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EICHLER-SHIMURA ISOMORPHISM AND GROUP COHOMOLOGY ON ARITHMETIC GROUPS

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Abstract. In this article, we give a group cohomological interpretation to the Eichler-Shimura isomorphism. For any quaternion algebra A over a totally real field with multiplicative group G, we interpret a weight $(k_1, k_2, \cdots, k_d)$-automorphic form of G as a $G(F)$-invariant homomorphism of $(\mathcal{G}_\infty, K_\infty)$-modules. Then the Eichler-Shimura isomorphism is given by the connection morphism provided by the natural exact sequences defining the $(\mathcal{G}_\infty, K_\infty)$-module of discrete series of weight $(k_1, k_2, \cdots, k_d)$.

Introduction

The Eichler-Shimura isomorphism establishes a bijection between the space of modular forms and certain cohomology groups with coefficients in a space of polynomials. More precisely, let $k \geq 2$ be an integer and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, then we have the following isomorphism of Hecke modules

$$M_k(\Gamma, \mathbb{C}) \oplus S_k(\Gamma, \mathbb{C}) \simeq H^1(\Gamma, V(k)^\vee), \tag{0.1}$$

where $V(k)^\vee$ is the dual of the $\mathbb{C}$-vector space of homogenous polynomials of degree $k-2$, $M_k(\Gamma, \mathbb{C})$ is the space of modular forms of weight k and $S_k(\Gamma, \mathbb{C}) \subset M_k(\Gamma, \mathbb{C})$ is the subspace of cuspidal modular forms (see [6, Thm. 8.4] and [4, Thm. 6.3.4]).

This isomorphism can be interpreted in geometric terms. Indeed, a modular form of weight k can be interpreted as a section of certain sheaf of differential forms on the open modular curve attached to Γ. With this in mind, the Eichler-Shimura isomorphism can be obtained comparing deRham and singular cohomology, noticing that the singular cohomology of the open modular curve is given by the group cohomology $H^\bullet(\Gamma, V(k)^\vee)$. The aim of this paper is to omit this geometric interpretation and to provide a new group cohomological interpretation.

Let us remark that the identification (0.1) provides an integral and rational structure to the space of modular forms, since the space of polynomials $V(k)$ has integral and rational models, namely, the space of polynomials with integer and rational coefficients.

The restriction of the Eichler-Shimura isomorphism to the spaces of cuspidal modular forms is given by the morphisms

$$\partial^\pm : S_k(\Gamma, \mathbb{C}) \longrightarrow H^1(\Gamma, V(k)^\vee),$$

where

$$\partial^\pm(f)(\gamma)(P) = \int_{z_0}^{\gamma z_0} P(1, -\tau) f(\tau) d\tau \pm \int_{z_0}^{\gamma z_0} P(1, \bar{\tau}) f(-\bar{\tau}) d(-\bar{\tau}).$$

for any z_0 in $\mathcal{H}$ the Poincaré hyperplane, $\gamma \in \Gamma$ and $P \in V(k)$. In fact, the morphism defining the cuspidal part of (0.1) is given by

$$\begin{array}{ccc} S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} & \longrightarrow & H^1(\Gamma, V(k)^\vee) \\ (f_1, \bar{f}_2) & \longmapsto & (\partial^+ + \partial^-)(f_1) + (\partial^+ - \partial^-)(f_2). \end{array}$$

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The image of $\partial^{\pm}$ lies in the subspaces $H^1(\Gamma, V(k)^\vee)^{\pm} \subset H^1(\Gamma, V(k)^\vee)$ where the natural action of $\omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ normalizing Γ acts by ± 1.

In a general setting, F is a totally real number field of degree d, G is the multiplicative group of a quaternion algebra A over F, and ϕ is a weight $\underline{k} = (k_1, \cdots, k_d)$ cuspidal automorphic form of G with level $\mathcal{U} \subset G(\mathbb{A}^\infty)$ and central character $\psi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$. We assume that $\psi_{\sigma_i}(x) = \mathrm{sign}(x)^{k_i}|x|^{\mu_i}$, for any archimedean place $\sigma_i : F \hookrightarrow \mathbb{R}$. In this scenario, by an automorphic form we mean a function on $\mathcal{H}^r \times G(\mathbb{A}^\infty)/\mathcal{U}$, where $\mathcal{H}$ is the Poincaré upperplane and r is the cardinal of the set Σ of archimedean places where A splits, with values in $\bigotimes_{\sigma_i \in \infty \setminus \Sigma} V_{\mu_i}(k_i)^\vee$ (see §1.2 for a precise definition of $V_{\mu_i}(k_i)$), that satisfies the usual transformation laws with respect to the weight-k_i-actions of $G(F)$. The interesting cohomology subgroups to consider are:

$$H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}}) = \bigoplus_{i=1}^{n} H^r(\Gamma_{g_i}, V_\psi(\underline{k})^\vee), \qquad \Gamma_{g_i} = G(F)^+ \cap g_i \mathcal{U} g_i^{-1},$$

where $V_\psi(\underline{k})$ is the tensor product of the polynomial spaces $V_{\mu_i}(k_j)$ $(j = 1, \cdots, d)$, $G(F)^+ \subseteq G(F)$ is the subgroup of totally positive elements, $\{g_i\}_{i=1,\cdots,n} \subset G(\mathbb{A}^\infty)$ is a set of representatives of the double coset space $G(F)^+\backslash G(\mathbb{A}^\infty)/\mathcal{U}$, and $\mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}} = C(G(\mathbb{A}^\infty)/\mathcal{U}, V_\psi(\underline{k})^\vee)$. Similarly as in the classical case, for any character $\varepsilon : G(F)/G(F)^+ \longrightarrow \pm 1$ we can define a morphism

$$\partial^\varepsilon : S_{\underline{k}}(\mathcal{U}, \psi) \longrightarrow H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}})(\varepsilon),$$

from the set $S_{\underline{k}}(\mathcal{U}, \psi)$ of automorphic cuspforms of weight $\underline{k}$, level $\mathcal{U}$ and central character ψ, to the ε-isotypical component of $H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}})$. Such a map is given by:

$$\partial^\epsilon \phi = \sum_{\gamma \in G(F)/G(F)^+} \epsilon(\gamma) \partial \phi^\gamma,$$

where $\partial\phi \in H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))$ is the class of the cocycle

$$(G(F)^+)^r \ni (g_1, g_2, \cdots, g_r) \longmapsto \int_{\tau_1}^{g_1\tau_1} \cdots \int_{g_1\cdots g_{r-1}\tau_r}^{g_1\cdots g_r\tau_r} P_\Sigma(1, -\underline{z}) \langle \phi(\underline{z}, g), P^\Sigma \rangle d\underline{z},$$

with $P_\Sigma \in \bigotimes_{j=1}^r V_{\mu_j}(k_j)$, $P^\Sigma \in \bigotimes_{j=r+1}^d V_{\mu_j}(k_j)$ and $\underline{z} = (z_1, \cdots, z_r)$, $(\tau_1, \cdots, \tau_r) \in \mathcal{H}^r$. Our result will provide a group cohomological interpretation to the morphisms ∂^ε, for any character ε.

Let $F_\infty \simeq \mathbb{R}^d$ be the product of the archimedean completions of F, let $\mathcal{G}_\infty$ be the Lie algebra of $G(F_\infty)$ and let $K_\infty \subseteq G(F_\infty)$ be a maximal compact subgroup. Then the $(\mathcal{G}_\infty, K_\infty)$-module generated by ϕ is isomorphic to $D_\psi(\underline{k})$, the tensor product of discrete series of weight k_j at archimedean places in Σ and polynomial spaces $V_{\mu_j}(k_j)$ at archimedean places not in Σ. This implies that any $\phi \in S_{\underline{k}}(\mathcal{U}, \psi)$ provides an element

$$\phi \in H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}}); \qquad \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}} := \mathrm{Hom}_{(\mathcal{G}_\infty, K_\infty)}(D_\psi(\underline{k}), \mathcal{A}^{\mathcal{U}}),$$

where $\mathcal{A}^{\mathcal{U}}$ is the $(\mathcal{G}_\infty, K_\infty)$-module of smooth admissible functions $f : G(\mathbb{A})/\mathcal{U} \to \mathbb{C}$. Our main result (Theorem 2.4) can be rewritten as follows:

Theorem 0.1. *There exists an exact sequence of $G(F)$-modules*

$$0 \to \mathcal{A}(V_\psi(\underline{k})(\varepsilon), \mathbb{C})^{\mathcal{U}} \to \mathcal{A}(I_1^\varepsilon(\underline{k}), \mathbb{C})^{\mathcal{U}} \to \mathcal{A}(I_2^\varepsilon(\underline{k}), \mathbb{C})^{\mathcal{U}} \to \cdots \to \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}} \to 0,$$

such that, up to an explicit constant, the morphism ∂^ϵ is given by the corresponding connection morphism

$$H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}}) \longrightarrow H^r(G(F), \mathcal{A}(V_\psi(\underline{k})(\varepsilon), \mathbb{C})^{\mathcal{U}}) \simeq H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}})(\varepsilon).$$

We obtain the above exact sequence from extensions of the $(\mathcal{G}_{\sigma_i}, K_{\sigma_i})$-modules of discrete series $D_{\mu_i}(k_i)$ at every place $\sigma_i \in \Sigma$. The archimedean local nature of these connection morphisms ∂^ε implies that the $G(\mathbb{A}^\infty)$-representation generated by $\partial^\varepsilon\phi$ coincides with the restriction to $G(\mathbb{A}^\infty)$ of π_ϕ, the automorphic representation attached to ϕ.

The image of a cuspidal automorphic representation π_ϕ through the morphisms ∂^ε is used in many papers to give a group cohomological construction of cyclotomic and anti-cyclotomic p-adic L-functions and Stickelberger elements attached to quadratic extensions of a totally real number field (see for instance [7], [5] and [1]). The explicit form of ∂^ε given in Theorem 2.4 provides the interpolation properties of these objects.

Another application is the construction of Stark-Heegner points. By means of the connection morphisms $\partial^{\varepsilon^{\pm 1}}$, where $(\varepsilon^+, \varepsilon^-)$ is a well chosen pair of characters, one can construct a complex torus $\mathbb{C}^{[L:\mathbb{Q}]}/\Lambda$ attached to a weight 2 automorphic representation π_ϕ with field of coefficients L. It is conjectured that such complex torus coincides with the abelian variety of GL_2-type attached to π_ϕ. In [3], we use the cohomological description of $\partial^{\varepsilon^{\pm 1}}$ to construct Stark-Heegner points in the complex torus, that we conjecture to be global points in the corresponding abelian variety. Such points are conjecturally defined over class fields of quadratic extensions of F and satisfy explicit reciprocity laws.

Notation. Throughout this paper, we will denote by $\int_{S^1} d\theta = \int_{\mathrm{SO}(2)} d\theta$ the Haar measure of $S^1 = \mathrm{SO}(2)$ such that $\mathrm{vol}(S^1) = \pi$.

Let F be a number field. For any place v of F, we denote by F_v its completion at v. Given a finite set of places S of F, we denote by F_S the product of completions at every place in S. We denote by F_∞ the product of completions at every archimedean place. Similarly, for any subset Σ of archimedean places, $F_{\infty\setminus\Sigma}$ will be the product of completions at every archimedean place not in Σ. We denote by $\mathbb{A}$ the ring of adeles of F. For any set S of places of F, we write $\mathbb{A}^S$ for the ring adeles outside S. Consistent with this notation, we denote by $\mathbb{A}^\infty$ the ring of finite adeles of F.

1. Discrete series

1.1. **Finite dimensional representations.** Let A be a quaternion algebra defined over a local field F. Let K/F be an extension where A splits. For any natural number $k \in \mathbb{N}$, let $\mathcal{P}_{k-2}^K \simeq \mathrm{Sym}^{k-2}(K^2)$ be the finite K-vector space of homogeneous polynomials of degree $k-2$. We have a well defined action of $\mathrm{GL}_2(K)$ on $\mathcal{P}_{k-2}^K$ given by

$$\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) * P\right)(x,y) := P\left((x,y)\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) = P\left(ax+cy, bx+dy\right). \tag{1.2}$$

If we fix an embedding

$$\iota : A \longrightarrow A \otimes_F K \simeq \mathrm{M}_2(K),$$

then $\mathcal{P}_{k-2}^K$ is equipped with an action of $A^\times$.

We denote by $\det : A \to F$ the reduced norm of A, and let us consider

$$A^+ = \{a \in A : \ \det(a) \in F^2\}.$$

We write $V(k)_K = \mathcal{P}_{k-2}^K \otimes \det^{\frac{2-k}{2}}$ with the natural action of A^+. It is clear that the centre of $A^\times$ acts trivially on $V(k)_K$. Notice that, if k is even, the action of A^+ on $V(k)_K$ extends to a natural action of A.

1.2. **Discrete series and exact sequences.** Assume that $F=\mathbb{R}$ and $A=\mathrm{M}_2(\mathbb{R})$. Let $\mathcal{G}L_2(\mathbb{R})$ be the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$. For any $k\in\mathbb{Z}$ and $\mu\in\mathbb{C}$, we define $I_\mu(k)$ as the $(\mathcal{G}L_2(\mathbb{R}),\mathrm{SO}(2))$-module of smooth admissible vectors in

$$\left\{f:\mathrm{GL}_2(\mathbb{R})^+\to\mathbb{C}:\; f\left(\left(\begin{array}{cc} t_1 & x\\ & t_2\end{array}\right)g\right)=\mathrm{sign}(t_1)^k(t_1t_2)^{\frac{\mu}{2}}\left(\frac{t_1}{t_2}\right)^{\frac{k}{2}}f(g)\right\}.$$

Write $V_\mu(k)=V(k)_{\mathbb{C}}\otimes\det^{\frac{\mu}{2}}$. If we assume that $k\geq 2$, we have the well defined morphism of $(\mathcal{G}L_2(\mathbb{R}),\mathrm{SO}(2))$-modules

$$\iota:V_\mu(k)\longrightarrow I_\mu(2-k);\qquad \iota(P)\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)=(ad-bc)^{\frac{2-k+\mu}{2}}P(c,d).$$

Moreover, we have a $(\mathcal{G}L_2(\mathbb{R}),\mathrm{SO}(2))$-invariant pairing (see [2, §2])

$$\langle\;,\;\rangle_I:I_\mu(k)\times I_{-\mu}(2-k)\longrightarrow\mathbb{C};\qquad (f,g)\longmapsto\int_{\mathrm{SO}(2)}f(\theta)g(\theta)d\theta,$$

providing the morphism of $(\mathcal{G}L_2(\mathbb{R}),\mathrm{SO}(2))$-modules

$$\bar\varphi:I_\mu(k)\longrightarrow V_{-\mu}(k)^\vee;\qquad \langle\bar\varphi(f),P\rangle_V:=\langle f,\iota(P)\rangle_I.$$

Composing with the natural $\mathrm{GL}_2(\mathbb{R})^+$-morphism

$$V_{-\mu}(k)^\vee\longrightarrow V_\mu(k);\qquad F\longmapsto P_F(x,y)=\langle F,(Yx-Xy)^{k-2}\rangle_{V(X,Y)},\tag{1.3}$$

we obtain a map

$$\begin{array}{rcl}\varphi:I_\mu(k) & \longrightarrow & V_\mu(k);\\ \varphi(f)(x,y) & = & \langle\bar\varphi(f),(Yx-Xy)^{k-2}\rangle_{V(X,Y)}=\displaystyle\int_{S^1}f(\theta)(y\sin\theta+x\cos\theta)^{k-2}d\theta\end{array}$$

Remark 1.1. Notice that we have the symmetry

$$\begin{array}{rcl}\langle F,P_G\rangle_V & = & \langle F,\langle G,(Yx-Xy)^{k-2}\rangle_{V(X,Y)}\rangle_{V(x,y)}\\ & = & (-1)^k\langle G,\langle F,(Xy-Yx)^{k-2}\rangle_{V(x,y)}\rangle_{V(X,Y)}=(-1)^k\langle G,P_F\rangle_V,\end{array}$$

for all $F,G\in V_\mu(k)^\vee$.

The kernel of φ is $D_\mu(k)$ the *Discrete Series* $(\mathcal{G}L_2(\mathbb{R}),O(2))$-module of weight k and central character $x\mapsto\mathrm{sign}(x)^k|x|^\mu$. This definition implies that $D_\mu(k)$ lies in the following exact sequence of $(\mathcal{G}L_2(\mathbb{R}),SO(2))$-modules:

$$0\longrightarrow D_\mu(k)\longrightarrow I_\mu(k)\stackrel{\varphi}{\longrightarrow}V_\mu(k)\longrightarrow 0.\tag{1.4}$$

Since any $g\in\mathrm{GL}_2(\mathbb{R})^+$ can be written uniquely as $g=u\cdot\tau(x,y)\cdot\kappa(\theta)$, where

$$u\in\mathbb{R}^+,\;\tau(x,y)=\left(\begin{array}{cc} y^{1/2} & xy^{-1/2}\\ & y^{-1/2}\end{array}\right)\in B,\;\kappa(\theta)=\left(\begin{array}{cc}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{array}\right)\in\mathrm{SO}(2),$$

we have that

$$I_\mu(k)=\bigoplus_{t\equiv k\pmod 2}\mathbb{C}f_t;\qquad f_t(u\cdot\tau(x,y)\cdot\kappa(\theta))=u^\mu y^{\frac{k}{2}}e^{ti\theta}.$$

The $(\mathcal{G}L_2(\mathbb{R}),\mathrm{SO}(2))$-module structure of $I_s(k)$ can be described as follows: Let $L,R\in\mathcal{G}L_2(\mathbb{R})$ be the *Maass differential operators* defined in [2, §2.2]

$$L=e^{-2i\theta}\left(-iy\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-\frac{1}{2i}\frac{\partial}{\partial\theta}\right),\quad R=e^{2i\theta}\left(iy\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{1}{2i}\frac{\partial}{\partial\theta}\right).$$

Then, the $(\mathcal{G}L_2(\mathbb{R}),\mathrm{SO}(2))$-module $I_s(k)$ is characterized by the relations:

$$Rf_t=\left(\frac{k+t}{2}\right)f_{t+2};\qquad Lf_t=\left(\frac{k-t}{2}\right)f_{t-2};\tag{1.5}$$

$$\kappa(\theta)f_t=e^{ti\theta}f_t;\qquad uf_t=u^\mu f_t,\tag{1.6}$$

for any $\kappa(\theta)\in\mathrm{SO}(2)$ and $u\in\mathbb{R}^+\subset\mathrm{GL}_2(\mathbb{R})^+$.

Write $z = x + iy$ and $\bar{z} = x - iy$. For $n \in \{0, 1, \cdots, k-2\}$, let us consider the elements $P_n \in V_\mu(k)$, $P_n(x,y) = z^n \bar{z}^{k-2-n}$. It is clear that $\{P_n\}_{n=0,\cdots,k-2}$ is a basis for the $\mathbb{C}$-vector space $V_\mu(k)$. We compute that

$$\begin{aligned}\varphi(f)(x,y) &= \int_{S^1} f(\theta)((2i^{-1})(z-\bar{z})\sin\theta + 2^{-1}(z+\bar{z})\cos\theta)^{k-2} d\theta \\ &= \int_{S^1} 2^{2-k} f(\theta)(ze^{-i\theta} + \bar{z}e^{i\theta})^{k-2} d\theta \\ &= 2^{2-k} \sum_{n=0}^{k-2} \binom{k-2}{n} P_n(x,y) \int_{S^1} f(\theta) e^{-i(2n-k+2)\theta} d\theta\end{aligned}$$

By orthogonality, we deduce that $\varphi(f_{2n-k+2}) = 2^{2-k}\pi\binom{k-2}{n} P_n(x,y)$.

Since $\kappa(\theta)P_n = e^{(2n-k+2)i\theta} P_n$, the morphism of $\mathbb{C}$-vector spaces

$$s : V_\mu(k) \longrightarrow I_\mu(k); \qquad s(P_n) = \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} f_{2n-k+2}, \tag{1.7}$$

defines a section of φ as $\mathrm{SO}(2)\mathbb{R}^+$-modules.

Remark 1.2. Since (1.3) is an isomorphism, we can define a non-degenerate $\mathrm{GL}_2(\mathbb{R})^+$-invariant bilinear pairing $V_\mu(k) \times V_{-\mu}(k) \to \mathbb{C}$

$$\langle P_F, Q \rangle = \langle F, Q \rangle_V, \qquad F \in V_{-\mu}(k)^\vee_{\mathbb{C}}, \quad Q \in V_{-\mu}(k),$$

which is symmetric or antisymmetric depending on the parity of k, by Remark 1.1. Moreover, by the definition of (1.3),

$$P(s,t) = \langle P, Q_{s,t} \rangle, \qquad Q_{s,t}(x,y) = (ys - xt)^{k-2}.$$

In particular,

$$P(-1, i) = i^{2-k}\langle P, P_0 \rangle = i^{k-2}\langle P_0, P \rangle. \tag{1.8}$$

Since s is a section of φ, we compute

$$\langle P, Q \rangle = \langle \bar{\varphi}(s(P)), Q \rangle_V = \langle s(P), \iota(Q) \rangle_I, \tag{1.9}$$

for all $P, Q \in V_\mu(k)$.

1.3. **The $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module of discrete series.** We want to give structure of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module to $I_\mu(k)$. Hence, we have to define the action of $\omega = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in O(2) \setminus \mathrm{SO}(2)$. That is to say, we have to define $\omega \in \mathrm{End}(I_\mu(k))$ such that

$$(i) \quad \omega f_t \in \mathbb{C} f_{-t}; \qquad (ii) \quad \omega^2 = 1; \qquad (iii) \quad \omega R = L\omega.$$

If we write $\omega f_t = \lambda(t) f_{-t}$, condition (ii) implies that $\lambda(t)\lambda(-t) = 1$. Moreover, condition (iii) implies that $\lambda(t) = \lambda(t+2)$. We obtain two possible $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module structures for $I_\mu(k)$: Letting $\lambda(t) = 1$ for all $t \equiv k \bmod 2$, or letting $\lambda(t) = -1$ for all $t \equiv k \bmod 2$. Write $I_\mu(k)^\pm$ for the $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module such that $\omega f_t = \pm f_{-t}$, respectively.

By abuse of notation, write also $V_\mu(k)$ and $V_\mu(k)_{\mathbb{R}}$ (in case $\mu \in \mathbb{R}$) for the $\mathrm{GL}_2(\mathbb{R})$-representations

$$V_\mu(k) = V_\mu(k)_{\mathbb{C}} = \mathcal{P}^{\mathbb{C}}_{k-2} \otimes |\det|^{\frac{2-k+\mu}{2}}, \qquad V_\mu(k)_{\mathbb{R}} = \mathcal{P}^{\mathbb{R}}_{k-2} \otimes |\det|^{\frac{2-k+\mu}{2}}.$$

Remark 1.3. With this $\mathrm{GL}_2(\mathbb{R})$-module structure, the pairing $\langle \cdot, \cdot \rangle$ introduced in Remark 1.2 is not a $\mathrm{GL}_2(\mathbb{R})$-invariant in general. In fact one can show that

$$\langle gP, Q \rangle = \mathrm{sign}(\det g)^k \langle P, g^{-1}Q \rangle.$$

Note that, for any $f \in I_\mu(k)^\pm$, we have that $\omega f(\theta) = \pm f(-\theta)$, hence we compute that,

$$\begin{aligned} \varphi(\omega f)(x,y) &= \int_{S^1} \omega f(\theta)(x\cos\theta + y\sin\theta)^{k-2} d\theta \\ &= \pm \int_{S^1} f(-\theta)(x\cos\theta + y\sin\theta)^{k-2} d\theta \\ &= \pm \int_{S^1} f(\theta)(x\cos\theta - y\sin\theta)^{k-2} d\theta = \pm\omega(\varphi(f))(x,y) \end{aligned}$$

This implies that the exact sequence of $(\mathcal{G}L_2(\mathbb{R}), SO(2))$-modules (1.4) provides the exact sequences of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules

$$0 \longrightarrow D_\mu(k) \stackrel{\iota}{\longrightarrow} I_\mu(k)^+ \longrightarrow V_\mu(k) \longrightarrow 0, \tag{1.10}$$

$$0 \longrightarrow D_\mu(k) \stackrel{\iota\circ I}{\longrightarrow} I_\mu(k)^- \longrightarrow V_\mu(k)(\varepsilon) \longrightarrow 0, \tag{1.11}$$

where $\varepsilon : \mathrm{GL}_2(\mathbb{R}) \to \pm 1$ is the character given by $\varepsilon(g) = \mathrm{sign}\det(g)$, $D_\mu(k)$ is the $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module with fixed action of ω given by $\omega f(\theta) = f(-\theta)$, and I is the automorphism of $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-modules

$$I : D_\mu(k) \longrightarrow D_\mu(k) : \quad I(f_t) = \mathrm{sign}(t) f_t.$$

Note that $\iota \circ I$ is a monomorphism of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules because $I(\omega f) = -\omega(I(f))$.

1.4. **Matrix coefficients.** Let us consider $A(\mathbb{C})$ the $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-module of admissible C^∞ functions $f : \mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{C}$. For any $f_0 \in I_{-\mu}(2-k)$, I claim that

$$\varphi_{f_0} : I_\mu(k) \longrightarrow A(\mathbb{C}), \qquad \varphi_{f_0}(f)(g_\infty) = \langle g_\infty f, f_0 \rangle_I, \qquad g_\infty \in \mathrm{GL}_2(\mathbb{R})^+,$$

provides a well defines morphism of $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-modules. Indeed, for any element of the Lie algebra $G \in \mathcal{G}L_2(\mathbb{R})$,

$$\begin{aligned} G\varphi_{f_0}(f)(g_\infty) &= \frac{d}{dt}(\varphi_{f_0}(g_\infty \exp(tG))) \mid_{t=0} = \frac{d}{dt}(\langle g_\infty \exp(tG) f, f_0 \rangle_I) \mid_{t=0} \\ &= \varphi_{f_0}(Gf)(g_\infty). \end{aligned}$$

1.5. **$\mathbb{R}$-structures of Discrete series.** As we can see in [2, §2.2], R and L are not in $\mathcal{G}L_2(\mathbb{R})$, they are *Caley transformations* in $\mathcal{G}L_2(\mathbb{C})$ of elements in $\mathcal{G}L_2(\mathbb{R})$. In fact, $\mathcal{G}L_2(\mathbb{R})$ is generated by

$$\begin{aligned} R + L &= -2y\sin(2\theta)\frac{\partial}{\partial x} + 2y\cos(2\theta)\frac{\partial}{\partial y} + \sin(2\theta)\frac{\partial}{\partial \theta}; \qquad u\frac{\partial}{\partial u}; \\ i(R-L) &= -2y\cos(2\theta)\frac{\partial}{\partial x} - 2y\sin(2\theta)\frac{\partial}{\partial y} + \cos(2\theta)\frac{\partial}{\partial \theta}; \quad \text{and} \quad \frac{\partial}{\partial\theta}. \end{aligned}$$

If we define $h_t := f_t + f_{-t} \in I_\mu(k)^\pm$ and $g_t := i(f_t - f_{-t}) \in I_\mu(k)^\pm$, it is easy to compute that

$$\begin{aligned} (R+L)h_t &= \left(\frac{k+t}{2}\right) h_{t+2} + \left(\frac{k-t}{2}\right) h_{t-2}, & \frac{\partial}{\partial\theta} h_t &= -t g_t, \\ i(R-L)h_t &= \left(\frac{k+t}{2}\right) g_{t+2} - \left(\frac{k-t}{2}\right) g_{t-2}, & \omega h_t &= \pm h_t \\ \kappa(\theta) h_t &= \cos(t\theta) h_t - \sin(t\theta) g_t, & \omega g_t &= \mp g_t. \end{aligned}$$

Hence the $\mathbb{R}$-vector space $I_\mu(k)^\pm_{\mathbb{R}} \subset I_\mu(k)^\pm$ generated by h_t and g_t defines a $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module over $\mathbb{R}$.

We check that the morphisms $\varphi : I_\mu(k)^+ \to V_\mu(k)_{\mathbb{C}}$ and $\varphi : I_\mu(k)^- \to V_\mu(k)(\varepsilon)_{\mathbb{C}}$ descend to morphisms of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules over $\mathbb{R}$

$$\varphi^+ : I_\mu(k)^+_{\mathbb{R}} \longrightarrow V_\mu(k)_{\mathbb{R}}, \qquad \varphi^- : I_\mu(k)^-_{\mathbb{R}} \longrightarrow V_\mu(k)_{\mathbb{R}}(\varepsilon).$$

Hence the kernel $D_\mu(k)_\mathbb{R} \subset D_\mu(k)$ of φ^+ defines a $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module over $\mathbb{R}$, generated by h_k, such that $D_\mu(k)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = D_\mu(k)$. Nevertheless, the automorphism of $(\mathcal{G}L_2(\mathbb{R}), SO(2))$-modules $I : D_\mu(k) \to D_\mu(k)$ does not descend to an automorphism of $(\mathcal{G}L_2(\mathbb{R}), SO(2))$-modules over $\mathbb{R}$ since $I(h_t) = -\mathrm{sign}(t) i g_t$. In fact,

$$I(D_\mu(k)_\mathbb{R}) = i D_\mu(k)_\mathbb{R} \subset D_\mu(k).$$

We obtain the exact sequences of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules over $\mathbb{R}$

$$0 \longrightarrow D_\mu(k)_\mathbb{R} \stackrel{\iota}{\longrightarrow} I_\mu(k)^+_\mathbb{R} \longrightarrow V_\mu(k)_\mathbb{R} \longrightarrow 0, \tag{1.12}$$

$$0 \longrightarrow D_\mu(k)_\mathbb{R} \stackrel{\iota \circ I}{\longrightarrow} i I_\mu(k)^-_\mathbb{R} \longrightarrow i V_\mu(k)_\mathbb{R}(\varepsilon) \longrightarrow 0. \tag{1.13}$$

2. Connection morphisms

In this section, we assume that G is the multiplicative group of a quaternion algebra that splits at the set of archimedean places Σ. Write $r = \#\Sigma$. Let us consider the $\mathbb{C}$-vector space $\mathcal{A}(\mathbb{C})$ of functions $f : G(\mathbb{A}) \longrightarrow \mathbb{C}$ such that:

- There exists an open compact subgroup $U \subseteq G(\mathbb{A}^\infty)$ such that $f(gU) = f(g)$, for all $g \in G(\mathbb{A})$.
- Under a fixed identification $G(F_\Sigma) \simeq \mathrm{GL}_2(\mathbb{R})^r$, $f\mid_{G(F_\Sigma)} \in C^\infty(\mathrm{GL}_2(\mathbb{R})^r, \mathbb{C})$.
- Fixing K_Σ, a maximal compact subgroup of $G(F_\Sigma)$ isomorphic to $O(2)^r$, we assume that any $f \in \mathcal{A}(\mathbb{C})$ is K_Σ-finite, namely, its right translates by elements of K_Σ span a finite-dimensional vector space.
- We assume that any $f \in \mathcal{A}(\mathbb{C})$ is $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the centre of the universal enveloping algebra of $G(F_\Sigma)$.

Write ρ for the action of $G(\mathbb{A})$ given by right translation, then $(\mathcal{A}(\mathbb{C}), \rho)$ defines a smooth $G(\mathbb{A}^\infty)$-representation and a $(\mathcal{G}_\infty, K_\infty)$-module, where $\mathcal{G}_\infty$ is the Lie algebra of $G(F_\Sigma)$ and $K_\infty = K_\Sigma \times G(F_{\infty\setminus\Sigma})$. Moreover, $\mathcal{A}(\mathbb{C})$ is also equipped with the $G(F)$-action:

$$(h \cdot f)(g) = f(h^{-1} g), \quad h \in G(F),$$

where $g \in G(\mathbb{A})$, $f \in \mathcal{A}(\mathbb{C})$. Let us fix an isomorphism $G(F_\Sigma) \simeq \mathrm{GL}_2(\mathbb{R})^r$ that maps K_Σ to $O(2)^r$ and let V be a $(\mathcal{G}_\infty, K_\infty)$-module. We define

$$\mathcal{A}(V, \mathbb{C}) := \mathrm{Hom}_{(\mathcal{G}_\infty, K_\infty)}(V, \mathcal{A}(\mathbb{C})),$$

endowed with the natural $G(F)$- and $G(\mathbb{A}^\infty)$-actions.

Remark 2.1. Note that if the $(\mathcal{G}_\infty, K_\infty)$-module V comes from a finite dimensional $G(F_\infty)$-representation V,

$$\mathcal{A}(V, \mathbb{C}) \simeq C(G(\mathbb{A}^\infty), \mathrm{Hom}(V, \mathbb{C})) = C(G(\mathbb{A}^\infty), V^\vee),$$

where $V^\vee$ is seen as a $G(F)$-module by means of the usual injection $G(F) \hookrightarrow G(F_\infty)$, and the action of $G(F)$ on $C(G(\mathbb{A}^\infty), V^\vee)$ is given by $(h * f)(g) = h(f(h^{-1}g))$.

Fix $\sigma \in \Sigma$, $\mu \in \mathbb{C}$ and let us consider $D_\mu(k)$, $V_\mu(k)$ and $V_\mu(k)(\varepsilon)$ as $(\mathcal{G}_\infty, K_\infty)$-modules by means of the projection $G(F_\infty) \to G(F_\sigma)$. The exact sequences (1.10) and (1.11) provide the connection morphisms

$$\partial_\sigma^{\epsilon_\sigma} : H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})) \longrightarrow H^{i+1}(G(F), \mathcal{A}(V \otimes V_\mu(k)(\epsilon_\sigma), \mathbb{C})),$$

for any of the two characters $\epsilon_\sigma : G(F_\sigma)/G(F_\sigma)^+ \to \pm 1$.

Let $V_\mathbb{R}$ be a $(\mathcal{G}_\infty, K_\infty)$-representation over $\mathbb{R}$ such that $V = V_\mathbb{R} \otimes \mathbb{C}$. This implies that we have a well defined complex conjugation on V by conjugating on the second factor. Thus, we have a complex conjugation on $\mathcal{A}(V, \mathbb{C})$, given by

$$\mathcal{A}(V, \mathbb{C}) \ni \phi \longmapsto \bar\phi \in \mathcal{A}(V, \mathbb{C}); \qquad \bar\phi(v) = \overline{\phi(\bar v)}.$$

Lemma 2.2. *Assume that $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for some $(\mathcal{G}_\infty, K_\infty)$-module $V_{\mathbb{R}}$ over $\mathbb{R}$, and let $\mu \in \mathbb{R}$. Then, for any $\phi \in H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C}))$, we have*

$$\overline{\partial_\sigma^{\epsilon_\sigma}(\phi)} = \epsilon_\sigma(c) \cdot \partial_\sigma^{\epsilon_\sigma}(\overline{\phi})$$

for any $c \in G(F_\sigma) \setminus G(F_\sigma)^+$.

Proof. We denote by $\mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1} \subset \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})$ the subspaces where complex conjugation acts by ± 1, respectively. Since exact sequences (1.10) and (1.11) descend to exact sequences (1.12) and (1.13), we obtain

$$0 \longrightarrow \mathcal{A}(V \otimes V_\mu(k)(\varepsilon_\sigma), \mathbb{C})^{\pm\varepsilon_\sigma} \longrightarrow \mathcal{A}(V \otimes I_\mu(k)^{\varepsilon_\sigma}, \mathbb{C})^{\pm\varepsilon_\sigma} \longrightarrow \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1} \longrightarrow 0.$$

Hence the connection morphism satisfies

$$\delta_\sigma^{\epsilon_\sigma}\left(H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1})\right) \subseteq H^{i+1}(G(F), \mathcal{A}(V \otimes V_\mu(k)(\varepsilon_\sigma), \mathbb{C})^{\pm\varepsilon_\sigma})$$

and the result follows. □

2.1. **Explicit computation of the connection morphisms.** Let us consider the section $s : V_\mu(k)_{\mathbb{C}} \to I_\mu(k)$ of (1.7).

We compute that

$$\begin{aligned}
\langle f_m, \iota P_n \rangle_I &= \int_{S^1} e^{mi\theta} P_n(-\sin\theta, \cos\theta) d\theta \\
&= \int_{S^1} i^{2n-k+2} e^{mi\theta} e^{ni\theta} e^{(n-k+2)i\theta} d\theta = \pi i^{2n-k+2} \delta(2n - k + 2 + m),
\end{aligned}$$

where $\delta(n)$ is the Dirac delta. Thus, $g_\infty^{-1} P = \sum_{n=0}^{k-2} \alpha_n(g_\infty) P_n$, where

$$\alpha_n(g_\infty) = \frac{i^{k-2-2n}}{\pi} \langle f_{k-2-2n}, \iota(g_\infty^{-1} P) \rangle_I = \frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2-2n}, \iota(P) \rangle_I.$$

Since $V_\mu(k)$ is generated by $\{P_0, \cdots, P_{k-2}\}$, we deduce that $\alpha_n = 0$ unless $n \in \{0, \cdots, k-2\}$. Since matrix coefficient morphism are $(\mathcal{G}_\sigma, K_\sigma)$-module morphisms by §1.4, we can compute on the one side

$$\begin{aligned}
R\alpha_n(g_\infty) &= \frac{i^{k-2-2n}}{\pi} \langle g_\infty (R f_{k-2-2n}), \iota(P) \rangle_I \\
&= (k - n - 1) \frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2n}, \iota(P) \rangle_I = (n + 1 - k) \alpha_{n-1}(g_\infty), \\
L\alpha_n(g_\infty) &= \frac{i^{k-2-2n}}{\pi} \langle g_\infty (L f_{k-2-2n}), \iota(P) \rangle_I \\
&= (n + 1) \frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2n-4}, \iota(P) \rangle_I = -(n + 1) \alpha_{n+1}(g_\infty).
\end{aligned}$$

On the other side, we have that $s(P_n) = \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} f_{2n-k+2}$. Hence,

$$\begin{aligned}
(n < k - 2) \; Rs(P_n) &= \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} (n + 1) f_{2n-k+4} = (k - 2 - n) s(P_{n+1}), \\
(n > 0) \; Ls(P_n) &= \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} (k - n - 1) f_{2n-k} = n s(P_{n-1}).
\end{aligned}$$

Assume that $\tilde{\phi} \in \mathcal{A}(V \otimes I_\mu(k), \mathbb{C})$ and the action of $(\mathcal{G}_\sigma, K_\sigma)$ on V is trivial. For any $f \in I_\mu(k)$ and $v \in V$, we will usually denote by $\tilde{\phi}_v(f)$ the expression $\tilde{\phi}(v \otimes f)$. We aim to compute

$$h_P(g_\infty) := \tilde{\phi}_v(s(g_\infty^{-1} P))(g_\infty, g), \qquad g_\infty \in G(F_\sigma)^+ \simeq \mathrm{GL}_2(\mathbb{R})^+,$$

for all $g\in G(\mathbb{A}^\sigma)$, $P\in V_\mu(k)$, and $v\in V$. Since $h_P(g_\infty)=\sum_{n=0}^{k-2}\alpha_n(g_\infty)\tilde\phi_v(s(P_n))(g_\infty,g)$, we compute

$$\begin{aligned}
Rh_P &= \sum_{n=0}^{k-2}\Big((R\alpha_n)\tilde\phi_v(s(P_n))+\alpha_n\tilde\phi_v(Rs(P_n))\Big)=\\
&= \sum_{n=0}^{k-2}(n+1-k)\alpha_{n-1}\tilde\phi_v(s(P_n))+\frac{k-1}{2^{2-k}\pi}\alpha_{k-2}\tilde\phi_v(f_k)+\sum_{n=0}^{k-3}(k-2-n)\alpha_n\tilde\phi_v(s(P_{n+1}))\\
&= \frac{k-1}{\pi}2^{k-2}\left(\alpha_{k-2}\tilde\phi_v(f_k)-\alpha_{-1}\tilde\phi_v(f_{2-k})\right)=\frac{k-1}{\pi}2^{k-2}\alpha_{k-2}\tilde\phi_v(f_k),\\
Lh_P &= \sum_{n=0}^{k-2}\Big((L\alpha_n)\tilde\phi_v(s(P_n))+\alpha_n\tilde\phi_v(Ls(P_n))\Big)=\\
&= \sum_{n=0}^{k-2}(-n-1)\alpha_{n+1}\tilde\phi_v(s(P_n))+\frac{k-1}{2^{2-k}\pi}\alpha_0\tilde\phi_v(f_{-k})+\sum_{n=1}^{k-2}n\alpha_n\tilde\phi_v(s(P_{n-1}))\\
&= \frac{k-1}{\pi}2^{k-2}\left(\alpha_0\tilde\phi_v(f_{-k})-\alpha_{k-1}\tilde\phi_v(f_{k-2})\right)=\frac{k-1}{\pi}2^{k-2}\alpha_0\tilde\phi_v(f_{-k}),
\end{aligned}$$

Notice that $R=e^{2i\theta}2iy(\frac{\partial}{\partial\tau}-\frac{1}{4y}\frac{\partial}{\partial\theta})$ and $L=-e^{-2i\theta}2iy(\frac{\partial}{\partial\bar\tau}-\frac{1}{4y}\frac{\partial}{\partial\theta})$ with $\tau=x+iy$. Since s is a morphism of $\mathrm{SO}(2)\mathbb{R}^+$-modules, h_P is a function of $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2)\mathbb{R}^+\simeq\mathcal{H}$, thus h_P is a function on τ and $\bar\tau$. Let us compute $\frac{\partial h_P}{\partial\tau}$ and $\frac{\partial h_P}{\partial\bar\tau}$: By (1.9) and (1.8),

$$\begin{aligned}
\frac{\partial h_P}{\partial\tau}(\tau,\bar\tau) &= \frac{y^{-1}e^{-2i\theta}}{2i}R(h_P)=\frac{(k-1)}{2\pi iye^{2i\theta}}i^{2-k}\langle g_\infty P_0,P\rangle\tilde\phi_v(f_k)\\
&= \frac{(k-1)}{2\pi iye^{2i\theta}}(g_\infty^{-1}P)(1,-i)\tilde\phi_v(f_k)=\frac{(k-1)}{2\pi i}P(1,-\tau)\frac{\tilde\phi_v(f_k)}{f_k}(\tau,\bar\tau,g),\\
\frac{\partial h_P}{\partial\bar\tau}(\tau,\bar\tau) &= \frac{-y^{-1}e^{2i\theta}}{2i}L(h_P)=\frac{(1-k)}{2\pi iye^{-2i\theta}}i^{k-2}\langle g_\infty\omega P_0,P\rangle\tilde\phi_v(f_{-k})\\
&= \frac{(1-k)}{2\pi iye^{-2i\theta}}(g_\infty^{-1}P)(1,i)\tilde\phi_v(f_{-k})=\frac{(1-k)}{2\pi i}P(1,-\bar\tau)\frac{\tilde\phi_v(f_{-k})}{f_{-k}}(\tau,\bar\tau,g),
\end{aligned}$$

by Remark 1.3, where $\tilde\phi_v(f_k)f_k^{-1}$ and $\tilde\phi_v(f_{-k})f_{-k}^{-1}$ are seen as functions of $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2)\mathbb{R}^+\simeq\mathcal{H}$. A similar (and classical) calculation shows that $\tilde\phi_v(f_k)f_k^{-1}$ and $\tilde\phi_v(f_{-k})f_{-k}^{-1}$ are holomorphic and anti-holomorphic, respectively.

For any $P\in V_\mu(k)$, $g\in G(\mathbb{A}^\sigma)$, $v\in V$ and $\phi\in\mathcal{A}(V\otimes D_\mu(k),\mathbb{C})$ the expressions

$$\omega_\phi(P,v,g)(\tau) := P(1,-\tau)\frac{\phi(v\otimes f_k)}{2\pi if_k}(\tau,g)d\tau, \tag{2.14}$$

$$\bar\omega_\phi(P,v,g)(\bar\tau) := P(1,-\bar\tau)\frac{\phi(v\otimes f_{-k})}{2\pi if_{-k}}(\bar\tau,g)d\bar\tau, \tag{2.15}$$

define holomorphic and anti-holomorphic forms in $\mathcal{H}$, respectively. Moreover, it is easy to check that

$$\omega_\phi(P,v,g)(\gamma^{-1}\tau)=\omega_{\gamma\phi}(\gamma P,v,\gamma g)(\tau),\qquad \bar\omega_\phi(P,v,g)(\gamma^{-1}\bar\tau)=\bar\omega_{\gamma\phi}(\gamma P,v,\gamma g)(\bar\tau),$$

for any $\gamma\in G(F)\cap G(F_\sigma)^+$. Assume that $c_\phi\in H^n(G(F),\mathcal{A}(V\otimes D_\mu(k),\mathbb{C}))$ is represented by the n-cocycle $\phi:G(F)^n\to\mathcal{A}(V\otimes D_\mu(k),\mathbb{C})$. Then $\partial_\sigma^{\epsilon_\sigma}(c_\phi)$ is represented by the $(n+1)$-cocycle $d^n\tilde\phi$, where $\tilde\phi(\underline{\gamma})\in\mathcal{A}(V\otimes I_\mu(k)_\mathbb{C},\mathbb{C})$ is any preimage of $\phi(\underline{\gamma})$ for all $\underline{\gamma}\in G(F)^n$. We consider the $(n+1)$-cocycle $\partial_\sigma^{\varepsilon_\sigma}(\phi)=d^n\tilde\phi-d^nb$, where $b(\underline{\gamma})(g)(v\otimes P)=\tilde\phi(\underline{\gamma})(v\otimes s(P))(1,g)$. We compute, for all

$P\in V_\mu(k)$, $v\in V$, $\underline{\gamma}=(\gamma_1,\cdots,\gamma_n)\in G(F)^n$, $\alpha\in G(F)$ and $g\in G(\mathbb{A}^\sigma)$,

$$\begin{aligned}\partial_\sigma^{\epsilon_\sigma}\phi(\alpha,\underline{\gamma})(g)(v\otimes P) &= \alpha\left((\tilde\phi-b)(\underline{\gamma})\right)(v\otimes s(P))(1,g)+\\ &+\sum_{i=1}^{n+1}(-1)^i(\tilde\phi-b)(\alpha\underline{\gamma}_i)(v\otimes s(P))(1,g),\end{aligned}$$

where $\alpha\underline{\gamma}_i=(\alpha,\gamma_1,\cdots,\gamma_{i-1}\gamma_i,\cdots,\gamma_n)$ for $i=1,\cdots,n$, and $\alpha\underline{\gamma}_{n+1}=(\alpha,\gamma_1,\cdots,\gamma_{n-1})$. Since $(\tilde\phi-b)(\underline{\gamma})(v\otimes s(P))=0$ by construction, we obtain

$$\begin{aligned}\partial_\sigma^{\epsilon_\sigma}\phi(\alpha,\underline{\gamma})(g)(v\otimes P) &= \alpha\left(\tilde\phi(\underline{\gamma})\right)(v\otimes s(P))(1,g)-\alpha\left(b(\underline{\gamma})\right)(v\otimes P)(g)\\ &= \tilde\phi(\underline{\gamma})(v\otimes s(P))(\alpha^{-1},\alpha^{-1}g)-\tilde\phi(\underline{\gamma})(v\otimes s(\alpha^{-1}P))(1,\alpha^{-1}g).\end{aligned}$$

Since $\tilde\phi(\underline{\gamma})(v\otimes f_k)=\phi(\underline{\gamma})(v\otimes f_k)$ and $\tilde\phi(\underline{\gamma})(v\otimes f_{-k})=\epsilon_\sigma(c)\phi(\underline{\gamma})(v\otimes f_{-k})$, we deduce from the above computations that, for any $\alpha\in G(F)\cap G(F_\sigma)^+$,

$$\begin{aligned}\partial_\sigma^{\epsilon_\sigma}\phi(\alpha,\underline{\gamma})(g)(v\otimes P) &= (k-1)\int_i^{\alpha^{-1}i}\omega_{\phi(\underline{\gamma})}(\alpha^{-1}P,v,\alpha^{-1}g)-\epsilon_\sigma(c)\bar\omega_{\phi(\underline{\gamma})}(\alpha^{-1}P,v,\alpha^{-1}g)\\ &= (1-k)\int_i^{\alpha i}\omega_{\alpha\phi(\underline{\gamma})}(P,v,g)-\epsilon_\sigma(c)\bar\omega_{\alpha\phi(\underline{\gamma})}(P,v,g).\end{aligned}$$

Remark 2.3. We have a well defined action of $G(F)/G(F)^+$ on $H^r(G(F)^+,M)$, for any $G(F)$-module M, given by

$$c^\gamma(\alpha_1,\cdots,\alpha_r)=\gamma\left(c(\gamma^{-1}\alpha_1\gamma,\cdots,\gamma^{-1}\alpha_r\gamma)\right),$$

for $\gamma\in G(F)$, and $\alpha_i\in G(F)^+$. The image of the restriction map

$$H^r(G(F),M)\longrightarrow H^r(G(F)^+,M)$$

lies in $H^0(G(F)/G(F)^+,H^r(G(F)^+,M))$.

Let $\psi:\mathbb{A}^\times/F^\times\to\mathbb{C}^\times$ be a Hecke character such that, for any archimedean place $\sigma_i:F\hookrightarrow\mathbb{R}$, $\psi_{\sigma_i}(x)=\mathrm{sign}(x)^{k_i}|x|^{\mu_i}$. Let $D_\psi(\underline{k})$ be the $(\mathcal{G}_\infty,K_\infty)$-module obtained by making the tensor product of $D_{\mu_i}(k_i)$ at the place σ_i, if $\sigma_i\in\Sigma$, and $V_{\mu_j}(k_j)$ at the place σ_j, if $\sigma_j\notin\Sigma$. An element of $D_\psi(\underline{k})$ is $f_{\underline{k}}\otimes P^\Sigma$, where $f_{\underline{k}}=\bigotimes_{\sigma_i\in\Sigma}f_{k_i}$, $f_{k_i}\in D_{\mu_i}(k_i)$ are the elements defined above, and $P^\Sigma\in\bigotimes_{\sigma_i\notin\Sigma}V_{\mu_i}(k_i)$. Let $V_\psi(\underline{k})$ be the $(\mathcal{G}_\infty,K_\infty)$-module obtained by making the tensor product of $V_{\mu_i}(k_i)$ at all the places σ_i. For any character $\epsilon:G(F_\infty)/G(F_\infty)^+\simeq G(F)/G(F)^+\to\pm1$, we denote by $H^r(G(F)^+,\mathcal{A}(V_\psi(\underline{k}),\mathbb{C}))(\epsilon)$ the ε-isotypical component, namely, the subspace of $H^r(G(F)^+,\mathcal{A}(V_\psi(\underline{k}),\mathbb{C}))$ such that the action of $G(F)/G(F)^+$ is given by the character. By the above remark, the restriction map provides an isomorphism

$$H^r(G(F),\mathcal{A}(V_\psi(\underline{k})(\varepsilon),\mathbb{C}))\simeq H^r(G(F)^+,\mathcal{A}(V_\psi(\underline{k}),\mathbb{C}))(\epsilon).$$

Using the above computations, we aim to give an explicit formula for the connection morphism:

Theorem 2.4. *Let ϕ be a weight $\underline{k}$ automorphic form of $G(\mathbb{A})$ with central character ψ. Then ϕ defines an element of $\phi\in H^0(G(F),\mathcal{A}(D_\psi(\underline{k}),\mathbb{C}))$. For a choice of signs at the places at infinity*

$$\epsilon:G(F_\infty)/G(F_\infty)^+\simeq G(F)/G(F)^+\to\pm1,$$

the composition of the connection morphisms δ_{ϵ_σ}, for $\sigma\in\Sigma$,

$$\partial_\epsilon:H^0(G(F),\mathcal{A}(D_\psi(\underline{k}),\mathbb{C}))\longrightarrow H^r(G(F),\mathcal{A}(V_\psi(\underline{k})(\epsilon),\mathbb{C}))\simeq H^r(G(F)^+,\mathcal{A}(V_\psi(\underline{k}),\mathbb{C}))(\epsilon),$$

can be computed as follows:

$$\partial_\epsilon\phi=\prod_{j+1}^r(1-k_j)\sum_{\gamma\in G(F)/G(F)^+}\epsilon(\gamma)\partial\phi^\gamma,$$

where $\partial\phi \in H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}),\mathbb{C}))$ *is the class of the cocycle*

$$(G(F)^+)^r \ni (g_1,g_2,\cdots,g_r) \longmapsto \int_{\tau_1}^{g_1\tau_1}\cdots\int_{g_1\cdots g_{r-1}\tau_r}^{g_1\cdots g_r\tau_r} P_\Sigma(1,-\underline{z})\frac{\phi(f_{\underline{k}}\otimes P^\Sigma)}{(2\pi i)^r f_{\underline{k}}}(\underline{z},1,g)d\underline{z},$$

for any $\underline{P} = P^\Sigma\otimes P_\Sigma \in V_{\underline{\mu}}(\underline{k})$, $\underline{z} = (z_1,\cdots,z_r), (\tau_1,\cdots,\tau_r)\in\mathcal{H}^r$.

Proof. Let $S\subset\Sigma$ be a subset of archimedean places such that $\#S = s < r$. Assume that $\sigma = \sigma_j \in \Sigma\setminus S$ and let k be its corresponding weight and $\mu = \mu_j$. Let $V = \bigotimes_{\sigma_i\in S'} D_{\mu_i}(k_i)\otimes\bigotimes_{\sigma_i\in\infty\setminus(\Sigma\setminus S)} V_{\mu_i}(k_i)(\varepsilon_{\sigma_i})$, where $S' = \Sigma\setminus(S\cup\{\sigma\})$. Thus, the composition of the connection morphisms corresponding to $\sigma_i\in S$, provides a morphism

$$\delta_S : H^0(G(F),\mathcal{A}(D_\psi(\underline{k}),\mathbb{C}))\longrightarrow H^s(G(F),\mathcal{A}(V\otimes D_\mu(k),\mathbb{C})).$$

By the previous computations, if $P\in V_\mu(k)$, $g\in G(\mathbb{A}^\sigma)$, $v\in V$,

$$\partial_\sigma^{\epsilon_\sigma}\partial_S\phi(\alpha,\underline{\gamma})(g)(v\otimes P) = (1-k)\int_i^{\alpha i}\omega_{\alpha\partial_S\phi(\underline{\gamma})}(P,v,g) - \epsilon_\sigma(c)\bar\omega_{\alpha\partial_S\phi(\underline{\gamma})}(P,v,g).$$

Notice that, letting $c\in(G(F)\cap G(F_S)^+)\setminus(G(F)\cap G(F_\sigma)^+)$, by means of the change of variables $\tau = g_\infty i\mapsto z = c\bar\tau = cg_\infty\omega i\in\mathcal{H}$, where $g_\infty\in \mathrm{GL}_2(\mathbb{R})^+$, we obtain that

$$\begin{aligned}\int_i^{\alpha i}\bar\omega_{\alpha\partial_S\phi(\underline{\gamma})}(P,v,g) &= \int_i^{\alpha i} P(1,-\bar\tau)\frac{\alpha\partial_S\phi(\underline{\gamma})(v\otimes f_{-k})(g_\infty,g)}{f_{-k}(g_\infty)}d\bar\tau\\ &= -\int_{c(-i)}^{c\alpha(-i)}(c*P)(1,-z)\frac{c\alpha\partial_S\phi(\underline{\gamma})(v\otimes f_k)(cg_\infty\omega,cg)}{f_k(cg_\infty\omega)}dz\\ &= -\int_{\tau_0}^{c\alpha c^{-1}\tau_0}\omega_{c\alpha\partial_S\phi(\underline{\gamma})}(c*P,v,cg),\end{aligned}$$

where $\tau_0 = c(-i)$, but in fact, this last expression does not depend on the choice of τ_0 because $\mathcal{H}$ is simply connected. Since $c\partial_S\phi(\underline{\gamma}) = \partial_S\phi(c\underline{\gamma}c^{-1})$ because $c\in G(F)\cap G(F_S)^+$, we obtain that

$$\partial_\sigma^{\epsilon_\sigma}\partial_S\phi(\alpha,\underline{\gamma}) = (1-k)(r(\alpha,\underline{\gamma})+\varepsilon(c)c^{-1}r(c\alpha c^{-1},c\underline{\gamma}c^{-1})),$$

where r is the cocycle

$$r(\alpha,\underline{\gamma})(g)(v\otimes P) = \int_i^{\alpha i}\omega_{\alpha\partial_S\phi(\underline{\gamma})}(P,v,g).$$

Applying a simple induction on S we obtain the desired result. ☐

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