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Spreads in Strongly Regular Graphs

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Dedicated to Hanfried Lenz on the occasion of his 80th birthday

Abstract. A spread of a strongly regular graph is a partition of the vertex set into cliques that meet Delsarte's bound (also called Hoffman's bound). Such spreads give rise to colorings meeting Hoffman's lower bound for the chromatic number and to certain imprimitive three-class association schemes. These correspondences lead to conditions for existence. Most examples come from spreads and fans in (partial) geometries. We give other examples, including a spread in the McLaughlin graph. For strongly regular graphs related to regular two-graphs, spreads give lower bounds for the number of non-isomorphic strongly regular graphs in the switching class of the regular two-graph.

Keywords: strongly regular graphs, graph colorings, partial geometries, spreads, linked designs, regular 2-graphs

1. Introduction

A spread in a geometry is a set of pairwise disjoint lines that cover all the points. For a partial geometry the point graph (or collinearity graph) is strongly regular and lines are cliques in the point graph that meet Delsarte's bound. We define a spread in a strongly regular graph as a partition of the vertex set into cliques that meet Delsarte's bound. So that a spread of a partial geometry provides a spread in its point graph. A spread in a strongly regular graph Γ corresponds to a coloring of the complement of Γ that meets Hoffman's bound for the chromatic number. In terms of a partition of the pairs of vertices it corresponds to an imprimitive three-class association scheme. The chromatic number of strongly regular graphs has been studied by the first author in [11]; some of his results have direct consequences for spreads. Imprimitive three class association schemes have been studied by Chang [6] and some results, presented here, can also be found in Chang's work.

Throughout Γ will denote a (n, k, λ, μ) strongly regular graph on n vertices with eigenvalues k, r and s ($k \geq r > s$) and multiplicities 1, f and g, respectively. Then the parameters satisfy the following basic equations:

$$\mu = \lambda - r - s = k + rs = (k-r)(k-s)/n, \ 1 + f + g = n, \ k + fr + gs = 0.$$

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If Γ is primitive (that is, Γ is neither a disjoint union of cliques or a complete multi-partite graph), then $0 < \mu < k$, $1 < r < k$ and $s < 0$. For these and other results on graphs, designs, finite geometries and association schemes, we refer to Cameron and Van Lint [5] or Van Lint and Wilson [17].

2. Delsarte-Cliques and Hoffman-Colorings

Delsarte [8] showed that a clique in Γ has at most $K = 1 - k/s$ vertices. Applied to the complement of Γ it yields that a coclique has at most

$$\overline{K} = 1 + \frac{n-k-1}{r+1} = \frac{n}{K}$$

vertices. We call a (co)clique that meets the Delsarte bound a *Delsarte-(co)clique.* (Many people call them Hoffman-(co)cliques. The bound for strongly regular graphs, however, was first given by Delsarte. Hoffman later generalized it to arbitrary regular graphs.) The following result is well known; see for example [2] p. 10.

LEMMA 2.1 *A (co)clique C of Γ is a Delsarte-(co)clique if and only if every vertex not in C is adjacent to a constant number of vertices of C.*

Clearly, if Γ has a spread, K and $n/K = \overline{K}$ must be integers. We call a parameter set for a strongly regular graph *feasible* for a spread if it satisfies these divisibility conditions. Note that if a parameter set is feasible for a spread, then so is the parameter set of the complement. Hoffman [16] (see also [17] p. 397 or [12]) proved that the chromatic number of Γ is at least $K = 1 - k/s$ (the bound holds for any graph with largest eigenvalue k and smallest eigenvalue s). We call a coloring meeting this bound a *Hoffman-coloring.* It is clear that each color class of a Hoffman-coloring of Γ is a coclique of size $n/K = \overline{K}$, so a Hoffman coloring of Γ is the same as a spread in the complement of Γ. Results from [11] on the chromatic number of strongly regular graphs have the following consequences for Hoffman-colorings.

THEOREM 2.2 *If Γ is primitive and admits a Hoffman-coloring then $kr \geq s^2$.*

Proof. Theorem 2.2.3 of [11] (see also [12]) states that if Γ is not the pentagon (which obviously has no Hoffman-coloring), the chromatic number is at least $1 - s/r$, so $K \geq 1 - s/r$. ■

COROLLARY 2.3 *For a fixed K there are only finitely many primitive strongly regular graphs with a Hoffman-coloring with K colors.*

Proof. The above inequality and $k + rs = \mu > 0$ give $-s \leq r(K-1) < (K-1)^2$. Hence $n = (k-r)(k-s)/\mu \leq k(k-s) = s^2K(K-1) < K(K-1)^5$. ■

In fact, by Theorem 4.1.2 of [11] the above statement holds for any coloring of a primitive strongly regular graph. If K is small, we can be more precise:

THEOREM 2.4 *Suppose Γ is a primitive strongly regular graph with a Hoffman-coloring with at most four colors. Then Γ has chromatic number* 3 *and Γ is the Lattice graph $L(3)$ (i.e. the line graph of $K_{3,3}$), or Γ has chromatic number* 4 *and Γ is $L(4)$, the complement of $L(4)$, the Shrikhande graph or one of the eleven* $(64, 18, 6, 4)$ *strongly regular graphs that are incidence graphs of three linked symmetric* 2-$(16, 6, 2)$ *designs.*

Proof. Theorem 4.3.1 of [11] gives all 4-colorable strongly regular graphs. Of these we take the primitive ones that meet Hoffman's bound. ■

For the definition of (and more about) linked symmetric designs we refer to Section 5.

3. Partial Geometries

Suppose Γ is geometric, that is, Γ is the point graph of a partial geometry G (say). Then the parameters of G are $K = 1 - k/s$ (= line size), $R = -s$ and $T = -r - k/s$. The lines of G are Delsarte-cliques of Γ, but not all Delsarte-cliques need to be lines. Thus if G has a spread, then so does Γ, but the converse needs not be true. This is illustrated by the partial geometry with parameters $(K, R, T) = (3, 2, 2)$, which has the complete 3-partite graph $K_{2,2,2}$ as point graph. However, a spread of Γ obviously gives a spread of G if all Delsarte-cliques of Γ are lines of G, in this case we will call Γ *faithfully geometric*.

An *ovoid* in G is a set C of pairwise non-collinear points so that every line intersects C in just one point. Thus C is a spread in the dual of G. It follows (for instance from Lemma 2.1) that C is a Delsarte-coclique of Γ, and conversely, each Delsarte-coclique corresponds to an ovoid. A partition of the points of G into ovoids is called a *fan* of G. So we have:

PROPOSITION 3.1 *If Γ is the point graph of a partial geometry G, then Γ has a Hoffman-coloring if and only if G has a fan.*

Many partial geometries with spreads and fans are known, leading to many examples of strongly regular graphs with spreads and Hoffman-colorings. To be more specific we distinguish, as usual, four types of partial geometries: the (dual) Steiner 2-designs, the (dual) nets, the generalized quadrangles and the proper partial geometries. For spreads and fans in generalized quadrangles we refer to a nice survey by Payne and Thas [21]. A fan in a dual Steiner 2-design is the same as a parallelism or resolution. Many such designs are known (see [19]). They exist for example for all feasible parameters with block size (= R) equal to 2, 3 or 4. Any two lines of a dual Steiner 2-design meet, so this geometry has no spread. A net is a partial geometry with $T = R - 1$; it is the same as a set of $R - 2$ MOLS (mutually orthogonal Latin squares) of order K. Nets clearly have spreads and it is also easy to see that a net has a fan if and only if the set of MOLS can be extended by one more square. See [1] for more about nets and Steiner systems. For spreads and fans in proper partial geometries we refer to [7].

Many pseudo-geometric graphs are not geometric. On the other hand, in some cases being (faithfully) geometric is forced by its parameters. This can lead to non-existence of strongly regular graphs with spreads or Hoffman-colorings for certain parameters.

PROPOSITION 3.2 *If $\mu = s^2$ (i.e. Γ has the parameters of the point graph of a dual Steiner 2-design) and if $2r > (s+1)(s^3+s-2)$, then Γ has no spread.*

Proof. By Neumaier [20], Γ is faithfully geometric to a dual Steiner 2-design, which has no spread. ■

Note that just the condition that $\mu = s^2$ is not enough to exclude spreads, since $K_{2,2,2}$ has spreads (but we know of no primitive counter example).

4. Three-Class Association Schemes

Suppose Γ is primitive and has a spread. We define on the vertices of Γ the relations R_0, R_1, R_2 and R_3 as follows: $\{x, y\} \in R_3$ if x and y are distinct vertices in the same clique of the spread and $\{x, y\} \in R_i$ if $\{x, y\} \notin R_3$ and the distance between x and y in Γ equals i ($i = 0, 1, 2$).

PROPOSITION 4.1 *The relations R_0, R_1, R_2, R_3 form an imprimitive 3-class association scheme with eigenmatrix*

$$P = \begin{bmatrix} 1 & k+k/s & n-k-1 & -k/s \\ 1 & r+1 & -r-1 & -1 \\ 1 & s+1 & -s-1 & -1 \\ 1 & r+k/s & -r-1 & -k/s \end{bmatrix}$$

and respective multiplicities 1, $f-\overline{K}+1$, g and $\overline{K}-1$. And conversely, a 3-class association scheme with eigenmatrix P gives rise to a strongly regular graph with eigenvalues k, r and s having a spread.

Proof. Let A_0, A_1, A_2, A_3 be the adjacency matrices of the relations $R_0, \ldots, R_3$. Then

$$A_0 = I, \ \sum_{i=0}^{3} A_i = J, \ A_3 + I = I_{\overline{K}} \otimes J_K$$

and $A = A_1 + A_3$ is the adjacency matrix of Γ. Since Γ is strongly regular, the span $\langle I, J, A\rangle$ is closed under multiplication. Lemma 2.1 implies that $AA_3 \in \langle I, J, A_3\rangle$. Therefore $\langle A_0, A_1, A_2, A_3\rangle$ is closed under multiplication, so represents an association scheme. The scheme is imprimitive since $R_3 \cup R_0$ is an equivalence relation. The i-th column of P contains the eigenvalues of A_i, which are straightforward for $i = 0$, 2 and 3. Next, observe that the eigenvectors of A_3 for the eigenvalue $-k/s$ are in $\langle J, A_3 + I\rangle$, that is, the coordinates are constant on each equivalence class. But from Lemma 2.1 it follows that A_2 has eigenvectors in the same space with eigenvalues $n-k-1$ or $-r-1$, so the eigenvalues of A_2 and A_3 correspond as given and the eigenvalues of A_1 follow. Conversely, for a scheme with eigenmatrix P, A_3 has only two distinct eigenvalues, so must represent a disjoint union of cliques and $A_1 + A_3$ has only three distinct eigenvalues, so represents a strongly regular graph Γ. Relation R_3 gives a partition of Γ into cliques, which must be Delsarte-cliques by Lemma 2.1. ■

Imprimitive 3-class association schemes are studied by Chang [6]. He calls the schemes considered here of Γ type.

Observe that, for each ℓ, the product $(P)_{1\ell}(P)_{2\ell}(P)_{3\ell}$ is positive and therefore the Krein parameter q_{13}^{2} is positive and hence Neumaier's absolute bound (see [2] p.51) gives $g \leq (f - \overline{K} + 1)(\overline{K} - 1)$. By use of $k + fr + gs = 0$ it follows easily that the latter inequality is equivalent to Theorem 2.2 applied to the complement of Γ. Chang derives the same inequality from the Krein condition and in the next section we shall give a direct proof and consider the case of equality.

The relation R_1 of the scheme is a distance-regular graph precisely when two vertices in R_3 have distance 3 in the graph R_1. In Γ this means that each vertex p has one neighbor in each clique of the spread not containing p. This is the case if and only if $-s(r+1) = k$, that is, if Γ is pseudo geometric for a partial geometry with $T = 1$ (i.e. a generalized quadrangle). The involved distance-regular graphs are antipodal covers of the complete graphs. Such graphs have been studied extensively by Godsil and Hensel [9].

5. Linked Symmetric Designs

A system of m *linked symmetric* (v, k', λ') *designs* is a collection $\{\Omega_0, \ldots, \Omega_m\}$ of disjoint sets and an incidence relation between each pair of sets such that:

1. For each pair Ω_i, Ω_j the incidence relation gives a symmetric 2-(v, k', λ') design.
2. For any three distinct sets $\Omega_i, \Omega_j, \Omega_k$ and for any two points $p \in \Omega_j$ and $q \in \Omega_k$, the number of elements in Ω_i incident with both p and q can take only two values x and y say, depending on whether p and q are incident or not.

Linked symmetric designs were introduced by Cameron [4]. (Though Cameron did not require that all designs have the same parameters, but for simplicity we do.) It follows that $(x - y)^2 = k' - \lambda'$ and $y(k' + x - y) = k'\lambda'$. The *incidence graph* of such a system has the union of $\Omega_0, \ldots, \Omega_m$ as vertex set; two vertices being adjacent whenever they belong to incident points of different sets. By definition we see that such a graph is strongly regular if and only if $m\lambda' = y(m-1)$. If so, it has a Hoffman-coloring (by Lemma 2.1) and the eigenvalues are $k = mk'$, $r = k'/m$ and $s = -k'$, and so the bound of Theorem 2.2 is tight. The next result states that the converse is also true. For convenience we use the formulation of the previous section.

THEOREM 5.1 *If Γ is a primitive strongly regular graph with a spread, then*

$$g \leq (f - \overline{K} + 1)(\overline{K} - 1)$$

and equality holds if and only if the complement of Γ is the incidence graph of a system of linked symmetric designs.

Proof. The proof is just the obvious generalization of the one of Theorem 4.2.7 in [11]. Let, as before, $A_0, \ldots, A_3$ be the adjacency matrices of the corresponding association scheme.

Define

$$E = -s(k-r)A_0 + (k-s)A_1 + (k+rs-s-s^2)A_3.$$

Then by use of the eigenmatrix P we find that $\text{rank}(E) \leq f - \overline{K} + 2$. (In fact, we choose $E = (k-r)(k-s)E_0 + (k-s)(r-s)E_1$, where E_0 and E_1 are the minimal indempotents of rank 1 and $f - \overline{K} + 1$, respectively.) We partition the matrices E and A_2 according to the spread:

$$E = \begin{bmatrix} E_{00} & \cdots & E_{0m} \\ \vdots & & \vdots \\ E_{m0} & \cdots & E_{mm} \end{bmatrix}, \ A_2 = \begin{bmatrix} A_{00} & \cdots & A_{0m} \\ \vdots & & \vdots \\ A_{m0} & \cdots & A_{mm} \end{bmatrix},$$

wherein $m = \overline{K} - 1$. Then $E_{ij} = (k-s)(J - A_{ij})$ for $i \neq j$, $E_{ii} = (k+rs-s-s^2)J - (s+1)(k-s)I$ and $A_{ii} = 0$ for $i, j = 0, \ldots, m$. It follows that E_{00} is non-singular and $E_{00}^{-1} \in \langle I, J \rangle$. Now $\text{rank}(E_{00}) \leq \text{rank}(E)$ gives $K \leq f - \overline{K} + 2$ and by use of $K\overline{K} = n = f + g + 1$ we find the required inequality. If equality holds, then $K = \text{rank}(E_{00}) = \text{rank}(E)$, which implies that $E_{ij} = E_{i0}E_{00}^{-1}E_{0j}$. By use of $A_{i0}J = A_{0i}J \in \langle J \rangle$ and the formulas above this leads to $A_{i0}A_{i0}^{\mathrm{T}} = A_{i0}A_{0i} \in \langle I, J \rangle$ in case $i = j$ and to $A_{i0}A_{0j} \in \langle J, A_{ij} \rangle$ for $i \neq j$. The first equation reflects that A_{i0} is the incidence matrix of a symmetric 2-design and the second equation gives by Theorem 2 of [4] that the 2-designs are linked. ∎

Sufficiently large systems of linked designs are known to exist if v is a power of 4. Mathon [18] proved that there are exactly twelve systems of three linked (16, 6, 2) designs, leading to eleven non-isomorphic incidence graphs. One of these graphs also comes from a fan in the generalized quadrangle with parameters (4, 6, 1), but the remaining ten are not geometric. These graphs are mentioned in Theorem 2.4. The theorem above excludes the existence of a (75, 42, 25, 21) strongly regular graph with a spread, indeed the complement would have a Hoffman-coloring with $kr = s^2$, but the corresponding system of 4 linked (15, 8, 4) designs does not exist, because $m\lambda' = 16$ is not divisible by $m - 1 = 3$. In fact, it is not known if a strongly regular graph with these parameters exists. Similarly it follows that no (96, 45, 24, 18) strongly regular graph with a spread exists.

6. Small Parameters

In this section we list the feasible parameters for strongly regular graphs with a spread up to 100 vertices and try to determine existence. First we consider some easy infinite families. Imprimitive strongly regular graphs obviously have spreads and Hoffman-colorings. The triangular graph $T(m)$ is the line graph of K_m and is geometric for a (trivial) dual Steiner system. It is primitive and feasible for a spread if $m \geq 5$ and even. Then $T(m)$ has no spreads (by Theorem 2.2 for example), but several Hoffman-colorings (corresponding to 1-factorizations of K_m). For $m \neq 8$, $T(m)$ is determined by its parameters, but there are three more graphs with the parameters of $T(8)$: the Chang graphs. They too have no spreads (again by Theorem 2.2) but several Hoffman-colorings (easy exercise). The Lattice graph $L(m)$ is the linegraph of $K_{m,m}$ and is geometric for a net. For each m, $L(m)$ has precisely two spreads and a number of Hoffman-colorings (corresponding to Latin squares of order m). For $m \neq 4$, $L(m)$ is determined by its parameters. There is one more graph with the parameters of $L(4)$: the Shrikhande graph. By Theorem 2.4 (or just by checking) it follows that the Shrikhande graph has Hoffman-colorings, but no spreads. All remaining feasible parameters of strongly regular graphs with a spread are listed in Table 1 (by feasible we mean that the parameters n, k, λ, μ, f, g ,K and $\overline{K}$ are positive integers that satisfy the basic equations). For each parameter set we indicate what is known about existence of a spread and a Hoffman-coloring, so that we do not need to consider the complementary parameter set. Most examples come from spreads and fans in nets (indicated by "net"), dual Steiner systems ("dss") or generalized quadrangles ("gq"). The abbreviation "abs" refers to the absolute bound for strongly regular graphs ($v \leq f(f+3)/2$) and "drg" means that the relation R_1 of the association scheme is a distance-regular graph. Most cases of non-existence come from results treated earlier. Two cases need more explanation:

PROPOSITION 6.1 *For the parameter sets* $(35, 18, 9, 9)$ *and* $(45, 12, 3, 3)$ *there exists no strongly regular graph with a spread.*

Proof. Consider the complement and assume existence of a (35, 16, 6, 8) strongly regular graph Γ with a Hoffman-coloring. Then $r = 2$, $s = -4$ and Γ has five color classes of size 7. The subgraph induced by three of these classes has a regular partition (i.e. each block matrix of the partitioned incidence matrix has constant row and column sum) with quotient matrix $4(J - I_3)$, so has the eigenvalue -4 with multiplicity at least 2. This implies that the bipartite subgraph Γ' induced by the remaining two color classes has at least twice the eigenvalue 2 (By Theorem 1.3.3 in [11] or Lemma 1.2 in [14]), and by interlacing, no eigenvalue between 2 and 4. Therefore the bipartite complement of Γ' is a cubic bipartite graph on 14 vertices for which the three largest eigenvalues are 3, 2 and 2. Bussemaker et al. [3] have enumerated all cubic graphs on 14 vertices, but none has the required property.

A (45, 12, 3, 3) strongly regular graph is pseudo geometric to a generalized quadrangle, and hence a spread would provide a distance regular antipodal 5-cover of K_9. Such a distance-regular graph does not exist; see [2] p. 152. ∎

The smallest unsolved case is a (36, 15, 6, 6) strongly regular graph with a Hoffman-coloring. Since there exist no two orthogonal Latin squares of order 6, such a graph cannot be geometric. Probably such a graph does not exist at all, since E. Spence has tested all strongly regular graphs with these parameters known to him (over 30000; see [22]) and found that none has a Hoffman-coloring.

7. Regular 2-Graphs

In this section we need some results from regular two-graphs, which we shall briefly explain (see [5] for more details). A *two graph* (Ω, Δ) consists of a finite set Ω, together with a set Δ of unordered triples (called *coherent* triples) from Ω, such that every 4-subset of Ω contains

Table 1. Feasible parameters for primitive strongly regular graphs with a spread (or Hoffman-coloring) on at most 100 vertices. The parameters of the triangular and the lattice graphs are left out. For each pair of complementary parameters, only the one with the smaller k is given.

n	k	λ	μ	r	s	K	$\overline{K}$	spread	Hoffman-coloring
25	12	5	6	2	−3	5	5	YES, net	YES, net
27	10	1	5	1	−5	3	9	YES, gq, drg	NO, 2.2, 2.4, 3.1
35	16	6	8	2	−4	5	7	YES, dss	NO, 6.1
36	15	6	6	3	−3	6	6	YES, net	?
40	12	2	4	2	−4	4	10	YES, gq, drg	NO, 2.4
45	12	3	3	3	−3	5	9	NO, 6.1, drg	YES, gq
49	18	7	6	4	−3	7	7	YES, net	YES, net
49	24	11	12	3	−4	7	7	YES, net	YES, net
63	22	1	11	1	−11	3	21	NO, abs	NO, abs, 2.2, 2.4
63	30	13	15	3	−5	7	9	YES, dss	?
64	18	2	6	2	−6	4	16	YES, gq, drg	YES, gq, 2.4, 5.1
64	21	8	6	5	−3	8	8	YES, net	YES, net
64	28	12	12	4	−4	8	8	YES, net	YES, net
64	30	18	10	10	−2	16	4	NO, abs, 2.2, 2.4	NO, abs
70	27	12	9	6	−3	10	7	?	YES, dss
75	32	10	16	2	−8	5	15	?	NO, 5.1
76	21	2	7	2	−7	4	19	NO, [13], drg	NO, [13], 2.2, 2.4
81	24	9	6	6	−3	9	9	YES, net	YES, net
81	32	13	12	5	−4	9	9	YES, net	YES, net
81	40	19	20	4	−5	9	9	YES, net	YES, net
85	20	3	5	3	−5	5	17	YES, gq, drg	?
95	40	12	20	2	−10	5	19	?	NO, 2.2
96	20	4	4	4	−4	6	16	YES, gq, drg	YES, gq
96	35	10	14	3	−7	6	16	?	?
96	45	24	18	9	−3	16	6	NO, 5.1	?
99	48	22	24	4	−6	9	11	YES, dss	?
100	27	10	6	7	−3	10	10	YES, net	YES, net
100	36	14	12	6	−4	10	10	YES, net	?
100	45	20	20	5	−5	10	10	?	?

an even number of triples from Δ. With any graph (Ω, E) we associate a two-graph (Ω, Δ) by defining three vertices coherent if they induce an odd number of edges. Two graphs (Ω, E) and (Ω, E') give rise to the same two-graph if and only if Ω can be partitioned into two parts $\Omega = \Omega_1 \cup \Omega_2$ such that $E \cap (\Omega_i \times \Omega_i) = E' \cap (\Omega_i \times \Omega_i)$ for $i = 1, 2$ and $E \cap (\Omega_1 \times \Omega_2) = (\Omega_1 \times \Omega_2) \setminus E'$. The operation that transforms E to E' is called Seidel switching and the corresponding graphs are called switching equivalent. The *descendant* (or *derived graph*) Γ_ω of (Ω, Δ) with respect to a point $\omega \in \Omega$ is the graph with vertex set $\Omega \setminus \{\omega\}$, where two vertices p and q are adjacent if $\{\omega, p, q\} \in \Delta$. Clearly the two-graph associated with $\Gamma_\omega + \omega$ is (Ω, Δ) again, thus there is a one-to-one correspondence between two-graphs and switching classes of graphs.

A two-graph (Ω, Δ) is *regular* if every pair of points from Ω is contained in a constant number a of coherent triples. Every descendant of a regular two-graph is a strongly regular graph with parameters $n = |\Omega| - 1$, $k = a$, and $\mu = a/2$. We will parameterize a regular two-graph with the eigenvalues r and s of a descendant ($a = -2rs$ and $|\Omega| = 1-(2r+1)(2s+1)$). Conversely, any strongly regular graph with $k = 2\mu$ (or $k = -2rs$) is a descendant of a regular two-graph. Often there are other strongly regular graphs associated to a regular two-graph (Ω, Δ). This is the case if the switching class of (Ω, Δ) contains a regular graph Γ. Then it follows that Γ is strongly regular and has also the eigenvalues r and s, but Γ has one more vertex and a different degree than a descendant. In fact, there are two possible values for the degree of Γ: $-2rs-r$ and $-2rs-s$.

A clique of (Ω, Δ) is a subset C of Ω, such that every triple of C is coherent. So if $\omega \in C$ then $C \setminus \{\omega\}$ is a clique in Γ_ω, hence $|C| \leq K+1 = 2r+2$ and from Lemma 2.1 it follows that every vertex of Γ_ω, not in C, is adjacent to r vertices of C. A spread in a regular two-graph is a partition of the point set into cliques of size $2r+2$.

PROPOSITION 7.1 *If a regular two-graph admits a spread, then the corresponding switching class contains a strongly regular graph of degree $-2rs-s$ with a spread.*

Proof. Take a graph Γ in the switching class of the regular two-graph (Ω, Δ) switched such that each (two-graph) clique of the spread corresponds to a (graph) clique of Γ (because the cliques are disjoint, we can always do so). Let C be such a clique. By considering the descendant with respect to a vertex of C it follows that every vertex of Γ, not in C is adjacent to $|C|/2$ vertices of C. Therefore Γ is regular of degree $|C|-1+(|\Omega|-|C|)/2 = -2rs-s$ and hence strongly regular. ■

For example for every odd prime power q, the unitary two-graph (Ω, Δ) with eigenvalues $r = (q-1)/2$ and $s = -(q^2+1)/2$ (see Taylor [23]) is defined on the q^3+1 absolute points of a unitary polarity in $PG(2, q^2)$. The non-absolute lines of the plane meet Ω in $q+1 = 2r+2$ points, that form a clique in (Ω, Δ) and one easily finds q^2-q+1 non-absolute lines that intersect each other outside Ω. So we have a spread in (Ω, Δ) and by the above proposition we obtain a strongly regular graph with a spread with parameter set:

$$(\,q^3+1,\ q(q^2+1)/2,\ (q^2+3)(q-1)/4,\ (q^2+1)(q+1)/4\,). \tag{1}$$

Notice that by Theorem 2.2 these graphs have no Hoffman-coloring. If we switch in Γ with respect to the union of some cliques, we again find a strongly regular graph with a spread with the same parameters, which may or may not be isomorphic to the Γ. There are 2^{q^2-q} such switchings possible and $|\mathrm{aut}(\Omega, \Delta)| = 2q^3(q^3+1)(q^2-1)$ (we restrict to the case that q is a prime), so then the number of non-isomorphic such strongly regular graphs is at least

$$\frac{2^{q^2-q-1}}{q^3(q^3+1)(q^2-1)}\,. \tag{2}$$

Also spreads in a descendant give switching partitions of (Ω, Δ), that produce (many) strongly regular graphs.

PROPOSITION 7.2 *If a descendant Γ_ω of a regular two-graph (Ω, Δ) admits $-s-1$ disjoint Delsarte-cliques, then the corresponding switching class contains a strongly regular graph of degree $-2rs - s$.*

Proof. Let Ω_1 be the set of vertices of the $-s-1$ Delsarte-cliques. Switch in $\Gamma_\omega + \omega$ with respect to $\Omega_1 \cup \{\omega\}$. Then we obtain (as follows easily by use of Lemma 2.1) a regular graph of degree $-2rs - s$ in the switching class of (Ω, Δ). ■

Since a spread in Γ_ω contains $-2s-1$ Delsarte-cliques, we have:

COROLLARY 7.3 *If Γ_ω has a spread, then there exist at least*

$$\frac{1}{|aut(\Omega, \Delta)|}\binom{-2s-1}{-s-1}$$

non-isomorphic strongly regular graphs of degree $-2rs - s$ in the switching class of (Ω, Δ).

Consider again the unitary two-graph. The q^2 non-absolute lines through a fixed absolute point ω form a spread Γ_ω. Thus there are at least

$$\frac{1}{2q^3(q^3+1)(q^2-1)}\binom{q^2}{(q^2-1)/2}$$

strongly regular graphs with parameters (1) and q prime. This number is bigger than the one given in (2), but here we don't know if the graphs have spreads. For $q = 5$, for example, we find at least six non-isomorphic $(126, 65, 28, 39)$ strongly regular graphs and at least two with a spread.

8. The McLaughlin Graph

The McLaughlin graph (for short $Mc\Gamma$) is the unique strongly regular graph with parameters $n = 275$, $k = 112$, $\lambda = 30$ and $\mu = 56$. It is the descendant of the (also unique) regular two-graph (Ω, Δ) with eigenvalues $r = 2$ and $s = -28$, see Goethals and Seidel [10]. For another discussion of $Mc\Gamma$ see [15]. The automorphism group of (Ω, Δ) is Conway's simple group Co_3 which acts 2-transitively on Ω and the point stabilizer is $McL.2$, the full automorphism group of $Mc\Gamma$. We shall now describe $Mc\Gamma$ explicitely by means of this group. Therefor we list six permutations of $\{1, \ldots, 275\}$ which generate $McL.2$ (of order 1796256000), and the indices of the 112 neighbors of vertex 1:

1. 1 2 3 4 118 106 141 210 228 10 109 12 113 147 85 91 39 219 229 37 100 96 220 105 35 86 161 208 235 43 41 236 44 52 25 250 20 47 17 48 31 249 30 33 132 104 38 40 128 95 133 34 84 151 140 156 123 131 126 273 87 62 79 88 81 102 98 124 92 127 99 265 94 103 129 260 257 252 63 237 65 234 251 53 15 26 61 64 248 253 16 69 177 73 50 22 175 67 71 21 184 66 74 46 24 6 165 169 11 162 142 160 13 218 227 209 152 5 194 197 188 185 57 68 174 59 70 49 75 176 58 45 51 233 224 241 204 274 221 55 7 111 212 239 214 244 14 199 201 196 54 117 181 180 183 56 190 186 187 112 27 110 163 164 107 166 255 216 108 215 217 266 268 125 97 130 93 254 259 154 153 261 155 101 122 158 159 121 256 157 267 270 193 119 263 150 120 264 148 262 149 272 226 137 238 231 223 28 116 8 240 143 245 145 170 168 171 114 18 23 139 243 207 135 247 203 115 9 19 246 206 271 134 82 29 32 80 205 144 211 136 269 222 146 213 230 225 89 42 36 83 78 90 178 167 189 77 275 179 76 182 200 195 198 72 172 191 173 242 192 232 202 60 138 258

2. 1 2 3 4 156 124 158 35 71 19 17 69 62 73 157 164 11 162 10 193 220 201 199 206 197 103 104 208 219 216 108 153 72 75 8 160 37 163 155 40 150 136 165 261 257 260 47 166 126 171 134 119 141 118 140 151 132 131 170 270 127 13 122 154 205 102 106 98 12 121 9 33 14 159 34 161 123 251 191 224 179 223 275 186 230 187 188 227 262 268 91 92 236 184

239 240 145 68 142 148 147 66 26 27 167 67 168 31 183 235 228 256 113 174 180 243 175 54 52 210 70 63 77 6 212 49
61 215 194 213 58 57 195 51 198 42 202 204 200 55 53 99 218 217 97 209 101 100 211 41 56 214 32 64 39 5 15 7 74 36
76 18 38 16 43 48 105 107 196 59 50 207 203 114 117 247 181 249 81 115 177 265 109 94 267 84 86 87 250 246 79 271
20 129 133 169 25 135 23 139 22 137 173 138 65 24 172 28 146 120 149 125 130 152 128 30 144 143 29 21 248 244 82
80 245 253 88 111 229 85 255 273 263 266 110 93 264 259 95 96 254 269 116 222 225 190 176 221 178 189 78 258 226
241 231 112 45 252 238 46 44 89 233 237 182 234 185 90 242 60 192 274 232 272 83

3. 1 2 3 5 4 34 29 28 27 40 12 11 37 19 131 129 119 18 14 124 121 122 36 126 120 127 9 8 7 125 123 32 130 6 35 23 13 39
38 10 92 42 107 97 98 99 108 101 102 103 93 94 110 56 55 54 193 216 215 258 217 77 149 158 153 154 164 155 163 159
157 72 150 148 166 76 62 116 89 80 82 81 90 105 104 100 112 115 79 83 109 41 51 52 95 96 44 45 46 86 48 49 50 85 84
113 43 47 91 53 111 87 106 114 88 78 117 132 17 25 21 22 31 20 30 24 26 128 16 33 15 118 133 172 145 136 138 137 139
160 162 161 168 144 135 173 151 74 63 73 147 152 65 66 68 169 71 64 70 140 142 141 69 67 165 75 170 143 156 167 171
134 146 260 264 261 266 201 205 206 195 214 196 257 267 186 189 220 187 230 221 226 57 229 181 183 197 247 239
240 178 244 242 243 179 180 238 208 209 210 241 246 245 182 59 58 61 228 227 188 191 232 223 225 224 192 219 218
194 190 231 222 233 263 255 259 265 207 199 200 211 203 204 202 213 212 198 254 262 256 270 273 268 248 235 250
184 60 236 174 176 249 234 175 237 177 185 253 272 251 271 269 252 274 275

4. 1 2 4 10 47 20 49 5 51 17 25 52 106 91 43 86 35 44 39 103 94 79 112 117 54 14 27 56 110 107 129 85 105 109 118 45 3
48 113 12 197 185 228 180 183 181 164 210 141 250 174 70 59 68 57 188 67 71 175 242 169 6 22 84 114 18 23 119 34 31
133 24 16 50 11 46 21 221 268 255 212 266 274 265 260 257 217 215 244 241 166 163 190 187 196 201 176 194 58 208
229 219 147 161 235 220 236 75 199 186 216 239 62 233 214 204 224 124 127 55 41 96 100 37 111 7 108 102 92 88 99
98 81 125 80 122 89 83 90 123 126 131 93 101 130 42 19 28 116 120 121 115 15 53 13 38 30 128 95 132 104 33 40 26 9 8
29 32 87 97 36 82 78 134 135 206 207 226 143 145 223 245 148 159 243 72 77 171 189 195 173 60 74 69 65 61 151 237
160 253 63 248 252 251 218 152 234 156 249 140 209 142 64 227 66 165 184 177 162 73 191 200 172 167 198 192 170
168 193 76 182 178 179 246 158 157 247 238 150 149 240 272 137 262 264 263 231 267 203 256 139 271 270 211 261
144 155 232 205 153 154 146 259 225 213 230 222 136 202 269 273 258 254 275 138

5. 1 3 2 10 94 123 112 8 46 106 34 109 119 91 114 31 52 95 6 113 126 133 128 87 25 18 99 120 148 168 156 262 175 151
210 231 164 169 257 193 183 263 228 265 129 131 4 118 103 84 79 48 23 39 110 92 98 107 86 90 158 194 207 256 267
121 38 124 11 105 104 97 16 115 54 125 101 226 236 145 185 240 272 140 260 141 66 187 238 137 62 163 149 229 170
181 245 68 190 180 150 188 219 71 250 47 85 56 20 127 27 88 12 30 21 116 44 17 40 28 24 223 184 57 143 74 186 171
147 264 165 67 249 63 237 195 269 138 234 162 220 157 230 167 213 192 196 154 259 155 69 135 200 189 77 73 218 160
227 206 174 50 108 41 9 5 36 15 132 26 53 89 78 43 33 117 22 93 122 100 96 130 37 19 81 55 7 102 29 111 32 83 13 14
42 45 35 80 49 82 51 242 252 274 191 61 248 197 270 208 179 212 198 224 217 216 255 246 199 59 233 173 211 214 182
222 159 161 166 76 144 258 241 139 70 201 247 254 215 153 266 273 209 268 261 142 152 134 177 235 203 271 243 172
239 64 75 232 178 58 72 205 136 225 176 221 65 204 60 253 251 146 275 244 202

6. 2 1 3 10 38 40 9 8 7 4 37 12 13 98 53 41 52 99 108 109 95 96 127 30 25 26 126 120 125 24 91 97 51 39 35 36 11 5 34 6 16
122 105 133 132 128 118 106 104 100 33 17 15 124 121 123 156 162 161 269 160 164 63 218 152 165 147 151 194 220
219 153 75 74 73 154 155 90 79 82 117 80 83 85 84 86 112 114 89 78 31 119 130 129 21 22 32 14 18 50 101 107 103 49
43 48 102 19 20 131 115 87 113 88 111 116 81 47 92 28 55 42 56 54 29 27 23 46 94 93 110 45 44 224 233 191 226 232
221 188 228 227 231 189 223 192 67 148 149 150 68 65 72 76 77 57 215 216 217 61 59 58 193 62 66 166 230 186 229 190
187 225 222 235 236 177 176 254 261 260 265 259 257 184 249 168 171 140 144 170 136 146 163 69 213 196 208 205
199 211 201 202 204 203 198 212 214 197 209 210 200 206 195 207 157 158 159 64 71 70 139 173 145 134 172 137 142
141 169 167 143 138 135 237 174 175 234 264 256 262 267 273 268 270 263 255 266 248 185 250 251 253 252 178 246
239 183 258 182 180 179 240 245 238 181 247 241 243 60 244 274 272 242 271 275

The 112 vertices adjacent to vertex 1 are:

9 15 22 27 30 32 42 43 46 51 58 60 63 65 71 76 78 79 81 82 83 85 89 90 93 96 99 104 107 111 114 116 122 125 131 133
134 136 137 138 139 142 143 146 149 153 157 161 165 168 172 173 174 177 178 179 181 185 190 191 192 195 200 201
202 203 204 205 207 209 211 212 216 218 221 222 223 226 228 230 232 233 234 236 238 240 241 242 243 244 246 248
249 251 252 253 254 258 259 260 262 263 266 267 268 269 270 271 272 273 274 275

THEOREM 8.1 *The McLaughlin graph admits a spread.*

Proof. The following ordering groups the vertices into 55 disjoint cliques of size 5 (i.e. Delsarte-cliques):

1	9	85	90	275;	2	7	138	141	274;	3	21	192	199	273;
4	14	213	218	272;	5	8	225	228	271;	6	20	202	226	257;
10	16	221	238	261;	11	17	229	252	258;	12	26	206	241	263;
13	18	223	235	266;	15	23	214	227	260;	19	24	216	234	265;
22	36	183	219	259;	25	32	184	215	270;	27	33	189	210	254;
28	35	198	236	267;	29	48	153	174	264;	30	49	154	181	256;
31	51	146	193	255;	34	45	163	243	269;	37	50	170	233	262;
38	56	171	248	268;	39	78	140	191	196;	40	79	167	177	217;
41	84	137	166	195;	42	86	143	162	187;	43	95	136	164	182;
44	82	155	169	201;	46	88	158	172	220;	47	83	145	157	194;
52	80	148	161	178;	53	94	160	168	244;	54	100	139	165	245;
55	104	135	150	185;	57	116	124	152	246;	58	106	126	144	242;
59	81	119	156	232;	60	91	121	149	208;	61	108	129	173	249;
62	105	120	179	253;	63	87	130	190	197;	64	99	115	134	188;
65	101	118	186	203;	66	98	133	200	239;	67	103	125	176	211;
68	117	123	209	230;	69	107	127	231	251;	70	89	110	151	205;
71	97	111	147	247;	72	109	131	212	237;	73	96	112	159	240;
74	102	114	142	224;	75	93	128	180	207;	76	92	132	175	204;
77	113	122	222	250.										

■

The above spread was found by a computer search. The search was stopped after five different spreads were found. At that point we had given up hope for completing the search. The order of Co_3 equals $2^{10}.3^7.5^3.7.11.23$, so by Corollary 7.3 there are at least 7715 non-isomorphic strongly regular graphs in the switching class of (Ω, Δ). Since $Mc\Gamma$ probably has many spreads and since the bound of Corollary 7.3 is very pessimistic, the actual number of non-isomorphic (276, 140, 58, 84) strongly regular graph is, no doubt, much bigger. It is to be expected that only relatively small collections of the $\binom{55}{27}$ possible switching sets coming from the spread above lead to isomorphic graphs. But because the corresponding permutations do not need to form a group it is not clear how to get a significantly better estimate in an easy way.

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