\{1\}$	-405.52	$\{2,5\}$	-239.77	$\{2,3,4\}$	-490.11
$\{2\}$	-237.61	$\{3,4\}$	-311.28	$\{2,3,5\}$	-492.03
$\{3\}$	-311.08	$\{3,5\}$	-313.38	$\{2,4,5\}$	-239.97
$\{4\}$	-000.21	$\{4,5\}$	-002.98	$\{3,4,5\}$	-313.58
$\{5\}$	-002.77	$\{1,2,3\}$	-869.53	$\{1,2,3,4\}$	-869.73
$\{1,2\}$	-620.21	$\{1,2,4\}$	-620.41	$\{1,2,3,5\}$	-871.63
$\{1,3\}$	-661.65	$\{1,2,5\}$	-622.34	$\{1,2,4,5\}$	-622.14
$\{1,4\}$	-405.72	$\{1,3,4\}$	-661.85	$\{1,3,4,5\}$	-664.06
$\{1,5\}$	-407.88	$\{1,3,5\}$	-663.86	$\{2,3,4,5\}$	-492.23
$\{2,3\}$	-489.91	$\{1,4,5\}$	-408.08	$\{1,2,3,4,5\}$	-871.83
$\{2,4\}$	-237.81				

Table 6.1

The core of this game is then defined by

$$C(v_\Gamma) = \{y \in \mathbb{R}^5 | \sum_{j=1}^{5} y_j = -871.83,\ \forall_{S \subset N_I \cup N_P} : \sum_{i \in S} y_i \geq x_S\}.$$

Next, note that for a Pareto optimal allocation $(d_i + R_i^* \boldsymbol{X}^N)_{i \in N} \in \mathcal{Z}_\Gamma(N)$ we have that

$$\begin{aligned}
m_1(d_1 + R_1^* \boldsymbol{X}^N) &= d_1 - 153.77, \\
m_2(d_2 + R_2^* \boldsymbol{X}^N) &= d_2 - 512.59, \\
m_3(d_3 + R_3^* \boldsymbol{X}^N) &= d_3 - 205.04, \\
m_4(d_4 + R_4^* \boldsymbol{X}^N) &= d_4 - 0.03, \\
m_5(d_5 + R_5^* \boldsymbol{X}^N) &= d_5 - 0.40.
\end{aligned}$$

Next, take $d^0 = (-229.65, 278.33, -46.81, -0.17, -1.70)$. Then the resulting payoffs equal $m_i(d_i^0 + R_i^* \boldsymbol{X}^N)_{i \in N} = (-383.42, -234.26, -251.85, -0.20, -2.10)$. It is easy to check that this allocation is in the core of the TU-game (N, v_Γ). Hence, $(d_i^0 + R_i^* \boldsymbol{X}^N)_{i \in N} \in \mathcal{C}(\Gamma)$.

So, since the core is nonempty, we know that if all agents cooperate then there exist allocations such that this cooperation is stable. Moreover, from the Pareto optimality of a core allocation it follows that the allocation risk exchange matrix is uniquely determined. A similar

argument, however, does not hold for the allocation transfer payments (i.e., the premiums that have to be paid). Since the number of core allocations will mostly be infinite, the number of premiums resulting in a core allocation will also be infinite. Consequently, the insurers still have to agree on the premiums that have to be paid. A possibility is considering existing premium calculation principles and check if they result in core allocations for insurance games. This approach is elaborated in the next subsection.

6.1.3 The Zero Utility Principle

Premium calculation principles indicate how to determine the premium for a certain risk. In the past, various of these principles were designed, for example, the net premium principle, the expected value principle, the standard deviation principle, the Esscher principle, and the zero utility principle (cf. GOOVAERTS, DE VYLDER and HAEZENDONCK (1984)). In this section we focus on the zero utility principle. A premium calculation principle determines a premium $\pi_i(\boldsymbol{X})$ for individual i for bearing the risk $\boldsymbol{X}$. The zero utility principle assigns a premium $\pi_i(\boldsymbol{X})$ to $\boldsymbol{X}$ such that the utility level of individual i, who bears the risk $\boldsymbol{X}$, remains unchanged when the wealth w_i of this individual changes to $w_i + \pi_i(\boldsymbol{X}) - \boldsymbol{X}$. Since individuals are expected utility maximizers this means that the premium $\pi_i(\boldsymbol{X})$ satisfies $U_i(w_i) = E(U_i(w_i + \pi_i(\boldsymbol{X}) - \boldsymbol{X}))$. Note that the premium of the risk $\boldsymbol{X}$ depends on the individual who bears this risk and his wealth w_i.

Now, let us return to insurance games and utilize the zero utility principle to determine the allocation transfer payments $d \in \mathbb{R}^{N_I \cup N_P}$. At first this might seem difficult since the zero utility principle requires initial wealths w_i which do not appear in our model of insurance games. The exponential utility functions, however, yield that the zero utility principle is independent of these initial wealths w_i. To see this, let $\Gamma \in IG(N_I, N_P)$ be an insurance game. Since utility functions are exponential we can rewrite the expression $U_i(w_i) = E(U_i(w_i + \pi_i(\boldsymbol{X}) - \boldsymbol{X}))$ as follows

$$w_i = U_i^{-1}(E(U_i(w_i + \pi_i(\boldsymbol{X}) - \boldsymbol{X}))) = w_i + \pi_i(\boldsymbol{X}) + U_i^{-1}(E(U_i(-\boldsymbol{X}))).$$

Hence, $\pi_i(\boldsymbol{X}) = -U_i^{-1}(E(U_i(-\boldsymbol{X}))) = -m_i(-\boldsymbol{X})$ which indeed is independent of the wealth w_i. Given this expression we can calculate the premium individuals receive for the risk they bear. For this, recall that for the Pareto optimal allocation risk exchange matrix R^* we have

$$r_{ij}^* = \begin{cases} \dfrac{\frac{1}{\alpha_i}}{\sum_{h \in S_I} \frac{1}{\alpha_h}} & \text{, if } i, j \in S_I, \\ \dfrac{\frac{1}{\alpha_i}}{\sum_{h \in S_I \cup \{j\}} \frac{1}{\alpha_h}} & \text{, if } i \in S_I \cup \{j\} \text{ and } j \in S_P, \\ 0 & \text{, otherwise.} \end{cases}$$

Consequently, the risk that insurer i bears equals $\sum_{j \in N_I \cup N_P} r_{ij}^* \boldsymbol{X}_j$. The premium he should receive for bearing this risk according to the zero utility principle equals

$$\begin{aligned}
\pi_i(\sum_{j\in N_I\cup N_P} r^*_{ij}\boldsymbol{X}_j) &= \pi_i(\sum_{j\in N_I\cup N_P}\sum_{k\in K} r^*_{ij}f_{jk}\boldsymbol{Y}_k)\\
&= -m_i(-\sum_{j\in N_I\cup N_P}\sum_{k\in K} r^*_{ij}f_{jk}\boldsymbol{Y}_k)\\
&= -\sum_{j\in N_I\cup N_P}\sum_{k\in K}\tfrac{1}{\alpha_i}\log\left(1-\tfrac{1}{\mu_k}\alpha_i r^*_{ij}f_{jk}\right)\\
&= -\sum_{j\in N_I}\sum_{k\in K_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right)\\
&\quad -\sum_{j\in N_P}\sum_{k\in K_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right),
\end{aligned}$$

where the third equality follows from expression (6.3) with $d_i = 0$.

Note that for these type of games the zero utility principle satisfies additivity, that is, $\pi_i(\sum_{j\in N_I\cup N_P} r^*_{ij}\boldsymbol{X}_j) = \sum_{j\in N_I\cup N_P}\pi_i(r^*_{ij}\boldsymbol{X}_j)$. As a consequence, we let the premium that insurer i has to pay for reinsuring the fraction r^*_{ji} of his own portfolio $\boldsymbol{X}_i$ at insurer j, equal the premium that insurer j wants to receive for bearing this risk, that is,

$$\pi_j(r^*_{ji}\boldsymbol{X}_i) = -m_j(-\sum_{k\in K} r^*_{ji}f_{ik}\boldsymbol{Y}_k) = -\sum_{k\in K_i}\tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right).$$

Then the premium insurer i receives in aggregate equals

$$\sum_{j\in N_I\cup N_P}\pi_i(r^*_{ij}\boldsymbol{X}_j) - \sum_{j\in N_I}\pi_j(r^*_{ji}\boldsymbol{X}_i).$$

Similarly, the premium that individual $i \in N_P$ has to pay for insuring his loss at insurer j equals the zero utility premium that this insurer wants to receive for bearing this risk. Hence, individual i pays insurer j an amount

$$\pi_j(r^*_{ji}\boldsymbol{X}_i) = -\sum_{k\in K_i}\tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum_{h\in N_I\cup\{i\}}\frac{1}{\alpha_h}}\right).$$

Because individuals are not allowed to bear (part of) the risk of any other individual/insurer he does not receive any premium. So in aggregate he receives

$$-\sum_{j\in N_I}\pi_j(r^*_{ji}\boldsymbol{X}_i).$$

Since

$$\sum_{i\in N_I}\left(\sum_{j\in N_I\cup N_P}\pi_i(r^*_{ij}\boldsymbol{X}_j) - \sum_{j\in N_I}\pi_j(r^*_{ji}\boldsymbol{X}_i)\right) - \sum_{i\in N_P}\sum_{j\in N_I}\pi_j(r^*_{ji}\boldsymbol{X}_i) = 0,$$

the zero utility principle yields an allocation transfer payments vector d^0 where

$$
\begin{aligned}
d_i^0 &= \sum_{j\in N_I\cup N_P} \pi_i(r_{ij}^* \boldsymbol{X}_j) - \sum_{j\in N_I} \pi_j(r_{ji}^* \boldsymbol{X}_i) \\
&= -\sum_{j\in N_I}\sum_{k\in K_j} \tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right) - \sum_{j\in N_P}\sum_{k\in K_j} \tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right) \\
&\quad + \sum_{j\in N_I}\sum_{k\in K_i} \tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right) \qquad (6.8)
\end{aligned}
$$

for all $i \in N_I$ and

$$
d_i^0 = -\sum_{j\in N_I} \pi_j(r_{ji}^* \boldsymbol{X}_i) = \sum_{j\in N_I}\sum_{k\in K_i} \tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum_{h\in N_I\cup\{i\}}\frac{1}{\alpha_h}}\right) \qquad (6.9)
$$

for all $i \in N_P$.

Example 6.4 Consider again the situation described in Example 6.2. Applying the zero utility principle gives for insurer 1

$$
\begin{aligned}
d_1^0 &= -1200\cdot 3\log\left(1-\tfrac{1}{5\cdot 17}\right) - 115\cdot 3\log\left(1-\tfrac{1}{0.5\cdot 17}\right) - 1\cdot 3\log\left(1-\tfrac{2}{5\cdot 39}\right) \\
&\quad -1\cdot 3\log\left(1-\tfrac{1}{0.5\cdot 21}\right) + 1800\cdot 10\log\left(1-\tfrac{1}{5\cdot 17}\right) + 10\cdot 10\log\left(1-\tfrac{1}{0.5\cdot 17}\right) \\
&\quad +1800\cdot 4\log\left(1-\tfrac{1}{5\cdot 17}\right) + 10\cdot 4\log\left(1-\tfrac{1}{0.5\cdot 17}\right) \\
&= 42.60 + 43.18 + 0.03 + 0.30 - 213.02 - 12.52 - 85.21 - 5.01 \\
&= -229.65.
\end{aligned}
$$

Similarly, we get for insurers 2 and 3 and individuals 4 and 5

$$
\begin{aligned}
d_2^0 &= 248.52 + 125.17 + 0.10 + 1.00 - 31.95 - 9.39 - 42.60 - 12.52 = 278.33 \\
d_3^0 &= 127.81 + 17.53 + 0.04 + 0.40 - 10.65 - 33.79 - 35.50 - 112.65 = -46.81 \\
d_4^0 &= -0.03 - 0.10 - 0.04 = -0.17 \\
d_5^0 &= -0.3 - 1.00 - 0.40 = -1.70.
\end{aligned}
$$

So, $d^0 = (-229.65, 278.33, -46.81, -0.17, -1.70)$. From Example 6.3 we know that the resulting allocation $(d_i^0 + R_i^* \boldsymbol{X}^N)_{i\in N}$ is in the core of the game.

In Example 6.4 it is seen that the allocation corresponding to the zero utility principle is a core allocation. The next theorem shows that this is not a coincidence.

Theorem 6.5 Let $\Gamma \in IG(N_I, N_P)$. If d^0 is the vector of transfer payments determined by the zero utility premium calculation principle and R^* is the Pareto optimal risk exchange matrix then $(d_i^0 + R_i^* \boldsymbol{X}^N)_{i\in N} \in \mathcal{C}(\Gamma)$.

PROOF: By Proposition 6.3 it suffices to show that $(m_i(d_i^0 + R_i^* \boldsymbol{X}^N)_{i\in N} \in C(v_\Gamma)$. Hence, we must show that $\sum_{i\in S} m_i(d_i^0 + R_i^* \boldsymbol{X}^N) \geq v_\Gamma(S)$ for all $S \subset N$. Since for $i \in N_I$ it holds that

$$\begin{aligned}
m_i(d_i^0 + R_i^* \boldsymbol{X}^N) = & \\
& -\sum_{j\in N_I}\sum_{k\in K_j} \tfrac{1}{\alpha_i} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h\in N_I} \frac{1}{\alpha_h}}\right) - \sum_{j\in N_P}\sum_{k\in K_j} \tfrac{1}{\alpha_i} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h\in N_I\cup\{j\}} \frac{1}{\alpha_h}}\right) \\
& + \sum_{j\in N_I}\sum_{k\in K_i} \tfrac{1}{\alpha_j} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I} \frac{1}{\alpha_h}}\right) \\
& + \sum_{j\in N_I}\sum_{k\in K_j} \tfrac{1}{\alpha_i} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h\in N_I} \frac{1}{\alpha_h}}\right) + \sum_{j\in N_P}\sum_{k\in K_j} \tfrac{1}{\alpha_i} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h\in N_I\cup\{j\}} \frac{1}{\alpha_h}}\right) \\
= & \sum_{j\in N_I}\sum_{k\in K_i} \tfrac{1}{\alpha_j} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I} \frac{1}{\alpha_h}}\right) \\
= & \sum_{k\in K_i} \tfrac{f_{ik}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I} \frac{1}{\alpha_h}}
\end{aligned}$$

and for $i \in N_P$ that

$$\begin{aligned}
m_i(d_i^0 + R_i^* \boldsymbol{X}^N) = & \\
& \sum_{j\in N_I}\sum_{k\in K_i} \tfrac{1}{\alpha_j} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}\right) + \sum_{k\in K_i} \tfrac{1}{\alpha_i} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}\right) \\
= & \sum_{j\in N_I\cup\{i\}}\sum_{k\in K_i} \tfrac{1}{\alpha_j} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}\right) \\
= & \sum_{k\in K_i} \tfrac{f_{ik}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}
\end{aligned}$$

we have for $S \subset N_I \cup N_P$ that

$$\begin{aligned}
\sum_{i\in S} m_i(d_i^0 + R_i^* \boldsymbol{X}^N) = & \sum_{i\in S_I}\sum_{k\in K_i} \tfrac{f_{ik}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I} \frac{1}{\alpha_h}} \\
& + \sum_{i\in S_P}\sum_{k\in K_i} \tfrac{f_{ik}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}} \\
\geq & \sum_{i\in S_I}\sum_{k\in K_i} \tfrac{f_{ik}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in S_I} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum_{h\in S_I} \frac{1}{\alpha_h}} \\
& + \sum_{i\in S_P}\sum_{k\in K_i} \tfrac{f_{ik}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum_{h\in S_I\cup\{i\}} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum_{h\in S_I\cup\{i\}} \frac{1}{\alpha_h}} \\
= & v_\Gamma(S),
\end{aligned}$$

where the inequality follows from Lemma A.6 with $c = 0$.

Example 6.5 Consider the insurance game introduced in Example 6.2. Now, let us take a closer look at the changes in insurer 1's utility when the allocation $(d_i^0 + R_i^* \boldsymbol{X}^N)_{i \in N}$ is realized. In the initial situation insurer 1 bears the risk $\boldsymbol{X}_1$ of his own insurance portfolio. The certainty equivalent of $\boldsymbol{X}_1$ equals

$$m_1(\boldsymbol{X}_1) = 1800 \cdot 3 \log\left(1 - \tfrac{1}{5 \cdot 3}\right) + 10 \cdot 3 \log\left(1 - \tfrac{1}{0.5 \cdot 3}\right) = -405.52.$$

To allocate the total risk in a Pareto optimal way, insurer 1 bears the fraction $r_{12}^* = \frac{3}{17}$ of the risk $\boldsymbol{X}_2$ of insurer 2. For this risk he receives a premium $\pi_1(\frac{3}{17}\boldsymbol{X}_2)$ determined by the zero utility principle. From the definition of the zero utility calculation principle it follows that $m_1(\boldsymbol{X}_1 + \frac{3}{17}\boldsymbol{X}_2 - \pi_1(\frac{3}{17}\boldsymbol{X}_2)) = -405.52$. So insurer 1's welfare does not change when he insures a part of the risk of insurer 2. A similar argument holds when he insures a part of the risks of the other agents. Hence

$$\begin{aligned} &m_1(\boldsymbol{X}_1 - \tfrac{3}{17}\boldsymbol{X}_2 + \pi_1(\tfrac{3}{17}\boldsymbol{X}_2) - \tfrac{3}{17}\boldsymbol{X}_3 + \pi_1(\tfrac{3}{17}\boldsymbol{X}_3) \\ &\quad - \tfrac{6}{39}\boldsymbol{X}_4 + \pi_1(\tfrac{6}{39}\boldsymbol{X}_4) - \tfrac{3}{21}\boldsymbol{X}_5 + \pi_1(\tfrac{3}{21}\boldsymbol{X}_5)) = -405.52. \end{aligned}$$

The increase in insurer 1's welfare arises only from the risks $\frac{10}{17}\boldsymbol{X}_1$ and $\frac{4}{17}\boldsymbol{X}_1$ he transfers to insurers 2 and 3, respectively:

$$\begin{aligned} &m_1(\tfrac{3}{17} - \pi_2(\tfrac{10}{17}\boldsymbol{X}_1) - \pi_3(\tfrac{4}{17}\boldsymbol{X}_1) + \boldsymbol{X}_1 - \tfrac{3}{17}\boldsymbol{X}_2 + \pi_1(\tfrac{3}{17}\boldsymbol{X}_2) - \tfrac{3}{17}\boldsymbol{X}_3 + p_1(\tfrac{3}{17}\boldsymbol{X}_3) \\ &\quad - \tfrac{6}{39}\boldsymbol{X}_4 + \pi_1(\tfrac{6}{39}\boldsymbol{X}_4) - \tfrac{3}{21}\boldsymbol{X}_5 + \pi_1(\tfrac{3}{21}\boldsymbol{X}_5)) = m_1(d_1^0 + R_1^* \boldsymbol{X}^N) = \quad -229.65 \ . \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad > \quad -405.52 \end{aligned}$$

The situation described in the example above is subsistent in the definition of the zero utility principle. This means that the welfare of an insurer always remains the same when he bears the risk of someone else in exchange for the zero utility principle based premium. An increase in welfare only arises when he transfers (a part of) his own risk to someone else. Consequently, the insurers' welfare does not increase when individuals insure their losses. Hence, the insurers' incentives to insure the individual's losses is low. To increase these incentives it may be better to utilize other premium calculation principles. One could, for example, consider subadditive premiums. In the next section we give another reason why it could be desirable that insurance companies employ subadditive premiums.

6.2 Subadditivity for Collective Insurances

In the insurance games defined in the previous section individual persons are not allowed to cooperate; they cannot redistribute the risk amongst themselves. Looking at the individuals' behavior in everyday life, this is a justified assumption. People who want to insure themselves against certain risks do so by contacting insurance companies, pension funds etc. We show, however, that when this restriction is abandoned then the mere fact that risk exchanges could take place between individuals implies that insurance companies have incentives to employ subadditive premiums. Whether or not such risk exchanges actually do take place is not important. As a consequence, collective insurances become cheaper for the individuals.

Let N_P be the set of individuals. A premium calculation principle π is called subadditive if for all subsets $S, T \subset N_P$ with $S \cap T = \emptyset$ it holds that $\pi(\boldsymbol{X}_S) + \pi(\boldsymbol{X}_T) \geq \pi(\boldsymbol{X}_S + \boldsymbol{X}_T)$. Here, $\boldsymbol{X}_S$ denotes the total loss of the coalition S. So, it is attractive for the individuals to take a collective insurance, since this reduces the total premium they have to pay.

Next, consider a game with agent set N_P only where the individuals are allowed to redistribute their risks. This situation can be described by an insurance game $\Gamma \in IG(N_P, \emptyset)$. So, the individuals N_P can now insure their losses among each other. Then we can associate with Γ the TU-game (N, v_Γ), with

$$v_\Gamma(S) = \max\left\{\sum_{i\in S} m_i(d_i + R_i r X^S) |\, (d_i + R_i \boldsymbol{X}^S)_{i\in S} \in \mathcal{Z}_\Gamma(S)\right\}$$

for all $S \subset N_P$. Note that this maximum is attained for Pareto optimal allocations $(d_i + R_i^* \boldsymbol{X}^S)_{i\in S} \in \mathcal{Z}_\Gamma(S)$ for coalition S. For this game, the value $v_\Gamma(S)$ can be interpreted as the maximum premium coalition S wants to pay for the insurance of the total risk $\boldsymbol{X}_S$. To see this, suppose that the coalition S can insure the loss $\boldsymbol{X}_S$ for a premium $\pi(\boldsymbol{X}_S)$ that exceeds the valuation of the risk $\boldsymbol{X}_S$, that is, $-\pi(\boldsymbol{X}_S) < v_\Gamma(S)$. Then for each allocation $y \in \mathbb{R}^S$ of the premium $-\pi(\boldsymbol{X}_S)$ there exists an allocation $(\tilde{d}_i + R_i^* \boldsymbol{X}^S) \in \mathcal{Z}_\Gamma(S)$ such that $E(U_i(\tilde{d}_i + R_i^* \boldsymbol{X}^S)) > U_i(y_i)$ for all $i \in S$. Indeed, let $(d_i + R_i^* \boldsymbol{X}^S)_{i\in S} \in \mathcal{Z}_\Gamma(S)$ be such that $\sum_{i\in S} m_i(d_i + R_i^* \boldsymbol{X}^S) = v_\Gamma(S)$. Define

$$\tilde{d}_i = d_i - m_i(d_i + R_i^* \boldsymbol{X}^S) + y_i + \tfrac{1}{\#S}\left(v_\Gamma(S) + \pi(\boldsymbol{X}_S)\right),$$

for all $i \in S$. Since

$$\begin{aligned}\sum_{i\in S} \tilde{d}_i &= \sum_{i\in S} d_i - \sum_{i\in S} m_i(d_i + R_i \boldsymbol{X}^S) + \sum_{i\in S} y_i + v_\Gamma(S) + \pi(\boldsymbol{X}_S) \\ &= \textstyle\sum_{i\in S} d_i \leq 0\end{aligned}$$

it follows that $(\tilde{d}_i + R_i \boldsymbol{X}^S)_{i\in S} \in \mathcal{Z}_\Gamma(S)$. Then by the linearity of m_i in $\tilde{d}_i$ (cf. expression (6.3)) we have for all $i \in S$ that

$$m_i(\tilde{d}_i + R_i^* \boldsymbol{X}^S) = y_i + \tfrac{1}{\#S}\left(v_\Gamma(S) + \pi(\boldsymbol{X}_S)\right) > y_i.$$

Hence, the members of S prefer the allocation $(\tilde{d}_i + R_i^* \boldsymbol{X}^S)_{i \in S}$ of $\boldsymbol{X}_S$ to an insurance of $\boldsymbol{X}_S$ and paying the premium $\pi(\boldsymbol{X}_S)$. Consequently, they will not pay more for the insurance of the risk $\boldsymbol{X}_S$ than the amount $-v_\Gamma(S)$. The next theorem shows that this maximum premium $-v_\Gamma(S)$ is subadditive, i.e., $-v_\Gamma(S) - v_\Gamma(T) \geq -v_\Gamma(S \cup T)$, or equivalently, $v_\Gamma(S) + v_\Gamma(T) \leq v_\Gamma(S \cup T)$, for all disjoint subcoalitions S and T of N_P.

Theorem 6.6 Let $S, T \subset N_P$ such that $S \cap T = \emptyset$. Then

$$v_\Gamma(S) + v_\Gamma(T) \;\leq\; v_\Gamma(S \cup T).$$

PROOF: Define for all $S \subset N_P$, all $j \in N_P$, and all $k \in K$

$$a_{jk}(S) = \tfrac{\mu_k}{f_{jk}} \sum_{h \in S} \tfrac{1}{\alpha_h}.$$

Recall from expression (6.7) that for all $S \subset N_P$ it holds that

$$\begin{aligned} v_\Gamma(S) &= \sum_{j \in S} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in S} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}} \sum_{h \in S} \frac{1}{\alpha_h}} \\ &= \sum_{j \in S} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log\left(1 - \frac{1}{a_{jk}(S)}\right)^{a_{jk}(S)} \end{aligned}$$

Now, take $S, T \subset N_P$ such that $S \cap T = \emptyset$. We have to show that $v_\Gamma(S) + v_\Gamma(T) \leq v_\Gamma(S \cup T)$.

$$\begin{aligned} & v_\Gamma(T \cup S) - v_\Gamma(S) - v_\Gamma(T) = \\ &= \sum_{j \in (T \cup S)} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log(1 - \tfrac{1}{a_{jk}(T \cup S)})^{a_{jk}(T \cup S)} - \sum_{j \in S} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log(1 - \tfrac{1}{a_{jk}(S)})^{a_{jk}(S)} \\ &\quad - \sum_{j \in T} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log(1 - \tfrac{1}{a_{jk}(T)})^{a_{jk}(T)} \\ &= \sum_{j \in S} \sum_{k \in K_j} \left(\tfrac{f_{jk}}{\mu_k} \log(1 - \tfrac{1}{a_{jk}(T \cup S)})^{a_{jk}(T \cup S)} - \tfrac{f_{jk}}{\mu_k} \log(1 - \tfrac{1}{a_{jk}(S)})^{a_{jk}(S)} \right) + \\ &\quad \sum_{j \in T} \sum_{k \in K_j} \left(\tfrac{f_{jk}}{\mu_k} \log(1 - \tfrac{1}{a_{jk}(T \cup S)})^{a_{jk}(T \cup S)} - \tfrac{f_{jk}}{\mu_k} \log(1 - \tfrac{1}{a_{jk}(T)})^{a_{jk}(T)} \right) \\ &= \sum_{j \in S} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log\left(\left(1 - \tfrac{1}{a_{jk}(T \cup S)}\right)^{a_{jk}(T \cup S)} \left(\tfrac{a_{jk}(S)}{a_{jk}(S) - 1}\right)^{a_{jk}(S)} \right) + \\ &\quad \sum_{j \in T} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log\left(\left(1 - \tfrac{1}{a_{jk}(T \cup S)}\right)^{a_{jk}(T \cup S)} \left(\tfrac{a_{jk}(T)}{a_{jk}(T) - 1}\right)^{a_{jk}(T)} \right) \\ &= \sum_{j \in S} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log\left(\left(1 - \tfrac{1}{a_{jk}(T) + a_{jk}(S)}\right)^{a_{jk}(T) + a_{jk}(S)} \left(\tfrac{a_{jk}(S)}{a_{jk}(S) - 1}\right)^{a_{jk}(S)} \right) + \\ &\quad \sum_{j \in T} \sum_{k \in K_j} \tfrac{f_{jk}}{\mu_k} \log\left(\left(1 - \tfrac{1}{a_{jk}(T) + a_{jk}(S)}\right)^{a_{jk}(T) + a_{jk}(S)} \left(\tfrac{a_{jk}(T)}{a_{jk}(T) - 1}\right)^{a_{jk}(T)} \right) \;\geq\; 0, \end{aligned}$$

where the second and the fourth equality follow from $S \cap T = \emptyset$ and the inequality follows from Lemma A.7 which says that

$$\left(1 - \frac{1}{a_{jk}(T)+a_{jk}(S)}\right)^{a_{jk}(T)+a_{jk}(S)} \left(\frac{a_{jk}(S)}{a_{jk}(S)-1}\right)^{a_{jk}(S)} \geq 1,$$

and

$$\left(1 - \frac{1}{a_{jk}(T)+a_{jk}(S)}\right)^{a_{jk}(T)+a_{jk}(S)} \left(\frac{a_{jk}(T)}{a_{jk}(T)-1}\right)^{a_{jk}(T)} \geq 1.$$

Recall that insurers do not benefit from insuring the risks of the individuals when utilizing the additive zero utility principle; this premium calculation principle yields the lowest premium for which insurers still want to exchange risks with the individuals (cf. Example 6.5). So, from a social point of view, it might be best to adopt a middle course and look for premiums where both insurers and individuals benefit from the insurance transaction. Interesting questions then remaining are: are these premiums additive or subadditive and do they yield core allocations?

6.3 Remarks

In this chapter (re)insurance problems are modelled as cooperative games with stochastic payoffs. In fact, we defined a game that dealt with both the insurance and the reinsurance problem simultaneously. We showed that there is only one allocation risk exchange matrix yielding a Pareto optimal distribution of the losses and that a core allocation results when insurance premiums are calculated according to the zero utility principle. Moreover, we explained why subadditive premium calculation principles might be attractive to use for insurance companies.

An issue only briefly mentioned in this paper concerns the insurers' behavior. What if an insurer is risk neutral or risk loving instead of risk averse? Thus, there is at least one insurer whose utility function is linear or of the form $u_i(t) = \beta_i e^{-\alpha_i t}$ ($t \in \mathbb{R}$) with $\beta_i > 0$, $\alpha_i < 0$. Although the proofs are not provided here, most of the results presented in this paper still hold for these situations. This means that the corresponding games have nonempty cores and that the zero utility principle still yields a core allocation. The result that does change is the Pareto optimal allocation of the risk. The allocations that are Pareto optimal when all insurers are risk averse are not Pareto optimal anymore when one or more insurers happen to be risk loving. In fact, they are the worst possible allocations of the risk one can think of. In that case, allocating all the risk to the most risk loving insurer is Pareto optimal. This would actually mean that only one insurance company is needed, since other insurance companies will ultimately reinsure their complete portfolios at this most risk loving insurer.

A Proofs

This appendix contains the proofs that are omitted in the text. Furthermore, it states some additional results referred to in the previous chapters. The appendix is divided in sections, each section contains the proofs belonging to one particular chapter only. Therefore, each section title is the same as the title of the corresponding chapter.

A.1 A Nucleolus for Stochastic Cooperative Games

Proposition 5.1 $IR_\Gamma(S)$ is a compact subset of $I_\Gamma(S)$ for each coalition $S \subset N$.

PROOF: Since $IR_\Gamma(S) \subset I_\Gamma(S) \subset \mathbb{R}^S \times \mathbb{R}^S$ it is sufficient to prove that $IR_\Gamma(S)$ is closed and bounded in $\mathbb{R}^S \times \mathbb{R}^S$. Since

$$IR_\Gamma(S) = \{(d, r) \in I_\Gamma(S) | \sum_{i \in S} d_i \leq 0\}$$

and $I_\Gamma(S)$ is closed by the weak continuity of $\succsim_i$ for all $i \in S$ it follows that $IR_\Gamma(S)$ is closed. To see that $IR_\Gamma(S)$ is bounded, define for each $i \in S$ and each $r_i \in [0, 1]$

$$\underline{d}_i(r_i) = \min\{d_i | d_i + r_i \boldsymbol{X}_S \succsim_i \boldsymbol{X}_{\{i\}}\}.$$

Note that $\underline{d}_i(r_i)$ exists by assumptions (C5) and (C3) and that $\underline{d}_i(r_i) + r_i \boldsymbol{X}_S \sim_i \boldsymbol{X}_{\{i\}}$. To show that $\min_{r_i \in [0,1]} \underline{d}_i(r_i)$ exists it suffices to show that $\underline{d}_i(r_i)$ is continuous in r_i. Therefore, consider the sequence $(r_i^k)_{k \in \mathbb{N}}$ with $r_i^k \in [0, 1]$ and $\lim_{k \to \infty} r_i^k = r_i$. By definition we have for all $k \in \mathbb{N}$ that $\underline{d}_i(r_i^k) + r_i^k \boldsymbol{X}_S \sim_i \boldsymbol{X}_{\{i\}}$. Hence, $\underline{d}_i(r_i^k) + r_i^k \boldsymbol{X}_S \sim_i \underline{d}_i(r_i) + r_i \boldsymbol{X}_S$ for all $i \in S$. Since $\succsim_i$ is weakly continuous it follows that

$$\lim_{k \to \infty} \left(\underline{d}_i(r_i^k) + r_i^k \boldsymbol{X}_S\right) = \lim_{k \to \infty} \underline{d}_i(r_i^k) + r_i \boldsymbol{X}_S \sim_i \underline{d}_i(r_i) + r_i \boldsymbol{X}_S$$

Then assumption (C2) implies that $\lim_{k\to\infty} \underline{d}_i(r_i^k) = \underline{d}_i(r_i)$. Consequently, $\underline{d}_i(r_i)$ is continuous in r_i and

$$\underline{d}_i = \min_{r_i \in [0,1]} \underline{d}_i(r_i)$$

exists and is finite for all $i \in S$.

Since $(d, r) \in IR_\Gamma(S)$ implies that $d_i + r_i \boldsymbol{X}_S \succsim_i \boldsymbol{X}_{\{i\}}$ for all $i \in S$ it follows by condition (C2) that $d_i \geq \underline{d}_i$ for all $i \in S$. Hence, $(d, r) \in IR_\Gamma(S)$ implies that

$$d \in \{\tilde{d} \in \mathbf{R}^S | \forall_{i \in S} : \tilde{d}_i \geq \underline{d}_i, \sum_{i \in S} \tilde{d}_i \leq 0\}$$

and $r \in \Delta^S$. Since both sets are bounded, we have that $IR_\Gamma(S)$ is bounded. ∎

Proposition 5.2 The set of Pareto optimal allocations $PO_\Gamma(S)$ is a compact subset of $I_\Gamma(S)$ for each coalition $S \subset N$.

PROOF: Since $PO_\Gamma(S) \subset IR_\Gamma(S)$ and $IR_\Gamma(S)$ is compact it is sufficient to show that $PO_\Gamma(S)$ is closed in $IR_\Gamma(S)$. Let $(d, r) \in IR_\Gamma(S)$ be such that $(d, r) \notin PO_\Gamma(S)$. Then there exists $(\bar{d}, \bar{r}) \in IR_\Gamma(S)$ such that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \succ_i d_i + r_i \boldsymbol{X}_S$ for all $i \in S$. Next, consider the set $\{(d', r') \in IR_\Gamma(S) | d'_i + r'_i \boldsymbol{X}_S \prec_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S\}$. By the weak continuity of $\succsim_i$ this set is open in $IR_\Gamma(S)$. Indeed, by the continuity of $\succsim_i$ we have that $\{\boldsymbol{Y} \in \prod_{i \in S} \mathcal{L}(\mathcal{X}_i) | \exists_{i \in S} : \boldsymbol{Y}_i \succsim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S\}$ is closed. Hence, Proposition 5.1 implies that $\{(d', r') \in IR_\Gamma(S) | \exists_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \succsim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S\}$ is closed in $I_\Gamma(S)$. Hence, it is also closed in $IR_\Gamma(S)$. Consequently, $\{(d', r') \in IR_\Gamma(S) | \forall_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \prec_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S\}$ must be open in $IR_\Gamma(S)$. Since (d, r) belongs to the latter set there exists an open neighbourhood O of (d, r) in $IR_\Gamma(S)$ such that $O \subset \{(d', r') \in IR_\Gamma(S) | \forall_{i \in S} : d'_i + r'_i \boldsymbol{X}_S \prec_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S\}$. This implies that $(\tilde{d}, \tilde{r}) \notin PO_\Gamma(S)$ whenever $(\tilde{d}, \tilde{r}) \in O$. Hence, $IR_\Gamma(S) \backslash PO_\Gamma(S)$ is open in $IR_\Gamma(S)$ and, consequently, $PO_\Gamma(S)$ is closed in $IR_\Gamma(S)$. ∎

Proposition 5.3 Let $\Gamma \in CG(N)$. Take $(d, r) \in PD_\Gamma(S)$ and $(\tilde{d}, \tilde{r}) \in NPD_\Gamma(S)$ such that $d_i + r_i \boldsymbol{X}_S \precsim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$ for all $i \in S$. Then there exists $(\hat{d}, \hat{r}) \in PO_\Gamma(S)$ such that

$$d_i + r_i \boldsymbol{X}_S \precsim_i \hat{d}_i + \hat{r}_i \boldsymbol{X}_S \precsim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$$

for all $i \in S$.

PROOF: Let $(d, r) \in PD_\Gamma(S)$ and $(\tilde{d}, \tilde{r}) \in NPD_\Gamma(S)$. Without loss of generality we may

assume that $(d,r) \in IR_\Gamma(S)$.[1] Take $\delta_i \in \mathbb{R}$ be such that $d_i + \delta_i + r_i \boldsymbol{X}_S \sim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$. Note that $\delta_i \geq 0$ by condition (C2). Next, take $\bar{r} \in \Delta^S$ and $t \in [0,1]$. Let $\bar{d}_i(\bar{r},t)$ be such that $\bar{d}_i(\bar{r},t) + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + t\delta_i + r_i \boldsymbol{X}_S$. Note that the allocation $(\bar{d}(\bar{r},t),\bar{r})$ is feasible if and only if $\sum_{i\in S} \bar{d}_i(\bar{r},t) \leq 0$. First, we show that $\bar{d}_i(\bar{r},t)$ is continuous in $(\bar{r},t)$. Let $((\bar{r}^k,t^k))_{k\in\mathbb{N}}$ be a convergent sequence with limit $(\bar{r},t)$. We have to show that $\lim_{k\to\infty} \bar{d}_i(\bar{r}^k,t^k) = \bar{d}_i(\bar{r},t)$. Note that $\bar{d}_i(\bar{r}^k,t^k) + \bar{r}_i^k \boldsymbol{X}_S \sim_i d_i + t^k\delta_i + r_i \boldsymbol{X}_S$ for all $k \in \mathbb{N}$. Define for $\varepsilon > 0$

$$\mathcal{O}_i^\varepsilon = \{\boldsymbol{Y} \in \mathcal{L}(\mathcal{X}_i) | d_i + t\delta_i + r_i \boldsymbol{X}_S - \varepsilon \prec_i \boldsymbol{Y} \prec_i d_i + t\delta_i + r_i \boldsymbol{X}_S + \varepsilon\}.$$

Since $t^k \to t$ there exists $K^\varepsilon \in \mathbb{N}$ such that $d_i + t^k\delta_i + r_i \boldsymbol{X}_S \in \mathcal{O}_i^\varepsilon$ for all $k > K^\varepsilon$. Consequently, we have that $\bar{d}_i(\bar{r}^k,t^k) + \bar{r}_i^k \boldsymbol{X}_S \in \mathcal{O}_i^\varepsilon$ for all $k > K^\varepsilon$. This implies that $\lim_{k\to\infty} \left(\bar{d}_i(\bar{r}^k,t^k) + \bar{r}_i^k \boldsymbol{X}_S\right) \in \cap_{\varepsilon>0} \mathcal{O}_i^\varepsilon$. So,

$$\lim_{k\to\infty} \left(\bar{d}_i(\bar{r}^k,t^k) + \bar{r}_i^k \boldsymbol{X}_S\right) = \lim_{k\to\infty} \bar{d}_i(\bar{r}^k,t^k) + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + t\delta_i + r_i \boldsymbol{X}_S.$$

Since $\bar{d}_i(\bar{r},t) + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + t\delta_i + r_i \boldsymbol{X}_S$ it follows from condition (C2) that $\lim_{k\to\infty} \bar{d}_i(\bar{r}^k,t^k) = \bar{d}_i(\bar{r},t)$.

Next, define $f(t) = \min_{\bar{r}\in\Delta^S} \sum_{i\in S} \bar{d}_i(\bar{r},t)$ for all $t \in [0,1]$. Then f is a continuous function. Moreover, since $(d,r) \in IR_\Gamma(S)$ and $\bar{d}_i(r,0) = d_i$ for all $i \in S$ it follows from the feasibility of (d,r) that $f(0) \leq \sum_{i\in S} \bar{d}_i(r,0) = \sum_{i\in S} d_i \leq 0$. Furthermore, since $d_i + \delta_i + r_i \boldsymbol{X}_S \sim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$ for all $i \in S$ and $(\tilde{d},\tilde{r}) \in NPD_\Gamma(S)$ it follows that $(d+\delta,r) \in NPD_\Gamma(S)$. This implies that $f(1) \geq 0$. For, if $f(1) < 0$ then there exists $r^* \in \Delta^S$ such that $\sum_{i\in S} \bar{d}_i(r^*,1) < 0$ and $\bar{d}_i(r^*,1) + r_i^* \boldsymbol{X}_S \sim_i d_i + \delta_i + r_i \boldsymbol{X}_S$ for all $i \in S$. Consequently, the allocation yielding the payoffs

$$\bar{d}_i(r^*,1) - \tfrac{1}{|S|} \sum_{i\in S} \bar{d}_i(r^*,1) + r_i^* \boldsymbol{X}_S$$

for each $i \in S$ is feasible and preferred to $d_i + \delta_i + r_i \boldsymbol{X}_S$ by all agents $i \in S$. Clearly, this contradicts the fact that $(d+\delta,r) \in NPD_\Gamma(S)$. Thus, $f(0) \leq 0 \leq f(1)$. The continuity of f then implies that there exists $\hat{t}$ such that $f(\hat{t}) = 0$.

Let $\hat{r} \in \Delta^S$ be such that $\sum_{i\in S} \bar{d}_i(\hat{r},\hat{t}) = 0$. Then the allocation $(\bar{d}(\hat{r},\hat{t}),\hat{r})$ is Pareto optimal. To see this, first note that $\sum_{i\in S} \bar{d}_i(\bar{r},\hat{t}) \geq 0$ for all $\bar{r} \in \Delta^S$. Second, note that the definition of $\bar{d}_i(\bar{r},t)$ implies that

$$\bar{d}_i(\hat{r},\hat{t}) + \hat{r}_i \boldsymbol{X}_S \sim_i \bar{d}_i(\bar{r},\hat{t}) + \bar{r}_i \boldsymbol{X}_S \tag{A.1}$$

for all $i \in S$ and all $\bar{r} \in \Delta^S$. Next, take $\bar{r} \in \Delta^S$. If $\sum_{i\in S} \bar{d}_i(\bar{r},\hat{t}) > 0$ then the allocation $(\bar{d}(\bar{r},\hat{t}),\bar{r})$ is not feasible. From expression (A.1) it then follows that there exists no feasible

[1] If $(d,r) \notin IR_\Gamma(S)$ then there exists $(d',r') \in IR_\Gamma(S)$ such that $d_i' + r_i' \boldsymbol{X}_S \succ_i d_i + r_i \boldsymbol{X}_S$ for all $i \in S$. If $\delta_i' < 0$ is such that $d_i' + \delta_i' + r_i' \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_S$ for all $i \in S$ then $(d'+\delta',r')$ is still a feasible allocation. Thus, $(d'+\delta',r') \in IR_\Gamma(S)$. Continuing the proof with the allocation (d,r) replaced by $(d'+\delta',r')$ would yield the same result.

allocation $(\bar{d}, \bar{r})$ which all agents $i \in S$ prefer to the allocation $(\bar{d}(\hat{r}, \hat{t}), \hat{r})$. If $\sum_{i \in S} \bar{d}_i(\bar{r}, \hat{t}) = 0$ then the allocation $(\bar{d}(\bar{r}, \hat{t}), \bar{r})$ is feasible. Moreover, an allocation $(\bar{d}, \bar{r})$ that all agents $i \in S$ prefer to $\bar{d}(\bar{r}, \hat{t}), \bar{r})$ must be infeasible by condition (C2) and expression (A.1). Hence, there exists no feasible allocation $(\bar{d}, \bar{r})$ which all agents $i \in S$ prefer to $(\bar{d}(\hat{r}, \hat{t}), \hat{r})$. Consequently, $(\bar{d}(\hat{r}, \hat{t}), \hat{r})$ is Pareto optimal. $0 \leq \hat{t} \leq 1$ then implies that

$$d_i + r_i \boldsymbol{X}_S \precsim_i \bar{d}_i(\hat{r}, \hat{t}) + \hat{r}_i \boldsymbol{X}_S \precsim_i d_i + \delta_i + r_i \boldsymbol{X}_S \sim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$$

for all $i \in S$.

Lemma A.1 $PO^*_S((d, r))$ is a nonempty compact subset of $PO_\Gamma(S)$.

PROOF: That $PO^*_S((d, r))$ is compact follows from the facts that $W_S((d, r))$ and $B_S((d, r))$ are closed by the continuity condition (C5) and $PO_\Gamma(S)$ is compact. To show that it is nonempty let us distinguish two cases.

First, let $B_S((d, r)) \neq \emptyset$. Then there exists $(d', r') \in IR_\Gamma(S)$ such that $d'_i + r'_i \boldsymbol{X}_S \succsim_i d_i + r_i \boldsymbol{X}_S$ for all $i \in S$. Since $(d', r') \in IR_\Gamma(S)$ we know from Proposition 5.3 that there exists $(\bar{d}, \bar{r}) \in PO_\Gamma(S)$ such that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \succsim_i d'_i + r'_i \boldsymbol{X}_S$ for all $i \in S$. Hence, $(\bar{d}, \bar{r}) \in PO_\Gamma(S)$ and $(\bar{d}, \bar{r}) \in B_S((d, r))$. Consequently, $(\bar{d}, \bar{r}) \in PO^*_S((d, r))$.

Second, let $B_S((d, r)) = \emptyset$. Take $(\tilde{d}, \tilde{r}) \in I_\Gamma(S)$ such that $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_N$ for all $i \in S$. From $B_S((d, r)) = \emptyset$ it follows that $(\tilde{d}, \tilde{r}) \in NPD_\Gamma(S)$. Proposition 5.3 then implies that there exists $(\bar{d}, \bar{r}) \in PO_\Gamma(S)$ such that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \precsim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$ for all $i \in S$. Hence, $(\bar{d}, \bar{r}) \in W_S((d, r))$ and, consequently, $(\bar{d}, \bar{r}) \in PO^*_S((d, r))$.

Lemma A.2 Let $(d, r) \in IR_\Gamma(N)$. Then $\overline{E}_S((d, r))$ is a compact subset of $\mathbb{R}$.

PROOF: We have to show that $\overline{E}_S((d, r))$ is closed and bounded. That $\overline{E}_S((d, r))$ is bounded follows from the compactness of $PO^*_S((d, r))$ and the fact that for each $(d', r') \in PO^*_S((d, r))$ the number δ_i is uniquely determined by conditions (C3) and (C5). To see that $\overline{E}_S((d, r))$ is closed, let $(\sum_{i \in S} \delta^k_i)_{k \in \mathbb{N}}$ be a convergent sequence[2] in $\overline{E}_S((d, r))$ with limit $\sum_{i \in S} \delta_i$. We have to show that $\sum_{i \in S} \delta_i \in \overline{E}_S((d, r))$. Therefore, let $((\bar{d}^k, \bar{r}^k))_{k \in \mathbb{N}}$ be a sequence in $PO^*_S((d, r))$ such that $\bar{d}^k_i + \bar{r}^k_i \boldsymbol{X}_S \sim_i d_i + \delta^k_i + r_i \boldsymbol{X}_N$ for all $i \in S$. Since $PO^*_S((d, r))$ is compact there exists a convergent subsequence $((\bar{d}^l, \bar{r}^l))_{l \in \mathbb{N}}$ with limit $(\bar{d}, \bar{r}) \in PO^*_S((d, r))$. Take $\bar{\delta}_i \in \mathbb{R}$ such that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + \bar{\delta}_i + r_i \boldsymbol{X}_N$ for all $i \in S$. Note that $\sum_{i \in S} \bar{\delta}_i \in \overline{E}_S((d, r))$. The

[2]Formally, it would be more correct to start with a convergent sequence $(a^k)_{k \in N}$ in $\overline{E}_S((d, r))$. Then $a^k \in \overline{E}_S((d, r))$ and the definition of $\overline{E}_S$ imply that there exist δ^k_i such that $d_i + \delta^k_i + r_i \boldsymbol{X}_N \sim_i d'_i + r'_i \boldsymbol{X}_S$ $(i \in S)$ for some $(d', r') \in PO^*_S((d, r))$ and $\sum_{i \in S} \delta^k_i = a^k$. Consequently, the sequence $(a^k)_{k \in \mathbb{N}}$ can be replaced by a sequence $(\sum_{i \in S} \delta^k_i)_{k \in \mathbb{N}}$.

proof is finished if we can show that $\delta_i = \bar{\delta}_i$ for all $i \in S$. Therefore, let $\varepsilon > 0$ and $i \in S$. Define

$$\mathcal{O}_i^\varepsilon = \{\boldsymbol{Y} \in \mathcal{L}(\mathcal{X}_i) | \bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \varepsilon \prec_i \boldsymbol{Y} \prec_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S + \varepsilon\}.$$

Since $\mathcal{O}_i^\varepsilon$ is open by the weak continuity of $\succsim_i$, $(\bar{d}^l, \bar{r}^l) \to (\bar{d}, \bar{r})$ and $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \in \mathcal{O}_i^\varepsilon$ there exists $L^\varepsilon \in \mathbb{N}$ such that $\bar{d}_i^l + \bar{r}_i^l \boldsymbol{X}_S \in \mathcal{O}_i^\varepsilon$ for all $l > L^\varepsilon$. This implies that $d_i + \delta_i^l + r_i \boldsymbol{X}_N \in \mathcal{O}_i^\varepsilon$ for all $l > L^\varepsilon$. Since $\varepsilon > 0$ was arbitrarily chosen it follows that

$$\lim_{l \to \infty} \left(d_i + \delta_i^l + r_i^l \boldsymbol{X}_N\right) = d_i + \delta_i + r_i \boldsymbol{X}_N \in \cap_{\varepsilon > 0} \mathcal{O}_i^\varepsilon.$$

Hence, $d_i + \delta_i + r_i \boldsymbol{X}_N \sim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S$. Since by definition it holds that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + \bar{\delta}_i + r_i \boldsymbol{X}_N$ it follows by assumption (C3) that $\delta_i = \bar{\delta}_i$. ∎

Lemma A.3 $\overline{E}_S((d,r))$ is upper semi continuous in (d,r) for all $(d,r) \in IR_\Gamma(N)$.

PROOF: Let $((d^k, r^k))_{k \in \mathbb{N}}$ be a sequence in $IR_\Gamma(N)$ converging to (d,r). Take $\sum_{i \in S} \delta_i^k \in \overline{E}_S((d^k, r^k))$ such that $\sum_{i \in S} \delta_i^k$ converges to $\sum_{i \in S} \delta_i$. For upper semi continuity to be satisfied it is sufficient to show that $\sum_{i \in S} \delta_i \in \overline{E}_S((d,r))$.

First, take $(\bar{d}^k, \bar{r}^k) \in PO_S^*((d^k, r^k))$ such that $\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S \sim_i d_i^k + \delta_i^k + r_i^k \boldsymbol{X}_N$ for all $i \in S$. Since $((\bar{d}^k, \bar{r}^k))_{k \in \mathbb{N}}$ is a sequence in the compact set $PO_\Gamma(S)$ there exists a convergent subsequence $((\bar{d}^l, \bar{r}^l))_{l \in \mathbb{N}}$ with limit $(\bar{d}, \bar{r}) \in PO_\Gamma(S)$. Moreover, it holds that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + \delta_i + r_i \boldsymbol{X}_N$ for all $i \in S$. To see this, take $\varepsilon > 0$ and $i \in S$. Define

$$\mathcal{O}_i^\varepsilon = \{\boldsymbol{Y} \in \mathcal{L}(\mathcal{X}_i) | \bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \varepsilon \prec_i Y \prec_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S + \varepsilon\}.$$

Since $\mathcal{O}_i^\varepsilon$ is open by the continuity of $\succsim_i$, $(\bar{d}^l, \bar{r}^l) \to (\bar{d}, \bar{r})$ and $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \in \mathcal{O}_i^\varepsilon$ there exists $L^\varepsilon \in \mathbb{N}$ such that $\bar{d}_i^l + \bar{r}_i^l \boldsymbol{X}_S \in \mathcal{O}_i^\varepsilon$ for all $l > L^\varepsilon$. This implies that $d_i^l + \delta_i^l + r_i^l \boldsymbol{X}_N \in \mathcal{O}_i^\varepsilon$ for all $l > L^\varepsilon$. Since $\varepsilon > 0$ is arbitrary we have that

$$\lim_{l \to \infty} \left(d_i^l + \delta_i^l + r_i^l \boldsymbol{X}_N\right) = d_i + \delta_i + r_i \boldsymbol{X}_N \sim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S.$$

The proof is finished if we can show that $(\bar{d}, \bar{r}) \in PO_S^*((d,r))$. Therefore, take $\varepsilon > 0$ and define

$$\begin{aligned}
W_S^\varepsilon((d,r)) &= \{(d', r') \in IR_\Gamma(S) | \forall_{i \in S} : d_i' + r_i' \boldsymbol{X}_S \precsim_i d_i + r_i \boldsymbol{X}_N + \varepsilon\}, \\
B_S^\varepsilon((d,r)) &= \{(d', r') \in IR_\Gamma(S) | \forall_{i \in S} : d_i' + r_i' \boldsymbol{X}_S \succsim_i d_i + r_i \boldsymbol{X}_N - \varepsilon\}, \\
PO_S^{*\varepsilon}((d,r)) &= (W_S^\varepsilon((d,r)) \cup B_S^\varepsilon((d,r))) \cap PO_\Gamma(S).
\end{aligned}$$

We refer to Figure A.1 for a graphical interpretation of these sets. Note that $W_S((d,r)) =$

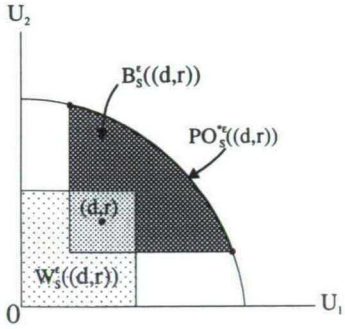


Figure A.1

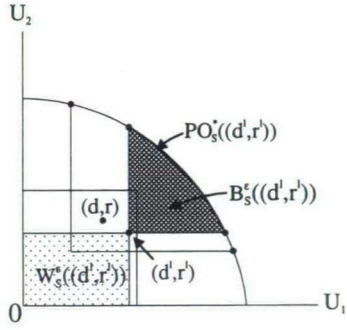


Figure A.2

$\cap_{\varepsilon>0} W_S^\varepsilon((d,r))$, $B_S((d,r)) = \cap_{\varepsilon>0} B_S^\varepsilon((d,r))$ and $PO_S^*((d,r)) = \cap_{\varepsilon>0} PO_S^{*\varepsilon}((d,r))$. Furthermore, define

$$\mathcal{O}^\varepsilon = \{\boldsymbol{Y} \in \prod_{i\in S} \mathcal{L}(\mathcal{X}_i) | \forall_{i\in S} : d_i + r_i \boldsymbol{X}_N - \varepsilon \prec_i \boldsymbol{Y}_i \prec_i d_i + r_i \boldsymbol{X}_N + \varepsilon\}.$$

Since $\mathcal{O}^\varepsilon$ is open by the weak continuity of $\succsim_i$, $(d^l, r^l) \to (d,r)$ and $(d_i + r_i \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^\varepsilon$ there exists $L^\varepsilon \in \mathbb{N}$ such that $(d_i^l + r_i^l \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^\varepsilon$ for all $l > L^\varepsilon$. This implies that $(d^l, r^l) \in W_S^\varepsilon((d,r))$ and $(d^l, r^l) \in B_S^\varepsilon((d,r))$ for all $l > L^\varepsilon$ (see also Figure A.2). Hence, $W_S((d^l, r^l)) \subset W_S^\varepsilon((d,r))$ and $B_S((d^l, r^l)) \subset B_S^\varepsilon((d,r))$ for all $l > L^\varepsilon$. Consequently, we have for all $l > L^\varepsilon$ that $PO_S^*((d^l, r^l)) \subset PO_S^{*\varepsilon}((d,r))$. In particular, we have $(\bar{d}^l, \bar{r}^l) \in PO_S^{*\varepsilon}((d,r))$ for all $l > L^\varepsilon$. Hence,

$$\lim_{l\to\infty} (\bar{d}^l, \bar{r}^l) = (\bar{d}, \bar{r}) \in \cap_{\varepsilon>0} PO_S^{*\varepsilon}((d,r)) \;=\; PO_S^*((d,r)).$$

Lemma A.4 $\overline{E}_S((d,r))$ is lower semi continuous in (d,r) for all $(d,r) \in IR_\Gamma(N)$.

PROOF: Let $((d^k, r^k))_{k\in\mathbb{N}}$ be a sequence converging to (d,r) and let $\sum_{i\in S} \delta_i \in \overline{E}_S((d,r))$. To prove lower semi continuity it suffices to show that there exists a sequence $(\sum_{i\in S} \delta_i^k)_{k\in\mathbb{N}}$ with $\sum_{i\in S} \delta_i^k \in \overline{E}_S((d^k, r^k))$ for all $k \in \mathbb{N}$ such that $\sum_{i\in S} \delta_i^k$ converges to $\sum_{i\in S} \delta_i$.

First, note that since $IR_\Gamma(N)$ is compact and $\overline{E}_S$ is upper semi continuous that

$$\overline{E}_S(IR_\Gamma(N)) = \cup_{(d,r)\in IR_\Gamma(N)} E_S((d,r))$$

is a compact subset of $\mathbb{R}$. Second, note that if there exists a sequence $(\sum_{i\in S} \delta_i^k)_{k\in\mathbb{N}}$ with $\sum_{i\in S} \delta_i^k \in \overline{E}_S((d^k, r^k))$ for each $k \in \mathbb{N}$ such that every convergent subsequence $(\sum_{i\in S} \delta_i^l)_{l\in\mathbb{N}}$ converges to $\sum_{i\in S} \delta_i$ then the compactness of $\overline{E}_S(IR_\Gamma(N))$ implies that the sequence $(\sum_{i\in S} \delta_i^k)_{k\in\mathbb{N}}$ converges to $\sum_{i\in S} \delta_i$.

Take $(\bar{d},\bar{r}) \in PO_S^*((d,r))$ such that $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + \delta_i + r_i \boldsymbol{X}_N$ for all $i \in S$. Let $\varepsilon > 0$ and define

$$\mathcal{O}^\varepsilon = \left\{ \boldsymbol{Y} \in \prod_{i\in S} \mathcal{L}(\mathcal{X}_i) \;\middle|\; \begin{array}{l} \forall_{i\in S} : \boldsymbol{Y}_i \succ_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \varepsilon \text{ or} \\ \forall_{i\in S} : \boldsymbol{Y}_i \prec_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S + \varepsilon \end{array} \right\}.$$

Note that $\mathcal{O}^\varepsilon$ is open by the weak continuity of $\succsim_i$ for each $i \in S$. Next, we show that if $(d_i^k + r_i^k \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^\varepsilon$ then there exists $(\bar{d}^k,\bar{r}^k) \in PO_S^*((d^k,r^k))$ such that $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^\varepsilon$. Therefore, let $k \in \mathbb{N}$ be such that $(d^k,r^k) \in \mathcal{O}^\varepsilon$ and let $(\tilde{d},\tilde{r}) \in I_\Gamma(S)$ be such that $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \sim_i d_i^k + r_i^k \boldsymbol{X}_N$ for all $i \in S$. We distinguish the following three cases.

First, suppose that $(\tilde{d},\tilde{r}) \in NPD_\Gamma(S)$. Since $(\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^\varepsilon$ it holds that $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \succ_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \varepsilon$ for all $i \in S$. From $(\bar{d} - \frac{1}{2}(\varepsilon,\varepsilon,\ldots,\varepsilon),\bar{r}) \in PD_\Gamma(S)$ and Proposition 5.3 it follows that there exists $(\bar{d}^k,\bar{r}^k) \in PO_\Gamma(S)$ such that

$$\bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \tfrac{1}{2}\varepsilon \precsim_i \bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S \precsim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S$$

for all $i \in S$. Thus, $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^\varepsilon$. Since $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \sim_i d_i^k + r_i^k \boldsymbol{X}_N$ for all $i \in S$ it holds that $(\bar{d}^k,\bar{r}^k) \in W_S((d^k,r^k))$. Hence, $(\bar{d}^k,\bar{r}^k) \in PO_S^*((d^k,r^k))$.

Second, suppose that $(\tilde{d},\tilde{r}) \in PD_\Gamma(S)$ and that $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \prec_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S + \varepsilon$ for all $i \in S$. Since the Pareto optimality of $(\bar{d},\bar{r})$ implies that $(\bar{d} + \frac{1}{2}(\varepsilon,\varepsilon,\ldots,\varepsilon),\bar{r}) \in NPD_\Gamma(S)$ it follows from Proposition 5.3 that there exists $(\bar{d}^k,\bar{r}^k) \in PO_\Gamma(S)$ such that

$$\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \precsim_i \bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S \precsim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S + \tfrac{1}{2}\varepsilon$$

for all $i \in S$. Thus, $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^\varepsilon$. Since $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \sim_i d_i^k + r_i^k \boldsymbol{X}_N$ for all $i \in S$ it holds that $(\bar{d}^k,\bar{r}^k) \in B_S((d^k,r^k))$. Hence, $(\bar{d}^k,\bar{r}^k) \in PO_S^*((d^k,r^k))$.

Finally, suppose that $(\tilde{d},\tilde{r}) \in PD_\Gamma(S)$ and that $\tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \succ_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \varepsilon$ for all $i \in S$. Then Proposition 5.3 implies that there exists $(\bar{d}^k,\bar{r}^k) \in PO_\Gamma(S)$ such that

$$\bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \varepsilon \precsim_i \tilde{d}_i + \tilde{r}_i \boldsymbol{X}_S \precsim_i \bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S$$

for all $i \in S$. Thus, $(\bar{d}^k,\bar{r}^k) \in B_S((d^k,r^k))$. Therefore we have that $(\bar{d}^k,\bar{r}^k) \in PO_S^*((d^k,r^k))$. Moreover, $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^\varepsilon$.

Now we are able to construct a sequence $(\sum_{i\in S} \delta_i^k)_{k\in\mathbb{N}}$ with $\sum_{i\in S} \delta_i^k \in PO_S^*((d^k,r^k))$ for each $k \in \mathbb{N}$ such that each convergent subsequence converges to $\sum_{i\in S} \delta_i$.

Let $(\varepsilon^m)_{m\in\mathbb{N}}$ be a strictly decreasing sequence such that $\varepsilon^m > 0$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} \varepsilon^m = 0$. Hence, $(\mathcal{O}^{\varepsilon^m})_{m\in\mathbb{N}}$ is a decreasing sequence in the sense that $\mathcal{O}^{\varepsilon^m} \subset \mathcal{O}^{\varepsilon^{m'}}$ if $m > m'$. Define $\mathcal{O}^0 = \cap_{\varepsilon>0} \mathcal{O}^\varepsilon$. From $(\bar{d},\bar{r}) \in PO_S^*((d,r))$ it follows that $(d_i + r_i \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^0$. Hence, $(d_i + r_i \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^\varepsilon$ for all $\varepsilon > 0$. Since (d^k,r^k) converges to (d,r) there exists $K^1 \in \mathbb{N}$ such that for all $k > K^1$ it holds that $(d_i^k + r_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^{\varepsilon^1}$. Next, take $k \in \mathbb{N}$. If $k \leq K^1$ then take $\sum_{i\in S} \delta_i^k \in \overline{E}_S((d^k,r^k))$ arbitrary. If $k > K^1$ we distinguish the following two cases.

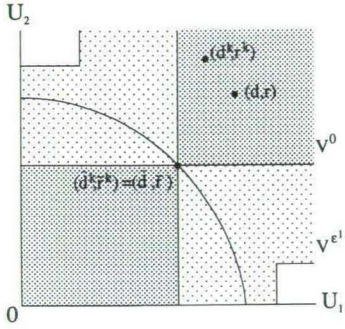


Figure A.3

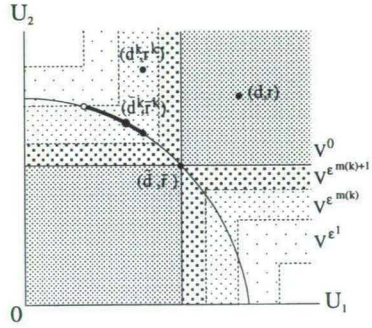


Figure A.4

In the first case, suppose that $(d_i^k + r_i^k \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^0$. Then $(\bar{d}, \bar{r}) \in PO_S^*((d^k, r^k))$ and $(\bar{d}_i + \bar{r}_i \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^0$. So, we can take $(\bar{d}^k, \bar{r}^k)$ equal to $(\bar{d}, \bar{r})$ (See Figure A.3).

In the second case, let $(d_i^k + r_i^k \boldsymbol{X}_N)_{i\in S} \notin \mathcal{O}^0$. Then there exists $m(k) \in \mathbb{N}$ such that $(d_i^k + r_i^k \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(k)}} \backslash \mathcal{O}^{\varepsilon^{m(k)+1}}$. Subsequently, take $(\bar{d}^k, \bar{r}^k) \in PO_S^*((d^k, r^k))$ such that $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(k)}}$ (See Figure A.4, where the bold printed curve represents the set of allocations that belong to both $PO_S^*((d^k, r^k))$ and $\mathcal{O}^{\varepsilon^{m(k)}}$). That such $(\bar{d}^k, \bar{r}^k)$ exists can be seen as follows. Let $(d', r') \in I_\Gamma(S)$ be such that $d_i' + r_i' \boldsymbol{X}_S \sim_i d_i^k + r_i^k \boldsymbol{X}_N$ for all $i \in S$. Then either $(d', r') \in PD_\Gamma(S)$ or $(d', r') \in NPD_\Gamma(S)$.

For the case that $(d', r') \in PD_\Gamma(S)$ then $(d_i^k + r_i^k \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(k)}}$ implies that $(d_i' + r_i' \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(k)}}$ and, consequently, that

$$d_i' + r_i' \boldsymbol{X}_S \succsim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S - \varepsilon^{m(k)}$$

for all $i \in S$ or

$$d_i' + r_i' \boldsymbol{X}_S \precsim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S + \varepsilon^{m(k)}$$

for all $i \in S$. If the first statement is true then it follows from Proposition 5.3 that there exists $(\bar{d}^k, \bar{r}^k) \in PO_\Gamma(S)$ such that $\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S \succsim_i d_i' + r_i' \boldsymbol{X}_S$ for all $i \in S$. This implies that $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(k)}}$ and $(\bar{d}^k, \bar{r}^k) \in PO_S^*((d^k, r^k))$. If the second statement is true then it follows from $(d', r') \in PD_\Gamma(S)$ and Proposition 5.3 that there exists $(\bar{d}^k, \bar{r}^k) \in PO_\Gamma(S)$ satisfying

$$d_i^k + r_i^k \boldsymbol{X}_N \sim_i d_i' + r_i' \boldsymbol{X}_S \precsim_i \bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S \precsim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S + \varepsilon^{m(k)}$$

for all $i \in S$. Hence, $(\bar{d}^k, \bar{r}^k) \in PO_S^*((d^k, r^k))$ and $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(k)}}$.

For the case that $(d', r') \in NPD_\Gamma(S)$ a similar argument holds.

Next, let δ_i^k be such that $\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S \sim_i d_i^k + r_i^k \boldsymbol{X}_N + \delta_i^k$ for all $i \in S$. Note that for $k > K^1$ we have $\sum_{i\in S} \delta_i^k \in \overline{E}_S((d^k, r^k))$ and $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^0$ if $(d_i^k + r_i^k \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^0$

and $(\bar{d}_i^k + \bar{r}_i^k \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(k)}}$ if $(d_i^k + r_i^k \boldsymbol{X}_N)_{i\in S} \notin \mathcal{O}^0$. Since $(\sum_{i\in S} \delta_i^k)_{k\in\mathbb{N}}$ is a sequence in the compact set $\overline{E}_S(IR_\Gamma(S))$ there exists a convergent subsequence $(\sum_{i\in S} \delta_i^l)_{l\in\mathbb{N}}$ with limit $\sum_{i\in S} \tilde{\delta}_i$. Corresponding to this convergent subsequence there is a sequence $(\bar{d}_i^l + \bar{r}_i^l \boldsymbol{X}_S)_{l\in\mathbb{N}}$ such that $(\bar{d}_i^l + \bar{r}_i^l \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^0$ if $(d_i^l + r_i^l \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^0$ and $(\bar{d}_i^l + \bar{r}_i^l \boldsymbol{X}_S)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(l)}}$ if $(d_i^l + r_i^l \boldsymbol{X}_N)_{i\in S} \notin \mathcal{O}^0$. Moreover, it holds that $\bar{d}_i^l + \bar{r}_i^l \boldsymbol{X}_S \sim_i d_i^l + r_i^l \boldsymbol{X}_N + \delta_i^l$ for all $i \in S$. This implies that $(d_i^l + r_i^l \boldsymbol{X}_N + \delta_i^l)_{i\in S} \in \mathcal{O}^0$ if $(d_i^l + r_i^l \boldsymbol{X}_N)_{i\in S} \in \mathcal{O}^0$ and $(d_i^l + r_i^l \boldsymbol{X}_N + \delta_i^l)_{i\in S} \in \mathcal{O}^{\varepsilon^{m(l)}}$ if $(d_i^l + r_i^l \boldsymbol{X}_N)_{i\in S} \notin \mathcal{O}^0$. From $\mathcal{O}^0 = \cap_{l\in\mathbb{N}} \mathcal{O}^{\varepsilon^{m(l)}}$ it follows that

$$\lim_{l\to\infty} (d_i^l + r_i^l \boldsymbol{X}_N + \delta_i^l)_{i\in S} = (d_i + r_i \boldsymbol{X}_N + \tilde{\delta}_i)_{i\in S} \in \mathcal{O}^0.$$

This implies that $d_i + r_i \boldsymbol{X}_N + \tilde{\delta}_i \sim_i \bar{d}_i + \bar{r}_i \boldsymbol{X}_S$ for all $i \in S$. Since $\bar{d}_i + \bar{r}_i \boldsymbol{X}_S \sim_i d_i + r_i \boldsymbol{X}_N + \delta_i$ for all $i \in S$ assumption (C2) implies that $\tilde{\delta}_i = \delta_i$ for all $i \in S$. Consequently, it holds that

$$\lim_{l\to\infty} \sum_{i\in S} \delta_i^l = \sum_{i\in S} \tilde{\delta}_i = \sum_{i\in S} \delta_i.$$

So, each convergent subsequence $(\sum_{i\in S} \delta_i^l)_{l\in\mathbb{N}}$ converges to $\sum_{i\in S} \delta_i$. Hence, $(\sum_{i\in S} \delta_i^k)_{k\in\mathbb{N}}$ converges to $\sum_{i\in S} \delta_i$, which completes the proof. ∎

Lemma A.5 The excess function $E_S((d,r))$ is continuous in (d,r) for each $(d,r) \in IR_\Gamma(S)$.

PROOF: Let $((d^k, r^k))_{k\in\mathbb{N}}$ be a sequence in $IR_\Gamma(N)$ converging to $(d,r) \in IR_\Gamma(N)$. We have to show that $\lim_{k\to\infty} E_S((d^k, r^k)) = E_S((d,r))$. Since $(E_S((d^k, r^k)))_{k\in\mathbb{N}}$ is a sequence in the compact set $\overline{E}_S(IR_\Gamma(N))$ there exists a convergent subsequence $(E_S((d^l, r^l)))_{l\in\mathbb{N}}$ with limit η. Note that the upper semi continuity of $\overline{E}_S$ implies that $\eta \in \overline{E}_S((d,r))$. Since $\overline{E}_S$ is lower semi continuous there exists a sequence $(\sum_{i\in S} \delta_i^l)_{l\in\mathbb{N}}$ such that $\sum_{i\in S} \delta_i^l \in \overline{E}_S((d^l, r^l))$ for all $l \in \mathbb{N}$ and $\lim_{l\to\infty} \sum_{i\in S} \delta_i^l = E_S((d,r))$. Then

$$\begin{aligned} \lim_{l\to\infty} E_S((d^l, r^l)) &\leq \lim_{l\to\infty} \textstyle\sum_{i\in S} \delta_i^l &&= E_S((d,r)) \\ &\leq \eta &&= \lim_{l\to\infty} E_S((d^l, r^l)). \end{aligned}$$

Hence, $\lim_{l\to\infty} \sum_{i\in S} E_S((d^l, r^l)) = E_S((d,r))$. Thus, every convergent subsequence of $(E_S((d^k, r^k)))_{k\in\mathbb{N}}$ converges to $E_S((d,r))$. The compactness of $\overline{E}_S(IR_\Gamma(N))$ then implies that $\lim_{k\to\infty} E_S((d^k, r^k)) = E_S((d,r))$. ∎

A.2 Insurance Games

Lemma A.6 Let

$$f(x) = \left(1 - \frac{1}{x+c}\right)^{x+c}$$

for $x > 1$ and $c \geq 0$. Then f is a non decreasing function in x.

PROOF: The result follows from

$$\begin{aligned}
\frac{df(x)}{dx} &= \left(1-\frac{1}{x+c}\right)^{x+c}\left(\frac{1}{x+c-1}+\log\left(1-\frac{1}{x+c}\right)\right)\\
&= \left(1-\frac{1}{x+c}\right)^{x+c}\left(\frac{1}{x+c-1}+\log\left(\frac{x+c-1}{x+c}\right)\right)\\
&\geq \left(1-\frac{1}{x+c}\right)^{x+c}\left(\frac{1}{x+c-1}+1-\frac{x+c}{x+c-1}\right)\\
&= \left(1-\frac{1}{x+c}\right)^{x+c}\left(\frac{1}{x+c-1}-\frac{1}{x+c-1}\right) = 0,
\end{aligned}$$

where the inequality follows from $\log(x) \geq 1 - \frac{1}{x}$ for $x > 1$.

Lemma A.7 Let

$$f(x) = \left(\frac{x}{x-1}\right)^{x}\left(1-\frac{1}{x+c}\right)^{x+c}$$

for $x > 1$ and $c \geq 0$. Then f is a non increasing function with $f(x) \geq 1$ for all $x > 1$.

PROOF: Since $\lim_{x\to\infty} f(x) = e^{-1}e^{1} = 1$ it is sufficient to prove that f is non increasing in x. This follows from

$$\begin{aligned}
\frac{df(x)}{dx} &= \left(\frac{x}{x-1}\right)^{x}\left(1-\frac{1}{x+c}\right)^{x+c} *\\
&\quad \left(\frac{-1}{x-1}+\log\left(1+\frac{1}{x-1}\right)+\frac{1}{x+c-1}+\log\left(1-\frac{1}{x+c}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{-1}{x-1}+\log\left(1+\frac{1}{x-1}\right)+\frac{1}{x+c-1}+\log\left(1-\frac{1}{x+c}\right) =\\
&= \frac{-1}{x-1}+\log\left(1+\frac{1}{x-1}\right)+\frac{1}{x+c-1}+\log\left(\frac{x+c-1}{x+c}\right)\\
&= \frac{x-1-(x+c-1)}{(x-1)(x+c-1)}+\log\left(\frac{x(x+c-1)}{(x-1)(x+c)}\right)\\
&= \frac{-c}{(x-1)(x+c-1)}+\log\left(\frac{x^2+cx-x}{x^2+cx-x-c}\right)\\
&= \frac{-c}{(x-1)(x+c-1)}+\log\left(1+\frac{c}{x^2+cx-x-c}\right)\\
&\leq \frac{-c}{(x-1)(x+c-1)}+\frac{c}{x^2+cx-x-c}\\
&= \frac{-c}{(x-1)(x+c-1)}+\frac{c}{(x-1)(x+c)} \leq 0,
\end{aligned}$$

where the first inequality follows from $\log(1+x) \leq x$ and the second inequality follows from $x > 1$ and $c \geq 0$.

Lemma A.8 Let $\Gamma \in IG(N_I, N_P)$. Let $S \subset N$ and $(d_i + R_i \boldsymbol{X}^S)_{i \in S} \in \mathcal{Z}_\Gamma(S)$. Then for all $i \in S$ it holds that

$$m_i(d_i + R_i \boldsymbol{X}^S) = \begin{cases} d_i + \sum_{k \in K} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k} \alpha_i r_{ii} f_{ik}\right), & \text{if } i \in S_P, \\ d_i + \sum_{j \in S} \sum_{k \in K} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k} \alpha_i r_{ij} f_{jk}\right), & \text{if } i \in S_I. \end{cases}$$

PROOF: Let $i \in S$. Then

$$\begin{aligned} m_i(d_i + R_i \boldsymbol{X}^S) &= U_i^{-1}\left(E(U_i(d_i - \sum_{j \in S} r_{ij} \boldsymbol{X}_j))\right) \\ &= -\tfrac{1}{\alpha_i} \log\left(\tfrac{1}{\beta_i} E(\beta_i e^{-\alpha_i (d_i - \sum_{j \in S} r_{ij} \boldsymbol{X}_j)})\right) \\ &= -\tfrac{1}{\alpha_i} \log\left(E\left(e^{-\alpha_i d_i} e^{\alpha_i \sum_{j \in S} \sum_{k \in K} r_{ij} f_{jk} \boldsymbol{Y}_k}\right)\right) \\ &= -\tfrac{1}{\alpha_i} \log\left(e^{-\alpha_i d_i} \prod_{j \in S} \prod_{k \in K} E(e^{\alpha_i r_{ij} f_{jk} \boldsymbol{Y}_k})\right) \\ &= -\tfrac{1}{\alpha_i} \log(e^{-\alpha_i d_i}) - \tfrac{1}{\alpha_i} \sum_{j \in S} \sum_{k \in K} \log\left(E(e^{\alpha_i r_{ij} f_{jk} \boldsymbol{Y}_k})\right) \\ &= d_i - \tfrac{1}{\alpha_i} \sum_{j \in S} \sum_{k \in K} \log\left(E(e^{\alpha_i r_{ij} f_{jk} \boldsymbol{Y}_k})\right) \\ &= d_i - \tfrac{1}{\alpha_i} \sum_{j \in S} \sum_{k \in K} \log\left(\int_0^\infty \mu_k e^{-t(\mu_k - \alpha_i r_{ij} f_{jk})} dt\right), \end{aligned}$$

where the fourth equality follows from the independence of the random losses $\boldsymbol{Y}_k$, $(k \in K)$. Since we implicitly assumed that the expected utility exists, we must have that $\mu_k - \alpha_i r_{ij} f_{jk} > 0$ for all $j \in S$ and all $k \in K$. Then

$$\begin{aligned} m_i(d_i + R_i \boldsymbol{X}^S) &= d_i - \tfrac{1}{\alpha_i} \sum_{j \in S} \sum_{k \in K} \log\left(\tfrac{\mu_k}{\mu_k - \alpha_i r_{ij} f_{jk}}\right) \\ &= d_i + \sum_{j \in S} \sum_{k \in K} \tfrac{-1}{\alpha_i} \log\left(\frac{1}{1 - \frac{1}{\mu_k} \alpha_i r_{ij} f_{jk}}\right) \\ &= d_i + \sum_{j \in S} \sum_{k \in K} \tfrac{1}{\alpha_i} \log\left(1 - \tfrac{1}{\mu_k} \alpha_i r_{ij} f_{jk}\right). \end{aligned}$$

Using $r_{ij} = 0$ for all $i \in S_P$ and all $j \in S$ with $i \neq j$ gives the desired result.

B Probability Theory

This appendix provides the reader with a brief introduction into probability theory. We will confine with explaining those terms that are needed to understand this monograph. Furthermore, we will refrain from any mathematical details on this subject. For an extensive discussion on measure theory and probability theory in particular we refer to BURRILL (1972) or FELLER (1950), (1966).

A stochastic variable describes a situation or experiment whose outcome is determined by chance. For instance, the number of points one obtains when throwing a die can be described by a stochastic variable. Before giving a general definition of a stochastic variable, let us elaborate on this example.

Example B.1 When throwing a fair die, the outcome can be any integer between 1 and 6. Let us denote these outcomes by the set Ω. So, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Obviously, one does not know the outcome of this experiment beforehand. What one does know, however, is the probability with which each outcome occurs. For instance, the chance that the experiment results in three points is $\frac{1}{6}$. In other words, the event that the die ends up with three dots faced upward occurs with probability $\frac{1}{6}$. This, of course, is only one particular event. We can think of many other events, for instance, the event that the number of points is even, or that the number of points is less than 4. Such events can be described by subsets of outcomes. For example, $E = \{2, 4, 6\} \subset \Omega$ describes the event that the outcome is an even number of points. Now, let $\mathcal{H}$ denote the set of all possible events. Thus,

$$\mathcal{H} = \{E | \, E \subset \Omega\}.$$

For each event $E \in \mathcal{H}$ we know the probability that this event occurs. We already noted that the event $E = \{3\}$ occurs with probability $\frac{1}{6}$. The probabilities with which these events occur

are described by a function $\mathbf{P} : \mathcal{H} \to [0,1]$. For this case this implies that $\mathbf{P}(\{\omega\}) = \frac{1}{6}$ for all $\omega \in \Omega$. Furthermore, we have that

$$\mathbf{P}(\text{the outcome is even}) \;=\; \mathbf{P}(\{2,4,6\}) \;=\; \tfrac{1}{2},$$

and

$$\mathbf{P}(\text{the outcome is less than } 4) \;=\; \mathbf{P}(\{1,2,3\}) \;=\; \tfrac{1}{2}.$$

Summarizing, the experiment of throwing a die is described by the triple $(\Omega, \mathcal{H}, \mathbf{P})$.

The triple $(\Omega, \mathcal{H}, \mathbf{P})$ in Example B.1 is called a probability space. In general, each chance experiment is described by a *probability space* $(\Omega, \mathcal{H}, \mathbf{P})$, where Ω is the outcome space, $\mathcal{H}$ is a σ-algebra on Ω, and $\mathbf{P}$ is a probability measure on the σ-algebra $\mathcal{H}$. The terms σ-algebra and probability measure need some further explanation. Let $\mathcal{H}$ be a set of events. Then $\mathcal{H}$ is called a *σ-algebra* on Ω if the following three conditions are satisfied:

(i) $\emptyset \in \mathcal{H}$,

(ii) if $E \in \mathcal{H}$ then $\Omega \backslash E \in \mathcal{H}$,

(iii) if $E_k \in \mathcal{H}$ for $k = 1, 2, \ldots$, then $\cup_{k=1}^{\infty} E_k \in \mathcal{H}$.

Example B.2 Let $\Omega = \{1,2,3,4\}$. An example of a σ-algebra on Ω is

$$\mathcal{H} = \{\emptyset, \{2\}, \{3\}, \{2,3\}, \{1,4\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3,4\}\}\,.$$

Example B.3 Let $\Omega = \mathbb{R}$. The smallest σ-algebra on $\mathbb{R}$ is $\mathcal{H}_1 = \{\emptyset, \mathbb{R}\}$. A σ-algebra that contains $\mathcal{H}_1$ is the σ- algebra $\mathcal{H}_2$ given by $\mathcal{H}_2 = \{\emptyset, A, \mathbb{R}\backslash A, \mathbb{R}\}$, with A any subset of $\mathbb{R}$. The largest σ-algebra on $\mathbb{R}$ one can construct consists of all subsets of $\mathbb{R}$, that is, $\mathcal{H}_3 = \{A \mid A \subset \mathbb{R}\}$. Note that $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3$.

A *probability measure* $\mathbf{P}$ is a function assigning to each event $E \in \mathcal{H}$ a nonnegative number such that

(i) $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}(\Omega) = 1$,

(ii) for any mutually disjoint events E_k, $k = 1, 2, \ldots$, it holds that $\mathbf{P}(\cup_{k=1}^{\infty} E_k) \;=\; \sum_{k=1}^{\infty} \mathbf{P}(E_k)$.

Condition (ii) implies that for any disjoint events E_1 and E_2, the probability that either of the two occurs equals the sum of the separate probabilities. For instance, consider in Example B.1 the events $\{3\}$ and $\{4\}$. The probability that one of the two events occurs equals $\mathbf{P}(\{3\}\cup\{4\}) = \mathbf{P}(\{3\}) + \mathbf{P}(\{4\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Note that the events must be disjoint. For if we consider the events $\{3\}$ and $\{3,4\}$ then

$$\tfrac{1}{3} = \mathbf{P}(\{3,4\}) = \mathbf{P}(\{3\} \cup \{3,4\}) \neq \mathbf{P}(\{3\}) + \mathbf{P}(\{3,4\}) = \tfrac{1}{6} + \tfrac{1}{3} = \tfrac{1}{2}.$$

Let $(\Omega, \mathcal{H}, \mathbf{P})$ be a probability space and let $E_1, E_2 \in \Omega$. The events E_1 and E_2 are called *independent* if

$$\mathbf{P}(E_1 \cap E_2) \;=\; \mathbf{P}(E_1)\mathbf{P}(E_2). \tag{B.1}$$

Independence implies that occurrence of the event E_1 gives no additional information on the occurrence of the event E_2, and the other way around.

Example B.4 Consider the experiment described in Example B.1. The events $E_1 = \{2\}$ and $E_2 = \{2,4,6\}$ are not independent because $\frac{1}{6} = \mathbf{P}(E_1 \cap E_2) \neq \mathbf{P}(E_1)\mathbf{P}(E_2) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. Indeed, if one knows that the event E_1 has occurred, then the outcome is even. Hence, one knows with certainty that the event E_2 has also occurred. Furthermore, if one knows that E_2 has occurred, that is, the outcome is even, then the probability that event E_1 occurs equals $\frac{1}{3}$ instead of $\frac{1}{6}$.

Now that we are familiar with probability spaces, we can introduce stochastic variables. For this purpose we need to introduce the Borel σ-algebra. The *Borel σ-algebra* $\mathcal{B}$ is the smallest σ-algebra on $\mathbb{R}$ containing all half open intervals $(a, b]$ with $-\infty < a < b < \infty$. Furthermore, an element $B \in \mathcal{B}$ is called a Borel-set. Exactly which sets the Borel σ-algebra contains is difficult to say. Besides all the half open intervals, it also contains all singletons $\{a\}$, $a \in \mathbb{R}$. Other examples of Borel sets are $(-\infty, b]$ with $b \in \mathbb{R}$, $[a, \infty)$ with $a \in \mathbb{R}$, and $[a, b]$ with $-\infty < a \leq b < \infty$.

A *stochastic variable* $\boldsymbol{X}$ is a measurable function assigning to each outcome $\omega \in \Omega$ a real number $\boldsymbol{X}(\omega)$. A function $\boldsymbol{X}$ is *measurable* with respect to $\mathcal{H}$ if for all $B \in \mathcal{B}$ it holds that $\{\omega \in \Omega | \boldsymbol{X}(\omega) \in B\} \in \mathcal{H}$. Measurability thus means that all outcomes ω that map into a Borel set B constitute an event $E \in \mathcal{H}$. Consequently, we can determine the probability that $\boldsymbol{X} \in B$ for every $B \in \mathcal{B}$. For ease of notation, the event $\{\omega \in \Omega | \boldsymbol{X}(\omega) \in B\}$ is often abbreviated to $\{\boldsymbol{X} \in B\}$.

Example B.5 Let us return to the experiment presented in Example B.1. The stochastic variable $\boldsymbol{X}$ defined by $\boldsymbol{X}(\omega) = \omega$ for all $\omega \in \Omega$ describes the experiment of throwing a die. So does the stochastic variable $\boldsymbol{Y}$ with $\boldsymbol{Y}(\omega) = 7 - \omega$ for all $\omega \in \Omega$. The difference, however, is in the interpretation of the realizations. If $x = 5$ is a realization of $\boldsymbol{X}$ this means that the outcome is five, but if $y = 5$ is a realization of $\boldsymbol{Y}$ then the outcome of the experiment is only two.

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two stochastic variables on the outcome set Ω. Then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are *independent* if for any two Borel sets $B_1, B_2 \in \mathcal{B}$ it holds that

$$\mathbb{P}((\boldsymbol{X}, \boldsymbol{Y}) \in B_1 \times B_2) = \mathbb{P}(\boldsymbol{X} \in B_1)\mathbb{P}(\boldsymbol{Y} \in B_2). \tag{B.2}$$

Independence of stochastic variables implies that the outcome of one stochastic variable provides no additional information on the outcome of another one.

A stochastic variable $\boldsymbol{X}$ is called *nonnegative* if $\boldsymbol{X}(\omega) \geq 0$ for all $\omega \in \Omega$ and is denoted by $\boldsymbol{X} \geq 0$. Similarly, we have that $\boldsymbol{X} \geq \boldsymbol{Y}$ if $\boldsymbol{X}(\omega) \geq \boldsymbol{Y}(\omega)$ for all $\omega \in \Omega$. Furthermore, if $\alpha, \beta \in \mathbb{R}$ we define the stochastic variable $\boldsymbol{Z} = \alpha\boldsymbol{X} + \beta\boldsymbol{Y}$ by $\boldsymbol{Z}(\omega) = \alpha\boldsymbol{X}(\omega) + \beta\boldsymbol{Y}(\omega)$ for all $\omega \in \Omega$.

Corresponding to each stochastic variable $\boldsymbol{X}$ we define the *probability distribution function* $F_{\boldsymbol{X}} : \mathbb{R} \to [0, 1]$ by

$$F_{\boldsymbol{X}}(t) = \mathbb{P}(\{\boldsymbol{X} \leq t\}) = \mathbb{P}(\{\omega \in \Omega | \ \boldsymbol{X}(\omega) \leq t\}), \tag{B.3}$$

for all $t \in \mathbb{R}$. So, $F_{\boldsymbol{X}}(t)$ denotes the probability that the value of $\boldsymbol{X}$ is less than or equal to t. Note that for every probability disitribution function $F_{\boldsymbol{X}}$ it holds that $\lim_{t\to-\infty} F_{\boldsymbol{X}}(t) = 0$, $\lim_{t\to\infty} F_{\boldsymbol{X}}(t) = 1$, and that $F_{\boldsymbol{X}}$ is continuous from the right. Furthermore, probability distribution functions are (weakly) increasing and therefore discontinuous in at most countably many points.

Example B.6 Consider the stochastic variable $\boldsymbol{X}$ defined in Example B.5. The probability distribution function of $\boldsymbol{X}$ is illustrated in Figure B.1.

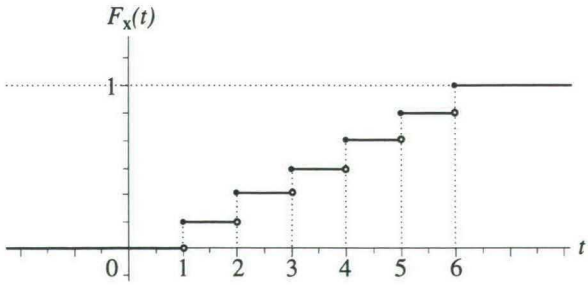


Figure B.1

A stochastic variable is uniquely determined by its probability distribution function. By this we mean the following. Given two probability spaces $(\Omega_1, \mathcal{H}_1, \mathbf{P}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbf{P}_2)$ and two stochastic variables $\boldsymbol{X} : \Omega_1 \to \mathbb{R}$ and $\boldsymbol{Y} : \Omega_2 \to \mathbb{R}$, we have that

$$\mathbf{P}_1(\{\omega \in \Omega_1 |\, \boldsymbol{X}(\omega) \in B\}) \;=\; \mathbf{P}_2(\{\omega \in \Omega_2 |\, \boldsymbol{Y}(\omega) \in B\})$$

for all $B \subset \mathcal{B}$ if and only if

$$F_{\boldsymbol{X}}(t) \;=\; F_{\boldsymbol{Y}}(t)$$

for all $t \in \mathbb{R}$. So, when working with stochastic variables, it suffices to know the distribution function; as a consequence, the underlying probability space is often left unspecified.

A stochastic variable $\boldsymbol{X}$ is called *degenerate* if it attains exactly one value with positive probability. So, there exists $y \in \mathbb{R}$ so that $\mathbf{P}(\{\boldsymbol{X} = x\} = 1$ if $x = y$ and $\mathbf{P}(\{\boldsymbol{X} = x\} = 0$ if $x \neq y$. In particular, we denote by $\mathbf{1}$ the degenerate stochastic variables with value 1. For ease of notation, we will use both x and $x\mathbf{1}$ to denote the degenerate stochastic variable $x\mathbf{1}$.

A stochastic variable $\boldsymbol{X}$ is called *discrete* if it can attain only countably many values. This means that there exists numbers $x_k \in \mathbb{R}$ and $p_k \in [0,1]$, $k = 1, 2, \ldots$, such that $\mathbf{P}(\{\boldsymbol{X} = x_k\}) = p_k$ for $k = 1, 2, \ldots$, and $\sum_{k=1}^{\infty} p_k = 1$. Note that for any Borel-set $B \in \mathcal{B}$ it holds that

$$\mathbf{P}(\{\boldsymbol{X} \in B\}) \;=\; \sum_{k \in \mathbb{N} : x_k \in B} p_k.$$

The stochastic variable $\boldsymbol{X}$, introduced in Example B.5 to describe the outcome of throwing a die, is an example of a discrete stochastic variable. $\boldsymbol{X}$ only attains six different values, namely, $x_k = k$, $k = 1, 2, \ldots, 6$. Furthermore, we have that $p_k = \frac{1}{6}$ for $k = 1, 2, \ldots, 6$.

A stochastic variable $\boldsymbol{X}$ is called *continuous* if there exists a continuous function $f_{\boldsymbol{X}} : \mathbb{R} \to \mathbb{R}$ such that

$$F_{\boldsymbol{X}}(t) \;=\; \int_{\infty}^{t} f_{\boldsymbol{X}}(t)dt, \tag{B.4}$$

for all $t \in \mathbb{R}$. The function $f_{\boldsymbol{X}}$ is called the *density function* of the stochastic variable $\boldsymbol{X}$. For any interval $[a, b]$ it then holds that

$$\mathbf{P}(\{\boldsymbol{X} \in [a, b]\}) \;=\; \int_a^b f_{\boldsymbol{X}}(t)dt.$$

Note that in the expression above we can replace the interval $[a, b]$ by the intervals $(a, b]$, $[a, b)$, and (a, b). Examples of continuous stochastic variables are provided at the end of this appendix.

When throwing a die for numerous times in a row, the average of all the outcomes will be around $3\frac{1}{2}$, that is, the average of the numbers 1 through 6. Therefore, we say that the

expectation of the outcome equals $3\frac{1}{2}$. If $\boldsymbol{X}$ is a discrete stochastic variable, the *expectation* of $\boldsymbol{X}$ is given by

$$E(\boldsymbol{X}) = \sum_{k=1}^{\infty} p_k x_k.$$

If $\boldsymbol{X}$ is continuous with density function $f_{\boldsymbol{X}}$, the *expectation* is given by

$$E(\boldsymbol{X}) = \int_{-\infty}^{\infty} t f_{\boldsymbol{X}}(t) dt.$$

Example B.7 Consider the following two lotteries $\boldsymbol{X}$ and $\boldsymbol{Y}$ given by

$$\boldsymbol{X} = \begin{cases} 0, & \text{with probability } \frac{1}{2} \\ 20,000, & \text{with probability } \frac{1}{2} \end{cases}$$

and

$$\boldsymbol{Y} = \begin{cases} -10,000, & \text{with probability } \frac{9}{11} \\ 100,000, & \text{with probability } \frac{2}{11} \end{cases}$$

So when playing the lottery $\boldsymbol{X}$, there is a 50% chance of winning \$ 20,000. But when playing the lottery $\boldsymbol{Y}$ there is about 18% change of winning no less than \$ 100,000. On the darker side though, there is also about 82% chance that one has to pay \$ 10,000. Both lotteries, however, have the same expected value of \$ 10,000.

Although both lotteries defined in Example B.7 yield the same expected revenue, most people will prefer the lottery $\boldsymbol{X}$ to the lottery $\boldsymbol{Y}$. The risk of paying \$ 10,000 in the second lottery is too high compared to \$ 100,000 one can win. So, when choosing between two different lotteries, not only the expected value plays a role, but also the variation in the different outcomes. A measure for the variation in the outcomes of a stochastic variable $\boldsymbol{X}$ is the *variance* $V(\boldsymbol{X})$. For discrete random variables $\boldsymbol{X}$ the variance is defined as

$$V(\boldsymbol{X}) = \sum_{k=1}^{\infty} p_k (x_k - E(\boldsymbol{X}))^2,$$

while for continuous stochastic variables with density function $f_{\boldsymbol{X}}$ it is defined as

$$V(\boldsymbol{X}) = \int_{-\infty}^{\infty} (t - E(\boldsymbol{X}))^2 f_{\boldsymbol{X}}(t) dt.$$

For the lotteries defined in Example B.7 we thus have that

$$V(\boldsymbol{X}) = \tfrac{1}{2}(10,000)^2 + \tfrac{1}{2}(10,000)^2 = 1 \cdot 10^8,$$

and

$$V(\boldsymbol{Y}) = \tfrac{9}{11}(-20,000)^2 + \tfrac{2}{11}(90,000)^2 = 17 \cdot 10^8.$$

As the variance shows, the outcomes for the lottery $\boldsymbol{Y}$ vary much more than for the lotter $\boldsymbol{X}$.

Other characteristic values of stochastic variables are quantiles. For example, the 10%-quantile of a stochastic variable $\boldsymbol{X}$ is the largest value ξ such that the outcome of $\boldsymbol{X}$ will be less than ξ with at most 10% chance. Of course, the quantile can be determined for any percentage between 0 and 100. Formally, the *α-quantile* of a stochastic variable $\boldsymbol{X}$ is defined by

$$\xi_\alpha(\boldsymbol{X}) = \sup\{t|\ \mathbf{P}(\boldsymbol{X} < t) \leq \alpha\}, \tag{B.5}$$

where $\alpha \in (0,1)$. In particular, the 0.5-quantile is called the *median* of $\boldsymbol{X}$. Figure B.2 shows how quantiles can be derived from the graph of the probability distribution function.

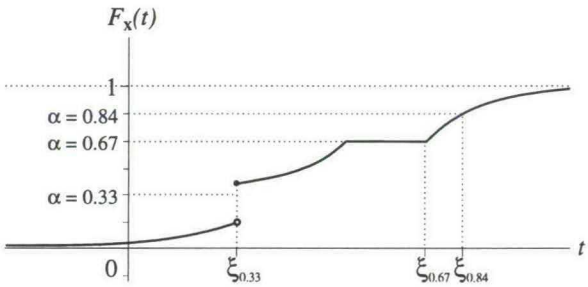


Figure B.2

Let $L^1(\boldsymbol{R})$ denote the set of all real valued stochastic variables with finite expectation, that is, $E(|\boldsymbol{X}|) < \infty$. Let $F_{\boldsymbol{X}}$ denote the probability distribution function of $\boldsymbol{X} \in L^1(\boldsymbol{R})$. Furthermore, define $\mathcal{F} = \{F_{\boldsymbol{X}}|\boldsymbol{X} \in L^1(\mathbf{R})\}$. Then $(\mathcal{F}, \rho)$ with

$$\rho(F,G) = \int_{-\infty}^{\infty} |F(t) - G(t)|e^{-|t|}dt \tag{B.6}$$

for all $F, G \in \mathcal{F}$ is a metric space. Next, let $(F_k)_{k\in\mathbf{N}}$ be a sequence in $\mathcal{F}$. Then the sequence $(F_k)_{k\in\mathbf{N}}$ *weakly converges* to $F \in \mathcal{F}$, denoted by $F_k \xrightarrow{w} F$, if $\lim_{k\to\infty} F_k(t) = F(t)$ for all $t \in \{t' \in \mathbf{R}|F \text{ is continuous in } t'\}$. Note that since probability distribution functions are continuous from the right, this limit is unique. Moreover, we have that $F_{\boldsymbol{X}_k} \xrightarrow{w} F_{\boldsymbol{X}}$ if and only if $\lim_{k\to\infty} \rho(F_{\boldsymbol{X}_k}, F_{\boldsymbol{X}}) = 0$.

We say that a sequence $(\boldsymbol{X}_k)_{k\in\mathbf{N}}$ of random variables in $L^1(\mathbf{R})$ converges to the random variable $\boldsymbol{X} \in L^1(\mathbf{R})$ if and only if the corresponding sequence $(F_{\boldsymbol{X}_k})_{k\in\mathbf{N}}$ of probability distribution functions weakly converges to the probability distribution function $F_{\boldsymbol{X}}$ of $\boldsymbol{X}$. The following theorem is known in the literature as one of the *Helly theorems*.

Theorem B.1 Let $(\boldsymbol{X}^k)_{k\in\mathbb{N}}$ be a sequence in $L^1(\mathbb{R})$ weakly converging to $\boldsymbol{X} \in L^1(\mathbb{R})$, i.e., $F_{\boldsymbol{X}^k} \xrightarrow{w} F_{\boldsymbol{X}}$. If $g : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function then $\lim_{k\to\infty} E(g(\boldsymbol{X}^k)) = E(g(\boldsymbol{X}))$.

The next two results are of particular interest for this monograph.

Lemma B.2 Let $\boldsymbol{X} \in L^1(\mathbb{R})$ and let $(d^k)_{k\in\mathbb{N}}$ and $(r^k)_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}$ and $[0,1]$, respectively. If $\lim_{k\to\infty} d^k = d$ and $\lim_{k\to\infty} r^k = r$ then $F_{d^k+r^k\boldsymbol{X}} \xrightarrow{w} F_{d+r\boldsymbol{X}}$.

PROOF: We have to show that $\lim_{k\to\infty} F_{d^k+r^k\boldsymbol{X}}(t) = F_{d+r\boldsymbol{X}}(t)$ for all continuity points t of $F_{d+r\boldsymbol{X}}$. Note that $F_{d^k+r^k\boldsymbol{X}}(t) = F_{\boldsymbol{X}}(\frac{t-d^k}{r^k})$ if $r^k \neq 0$. We distinguih two cases, $r = 0$ and $r > 0$.

First, suppose that $r = 0$. Then $F_{d+r\boldsymbol{X}}(t) = 0$ if $t < d$ and $F_{d+r\boldsymbol{X}}(t) = 1$ if $t \geq d$. Take $t > d$ so that t is a continuity point of $F_{d+r\boldsymbol{X}}$. Let $\varepsilon > 0$. Since $d^k \to d$ there exists $K \in \mathbb{N}$ such that $t - d^k > \varepsilon$ for $k \geq K$. Since $r^k \to 0$ we have that $\frac{t-d^k}{r^k} \to \infty$. Hence, $\lim_{k\to\infty} F(\frac{t-d^k}{r^k}) = 1$. Similarly, one can show that $\lim_{k\to\infty} F(\frac{t-d^k}{r^k}) = 0$ if $t < d$.

Next, suppose that $r \neq 0$. Let $t \in \mathbb{R}$ be such that $\frac{t-d}{r}$ is a continuity point of $F_{\boldsymbol{X}}$. Since $\frac{t-d^k}{r^k} \to \frac{t-d}{r}$ it follows that $\lim_{k\to\infty} F(\frac{t-d^k}{r^k}) = F(\frac{t-d}{r})$. ∎

Lemma B.3 Let $\boldsymbol{X} \in L^1(\mathbb{R})$ and let $(d^k)_{k\in\mathbb{N}}$ and $(r^k)_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}$ and $[0,1]$, respectively. If $F_{d^k+r^k\boldsymbol{X}} \xrightarrow{w} G$ then there exist numbers $d \in \mathbb{R}$ and $r \in [0,1]$ such that $F_{d^k+r^k\boldsymbol{X}} \xrightarrow{w} F_{d+r\boldsymbol{X}}$.

PROOF: Since $(r^k)_{k\in\mathbb{N}}$ is a sequence in the compact set $[0,1]$ there exists a convergent subsequence $(r^l)_{l\in\mathbb{N}}$. Let us denote its limit by $r \in [0,1]$. Lemma B.2 then implies that $F_{r^l\boldsymbol{X}} \xrightarrow{w} F_{r\boldsymbol{X}}$. Since $F_{d^k+r^k\boldsymbol{X}} \xrightarrow{w} G$ we also have that $F_{d^l+r^l\boldsymbol{X}} \xrightarrow{w} G$. Next, we show that the sequence $(d^l)_{l\in\mathbb{N}}$ is bounded. This implies that the sequence $(d^l)_{l\in\mathbb{N}}$ has a convergent sequence $(d^m)_{m\in\mathbb{N}}$. If we denote its limit by $d \in \mathbb{R}$ it follows from Lemma B.2 that $F_{d^m+r^m\boldsymbol{X}} \xrightarrow{w} F_{d+r\boldsymbol{X}}$. Since the limit G is unique, it also holds that $F_{d^k+r^k\boldsymbol{X}} \xrightarrow{w} F_{d+r\boldsymbol{X}}$.

So, we are left to prove that $(d^l)_{l\in\mathbb{N}}$ is a bounded sequence. Therefore, suppose that it is not bounded, then there exists a subsequence $(d_m)_{m\in\mathbb{N}}$ with either $d^m \to \infty$ or $d^m \to -\infty$.

Let us start with considering the first possibility, that is, $d^m \to \infty$. Note that without loss of generality we may assume that $d^1 \leq d^2 \leq d^3 \leq \ldots$. Take $t \in \mathbb{R}$ such that t is a continuity point of $F_{r\boldsymbol{X}}$ and $F_{r\boldsymbol{X}}(t) < 1$. Let $\varepsilon > 0$ be such that $F_{r\boldsymbol{X}}(t) + \varepsilon < 1$. Since $F_{r^m\boldsymbol{X}} \xrightarrow{w} F_{r\boldsymbol{X}}$ there exists $M_1 \in \mathbb{N}$ such that $F_{r^m\boldsymbol{X}}(t) < F_{r\boldsymbol{X}}(t) + \varepsilon$ for all $m \geq M_1$.

Let $\tau \in \mathbb{R}$ be a continuity point of G. Since $d^m \to \infty$ there exists $M_2 \geq M_1$ such that $t + d^m > \tau$ for all $m \geq M_2$. From $F_{r^m\boldsymbol{X}}(t) < F_{r\boldsymbol{X}}(t) + \varepsilon$ and $F_{d^m+r^m\boldsymbol{X}}(\tau) = F_{r^m\boldsymbol{X}}(\tau - d^m) < F_{r^m\boldsymbol{X}}(t)$ it then follows that $F_{d^m+r^m\boldsymbol{X}}(\tau) < F_{r\boldsymbol{X}}(t) + \varepsilon$ for all $m \geq M_2$. Hence, $\lim_{m\to\infty} F_{d^m+r^m\boldsymbol{X}}(\tau) = G(\tau)$ implies that $G(\tau) < F_{r\boldsymbol{X}}(t) + \varepsilon$. Since τ is an arbitrary

continuity point of G and $F_{r\boldsymbol{X}}(t) + \varepsilon < 1$ it holds that $\lim_{\tau\to\infty} G(\tau) < 1$. But this contradicts the fact that G is a probability distribution function.

Next, consider the case that $d^m \to -\infty$. Again, we may assume without loss of generality that the sequencing is monotonic, i.e., $d^1 \geq d^2 \geq d^3 \geq \ldots$. Take $t \in \mathbb{R}$ such that t is a continuity point of $F_{r\boldsymbol{X}}$ and $F_{r\boldsymbol{X}}(t) > 0$. Let $\varepsilon > 0$ be such that $F_{r\boldsymbol{X}}(t) - \varepsilon > 0$. Since $F_{r^m\boldsymbol{X}} \xrightarrow{w} F_{r\boldsymbol{X}}$ there exists $M_1 \in \mathbb{N}$ such that $F_{r^m\boldsymbol{X}}(t) > F_{r\boldsymbol{X}}(t) - \varepsilon$ for all $m \geq M_1$.

Let $\tau \in \mathbb{R}$ be a continuity point of G. Since $d^m \to -\infty$ there exists $M_2 \geq M_1$ such that $t + d^m < \tau$ for all $m \geq M_2$. From $F_{r^m\boldsymbol{X}}(t) > F_{r\boldsymbol{X}}(t) - \varepsilon$ and $F_{d^m+r^m\boldsymbol{X}}(\tau) = F_{r^m\boldsymbol{X}}(\tau - d^m) > F_{r^m\boldsymbol{X}}(t)$ it then follows that $F_{d^m+r^m\boldsymbol{X}}(\tau) > F_{r\boldsymbol{X}}(t) - \varepsilon$ for all $m \geq M_2$. Hence, $\lim_{m\to\infty} F_{d^m+r^m\boldsymbol{X}}(\tau) = G(\tau)$ implies that $G(\tau) > F_{r\boldsymbol{X}}(t) - \varepsilon$. Since τ is an arbitrary continuity point of G and $F_{r\boldsymbol{X}}(t) - \varepsilon > 0$ it holds that $\lim_{\tau\to-\infty} G(\tau) > 0$, which contradicts the fact that G is a probability distribution function. ∎

We end this appendix with two types of continuous probability distributions that are frequently used in this monogroaph.

Example B.8 A continuous stochastic variable $\boldsymbol{X}$ is called uniformly distributed on the interval (a, b), denoted by $\boldsymbol{X} \sim U(a, b)$, if

$$F_{\boldsymbol{X}}(t) = \begin{cases} 0, & \text{if } t \leq a \\ \frac{t-a}{b-a}, & \text{if } a < t < b \\ 1, & \text{if } b \leq t. \end{cases} \tag{B.7}$$

A corresponding density function equals $f_{\boldsymbol{X}}(t) = \frac{1}{b-a}$ for $t \in (a, b)$ and zero otherwise. The interpretation of the uniform distribution is that each outcome in the interval (a, b) occurs equally likely.

Example B.9 A continuous stochastic variable $\boldsymbol{X}$ is exponentially distributed with parameter λ on $[0, \infty)$, denoted by $\boldsymbol{X} \sim \text{Exp}(\lambda)$, if

$$F_{\boldsymbol{X}}(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1 - e^{-\lambda t}, & \text{if } t > 0. \end{cases} \tag{B.8}$$

A density function of the exponential distribution equals $f_{\boldsymbol{X}}(t) = \lambda e^{-\lambda t}$ for $t \in (0, \infty)$ and zero otherwise. The exponential distribution is often used to describe the time between two arrivals in a queueing model or to describe the lifetime of technical components like light bulbs.

References

ALEGRE, A., and M. MERCÈ CLARAMUNT: "Allocation of Solvency Cost in Group Annuities: Actuarial Principles and Cooperative Game Theory," *Insurance: Mathematics and Economics*, **17** (1995), 19-34.

ANAND, L., N. SHASHISHEKHAR, D. GHOSE, and U. PRASAD: "A Survey of Solution Concepts in Multicriteria Games," *Journal of the Indian Institute of Science*, **75** (1995), 141-174.

AUMANN, R.: "Linearity of Unrestricted Transferable Utilities," *Naval Research Logistics Quarterly*, **7** (1960), 281-284.

AUMANN, R., and B. PELEG: "Von Neumann-Morgenstern Solutions to Cooperative Games without Sidepayments," *Bulletin of the American Mathematical Society*, **66** (1960), 173-179.

BERGSTRESSER, K., and P. YU: "Domination Structures and Multi-Criteria Problems in n-Person Games," *Theory and Decision*, **8** (1977), 5-48.

BONDAREVA, O.: "Some Applications of Linear Programming Methods to the Theory of Cooperative Games (in Russian)," *Problemi Kibernet*, **10** (1963), 119-139.

BORCH, K.: "Equilibrium in a Reinsurance Market," *Econometrica*, **30** (1962a), 424-444.

BORCH, K.: "Application of Game Theory to Some Problems in Automobile Insurance," *Astin Bulletin*, **2** (1962b), 208-221.

BÜHLMANN, H.: "An Economic Premium Principle," *Astin Bulletin*, **11** (1980), 52-60.

BÜHLMANN, H.: "The General Economic Premium Principle," *Astin Bulletin*,**14** (1984), 13-21.

BURRILL, C.: *Measure, Integration, and Probability*. New York: McGraw-Hill, 1972.

CHARNES, A., and D. GRANOT: "Prior Solutions: Extensions of Convex Nucleolus Solutions to Chance-Constrained Games," *Proceedings of the Computer Science and Statistics Seventh Symposium at Iowa State University* (1973), 323-332.

CHARNES, A., and D. GRANOT: "Coalitional and Chance-Constrained Solutions to N-Person Games I," *SIAM Journal on Applied Mathematics*, **31** (1976), 358-367.

CHARNES, A., and D. GRANOT: "Coalitional and Chance-Constrained Solutions to N-Person Games II," *Operations Research*, **25** (1977), 1013-1019.

CURIEL, I., G. PEDERZOLI, and S. TIJS: "Sequencing Games," *European Journal of Operational Research*, **40** (1989), 344-351.

DEBREU, G.: *Theory of Value*, New Haven: Yale University Press, 1959.

EECKHOUDT, L., and C. GOLLIER: *Risk: Evaluation, Management, and Sharing*, New York: Harvester Wheatsheaf, 1995.

FELLER, W.: *An Introduction to Probability Theory and its Applications*, Vol. I. New York: Wiley, 1950.

FELLER, W.: *An Introduction to Probability Theory and its Applications*, Vol. II. New York: Wiley, 1966.

GOOVAERTS, M., F. DE VYLDER, and J. HAEZENDONCK: *Insurance Premiums*. Amsterdam: North Holland, 1984.

GRANOT, D.: "Cooperative Games in Stochastic Characteristic Function Form," *Management Science*, **23** (1977), 621-630.

HAMERS, H.: *Sequencing and Delivery Situations: a Game Theoretic Approach*, CentER Dissertation Series, Tilburg University, **7** 1996.

HAMERS, H., P. BORM and S. TIJS: "On Games Corresponding to Sequencing Situations with Ready Times," *Mathematical Programming*, **70** (1993) 1-13.

HAMERS, H., J. SUIJS, P. BORM, and S. TIJS: "The Split Core for Sequencing Games," *Games and Economic Behavior*, **15** (1996), 165-176.

HARSANYI, J.: "A Simplified Bargaining Model for the n-Person Cooperative Game," *International Economic Review*, **4** (1963), 194-220.

HSIAO, C., and T. RAGHAVAN: "Shapley Value for Multi-Choice Cooperative Games," *Games and Economic Behavior*, **5** (1993), 240-256.

ICHIISHI, T.: "Supermodularity: Applications to Convex Games and the Greedy Algorithm for LP," *Journal of Economic Theory*, **25** (1981), 283-286.

KALAI, E., and D. SAMET: "Monotonic Solutions to General Cooperative Games," *Econometrica*, **53** (1985), 307-327.

KANEKO, M.: "On Transferable Utility," *International Journal of Game Theory*, **5** (1976), 183-185.

LEMAIRE, J.: "Cooperative Game Theory and its Insurance Applications," *Astin Bulletin*, **21** (1991), 17-40.

LITTLECHILD, S.: "A Simple Expression for the Nucleolus in a Special Case," *International Journal of Game Theory*, **3** (1974), 21-29.

MASCHLER, M., and G. OWEN: "The Consistent Shapley Value for Hyperplane Games," *International Journal of Game Theory*, **18** (1989), 389-407.

MASCHLER, M., and G. OWEN: "The Consistent Shapley Value for Games without Side Payments," in *Rational Interaction* ed. by R. Selten. Berlin: Springer-Verlag, 1992.

MASCHLER, M., J. POTTERS and S. TIJS: "The General Nucleolus and the Reduced Game Property'" *International Journal of Game Theory*, **21** (1992), 85-106.

MAGILL, M., and W. SHAFER: "Incomplete Markets," in *Handbook of Mathematical Economics* IV, ed. by W. Hildenbrand and H. Sonnenschein. Amsterdam: Elsevier Science Publishers, 1991.

MUTO, S., M. NAKAYAMA, J. POTTERS, and S. TIJS: "On Big Boss Games," *The Economic Studies Quarterly*, **39** (1988), 303-321.

NEUMANN, J. VON, and O. MORGENSTERN: *Theory of Games and Economic Behavior*. Princeton: Princeton University Press, 1944.

NOUWELAND, A. VAN DEN, J. POTTERS, S. TIJS, and J. ZARZUELO: "Cores and Related Solution Concepts for Multi-choice Games," *ZOR - Mathematical Methods of Operation Research*, **41** (1995), 289-311.

OTTEN, G., B. PELEG, P. BORM and S. TIJS: "The MC-value for monotonic NTU-games," *CentER Discussion Paper, Tilburg University*, **9436** (1994) (to appear in *International Journal of Game Theory*).

OWEN, G.: "On the Core of Linear Production Games," *Mathematical Programming*, **9** (1975), 358-370.

POTTERS, J., and S. TIJS: "The Nucleolus of Matrix Games and Other Nucleoli," *Mathematics of Operations Research*, **17** (1992), 164-174.

SCARF, H.: "The Core of an N Person Game," *Econometrica*, **35** (1967), 50-69.

SCHMEIDLER, D.: "The Nucleolus of a Characteristic Function Game," *SIAM Journal of Applied Mathematics*, **17** (1969), 1163-1170.

SHAPLEY, L.: "A Value for n-Person Games," in *Contributions for the Theory of Games II*, ed. by A. Tucker and H. Kuhn. Princeton: Princeton University Press, 1953.

SHAPLEY, L.: "On Balanced Sets and Cores," *Naval Research Logistics Quarterly*, **14** (1967), 453-460.

SHAPLEY, L.: "Utility Comparisons and the Theory of Games," in *La Decision, Aggregation et Dynamique des Orders de Preference*. Paris: Edition du Centre National de la Recherche Scientifique, 1969.

SHAPLEY, L.: "Cores of Convex Games," *International Journal of Game Theory*, **1** (1971), 11-26.

SHAPLEY, L., and M. SHUBIK: "On Market Games," *Journal of Economic Theory*, **1** (1969), 9-25.

SHARKEY, W.: "Convex Games without Side Payments," *International Journal of Game Theory*, **10** (1981), 101-106.

SUIJS, J.: "A Nucleolus for Stochastic Cooperative Games," *CentER Discussion Paper, Tilburg University*, **9690** (1996).

SUIJS, J., and P. BORM: "Stochastic Cooperative Games: Superadditivity, Convexity, and Certainty Equivalents," (to appear in *Games and Economic Behavior*) revised version of "Cooperative Games with Stochastic Payoffs: Deterministic Equivalents," *FEW Research Memorandum, Tilburg University*, **713** (1996).

SUIJS, J., A. DE WAEGENAERE, and P. BORM: "Stochastic Cooperative Games in Insurance and Reinsurance," *CentER Discussion Paper, Tilburg University*, **9653** (1996) (to appear in *Insurance: Mathematics & Economics*).

SUIJS, J., P. BORM, A. DE WAEGENAERE and S. TIJS: "Cooperative Games with Stochastic Payoffs," *CentER Discussion Paper, Tilburg University*, **9588** (1995) (to appear in *European Journal of Operational Reseacrh*).

VILKOV, V.: "Convex Games without Side Payments (in Russian)," *Vestnik Leningradskiva Universitata*, **7** (1977), 21-24.

WILSON, R.: "The Theory of Syndicates," *Econometrica*, **36** (1968), 119-132.

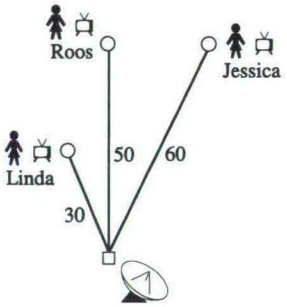
Samenvatting

Coöperatieve speltheorie beschrijft vanuit een wiskundig standpunt situaties waarbij meerdere personen betrokken zijn die voordeel kunnen behalen door met elkaar samen te werken. Ter illustratie beschouwen we het volgende voorbeeld.

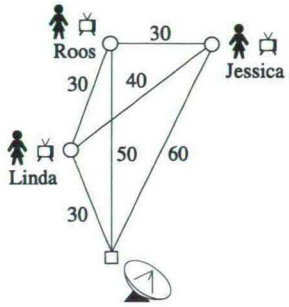
Ter verhoging van hun levenstandaard willen de inwoners van een dorp hun woningen voorzien van een kabelaansluiting voor de televisie zodat ze elke avond hun favoriete soapserie kunnen volgen. Het tv-signaal wordt geleverd door een kabelmaatschappij waarmee elke woning verbonden dient te worden. Vanzelfsprekend is het aanleggen van een verbinding niet kosteloos. Elke verbinding heeft een eigen kostprijs afhankelijk van bijvoorbeeld de lengte van de verbinding. Laten we nu voor het gemak veronderstellen dat er slechts drie inwoners zijn, genaamd Linda, Roos en Jessica. Een manier om dan zo'n netwerk te maken is door middel van drie directe verbindingen tussen de maatschappij en elke woning, zoals afgebeeld in Figuur 1. De totale kosten van dit net zijn gelijk aan fl. 140, waarbij het duidelijk moge zijn dat Linda voor haar verbinding fl. 30 moet betalen, Roos fl. 50 en Jessica fl. 60.

Door met elkaar samen te werken kunnen Linda, Roos en Jessica echter een goedkoper net aanleggen dat iedereen verbindt met de kabelmaatschappij. Naast de directe verbinding tussen een woning en de kabelmaatschappij, is het namelijk ook mogelijk om woningen met elkaar te verbinden. Deze verbindingen, en de daarbij behorende kosten, zijn weergegeven in Figuur 2.

Laten we nu de situatie voor Linda en Roos eens nader beschouwen. In de huidige , waarbij zowel Linda als Roos een directe verbinding met de kabelmaatschappij hebben, zijn de totale kosten fl. 80. Door met elkaar samen te werken kunnen ze echter een goedkoper net aanleggen. Immers, als Roos gebruik mag maken van Linda's verbinding met de kabelmaatschappij, dan volstaat een verbinding tussen Linda en Roos om Roos kabel-tv te geven. De totale kosten zijn dan fl. 60, namelijk fl. 30 voor de verbinding tussen Linda en de kabel-



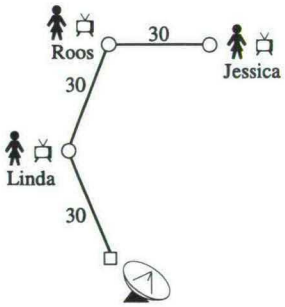
Figuur 1



Figuur 2

maatschappij en fl. 30 voor de verbinding tussen Roos en Linda. Dus Linda en Roos kunnen fl. 20 uitsparen door met elkaar samen te werken.

Hetzelfde geldt voor Linda, Roos en Jessica samen. Door van elkaars verbindingen gebruik te maken kunnen ze een goedkoper net aanleggen. Het goedkoopste net dat ze kunnen aanleggen is afgebeeld in Figuur 3. Hier is Linda direct verbonden met de kabelmaatschappij, Roos is verbonden met Linda en Jessica is verbonden met Roos. De kosten van dit net bedragen fl. 90. Dus door samen te werken kunnen Linda, Roos en Jessica tegen lagere kosten kabel-tv krijgen. De vraag die rest is, wie wat betaalt van die fl. 90.



Figuur 3

Het hierboven beschreven probleem, kan beschouwd worden als een coöperatief spel.

De spelers van dit spel zijn Linda, Roos en Jessica en de karakteristieken van dit spel zijn de minimale kosten waartegen een groep van spelers kabel-tv kan aanleggen. Voor Linda bedragen deze kosten fl. 30, de kosten van een directe verbinding met de kabelmaatschappij. Laten we deze kosten noteren met c_L. Evenzo geldt dat $c_R =$ fl. 50, $c_J =$ fl. 60, $c_{LR} =$ fl. 60, $c_{LJ} =$ fl. 70 en $c_{RJ} =$ fl. 80. Tenslotte zijn de kosten voor Linda, Roos en Jessica tesamen gelijk aan $c_{LRJ} =$ fl. 90.

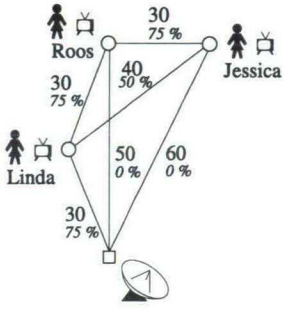
Het doel van de coöperatieve speltheorie is nu te bepalen wie met wie zal gaan samenwerken en hoe de daarbij behorende kosten verdeeld zullen worden. De meeste aandacht gaat hierbij echter uit naar de laatstgenoemde doelstelling: wat is een eerlijke verdeling van de kosten onder de aanname dat alle spelers met elkaar samen willen werken?

Laten we weer terugkeren naar het voorbeeld met Linda, Roos en Jessica. Een manier om de kosten te verdelen is de volgende: de kosten van een verbinding worden gelijk verdeeld over de gebruikers van deze verbinding. De verbinding tussen Linda en de kabelmaatschappij kost fl. 30 en wordt gebruikt door zowel Linda als Roos als Jessica. Verdelen we de kosten gelijk over deze gebruikers, dan betekent dit dat ieder fl. 10 bijdraagt. Van de verbinding tussen Roos en Linda maken alleen Roos en Jessica gebruik. Zij betalen dus ieder de helft van de kosten; fl. 15 voor Roos en fl. 15 voor Jessica. De verbinding tussen Roos en Jessica wordt alleen door Jessica gebruikt. De kosten van deze verbinding zijn dus voor haar rekening. In totaal betaalt Linda dan fl. 10, Roos fl. 25 en Jessica fl. 55.

Een belangrijke eigenschap van deze verdeling is dat iedereen bereid is de samenwerking in stand te houden. Zo kost voor Jessica de goedkoopste verbinding met de kabelmaatschappij fl. 60, wat meer is dan de 55 gulden die ze nu moet bijdragen. Hetzelfde geldt voor Linda en Roos samen. Nu moeten zij fl. 10 + fl. 25 = fl. 35 bijdragen. Dit is echter minder dan fl. 60, wat de laagste kosten zijn om Linda en Roos met de kabelmaatschappij te verbinden. Kortom, deze verdeling van de kosten maakt de samenwerking stabiel: er is geen groep van spelers die tegen lagere kosten een kabel-tv netwerk kunnen aanleggen dan het bedrag dat zij nu moeten betalen. Dergelijke verdelingen worden in de speltheorie core-allocaties genoemd. Naast de verdeling fl. 10, fl. 25, en fl. 55 zijn er in dit voorbeeld nog meer core-allocaties, zoals fl. 20, fl. 30 en fl. 40 voor achtereenvolgens Linda, Roos en Jessica of de verdeling waarbij ieder fl. 30 betaalt.

Een belangrijke aanname in dit voorbeeld is dat de kosten van elke verbinding met zekerheid bekend zijn. Linda, Roos en Jessica weten vooraf - wanneer zij beslissen welk netwerk ze gaan aanleggen - exact hoeveel elke verbinding kost. Er is dus geen onzekerheid. Dit maakt het relatief eenvoudig om het goedkoopste netwerk te bepalen. De aangelegde verbindingen zouden echter ook stuk kunnen gaan. In dat geval moet de verbinding opnieuw worden aangelegd voor er weer een tv-signaal van de kabelmaatschappij ontvangen kan worden. Aangezien vooraf niet bekend is of een verbinding stuk gaat of niet, zijn de kosten van een verbinding dus ook niet meer met zekerheid bekend. Beschouwen we nu een periode van twee jaar, dan is van elke verbinding bekend wat de directe aanlegkosten zijn én de kans dat deze

verbinding na een periode van een jaar stuk is. De situatie die dan ontstaat zou kunnen zijn zoals afgebeeld in Figuur 4, waarbij het cursief afgedrukte percentage bij elke verbinding de kans is dat deze verbinding na een jaar stuk is.

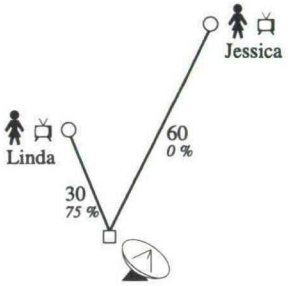


Figuur 4

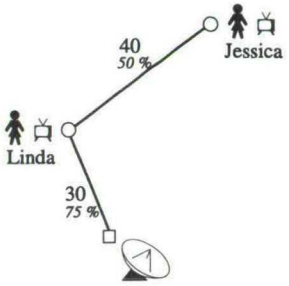
De verbinding tussen Linda en Jessica bijvoorbeeld, kost fl. 40 en gaat met 50% kans na een jaar kapot. Wanneer de verbinding stuk gaat moet deze opnieuw worden aangelegd voor fl. 40. De totale kosten bedragen dan fl. 80. Over een periode van twee jaar kost de verbinding tussen Linda en Jessica dus fl. 40 met 50% kans en fl. 80 met 50% kans.

Aangezien de kosten nu niet meer met zekerheid bekend zijn, wordt het moeilijker om te bepalen welk netwerk het goedkoopste is. Naast de hoogte van de kosten speelt nu ook het risico in de kosten een rol, dat wil zeggen, welke kosten komen met welke kans voor. Laten we ter illustratie Linda en Jessica beschouwen en twee mogelijke manieren waarop zij een verbinding met de kabelmaatchappij kunnen maken.

De eerste manier verbindt zowel Linda als Jessica direct met de kabelmaatschappij en is afgebeeld in Figuur 5. De kosten van dit netwerk bedragen fl. 30 + fl. 60 = fl. 90 met 25% kans en fl. 60 + fl. 60 = fl. 120 met 75% kans. De tweede manier is afgebeeld in Figuur 6. De kosten van dit netwerk bedragen fl. 30 + fl. 40 = fl. 70 met 12,5% kans, fl. 60 + fl. 40 = fl. 100 met 37,5% kans, fl. 30 + fl. 80 = fl. 110 met 12,5% kans, en fl. 60 + fl. 80 = fl. 140 met 37,5% kans. Het is niet direct te zeggen welke van deze twee het 'goedkoopste' is. Dit hangt vooral af van hoeveel risico Linda en Jessica bereid zijn om te nemen. Willen zij veel risico nemen, dan kiezen zij wellicht voor het tweede netwerk omdat deze met 12,5% kans slechts fl. 70 kost. Willen zij daarentegen maar weinig risico nemen, dan ligt het eerste netwerk misschien meer voor de hand omdat deze maximaal fl. 120 kost. Kortom, de beslissing over welk netwerk zij aanleggen wordt nu mede bepaald door het risico dat Linda en Jessica bereid zijn om te nemen.



Figuur 5



Figuur 6

De situatie weergegeven in Figuur 4 geeft net als de situatie in Figuur 2 aanleiding tot een coöperatief spel. De kosten waartegen de spelers nu kabel-tv kunnen aanleggen zijn echter geen vaste bedragen meer maar kansvariabelen, ook wel stochastische variabelen genoemd. Dergelijke spelen worden daarom coöperatieve spelen met stochastische uitbetalingen genoemd en zijn het onderwerp van dit proefschrift.

Een coöperatief spel met stochastische uitbetalingen beschrijft vanuit een wiskundig standpunt een situatie waarbij meerdere personen betrokken zijn die voordeel kunnen behalen door met elkaar samen te werken. Het voordeel dat men kan behalen is nu echter niet meer met zekerheid bekend, maar wordt beschreven met behulp van kansvariabelen. In de wetenschappelijke literatuur is deze tak van de coöperatieve speltheorie nauwelijks onderzocht. Het doel van dit proefschrift is dan ook om voor deze spelen een theorie op te bouwen, waarbij de (traditionele) coöperatieve speltheorie als leidraad dient. De nadruk ligt hierbij op het verdelen van de opbrengsten/kosten onder de aanname dat iedereen bereid is met elkaar samen te werken.

Het proefschrift is als volgt opgebouwd. Hoofdstuk 2 geeft een beknopte inleiding in de coöperatieve speltheorie. Het behandelt enkele basisbegrippen die nodig zijn voor een goed begrip van de daaropvolgende hoofdstukken.

Hoofdstuk 3 introduceert coöperatieve spelen met stochastische uitbetalingen. Naast de formele definitie worden enkele voorbeelden gepresenteerd op het gebied van productie, machinevolgorde problemen en de financiële markten. Verder wordt besproken hoe men de preferenties van de spelers over kansvariabelen kan beschrijven en hoe men uit deze preferenties het gedrag ten opzichte van risico kan afleiden.

Hoofdstuk 4 introduceert enkele nieuwe begrippen voor coöperatieve spelen met uitbetalingen. Het betreft hier de core, superadditiviteit en convexiteit. De core van een coöperatief

spel bevat stabiele verdelingen van de opbrengsten/kosten. De begrippen superadditiviteit en convexiteit zeggen iets over de wijze waarop de opbrengst van samenwerking afhangt van de grootte van de groep die samenwerkt. Zo kunnen in een superadditief coöperatief spel twee afzonderlijke groepen hogere opbrengsten behalen door samen een groep te vormen.

Hoofdstuk 5 introduceert de nucleolus. De nucleolus beschrijft een methode om de opbrengsten te verdelen. Gegeven een verdeling van de opbrengsten/kosten wordt voor elke groep van spelers bepaald hoe ontevreden deze groep is met deze verdeling. Deze mate van ontevredenheid wordt uitgedrukt met een getal, ook wel de klacht genaamd. Hoe hoger het getal, hoe hoger de klacht van een groep is. De verdeling die de nucleolus dan voorschrijft is die verdeling waarvoor de grootste klacht minimaal is. Verder laten we zien dat wanneer er voor een bepaald spel core-allocaties bestaan, dan resulteert de nucleolus ook in een core-allocatie.

Tenslotte wordt in Hoofdstuk 6 een toepasssing van een theorie in de verzekeringswereld besproken. Een verzekering wordt hierbij geïnterpreteerd als samenwerking tussen een individu en een verzekeringsmaatschappij. We laten voor deze spelen zien dat er core-allocaties bestaan. Bovendien tonen we aan dat het zogeheten 'zero-utility principle' voor het berekenen van verzekerinspremies tot een core-allocatie leidt.

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graduation he became a Ph.D. student at CentER, the Graduate School of Tilburg University. As of March 1998, he will be taking up a position at the CentER Accounting Research Group.

Cooperative game theory is a mathematical tool to analyze situations involving several individuals who can obtain certain benefits by cooperating. The main questions this theory addresses are who will cooperate with whom and how will the corresponding benefits be divided. Most results of cooperative game theory, however, only apply to cases where these benefits are deterministic. In this thesis we abandon this assumption and allow for stochastic benefits. Examples of such situations can be found in linear production situations, sequencing and financial markets. The aim of this work is then to develop a theory on games with stochastic payoffs using traditional game theory as a guideline.

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