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On the Cores of Cooperative Games and the Stability of the Weber Set[1]

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Abstract: In this paper conditions are given guaranteeing that the Core equals the D-core (the set of unDominated imputations). Under these conditions, we prove the non-emptiness of the intersection of the Weber set with the imputation set. This intersection has a special stability property: it is externally stable. As a consequence we can give a new characterization (th. 3.2) for the convexity of a cooperative game in terms of its stability (von Neumann-Morgenstern solutions) using the Weber set.

1 Introduction

We will follow the notations and terminologies of Driessen's book [1988]. Let $N=\{1,\dots,n\}$ be a finite set, the players set. A cooperative n-person game in characteristic function form is an ordered pair (N, v), where $v\colon 2^N \to R$ is a real-valued function on the set 2^N of all subsets of N such that $v(\emptyset)=0$. The class of all cooperative n-person games with player set N will be denoted by G^N. By $I^*(v)$ and $I(v)$ will be denoted the classical pre-imputation and imputation sets, i.e., $I^*(v):=\{x\in R^n/x_1+\cdots+x_n=x(N)=v(N)\}$ and $I(v)=\{x\in I^*(v)/x_i\geq v(i)$ $\forall i=1,\dots,n\}$. Given a vector $x\in R^n$ and a coalition S, $x(S):=\sum_{i\in S}x_i$ if $S\neq\emptyset$ and $x(\emptyset)=0$.

For each permutation $\theta\in S_n$ over N, we define the marginal worth vector, $m_\theta^v\in R^n$ such that $m_\theta^v(i)=v(P_{\theta,i}\cup\{i\})-v(P_{\theta,i})$ where $P_{\theta,i}=\{j\in N/\theta(j)<\theta(i)\}$ is the set of predecessors of i by θ. The convex hull of the marginal worth vectors is called the Weber set, $W(v):=\text{convex}\{m_\theta^v\}_{\theta\in S_n}$. Each marginal worth vector is an efficient payoff vector (a pre-imputation): $m_\theta^v(N)=v(N)$. It is well-known that the Weber set always contains the core of a game, which is defined by

$$C(v):=\{x\in I(v)/x(S)\geq v(S)\quad \forall S\in 2^N\} \tag{1}$$

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Given two imputations $x, y \in I(v)$ we say that x dominates y, in short $x \text{ dom}^v y$, if there exists an $S \in 2^N$, $2 \leq |S| \leq n-1$ such that:

1. $x_i > y_i, \quad \forall i \in S$
2. $x(S) \leq v(S)$

If we need to specify the game or/and the coalition used in a domination then we will write $x \; dom_S^v \; y$. If no confusion arises then we will use the notation: $x \; dom \; y$ or $x \; dom_S \; y$.

For $X \subseteq I(v)$ we will denote by $\text{Dom}^v X$ (or, $\text{Dom}X$), the set of all imputations dominated by some imputation of the set X, i.e.,

$$Dom^v X = \{y \in I(v) / \exists x \in X, \; x \; dom^v \; y\}.$$

A set of imputations $V \subseteq I(v)$ is a stable set (or von Neumann-Morgenstern solution) if it is internally stable: $V \cap Dom V = \emptyset$ and externally stable $V \cup Dom V = I(v)$. Therefore, stable sets are the fixed points of function $H: 2^{I(v)} \rightarrow 2^{I(v)}: H(X) = I(v) \backslash Dom X$, see Lucas (1992).

2 The Core and the D-Core of a Game

There exist in the literature two core concepts. One core, $C(v)$, is the solution set of a system of linear inequalities (1). The second core, called the unDominated core, $DC(v)$, is precisely the set of undominated imputations. Formally,

$$DC(v) := I(v) \backslash Dom I(v), \quad \text{for any } v \in G^N.$$

The core is always included in the undominated core, $C(v) \subseteq DC(v)$. Nevertheless there are games where both concepts are different (see example 3.1).

If a game has a nonempty undominated core (otherwise both are the same) it is easy to see that the weak condition,

$$\forall S \subseteq N \; v(S) + \sum_{i \in N \backslash S} v(i) \leq v(N), \tag{2}$$

is a characterization for the coincidence of both cores. Formally we have

Proposition 2.1: Let $v \in G^N$ such that $DC(v) \neq \emptyset$. Then the following statements are equivalent:

1. $C(v) = DC(v)$

2. $\forall S \subseteq N v(S) + \sum_{i \in N \backslash S} v(i) \leq v(N)$. □

It is well-known that superadditivity or balancedness are sufficient conditions for the coincidence between the core and the D-core. Nevertheless, they can also be easily deduced from the above proposition.

We would like to point out that condition (2) is equivalent to the fact that in the 0-normalized game, $v_0(S) := v(S) - \sum_{i \in S} v(i), \forall S \subseteq N$, the grand coalition is the largest in value, i.e.

$$\forall S \subseteq N v_0(S) \leq v_0(N) \tag{3}$$

A game will be called a 0-normalized N-monotonic game, if it satisfies condition (2) or the equivalent condition (3) and we will denote this class of games by

$$Z^N := \left\{ v \in G^N / \forall S \subseteq N \; v(S) + \sum_{i \in N \setminus S} v(i) \leq v(N) \right\} \tag{4}$$

This class of games, Z^N, will play an important role in the next section. Let us remark that if $v \in Z^N$, then $I(v) \neq \emptyset$. Moreover, superadditive games (i.e. $v \in G^N$ such that $\forall S, T \in 2^N$ if $S \cap T = \emptyset$ then $v(S) + v(T) \leq v(S \cup T)$), balanced games (i.e. $v \in G^N$ such that $C(v) \neq \emptyset$), convex games (i.e. $v \in G^N$ such that $\forall S, T \in 2^N$ $v(S) + v(T) \leq v(S \cup T) + (S \cap T)$) and 0-monotonic games ($v \in G^N$ such that i.e. $\forall S \subseteq T \; v(S) + \sum_{i \in T \setminus S} v(i) \leq v(T)$) are all included in Z^N.

In spite of the fact that the core and the undominated core are in general not equal, we are going to prove that under the non-emptiness condition $DC(v) \neq \emptyset$, the undominated core can be seen as the core of a new associated game. Compactness and convexity of the undominated core will be a direct consequence of the following theorem.

Theorem 2.1: For any game $v \in G^N$ such that $DC(v) \neq \emptyset$ follows $DC(v) = C(v')$ where $v'(S) := min(v(S), v(N) - \sum_{i \in N \setminus S} v(i)) \; \forall S \in 2^N \setminus \{\emptyset\}$ and $v'(\emptyset) = 0$.

Proof: Let us point out that $v'(N) = v(N)$, $I(v') \neq \emptyset$ and, given $i \in N$, $v'(i) = v(i)$ if and only if $I(v) \neq \emptyset$. As a consequence, if $I(v) \neq \emptyset$ then $I(v') = I(v)$ and the converse is also true. Moreover, $DC(v') = C(v')$ because $v' \in Z^N$.

By assumption, $DC(v) \neq \emptyset$, which implies $I(v) \neq \emptyset$ or $I(v) = I(v')$.

Let $x \in Dom^v I(v)$. Then there exists a $y \in I(v)$ such that $y \; dom^v \; x$ or equivalently there exists an $S \in 2^N$, $2 \leq |S| \leq n-1$, such that $y_i > x_i \; \forall i \in S$ and $y(S) \leq v(S)$. By the efficiency of the vector y, we know

$$y(S) = y(N) - \sum_{i \in N \setminus S} y_i \leq v(N) - \sum_{i \in N \setminus S} v(i).$$

Then $y(S) \leq v'(S)$ and $y \in I(v')$ which implies that $x \in Dom^{v'} I(v')$. So $Dom^v I(v) \subseteq Dom^{v'} I(v')$.

The converse inclusion obviously holds and therefore $Dom^{v}I(v) = Dom^{v'}I(v')$ which implies $DC(v) = DC(v') = C(v')$. □

To end this paragraph we are going to study which is the behaviour of two well-known results devoted to the core $C(v)$ when we replace the core by the undominated core $DC(v)$. First, we will see that the coincidence between the undominated core and the Weber set is a characterization of the convexity of the game.

Corollary 2.1: Let $v \in G^N$. The following statements are equivalent:

1. v is a convex game.
2. $DC(v) = W(v)$.

Proof:

$1 \to 2$) See Shapley *(1971)* and take into account that $C(v) \neq \emptyset$ for convex games.
$2 \to 1$) *If* $DC(v) = W(v)$, then $DC(v) \neq \emptyset$ and $W(v) \subseteq I(v)$ which implies $v \in Z^N$. Using theorem *2.1*, $DC(v) = C(v) = W(v)$ or equivalently v is convex. □

On the other hand, Weber (1978) proved that the core is always included in the Weber set. This result is not preserved if we replace the core by the undominated core, as we can see in the next example due to an anonymous referee.

Take the 4-person game v with $v(S) = 0$ if $|S| \leq 1$, $v(S) = 1$ if $|S| = 2$ and $S \neq \{3,4\}$, $v(3,4) = -2$, $v(1,2,3) = v(1,2,4) = 2$, $v(1,3,4) = v(2,3,4) = -1$ and $v(N) = 0$. Then the imputation set only cointains the zero allocation and $DC(v) = \{0,0,0,0\}$. Furthermore, if player 1 or 2 enters a non-empty coalition then the value of that coalition increases, implying that the sum of the payoffs to player 1 and player 2 is positive in any marginal worth vector and, therefore, in any element of the Weber set. This proves that the zero allocation is not an element of the Weber set.

3 Stability and the Weber Set

It is a well-known result that for a convex cooperative game the core is the unique stable set, but the reverse is not true in general. Since for convex games the Weber set is equal to the core, we obtain, in the convex case, the stability of the Weber set. The main objective of this paragraph is to characterize the convexity of a game in terms of the stability of its Weber set.

To show this we are going to study a stability property of the Weber set.

First of all we will prove that if a game is N-monotonic (i.e. $\forall S \subseteq N\ v(S) \leq v(N)$) then its positive Weber set $W_+(v) := W(v) \cap R^n_+$ is non-empty.

From this result we will be able to show: if a game $v \in Z^N$, then the intersection between the imputation set and the Weber set is non-empty, externally stable and $C(v) \cup Dom(I(v) \cap W(v)) = I(v)$ which means that each imputation either belongs to the core or else it is dominated by some imputation of the Weber set, but never both.

With the aid of these results a new equivalence will be proved: a game is convex if and only if the Weber set is a stable set (the unique one). In fact we will be able to simplify the above characterization for the convexity of a game, only requiring the internal stability of the Weber set.

Lemma 3.1: Let $v \in G^N$ such that $v(S) \leq v(N)\ \forall S \in 2^N$ (in particular, $0 = v(\emptyset) \leq v(N)$.) Then

$$W_+(v) \neq \emptyset.$$

Proof: If $W_+(v) \neq \emptyset$ then, by the hyperplane separation theorem, we know that there exists an $\alpha \in R^n$ such that $\langle m_\theta^v, \alpha \rangle > \langle x, \alpha \rangle\ \forall \theta \in S_n$ and $\forall x \in R^n_+$.

Let σ be a permutation such that, $\alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(n)}$. Then:

$$\begin{aligned}\langle m^v_{\sigma^{-1}}, \alpha \rangle &= \alpha_{\sigma(n)} v(N) + \sum_{i=1}^{n-1} (\alpha_{\sigma(i)} - \alpha_{\sigma(i+1)})\, v(\sigma(1), \ldots \sigma(i)) \\ &\leq \alpha_{\sigma(n)} \cdot v(N) + \sum_{i=1}^{n-1} (\alpha_{\sigma(i)} - \alpha_{\sigma(i+1)}) \cdot v(N) = \alpha_{\sigma(1)} \cdot v(N) = \\ &= \langle v(N) \cdot e_{\sigma(1)}, \alpha \rangle\end{aligned}$$

where $v(N) e_{\sigma(1)} \in R^n_+$ and $e_{\sigma(1)}$ is the corresponding canonical vector i.e.

$$[e_{\sigma(1)}]_j = \begin{cases} 0 & j \neq \sigma(1), \\ 1 & j = \sigma(1). \end{cases} \quad \square$$

As a consequence of the above result we can prove.

Lemma 3.2: Let $v \in Z^N$. Then

$$W(v) \cap I(v) \neq \emptyset.$$

Proof: Defining the zero normalized associated game $v_0(S) := v(S) - \sum_{i \in S} v(i)$, we can show in a straightforward way: $v_0(i) = 0\ \forall i \in N$, $v_0(N) = v(N) - \sum_{i \in N} v(i) \geq 0$ and $v_0(S) \leq v_0(N)$, $\forall S \in 2^N$. Using lemma 3.1, $W_+(v_0) \neq \emptyset$ or equivalently $W(v_0) \cap I(v_0) \neq \emptyset$. Now using the fact that $I(v) = I(v_0) + \{(v(1), \ldots, v(n))\}$ and $W(v) = W(v_0) + \{(v(1), \ldots, v(n))\}$ we obtain $I(v) \cap W(v) \neq \emptyset$. $\square$

Once established these previous lemmas we state the main result of this paragraph.

Theorem 3.1: Let $v \in Z^N$. Then

1. $C(v) \cup Dom(I(v) \cap W(v)) = I(v)$.
2. $I(v) \cap W(v)$ is externally stable.

Proof:

1) Note first that we only have to prove that $C(v_0) \cup Dom^{v_0}(I(v_0) \cap W(v_0)) = I(v_0)$. In fact, we only have to show that $I(v_0) \subseteq C(v_0) \cup Dom^{v_0}(I(v_0) \cap W(v_0))$. Let $x \in I(v_0) \setminus C(v_0)$ we have to prove that $x \in Dom^{v_0}(I(v_0) \cap W(v_0))$. Since $x \notin C(v_0)$, let T be a coalition of minimal cardinality such that $x(T) < v_0(T)$. Let us define w: $2^{N \setminus T} \to R$, $w(S) = v_0(T \cup S) - v_0(T)$. Let us point out that w is $N \setminus T$-monotonic, which means $w(S) = v_0(T \cup S) - v_0(T) \leq w(N \setminus T) = v_0(N) - v_0(T)\ \forall S \subseteq N \setminus T$. This is true by the assumption that $v \in Z^N$. By lemma *3.1* $W_+(w) \neq \varnothing$ and let $z \in W_+(w)$. We define $y \in R^n$ as follows:

$$y_i := \begin{cases} x_i + \alpha & \text{if} \quad i \in T \quad \text{where} \quad \alpha = \dfrac{v_0(T) - x(T)}{|T|} \\ z_i & \text{if} \quad i \notin T \end{cases}$$

Let us point out the following properties for the vector y.

1. $y \in I(v_0)$
   If $i \in T$ then $y_i = x_i + \alpha > x_i \geq v_0(i) = 0$.
   If $i \notin T$ then $y_i = z_i \geq 0 = v_0(i)$ because $z \in W_+(w)$.
2. $y(T) = v_0(T)$.
3. $y_{\setminus T} \in W(v_{0_{|T}})$.
   Since T is a minimal cardinality coalition such that $x(T) < v_0(T)$, then $\forall S \subset T$ we know $x(S) \geq v_0(S)$ and then $y(S) = x(S) + |S|\alpha \geq x(S) \geq v_0(S)$. We know $y_{|T} \in C(v_{0_{|T}}) \subseteq W(v_{0_{|T}})$.
4. $y\ dom_T^{v_0}\ x$.
5. $y \in W(v_0)$.
   By *3* we know $y_{|T} \in W(v_{0_{|T}})$ and also $y_{|N \setminus T} = z \in W(w)$ where $w(S) = v_0(T \cup S) - v_0(T)$ for every $S \subseteq N \setminus T$. Using Weber's original argument when proving the inclusion $C(v) \subseteq W(v)$ (Weber (1978)), we know
   $y_{|T} = \sum_{\theta \in S_{|T|}} \lambda_\theta m_\theta^{v_{0|T}}$ and $y_{|N \setminus T} = \sum_{\hat{\theta} \in S_{|N \setminus T|}} \mu_{\hat{\theta}} m_{\hat{\theta}}^{w}$ where $\lambda_\theta \geq 0$, $\mu_{\hat{\theta}} \geq 0$, $\sum_{\theta \in S_{|T|}} \lambda_\theta = 1$ and $\sum_{\hat{\theta} \in S_{|N \setminus T|}} \mu_{\hat{\theta}} = 1$. For every $\theta \in S_{|T|}$ and $\hat{\theta} \in S_{|N \setminus T|}$ we can define a natural way a permutation σ appending both, $\sigma = (\theta, \hat{\theta})$ and we will obtain easily $y = \sum_{\substack{\sigma = (\theta, \hat{\theta}) \\ \theta \in S_T, \hat{\theta} \in S_{N \setminus T}}} (\lambda_\theta \cdot \mu_{\hat{\theta}})\, m_\sigma^{v_0}$ or equivalently $y \in W(v_0)$.

We have just proved the existence of $y \in I(v_0) \cap W(v_0)$ such that $y\ dom_T^{v_0}\ x$ and then $x \in Dom^{v_0}(I(v_0) \cap W(v_0))$.

2) Since $I(v) = C(v) \cup Dom(I(v) \cap W(v)) \subseteq (W(v) \cap I(v)) \cup Dom(I(v) \cap W(v))$ we have obtained the external stability of $I(v) \cap W(v)$. □

As a consequence of the above result we can give a new proof for the well-known result about the stability of the core for convex cooperative games proved by Shapley (1971).

Corollary 3.1: If a game $v \in G^N$ is convex, then $C(v)$ is (the unique) stable set.

Proof: We have only to point that a convex game is 0-normalized N-monotonic (i.e. $v \in Z^N$) and $W(v) \subseteq I(v)$. We know that, in general, $C(v)$ is internally stable. By theorem 3.1 $W(v)$ is externally stable. Since $W(v) = C(v)$, due to the convexity of the game v, we have finished the proof. □

For those games where $W(v) \subseteq I(v)$ it can be checked that $v \in Z^N$ and consequently we can apply theorem 3.1. The consequence will be that the Weber set has to be externally stable.

Corollary 3.2: The Weber set $W(v)$ is externally stable if the game $v \in G^N$ satisfies one of the following equivalent conditions:

1. $W(v) \subseteq I(v)$
2. $\forall i \in N \ \forall S \subseteq N \backslash i \ \ v(S) + v(i) \leq v(S \cup i)$
3. v is 0-monotonic □

Let us point out that the converse implication in the above corollary obviously holds. In fact, if the Weber set is externally stable then implicitly, in the definition of external stability, we are taking the inclusion of the Weber set in the imputation set. The above corollary can be summarized then as follows: the Weber set is externally stable in the largest class of games where the external stability makes sense (i.e. the 0-monotonic class of games).

We can prove now the second main result of this paragraph. The internal stability of the Weber set is a necessary and sufficient condition for the convexity of a game.

Theorem 3.2: Let $v \in G^N$. The following statements are equivalent:

1. v is a convex game.
2. $W(v)$ is internally stable.
3. $W(v)$ is a stable set.

Proof:

$1 \rightarrow 2)$ It is well-known since for convex games $C(v) = W(v)$ and the core is the only stable set.

$2 \rightarrow 3)$ By the definition of the internal stability, $W(v) \subseteq I(v)$, which implies that $v \in Z^N$. By Corollary 3.2, $W(v)$ is also externally stable, and then $W(v)$ is a stable set.

$3 \rightarrow 1)$ If $W(v)$ is a stable set then $W(v) \subseteq I(v)$. We can use now theorem 3.1 to see $C(v) \cup Dom(W(v)) = I(v)$. But $C(v) \cap Dom(W(v)) = \varnothing$, since $C(v) \subseteq DC(v)$.

Then $C(v) = I(v) \backslash Dom(W(v)) = W(v)$ using the fact that $W(v)$ is stable. But $C(v) = W(v)$ implies that v is convex. □

Let us finish the paper by connecting paragraphs 2 and 3. As we see in the second paragraph, some games have $C(v) = \varnothing$ and $DC(v) \neq \varnothing$. We want to describe some games v where $DC(v)$ is the unique stable set and $DC(v) \neq C(v)$ (of course, $C(v) = \varnothing$).

Corollary 3.3: Let $v \in G^N$ such that $DC(v) \neq \varnothing$. If v' is convex (see for the definition of v' theorem 2.1), then

$DC(v)$ is the unique stable set of v.

Proof: $I(v) \neq \varnothing$ since $DC(v) \neq \varnothing$ and then $I(v) = I(v')$. In this case the identity map: $Id: I(v) \to I(v') = I(v)$ is a domination isomorphism i.e. $\forall x, y \in I(v)$ $x\ dom_S^v\ y$ if and only if $x\ dom_S^{v'}\ y$ as the reader can easily check. By hypothesis v' is convex and then $C(v')$ is the unique stable set for v'. But using theorem 2.1 we know $C(v') = DC(v)$ and then $DC(v)$ is the unique stable set of v' and using the domination isomorphism we can obtain that $DC(v)$ is the unique stable set of v. □

Let us show now an example where the above result can be used to check stable sets. Nevertheless, the applicability of the above corollary related to the class of non-convex games with convex derived game v' it is to be expected rather limited.

Example 3.1: Let $v \in G^{\{123\}}$ such that $v(1) = v(2) = 0$, $v(3) = 1$, $v(12) = 2$, $v(13) = 1$, $v(23) = 1$, $v(123) = 2$.

This game has an empty core, is non convex and $DC(v) \neq \varnothing$. Checking the game v': $v'(1) = v'(2) = 0$, $v'(3) = 1$, $v'(12) = \min\{v(12), v(N) - v(3)\} = \min\{2, 1\} = 1$, $v'(13) = v'(23) = 1$ and $v'(123) = 2$ it is easy to see that v' is convex. As a consequence, $DC(v) = [(1, 0, 1), (0, 1, 1)]$ is the unique stable set for v.

Let us point out that if want to use the Gillies' method to make the original game, v, superadditive using partitions i.e.

$$v^G(S) = \max \left\{ \sum_{i \in \mathscr{P}} v(P_i) / \{P_i\}_{i \in \mathscr{P}} \text{ partitions of } S \right\}$$

then the value of the grand coalition will change to 3. As a consequence both games will have different imputation sets.

Another approach to solve the example is to make use of the monotonic cover of v i.e.

$$v^M(S) = \max_{T \subseteq S} \{v(T)\}$$

but in our example the game is monotonic and then the monotonic cover does not give more information about its stable sets.

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