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CHARACTERIZING BLOCK GRAPHS IN TERMS OF THEIR VERTEX-INDUCED PARTITIONS

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Abstract. Block graphs are a generalization of trees that arise in areas such as metric graph theory, molecular graphs, and phylogenetics. Given a finite connected simple graph $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$, we will show that the (necessarily unique) smallest block graph with vertex set V whose edge set contains E is uniquely determined by the V-indexed family $\mathbf{P}_G = (\pi_v)_{v \in V}$ of the partitions π_v of the set V into the set of connected components of the graph $(V, \{e \in E : v \notin e\})$. Moreover, we show that an arbitrary V-indexed family $\mathbf{P} = (\mathbf{p}_v)_{v \in V}$ of partitions $\mathbf{p}_v$ of the set V is of the form $\mathbf{P} = \mathbf{P}_G$ for some connected simple graph $G = (V, E)$ with vertex set V as above if and only if, for any two distinct elements $u, v \in V$, the union of the set in $\mathbf{p}_v$ that contains u and the set in $\mathbf{p}_u$ that contains v coincides with the set V, and $\{v\} \in \mathbf{p}_v$ holds for all $v \in V$. As well as being of inherent interest to the theory of block graphs, these facts are also useful in the analysis of compatible decompositions of finite metric spaces.

Keywords: block graph and vertex-induced partition and phylogenetic combinatorics and compatible decompositions and strongly compatible decomposition

1. Introduction

A *block graph* is a graph in which every maximal 2-connected subgraph or *block* is a clique [1, 8]. Block graphs are a natural generalization of trees, and they arise in areas such as metric graph theory [1], molecular graphs [2] and phylogenetics [7]. They have been characterised in various ways, for example, as certain intersection graphs [8], in terms of distance conditions [2, 9] and also by forbidden graph configurations [1]. Here we shall present an alternative approach to describing the set of block graphs.

More specifically, given a finite set V we call any partition of V a *V-partition*, and we define a V-indexed family of V-partitions $\mathbf{P}_V = (\mathbf{p}_v)_{v \in V}$ to be a *compatible family of V-partitions* if, for any two distinct elements $u, v \in V$, the union of the set in $\mathbf{p}_v$ that contains u and the set in $\mathbf{p}_u$ that contains v coincides with the set V, and $\{v\} \in \mathbf{p}_v$ holds for all $v \in V$. In addition, we let $\mathbf{P}(V)$ denote the set of all compatible families of V-partitions. Note that compatibility of partitions is a concept that naturally arises when analyzing phylogenetic trees (cf. e.g. [10]). In

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particular, if a V-indexed family of V-partitions is compatible, then every pair of partitions in this family is *strongly compatible* in the sense defined in [7].

In this note, we show that the map that takes each finite connected simple graph $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$ to the V-indexed family $\mathbf{P}_G := \big(\pi_v\big)_{v \in V}$ of the partitions π_v of the set V into the set of connected components of the graph $(V, \{e \in E : v \notin e\})$ induces a bijection from the set of connected block graphs with vertex set V onto the set $\mathbf{P}(V)$. We prove this in Theorem 1 below. In particular, defining two graphs G and G' with vertex set V to be *block-equivalent* if and only if the "smallest" block graphs that contain G and G' coincide, it immediately follows that the set of block-equivalence classes of connected simple graphs G with that vertex set V is in bijective correspondence with the set $\mathbf{P}(V)$.

As well as contributing to the tasks of phylogenetic combinatorics outlined in [5], this result is part of a broader investigation into so-called *compatible decompositions* and *block realizations* of finite metric spaces [3, 4] which was first mentioned in [6, Section 4]. In particular, it is key to proving that there is a unique "finest" compatible decomposition of any finite metric space (cf. [3, p.1619] for a more precise statement of this result).

The rest of this note is organised as follows. After presenting some preliminaries in the next section, we prove our main result.

2. Preliminaries

From now on, we will consider connected simple graphs G with a fixed finite vertex set V. Following [4], we will use the following notations and definitions.

Given any set Y, we denote
– by $Y - y$ the complement $Y - \{y\}$ of a one-element subset $\{y\}$ of Y,
– and by $\mathbf{p}[y]$, for any Y-partition $\mathbf{p}$ and any element $y \in Y$, that subset $Z \in \mathbf{p}$ of Y which contains y.

Further, given a simple graph G with vertex set V and edge set $E \subseteq \binom{V}{2}$, we denote
– by $\pi_0(G)$ the V-partition formed by the connected components of G,
– by $[G]$ the smallest block graph with vertex set V that contains G as a subgraph, i.e., the graph $(V, [E])$ with vertex set V whose edge set $[E]$ is the union of E and all 2-subsets $\{u, v\}$ of V that are contained in a circuit of G (i.e., a connected subgraph of G all of whose vertices have degree 2) (see e.g. [8]),
– by $G[v] := \pi_0(G)[v]$, for any vertex $v \in V$ of G, the connected component of G containing v,
– by $G^{(v)}$ the largest subgraph of G with vertex set V for which v is an isolated vertex, that is, the graph with vertex set V and edge set $\{e \in E : v \notin e\}$,
– and by $\mathbf{P}_G$ the V-indexed family

$$\mathbf{P}_G := \big(\pi_0(G^{(v)})\big)_{v \in V} \tag{1}$$

of partitions of V.

3. Main result

We now state and prove our main result:

Theorem 1: *Associating to each connected simple graph $G = (V, E)$ with vertex set V the V-indexed family $\mathbf{P}_G$ as defined above, induces a one-to-one map from the set $\mathbf{B}(V)$ of connected block graphs with vertex set V (or, equivalently, from the set of block-equivalence classes of connected simple graphs G with that vertex set) onto the set $\mathbf{P}(V)$ whose inverse is given by associating, to each family $\mathbf{P} = (\mathbf{p}_v)_{v\in V}$ in $\mathbf{P}(V)$, the graph $B_{\mathbf{P}} := (V, E_{\mathbf{P}})$ with vertex set V and edge set*

$$E_{\mathbf{P}} := \big\{\{u, v\} \in \binom{V}{2} : \forall_{w\in V-\{u,v\}}\mathbf{p}_w[u] = \mathbf{p}_w[v]\big\}.$$

In particular, given a connected graph $G = (V, E)$, the edge set $[E]$ of the associated block graph $[G]$ coincides with the set of all 2-subsets $\{u, v\}$ of V for which $G^{(w)}[u] = G^{(w)}[v]$ holds for all $w \in V - \{u, v\}$. And, given any family $\mathbf{P} = (\mathbf{p}_v)_{v\in V} \in \mathbf{P}(V)$, one has $\pi_0(B^{(v)}_{\mathbf{P}}) = \mathbf{p}_v$ for every element $v \in V$.

Proof: It is easy to see that, given any connected simple graph $G = (V, E)$ with vertex set V, the V-indexed family $\mathbf{P}_G = \big(\pi_0(G^{(v)})\big)_{v\in V}$ is a compatible family of V-partitions: Indeed, one has obviously $\pi_0(G^{(v)})[v] = \{v\}$ for every $v \in V$, and one has $\pi_0(G^{(v)})[u] \cup \pi_0(G^{(u)})[v] = V$ for any two distinct elements v, u in V as, given any vertex $w \in V$, there must exist a path $\mathfrak{p} = (u_0 := u, u_1, \ldots, u_k := w)$ connecting u and w in G implying that $w \in \pi_0(G^{(v)})[u]$ holds in case $v \notin \{u_1, u_2 \ldots, u_k\}$ and $w \in \pi_0(G^{(u)})[v]$ in case $v \in \{u_1, u_2, \ldots, u_k\}$.

We also have $[E] \subseteq E_{\mathbf{P}_G}$ for every connected graph $G = (V, E)$, that is, $G^{(w)}[u] = G^{(w)}[v]$ holds for every edge $\{u, v\} \in [E]$ and all $w \in V - \{u, v\}$ because this holds clearly for every edge $\{u, v\} \in E$, and it holds also for any two elements u, v that are contained in a circuit of G as, given any vertex $w \in V - \{u, v\}$, at least one of the two arcs of that circuit connecting u and v provides a path in $G^{(w)}$ connecting these two vertices in that graph.

And we have $E_{\mathbf{P}_G} \subseteq [E]$, that is, every 2-subset $\{u, v\}$ of V with $G^{(w)}[u] = G^{(w)}[v]$ for all $w \in V - \{u, v\}$ is either an element of E or contained in the vertex set of a circuit of G: Indeed, employing induction relative to the length k of a shortest path $\mathfrak{p} = (u_0 := u, u_1, \ldots, u_k := v)$ from u to v in G, there is nothing to prove in case $k = 1$. And in case $k = 2$, a circuit of G containing u and v can be found by concatenating p with a shortest path $\mathfrak{p}' = (u'_0 := u, u'_1, \ldots, u'_{k'} := v)$ from u to v in $G^{(u_1)}$ which must exist in view of $G^{(u_1)}[u] = G^{(u_1)}[v]$.
And, finally, in case $k > 2$, we first observe that $G^{(w)}[u_{k-1}] = G^{(w)}[u]$ holds for all $w \in V - \{u, u_{k-1}\}$. Indeed, in view of $\{u_{k-1}, v\} \in E$, we have

$$G^{(w)}[u_{k-1}] = G^{(w)}[v] = G^{(w)}[u]$$

for all $w \in V - \{u, v, u_{k-1}\}$, and we have also $G^{(w)}[u] = G^{(w)}[u_{k-1}]$ for $w := v$ in view of the fact that $(u, u_1, \ldots, u_{k-1})$ is a path in $G^{(v)}$ connecting u and u_{k-1}.

So, as $k > 2$ implies that $\{u, u_{k-1}\} \notin E$ must hold, our induction hypothesis implies that there must exist a circuit $\mathfrak{c}_0 = (C, F)$ in G with vertex set $C \subseteq V$ and edge set $F \subseteq E$ that passes through u and u_{k-1}, i.e., with $u, u_{k-1} \in C$. Furthermore, there must exist a shortest path $(v_0 := v, v_1, \ldots, v_j := u)$ connecting v and u in $G^{(u_{k-1})}$. Now, let i denote the smallest index in $\{0, 1, \ldots, j\}$ with $v_i \in C$ which must exist in view of $v_j = u \in C$. In case $i = 0$, we have $v = v_0 \in C$ and $u \in C$ implying that C is a circuit in G that passes through u and v, as required.
Otherwise, we may view $\mathfrak{c}_0$ as the concatenation of two edge-disjoint paths,

(i) the path $\mathfrak{p}_0$ from u_{k-1} to v_i not passing through u (unless $v_i = u$) and

(ii) the path $\mathfrak{p}_1$ from v_i back to u_{k-1} passing through u,

and then note that, replacing the path $\mathfrak{p}_0$ by the path $\mathfrak{p}_0' = (u_{k-1}, v, v_1, \ldots, v_i)$ (that is, concatenating $\mathfrak{p}_0'$ rather than $\mathfrak{p}_0$ with the path $\mathfrak{p}_1$), we obtain a new circuit $\mathfrak{c}_1$ in G that, starting, say, in u_{k-1}, runs along $\mathfrak{p}_0'$ via v over to v_i and then follows the path $\mathfrak{p}_1$ from v_i via u back to u_{k-1} and, thus, passes through both, u and v, as required.
This shows that the map from $\mathbf{B}(V)$ into the set $\mathbf{P}(V)$ given by associating to each connected simple graph $G = (V, E)$ with vertex set V the V-indexed family $\mathbf{P}_G$ is a well-defined injective map, and that $B_{\mathbf{P}_G} = (V, E_{\mathbf{P}_G}) = (V, [E]) = [G]$ holds for every connected graph $G = (V, E)$.

To establish the theorem, it therefore remains to show that, conversely, $\mathbf{P}_{B_\mathbf{P}} = \mathbf{P}$ holds for every compatible family $\mathbf{P}$ of V-partitions. So, assume that $\mathbf{P}$ is a fixed compatible family $\mathbf{P} = (\mathbf{p}_v)_{v \in V}$ of V-partitions. We have to show that $\mathbf{p}_v[u] = B_\mathbf{P}^{(v)}[u]$ holds for any two distinct elements $u, v \in V$. To this end, let us say that an element $w \in V$ **separates** two elements $u, v \in V$ (relative to $\mathbf{P}$) or, for short, that "$u|w|v$" holds if and only if $w \neq u, v$ and $\mathbf{p}_w[u] \neq \mathbf{p}_w[v]$ (and, therefore, also $u \neq v$) holds. Clearly, one has $\{u, v\} \in E_\mathbf{P}$ for two distinct elements $u, v \in V$ if and only if there is no $w \in V - \{u, v\}$ that separates u and v. So, we also have $B_\mathbf{P}^{(v)}[u] \subseteq \mathbf{p}_v[u]$ for any two distinct elements $u, v \in V$ since, otherwise, there would exist $u', u'' \in B_\mathbf{P}^{(v)}[u]$ with $\{u', u''\} \in E_\mathbf{P}$, but $\mathbf{p}_v[u'] \neq \mathbf{p}_v[u'']$.

To establish the converse, note that the following also holds:

Lemma 1. *Given any three distinct elements $u, v, w \in V$, the following nine assertions all are equivalent:*

- **(i)** *$w \in V$ separates $u, v \in V$, i.e., $\mathbf{p}_w[u] \neq \mathbf{p}_w[v]$ or, equivalently, "$u|w|v$" holds,*
- **(ii)** *$\mathbf{p}_w[u]$ is a proper subset of $\mathbf{p}_v[w]$,*
- **(iii)** *$\mathbf{p}_w[u]$ is a proper subset of $\mathbf{p}_v[u]$,*
- **(iv)** *$\mathbf{p}_w[u]$ is a subset of $\mathbf{p}_v[u]$,*
- **(v)** *$v \notin \mathbf{p}_w[u]$ holds,*
- **(vi)** *$\mathbf{p}_w[v]$ is a proper subset of $\mathbf{p}_u[w]$,*
- **(vii)** *$\mathbf{p}_w[v]$ is a proper subset of $\mathbf{p}_u[v]$,*

(viii) $\mathbf{p}_w[v]$ *is a subset of* $\mathbf{p}_u[v]$,
(ix) $u \notin \mathbf{p}_w[v]$ *holds,*

and they all imply that also

(x) $w \in \mathbf{p}_v[u] \cap \mathbf{p}_u[v]$

must hold.

Remark: Note that, while the last assertion (x) follows indeed from the former nine assertions, it is not equivalent to them – as e.g. the binary tree with the three leaves u, v, w immediately shows.

Proof: It is clear that, in view of $V = \mathbf{p}_w[v] \cup \mathbf{p}_v[w]$ and $w \notin \mathbf{p}_w[u]$, we have

$$\mathbf{p}_w[u] \neq \mathbf{p}_w[v] \Rightarrow \mathbf{p}_w[u] \cap \mathbf{p}_w[v] = \emptyset \Rightarrow \mathbf{p}_w[u] \subseteq V - (\mathbf{p}_w[v] \cup \{w\})$$

$$\Rightarrow \mathbf{p}_w[u] \subsetneq \mathbf{p}_v[w] \Rightarrow \mathbf{p}_w[u] \subseteq \mathbf{p}_v[w] \Rightarrow v \notin \mathbf{p}_w[u] \Rightarrow \mathbf{p}_w[u] \neq \mathbf{p}_w[v].$$

So, all these assertions must be equivalent to each other, and they imply also that $u \in \mathbf{p}_w[u] \subseteq \mathbf{p}_v[w]$ and, hence, $\mathbf{p}_v[w] = \mathbf{p}_v[u]$ and, therefore, also $w \in \mathbf{p}_v[w] = \mathbf{p}_v[u]$ must hold. In other words, the implications listed above yield that

$$\textbf{(i)} \iff \textbf{(ii)} \iff \textbf{(iii)} \iff \textbf{(iv)} \iff \textbf{(v)} \Longrightarrow w \in \mathbf{p}_v[u]$$

holds. And, switching u and v, we also get

$$\textbf{(i)} \iff \textbf{(vi)} \iff \textbf{(vii)} \iff \textbf{(viii)} \iff \textbf{(ix)} \Longrightarrow w \in \mathbf{p}_u[v]$$

and, therefore, also "**(i)** $\Rightarrow$ **(x)**", as claimed. ∎

Clearly, the lemma implies

(1) Given any four elements $u, u', v, v' \in V$ with $u' \neq v'$ and $u \neq v$, one has $\mathbf{p}_{v'}[u'] \subseteq \mathbf{p}_v[u]$ if and only if $\mathbf{p}_v[u'] = \mathbf{p}_v[u]$ and either $v = v'$ or $v|v'|u'$ holds.

Indeed, $\mathbf{p}_{v'}[u'] \subseteq \mathbf{p}_v[u]$ implies $u' \in \mathbf{p}_v[u]$ as well as $v \notin \mathbf{p}_{v'}[u']$ and, therefore, $\mathbf{p}_v[u'] = \mathbf{p}_v[u]$ as well as $v = v'$ or $v|v'|u'$ in view of "**(v)** $\Rightarrow$ **(i)**" while, conversely, $\mathbf{p}_v[u'] = \mathbf{p}_v[u]$ and $v = v'$ or $v|v'|u'$ implies $\mathbf{p}_{v'}[u'] \subseteq \mathbf{p}_v[u'] = \mathbf{p}_v[u]$. ∎

(2) Given any three **distinct** elements $u, v, w \in V$, one has $\mathbf{p}_u[w] \neq \mathbf{p}_v[w]$.

Indeed, one has $\mathbf{p}_u[w] \neq \mathbf{p}_v[w]$ for any three distinct elements u, v, w in V as $\mathbf{p}_u[w] = \mathbf{p}_v[w]$ would imply $u \notin \mathbf{p}_v[w]$ as well as $v \notin \mathbf{p}_u[w]$ and, therefore, $u|v|w$ as well as $v|u|w$ or, equivalently, $w \notin \mathbf{p}_v[u]$ and $w \notin \mathbf{p}_u[v]$ in contradiction to

$V = \mathbf{p}_v[u] \cup \mathbf{p}_u[v]$. ■

(3) Next, one has $\{u,v\} \in E_{\mathbf{P}}$ for two distinct elements $u,v \in V$ if and only if $\mathbf{p}_v[u]$ is a minimal set in the collection

$$\mathbf{P}_{\mathbf{p}_u[v]\cap\mathbf{p}_v[u]}[u] := \{\mathbf{p}_w[u] : w \in \mathbf{p}_u[v] \cap \mathbf{p}_v[u]\}$$

of subsets of V or, equivalently, in the collection

$$\mathbf{P}_{\mathbf{p}_u[v]}[u] := \{\mathbf{p}_w[u] : w \in \mathbf{p}_u[v]\}$$

or, still equivalently, in

$$\mathbf{P}[u] := \{\mathbf{p}_w[u] : w \in V - u\}.$$

Indeed, our definitions and the facts collected above imply that

$$\begin{aligned} \{u,v\} \notin E_{\mathbf{P}} &\iff \exists_{w \in V-\{u,v\}}\ \mathbf{p}_w[u] \neq \mathbf{p}_w[v] && \text{(by definition)} \\ &\iff \exists_{w \in \mathbf{p}_v[u]\cap\mathbf{p}_u[v]}\ \mathbf{p}_w[u] \subsetneq \mathbf{p}_v[u] && \text{(in view of “}\textbf{(i)} \Rightarrow \textbf{(iii)}\text{”)} \\ &\iff \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]\cap\mathbf{p}_v[u]}[u]\big) \end{aligned}$$

holds for any two distinct elements $u,v \in V$,

$$\mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]\cap\mathbf{p}_v[u]}[u]\big) \Longrightarrow \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]}[u]\big) \Longrightarrow \mathbf{p}_v[u] \notin \min\big(\mathbf{P}[u]\big)$$

holds for trivial reasons, and the last remaining implication

$$\mathbf{p}_v[u] \notin \min\big(\mathbf{P}[u]\big) \Longrightarrow \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]\cap\mathbf{p}_v[u]}[u]\big)$$

follows from the fact that $w \in V - u$ and $\mathbf{p}_w[u] \subsetneq \mathbf{p}_v[u]$ implies $w \neq u,v$ as well as $u|w|v$ and, therefore, also $w \in \mathbf{p}_u[v] \cap \mathbf{p}_v[u]$ in view of “**(i)**$\Rightarrow$**(x)**”, implying that also

$$\mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]\cap\mathbf{p}_v[u]}[u]\big) \iff \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]}[u]\big)$$

must hold. So,

$$\begin{aligned} \{u,v\} \in E_{\mathbf{P}} &\iff \mathbf{p}_v[u] \in \min\big(\mathbf{P}_{\mathbf{p}_u[v]\cap\mathbf{p}_u[v]}[u]\big) \\ &\iff \mathbf{p}_v[u] \in \min\big(\mathbf{P}_{\mathbf{p}_u[v]}[u]\big) \\ &\iff \mathbf{p}_v[u] \in \min\big(\mathbf{P}[u]\big) \end{aligned}$$

must hold, as claimed. ■

(4) Next, given three distinct elements $u,v,w \in V$ with $\{u,w\},\{w,v\} \in E_{\mathbf{P}}$, one has $\{u,v\} \in E_{\mathbf{P}}$ if and only if $\mathbf{p}_w[u] = \mathbf{p}_w[v]$ holds.

Indeed, $\{u,w\},\{w,v\} \in E_{\mathbf{P}}$ implies that $\mathbf{p}_{w'}[u] = \mathbf{p}_{w'}[w] = \mathbf{p}_{w'}[v]$ holds for all $w' \in V - \{u,v,w\}$ and that, therefore, $\{u,v\} \in E_{\mathbf{P}}$ or, equivalently, “$\forall_{w' \in V-\{u,v\}} \mathbf{p}_{w'}[u] = \mathbf{p}_{w'}[v]$” holds if and only if one has $\mathbf{p}_{w'}[u] = \mathbf{p}_{w'}[v]$ also for the only element

$w' \in V - \{u, v\}$ not in $V - \{u, v, w\}$, i.e., for $w' := w$. ■

(5) And finally, given any two distinct elements $u, v \in V$, and any sequence $\mathfrak{p} := (u_0 := u, u_1, \ldots, u_n := v)$ of elements of V such that

$$\mathbf{p}_{u_1}[u] \subsetneq \mathbf{p}_{u_2}[u] \subsetneq \cdots \subsetneq \mathbf{p}_{u_n}[u] = \mathbf{p}_v[u]$$

is a **maximal** chain of subsets of $\mathbf{p}_v[u]$ in

$$\mathbf{P}^{\subseteq \mathbf{p}_v[u]}[u] := \{\mathbf{p}_w[u] : w \in V - u, \mathbf{p}_w[u] \subseteq \mathbf{p}_v[u]\}$$

ending with $\mathbf{p}_v[u] = \mathbf{p}_{u_n}[u]$, the sequence $\mathfrak{p}$ forms a path from u to v in the graph $B_{\mathbf{P}} = (V, E_{\mathbf{P}})$, i.e., the 2-subsets $\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{n-1}, u_n\}$ of V are all contained in $E_{\mathbf{P}}$. Moreover, one has $u_i|u_j|u_k$ for all $i, j, k \in \{0, 1, \ldots, n\}$ with $i < j < k$ and, therefore, also $u_1, \ldots, u_{n-1} \in \mathbf{p}_u[v] \cap \mathbf{p}_v[u]$. In particular, we must have $u|u_j|v$ for all $j \in \{1, \ldots, n-1\}$ and $\mathbf{p}_{u_j}[u] = \mathbf{p}_{u_j}[u_i]$ and $\mathbf{p}_{u_i}[v] = \mathbf{p}_{u_i}[u_j]$ for all $i, j = 1, \ldots, n$ with $i < j$.

Indeed, our assumption that $\mathbf{p}_{u_j}[u] \subsetneq \mathbf{p}_{u_k}[u]$ holds for all $j, k \in \{1, 2, \ldots, n\}$ with $j < k$ implies, in view of "(**iii**) $\Rightarrow$ (**i**)" that also $u|u_j|u_k$ and, therefore, also $\mathbf{p}_{u_k}[u_j] = \mathbf{p}_{u_k}[u]$ must hold for all $j, k = 1, 2, \ldots, n$ with $j < k$. In consequence, we must also have $\mathbf{p}_{u_j}[u_i] = \mathbf{p}_{u_j}[u] \subsetneq \mathbf{p}_{u_k}[u] = \mathbf{p}_{u_k}[u_i]$ and, therefore, also $u_i|u_j|u_k$ as well as $\mathbf{p}_{u_k}[u_i] = \mathbf{p}_{u_k}[u_j]$ for all $i, j, k \in \{0, 1, \ldots, n\}$ with $i < j < k$. In particular, we must have $u|u_j|v$ for all $j \in \{1, \ldots, n-1\}$ and, hence, $u_1, \ldots, u_{n-1} \in \mathbf{p}_u[v] \cap \mathbf{p}_v[u]$ and $\mathbf{p}_{u_j}[u] = \mathbf{p}_{u_j}[u_i]$ and $\mathbf{p}_{u_i}[v] = \mathbf{p}_{u_i}[u_j]$ for all $i, j = 1, \ldots, n$ with $i < j$, as claimed.

To establish the remaining claim that $\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{n-1}, u_n\} \in E_{\mathbf{P}}$ also holds, note first that $\mathbf{p}_{u_1}[u]$ is, by assumption, a minimal set in the set system $\mathbf{P}^{\subseteq \mathbf{p}_v[u]}[u]$ and, therefore, also in $\mathbf{P}[u]$ as $w \in V - u$ and $\mathbf{p}_w[u] \subseteq \mathbf{p}_{u_1}[u]$ implies $\mathbf{p}_w[u] \subseteq \mathbf{p}_v[u]$ or, equivalently, $\mathbf{p}_w[u] \in \mathbf{P}^{\subseteq \mathbf{p}_v[u]}[u]$ and therefore, in view of the minimality of $\mathbf{p}_{u_1}[u]$ in $\mathbf{P}^{\subseteq \mathbf{p}_v[u]}[u]$, also $\mathbf{p}_w[u] = \mathbf{p}_{u_1}[u]$ or, equivalently, $w = u_1$. So, $\{u_0, u_1\} \in E_{\mathbf{P}}$ must hold.

Similarly, our choice of the elements $u_0, u_1, \ldots, u_n$ implies also that

$$\mathbf{p}_{u_i}[u] \in \min\{\mathbf{p}_w[u] : w \in V - u \text{ and } \mathbf{p}_{u_{i-1}}[u] \subsetneq \mathbf{p}_w[u] \subseteq \mathbf{p}_v[u]\} \tag{2}$$

must hold for all $i = 2, 3, \ldots, n$ and, therefore, also

$$\mathbf{p}_{u_i}[u] = \mathbf{p}_{u_i}[u_{i-1}] \in \min\left(\mathbf{P}_{\mathbf{p}_{u_i}[u_{i-1}] \cap \mathbf{p}_{u_{i-1}}[u_i]}[u_{i-1}]\right) \tag{3}$$

as $w \in \mathbf{p}_{u_i}[u_{i-1}] \cap \mathbf{p}_{u_{i-1}}[u_i]$ and $\mathbf{p}_w[u_{i-1}] \subsetneq \mathbf{p}_{u_i}[u] = \mathbf{p}_{u_i}[u_{i-1}]$ would imply $u_i \notin \mathbf{p}_w[u_{i-1}]$ and $w \notin \mathbf{p}_{u_{i-1}}[u]$ (in view of $w \in \mathbf{p}_{u_{i-1}}[u_i] = \mathbf{p}_{u_{i-1}}[v] \neq \mathbf{p}_{u_{i-1}}[u]$) and, therefore, $u_{i-1}|w|u_i$ as well as $u|u_{i-1}|w$ which, in turn, would imply

$$\mathbf{p}_{u_{i-1}}[u] \subsetneq \mathbf{p}_w[u] = \mathbf{p}_w[u_{i-1}] \subsetneq \mathbf{p}_{u_i}[u] \subseteq \mathbf{p}_v[u]$$

in contradiction to (2). So, (3) or, equivalently, $\{u_{i-1}, u_i\} \in E_{\mathbf{P}}$ must hold also for all $i \in \{2, \ldots, n\}$. ■

Now, to finalize the proof of our main result, it suffices to note that, with $\mathbf{P} =$

$(\mathbf{p}_v)_{v\in V} \in \mathbf{P}(V)$ as above, one has $\mathbf{p}_v[u] \subseteq B_{\mathbf{P}}^{(v)}[u]$ for any two distinct elements $u, v \in V$. Yet, given any further element $u' \in \mathbf{p}_v[u]$, Assertion (**5**) implies that there exist two paths $\mathfrak{p} := (u_0 := u, u_1, \dots, u_n := v)$ and $\mathfrak{p}' := (u'_0 := u', u'_1, \dots, u'_{n'} := v)$ connecting u and u' with v in $B_{\mathbf{P}}$, and Assertion (**4**) implies that also either $u_{n-1} = u'_{n'-1}$ or $\{u_{n-1}, u'_{n'-1}\} \in E_{\mathbf{P}}$ holds, implying that there exists also a path in $B_{\mathbf{P}}^{(v)}$ from u to u'.

This finishes the proof of the theorem. ■

Remark: It might also be worth noting that a compatible family of V-partitions $\mathbf{P} = (\mathbf{p}_v)_{v\in V}$ is fully encoded by the ternary relation "..|..|.."$\subseteq V^3$ as $\mathbf{p}_v[u]$ apparently coincides, for any two distinct elements $u, v \in V$, with the set of all $w \in V - v$ for which $u|v|w$ does not hold. Consequently, one can also record the specific properties an arbitrary ternary relation "..|..|.."$\subseteq V^3$ must satisfy to correspond to some $\mathbf{P} \in \mathbf{P}(V)$ – a simple task that we leave as an exercise to the reader.

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