Marketable permits in a stochastic dynamic model of the firm

Hartl, R.F.; Kort, P.M.

Published in:
Journal of Optimization Theory and Applications

Publication date:
1996

JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 89, No. 1, pp. 129-155, APRIL 1996

Marketable Permits in a Stochastic Dynamic Model of the Firm[1]

R. F. HARTL[2] AND P. M. KORT[3]

Communicated by G. Leitmann

Abstract. This contribution attempts to determine the effects of environmental regulation on the growth of an individual firm. Here, it is assumed that the firm revenue is stochastic. The government tries to reduce pollution by creating a market on which the firm has to buy permits in order to be allowed to pollute the environment.

Pollution is an inevitable byproduct of the firm production process, and in our model the firm is offered two ways to deal with it. The first is to buy marketable permits, and the second is to clean up pollution which can be achieved through investing in abatement capital stock.

It turns out that the firm optimal trajectory consists of at most seven different policies. They can be depicted in a feedback diagram from which we can conclude that, provided that the firm never faces a shortage of cash, productive and abatement capital stocks ultimately reach their equilibrium levels where marginal revenue equals marginal costs.

Key Words. Optimal control, stochastic control, Hamilton-Jacobi-Bellman equation, environmental problems, dynamics of the firm, marketable permits.

1. Introduction

This paper is concerned with two streams of research. The first deals with stochastic dynamic models of the firm; the second deals with the impact

[1]This paper was presented at EURO XIII, July 19-22, 1994 in Glasgow, Scotland. The research of the second author has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences. Thanks are due to M. Stimming for valuable comments and suggestions and to A. Van Den Elzen for his corrections.
[2]Professor, Institute of Management, University of Vienna, Vienna, Austria.
[3]Research Fellow, Department of Econometrics and CentER, Tilburg University, Tilburg, Netherlands.

of environmental regulation on dynamic firm behavior. In the first area, our research finds its origin in the work by Bensoussan and Lesourne (Ref. 1), who studied the optimal investment behavior of a self-financing dividend-maximizing firm with stochastic revenues. This model does not include environmental regulation. In the second area, our model is related to the works of Kort, Van Loon, and Luptacik (Ref. 2) and Xepapadeas (Ref. 3), where the effects of environmental regulation on the optimal dynamic firm behavior are analyzed. However, these models do not include any uncertainties.

Primarily, this paper extends the stochastic dynamic model of the firm of Bensoussan and Lesourne (Ref. 1) by letting the firm deal with environmental regulation in the form of a marketable permit system. In this system, the firm needs to buy permits in order to be allowed to pollute the environment. As soon as its pollution decreases, the firm can sell those permits that are not necessary anymore to support the firm's pollution output.

The organization of the paper is as follows. In Section 2, the dynamic optimization model is presented. Section 3 contains the derivation and some characteristics of the feasible policies. The optimal solution is presented in Section 4 with feedback diagrams and economic interpretations. Section 5 concludes the paper. Appendices A and B (Sections 6 and 7) contain the mathematical details of Section 3, while the proofs of the propositions stated in Section 4 can be found in Appendix C (Section 8).

2. Model

2.1. Production and Abatement Process. An empirical study by Jorgenson and Wilcoxen (Ref. 4) shows that there are three possible firm responses to environmental regulation. First, the firm may substitute less polluting inputs for more polluting ones. In Hartl and Kort (Ref. 5), the dynamic firm behavior is studied, while the firm uses this possibility to respond to environmental regulation in the form of a pollution standard. The second response is redesigning the production process to reduce emissions; the third response is the use of special devices to treat wastes after they have been generated. The latter approach is commonly known as end-of-pipe abatement and involves investment in costly new equipment for pollution abatement. Jorgenson and Wilcoxen (Ref. 4) concluded that investment in abatement equipment has the largest impact; for this reason, we choose to model the end-of-pipe abatement here.

Let us assume that the firm produces with one type of capital stock. We define the following variables: $K_1(t)$ = productive capital stock at time t, $K_2(t)$ = abatement capital stock at time t, $I_1(t)$ = investment in the productive capital stock at time t, $I_2(t)$ = investment in the abatement capital stock at time t.

We assume that there is no depreciation and that investment is irreversible, i.e.,

$$\dot{K}_1 = I_1, \qquad \dot{K}_2 = I_2, \tag{1}$$

$$I_1 \geq 0, \qquad I_2 \geq 0, \tag{2}$$

where a dot over a variable stands for differentiating this variable w.r.t. time. We omit the time argument if no confusion can be caused.

Producing with capital stock K_1 causes emissions E which can be influenced by utilizing the abatement capital stock K_2. We define $E(K_1, K_2)$ = pollution flow.

It is reasonable to assume that: emissions can never be reduced to zero unless no production takes place; emissions are increasing and convex in the productive capital stock; abatement is subject to decreasing returns to scale; abatement capital has a higher marginal effectiveness for a higher productive capital stock; emissions are convex in both capital stocks. Therefore,

$$E(K_1, K_2) > 0, \text{ for } K_1 > 0 \text{ and } K_2 \geq 0, \qquad E(0, K_2) = 0, \tag{3a}$$

$$E_{K_1} > 0, \qquad E_{K_1K_1} > 0, \tag{3b}$$

$$E_{K_2} < 0 \text{ and } E_{K_2K_2} > 0, \qquad \text{for } K_1 > 0, \tag{3c}$$

$$E_{K_1K_2} < 0, \qquad E_{K_1K_1}E_{K_2K_2} - (E_{K_1K_2})^2 > 0. \tag{3d}$$

In Xepapadeas (Ref. 3) and Kort, Van Loon, and Luptacik (Ref. 2), the pollution function is separable in K_1 and K_2. This implies that a given amount of abatement capital stock leads to a given reduction of pollution, irrespective of the current level of pollution. But in reality, it holds that more abatement investments are required to reduce pollution with some fixed amount as the level of pollution shrinks. Our nonseparable pollution function satisfies this property and therefore is more realistic than the pollution functions in the above-mentioned works.

2.2. Environmental Regulation. Well-known instruments that can be applied by the government to control pollution are imposing an upper limit (standard) on pollution, a pollution tax rate, subsidizing firm attempts to reduce its own pollution, and creating a market on which the firm needs to buy permits in order to be allowed to pollute the environment. Here, we consider a firm facing a pollution permit system. Responses of the firm to other forms of environmental regulations are studied, e.g., in Helfand (Ref. 6, standard), Kort, Van Loon, and Luptacik (Ref. 2, tax subsidies), and

Xepapadeas (Ref. 3, tax and standard). We assume a pollution permit system in the sense that: every time a firm increases pollution, it needs to buy permits; or alternatively every time it decreases pollution, it sells permits; one permit is associated with one unit of pollution; once a permit is bought, it remains valid forever; i.e., not only can a permit be used to cover one unit of pollution tomorrow, but it can cover also one unit of pollution over 5 years, 10 years, etc. It is of interest to relax this assumption, i.e., to consider permits that are valid only for a limited period of time.

The price of one permit is assumed to be fixed: p = price of one pollution permit. $E(K_1(0), K_2(0))$ is the initial emission rate, and we assume that the firm owns at time zero just enough permits to cover this emission rate. Should the firm never deviate from $E(K_1(0), K_2(0))$ during the remaining planning period, then the firm would not have to buy any more permits, so expenditures on permits would be zero. Only if the firm changes its emission rate [say positively, i.e., $\dot{E}>0$] does the firm need to buy more permits to support its higher rate of emissions. Hence, the firm's expense on the permit market at time t equals

$$p\dot{E}=p(E_{K_1}\dot{K}_1+E_{K_2}\dot{K}_2)=p(E_{K_1}I_1+E_{K_2}I_2). \tag{4}$$

Notice that spending turn into earning as soon as emissions are reduced.

2.3. Revenue. Revenue from selling the goods produced with capital stock K_1 is assumed to be subject to random fluctuations,

$$R(K_1)dt=S(K_1)\{dt+\sigma dB\}; \tag{5}$$

here, $R(K_1)$ = revenue from producing with capital stock K_1; B = standard Wiener process with independent increments dB, normally distributed with mean zero and variance dt; $S(K_1)$ = expected revenue from producing with capital stock K_1, with $S(0)=0$, $S'>0$, $S''<0$; σ = positive constant.

This general representation of uncertainty includes the cases of demand uncertainty, stochastic product price, input price uncertainty, or random distributions in the production function or cost function (Kobila, Ref. 7).

2.4. Financial Accounts. We assume that the firm finances its assets only by retained earnings, so that attracting debt money and issuing new equity are excluded. Admittedly, this assumption is restrictive and the reason for omitting external finance is mainly technical. On the other hand, it can be argued that internal finance has been the dominant source of finance historically as well as during the post-World War II era; see Judd and Petersen (Ref. 8, p. 374).

The firm's assets consist of the cash balance and productive and abatement capital stock. Fixing the value per unit of a capital good at one unit of money, we arrive at the following balance sheet equation:

$$M + K_1 + K_2 = X, \tag{6}$$

where X = equity and M = cash balance.

For technical reasons to be explained later, we introduce adjustment costs on abatement investments. Equity increases with revenue and decreases through expenses on the permit market, distributing dividends and adjustment costs on abatement investments,

$$dX = [R(K_1) - p\dot{E} - D - A(I_2)]dt; \tag{7}$$

here, D = dividends and $A(I_2)$ = adjustment costs on abatement investments, with $A(0) = 0$, $A' > 0$, $A'' > 0$.

Now, (1) and (4)–(7) lead to the following state equation for M:

$$dM = [S(K_1) - I_1(1 + pE_{K_2}) - I_2(1 + pE_{K_1}) - D - A(I_2)]dt$$
$$+ \sigma S(K_1)dB. \tag{8}$$

Further, it is assumed that the firm does not spend on investment and dividend more than the expected revenue net from pollution expenses,

$$D + I_1 + I_2 + A(I_2) \leq S(K_1) - p\dot{E}, \tag{9a}$$

which can also be written as

$$D + I_1(1 + pE_{K_1}) + I_2(1 + pE_{K_2}) + A(I_2) \leq S(K_1). \tag{9b}$$

This restriction makes clear why we have to introduce adjustment costs on abatement investments. If we skip $A(I_2)$ in this restriction, we see that I_2 is unbounded in case $1 + pE_{K_2}$ is negative.

2.5. Complete Model. Before we present the whole model, we define the terminal time T as an absorbing time at which the firm is no longer able to meet short-term commitments (i.e., cash balances are null or negative) and is considered bankrupt,

$$T = \inf\{t \mid M(t) \leq 0\}. \tag{10}$$

We assume that the firm behaves so as to maximize the shareholders' value of the firm. This value consists of the discounted dividend stream over the planning period.

With $\mathscr{E}$ denoting the expectation operator and i denoting the shareholder's time preference rate, $i > 0$ and constant, the stochastic dynamic model of the firm can now be summarized as follows:

$$\max_{I_1, I_2, D} \mathscr{E}\left\{\int_0^T \exp(-it)D\,dt\right\}, \tag{11a}$$

$$\text{s.t.} \quad dK_1 = I_1\,dt, \tag{11b}$$

$$dK_2 = I_2\,dt, \tag{11c}$$

$$dM = [S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - D - A(I_2)]\,dt \tag{11d}$$

$$+ \sigma S(K_1)dB, \tag{11e}$$

$$D \geq 0, \qquad I_1 \geq 0, \qquad I_2 \geq 0,$$

$$S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - D - A(I_2) \geq 0, \tag{11f}$$

$$T = \inf[t \mid M(t) \leq 0], \tag{11g}$$

$$K_1(0) = K_{10}, \qquad K_2(0) = K_{20}, \qquad M(0) = M_0, \tag{11h}$$

where K_{10}, K_{20}, M_0 are given initial values of capital and cash.

3. Solution Using Stochastic Control Theory

As usual, we define the value function $V(M, K_1, K_2)$ as

$$V(M(t), K_1(t), K_2(t)) = \max_{I_1, I_2, D} \mathscr{E}\left\{\int_t^T \exp(-i(s-t))D\,ds\right\}, \tag{12}$$

where all the above constraints have to be satisfied. We assume that $V(M, K_1, K_2)$ is C^2.

3.1. HJB Equation and KT Conditions. The Hamilton–Jacobi–Bellman equation (HJB) for this problem is (see, e.g., Ref. 9)

$$\begin{aligned} iV = {} & (\sigma^2/2)S^2(K_1)V_{MM} \\ & + \max_{(11e),(11f)} [D + V_M(S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) \\ & - D - A(I_2)) + V_{K_1}I_1 + V_{K_2}I_2], \end{aligned} \tag{13}$$

with boundary condition

$$V(0, K_1, K_2) = 0. \tag{14}$$

Using the Lagrangian

$$L = D + V_M[S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - D - A(I_2)] + V_{K_1}I_1 + V_{K_2}I_2 + \lambda_1 I_1 + \lambda_2 I_2 + \lambda_3 D + \lambda_4[S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - D - A(I_2)], \tag{15}$$

the application of the Kuhn–Tucker conditions to the maximization problem in the HJB equation (13) yields

$$L_D = 1 - V_M + \lambda_3 - \lambda_4 = 0, \tag{16}$$

$$L_{I_1} = V_{K_1} - (V_M + \lambda_4)(1 + pE_{K_1}) + \lambda_1 = 0, \tag{17}$$

$$L_{I_2} = V_{K_2} - (V_M + \lambda_4)[1 + pE_{K_2} + A'(I_2)] + \lambda_2 = 0, \tag{18}$$

with the complementary slackness conditions

$$\lambda_1 I_1 = \lambda_2 I_2 = \lambda_3 D = 0, \tag{19}$$

$$\lambda_4[S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - D - A(I_2)] = 0. \tag{20}$$

For fixed values of K_1, K_2, M, the maximization problem in the HJB equation can also be sketched graphically in the (K_1, K_2, M)-space, see Fig. 1. The optimum can lie either in the corners (Cases 1, 2, 3, 11), or the edges (Cases 4, 5, 6, 13, 14, 15), or on one of the facets (Cases 7, 8, 9, 10), or in the interior (Case 12) of the set of feasible points (D, I_1, I_2) satisfying

$$D \geq 0, \qquad I_1 \geq 0, \qquad I_2 \geq 0,$$

$$S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - D - A(I_2) \geq 0.$$

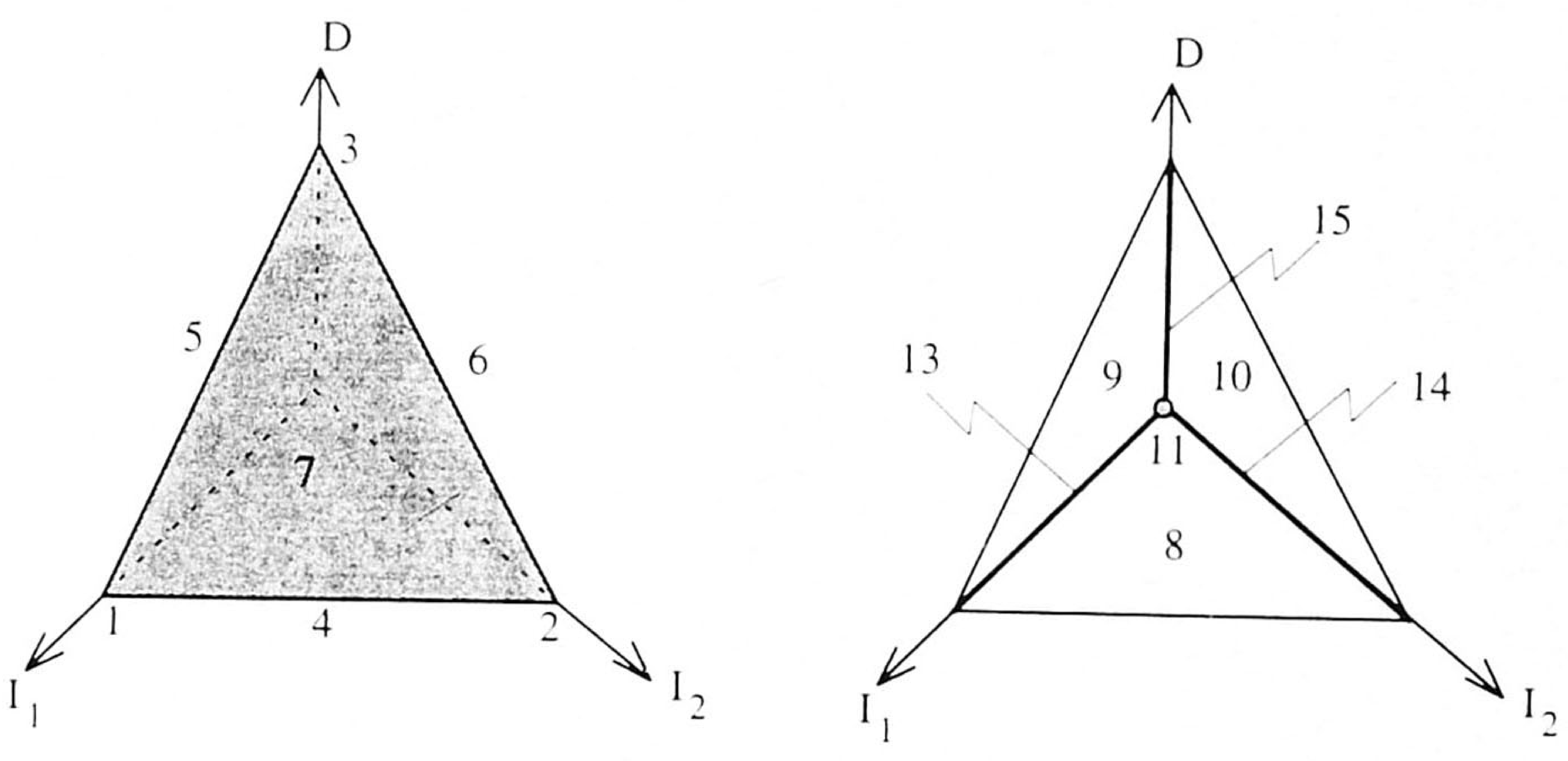

Fig. 1. Fifteen cases in the maximization problem. Case 12 refers to the interior.

In Appendix A (Section 6), we analyze these cases separately, determine conditions under which they can be optimal, and whenever possible, solve the HJB equation analytically.

3.2. Feasible Cases in the Static Maximization Problem of the HJB Equation. From Appendix A (Section 6), we obtain that the firm has the following candidate policies for optimality:

Case 1: I_1-policy,

$$I_1(1+pE_{K_1})=S(K_1), \qquad I_2=0, \qquad D=0;$$

Case 2: I_2-policy,

$$I_1=0, \qquad I_2(1+pE_{K_2})+A(I_2)=S(K_1), \qquad D=0;$$

Case 3: D-policy,

$$I_1=0, \qquad I_2=0, \qquad D=S(K_1);$$

Case 4: I_1/I_2-policy,

$$I_1(1+pE_{K_1})+I_2(1+pE_{K_2})+A(I_2)=S(K_1), \qquad D=0;$$

Case 6: I_2/D-policy,

$$I_1=0, \qquad I_2(1+pE_{K_2})+A(I_2)+D=S(K_1);$$

Case 11: M-policy,

$$I_1=I_2=D=0;$$

Case 14: I_2/M-policy,

$$I_1=0, \qquad I_2>0, \qquad I_2(1+pE_{K_2})+A(I_2)<S(K_1), \qquad D=0.$$

Depending on the values of the state variables M, K_1, K_2, one of these policies is optimal. The conditions under which these policies are optimal can be found in Appendix A (Section 6). In what follows, we denote the region in the (M, K_1, K_2)-space where the I_1-policy is optimal by I_1-region, the region where the I_2-policy is optimal by I_2-region, and so on.

Table 1. Feasible regions and regions which can satisfy the boundary condition, i.e., which can reach the (K_1, K_2)-plane defined by $M=0$. Regions that can be adjacent to each other are denoted by P; N means not possible; H means hairline case. Hairline cases can be omitted in the sequel, since they will not appear as two-dimensional surfaces in the (M, K_1, K_2)-space, but as lines or points.

	Case	C1	C2	C3	C4	C6	C11	
Case	Region	I_1	I_2	D	I_1/I_2	I_2/D	M	Boundary condition
C1	I_1							No
C2	I_2	N						No
C3	D	P	N					Yes
C4	I_1/I_2	P	P	H				No
C6	I_2/D	H	P	P	P			Yes
C11	M	P	N	P	H	H		Yes
C14	I_2/M	H	P	H	P	P	P	Yes

In Table 1, we show regions in which the boundary condition (14) can be satisfied, i.e., which regions can be adjacent to the plane $M=0$. Furthermore, we also show which regions can be bounded to each other. The proofs of these results can be found in Appendix B (Section 7).

4. Different Cases and Corresponding Regions in the Feedback Diagrams

Using the above results, we can derive the values of the state variables K_1, K_2, M for which the cases listed in Section 3.2 can occur. The corresponding proofs can be found in Appendix C (Section 8).

Proposition 4.1. Assume that K_2 is such that the D-region exists for $M\to\infty$ and K_1 finite. Then, we obtain the following results:

(i) Assume that the parameter values satisfy the following inequality:

$$1/i-\sigma/\sqrt{2i}>0, \qquad \text{i.e., } \sigma^2 i<2, \tag{21}$$

i.e., the discount rate and/or the stochastic disturbance is not too large. Under this scenario, only the M-region can exist for $M=0$ and $K_1<\tilde{K}_1(K_2)$, where $\tilde{K}_1(K_2)$ is implicitly defined by

$$1+pE_{K_2}(\tilde{K}_1, K_2)+A'(0)=0. \tag{22}$$

(ii) The boundary between the M-region and the D-region is given by $M=\rho S(K_1)$, where ρ satisfies the following equation:

$$\exp[(r_1-r_2)\rho]=[1-r_2(1/i-\sigma/\sqrt{2i})]/[1-r_1(1/i-\sigma/\sqrt{2i})]$$
$$=[2i-r_2(2-\sigma\sqrt{2i})]/[2i-r_1(2-\sigma\sqrt{2i})], \qquad (23)$$

in which

$$r_1=[\sqrt{1+2\sigma^2 i}-1]/\sigma^2, \qquad r_2=[-\sqrt{1+2\sigma^2 i}-1]/\sigma^2.$$

(iii) For K_2 fixed, the boundary between the I_1-region and the D-region increases in the (M, K_1)-plane and lies below a horizontal asymptote which is situated on the level $K_1^*(K_2)$, where $K_1^*(K_2)$ is implicitly defined by

$$S'(K_1^*)=i(1+pE_{K_1}(K_1^*, K_2)). \qquad (24)$$

The D-region is situated above this boundary.

Consider the (M, K_1)-plane for K_2 fixed. At the intersection point $(\bar{M}, \bar{K}_1)$ of the boundary between the I_1-region and the D-region and the boundary between the M-region and the D-region, it must hold that

$$[S'(K_1)/(1+pE_{K_1})][1/i-\sigma/\sqrt{2i}-\rho]=1. \qquad (25)$$

(iv) The boundary between the D-region and the I_2/D-region is given by $\tilde{K}_1(K_2)$, while the D-region is situated below this boundary. Hence, for K_2 fixed, this boundary is a horizontal line in the (M, K_1)-plane.

(v) Also, the boundary between the M-region and the I_2/M-region is given by $\tilde{K}_1(K_2)$.

(vi) Consider the (M, K_1)-plane for K_2 fixed. The boundary between the M-region and I_1-region starts at the origin and ends at the intersection point of the boundaries between the M-region and D-region, and between the I_1-region and D-region.

From properties (iii) and (iv) of the above proposition, we conclude that, when the D-region really exists for $M\to\infty$, it will exist for $K_1\in[K_1^*(K_2), \tilde{K}_1(K_2))$. With this in mind, we formulate the following proposition.

Proposition 4.2. Define K_2^* such that

$$K_1^*(K_2^*)=\tilde{K}_1(K_2^*). \qquad (26)$$

Then, it holds that

$$\tilde{K}_1(K_2)>K_1^*(K_2), \qquad \text{for } K_2>K_2^*.$$

Hence, a necessary condition for the D-region to exist for $M \to \infty$ and finite K_1 is that $K_2 > K_2^*$.

Of course, Proposition 4.2 provides only a necessary condition for a dividend policy to be optimal for $M \to \infty$ and finite K_1, and thus for Proposition 4.1 to be valid. Notice also that, from an economic point of view, it is unlikely that investing is optimal for $K_1 \in (K_1^*, \tilde{K}_1)$. Namely, if we consider the productive investment, we can argue that the expected return of marginal productive investment [i.e., $S'/(1+pE_{K_1})$], falls below the discount rate when $K_1 > K_1^*$; this is an economically plausible reason for productive investment not to be optimal. Furthermore, the cash flow of marginal abatement investment [i.e., $-(1+pE_{K_2}+A')$], is negative for $K_1 < \tilde{K}_1$, so abatement investment will not be optimal when the firm finds itself in the interval $(K_1^*, \tilde{K}_1)$.

Hence, due to economic arguments, we can exclude the policies I_1, I_1/I_2, I_2, I_2/D, I_2/M for $K_1 \in (K_1^*, \tilde{K}_1)$. Also, it does not make much sense to perform a cash policy when M is already infinite (very large). This implies that the only economically reasonable policy is the dividend policy when $M \to \infty$ and $K_1 \in (K_1^*, \tilde{K}_1)$.

In order to construct the feedback diagrams in the (M, K_1)-plane, we use the above information and Table 1, showing which regions can be bounded to each other. We consider three different K_2-values:

(I) $K_2 \geq K_2^*$;
(II) K_2 is low;
(III) intermediate values of K_2.

4.1. Results in the (K_1, M)-Diagram. Due to space restrictions, we consider only the most interesting case where[4]

$$S'(0)[1/i - \sigma/\sqrt{2i} - \rho]/[1+pE_{K_1}(0, K_2)] > 1. \tag{27}$$

Feedback Diagram for $K_2 \geq K_2^*$. From Propositions 4.1 and 4.2 and Table 1, we derive Fig. 2, which shows that the firm keeps its cash if the amount of productive capital stock is high enough while the cash situation is poor. The firm invests, if the stock of productive capital goods is low while there is plenty of cash to limit the risk of bankruptcy. The firm distributes dividends, if the expected marginal revenue is too small to justify

[4]If $S'(0)[1/i - \sigma/\sqrt{2i} - \rho]/[1+pE_{K_1}(0, K_2)] < 1$ or even $\sigma^2 i > 2$, then the solution changes as summarized by Van Hilten, Kort, and Van Loon (1993) for the original Bensoussan and Lesourne model (1980).

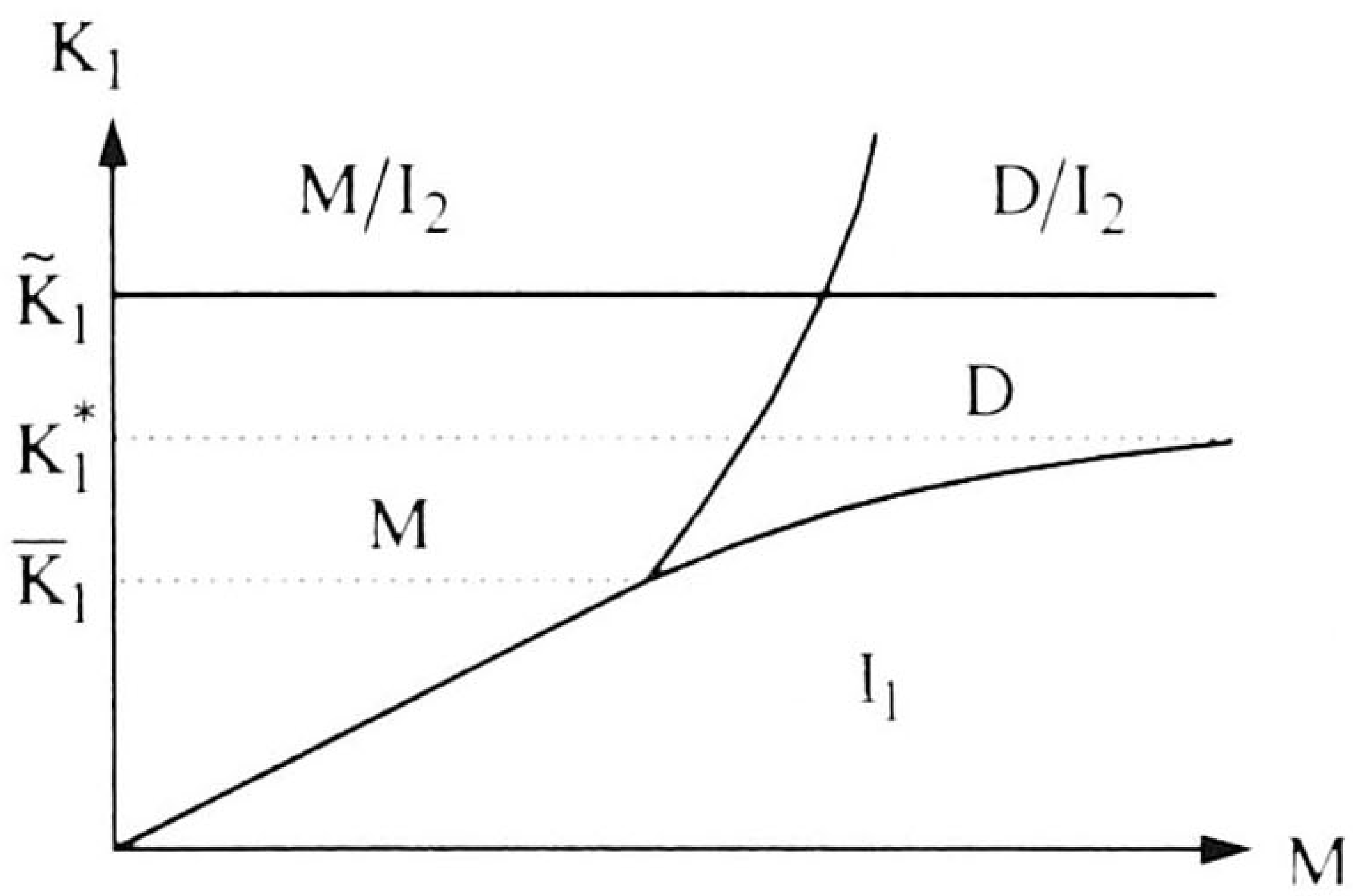

Fig. 2. Optimal solution when $K_2 \geq K_2^*$.

additional growth and the amount of cash available is high enough to guarantee a sufficiently safe situation.

If the firm invests in abatement capital stock, it incurs expenses on acquisition I_2 and adjustment costs $A(I_2)$. On the other hand, pollution decreases, so that the firm can sell permits which gives a revenue $-pE_{K_2}I_2$. Hence, the cash flow from abatement investments equals $-pE_{K_2}I_2 - I_2 - A(I_2)$.

For $K_1 > \tilde{K}_1$, abatement investment leads to a positive cash flow when I_2 is determined such that

$$-pE_{K_2} - 1 - A'(I_2) = 0. \tag{28}$$

The reason is that

$$-pE_{K_2}I_2 - I_2 - A(I_2) > -I_2[pE_{K_2} + 1 + A'(I_2)] = 0. \tag{29}$$

This leads to the conclusion that, for $K_1 > \tilde{K}_1$, the firm can earn money by investing in the abatement capital stock. In Fig. 2, this money is used to increase the cash balance when M is relatively low, while it is used for paying dividends when M is large.

As mentioned already, for $K_1 > K_1^*$ the expected marginal return of productive investment falls below the shareholder's time preference rate; therefore, it is not optimal to invest productively in this case.

Since $\tilde{K}_1$ exceeds K_1^*, it is not optimal to invest in the productive capital stock if $K_1 > K_1^*$. Therefore, the region I_1/I_2 does not occur here.

From Table 1, we derive that, when there is an I_2-region, it should occur somewhere above the line $K_1 = \tilde{K}_1$. But in the I_2-region, cash flow of abatement investment is negative, since it holds that

$$S(K_1) = I_2(1 + pE_{K_2}) + A(I_2) > 0$$

$$\Rightarrow -I_2(1 + pE_{K_2}) - A(I_2) < 0. \tag{30}$$

Another reason for the abatement investment to be optimal could be that it makes investment in productive capital stock cheaper. This is so, since the firm needs to buy less permits if K_2 is large when the firm invests in productive capital stock, because $E_{K_1K_2}<0$. But the firm does not invest in productive capital stock anymore when $K_1>\tilde{K}_1$. Also in the future this will be the case, even when we take into account that $\tilde{K}_1$ increases when K_2 increases, because from Fig. 2 we derive that just below $\tilde{K}_1$ we have $I_1=I_2=0$. $I_1=0$ indicates that there is still no productive investment, while $I_2=0$ implies that the increase of $\tilde{K}_1$ is stopped.

We conclude that we cannot find an economic reason for the I_2-policy to be optimal. Therefore, the I_2-region does not occur in Fig. 2.

Concerning Fig. 2, we have to remark also that the point $(\bar{M}, \bar{K}_1)$ does not exist when it holds that [cf. (25)]

$$S'(0)[1/i-\sigma/\sqrt{2i}-\rho]/[1+pE_{K_1}(0, K_2)]<1. \tag{31}$$

But nonexistence of $(\bar{M}, \bar{K}_1)$ would imply that it becomes optimal to pay dividends for low levels of M and K_1 [cf. Van Hilten, Kort, and Van Loon (Ref. 10, p. 404)]. This kind of behavior is only realistic in extreme situations; therefore, we choose to disregard such a case here.

Feedback Diagram for a Low Value of K_2. From Propositions 4.1 and 4.2 and Table 1, we derive Fig. 3. The firm carries out a cash policy with the aim of preventing bankruptcy. Bankruptcy occurs when the cash balance becomes negative. For $K_1>\tilde{K}_1$, extra cash money can be generated by abatement investments. Hence, besides carrying out a cash policy, the firm can also try to prevent bankruptcy by investing in abatement capital stock (I_2/M). Therefore, when K_1 is so large that investing in abatement capital stock generates a positive cash flow, the I_2/M-region need not be as large as the M-region, which would be optimal in the absence of abatement investment as extra instrument to keep the cash balance positive.

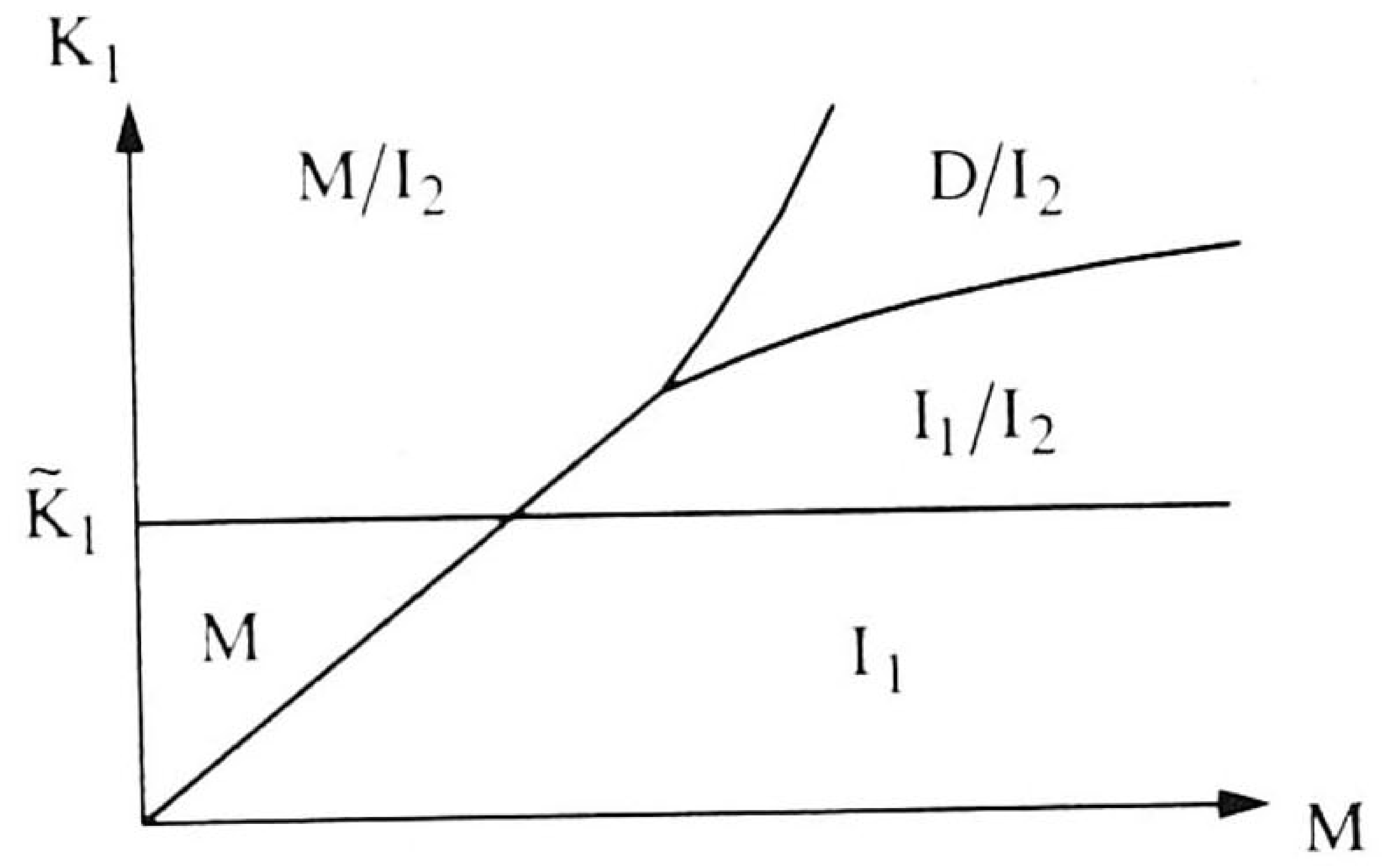

Fig. 3. Optimal solution when K_2 is low.

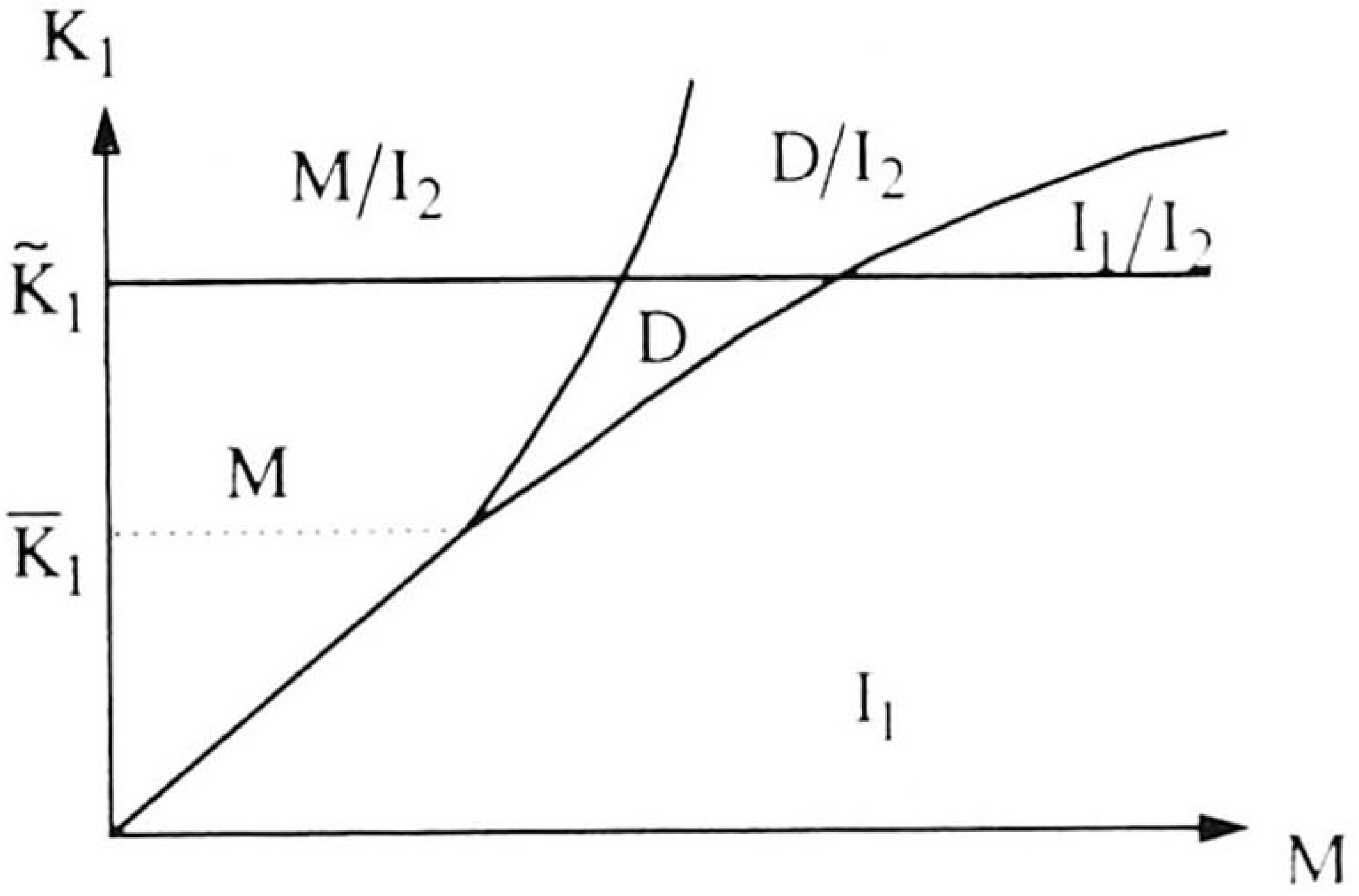

Fig. 4. Optimal solution for an intermediate value of K_2 that is below K_2^*.

Observe that Fig. 3 is drawn for a lower value of K_2 than Fig. 2. Therefore, for the same value of K_1, abatement investment generates more cash flow in Fig. 3 than in Fig. 2. This implies that, in Fig. 3, the I_2/M-region need not be that large to prevent bankruptcy; i.e., the $(I_2/M)-(I_2/D)$ boundary occurs for relatively low values of M.

As said already, K_2 is lower in Fig. 3 than in Fig. 2. This implies that, for the same value of K_1, E_{K_1} is larger in Fig. 3. Therefore, in Fig. 3, more permits have to be bought when the firm invests in productive capital stock, which makes productive investment more costly. We conclude that the regions where I_1 is positive will be smaller in Fig. 3.

Feedback Diagram for an Intermediate Value of K_2. Knowing Figs. 2 and 3, it is not so difficult to draw the feedback diagram in this case (K_2 below K_1^*). This is done in Fig. 4.

4.2. Feedback Diagram in the (K_2, K_1)-Plane for M Relatively Large. Due to Figs. 2, 3, 4, it is easy to draw a feedback diagram in the (K_2, K_1)-plane for a fixed large value of M. This is done in Fig. 5, which

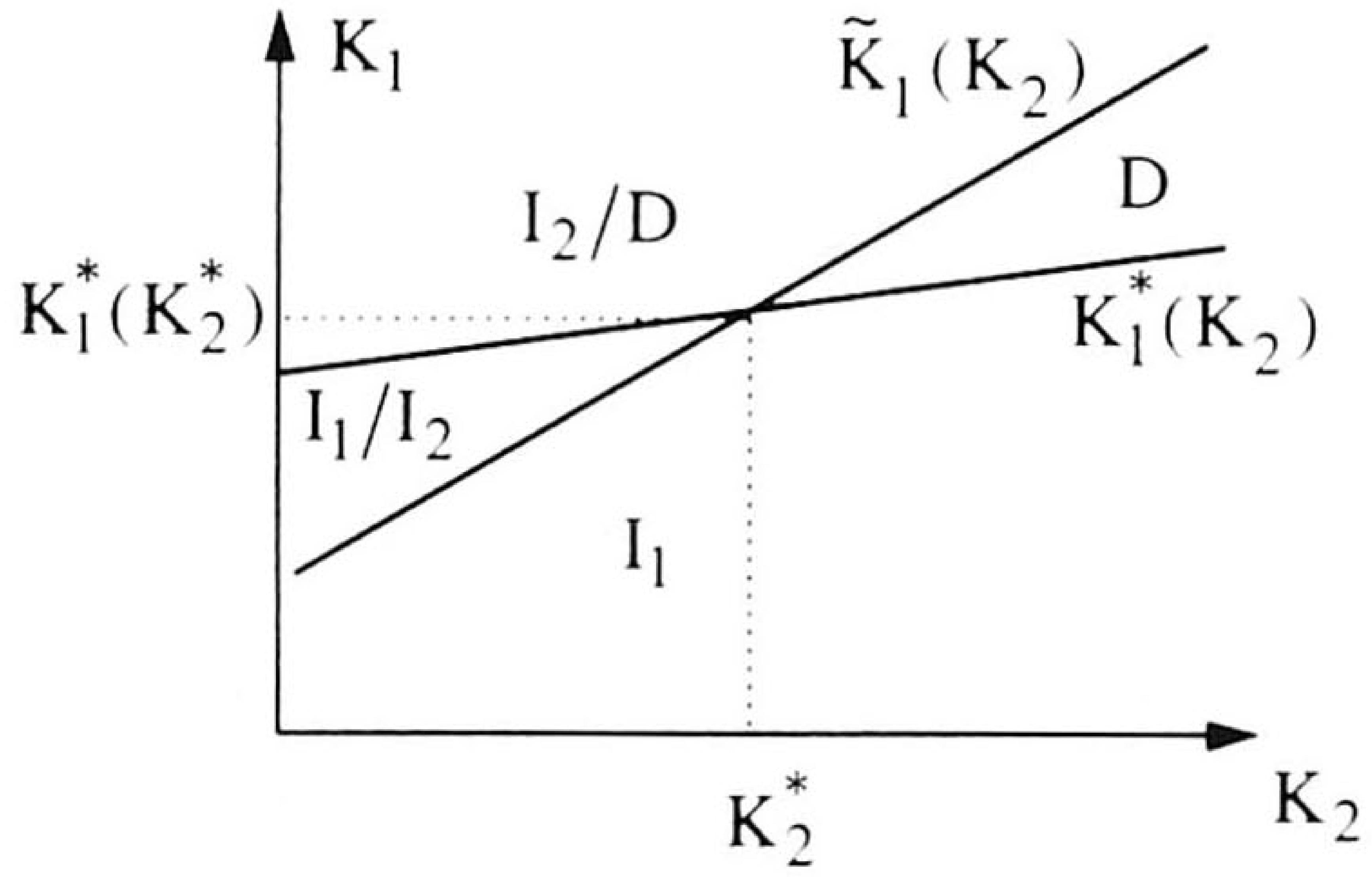

Fig. 5. Feedback diagram in (K_2, K_1)-plane for large value of M when (27) holds.

shows that the firm invests more in productive capital stock if K_2 is large, because then marginal pollution costs pE_{K_1} are lower.

$\tilde{K}_1(K_2)$ is a boundary below which no abatement investment takes place. Above it, K_1 is large so that pollution is also large, implying that it is worthwhile for the firm to spend money on abatement. Provided that there is enough cash money for all t and that the firm starts out such that

$$K_1(0) < K_1^*(K_2^*) \quad \text{and} \quad K_2(0) < K_2^*,$$

the firm ends up in $(K_2^*, K_1^*(K_2^*))$. Here, it holds that

$$S'(K_1^*) = i[1 + pE_{K_1}(K_1^*, K_2^*)], \tag{32}$$

$$1 + pE_{K_2}(K_1^*, K_2^*) + A'(0) = 0. \tag{33}$$

Equation (32) implies that, in (K_2^*, K_1^*), the expected return of a marginal productive investment expenditure equals the shareholder's time preference rate i. Equation (33) implies that the cash flow of a marginal abatement investment expenditure equals zero. Further growth is optimal only when the expected return of marginal productive investment at least equals i [i.e., K_1 remains below or on the curve $K_1^*(K_2)$] and when the cash flow of marginal abatement investment is nonnegative [i.e., K_1 remains above or on the curve $\tilde{K}_1(K_2)$]. From Fig. 5, we obtain that this is not possible; therefore, it is optimal for the firm to remain in (K_2^*, K_1^*), once this point has been reached.

If at a certain point of time the cash balance is low, it can be necessary to use part of the revenue, obtained from selling products and marketable permits, for increasing the cash balance in order to prevent bankruptcy (cf. Figs. 2, 3, 4), rather than using it for investments. Hence, due to liquidity problems, the investments program that leads to (K_2^*, K_1^*) can be delayed considerably. In fact, there is even a probability that (K_2^*, K_1^*) is not reached at all, which is due to the bankruptcy risk.

Since in real life the price of a permit will change over time, it is important to know what happens when the price p increases. From the definition of $K_1^*(K_2)$ in (24), we derive that K_1^* decreases when p increases, so that the firm will reduce investments in productive capital stock. This is easy to understand, since environmental costs are higher when the permit price is large.

If we look at the definition of $\tilde{K}_1(K_2)$ in (22), we see also that $\tilde{K}_1$ decreases when p increases, implying that abatement investments increase. This is the proper thing to do, because the cash inflow due to abatement investments, which is $-pE_{K_2}$, increases when the permit price rises.

Concerning the equilibrium (32)–(33), comparative statics analysis shows that an increase of p results in a decrease of $K_1^*(K_2^*)$ and an increase of K_2^* in the case where the mixed derivative $E_{K_1K_2}$ is not too low,

$$dK_1^*(K_2^*)/dp = (i/\Delta)(E_{K_1}E_{K_2K_2} - E_{K_1K_2}E_{K_2}) < 0, \tag{34}$$

$$dK_2^*/dp = [-S''E_{K_2} + ip(E_{K_1K_2}E_{K_2} - E_{K_1K_2}E_{K_1})]/p\Delta > 0, \tag{35}$$

where

$$\Delta = S''E_{K_2K_2} - ip(E_{K_1K_1}E_{K_2K_2} - E_{K_1K_2}^2) < 0. \tag{36}$$

5. Conclusions

This paper combines two different streams of research: stochastic dynamics of the firm [Bensoussan and Lesourne (Ref. 1)] and optimal firm behavior under environmental regulation [Xepapadeas (Ref. 3)]. Here, environmental regulation is present in the form of a marketable permits system. This implies that the firm needs to buy permits in order to be allowed to pollute the environment.

Pollution arises as an inevitable byproduct of the firm's production process. The firm owns two different sorts of capital goods. The first is used to produce goods, and the second is meant to clean pollution after it has been generated in the sense of an end-of-pipe technology.

It turns out that we can determine seven candidate policies for optimality. Dependent on the current level of the state variables, it is optimal for the firm to apply one of them. If the firm lives long enough, it will reach its equilibrium where marginal revenue and marginal costs of productive and abatement capital stocks balance. It is also possible that the firm goes bankrupt before the equilibrium is reached, due to shortage of cash. Of course, bankruptcy is also possible when the firm is already in the equilibrium.

Compared to a deterministic framework, here the solution consists of a feedback diagram which indicates the optimal policy given the current state, rather than time functions of state and control variables. Furthermore, here the terminal time is determined endogenously by the bankruptcy time. Since the bankruptcy time is defined as the moment of time that cash becomes negative, it is possible to analyze the firm's cash decision, which cannot be done in a deterministic model.

6. Appendix A: Feasible Cases in the Static Maximization Problem of the HJB Equation

For reasons of space limitations, we state only the main results here. Details can be found in a full-length working paper (Ref. 11) which can be obtained from the authors. We state its solution only in the few cases (Cases 3 and 11) where the HJB equation can be solved analytically. In other cases, it cannot be solved analytically.

Case 1: I_1-Region. This case is defined by

$$D=0, \qquad I_1>0, \qquad I_2=0,$$

$$S(K_1)-I_1(1+pE_{K_1})=0.$$

From complementary slackness, we obtain

$$\lambda_1=0;$$

and from the other Kuhn–Tucker conditions, i.e.,

$$1-V_M+\lambda_3-\lambda_4=0,$$

$$V_{K_1}-(V_M+\lambda_4)[1+pE_{K_1}]=0,$$

$$V_{K_2}-(V_M+\lambda_4)[1+pE_{K_2}+A'(0)]+\lambda_2=0,$$

it follows that this case is optimal if

$$V_{K_1}/(1+pE_{K_1})\geq \max\{1, V_M\},$$

$$[V_{K_1}/(1+pE_{K_1})][1+pE_{K_2}+A'(0)]\geq V_{K_2}.$$

The control I_1 is given by

$$I_1=S(K_1)/(1+pE_{K_1}).$$

The boundary condition $V(0, K_1, K_2)=0$ is not satisfied here, since $V_{K_1}>0$. Thus, the I_1-region cannot be adjacent to the plane $M=0$ in the (K_1, K_2, M)-space.

Case 2: I_2-Region. This case is defined by

$$D=0, \qquad I_1=0, \qquad I_2>0,$$

$$S(K_1)-I_2(1+pE_{K_2})-A(I_2)=0,$$

i.e.,

$$\lambda_2=0.$$

From the Kuhn–Tucker conditions, it follows that this case is optimal if

$$V_{K_2}/[1+pE_{K_2}+A'(I_2)]\geq \max\{1, V_M, V_{K_1}/(1+pE_{K_1})\}.$$

The control I_2 is defined by

$$S(K_1) - I_2(1 + pE_{K_2}) - A(I_2) = 0.$$

The boundary condition $V(0, K_1, K_2) = 0$ is not satisfied here, since $V_{K_2} > 0$.

Case 3: D-Region. This case is defined by

$$D > 0, \qquad I_1 = 0, \qquad I_2 = 0,$$

$$S(K_1) - D = 0,$$

i.e.,

$$\lambda_3 = 0.$$

From the Kuhn–Tucker conditions, it follows that this case is optimal if

$$1 \geq \max\{V_M, V_{K_1}/(1 + pE_{K_1})\},$$

$$V_{K_2} \leq 1 + pE_{K_2} + A'(0).$$

The control D is defined by

$$D = S(K_1).$$

In this region, the HJB equation (13), i.e.,

$$iV = S(K_1) + (\sigma^2/2)S^2(K_1)V_{MM},$$

can be solved to yield

$$V = S(K_1)/i + c_1(K_1, K_2)\exp[M\sqrt{2i}/\sigma S(K_1)] + c_2(K_1, K_2)\exp[-M\sqrt{2i}/\sigma S(K_1)].$$

The boundary condition $V(0, K_1, K_2) = 0$ can be satisfied here.

Case 4: I_1/I_2-Region. This case is defined by

$$D = 0, \qquad I_1 > 0, \qquad I_2 > 0,$$

$$S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - A(I_2) = 0,$$

i.e.,

$$\lambda_1 = \lambda_2 = 0.$$

From the Kuhn–Tucker conditions, it follows that this case is optimal if

$$V_{K_1}/(1 + pE_{K_1}) \geq \max\{V_M, 1\},$$

$$[V_{K_1}/(1 + pE_{K_1})][1 + pE_{K_2} + A'(I_2)] = V_{K_2}.$$

The controls I_1, I_2 are defined by

$$S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - A(I_2) = 0$$

and the last equality. The boundary condition $V(0, K_1, K_2) = 0$ is not satisfied here, since $V_{K_1} > 0$.

Case 5: I_1/D-Region. This case is defined by

$$D > 0, \qquad I_1 > 0, \qquad I_2 = 0,$$

$$S(K_1) - I_1(1 + pE_{K_1}) - D = 0,$$

i.e.,

$$\lambda_1 = \lambda_3 = 0.$$

From the Kuhn–Tucker conditions, it follows that this case can be optimal only if

$$V_{K_1}/(1 + pE_{K_1}) = 1 \geq \max\{V_M, V_{K_2}/[1 + pE_{K_2} + A'(0)]\}.$$

Here, the marginal value of the dividend payout 1 equals the marginal value of productive investment $V_{K_1}/(1 + pE_{K_1})$. Since this equality contains only state variables, Case 5 will be a lower-dimensional surface, rather than a three-dimensional subset of the (M, K_1, K_2)-space. Therefore, it is a hairline case which can be omitted. In the feedback diagrams to come, this case represents the boundary line between the I_1-region and D-region.

Case 6: I_2/D-Region. This case is defined by

$$D > 0, \qquad I_1 = 0, \qquad I_2 > 0,$$

$$S(K_1) - I_2(1 + pE_{K_2}) - D - A(I_2) = 0,$$

i.e.,

$$\lambda_2 = \lambda_3 = 0.$$

From the Kuhn–Tucker conditions, it follows that this case is optimal if

$$1 \geq \max\{V_M, V_{K_1}/(1 + pE_{K_1})\}.$$

The control I_2 is defined by

$$V_{K_2} = 1 + pE_{K_2} + A'(I_2),$$

and D is defined by

$$S(K_1) - I_2(1 + pE_{K_2}) - D - A(I_2) = 0.$$

A necessary condition for the boundary condition to be satisfied [i.e., $V(0, K_1, K_2)=0$] is that $V_{K_2}=0$, i.e.,

$$1+pE_{K_2}+A'(I_2)=0, \qquad \text{when } M=0.$$

The following four cases (Cases 7, 8, 9, 10) are hairline cases which can easily be checked as shown in Case 5 above.

Case 7: $I_1/I_2/D$-Region. This hairline case is defined by

$$D>0, \qquad I_1>0, \qquad I_2>0,$$

$$S(K_1)-I_1(1+pE_{K_1})-I_2(1+pE_{K_2})-D-A(I_2)=0.$$

Case 8: $I_1/I_2/M$-Region. This hairline case is defined by

$$D=0, \qquad I_1>0, \qquad I_2>0,$$

$$S(K_1)-I_1(1+pE_{K_1})-I_2(1+pE_{K_2})-A(I_2)>0.$$

Case 9: $I_1/D/M$-Region. This hairline case is defined by

$$D>0, \qquad I_1>0, \qquad I_2=0,$$

$$S(K_1)-I_1(1+pE_{K_1})-D>0.$$

Case 10: $I_2/D/M$-Region. This hairline case is defined by

$$D>0, \qquad I_1=0, \qquad I_2>0,$$

$$S(K_1)-I_2(1+pE_{K_2})-D-A(I_2)>0.$$

Case 11: M-Region. This case is defined by

$$D=0, \qquad I_1=0, \qquad I_2=0,$$

$$S(K_1)>0,$$

i.e.,

$$\lambda_4=0.$$

From the Kuhn–Tucker conditions, it follows that this case is optimal if

$$V_M \geq \max\{1, V_{K_1}/(1+pE_{K_1})\},$$

$$V_M(1+pE_{K_2}+A'(0)) \geq V_{K_2}.$$

In this region, the HJB equation

$$iV=(\sigma^2/2)S^2(K_1)V_{MM}+V_M S(K_1)$$

can be solved analytically to yield

$$V = k_1(K_1, K_2)\exp[Mr_1/S(K_1)] + k_2(K_1, K_2)\exp[Mr_2/S(K_1)],$$

with

$$r_1 = [\sqrt{1+2\sigma^2 i} - 1]/\sigma^2, \qquad r_2 = [-\sqrt{1+2\sigma^2 i} - 1]/\sigma^2.$$

The boundary condition $V(0, K_1, K_2) = 0$ can be satisfied here.

The following two cases (Cases 12 and 13) are hairline cases which can be omitted.

Case 12: $I_1/I_2/D/M$-Region. This hairline case is defined by

$$D > 0, \qquad I_1 > 0, \qquad I_2 > 0,$$

$$S(K_1) - I_1(1 + pE_{K_1}) - I_2(1 + pE_{K_2}) - D - A(I_2) > 0,$$

i.e.,

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.$$

Case 13: I_1/M-Region. This hairline case is defined by

$$D = 0, \qquad I_1 > 0, \qquad I_2 = 0,$$

$$S(K_1) - I_1(1 + pE_{K_1}) > 0,$$

i.e.,

$$\lambda_1 = \lambda_4 = 0.$$

Case 14: I_2/M-Region. This case is defined by

$$D = 0, \qquad I_1 = 0, \qquad I_2 > 0,$$

$$S(K_1) - I_2(1 + pE_{K_2}) - A(I_2) = 0,$$

i.e.,

$$\lambda_2 = \lambda_4 = 0.$$

From the Kuhn–Tucker conditions, it follows that this case is optimal if

$$V_M \geq \max\{1, V_{K_1}/(1 + pE_{K_1})\},$$

$$V_M[1 + pE_{K_2} + A'(I_2)] = V_{K_2}.$$

A necessary condition for the boundary condition $V(0, K_1, K_2) = 0$ to be satisfied is that $V_{K_2} = 0$, i.e.,

$$1 + pE_{K_2} + A'(I_2) = 0, \qquad \text{when } M = 0.$$

Case 15: D/M-Region. This case is defined by

$$D > 0, \qquad I_1 = 0, \qquad I_2 = 0,$$

$$S(K_1) - D > 0,$$

i.e.,

$$\lambda_3 = \lambda_4 = 0.$$

It is easily checked that this case is a hairline case, which can be omitted.

These results are summarized in Table 1 of Section 3.2.

7. Appendix B: Possibilities of Regions Adjacent to Each Other

Due to space limitations, we demonstrate only the combinations of Cases 1 and 2, Cases 1 and 3, and Cases 1 and 6 in order to have an example for each of the cases where adjacency is impossible, or possible, or a hairline case. The analysis of the other combinations can be found in our working paper (Ref. 11).

Cases 1 and 2. It is not possible that these cases are adjacent to each other, since at the possible boundary the conditions

$$[V_{K_1}/(1 + pE_{K_1})][1 + pE_{K_2} + A'(0)] \geq V_{K_2},$$

$$[V_{K_1}/(1 + pE_{K_1})][1 + pE_{K_2} + A'(I_2)] \leq V_{K_2}, \qquad \text{for } I_2 \geq 0,$$

would have to hold which are possible only for $I_2 = 0$; therefore $S(K_1) = 0$, which is a contradiction to $K_1 = 0$.

Cases 1 and 3. For Case 1, we have

$$D = 0, \qquad I_1 > 0, \qquad I_2 = 0,$$

$$S(K_1) - I_1(1 + pE_{K_1}) = 0,$$

while for Case 3 we have

$$D > 0, \qquad I_1 = 0, \qquad I_2 = 0,$$

$$S(K_1) - D = 0.$$

Thus, on the boundary, it must hold that

$$V_{K_1}/(1 + pE_{K_1}) = 1 \geq V_M,$$

$$1 + pE_{K_2} + A'(0) \geq V_{K_2}.$$

This is a possible case, since we have three independent equations for K_1, K_2, M, I_1, D.

Cases 1 and 6. The optimality conditions for both cases imply that $I_2=0$ on this boundary. Then, we have for Case 1 that

$$D=0, \qquad I_1>0, \qquad I_2=0,$$
$$S(K_1)-I_1(1+pE_{K_1})=0,$$

while for Case 6 we have

$$D>0, \qquad I_1=0, \qquad I_2=0,$$
$$S(K_1)-D=0.$$

On the boundary, it holds also that

$$V_{K_1}/(1+pE_{K_1})=1\geq V_M,$$
$$V_{K_2}=1+pE_{K_2}+A'(0).$$

This is a hairline case, since we have four equations for K_1, K_2, M, I_1, D. In each of the diagrams (K_1, M), (K_2, M), (K_1, K_2), the boundary between Cases 1 and 6 appears as a point rather than a curve, which is why we need not consider this combination any further.

The other combinations can be investigated in the same way and yield the results summarized in Table 1 of Section 3.2.

8. Appendix C: Proof of Propositions

Proof of Proposition 4.1. In the D-region, the value of the firm is given by

$$V=S(K_1)/i+c_1(K_1, K_2)\exp[M\sqrt{2i}/\sigma S(K_1)]$$
$$+c_2(K_1, K_2)\exp[-M\sqrt{2i}/\sigma S(K_1)].$$

Due to economic arguments, it is clear that the D-region exists for $M\rightarrow\infty$ and some finite K_1. If we keep in mind that V is finite, we can conclude that

$$c_1(K_1, K_2)=0.$$

(i) For $M=0$, the boundary condition $V(0, K_1, K_2)=0$ applies, which leads to $V_{K_1}=V_{K_2}=0$. Therefore, from Appendix A, we obtain that the only candidates that can include the K_1-axis in the (M, K_1)-plane for fixed K_2 are the regions D, M, I_2/D, I_2/M summarized in Table 1.

If $K_1 < \tilde{K}_1(K_2)$ it holds that

$$1 + pE_{K_2} + A'(0) > 0.$$

For the regions I_2/D and I_2/M, it holds that $V_{K_2} = 0$ implies

$$1 + pE_{K_2} + A'(I_2) = 0,$$

which contradicts the above inequality. Hence, these regions cannot occur for $M = 0$ and $K_1 < \tilde{K}_1(K_2)$.

Let us assume that the D-region exists for $M = 0$. If we keep in mind that $c_1(K_1, K_2) = 0$, we obtain

$$0 = S(K_1)/i + c_2(K_1, K_2)$$
$$\Rightarrow c_2(K_1, K_2) = -S(K_1)/i.$$

Then, we get the following value of the firm for the D-region:

$$V = S(K_1)/i - [S(K_1)/i] \exp[-M\sqrt{2i}/\sigma S(K_1)],$$

which implies

$$V_M = (\sqrt{2i}/\sigma i) \exp[-M\sqrt{2i}/\sigma S(K_1)]$$
$$\Rightarrow V_M|_{M=0} = (\sqrt{2i}/\sigma i) > 1, \qquad \text{for } 1/i - \sigma/\sqrt{2i} > 0.$$

But a dividend policy is optimal only if $V_M \leq 1$ (see Appendix A), so the D-region cannot occur for $M = 0$ in this scenario. Hence, the only candidate left that can occur for $M = 0$ and $K_1 < \tilde{K}_1(K_2)$ is the M-region, where it holds that

$$V = k_1(K_1, K_2) \exp[r_1 M/S(K_1)] + k_2(K_1, K_2) \exp[r_2 M/S(K_1)].$$

Satisfaction of the boundary condition $V(0, K_1, K_2) = 0$ leads to

$$k_1(K_1, K_2) = -k_2(K_1, K_2).$$

(ii) Denote the boundary between the M-region and the D-region by

$$M = \rho(K_1, K_2) S(K_1).$$

At this boundary, two conditions must hold: (a) equality of V in both regions; (b) $V_M = 1$ for both regions. Working on these conditions leads to the following equality [Van Hilten, Kort, and Van Loon, (Ref. 10, p. 390)]:

$$\exp[(r_1 - r_2)\rho(K_1, K_2)]$$
$$= [1 - r_2(1/i - \sigma/\sqrt{2i})]/[1 - r_1(1/i - \sigma/\sqrt{2i})].$$

We conclude that ρ is independent of K_1 and K_2, so that the boundary between the M-region and D-region is given by

$$M = \rho S(K_1).$$

According to Van Hilten, Kort, and Van Loon (Ref. 10, pp. 398–399), the functions $k_1(K_1, K_2)$ and $c_2(K_1, K_2)$ are given by the following expressions:

$$k_1(K_1, K_2) = S(K_1)/[r_1 \exp(r_1\rho) - r_2 \exp(r_2\rho)],$$

$$c_2(K_1, K_2) = [\sigma S(K_1)/\sqrt{2i}] \exp(\rho\sqrt{2i}/\sigma).$$

These expressions are used in the proofs stated below.

(iii) See Van Hilten, Kort, and Van Loon (Ref. 10, pp. 399–400), where the proof has to be adjusted slightly for the fact that now, on the boundary between the I_1-region and the D-region, it holds that

$$V_{K_1} = 1 + pE_{K_1}.$$

(iv) On the boundary between the D-region and the I_2/D-region, it holds that

$$V_{K_2} = 1 + pE_{K_2} + A'(0).$$

In the D-region, we have

$$V = S(K_1)/i - [\sigma S(K_1)/\sqrt{2i}] \exp\{[\rho - M/S(K_1)](\sqrt{2i}/\sigma)\}.$$

We see that, in the D-region, it holds that $V_{K_2} = 0$, so that the boundary between the regions D and I_2/D is given by $\tilde{K}_1(K_2)$, while the D-region is situated below this boundary. The latter is obtained from the fact that, for the D-region, $V_{K_2} = 0$ leads to

$$1 + pE_{K_2} + A'(0) \geq 0.$$

(v) On the boundary between the M-region and the I_2/M-region, it holds that

$$V_{K_2} = V_M[1 + pE_{K_2} + A'(0)].$$

In the M-region, we have

$$V = \{S(K_1)/[r_1 \exp(r_1\rho) - r_2 \exp(r_2\rho)\}$$
$$\times \{\exp[r_1 M/S(K_1)] - \exp[r_2 M/S(K_1)]\}.$$

Also, here we have that

$$V_{K_2} = 0;$$

since $V_M \geq 1$ in both the M-region and I_2/M-region, we can conclude that, on this boundary, K_1 must equal $\tilde{K}_1(K_2)$.

(vi) See Van Hilten, Kort, and Van Loon (Ref. 10, pp. 400–401). □

Proof of Proposition 4.2. From the implicit definitions of $K_1^*(K_2)$ and $\tilde{K}_1(K_2)$, we obtain

$$dK_1^*(K_2)/dK_2 = ipE_{K_1K_2}/(S'' - ipE_{K_1K_1}) > 0,$$

$$d\tilde{K}_1(K_2)/dK_2 = -E_{K_2K_2}/E_{K_1K_2} > 0.$$

Since

$$E_{K_1K_1}E_{K_2K_2} - E^2_{K_1K_2} > 0,$$

it holds that

$$d\tilde{K}_1(K_2)/dK_2 > dK_1^*(K_2)/dK_2, \qquad \text{if } K_1^*(K_2) = \tilde{K}_1(K_2).$$

This implies that

$$\tilde{K}_1(K_2) > K_1^*(K_2), \qquad \text{for } K_2 > K_2^*.$$ □

References

1. Bensoussan, A., and Lesourne, J., *Optimal Growth of a Self-Financing Firm in an Uncertain Environment*, Applied Stochastic Control in Economics and Management Science, Edited by A. Bensoussan et al., North Holland, Amsterdam, Netherlands, pp. 235–269, 1980.
2. Kort, P. M., Van Loon, P. J. J. M., and Luptacik, M., *Optimal Dynamic Environmental Policies of a Profit Maximizing Firm*, Journal of Economics, Vol. 54, pp. 195–225, 1991.
3. Xepapadeas, A. P., *Environmental Policy, Adjustment Costs, and Behavior of the Firm*, Journal of Environmental Economics and Management, Vol. 23, pp. 258–275, 1992.
4. Jorgenson, D. W., and Wilcoxen, P. J., *Environmental Regulation and US Economic Growth*, Rand Journal of Economics, Vol. 21, pp. 314–340, 1990.
5. Hartl, R. F., and Kort, P. M., *Optimal Input Substitution of a Firm Facing an Environmental Constraint*, Working Paper No. 5, Faculty of Economics, Magdeburg University, 1994, forthcoming in: Optimal Control Applications and Methods.
6. Helfand, G. E., *Standards versus Standards: The Effects of Different Pollution Restrictions*, American Economic Review, Vol. 81, pp. 622–634, 1991.
7. Kobila, T. O., *Partial Investment under Uncertainty*, Stochastic Models and Option Values, Edited by D. Lund and B. Oksendal, North Holland, Amsterdam, Netherlands, pp. 167–186, 1991.
8. Judd, K. L., and Petersen, B. C., *Dynamic Limit Pricing and Internal Finance*, Journal of Economic Theory, Vol. 39, pp. 369–399, 1986.

9. BENSOUSSAN, A., *Stochastic Control by Functional Analysis Methods*, North Holland, Amsterdam, Netherlands, 1982.
10. VAN HILTEN, O., KORT, P. M., and VAN LOON, P. J. J. M., *Dynamic Policies of the Firm: An Optimal Control Approach*, Springer, Berlin, Germany, 1993.
11. HARTL, R. F., and KORT, P. M., *Marketable Permits in a Stochastic Dynamic Model of the Firm*, Working Paper, Magdeburg University/Tilburg University, 1994.