A FRAMEWORK FOR FORCING CONSTRUCTIONS AT SUCCESSORS OF SINGULAR CARDINALS

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ABSTRACT. We describe a framework for proving consistency results about singular cardinals of arbitrary cofinality and their successors. This framework allows the construction of models in which the Singular Cardinals Hypothesis fails at a singular cardinal κ of uncountable cofinality, while κ^+ enjoys various combinatorial properties.

As a sample application, we prove the consistency (relative to that of ZFC plus a supercompact cardinal) of there being a strong limit singular cardinal κ of uncountable cofinality where SCH fails and such that there is a collection of size less than 2^{κ^+} of graphs on κ^+ such that any graph on κ^+ embeds into one of the graphs in the collection.

Introduction

The class of uncountable regular cardinals is naturally divided into three disjoint classes: the successors of regular cardinals, the successors of singular cardinals and the weakly inaccessible cardinals. When we consider a combinatorial question about uncountable regular cardinals, typically these classes require separate treatment and very frequently the successors of singular cardinals present the hardest problems. In particular there are subtle constraints (for example in cardinal arithmetic) on the combinatorics of successors of singular cardinals, and consistency results in this area often involve large cardinals.

To give some context for our work, we review a standard strategy for proving consistency results about the successors of regular cardinals. This strategy involves iterating $<\kappa$-closed κ^+-cc forcing with $<\kappa$-supports for some regular cardinal κ, with the plan that the whole iteration will also enjoy the κ^+-chain condition. The κ^+-chain condition of the iteration will of course ensure that all cardinals are preserved, and is also very helpful in the "catch your tail" arguments which frequently appear in iterated forcing constructions.

When $\kappa = \omega$ this proof strategy is completely straightforward as any finite support iteration of ccc forcing is ccc; for regular $\kappa > \omega$ we need to assume that $\kappa^{<\kappa} = \kappa$ and that the iterands have some strong form of κ^+-cc and some other properties in order to ensure κ^+-cc for the iteration (this issue is extensively discussed in §1 below). When κ is singular the strategy is no longer available, and this is one difficulty among many in proving consistency results involving singular cardinals and their successors.

Džamonja and Shelah [7] introduced a new idea, which we briefly describe. Initially κ is a supercompact cardinal whose supercompactness is indestructible under $<\kappa$-directed closed forcing. The final model is obtained by a two-step forcing. The first step, $\mathbb{P}$, is a $<\kappa$-directed closed and κ^+-cc iteration, whilst the second step is Prikry forcing, Pr_U, defined from a normal measure U on κ in $V[G]$, where G is a $\mathbb{P}$-generic filter over V. The iteration $\mathbb{P}$ is designed to anticipate and deal with Pr_U-names for subsets of κ^+, so that after forcing over $V[G]$ with Pr_U we obtain the desired consistency result. Džamonja and Shelah used this method to obtain the consistency, relative to that of a supercompact cardinal, of the existence for κ singular strong limit of cofinality ω, of a family of κ^{++} many graphs on κ^+ which are jointly

universal for all graphs on κ^+. In this model $2^\kappa = 2^{\kappa^+}$ and this value can be made arbitrarily large.

A well known early interaction between model theory and set theory gives one that if $2^\kappa = \kappa^+$ there is a saturated graph (*i.e.*, model of the theory of graphs) on κ^+ and any such graph is universal. So the point of results such as that of Džamonja and Shelah is to address the possibility of having small universal families of graphs on κ^+ when $2^\kappa > \kappa^+$.

In this paper we build a similar framework in which the final forcing step is a version of Radin forcing, and changes the cofinality of κ to become some uncountable cardinal less than κ. After building the framework, we prove a version of the result on universal graphs mentioned in the last paragraph.

Theorem 7.10. *Suppose κ is a supercompact cardinal, $\lambda < \kappa$ is a regular cardinal and Θ is a cardinal with $\mathrm{cf}(\Theta) \geq \kappa^{++}$ and $\kappa^{+3} \leq \Theta$. There is a forcing extension in which cofinally many cardinals below κ, κ itself and all cardinals greater than κ are preserved, $\mathrm{cf}(\kappa) = \lambda$, $2^\kappa = 2^{\kappa^+} = \Theta$ and there is a universal family of graphs on κ^+ of size κ^{++}.*

The need for large cardinals in the broad context of combinatorics at singular cardinals and their successors is at least partially explained by the theory of core models and covering lemmas. If there is no inner model with a Woodin cardinal then there is an inner model K with many strong combinatorial properties (for example GCH and square hold), and such that $\kappa^+ = (\kappa^+)^K$ for every singular cardinal κ. This resemblance between V and K in the absence of inner models with large enough cardinals exerts a strong influence on the combinatorics of κ^+ in V, for example it implies that $\square_\kappa$ holds in V under this hypothesis.

In the instance of the results on universal graphs we obtain a model in which κ is a singular strong limit cardinal where $2^\kappa > \kappa^+$, i.e. the singular cardinal hypothesis fails. It is known through work of Gitik, Mitchell, Shelah and Woodin that the consistency strength of the singular cardinal hypothesis failing alone is exactly that of a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$. Specifically, in one direction Woodin gave a forcing construction of a model in which the singular cardinal hypothesis fails from a certain large cardinal hypothesis and Gitik showed this hypothesis follows from that of the existence of a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$ ([9]). In the other direction Gitik showed ([10]), building on Mitchell's Covering Lemma

([19]) and ideas of Shelah developed in the course of his pcf theory ([23]), that if the singular cardinal hypothesis fails there is a model in which there is a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$. Thus some large cardinal hypotheses are necessary for these results on universal graphs that we prove.

This paper is in some sense a sequel to the work by Džamonja and Shelah [7], and so we briefly record some of the innovations introduced here.

- Prikry forcing at κ is homogeneous and adds no bounded subsets of κ, while Radin forcing has neither of these features. This entails major changes in the analysis of names for κ^+ in the final model, the proof of κ^+-cc for the main iteration, and the proof that the final model has a small jointly universal family.

- The arguments of [7] involve a complex iteration scheme which is used to build a Prikry forcing Pr_U, and would have become even more complex if we had used it to build a suitable Radin forcing. In this paper we use diamond sequences to achieve similar goals.

- One of the central points is that our main iteration enjoys a strong form of the κ^+-chain condition. None of the standard preservation theorems were quite suitable to show this, so we took a detour into iteration theory to formulate and prove a suitable preservation theorem.

The paper is organized as follows: §1 shows that cardinals are preserved in $<\kappa$-support iterations of certain types of $<\kappa$-closed, κ^+-stationary chain condition forcings (and as a spin-off we give a generalized Martin's axiom for these forcings). §2 is on preservation of diamond through forcing iterations. §3 collects relevant material on Radin forcing. §4 describes the long Mathias forcing and its variant for adding Radin names for universal graphs. §5 contains the proof of the stationary κ^+-chain condition for the forcings of §4. §6 gives the main iteration. §7 gets a small family of universal graphs at a cardinal of uncountable singular cofinality.

The problem of the existence of a small family of universal graphs has some independent interest, and is also a natural test question for our forcing framework. In [3] the methods of this paper are applied to prove a result parallel to Theorem 7.10 concerning graphs on $\aleph_{\omega+1}$. As part of his thesis work in progress, Jacob Davis has also obtained a parallel result for graphs on $\aleph_{\omega_1+1}$.

Notation. Card and On are the classes of cardinals and ordinals respectively. The size of a set A is denoted by either $\overline{\overline{A}}$ or $|A|$. If X is a set and κ is a cardinal then $\mathscr{P}(X)$ is the set of all subsets of X and $[X]^\kappa$ is the set of all subsets of X of size κ. We define $[X]^{<\kappa}$, etc., in the obvious way. If X, Y are sets then $^X Y$ is the set of functions from X to Y. As in [15], Definition (VII.6.1), if I and J are sets and κ is a cardinal we write $\mathrm{Fn}(I, J, \kappa)$ for the set of partial functions from I to J of size less than κ, $\{p \mid p \text{ is a function } \& \ \overline{\overline{p}} < \kappa \ \& \ \mathrm{dom}(p) \subseteq I \ \& \ \mathrm{rge}(p) \subseteq J\}$.

We designate names for the projection functions on cartesian products. If X, Y are sets we define $\pi_0 : X \times Y \longrightarrow X$ and $\pi_1 : X \times Y \longrightarrow Y$ by, if $x \in X$ and $y \in Y$ then $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$.

We write $f \cdot g$ for the composition of the functions f and g. When f and g agree on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ we write $f + g$ for the unique function h such that $\mathrm{dom}(h) = \mathrm{dom}(f) \cup \mathrm{dom}(g)$, $h \restriction \mathrm{dom}(f) = f$ and $h \restriction \mathrm{dom}(g) = g$.

If $f : A \times B \longrightarrow C$ is a function, $a \in A$ and $b \in B$ we write $f(a, .)$ for the function $g : B \longrightarrow C$ such that $g(d) = f(a, d)$ for all $d \in B$ and $f(., b)$ for the function $h : A \longrightarrow C$ such that $h(c) = f(c, b)$ for all $c \in A$.

If X is a set of ordinals we write $\mathrm{ssup}(X)$ for the *strong supremum of X*, that is the least ordinal α such that $X \subseteq \alpha$, and $\mathrm{cl}(X)$ for the closure of X in the order topology. If α, β are ordinals we write $\alpha\beta$ for their ordinal multiplication product.

One *non-standard* piece of notation which will be useful and which we define here is the following. If $y = \langle y_\tau \mid \tau < \mathrm{lh}(y) \rangle$ is a sequence of sets and $y_0 \in \mathrm{Card}$ we often write κ_y for y_0, the first entry in y.

If μ, κ are regular cardinals with $\mu < \kappa$ we write, on occasion, S^κ_μ for $\{\xi < \kappa \mid \mathrm{cf}(\xi) = \mu\}$, and similarly for $S^\kappa_{<\mu}$, $S^\kappa_{\geq\mu}$ and so on. If S is a stationary subset of κ we say that $T \subseteq S$ is a *club relative to S* if there is some club $C \subseteq \kappa$ such that $T = S \cap C$.

A function $g : \kappa \longrightarrow \kappa$ is *regressive* on a set $S \subseteq \kappa$ if for every $\varepsilon \in S$ we have $g(\varepsilon) < \varepsilon$.

A subset X of a partial order $(\mathbb{P}, \leq)$ is *linearly ordered* if for all p, $q \in X$ either $p \leq q$ or $q \leq p$, *directed* if every finite subset of X has a lower bound in X, and *centred* if every finite subset of X has a lower bound in $\mathbb{P}$. A partial order $(\mathbb{P}, \leq)$ is $<\kappa$-*closed* if every linearly ordered subset of size $<\kappa$ of $\mathbb{P}$ has a lower bound, $<\kappa$-*directed closed* if every

directed subset of size $<\kappa$ of $\mathbb{P}$ has a lower bound, and $<\kappa$-*compact* if every centred subset of size $<\kappa$ of $\mathbb{P}$ has a lower bound. $\mathbb{P}$ is *countably compact* if and only if it is $<\omega_1$-compact. Clearly any $<\kappa$-compact partial order is $<\kappa$-directed closed, and any $<\kappa$-directed closed partial order is $<\kappa$-closed.

If $(\mathbb{P}, \leq)$ is a partial order and q, $r \in \mathbb{P}$ we write $q \parallel r$ to mean that q and r are compatible in $\mathbb{P}$, that is there is some $p \in \mathbb{P}$ such that $p \leq q, r$. If $(\mathbb{Q}, \leq)$ is a sub-partial order of $(\mathbb{P}, \leq)$ and q, $r \in \mathbb{Q}$ we write $q \parallel_{\mathbb{Q}} r$ to mean q and r are compatible *in* $\mathbb{Q}$, that is there is some $p \in \mathbb{Q}$ such that $p \leq q, r$. A partial order is *splitting* if every element has two incompatible extensions.

A partial order $(\mathbb{P}, \leq)$ is *well-met* if every compatible pair of elements has a greatest lower bound: *i.e.*, for all p, $q \in \mathbb{P}$ if $p \parallel q$ then there is some $r \in \mathbb{P}$ such that $r \leq p, q$ and for all $s \in \mathbb{P}$ with $s \leq p, q$ we have $s \leq r$.

For a regular cardinal κ, a partial order $\mathbb{P}$ has the κ^+-*stationary chain condition* (abbreviated later as the κ^+-*stationary cc*) if and only if for every sequence $\langle p_i \mid i < \kappa^+ \rangle$ of conditions in $\mathbb{P}$ there is a club set $C \subseteq \kappa^+$ and there is a regressive function f on $C \cap S_\kappa^{\kappa^+}$ such that for all α, $\beta \in C \cap S_\kappa^{\kappa^+}$ with $f(\alpha) = f(\beta)$ the conditions p_α and p_β are compatible. We note that by an easy application of Fodor's lemma, this property implies that $\mathbb{P}$ enjoys the strengthened form of the κ^+-Knaster property in which any κ^+-sequence of conditions has a stationary subsequence of pairwise compatible conditions. A stronger notion (see Lemma (1.5) and Example (1.6) below) is that $\mathbb{P}$ is κ-*linked* if $\mathbb{P}$ is the union of κ many sets of pairwise compatible elements.

When forcing with a partial order $(\mathbb{P}, \leq)$ over a model V we take the notions of *names* and *canonical names* to be as in [15], writing $\dot{x}$ for a $\mathbb{P}$-name in V and $\hat{y}$ for a standard $\mathbb{P}$-name for $y \in V$. We will freely use the well-known *Maximum Principle*, which states that if $\Vdash \exists x \phi(x, \hat{y}, \dot{G})$ then there is a name $\dot{x}$ such that $\Vdash \phi(\dot{x}, \hat{y}, \dot{G})$. Usually our usage should be clear: as is customary we take the Boolean truth values $\mathbb{0}$, $\mathbb{1}$ to be identified with the (ordinal) elements of $\{0, 1\}$. Thus forcing names for truth values are names for ordinals < 2. In the context of forcing, if p, $q \in \mathbb{P}$ then $p \leq q$ means that p is stronger than q.

Cohen forcing to add λ many subsets of a regular cardinal κ, which we denote by $\mathrm{Add}(\kappa, \lambda)$, is the collection of partial functions of size less than κ from λ to 2, $\mathrm{Fn}(\lambda, 2, \kappa)$, ordered by reverse inclusion.

The Laver iteration $\mathbb{L}$ for a supercompact cardinal κ is a κ-cc forcing poset of cardinality κ which make the supercompactness of κ indestructible under $<\kappa$-directed closed forcing [16].

If κ^+ is a successor cardinal a κ^+-*tree* is a tree of size κ^+, height κ^+ and each level of size at most κ. A *binary κ^+-tree* is a κ^+-tree such that each point in the tree has exactly two successors in the tree order. If Υ is a cardinal greater than κ^+ a κ^+-*Kurepa tree with Υ-many branches* is a κ^+-tree with Υ-many branches of length κ^+. If there is such a tree there is one which is binary and a subtree of the complete binary tree $^{<\kappa^+}2$ (see, for example, [15], Chapter (2)). In a slight abuse of terminology, we refer here to binary κ^+-Kurepa subtrees of $^{<\kappa^+}2$ as binary κ^+-Kurepa trees.

The *usual forcing to add a κ^+-Kurepa tree with Υ-many branches*, due to Stewart [26], is the forcing notion in which conditions are pairs (t,f) consisting of a binary sub-tree, t, of $^{<\kappa^+}2$ of successor height, say $\gamma+1 < \kappa^+$, each level of size at most κ and each point in the tree having a successor of height γ, and a bijection f between a subset of Υ and the set of points of height γ. The ordering is that $p = (t^p, f^p) \leq q = (t^q, f^q)$ if t^q is obtained by, in Kunen's [15] vivid précis, 'sawing off t^p parallel to the ground': *i.e.* there is some $\alpha < \kappa^+$ such that $t^q = t^p \cap {}^\alpha 2$, $\mathrm{dom}(f^q) \subseteq \mathrm{dom}(f^p)$ and $f^q(\xi) <^p f^p(\xi)$, where $<^p$ is the tree order on t^p. This forcing has the κ^{++}-chain condition and is $<\kappa^+$-directed closed with greatest lower bounds: every directed set of conditions of size at most κ has a greatest lower bound.

We recall here in one place, for the reader's convenience, the definitions of various variants of the $\diamondsuit$ principle. The reader can also consult, for example, [15] and Rinot's article [21] for further information about these principles. Let κ be a regular cardinal and B a stationary subset of κ.

A sequence $\langle A_\alpha \mid \alpha \in B \rangle$ is a $\diamondsuit_\kappa(B)$-*sequence* if $A_\alpha \subseteq \alpha$ for each $\alpha \in B$ and whenever $A \subseteq \kappa$ the sequence frequently predicts A correctly: $\{\alpha \in B \mid A \cap \alpha = A_\alpha\}$ is a stationary subset of κ.

A sequence $\langle \mathcal{A}_\alpha \mid \alpha \in B \rangle$ is a $\diamondsuit^*(B)$-*sequence* if $\mathcal{A}_\alpha \in [\mathcal{P}(\alpha)]^{\leq \overline{\overline{\alpha}}}$ for each $\alpha < \kappa$ and whenever $A \subseteq \kappa$ there is a closed unbounded subset C of κ such that $C \cap B \subseteq \{\alpha \in B \mid A \cap \alpha \in \mathcal{A}_\alpha\}$.

A sequence $\langle \mathcal{A}_\alpha \mid \alpha \in B \rangle$ is a $\diamondsuit^+(B)$-*sequence* if $\mathcal{A}_\alpha \in [\mathcal{P}(\alpha)]^{\leq \overline{\overline{\alpha}}}$ for each $\alpha < \kappa$ and whenever $A \subseteq \kappa$ there is a closed unbounded subset C of κ such that $C \cap B \subseteq \{\alpha \in B \mid A \cap \alpha, \ C \cap \alpha \in \mathcal{A}_\alpha\}$.

We say that $\diamondsuit_\kappa(B)$ *holds* (*resp.* $\diamondsuit_\kappa^*(B)$ *holds*, $\diamondsuit_\kappa^+(B)$ *holds*) if there is a $\diamondsuit_\kappa(B)$-sequence (*resp.* a $\diamondsuit_\kappa^*(B)$-sequence, a $\diamondsuit_\kappa^+(B)$-sequence).

When B is the whole of κ, *i.e.*, $B = \{\alpha \,|\, \alpha < \kappa\}$, we omit mention of it. So, for example, a $\diamondsuit_\kappa$-sequence is a $\diamondsuit_\kappa(\{\alpha \,|\, \alpha < \kappa\})$-sequence and so on.

A $\diamondsuit_\kappa(B)$-sequence *on* $\kappa \times \kappa$ is a sequence $\langle A_\alpha \,|\, \alpha \in B \rangle$ such that $A_\alpha \subseteq \alpha \times \alpha$ for each $\alpha \in B$ and whenever $A \subseteq \kappa \times \kappa$ one has that $\{\alpha \in B \,|\, A \cap \alpha \times \alpha = A_\alpha\}$ is stationary in κ.

Clearly, if there is a $\diamondsuit_\kappa(B)$-sequence then there is a $\diamondsuit_\kappa(B)$-sequence on $\kappa \times \kappa$ – the closure points of any enumeration of $\kappa \times \kappa$ in order type κ form a closed unbounded subset of κ. We talk similarly of $\diamondsuit_\kappa^*(B)$-sequences and $\diamondsuit_\kappa^+(B)$-sequences *on* $\kappa \times \kappa$.

If $\mathcal{G} = (X, E)$ and $\mathcal{H} = (Y, F)$ are graphs then $f : \mathcal{G} \longrightarrow \mathcal{H}$ is an *embedding of $\mathcal{G}$ into $\mathcal{H}$ as an induced subgraph* if $\forall x_0 \in X \ \forall x_1 \in X \,(\, x_0 \ E \ x_1 \longleftrightarrow f(x_0) \ F \ f(x_1) \,)$. Since this is the only kind of graph embedding which concerns us we will simply call them *embeddings*.

Let κ be a regular cardinal. We say that a family $\mathcal{F}$ of graphs on κ is *jointly universal* for graphs of size κ if for every graph $\mathcal{G} = (\kappa, E)$ there is some $\mathcal{H} = (\kappa, F) \in \mathcal{F}$ and some embedding $f : \mathcal{G} \longrightarrow \mathcal{H}$ of $\mathcal{G}$ into $\mathcal{H}$. We say that $\mathcal{F}$ is a *small* universal family if $\overline{\overline{\mathcal{F}}} < 2^\kappa$.

1. Some iterated forcing theory

In the classical exposition of iterated forcing Baumgartner [1, §4] wrote

> The search for extensions of MA for larger cardinals has proved to be rather difficult.

One reason for this is that for κ regular and uncountable, an iteration of $<\kappa$-closed and κ^+-cc forcing posets with supports of size less than κ does not in general have the κ^+-cc. For example, a construction due to Mitchell described in a paper of Laver and Shelah [17] shows that in L there is an iteration of length ω of countably closed $\aleph_2$-cc forcing posets such that the inverse limit at stage ω does not have the $\aleph_2$-cc.

The literature contains several preservation theorems for iterations involving strengthened forms of closure and chain condition, along with corresponding forcing axioms. The first results in this direction are in unpublished work by Laver [1, §4]. Baumgartner [1] proved that under CH an iteration with countable supports of countably compact

$\aleph_1$-linked forcing posets is $\aleph_2$-cc, and proved the consistency of some related forcing axioms. Shelah [22] proved that under CH an iteration with countable supports of posets which are countably closed and well-met and which enjoy the $\aleph_2$-stationary cc also enjoys the $\aleph_2$-stationary cc. Shelah also proved more general results for certain iterations of κ^+-stationary cc posets with supports of size less than κ, and proved the consistency of a number of related forcing axioms.

The main result of this section is a common generalisation of the results of Baumgartner and Shelah quoted above. In order to state the theorem we require a definition.

Definition 1.1. Let $(\mathbb{P}, \leq)$ be a partial order.

- Two descending sequences $\langle q_i \,|\, i < \omega \rangle$ and $\langle r_i \,|\, i < \omega \rangle$ from $\mathbb{P}$ are *pointwise compatible* if for each $i < \omega$ we have $q_i \parallel r_i$.

- $(\mathbb{P}, \leq)$ is *countably parallel-closed* if each pair of pointwise compatible descending ω-sequences has a common lower bound.

Theorem 1.2. *Let κ be an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$. Every iteration of countably parallel-closed, $<\kappa$-closed, κ^+-stationary cc forcing with supports of size less than κ has the κ^+-stationary cc.*

Proof. Let $\mathbb{P} = \langle \langle \mathbb{P}_\xi \,|\, \xi \leq \chi \rangle, \langle \dot{\mathbb{Q}}_\xi \,|\, \xi < \chi \rangle \rangle$ be a $<\kappa$-support iteration of forcings such that for each $\xi < \chi$ it is forced by $\mathbb{P}_\xi$ that $\dot{\mathbb{Q}}_\xi$ is countably parallel-closed, $<\kappa$-closed and κ^+-stationary cc. We prove, by induction on ξ, and following the proof of Lemma (1.3) of [22] closely, that for each $\xi \leq \chi$, $\mathbb{P}_\xi$ has the stationary κ^+-chain condition.

Suppose that $\mathbb{P}_\varepsilon$ has the stationary κ^+-chain condition for all $\varepsilon < \xi$. Let $\langle p_i \,|\, i < \kappa^+ \rangle \in {}^{\kappa^+}\mathbb{P}_\xi$. We construct for each $i < \kappa^+$ a decreasing ω-sequence $\langle p_i^n \,|\, n < \omega \rangle$ such that $p_i^0 = p_i$ and $p_i^n \in \mathbb{P}_\xi$ for each $n < \omega$.

Induction step $n + 1$. Suppose we have already defined $\langle p_i^n \,|\, i < \kappa^+ \rangle$. For each $i < \kappa^+$ and $\varepsilon < \xi$ we have $\Vdash_{P_\varepsilon} \text{``} p_i^n(\varepsilon) \in \dot{\mathbb{Q}}_\varepsilon\text{''}$ and $p_i^n(\varepsilon) = \mathbb{1}_{\dot{q}_\varepsilon}$ if $\varepsilon \notin \operatorname{supp}(p_i^n)$. As $\Vdash_{P_\varepsilon} \text{``} \dot{\mathbb{Q}}_\varepsilon$ has the κ^+-stationary cc'' and $\langle p_i^n(\varepsilon) \,|\, i < \kappa^+ \rangle$ names a κ^+-sequence of elements of $\dot{\mathbb{Q}}_\varepsilon$, we may find $\mathbb{P}_\varepsilon$-names $\dot{C}_\varepsilon^n$ for a club subset of κ^+ and $\dot{g}_\varepsilon^n$ for a regressive function on $\dot{C}_\varepsilon^n$ witnessing the κ^+-stationary cc for the sequence named by $\langle p_i^n(\varepsilon) \,|\, i < \kappa^+ \rangle$.

As $\mathbb{P}_\varepsilon$ has the κ^+-chain condition, if $\Vdash_{\mathbb{P}_\varepsilon} \text{`` } \dot{D}$ is a club subset of $\kappa$ '' there is some club $D' \in V$ such that $\Vdash_{\mathbb{P}_\varepsilon} \text{`` } D' \subseteq \dot{D}$ ''. So we may as well assume that $\dot{C}_\varepsilon^n = \hat{C}_\varepsilon^n$ for some club $C_\varepsilon^n \in V$.

For each $i < \kappa^+$, dealing with each $\varepsilon \in \mathrm{supp}(p_i^n)$ inductively and using the $<\kappa$-closure of $\mathbb{P}_\xi$, find some $p_i^{n+1} \leq p_i^n$ such that

$$\forall \varepsilon \in \mathrm{supp}(p_i^n) \ p_i^{n+1} \restriction \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{`` } \dot{g}_\varepsilon^n(i) = \rho_\varepsilon^n(i) \text{ ''}$$

for some ordinal $\rho_\varepsilon^n(i) < i$. Let $\rho_\varepsilon^n(i) = 0$ for $\varepsilon \notin \mathrm{supp}(p_i^n)$.

Let $\{\varepsilon_\alpha \,|\, \alpha < \mu\}$, for some $\mu \leq \kappa^+$, enumerate $\bigcup\{\mathrm{supp}(p_i^n) \,|\, n < \omega \ \& \ i < \kappa^+\}$. For each $\varepsilon < \xi$ let $C_\varepsilon = \bigcap_{n<\omega} C_\varepsilon^n$ and let $C = \{i < \kappa^+ \,|\, \forall \alpha < i \ (i \in C_{\varepsilon_\alpha})\}$.

Claim 1.3. *Using the hypothesis that $\kappa^{<\kappa} = \kappa$, we can find a club $E \subseteq C$ and a regressive function k on $S_\kappa^{\kappa^+} \cap E$ such that if $i, i' \in S_\kappa^{\kappa^+} \cap E$, $k(i) = k(i')$ and $i < i'$ then*

(1) $\bigcup\{\mathrm{supp}(p_i^n) \,|\, n < \omega\} \cap \{\varepsilon_\gamma \,|\, \gamma < i\} =$

   $\bigcup\{\mathrm{supp}(p_{i'}^n) \,|\, n < \omega\} \cap \{\varepsilon_\gamma \,|\, \gamma < i'\}$

(2) $\bigcup\{\mathrm{supp}(p_i^n) \,|\, n < \omega\} \subseteq \{\varepsilon_\gamma \,|\, \gamma < i'\}$

(3)   *If $\gamma < i'$, $n < \omega$ and $\varepsilon_\gamma \in \mathrm{supp}(p_{i'}^n)$ then $\rho_{\varepsilon_\gamma}^n(i) = \rho_{\varepsilon_\gamma}^n(i')$.*

(Note that, by (1), for every γ to which clause (3) applies we have $\gamma < i$ and $\varepsilon_\gamma \in \bigcup\{\mathrm{supp}(p_i^n) \,|\, n < \omega\}$.)

Proof. We start by making some auxiliary definitions.

Let $H = \{h \,|\, \mathrm{dom}(h) \in [\kappa^+]^{<\kappa} \text{ and } h : \mathrm{dom}(h) \longrightarrow {}^\omega\kappa^+\}$. By the hypothesis that $\kappa^{<\kappa} = \kappa$ we have that $\overline{\overline{H}} = \kappa^+$. So we can enumerate H as, say, $\langle h_\eta \,|\, \eta < \kappa^+ \rangle$.

Noticing that $H \subset [\kappa^+ \times {}^\omega\kappa^+]^{<\kappa}$, for $i < \kappa^+$, define $H_i = H \cap [i \times {}^\omega i]^{<\kappa}$. Since $\kappa^{<\kappa} = \kappa$ we have, for each $i < \kappa$, that $\overline{\overline{H_i}} = \kappa$.

So we can define $f : \kappa^+ \longrightarrow \kappa^+$ by, for $i < \kappa^+$, letting $f(i) =$ the least $\tau < \kappa^+$ such that $H_i \subseteq \{h_\eta \,|\, \eta < \tau\}$.

Also, for each $i < \kappa^+$, set $F_i = \{\gamma < \kappa^+ \,|\, \varepsilon_\gamma \in \bigcup\{\mathrm{supp}(p_i^n) \,|\, n < \omega\}\}$ and $D_i = F_i \cap i$.

Let $E = \{i < \kappa^+ \,|\, f``i \cup \bigcup\{F_j \,|\, j < i\} \subseteq i\}$. It follows that E is a closed unbounded subset of κ^+.

In order to see this let us define for $n < \omega$ the functions $g_n : \kappa^+ \longrightarrow \kappa^+$ by $g_0(i)$ is the least ordinal $\bar{i}$ such that $f``i \cup \bigcup\{F_j \,|\, j < i\} \subseteq \bar{i}$, and $g_{n+1}(i) = g_0(g_n(i))$ for $n < \omega$. Then for any $i < \kappa^+$ we have that $i < \bigcup_{n<\omega} g_n(i) \in E$. This shows that E is unbounded. Moreover, if $B \subseteq E$ is of limit order type less than κ^+ and $d \in \sup(B) \cup \bigcup\{F_j \,|\, j < \sup(B)\}$

then there is some $b \in B$ such that $d \in b \cup \bigcup \{F_j \mid j < b\}$ and hence $f(d) < b < \sup(B)$.

Now define, for each $i < \kappa^+$, the function $k_i : D_i \longrightarrow {}^\omega i$ by letting $k_i(\gamma)(n) = \rho^n_{\varepsilon_\gamma}(i)$ if $\varepsilon_\gamma \in \operatorname{supp}(p^n_i)$, and $= 0$ otherwise.

So for each $i < \kappa^+$ we have $k_i \in H_i$.

Define $k : \kappa^+ \longrightarrow \kappa^+$ by setting $k(i) =$ that η such that $k_i = h_\eta$.

For each $i < \kappa^+$ we have $k_i = h_{k(i)}$ and thus $k(i) < f(i)$.

Now if $i \in E$ and $\operatorname{cf}(i) = \kappa$, simply because k_i is a function of size $<\kappa$, there is some $i^* < i$ such that $k_i \in H_{i^*}$. Hence for such i we have $k(i) < f(i^*)$. We also have $f(i^*) < i$ since $i \in E$.

Hence if $i \in E$ and $\operatorname{cf}(i) = \kappa$ we have $k(i) < i$ and so we have shown that k is regressive on $E \cap S^{\kappa^+}_\kappa$.

Moreover, if $i, i' \in E \cap S^{\kappa^+}_\kappa$ and $k(i) = k(i')$ we have that $k_i = k_{i'}$, and hence that (1) and (3) hold. Finally, if $i < i'$ as well we have, since $i' \in E$, that $F_i \subseteq i'$ and hence (2) holds. ▲$_{1.3}$

With Claim (1.3) in hand, suppose that $i < i' < \kappa^+$, $i, i' \in S^{\kappa^+}_\kappa \cap E$ and $k(i) = k(i')$. We now construct, by an induction of length ξ, a condition $q \in \mathbb{P}_\xi$ which is a common refinement of p^n_i and $p^n_{i'}$ for all n, and hence of p_i and $p_{i'}$. The support of q will be the union of $\bigcup \{\operatorname{supp}(p^n_i) \mid n < \omega\}$ and $\bigcup \{\operatorname{supp}(p^n_{i'}) \mid n < \omega\}$.

For $\sigma \in \bigcup \{\operatorname{supp}(p^n_i) \mid n < \omega\} \setminus \bigcup \{\operatorname{supp}(p^n_{i'}) \mid n < \omega\}$, as $\Vdash_{\mathbb{P}_\sigma}$ " $\dot{\mathbb{Q}}_\sigma$ is countably closed ", take $q(\sigma)$ to be any r such that $q \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ " $\forall n < \omega \ (r \le p^n_i(\sigma))$ ". Similarly, for $\sigma \in \bigcup \{\operatorname{supp}(p^n_{i'}) \mid n < \omega\} \setminus \bigcup \{\operatorname{supp}(p^n_i) \mid n < \omega\}$, take $q(\sigma)$ to be any r such that $q \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ " $\forall n < \omega \ (r \le p^n_{i'}(\sigma))$ ".

Finally, if $\sigma \in \bigcup \{\operatorname{supp}(p^n_i) \mid n < \omega\} \cap \bigcup \{\operatorname{supp}(p^n_{i'}) \mid n < \omega\}$ then $\sigma = \varepsilon_\gamma$ for some $\gamma < i$ (by conditions (1) and (2) above), for each $n < \omega$ we have $\rho^n_{\varepsilon_\gamma}(i) = \rho^n_{\varepsilon_\gamma}(i')$ (by (3) above), and $i, i' \in C_{\varepsilon_\gamma}$ (by the construction of C and the choice of E).

By construction, for each $n < \omega$ we have $q \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ " $p^n_i(\sigma) \parallel_{\dot{\mathbb{Q}}_\sigma} p^n_{i'}(\sigma)$ ". By the fact that $\Vdash_{\mathbb{P}_\sigma}$ " $\dot{\mathbb{Q}}_\sigma$ is countably parallel-closed ", we can choose $q(\sigma)$ to be some r such that $q \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ " $\forall n < \omega \ (r \le p^n_i(\sigma), \ p^n_{i'}(\sigma))$ ". Thus for all $n < \omega$ we have that $q \restriction (\sigma + 1) \le p^n_i \restriction (\sigma + 1), p^n_{i'} \restriction (\sigma + 1)$. ▲$_{1.2}$

We remark that the previous theorem allows us to give 'generalised Martin's axiom' forcing axioms similar to those formulated by Baumgartner [1] and Shelah [22]. One example is given by the following theorem.

Theorem 1.4. *Let κ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$, and let $\lambda > \kappa^+$ be a cardinal such that $\gamma^{<\kappa} < \lambda$ for every $\gamma < \lambda$. Then there is a $<\kappa$-closed and κ^+-stationary cc forcing poset $\mathbb{P}$ such that if G is $\mathbb{P}$-generic, then in $V[G]$ we have $2^\kappa = \lambda$ and the following forcing axiom holds:*

For every poset $\mathbb{Q}$ which is $<\kappa$-closed, countably parallel-closed and κ^+-stationary cc, every $\gamma < \lambda$ and every sequence $\langle D_i \,|\, i < \gamma \rangle$ of dense subsets of $\mathbb{Q}$ there is a filter on $\mathbb{Q}$ which meets each set D_i.

As in [1] and [22], there are variations with weaker hypotheses on λ, yielding weaker forcing axioms which only apply to posets $\mathbb{Q}$ of bounded size.

The following straightforward lemma indicates some connections between the hypotheses used in [1] and [22], and the ones used here.

Lemma 1.5. *Let $\mathbb{P}$ be a forcing poset.*

(1) *If $\mathbb{P}$ is well-met and countably closed then it is countably compact.*

(2) *If $\mathbb{P}$ is countably compact then it is countably parallel-closed.*

(3) *If $\mathbb{P}$ is κ-linked then it is κ^+-stationary cc.*

Proof. For (1), note that by an easy induction any finite subset of a well-met poset with a common lower bound has a greatest lower bound. Now suppose that $C \subseteq \mathbb{P}$ is a countable centred set and enumerate it as $\langle p_n \,|\, n < \omega \rangle$, then let q_m be the greatest lower bound for $\langle p_n \,|\, n \le m \rangle$; the conditions q_m form a decreasing sequence which has a lower bound by countable closure.

For (2), suppose $(\mathbb{P}, \le)$ is countably compact and that $Q = \langle q_i \,|\, i < \omega \rangle$ and $R = \langle r_j \,|\, j < \omega \rangle$ are two pointwise compatible descending sequences from $\mathbb{P}$. If $i_0 < i_1 < \cdots < i_m < \omega$ and $j_0 < j_1 < \cdots < j_n < \omega$ let $k = \max(\{i_m, j_n\})$ and note that since $q_k \parallel r_k$ there is some $p \in \mathbb{P}$ such that for all $l \le m$ and all $e \le n$ we have $p \le q_{i_l}, r_{j_e}$. That is, any finite subset of $Q \cup R$ has a lower bound. Hence, by the countable compactness of $(\mathbb{P}, \le)$, we have that $Q \cup R$ has a lower bound. Thus we have shown that $(\mathbb{P}, \le)$ is countably parallel-closed.

For (3), suppose $\mathbb{P}$ is κ-linked and that $\langle A_\gamma \,|\, \gamma < \kappa \rangle$ is a partition of $\mathbb{P}$ such that each A_γ is a pairwise compatible subset. Let $\langle p_i \,|\, i < \kappa^+ \rangle \subseteq \mathbb{P}$. Define $f : S_\kappa^{\kappa^+} \longrightarrow \kappa$ by $f(i) = \gamma$ if and only if $p_i \in A_\gamma$ for $i \in S_\kappa^{\kappa^+}$. Then f is a regressive function and if $f(i) = f(j)$ we have that p_i and p_j are compatible.   ▲$_{1.5}$

One cannot reverse Lemma (1.5.(3)). For example, it is a folklore result that $\mathrm{Add}(\kappa, \lambda)$ is κ^+-stationary cc if $\kappa^{<\kappa} = \kappa$ but is not κ-linked if $2^\kappa < \lambda$. We give proofs of these facts for completeness. The reader may also find the proof of the former useful as a dry run for the considerably more elaborate proof in §5 that the forcing $Q(w)$ of §4 is κ^+-stationary cc; the notation used here closely mirrors that used there.

Example 1.6. Let κ and λ be cardinals, with κ regular, $\kappa^{<\kappa} = \kappa$ and $2^\kappa < \lambda$. Then $\mathrm{Add}(\kappa, \lambda)$ is not κ-linked, however it is κ^+-stationary cc.

Proof. In order to see that $\mathrm{Add}(\kappa, \lambda)$ is not κ-linked let us suppose that $\langle B_i \,|\, i < \kappa \rangle$ is a collection of κ many pairwise compatible subsets of $\mathrm{Fn}(\lambda, 2, \kappa)$. For each $i < \kappa$ and $\alpha < \lambda$ we have that if p, $q \in B_i$ and $\alpha \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ then $p(\alpha) = q(\alpha)$. Hence for each $i < \kappa$ there is a function $f_i \in {}^\lambda 2$ such that for each $p \in B_i$ we have $p \subseteq f_i$.

For each $\alpha < \lambda$ define $d_\alpha : \kappa \longrightarrow 2$ by $d_\alpha(i) = f_i(\alpha)$. Since $2^\kappa < \lambda$ there are α, $\beta < \lambda$ such that $d_\alpha = d_\beta$. Now consider $p \in \mathrm{Fn}(\lambda, 2, \kappa)$ such that α, $\beta \in \mathrm{dom}(p)$ and $p(\alpha) \neq p(\beta)$. Then for each $i < \kappa$ we cannot have that both $p(\alpha) = f_i(\alpha)$ and $p(\beta) = f_i(\beta)$. Hence $\bigcup\{ B_i \,|\, i < \kappa \} \neq \mathrm{Fn}(\lambda, 2, \kappa)$.

We now verify that $\mathrm{Add}(\kappa, \lambda)$ is κ^+-stationary cc.

Suppose $\langle p^i \,|\, i < \kappa^+ \rangle$ is a collection of conditions in $\mathrm{Add}(\kappa, \lambda)$. For each $i < \kappa^+$ let $t^i = \mathrm{dom}(p^i)$. Let $\{ \alpha_\gamma \,|\, \gamma < \gamma^* \}$ be an enumeration of $\bigcup\{ t^i \,|\, i < \kappa^+ \}$, for some $\gamma^* \leq \kappa^+$.

Next, for each $i < \kappa^+$ let $\{ \alpha_\gamma^i \,|\, \gamma < \gamma^i \}$, for some $\gamma^i < \kappa$, be the increasing enumeration of t^i, let $\theta^i = \mathrm{ssup}(\{ \gamma \,|\, \alpha_\gamma \in t^i \})$, let $T^i = \{ \gamma < i \,|\, \alpha_\gamma \in t^i \}$, and let $q^i \in \mathrm{Fn}(T^i, 2, \kappa)$ be defined by $q^i(\gamma) = p^i(\alpha_\gamma)$.

So each $\gamma < \kappa^+$, each $\alpha_\gamma^i < \lambda$, each $\theta^i < \kappa^+$, each $T^i \in [i]^{<\kappa}$ and each $q^i \in \mathrm{Fn}(T^i, 2, \kappa) \subseteq \mathrm{Fn}(i, 2, \kappa)$.

For $i \in [\kappa, \kappa^+]$ let $H_i = [i]^{<\kappa} \times \mathrm{Fn}(i, 2, \kappa) \times \kappa$, and write H for H_{κ^+}. Let h^* be an injection from H into κ^+. Define $k : [\kappa, \kappa^+) \longrightarrow \kappa^+$ by setting $k(i)$ to be the least $i^* < \kappa^+$ such that $H_i \subseteq h^{*-1} "i^*$.

Let $\tilde{C} = \{j < \kappa^+ \,|\, \forall i < j \,(\theta^i, k(i) < j)\}$. As the intersection of the sets of closure points of the two given functions, $\tilde{C}$ is a club subset of κ^+.

Let $h(i) = h^*(T^i, q^i, \mathrm{otp}(t^i))$ for $i \in \tilde{C} \cap S_\kappa^{\kappa^+}$, and $h(i) = 0$ otherwise.

We have that $h^{*-1}(h(i)) \in H_i$ for all $i \in [\kappa, \kappa^+)$. If $i \in \widetilde{C} \cap S_\kappa^{\kappa^+}$, since $\overline{h^{*-1}(h(i))} < \kappa$, there is some $i' < i$ such that $h^{*-1}(h(i)) \in H_{i'}$, and hence there is some $\widetilde{i} < i$ such that $h(i) < k(\widetilde{i})$.

Hence, as i is a closure point of k, we have $h(i) < i$ for all nonzero $i < \kappa^+$.

Now suppose that $i, j \in \widetilde{C} \cap S_\kappa^{\kappa^+}$, $i < j$ and $h(i) = h(j)$. So we have $T^i = T^j$, $q^i = q^j$ and $\mathrm{otp}(t^i) = \mathrm{otp}(t^j)$.

Set $t = t^i \cap t^j$.

Lemma 1.7.   (a) $(t^j \setminus t^i) \cap \{\alpha_\gamma \,|\, i \leq \gamma < j\} = \emptyset$, and

   (b) $t \subseteq \{\alpha_\gamma \,|\, \gamma < i\}$.

Proof. Suppose $\alpha_\gamma \in t^j$. If $\gamma < j$ then $\gamma \in T^j$. But $T^j = T^i$, so $\gamma \in T^i$. Hence $\gamma < i$ and $\alpha_\gamma \in t^i$, proving (a). If $\alpha_\gamma \in t^i$ then $\gamma < \theta^i < j$. (For the definition of θ^i immediately gives that $\gamma < \theta^i$; and since $i, j \in \widetilde{C}$ and $i < j$ one has that $\theta^i < j$.) Thus if $\alpha_\gamma \in t^i \cap t^j$ we have $\gamma < i$ by (a). So (b) holds.   ▲$_{1.7}$

By Lemma (1.7(b)) we have for $\alpha_\gamma \in t$ that $p^i(\alpha_\gamma) = q^i(\gamma) = q^j(\gamma) = p^j(\alpha_\gamma)$, and hence p^i and p^j agree on the intersection of their domains, and thus are compatible conditions.   ▲$_{1.6}$

2. Preserving diamond under forcing

In this section we give an account of how versions of the diamond principle at a regular cardinal χ are preserved by certain forcing posets. We are most interested in the situation where the forcing poset is an iteration $\mathbb{P}_\chi$ of length χ, and we can find a diamond sequence $\langle S_\alpha \,|\, \alpha < \chi \rangle \in V[G_\chi]$ such that $S_\alpha \in V[G_\alpha]$ for all α. We will use the results of this section in §6 and §7.

The following result is well-known. For the reader's convenience we will sketch a proof. Similar arguments appear, for example, in the proof that ♣ does not imply $\diamondsuit_{\omega_1}$ in [24], §I.7 and in a preservation theorem for $\diamondsuit_{\omega_1}^+$ in [4], §12.

Theorem 2.1. *Let χ be a regular uncountable cardinal, let $\mathbb{P}$ be a forcing poset of cardinality χ which preserves the regularity of χ and let G be $\mathbb{P}$-generic over V.*

(1) If $\diamondsuit_\chi$ holds in V and $\mathbb{P}$ preserves stationary subsets of χ then $\diamondsuit_\chi$ holds in $V[G]$.

(2) If $\diamondsuit_\chi^*$ (resp $\diamondsuit_\chi^+$) holds in V then $\diamondsuit_\chi^*$ (resp $\diamondsuit_\chi^+$) holds in $V[G]$.

Proof. (1). Enumerate the elements of $\mathbb{P}$ as p_i for $i < \chi$, fix in V a diamond sequence $\langle T_\alpha \subseteq \alpha \times \alpha \,|\, \alpha < \chi \rangle$ which guesses subsets of $\chi \times \chi$, and in $V[G]$ define $S_\alpha = \{\eta < \alpha \,|\, \exists i < \alpha \; (p_i \in G \text{ and } (i, \eta) \in T_\alpha)\}$. Now if $S = \dot{S}^G \subseteq \chi$ we let $T = \{(i, \eta) \,|\, p_i \Vdash \hat{\eta} \in \dot{S}\}$, and observe that $\{\alpha \,|\, T \cap \alpha \times \alpha = T_\alpha\}$ is stationary in V (hence also in $V[G]$) and $E = \{\alpha \,|\, \forall \eta \in S \cap \alpha \; \exists i < \alpha \; p_i \in G \text{ and } p_i \Vdash \hat{\eta} \in \dot{S}\}$ is club in $V[G]$; at any point $\alpha \in E$ such that $T \cap \alpha \times \alpha = T_\alpha$ we have that $S \cap \alpha = S_\alpha$. Hence $\langle S_\alpha \,|\, \alpha < \chi \rangle$ is a $\diamondsuit_\chi$-sequence in $V[G]$.

(2). The proofs for $\diamondsuit_\chi^*$ and $\diamondsuit_\chi^+$ are very similar. The key difference is that the hypothesis that the regularity of χ is preserved suffices to see that E is club in $V[G]$, and the stronger form of guessing occurring in the $\diamondsuit_\chi^*$ and $\diamondsuit_\chi^+$ gives a club from V with which to intersect it. $\blacktriangle_{2.1}$

Observation 2.2. For future reference we note that:

(1) An entirely similar argument shows that if B is a stationary subset of χ and $\mathbb{P}$ preserves stationary subsets of B, then $\mathbb{P}$ preserves $\diamondsuit_\chi(B)$. If $\mathbb{P}$ preserves the stationarity of B then $\mathbb{P}$ preserves $\diamondsuit_\chi^*(B)$ and $\diamondsuit_\chi^+(B)$.

(2) In both parts of Theorem (2.1), the value of the diamond sequence in $V[G]$ at α is computed in a uniform way from $G \upharpoonright \alpha$ where $G \upharpoonright \alpha = G \cap \{p_i \,|\, i < \alpha\}$. In fact the restriction of the diamond sequence to $\alpha + 1$ (*i.e.*, $\langle S_i \,|\, i \leq \alpha \rangle$) can be computed in a uniform way from $G \upharpoonright \alpha$.

(3) In the proof of part (1) of Theorem (2.1), let A be the stationary set in $V[G]$ given by $A = \{\alpha \,|\, S \cap \alpha = S_\alpha\}$. For every $\alpha \in E$, $S \cap \alpha \in V[G \upharpoonright \alpha]$ and so easily $E \cap \alpha \in V[G \upharpoonright \alpha]$. Also, by the preceding remarks, $A \cap \alpha \in V[G \upharpoonright \alpha]$. If we now let $H = A \cap E$ then H is a stationary set on which S is guessed correctly and additionally $H \cap \alpha \in V[G \upharpoonright \alpha]$ for all α in the closure of H.

Proposition 2.3. *Let κ, χ be regular cardinals with $\kappa < \chi$. Let $\mathbb{P} = \langle \langle \mathbb{P}_\xi \,|\, \xi \leq \chi \rangle, \langle \dot{\mathbb{Q}}_\xi \,|\, \xi < \chi \rangle \rangle$ be a forcing iteration with $<\kappa$-supports with $\overline{\overline{\mathbb{P}_\xi}} < \chi$ for each $\xi < \chi$. Let G be $\mathbb{P}$-generic over V.*

Suppose A is a stationary subset of $S^{\chi}_{\geq \kappa}$ and $\Diamond_{\chi}(A)$ holds. Then there is a sequence $\langle \dot{S}_{\xi} \,|\, \xi \in A \rangle$ such that

(1) *For all $\xi \in A$, $\dot{S}_{\xi}$ is a $\mathbb{P}_{\xi}$-name for a subset of ξ.*

(2) *If in $V[G]$ we define S_{ξ} as the interpretation of $\dot{S}_{\xi}$ by G_{ξ} then the sequence $\langle S_{\xi} \,|\, \xi \in A \rangle$ has the following strengthened form of the $\Diamond_{\chi}(A)$-property: for every $S \subseteq \chi$ there is a stationary set $H \subseteq A$ such that $S \cap \xi = S_{\xi}$ for all $\xi \in H$, and in addition $H \cap \xi \in V[G_{\xi}]$ for all ξ in the closure of H.*

Proof. We start by observing that by arguments as in Baumgartner's survey paper on iterated forcing [1, §2] the poset $\mathbb{P}$ is χ-cc. We sketch the proof briefly: given a sequence $\langle p_i \,|\, i < \chi \rangle$ of elements of $\mathbb{P}$ we apply Fodor's theorem and the bound on the size of initial segments to find a stationary set $U \subseteq S^{\chi}_{\kappa}$ and an ordinal $\eta < \chi$ such that $\mathrm{supp}(p_i) \cap i \subseteq \eta$ and $p_i \restriction \eta$ is constant for $i \in U$, then find $i, j \in U$ such that $\mathrm{supp}(p_i) \subseteq j$ and argue that p_i is compatible with p_j.

Noting that $|\mathbb{P}| = \chi$, we enumerate $\mathbb{P}$ as $\langle p_{\xi} \,|\, \xi < \chi \rangle$ and identify each $\mathbb{P}_{\xi}$ with the set of conditions $p \in \mathbb{P}$ such that $\mathrm{supp}(p) \subseteq \xi$ for $\xi < \chi$.

Let F be $\{ \xi < \chi \,|\, \forall \varepsilon < \xi \,(\mathrm{supp}(p_{\varepsilon}) < \xi \ \& \ \mathbb{P}_{\xi} = \{ p_{\varepsilon} \,|\, \varepsilon < \xi \}) \}$. Since $\mathbb{P}$ is an iteration with $<\kappa$-support, F is club relative to $S^{\chi}_{\geq \kappa}$. For let us define $h : \chi \longrightarrow \chi$ by setting $h(\varepsilon) =$ the least ξ such that $\mathrm{supp}(p_{\varepsilon}) \subseteq \xi$ and $\mathbb{P}_{\varepsilon} \subseteq \{ p_{\zeta} \,|\, \zeta < \xi \}$, and define C to be the set of closure points of h.

For each $\xi \in C \cap S^{\chi}_{\geq \kappa}$ and $\varepsilon < \xi$ we have that $\mathrm{supp}(p_{\varepsilon}) \subseteq h(\varepsilon) < \xi$ and, since $\mathrm{cf}(\xi) \geq \kappa$ and the size of the support of each condition is less than κ, there is some $\gamma \in C \cap \xi$ such that $h(\varepsilon) < \gamma$ and so $\mathbb{P}_{\varepsilon} \subseteq \{ p_{\zeta} \,|\, \zeta < \gamma \}$. Hence $\xi \in F$. Thus $A \setminus F$ is non-stationary.

Since $\mathbb{P}$ is χ-cc it preserves stationary subsets of χ, and we may therefore appeal to Theorem (2.1) and Observation (2.2) to obtain a sequence $\langle \dot{S}_{\xi} \,|\, \xi \in A \cap F \rangle$ such that

(1) For all $\xi \in A \cap F$, $\dot{S}_{\xi}$ is a $\mathbb{P}_{\xi}$-name for a subset of ξ.

(2) If in $V[G]$ we define S_{ξ} as the realisation of $\dot{S}_{\xi}$ by G_{ξ}, then for every $S \subseteq \kappa$ there is a stationary set $H \subseteq A \cap F$ such that $S \cap \xi = S_{\xi}$ for all $\xi \in H$ and additionally $H \cap \xi \in V[G_{\xi}]$ for all ξ in the closure of H.

To finish the proof we fill in the missing values by defining $\dot{S}_\xi = \hat{\emptyset}$ for $\xi \in A \setminus F$. $\blacktriangle_{2.3}$

3. Radin forcing

As we commented in the preamble, our proof involves Radin forcing. There are several accounts of this forcing in the literature (see [20], [18], [5], [11] and [13]), each subtly different from the others, and it turns out that it does matter which version of the forcing we use. Although the proofs of the various properties of the forcing are easiest, or at least slickest, for the versions given in Cummings-Woodin ([5]) and in Gitik ([11]), using either of these here creates technical difficulties. Consequently, we shall define and use a version of Radin forcing which, except for one small alteration to which we draw attention below, closely follows that of Mitchell in [18].

First of all, we give, by induction on $\kappa \in \mathrm{Card}$, the definition of the set of *ultrafilter sequences at* κ.

Definition 3.1. Let κ be a cardinal and let $\mathcal{U}_\kappa$ be the collection of ultrafilter sequences at cardinals smaller than κ. A sequence $u = \langle u_\tau \mid \tau < \mathrm{lh}(u) \rangle$ is an *ultrafilter sequence at* κ if $\mathrm{lh}(u)$ is a non-zero ordinal, $u_0 = \kappa$, so that, using the notation of the introduction, $\kappa_u = \kappa$, and, for $\tau \in (0, \mathrm{lh}(u))$, each u_τ is a κ-complete ultrafilter on V_κ with $\mathcal{U}_\kappa \in u_\tau$ and satisfies the following *normality* and *coherence* conditions (with respect to u):

(normality) if $f : \mathcal{U}_\kappa \longrightarrow V_\kappa$ and $\{ w \in \mathcal{U}_\kappa \mid f(w) \in V_{\kappa_w} \} \in u_\tau$ then there is some $x \in V_\kappa$ such that $\{ w \in \mathcal{U}_\kappa \mid f(w) = x \} \in u_\tau$.

(coherence 1) if $f : \mathcal{U}_\kappa \longrightarrow \kappa$ and $\{ w \in \mathcal{U}_\kappa \mid f(w) < \mathrm{lh}(w) \} \in u_\tau$ there is $\sigma < \tau$ such that $u_\sigma = \{ X \subseteq V_\kappa \mid \{ w \in \mathcal{U}_\kappa \mid X \cap V_{\kappa_w} \in w_{f(w)} \} \in u_\tau \}$.

(coherence 2) if $\sigma < \tau$ and $X \in u_\sigma$ then $\{ w \in \mathcal{U}_\kappa \mid \exists \bar{\sigma} < \mathrm{lh}(w)\ X \cap V_{\kappa_w} \in w_{\bar{\sigma}} \} \in u_\tau$.

Definition 3.2. The class $\mathcal{U}$ is the class of all ultrafilter sequences on any cardinal: $\mathcal{U} = \bigcup_{\kappa \in \mathrm{Card}} \mathcal{U}_\kappa$.

Definition 3.3. For clarity we re-iterate the following special cases of notation of Definition (3.1): $\mathcal{U}_{\kappa^+}$ is the set of ultrafilter sequences at cardinals less than or equal to κ and $\mathcal{U}_{\kappa^+} \setminus \mathcal{U}_\kappa$ is the set of ultrafilter sequences at κ.

Observation 3.4. As Mitchell comments in [18], by the coherence properties, if $u \in \mathcal{U}$ and $\mathrm{lh}(u) \leq \kappa_u$ then for $\tau \in (0, \mathrm{lh}(u))$ one has that u_τ concentrates on $\{w \in \mathcal{U}_{\kappa_u} \mid \mathrm{lh}(w) = \tau\}$ (i.e., $\{w \in \mathcal{U}_{\kappa_u} \mid \mathrm{lh}(w) = \tau\} \in u_\tau$).

Observation 3.5. We need some large cardinal assumption in order to construct non-trivial ultrafilter sequences. For the purposes of this paper we can use a construction due to Radin [20]. Let $j : V \to M$ witness that κ is 2^κ-supercompact. Derive a sequence u by setting $u_0 = \kappa$ and then $u_\alpha = \{X \subseteq V_\kappa \mid u \restriction \alpha \in j(X)\}$ for $\alpha > 0$. One can verify that for every $\alpha < (2^\kappa)^+$ we have $u \restriction \alpha \in \mathcal{U}_\kappa$. In fact we will only need ultrafilter sequences of length less than κ in the sequel.

Definition 3.6. If $u \in \mathcal{U}$ then $\mathcal{F}(u) = \bigcap_{\tau \in (0, \mathrm{lh}(u))} u_\tau$.

Definition 3.7. A *pair* is some (u, A) with $u \in \mathcal{U}$ and $A \in \mathcal{F}(u)$ if $\mathrm{lh}(u) > 1$ and $A = \emptyset$ if $\mathrm{lh}(u) = 1$.

(Note that our notation thus far is marginally, but inessentially, different from that of [18]).

Definition 3.8. Let $w \in \mathcal{U}$. $\mathbb{R}_w$, *Radin forcing at w*, has as conditions sequences of pairs $\langle (u_0, B_0), \ldots, (u_n, B_n) \rangle$ such that $u_n = w$, and, writing κ_i for κ_{u_i}, $(u_i, B_i) \in \mathcal{U}_{\kappa_{i+1}}$ and $B_{i+1} \cap V_{\kappa_i^+} = \emptyset$ for $i < n$.

Let $p = \langle (u_0, B_0), \ldots, (u_n, B_n) \rangle$ and $q = \langle (v_0, D_0), \ldots, (v_m, D_m) \rangle \in \mathbb{R}_w$. Then $q \leq p$ (q refines p) if $m \geq n$ and

  (i) For every $i \leq n$ there is $j \leq m$ such that $u_i = v_j$ and $D_j \subseteq B_i$.

  (ii) For every $j \leq m$, either $v_j = u_i$ for some i or for the least i such that $\kappa_{v_j} < \kappa_{u_i}$, $v_j \in B_i$ and $D_j \subseteq B_i$.

We also define $q \leq^* p$ if $\langle u_i \mid i \leq n \rangle = \langle v_i \mid i \leq n \rangle$ and for each $i \leq n$ we have $D_i \subseteq B_i$. We say that q is a *(Radin-)direct extension* of p. Thus $q \leq^* p$ implies $q \leq p$.

Definition 3.9. Let $p = \langle (u_0, B_0), \ldots, (u_n, B_n) \rangle$ be a condition in $\mathbb{R}_{u_n}$. A pair (u, B) *appears in* p if there is some $i \leq n$ such that $(u, B) = (u_i, B_i)$. Similarly an ultrafilter sequence u *appears in* p if there is some $i \leq n$ such that $u = u_i$.

Thus clause (ii) in the definition of $\leq$ in Definition (3.8) reads: if v_j does not appear in p and i is minimal such that $\kappa_{v_j} < \kappa_{u_i}$ then $v_j \in B_i$ and $D_j \subseteq B_i$.

Observation 3.10. [18] omits the last clause in the definition of what it is to be a condition. However our conditions form a dense subset of the conditions as defined in [18] and the facts that we quote from [18] are also true of our forcing. This minor change is advantageous for technical reasons in order to make the proof below run smoothly.

To help orient the reader, we record a few remarks (without proof) about the nature of the generic object for the forcing poset $\mathbb{R}_w$. This generic object is best viewed as a sequence $\langle u_i \,|\, i < \delta \rangle$ where $u_i \in \mathcal{U}_{\kappa_w}$ and $\langle \kappa_{u_i} \,|\, i < \delta \rangle$ is increasing and continuous. When $\mathrm{lh}(w) > 1$ the sequence $\langle \kappa_{u_i} \,|\, i < \delta \rangle$ is cofinal in κ_w, and we will view this sequence as enumerating a club set in κ_w which we call the *Radin-generic club set*. When $i = 0$ or i is a successor ordinal then $\mathrm{lh}(u_i) = 1$, otherwise $\mathrm{lh}(u_i) > 1$. The translation between the generic sequence and the generic filter is given by the following.

Definition 3.11. Let G be an $\mathbb{R}_w$-generic filter over V. The sequence $\langle u_i \,|\, i < \delta \rangle$ is the *corresponding generic sequence* if it enumerates $\{ u \in \mathcal{U}_{\kappa_w} \,|\, \exists p \in G \ u \text{ appears in } p \}$, i.e., the set of $u \in \mathcal{U}_{\kappa_w}$ which appear in some condition in G.

Lemma 3.12. *Let G be an $\mathbb{R}_w$-generic filter over V and $\langle u_i \,|\, i < \delta \rangle$ the corresponding generic sequence. Then*

$$G = \{ p \in \mathbb{R}_w \,|\, \forall u \in \mathcal{U}_{\kappa_w} \, (u \text{ appears in } p \longrightarrow \exists i < \delta \ u = u_i) \, \& $$
$$\forall i < \delta \, \exists q \leq p \, (u_i \text{ appears in } q) \},$$

i.e. G is the set of conditions $p \in \mathbb{R}_w$ such that every $u \in \mathcal{U}_{\kappa_w}$ which appears in p is among the sequences u_i, and every sequence u_i appears in some extension of p.

Definition 3.13. A *lower part* for $\mathbb{R}_w$ is a condition in $\mathbb{R}_u$ for some $u \in \mathcal{U}_{\kappa_w}$.

We note that any condition in $\mathbb{R}_w$ has the form $y^\frown(w, B)$ where y is empty or a lower part for $\mathbb{R}_w$, and $B \in \mathcal{F}(w)$. In the case when y is non-empty we will say that y is *the lower part of* $y^\frown(w, B)$. It is easy to see that any two conditions with the same lower part are compatible, so that $\mathbb{R}_w$ is the union of κ_w-many κ_w-complete filters and in particular it enjoys the κ_w^+-chain condition.

A key point is that below a condition of the form $y^\frown(w, B)$ with y non-empty, everything up to the last ultrafilter sequence appearing in y is controlled by pairs in y. More formally:

Definition 3.14. Let $y = \langle (u_i, B_i) \,|\, i \leq n \rangle$ be a lower part for $\mathbb{R}_w$. Then $\kappa_y = \kappa_{u_n}$ and $\mathbb{R}_y = \{ q \in \mathbb{R}_{u_n} \,|\, q \leq y \}$.

Lemma 3.15. *Let $p = y^\frown(w, B)$ be a condition in $\mathbb{R}_w$ with y non-empty. Then the subforcing $\{ q \in \mathbb{R}_w \,|\, q \leq p \}$ is isomorphic to the product $\mathbb{R}_y \times \{ q \in \mathbb{R}_w \,|\, q \leq (w, B \setminus \mathcal{U}_{\kappa_y^+}) \}$.*

It follows from these considerations that if $\langle u_i \,|\, i < \delta \rangle$ is a generic sequence for $\mathbb{R}_w$ and $\zeta < \delta$, then $\langle u_i \,|\, i < \zeta \rangle$ is a generic sequence for $\mathbb{R}_{u_\zeta}$. More generally, if there is a condition in the generic filter with lower part y then the generic sequence induces an $\mathbb{R}_y$-generic object in the natural way.

Definition 3.16. Let y be a lower part for $\mathbb{R}_w$ and let G be $\mathbb{R}_w$-generic over V. We say that y *conforms with* G if and only if y is the lower part of some condition in G.

The following lemma is a "local" version of Lemma (3.12).

Lemma 3.17. *Let G be an $\mathbb{R}_w$-generic filter over V and let $\langle u_i \,|\, i < \delta \rangle$ be the corresponding generic sequence. Let $y = \langle (v_k, B_k) \,|\, k \leq n \rangle$ be a lower part. Then the following are equivalent:*

(1) y conforms with G.

(2) For every $k \leq n$ the sequence v_k appears among the sequences u_i, and for every i with $\kappa_{u_i} \leq \kappa_{v_n}$ the ultrafilter sequence u_i appears in some extension of y in $\mathbb{R}_y$.

Moreover, if $i < \delta$ then u_i appears in some lower part y' which conforms with G and with $\kappa_{y'} = \max(\{ \kappa_y, \kappa_{u_i} \})$, and such that if $y^\frown(w, B) \in G$ there is some $B' \in \mathcal{F}(w)$ such that $y'^\frown(w, B') \leq y^\frown(w, B)$.

The following result by Radin captures a key property of $\mathbb{R}_w$

Theorem 3.18 (Radin)**.** *(The Prikry property for Radin forcing.) Let $p \in \mathbb{R}_w$ and let ϕ be a sentence in the forcing language. Then either there is some $p' \leq^* p$ such that $p' \Vdash_{\mathbb{R}_w} \phi$ or there is some $p' \leq^* p$ such that $p' \Vdash_{\mathbb{R}_w} \neg\phi$.*

Combining Theorem (3.18) and Lemma (3.15), we obtain a lemma (due to Radin) which will be very important in the proof of κ^+-cc for the main iteration. Recall that by our conventions a name for a truth value is just a name for an ordinal which is either 0 (false) or 1 (true).

Lemma 3.19 (Radin). *Let $y^\frown(w, B)$ be a condition in $\mathbb{R}_w$ with y non-empty, and let $\dot{b}$ be an $\mathbb{R}_w$-name for a truth value. Then there exist $C \subseteq B$ with $C \in \mathcal{F}(w)$ and an $\mathbb{R}_y$ name for a truth value $\dot{c}$ such that whenever G is $\mathbb{R}_w$-generic with $y^\frown(w, C) \in G$ and G' is the induced generic object for $\mathbb{R}_y$, then $(\dot{b})^G = (\dot{c})^{G'}$.*

Proof. Using the factorisation given by Lemma (3.15) and the Product Lemma, we may view $\dot{b}$ as an $\mathbb{R}^*$-name for an $\mathbb{R}_y$-name for a truth value where $\mathbb{R}^* = \{\, q \in \mathbb{R}_w \mid q \leq (w, B \setminus \mathcal{U}_{\kappa_y^+}) \,\}$. Since $|\mathbb{R}_y| < \kappa$ and a name for a truth value amounts to an antichain in $\mathbb{R}_y$, there are fewer than κ many $\mathbb{R}_y$-names for truth values. Since $\mathcal{F}(w)$ is κ-complete we may appeal to the Prikry property for $\mathbb{R}^*$, and shrink $(w, B \setminus \mathcal{U}_{\kappa_y^+})$ to (w, C) in order to decide which $\mathbb{R}_y$-name $\dot{c}$ is in question. $\blacktriangle_{3.19}$

In the situation of Lemma (3.19), we will sometimes say that the condition $y^\frown(w, C)$ *reduces* the $\mathbb{R}_w$-name $\dot{b}$ to the $\mathbb{R}_y$-name $\dot{c}$. Similar arguments (which we omit) about reducing names for sets of ordinals give another important result.

Theorem 3.20 (Radin). *Let $w \in \mathcal{U}$, let G be $\mathbb{R}_w$-generic over V and let $\langle u_j \mid j < \delta \rangle$ be the corresponding generic sequence. For every $\alpha < \kappa_w$, if i is largest such that $\kappa_{u_i} \leq \alpha$, then every subset of α lying in $V[G]$ lies in the $\mathbb{R}_{u_i}$-generic extension given by $\langle u_j \mid j < i \rangle$.*

It follows readily that forcing with $\mathbb{R}_w$ preserves all cardinals. However cofinalities may change, and the general situation is a little complicated. The main point for us is given by the following result.

Theorem 3.21 (Radin). *Let $w \in \mathcal{U}$, let G be $\mathbb{R}_w$-generic over V and assume that $\mathrm{lh}(w)$ is a regular cardinal λ with $\lambda < \kappa_w$. Let G be $\mathbb{R}_w$-generic. If λ is not a limit point of the Radin-generic club, then $V[G] \models \lambda$ is regular and $\mathrm{cf}(\kappa) = \lambda$.*

We note that the condition on λ in this theorem is easy to arrange by working below a suitable condition in $\mathbb{R}_w$. For example we may arrange that the least point of the generic club is greater than λ.

We will also require a characterisation of Radin-genericity which is due to Mitchell.

Theorem 3.22 (Mitchell). *Let $\langle u_i \mid i < \delta \rangle$ be a sequence of ultrafilter sequences in some outer model of V. Then the following are equivalent:*

(1) The sequence $\langle u_i \mid i < \delta \rangle$ is $\mathbb{R}_w$-generic.

(2) *For every $j < \delta$ the sequence $\langle u_i \mid i < j \rangle$ is $\mathbb{R}_{u_j}$-generic, and if $\mathrm{lh}(w) > 1$ then $\mathcal{F}(w) = \{ X \in V \mid \exists j < \delta \; \forall i \; j < i < \delta \implies u_i \in X \}$ – i.e., if $\mathrm{lh}(w) > 1$ then $\mathcal{F}(w)$ is the tail filter generated on the V-powerset of V_κ by the generic sequence.*

Definition 3.23. Sequences which satisfy the Mitchell criterion from Theorem (3.22) are sometimes called *geometric sequences*.

The following considerations will play a central role in the proof of Theorem (7.10).

Observation 3.24. Suppose that $V \subseteq V'$ with $V_\kappa^V = V_\kappa^{V'}$ (*e.g.*, V' might be a generic extension of V by $<\kappa$-closed forcing). Suppose that in V' there is an ultrafilter sequence w' such that $\mathrm{lh}(w) = \mathrm{lh}(w')$ and $w(\alpha) = w'(\alpha) \cap V$ for $0 < \alpha < \mathrm{lh}(w)$. Let $\langle u_i \mid i < \delta \rangle$ be an $\mathbb{R}_{w'}$-generic sequence over V'.

- By the Mitchell criterion for genericity and the observation that $\mathcal{F}(w) \subseteq \mathcal{F}(w')$, the sequence $\langle u_i \mid i < \delta \rangle$ is also an $\mathbb{R}_w$-generic sequence over V.

Lemma 3.25. *Let G be an $\mathbb{R}_w$-generic filter over V and let $\langle u_i \mid i < \delta \rangle$ be the corresponding generic sequence. Let $y = \langle (v_k, B_k) \mid k \leq n \rangle$ be a lower part. Then the following are equivalent:*

(1) *y conforms with G.*

(2) *For every k the sequence v_k appears among the sequences u_i, and for every i with $\kappa_{u_i} \leq \kappa_{v_n}$ either u_i appears among the sequences v_k or $u_i \in B_k$ for the least k such that $\kappa_{u_i} < \kappa_{v_k}$.*

Proof. It is immediate that the first condition implies the second. For the converse direction, by Lemma (3.17), we need to verify that every sequence u_i with $\kappa_{u_i} \leq \kappa_{v_n}$ appears in some element of $\mathbb{R}_y$. This is immediate when u_i appears among the v_k, so we assume that it does not and let k be least with $\kappa_{u_i} < \kappa_{v_k}$. If $\mathrm{lh}(u_i) = 1$ then we may extend y by shrinking B_k to $B_k \setminus \mathcal{U}_{\kappa_{u_i}^+}$ and inserting the pair $(u_i, \emptyset)$. If $\mathrm{lh}(u_i) > 1$ then i is limit and by hypothesis $u_j \in B_k$ for all large $j < i$, so that (by the geometric condition from part (2) of Theorem (3.22)) $B_k \cap V_{\kappa_{u_i}} \in \mathcal{F}(u_i)$. In this case we may extend y by shrinking B_k to $B_k \setminus \mathcal{U}_{\kappa_{u_i}^+}$ and inserting the pair $(u_i, B_k \cap V_{\kappa_{u_i}})$.   ▲3.25

A very similar argument gives:

Lemma 3.26. *Let G be an $\mathbb{R}_w$-generic filter over V and let $\langle u_i \,|\, i < \delta \rangle$ be the corresponding generic sequence. Let $y^\frown(w, B)$ be a condition. Then $y^\frown(w, B) \in G$ if and only if y conforms with G and $u_i \in B$ for all i such that $\kappa_{u_i} > \kappa_y$.*

For use later we make a technical definition which is motivated by the Mitchell criterion for genericity, and will be used in the definition of the main iteration.

Definition 3.27. For $y = \langle (u_i, B_i) \,|\, i < n \rangle$ a lower part for $\mathbb{R}_w$, $A = \langle A_\rho \,|\, \rho < \rho^* \rangle$ a sequence of subsets of V_{κ_w} and $\eta < \kappa_w$ we say y is *harmonious with A past* or *above η*, if for each i with $i < n$ one of the following conditions holds.

- $\kappa_{u_i} < \eta$.

- $\kappa_{u_i} = \eta$, $u_i = \langle \eta \rangle$ and $B_i = \emptyset$.

- $\kappa_{u_i} > \eta$, and $\{u_i\} \cup B_i \subseteq \bigcup_{\rho < \rho^*} A_\rho \setminus \mathcal{U}_{\eta^+}$.

Observation 3.28. We record some remarks about the preceding definition.

- If a lower part y is harmonious with A past η it divides rather strictly into a part below η and a part above: there is no (u_i, B_i) appearing in y such that $\eta \leq \kappa_{u_i}$ while $\kappa_v \leq \eta$ for some $v \in B_i$.

- Definition (3.27) depends only on the set $\bigcup_{\rho < \rho^*} A_\rho$ rather than the sequence A itself. It is phrased in this way to avoid encumbering later definitions with union signs.

The following lemma shows how to thin a lower part conforming with a Radin generic sequence to one which still conforms and which is also harmonious past some η with a sequence of sets whose union contains the interval of the generic sequence consisting of those measure sequences with critical point at least η and which it can 'see'. This, also, is useful for the proof of Theorem (7.10).

Lemma 3.29. *Let G be an $\mathbb{R}_w$-generic filter over V and let $\langle u_i \,|\, i < \delta \rangle$ be the corresponding generic sequence. Let $A = \langle A_\rho \,|\, \rho < \rho^* \rangle$ be such that each A_ρ is a set of measure sequences with a common critical point $\kappa_\rho < \kappa$, let $a = \{\kappa_\rho \,|\, \rho < \rho^*\}$ and let $D = \bigcup_{\rho < \rho^*} A_\rho$.*

Suppose that η is a successor point in the generic club set C such that $u_i \in D$ for all i such that $\eta \leq \kappa_{u_i} < \mathrm{ssup}(a)$.

Let y be a lower part conforming with G with $\kappa_y < \mathrm{ssup}(a)$, and suppose that $\langle \eta \rangle$ appears in y. Then there is a lower part y' such that

(1) y' is a direct extension of y in $\mathbb{R}_y$,

(2) y' conforms with G, and

(3) y' is harmonious with A past η.

Proof. By the definition of conformity all sequences appearing in y have the form u_i for some i. We will obtain y' by some judicious shrinking of the measure one sets appearing as the second entries of pairs in y.

Consider the pairs (v, B) appearing in y. If $\kappa_v < \eta$ there is no problem, and if $\kappa_v = \eta$ then (as η is a successor point in C) we have $v = \langle \eta_\varepsilon \rangle$ and $B = \emptyset$, so again there is no problem.

Suppose now that (v, B) appears in y and $\kappa_v > \eta$. By the definition of conditionhood, B contains no measure sequence u with $\kappa_u \leq \eta$. Let j be such that $v = u_j$, and recall that $\mathrm{lh}(v) = 1$ and $B = \emptyset$ for j a successor, and $\mathrm{lh}(v) > 1$ and $B \in \mathcal{F}(v)$ for j a limit.

By hypothesis we have $u_i \in D$ for all i such that $\kappa_{u_i} \geq \eta$; it follows by the "geometric" criterion for genericity (see Theorem (3.22)) that if j is a limit then $D \cap V_{\kappa_v} \in \mathcal{F}(v)$. So when $v = v_j$ for j a limit, we may shrink B to obtain a set $B' = B \cap D$ with $B' \in \mathcal{F}(v)$. We define y' to be the resulting lower part. $\blacktriangle_{3.29}$

Lemma 3.30. *If y is a lower part which is harmonious with A past η and $y' \in \mathbb{R}_y$, then y' is harmonious with A past η.*

Proof. We consider each pair (u, B) appearing in y'. If u already appears in y then we have that (u, B') appears in y for some B' such that $B \subseteq B'$, and it is easy to see that in all cases the harmoniousness conditions are satisfied. If u does not appear in y then let (v, C) be the unique pair appearing in y such that $u \in C$, and consider the various cases of the definition for the pair (v, C): if $\kappa_v < \eta$ then $\kappa_u < \eta$ and we are done, the case $\kappa_v = \eta$ cannot occur as $C \neq \emptyset$, and if $\kappa_v > \eta$ then $\{u\} \cup B \subseteq C \subseteq \bigcup_{\rho < \rho^*} A_\rho \setminus \mathcal{U}_{\eta^+}$. $\blacktriangle_{3.30}$

4. The definition of $Q(w)$, and useful properties of it.

Suppose that U is a measure on a measurable cardinal κ. Recall that κ-*Mathias forcing* using U has as conditions pairs (s, S) with $s \in [\kappa]^{<\kappa}$, $S \in U$ and $\mathrm{ssup}(s) \leq \min(S)$ and ordering given by $(t, T) \leq (s, S)$ if

$s = t \cap \mathrm{ssup}(s)$ and $T \cup (t \setminus s) \subseteq S$. This forcing preserves cardinals as it is $<\kappa$-directed closed and has the κ^+-chain condition. If G is a generic filter over V for the forcing and $y = \bigcup \{ s \mid \exists S \in U \; (s, S) \in G \}$ then for all $S \in \mathcal{P}(\kappa)$ we have that $S \in U$ if and only if there is some $\alpha < \kappa$ such that $y \setminus \alpha \subseteq S$. Consequently, the forcing is also known as the forcing to *diagonalise* U.[1]

We also recall Ellentuck's topologically inspired notation [8] that if (s, S) is a condition in this forcing then $[s, S] = \{ y \in [\kappa]^\kappa \mid s \subseteq y \subseteq s \cup S \}$. Using this notation we have that $(t, T) \le (s, S)$ if and only if $[t, T] \subseteq [s, S]$.

In this section we define two forcing posets. The first poset $M(w)$ is the analogue of the κ-Mathias forcing for ultrafilter sequences w; it will diagonalise the filter $\mathcal{F}(w)$. The second poset $Q(w)$ is the variant of $M(w)$ which is tailored, in the style of the analogous forcing from [6], to deal with Radin forcing names for graphs, and binary relations more generally, on κ^+.

One can view κ-Mathias forcing as adding a subset y of κ which is potentially a member of U' for some normal measure U' extending U in some generic extension. In the same spirit, $M(w)$ is designed to add a set $A \subseteq V_{\kappa_w}$ which is potentially a member of $\mathcal{F}(w')$ for some sequence w' in a $< \kappa_w$-closed generic extension where $\mathrm{lh}(w') = \mathrm{lh}(w)$ and $w_i' \supseteq w_i$ for $0 < i < \mathrm{lh}(w)$.

For the rest of this section we fix an ultrafilter sequence w with $\mathrm{lh}(w) > 1$, and write κ for κ_w.

Definition 4.1. $M(w)$ is the forcing with conditions $p = (A, B)$, with:

- $A = \langle A_\rho \mid \rho < \rho^p \rangle$, where $\rho^p < \kappa$,

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[1] The earliest uses of this forcing of which we have evidence are in unpublished work from 1992 by Shizuo Kamo [14] and Tadatoshi Miyamoto, independently, on the splitting number for uncountable cardinals. See, for example, the reports on this work in [27] and [28]. The latter paper, in particular, gives a full account of their principal result, that it is consistent relative to there being a supercompact cardinal that there is an uncountable cardinal κ with splitting number $\mathfrak{s}_\kappa$ greater than κ^+, and its proof via iterating this forcing. Unfortunately we have not had access to the preprint [14] itself. The forcing has also at times been named long Prikry forcing.

A rather similar forcing, however, was given by Henle [12] in the early 1980s in the choiceless context of cardinals with strong partition properties. Henle called his forcing Radin-like forcing – the descriptor referring to properties of the subset of the cardinal added rather than the form of the conditions.

- $\forall \rho < \rho^p \, (A_\rho \subseteq \mathcal{U}_\kappa \ \& \ A_\rho \neq \emptyset \ \& \ \exists \kappa_\rho < \kappa \, \forall u \in A_\rho \ \kappa_u = \kappa_\rho)$,

- $\forall \rho < \rho^p \, \forall u \in A_\rho \, \forall \tau \in (0, \mathrm{lh}(u)) \ u \restriction \tau \in A_\rho$,

- $\langle \kappa_\rho \mid \rho < \rho^p \rangle$ is strictly increasing,

- $B \in \mathcal{F}(w)$,

- $\forall v \in B \, \forall \tau \in (0, \mathrm{lh}(v)) \ (v \restriction \tau \in B)$, and

- $\mathrm{ssup}(\{\kappa_\rho \mid \rho < \rho^p\}) \leq \min(\{\kappa_v \mid v \in B\})$.

If p, $q \in M(w)$ then $q \leq p$ if $A^p = A^q \restriction \rho^p$ and $B^q \cup \bigcup_{\rho \in [\rho^p, \rho^q)} A^q_\rho \subseteq B^p$, while $q \leq^* p$ if $q \leq p$ and $A^q = A^p$. If p, $q \in M(w)$ and $q \leq^* p$ we say that q is a *direct extension* of p.

Definition 4.2. Set $[A^p, B^p]$ to be

$$\{\, \langle D^\rho \mid \rho < \kappa \rangle \mid \forall \rho < \kappa \ (\, D^\rho \subseteq \mathcal{U}_\kappa \ \& \ \exists \kappa_\rho < \kappa \, \forall u \in D^\rho \, (\kappa_u = \kappa_\rho) \ \& $$
$$\forall u \in D^\rho \, \forall \tau \in (0, \mathrm{lh}(u)) \ (u \restriction \tau \in D^\rho) \,) \ \& \ \langle \kappa_\rho \mid \rho < \kappa \rangle \text{ is strictly}$$
$$\text{increasing} \ \& \ \langle D^\rho \mid \rho < \rho^p \rangle = A^p \ \& \ \bigcup \{ D^\rho \mid \rho \in [\rho^p, \kappa) \} \subseteq B^p \,\}.$$

Then $(A^q, B^q) \leq (A^p, B^p)$ if and only if $[A^q, B^q] \subseteq [A^p, B^p]$.

It is useful to have a name for the set of cardinals which are the first elements of the various ultrafilter sequences appearing anywhere in A^p for $p \in M(w)$. Accordingly we make the following definition.

Definition 4.3. If $p = (A, B) \in M(w)$ let $a^p = \{ \kappa_\rho \mid \rho < \rho^p \}$.

Now we move on to the definition of the forcing $Q(w)$. This is carried out under the following running combinatorial assumption.

Setting 4.4. Suppose $\mathcal{T}$ is a binary κ^+-Kurepa tree with Υ many branches.

Definition 4.5. Let $\langle b_\alpha \mid \alpha < \Upsilon \rangle$ enumerate a set of branches through $\mathcal{T}$. Let $\langle \dot{E}_\alpha \mid \alpha < \Upsilon \rangle$ be a list of canonical $\mathbb{R}_w$-names for binary relations on κ^+. We will use the sequences $\langle b_\alpha \mid \alpha < \Upsilon \rangle$ and $\langle \dot{E}_\alpha \mid \alpha < \Upsilon \rangle$ as parameters in the definition of the forcing $Q(w)$.

$Q^*(w)$ is the forcing with conditions $p = (A, B, t, f)$ satisfying the following four clauses.

(1) $(A, B) \in M(w)$ (see Definition (4.1)). We set $a = a^{(A,B)}$.

(2) $t \in [(a \cap \sup(a)) \times \Upsilon]^{<\kappa}$ and $f = \langle f^\eta_\alpha \mid (\eta, \alpha) \in t \rangle$. For $\eta \in a \cap \sup(a)$, set $t^\eta = \{ \alpha \mid (\eta, \alpha) \in t \}$.

(3) $\forall \eta \in a \cap \sup(a)\ \forall \alpha \in t^\eta\ d^\eta_\alpha = \mathrm{dom}(f^\eta_\alpha) \in [\kappa^+]^{<\kappa}$.

(4) $\forall \eta \in a \cap \sup(a)\ \forall \alpha \in t^\eta\ \forall \zeta \in d^\eta_\alpha\ \exists \nu < \kappa\ \ f^\eta_\alpha(\zeta) = (b_\alpha \restriction \zeta, \nu)$.

If p, $q \in Q^*(w)$ then $q \leq p$ if $[A^q, B^q] \subseteq [A^p, B^p]$, $t^p \subseteq t^q$ and $\forall(\eta, \alpha) \in t^p\ (f^\eta_\alpha)^p \subseteq (f^\eta_\alpha)^q$; and $q \leq^* p$ if $q \leq p$ and $A^q = A^p$, $t^q = t^p$ and $f^q = f^p$. (If $q \leq^* p$ we say q is a *direct extension* of p.)

We write $Q(w)$ for the suborder of $Q^*(w)$ consisting of conditions which also satisfy:

(5) for all $\eta \in a \cap \sup(a)$, for all α, $\beta \in t^\eta$, for every lower part y for $\mathbb{R}_w$ harmonious with A past η, and for all ζ, $\zeta' \in d^\eta_\alpha \cap d^\eta_\beta$ we have:

$$f^\eta_\alpha(\zeta) = f^\eta_\beta(\zeta) \neq f^\eta_\alpha(\zeta') = f^\eta_\beta(\zeta') \implies$$
$$y^\frown(w, B) \Vdash_{\mathbb{R}_w} \text{`` } \zeta \dot{E}_\alpha \zeta' \longleftrightarrow \zeta \dot{E}_\beta \zeta' \text{ ''}.$$

Observation 4.6. Observe that in clause (5), if $y = \langle (u_i, B_i) \mid i \leq n \rangle$ and y is harmonious with A past η for some $\eta \in a$, then $\kappa_{u_n} \leq \sup(a)$.

Observation 4.7. It is important to remember that $Q(w)$ and $Q^*(w)$ depend on the tree $\mathcal{T}$ and the sequences $\langle b_\alpha \mid \alpha < \Upsilon \rangle$ and $\langle \dot{E}_\alpha \mid \alpha < \Upsilon \rangle$ even though their names do not make this dependence explicit.

The poset $Q(w)$ is designed to add a sequence A^* which diagonalises $\mathcal{F}(w)$ in a sense made precise in Corollary (4.9) below, together with various objects that can potentially be understood as $\mathbb{R}_{w^*}$-names where w^* is an ultrafilter sequence existing in some $<\kappa$-directed closed forcing extension of V, with $\mathrm{lh}(w) = \mathrm{lh}(w^*)$, $w^*_\tau \cap V = w_\tau$ for $0 < \tau < \mathrm{lh}(w)$ and $\bigcup A^* \in \mathcal{F}(w^*)$.

The aim is that if $\langle u_i \mid i < \eta \rangle$ is an $\mathbb{R}_{w^*}$-generic sequence (where we note that by the Mitchell criterion this sequence is also $\mathbb{R}_w$-generic) and $u_i \in \bigcup A^*$ for all i such that $\kappa_{u_i} \geq \eta$ (along with various other technical conditions), then the "η-coordinate" in the poset $Q(w)$ will add an $\mathbb{R}_{w^*}$ name for a binary relation of size κ^+ together with embeddings of all the relations named by the names $\dot{E}_\alpha$ into this relation.

We are particularly interested in the posets $Q(w)$ for certain specific lists of relations, for example when $\langle \dot{E}_\alpha \mid \alpha < \Upsilon \rangle$ is a list of canonical names for all graphs on κ^+. In this case the "η-coordinate" will add a name for a graph whose realization will be universal for the graphs named by the names in the list.

Lemma 4.8. *For every $\varepsilon < \kappa$, the set of $q \in Q(w)$ such that $\varepsilon < \kappa_\rho^q$ for some $\rho < \rho^q$ is dense and open.*

Proof. It is immediate that this set is open, so we only need to verify that it is dense. Let $p \in Q(w)$ and let $\rho = \rho^p$. Find $v \in B^p$ such that $\kappa_v > \max(\{\varepsilon, \sup(a^p)\})$ and note that $v \upharpoonright \tau \in B^p$ for $0 < \tau \leq \mathrm{lh}(v)$. Now let $\rho^q = \rho + 1$, $A^q \upharpoonright \rho = A^p$, $A_\rho^q = \{v \upharpoonright \tau \mid 0 < \tau \leq \mathrm{lh}(v)\}$, $B^q = \{v' \in B^p \mid \kappa_{v'} > \kappa_v\}$, $t^q = t^p$ and $f^q = f^p$.

We claim that $q \in Q(w)$ and $q \leq p$. The only non-trivial point is to check that q satisfies Clause (5) in the definition of $Q(w)$. Let $\eta, \alpha, \beta, \zeta, \zeta'$ be such that $f_\alpha^\eta(\zeta) = f_\beta^\eta(\zeta) \neq f_\alpha^\eta(\zeta') = f_\beta^\eta(\zeta')$, and note that since $(\eta, \alpha) \in t^q = t^p$ we have that $\eta \in a^p \cap \sup(a^p)$.

Let y be harmonious with A^q above η. By the definition of harmonious, Definition (3.27), we have that $\kappa_y \leq \kappa_v$.

If $\kappa_y < \kappa_v$ then clearly y is harmonious with A^p above η, and we are done by Clause (5) for p and the remark that $y^\frown(w, B^q) \leq y^\frown(w, B^p)$.

If $\kappa_y = \kappa_v$ then we have $y = y'^\frown(v \upharpoonright \tau, D)$ for some $\tau \leq \mathrm{lh}(v)$, where $\kappa_{y'} < \kappa_v$, y' is harmonious with A^p and also $D \subseteq \bigcup_{\rho < \rho^p} A_\rho^p$. By the choice of κ_v to be strictly greater than $\sup(a^p)$ we see that D is bounded in V_{κ_v} and so cannot be of measure 1 for any measure on v, hence necessarily $\tau = 1$ and $D = \emptyset$, so that actually $v \upharpoonright \tau = \langle \kappa_v \rangle$. It follows that $y^\frown(w, B^q) \leq y'^\frown(w, B^p)$, and again we are done by Clause (5) for p.

$\blacktriangle_{4.8}$

Corollary 4.9. *If G is $Q(w)$-generic over V and A^* is the union of the sequences A^p for $p \in G$, then A^* is a κ-sequence and for every $B \in \mathcal{F}(w)$ there is $\rho^* < \kappa$ such that $A_\rho^* \subseteq B$ for $\rho^* < \rho < \kappa$.*

The proof of Lemma (4.8) also gives the following result.

Corollary 4.10. *The set of $p \in Q(w)$ such that ρ_p is a successor ordinal is dense in $Q(w)$.*

Notation 4.11. For $\eta < \kappa$, $\alpha < \Upsilon$ and $\zeta < \kappa^+$ let $\mathcal{D}_{\zeta,\alpha}^\eta = \{p \in Q(w) \mid \alpha \in (t^\eta)^p \ \& \ \zeta \in (d_\alpha^\eta)^p\}$.

Lemma 4.12. *Suppose $p \in Q(w)$, $\eta \in a^p$, $\alpha < \Upsilon$ and $\zeta < \kappa^+$. Then there is $q \in \mathcal{D}_{\zeta,\alpha}^\eta$ with $q \leq p$, $A^q = A^p$, $B^q = B^p$, $(t^\eta)^q = (t^\eta)^p \cup \{\alpha\}$ and $(d_\alpha^\eta)^q = (d_\alpha^\eta)^p \cup \{\zeta\}$. (In the last clause we formally take $(d_\alpha^\eta)^p = \emptyset$ if $\alpha \notin (t^\eta)^p$.)*

Proof. If $p \in \mathcal{D}^{\eta}_{\zeta,\,\alpha}$ then take $q = p$ and there is nothing more to do.

So suppose that $p \notin \mathcal{D}^{\eta}_{\zeta,\,\alpha}$. If $\alpha \notin (t^{\eta})^p$ observe that $\langle A^p, B^p, t^p \cup \{(\eta,\alpha)\}, f^p{}^{\frown}\langle \emptyset \rangle \rangle$ is a condition in $Q(w)$ and refines p. (The concatenated 1-tuple consisting of the empty set records that the value of the new f-part of the condition $\langle A^p, B^p, t^p \cup \{(\eta,\alpha)\}, f^p{}^{\frown}\langle \emptyset \rangle \rangle$ at the coordinate (η, α) is the empty function.) Consequently we may assume without loss of generality that $\alpha \in (t^{\eta})^p$ and $\zeta \notin (d^{\eta}_{\alpha})^p$.

Pick $\nu \in \kappa \setminus \bigcup_{(\eta,\beta) \in t^p} \mathrm{rge}((\pi_1 \cdot f^{\eta}_{\beta})^p)$, let $(f^{\eta}_{\alpha})^q = (f^{\eta}_{\alpha})^p \cup \{\langle \zeta, (b_{\alpha} \restriction \zeta, \nu) \rangle\}$ and set $(f^{\eta'}_{\beta})^q = (f^{\eta'}_{\beta})^p$ for $(\eta',\beta) \in t^p$ with $(\eta',\beta) \neq (\eta,\alpha)$. Then $q = \langle A^p, B^p, t^p, \langle (f^{\eta}_{\beta})^q \mid (\eta,\beta) \in t^p \rangle \rangle$ is clearly an element of $Q^*(w)$ and refines p.

Moreover, q satisfies (5) of the definition of $Q(w)$. In brief this is true because q inherits the truth of (5) from p for every collection of data for instances of (5) except one, and for that collection we have ensured (5) holds vacuously.

In more detail, suppose $\eta' \in a^{(A^q,B^q)} \cap \sup(a^{(A^q,B^q)})$, $\beta, \gamma \in (t^{\eta'})^q$, y is a lower part for $\mathbb{R}_w$ harmonious with A^q past η' and $\zeta', \zeta'' \in (d^{\eta'}_{\beta})^q \cap (d^{\eta'}_{\gamma})^q$ Either $\eta' \in a^{(A^p,B^p)} \cap \sup(a^{(A^p,B^p)})$, $\beta, \gamma \in (t^{\eta'})^p$, y is a lower part for $\mathbb{R}_w$ harmonious with A^p past η' and $\zeta', \zeta'' \in (d^{\eta'}_{\beta})^p \cap (d^{\eta'}_{\gamma})^p$, and hence

$$f^{\eta'}_{\gamma}(\zeta') = f^{\eta'}_{\beta}(\zeta') \neq f^{\eta'}_{\gamma}(\zeta'') = f^{\eta'}_{\beta}(\zeta'') \implies$$
$$y{}^{\frown}(w, B^p) \Vdash_{\mathbb{R}_w} \text{``} \zeta' \dot{E}_{\gamma} \zeta'' \longleftrightarrow \zeta' \dot{E}_{\beta} \zeta'' \text{''}.$$

Or $\eta' = \eta$, $\gamma = \alpha$ and $\zeta'' = \zeta$ and we have that ν was chosen specifically so that $(f^{\eta}_{\alpha})^q(\zeta) \neq (f^{\eta}_{\beta})^p(\zeta')$ and hence the antecedent of the instance of the implication in (5) of the definition of $Q(w)$ is false and so for this collection of data (5) holds. $\blacktriangle_{4.12}$

Corollary 4.13. *For each $\eta < \kappa$, each $\alpha < \Upsilon$ and each $\zeta \in \kappa^+$, the set $\mathcal{D}^{\eta}_{\zeta,\alpha} = \{p \in Q(w) \mid \alpha \in (t^{\eta})^p \ \& \ \zeta \in (d^{\eta}_{\alpha})^p\}$ is dense and open in $\{p \in Q(w) \mid \eta \in a^p\}$.*

Proof. Immediate from Lemma (4.12). $\blacktriangle_{4.13}$

Corollary 4.14. *If G is $Q(w)$-generic over V, $\eta \in \bigcup\{a^p \mid p \in G\}$ and $\alpha < \Upsilon$ then $\bigcup\{f^{p,\eta}_{\alpha} \mid p \in G, (\eta,\alpha) \in t^p\}$ is a function with domain κ^+.*

Lemma 4.15. $Q(w)$ *has splitting.*

Proof. This is proved using an argument very similar to the one for Lemma (4.12). As observed there, given a condition p we can easily find a condition in $Q(w)$ refining it with a non-empty t-part, so without loss of generality we may as well assume $t^p \neq \emptyset$. Let $(\eta, \alpha) \in t^p$. Now pick $\zeta \in \kappa^+ \setminus (d^\eta_\alpha)^p$ and working exactly as before we can define two extensions q_0, q_1 of p which are both in $\mathcal{D}^\eta_{\zeta, \alpha}$ but satisfy $(f^\eta_\alpha)^{q_0}(\zeta) \neq (f^\eta_\alpha)^{q_1}(\zeta)$, simply by choosing, at the ultimate stage of that argument, two distinct elements ν_0 and ν_1 of $\kappa \setminus \bigcup_{(\eta, \beta) \in t^p} \mathrm{rge}((\pi_1 \cdot f^\eta_\beta)^p)$, where previously we merely picked one. Then q_0 and q_1 are incompatible.

▲$_{4.15}$

Lemma 4.16.   *(1) The forcing poset $Q(w)$ is $<\kappa$-compact.*

 (2) Descending sequences from $Q(w)$ of length less than κ have greatest lower bounds.

Proof. We start by proving (1). Let $C = \{q_i \mid i < \mu\}$ be a centred subset of $Q(w)$ for some $\mu < \kappa$, and fix for each finite set $Y \subseteq \mu$ some condition $s_Y \in Q(w)$ such that $s_Y \leq q_i$ for all $i \in Y$. We will define a quadruple (A^*, B^*, t^*, f^*) of the appropriate type, and prove that it is a condition in $Q(w)$ and forms a lower bound for C.

Let $\rho^* = \sup\{\rho^{q_i} \mid i < \mu\}$. By the hypothesis that C is centred it is easy to see that there is a unique sequence $A^* = \langle A_\rho \mid \rho < \rho^* \rangle$ such that $A^* \upharpoonright \rho^{q_i} = A^{q_i}$ for all $i < \mu$. For if $i < j < \mu$ there is some $p \in Q(w)$ such that $p \leq q^i$, q^j and hence by the definition of $\leq$ (see Definition (4.1)) one has that $A^p \upharpoonright \rho^{q_i} = A^{q_i}$ and $A^p \upharpoonright \rho^{q_j} = A^{q_j}$; thus either A^{q_i} is an initial segment of A^{q_j} or vice versa. We write κ_ρ for the common value of κ_v for sequences $v \in A_\rho$, and a^* for $\{\kappa_\rho \mid \rho < \rho^*\}$.

Let $B^* = \bigcap_{Y \in [\mu]^{<\omega}} B^{s_Y}$. By the completeness of the filter $\mathcal{F}(w)$ we see that $B^* \in \mathcal{F}(w)$. It also follows from the definition of $Q(w)$ that $\kappa_\rho < \kappa_v$ for every $\rho < \rho^*$ and every $v \in B^*$, for if $\kappa_\rho \in a^*$ there is some $i < \mu$ such that $\kappa^\rho \in a^{q_i}$ and hence for all $v \in B^{q_i}$ one has $\kappa_\rho < \kappa_v$, and thus, as $B^* \subseteq B^{q_i}$ one has $\kappa_\rho < \kappa_v$.

Let $t^* = \bigcup_{i < \mu} t^{q_i}$. Clearly $t^* \subseteq (a^* \cap \sup(a^*)) \times \Upsilon$ and $|t^*| < \kappa$.

For each $(\eta, \alpha) \in t^*$, let $d^{*, \eta}_\alpha = \bigcup \{d^{q_i, \eta}_\alpha \mid (\eta, \alpha) \in t^{q_i}, i < \mu\}$.

By the hypothesis that C is centered it is easy to see that there is a unique sequence of functions $f^* = \langle f^{*, \eta}_\alpha \mid (\eta, \alpha) \in t^* \rangle$ with $\mathrm{dom}(f^{*, \eta}_\alpha) = d^{*, \eta}_\alpha$, and $f^{*, \eta}_\alpha(\zeta) = f^{q_i, \eta}_\alpha(\zeta)$ for all $\zeta \in d^{q_i, \eta}_\alpha$.

It should be clear that (A^*, B^*, t^*, f^*) is a condition in $Q^*(w)$, and that if (A^*, B^*, t^*, f^*) is a condition in $Q(w)$ then it forms a lower bound for C. So to finish the proof, we must verify clause (5) in the definition of $Q(w)$.

Let $\eta \in a^* \cap \sup(a^*)$ with $\eta < \sup(a^*)$, let $\alpha, \beta \in t^{*,\eta}$, let $y = \langle (u_i, B_i) \mid i \leq n \rangle$ be a lower part harmonious with A^* past η, and let ζ, $\zeta' \in d_\alpha^{*,\eta} \cap d_\beta^{*,\eta}$ be such that $f_\alpha^{*,\eta}(\zeta) = f_\beta^{*,\eta}(\zeta) \neq f_\alpha^{\eta,*}(\zeta') = f_\beta^{*,\eta}(\zeta')$.

Choose a finite set Y large enough that

- There is $i_0 \in Y$ with $\eta \in a^{q_{i_0}}$, $\eta < \sup(a^{q_{i_0}})$ and $\kappa_{u_n} \leq \sup(a^{q_{i_0}})$.

- There is $i_1 \in Y$ with $(\eta, \alpha) \in t^{q_{i_1}}$, and $\zeta \in d_\alpha^{q_{i_1}, \eta}$.

- There is $i_2 \in Y$ with $(\eta, \beta) \in t^{q_{i_2}}$, and $\zeta \in d_\beta^{q_{i_2}, \eta}$.

- There is $i_3 \in Y$ with $(\eta, \alpha) \in t^{q_{i_3}}$, and $\zeta' \in d_\alpha^{q_{i_3}, \eta}$.

- There is $i_4 \in Y$ with $(\eta, \beta) \in t^{q_{i_4}}$, and $\zeta' \in d_\alpha^{q_{i_4}, \eta}$.

Now consider the condition s_Y. By the construction $A^{q_{i_0}}$ is an initial segment of A^{s_Y} and so by the choice of i_0 one has that y is harmonious with A^{s_Y} past η. Since $s_Y \in Q(w)$ it follows easily that

$$y ^\frown (w, B^{s_Y}) \Vdash_{\mathbb{R}_w} \text{`` } \zeta \dot{E}_\alpha \zeta' \longleftrightarrow \zeta \dot{E}_\beta \zeta' \text{ ''}.$$

By construction $B^* \subseteq B^{s_Y}$, and so $y ^\frown (w, B^*)$ also forces this equivalence and we are done.

In order to see that (2) holds, suppose that C as in the proof of (1) is a descending sequence of conditions. Then (A^*, B^*, t^*, f^*) is a greatest lower bound for C. $\blacktriangle_{4.16}$

Corollary 4.17. $Q(w)$ *is both $<\kappa$-directed closed and countably compact.*

The following result is useful in the proof, which appears in the next section, that $Q(w)$ has the stationary κ^+-chain condition.

Lemma 4.18. *Let $p = (A, B, t, f) \in Q(w)$. Let $\boldsymbol{t} = \bigcup \{ t^\eta \mid \eta \in a^p \}$ and let $\boldsymbol{d} = \bigcup \{ d_\alpha^\eta \mid \eta \in a^p \,\&\, \alpha \in t^\eta \}$. Then there is $q \in Q(w)$ with $q \leq p$, $A^q = A^p$, $B^q = B^p$, $t^\eta = \boldsymbol{t}$ for all $\eta \in a^p$ and $(d_\alpha^\eta)^q = \boldsymbol{d}$ for all $\eta \in a^p$ and $\alpha \in t^\eta$.*

Proof. As in the second paragraph of the proof of Lemma (4.12), we have that $r = \langle A^p, B^p, t^p \cup \{ (\eta, \alpha) \mid \eta \in a^p \,\&\, \alpha \in \boldsymbol{t} \setminus t^\eta \}, f^p ^\frown \langle \emptyset \mid \eta \in$

$a^p \ \& \ \alpha \in t \setminus t^\eta \rangle \rangle$ is a condition in $Q(w)$ which refines p and is such that $a^r = a^p$ and $(t^\eta)^r = t$ for all $\eta \in a^r$.

Now enumerate $D = \{(\eta, \alpha, \zeta) \,|\, \underline{\eta \in a^p} \ \& \ \alpha \in t \ \& \ \zeta \in d \setminus d^\eta_\alpha\}$ as $\{(\eta_i, \alpha_i, \zeta_i) \,|\, i < \gamma\}$ for some $\gamma \leq \overline{a^p \times t \times d}$.

If $D = \emptyset$ then p itself satisfies the properties required for q.

Otherwise, carry out an induction on γ, using Lemma (4.12) at initial and successor stages and Lemma (4.16.(2)) at limits, to construct a descending sequence of conditions $\langle r_i \,|\, i < \gamma \rangle$ with $r_0 \leq r$ and such that $\zeta_i \in (d^{\eta_i}_{\alpha_i})^{r_i}$ for each $i < \gamma$.

Finally, if γ is a limit ordinal use Lemma (4.16.(2)) to choose a lower bound q for $\langle r_i \,|\, i < \gamma \rangle$, and if $\gamma = \gamma' + 1$ is a successor set $q = r_{\gamma'}$.

$$\blacktriangle_{4.18}$$

We now introduce the notion of a *weakening* of a condition.

Definition 4.19. Let p be a condition in $Q(w)$ such that ρ^p is a successor ordinal and let $\rho^* < \rho^p$. We define the ρ^*-*weakening of p* to be the quadruple $r = (A^r, B^r, t^r, f^r)$ where $A^r = A^p \upharpoonright \rho^*$, $a^r = \{\kappa^p_\rho \,|\, \rho < \rho^*\}$, $B^r = B^p \cup \bigcup\{A^p_\rho \,|\, \rho^* \leq \rho < \rho^p\}$, $t^r = \{(\eta, \alpha) \in t^p \,|\, \eta \in a^r \cap \sup(a^r)\}$ and $f^r = \langle f^{p, \eta}_\alpha \,|\, (\eta, \alpha) \in t^r \rangle$.

Lemma 4.20. *Let $p \in Q(w)$ with ρ^p a successor ordinal, let $\rho^* < \rho^p$ and let r be the ρ^*-weakening of p. Then*

(1) $r \in Q(w)$,

(2) $p \leq r$,

(3) and for every $\eta \in a^r$ and every lower part y which is harmonious with A^r past η, if $f^{p,\eta}_\alpha(\zeta) = f^{p,\eta}_\beta(\zeta) \neq f^{p,\eta}_\alpha(\zeta') = f^{p,\eta}_\beta(\zeta')$, then $y^\frown(w, B^r) \Vdash \text{``} \zeta \dot{E}_\alpha \zeta' \Longleftrightarrow \zeta \dot{E}_\beta \zeta'. \text{''}$

Proof. It is easy to see that $r \in Q^*(w)$, and that p refines r in $Q^*(w)$. Since $f^{p,\eta}_\alpha = f^{r,\eta}_\alpha$ when $(\eta, \alpha) \in t^r$, Clause (3) in the conclusion implies that r satisfies Clause (5) in Definition (4.5) (and hence that r is in $Q(w)$). It will therefore suffice to verify Clause (3).

Suppose for a contradiction that $y^\frown(w, B^r)$ does not force the desired equivalence, then there is an extension $y'^\frown(w, B)$ forcing that the equivalence is false. Shrinking B if necessary we may assume that $B \subseteq B^p$.

We will break up y' as $y_0 {}^\frown y_1 {}^\frown y_2$, where $y_0 \in \mathbb{R}_y$, y_1 consists of pairs (u, C) such that u is drawn from $\bigcup \{ A_\rho^p \mid \rho^* \le \rho < \rho^p \}$, and y_2 consists of pairs (u, C) such that u is drawn from B^p. This is possible because, by the definition of extension in $\mathbb{R}_w$, all pairs (u, C) in y' with $\kappa_u > \kappa_y$ have $u \in B^r$.

We claim that $y_0 {}^\frown y_1$ is harmonious with A^p above η. An appeal to Lemma (3.30) shows that y_0 is harmonious with A^r above η, which handles the pairs appearing in y_0. Let (u, C) be a pair appearing in y_1, and observe that $\eta \in a^r = \{ \kappa_\rho^p \mid \rho < \rho^* \}$, while $u \in \{ A_\rho^p \mid \rho^* \le \rho < \rho^p \}$, so that $\eta < \kappa_{\rho^*} \le \kappa_u$. By the definition of the ordering of $\mathbb{R}_w$ we have $C \subseteq B^r$, and since $\kappa_u = \kappa_\rho$ for some $\rho < \rho^p$ in fact $C \subseteq \bigcup \{ A_\rho^p \mid \rho^* \le \rho < \rho^p \}$. In particular as $\eta < \kappa_{\rho^*}$ we have that $C \cap V_{\eta+1} = \emptyset$.

We claim that by shrinking measure one sets appearing in y_2 (if necessary) we may assume that

$$ y' {}^\frown (w, B) = y_0 {}^\frown y_1 {}^\frown y_2 {}^\frown (w, B) \le y_0 {}^\frown y_1 {}^\frown (w, B^p). $$

For pairs (u, C) appearing in y_2 we have that $u \in B^p$ and $C \subseteq B^r = B^p \cup \bigcup \{ A_\rho^p \mid \rho^* \le \rho < \rho^p \}$; since ρ^p is a successor ordinal we have that $\kappa_v \le \kappa_{\rho^p - 1} < \kappa_u$ for all $v \in \bigcup \{ A_\rho^p \mid \rho^* \le \rho < \rho^p \}$, so that we may shrink C to obtain a pair (u, D) with $D \subseteq B^p$.

Since $y_0 {}^\frown y_1$ is harmonious with A^p above η and p is a condition, $y_0 {}^\frown y_1 {}^\frown (w, B^p)$ forces the equivalence $\zeta \dot{E}_\alpha \zeta' \iff \zeta \dot{E}_\beta \zeta'$. This is a contradiction as $y' {}^\frown (w, B)$ is an extension of $y_0 {}^\frown y_1 {}^\frown (w, B^p)$ and forces that the same equivalence fails. $\blacktriangle_{4.20}$

Corollary 4.21. *Let p and q be conditions in $Q(w)$ with $p \le q$ and ρ_p a successor ordinal. Let $\rho^q \le \rho^* < \rho^p$ and let r be the ρ^*-weakening of p. Then $p \le r \le q$.*

Proof. By Lemma (4.20) we already know that $p \le r$ so we need only to check that $r \le q$. It is routine to check that $A^r \upharpoonright \rho^q = A^p \upharpoonright \rho^q = A^q$, $t^r \supseteq t^q$, $f_\alpha^{r,\eta} = f_\alpha^{p,\eta} \supseteq f_\alpha^{q,\eta}$ for all $(\eta, \alpha) \in t^q$, and finally $B^r \cup \bigcup \{ A_\rho^r \mid \rho^q \le \rho < \rho^r \} = B^p \cup \bigcup \{ A_\rho^p \mid \rho^q \le \rho < \rho^p \} \subseteq B^q$. $\blacktriangle_{4.21}$

5. $Q(w)$ HAS THE STATIONARY κ_w^+–CHAIN CONDITION

For the duration of this section we fix an ultrafilter sequence w with $\mathrm{lh}(w) > 1$, and let $\kappa = \kappa_w$. As in the previous section, let $\mathcal{T}$ be a κ^+-Kurepa tree, let $\langle b_\alpha \mid \alpha < \Upsilon \rangle$ be an enumeration of a set of branches through $\mathcal{T}$ and let $\langle \dot{E}_\alpha \mid \alpha < \Upsilon \rangle$ be a list of canonical $\mathbb{R}_w$-names for binary relations on κ^+. Let $Q = Q(w)$.

Notation 5.1. For each $\alpha < \kappa^+$ let $\mathcal{T}_\alpha = \{\, x \in \mathcal{T} \mid \mathrm{lh}(x) = \alpha \,\}$ and let $e_\alpha : \mathcal{T}_\alpha \longrightarrow \overline{\overline{\mathcal{T}_\alpha}}$ be a bijection enumerating $\mathcal{T}_\alpha$.

As $\mathcal{T}$ is a κ^+-Kurepa tree one has for $\alpha < \kappa^+$ that $\overline{\overline{\mathcal{T}_\alpha}} \leq \kappa$.[2]

Notation 5.2. For $\alpha < \beta < \Upsilon$ let $\Delta(\alpha, \beta)$ be the least ζ such that $b_\alpha \restriction \zeta \neq b_\beta \restriction \zeta$.

Notation 5.3. If $p = (A, B', t, f) \in Q$ and there are $t \in [\Upsilon]^{<\kappa}$ and $d \in [\kappa^+]^{<\kappa}$ such that $t = t^\eta$ for all $\eta \in a$ and $d = \mathrm{dom}(f_\alpha^\eta)$ for all $\eta \in a$ and $\alpha \in t$, let Z^p be the set of all tuples $(\eta, y, x, \zeta, \zeta', \delta, \delta')$ where $\eta \in a \cap \sup(a)$, y is a lower part harmonious with A past η, $x \in \mathcal{T}$, $\zeta < \zeta' < \kappa^+$, $\zeta, \zeta' \in d$ and $\delta, \delta' < \kappa$.

Lemma 5.4. *Suppose $p' = (A, B', t, f) \in Q$ and there are $t \in [\Upsilon]^{<\kappa}$ and $d \in [\kappa^+]^{<\kappa}$ such that $t = t^\eta$ for all $\eta \in a$ and $d = \mathrm{dom}(f_\alpha^\eta)$ for all $\eta \in a$ and $\alpha \in t$.*

Then there is some $B \in \mathcal{F}(w)$ with $B \subseteq B'$, so that $p = (A, B, t, f) \leq p'$, and such that whenever $z = (\eta, y, x, \zeta, \zeta', \delta, \delta') \in Z^p$ and $\alpha \in t$ is such that $x = b_\alpha \restriction \zeta'$, $f_\alpha^\eta(\zeta) = (x \restriction \zeta, \delta)$ and $f_\alpha^\eta(\zeta') = (x, \delta')$ there is an $\mathbb{R}_y$-name $\dot\sigma_z$ such that $y^\frown(w, B)$ forces that $\dot\sigma_z$ is the truth value of the assertion that $\zeta \dot E_\alpha \zeta'$.

Proof. First of all notice that $Z^p = Z^{p'}$. For each $z \in Z^p$ we may define an $\mathbb{R}_w$-name $\dot\tau_z$ such that for all $\alpha \in t$ with $f_\alpha^{\eta,i}(\zeta) = (x \restriction \zeta, \delta)$ and $f_\alpha^{\eta,i}(\zeta') = (x, \delta')$, $y^\frown(w, \overline{B}^i)$ forces that $\dot\tau_z$ is the truth value of the assertion that $\zeta \dot E_\alpha \zeta'$. (Condition (5) in the definition of being a condition shows $\dot\tau_z$ does not depend on α.)

By Lemma (3.19), for each $z \in Z^p$ there is a set $B_z \subseteq \overline{B}$ such that $y^\frown(w, B_z)$ reduces $\dot\tau_z$ to an $\mathbb{R}_y$-name $\dot\sigma_z$. As $\overline{\overline{Z^p}} < \kappa$ any $B \in \mathcal{F}(w)$ such that $B \subseteq \bigcap_{z \in Z^p} B_z$ suffices.   ▲5.4

Proposition 5.5. *Q has the stationary κ^+-chain condition.*

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[2]We remark that we use Kurepa trees in the definition of the $Q(w)$, rather than arbitrary κ^+-trees with Υ-many branches, because the fact that each $\mathcal{T}_\alpha$ has size at most κ saves us a little work in the proof of the κ^+-stationary chain condition given in this section – see the definitions of the $g^{\eta,i}$ and $X^{\eta,i}$ and their roles in the definition of the function h below. In [3] an analogous chain condition argument is given without this nicety.

Proof. Let $\{p'^i \,|\, i < \kappa^+\} \in [Q]^{\kappa^+}$. Let $p'^i = (A^i, B'^i, t'^i, f'^i)$ for each $i < \kappa^+$, write a^i for a^{p^i}, ρ^i for ρ^{p^i}, and for each $\rho < \rho^i$ write κ^i_ρ for $(\kappa_\rho)^{p^i}$.

We start by tidying up, for each $i < \kappa^+$, the collection of domains of constituents of the f'^i. We want to emphasize that this step in the argument is not strictly necessary. At the cost of a more elaborate case analysis below, proofs of the stationary chain condition can be given which do *not* rely on the forcing having the property of having greatest lower bounds for descending sequences of length less than κ, a property which we do use in carrying out this tidying up.

By applying Lemma (4.18) and then Lemma (5.4) we can find $\{p^i \,|\, i < \kappa^+\} \in [Q]^{\kappa^+}$ such that for each $i < \kappa^+$ we have $p^i = (A^i, B^i, t^i, f^i) \leq p'^i$ and there is some $t^i \in [\Upsilon]^{<\kappa}$ and some $d^i \in [\kappa^+]^{<\kappa}$ such that

- $t^i = t^{\eta,i}$ for all $\eta \in a^i$,

- $d^i = \mathrm{dom}(f^{\eta,i}_\alpha)$ for all $\eta \in a^i$ and $\alpha \in t^i$, and

- B^i has the property stated in the conclusion of Lemma (5.4).

Now we make a plethora of auxiliary definitions for each i, set out in the table below.

Let $\{\alpha_\gamma \,|\, \gamma < \gamma^*\}$ be an enumeration of $\bigcup\{t^i \,|\, i < \kappa^+\}$, for some $\gamma^* \leq \kappa^+$. Let $\{\alpha^i_\gamma \,|\, \gamma < \gamma^i\}$ be the increasing enumeration of t^i, for some $\gamma^i < \kappa$, for each $i < \kappa^+$.

Next, for each $i < \kappa^+$ and $\eta \in a^i$ let

- $\theta^i_0 = \mathrm{ssup}(\{\gamma \,|\, \alpha_\gamma \in t^i\})$, $\theta^i_1 = \mathrm{ssup}(d^i)$,

- $\Gamma^i = \{\Delta(\alpha, \alpha') \,|\, \alpha,\, \alpha' \in t^i\}$, $\nu^i = \mathrm{ssup}(\Gamma^i)$,

- $T^i = \{\gamma < i \,|\, \alpha_\gamma \in t^i\}$, $\Delta^i = \Gamma^i \cap i$, $D^i = d^i \cap i$,

- $Y^{\eta,i} = \{y \,|\, y \text{ is a lower part harmonious with } A^i \text{ past } \eta\}$.

- $g^{\eta,i}(\zeta, \gamma) = (e_\zeta(\pi_0(f^{\eta,i}_{\alpha_\gamma}(\zeta))), \pi_1(f^{\eta,i}_{\alpha_\gamma}(\zeta)))$ for $\gamma \in T^i$ and $\zeta \in D^i$.

- $X^{\eta,i} = \{\langle y, \varepsilon, \zeta, \zeta', F^\eta_\varepsilon, \dot\sigma \rangle \,|\, y \in Y^{\eta,i} \ \& \ \varepsilon < \gamma^i \ \& \ \zeta,\, \zeta' \in D^i \ \& \ \zeta < \zeta' \ \& \text{ on setting } \alpha = \alpha^i_\varepsilon \text{ one has } F^\eta_\varepsilon : D^i \longrightarrow \kappa \times \kappa \text{ is given by} F^\eta_\varepsilon(\zeta^*) = (e_{\zeta^*}(\pi_0(f^{\eta,i}_\alpha(\zeta^*))), \pi_1(f^{\eta,i}_\alpha(\zeta^*))) \text{ for } \zeta^* \in D^i \ \& \text{ letting } x = b_\alpha \restriction \zeta', f^{\eta,i}_\alpha(\zeta) = (x \restriction \zeta, \delta) \text{ and } f^{\eta,i}_\alpha(\zeta') = (x, \delta'), \text{ we have } \dot\sigma = \dot\sigma^i_{(y,x,\eta,\zeta,\zeta',\delta,\delta')} \}$.

Recall that π_l is projection onto the lth co-ordinate, so in the definition of $g^{\eta,i}$, for example, $\pi_0(b_{\alpha_\gamma} \restriction \zeta, \nu) = b_{\alpha_\gamma} \restriction \zeta$ and $\pi_1(b_{\alpha_\gamma} \restriction \zeta, \nu) = \nu$, while, as per Notation (5.1), e_ζ is the function enumerating the ζth level of the κ^+-Kurepa tree $\mathcal{T}$, so that $e_\zeta(b_{\alpha_\gamma} \restriction \zeta) \in \kappa$. Thus $g^{\eta,i} : D^i \times T^i \longrightarrow \kappa \times \kappa$ and $\mathrm{dom}(g^{\eta,i}(.,\gamma)) = D^i$ for $\gamma \in T^i$.

Notice, also, that the various F_ε^η, $\dot\sigma$, x, δ and δ' appearing in the definition of $X^{\eta,i}$ are uniquely determined by i, η and the $\langle y, \varepsilon, \zeta, \zeta' \rangle$.

For the convenience of the reader we record the types of the objects which we have just defined. $\alpha_\gamma \in \Upsilon$ for $\gamma < \gamma^* \leq \kappa^+$, $\alpha_\gamma^i \in \Upsilon$ for $\gamma < \gamma^i < \kappa$, $\theta_0^i < \kappa^+$, $\Gamma^i \in [\kappa^+]^{<\kappa}$, $\nu^i < \kappa^+$, $\theta_1^i < \kappa^+$, $T^i \in [i]^{<\kappa}$, $\Delta^i \in [i]^{<\kappa}$, $D^i \in [i]^{<\kappa}$, $Y^{\eta,i} \in V_\kappa$, and $g^{\eta,i}$ is a partial function of size less than κ from $i \times i$ to $\kappa \times \kappa$.

We next argue similarly to the proof of Claim (1.3) and define a regressive function on a set which is club relative to $S_\kappa^{\kappa^+}$. Recall from the section on notation that if I and J are sets and κ is a cardinal then $\mathrm{Fn}(I, J, \kappa)$ is the set of partial functions from I to J of size less than κ.

Let h^* be an injection from $V_\kappa \times ([\kappa^+]^{<\kappa} \times \mathrm{Fn}(\kappa^+ \times \kappa^+, \kappa \times \kappa, \kappa) \times [\kappa^+]^{<\kappa} \times [\kappa^+]^{<\kappa} \times \kappa \times (V_\kappa \times \kappa \times \kappa^+ \times \kappa^+ \times V_\kappa \times \mathrm{Fn}(\kappa^+, \kappa \times \kappa, \kappa))^{<\kappa}) \times \kappa$ into κ^+. Let us write H for the domain of h^*.

For $\kappa \leq i < \kappa^+$ let H_i be defined similarly to H with i in place of κ^+: $H_i = V_\kappa \times ([\tau]^{<\kappa} \times \mathrm{Fn}(\tau \times \tau, \kappa \times \kappa, \kappa) \times [\tau]^{<\kappa} \times [\tau]^{<\kappa} \times \kappa \times (V_\kappa \times \kappa \times \tau \times \tau \times V_\kappa \times \mathrm{Fn}(\tau, \kappa \times \kappa, \kappa))^{<\kappa}) \times \kappa$.

Define $k : [\kappa, \kappa^+) \longrightarrow \kappa^+$ by $k(i)$ is the least $i^* < \kappa^+$ such that $H_i \subseteq h^{*-1}``\tau^*$.

Let $\widetilde{C} = \{ j < \kappa^+ \mid \forall i < j \, (\theta_0^i, \theta_1^i, \nu^i, k(i) < j) \}$.

As the intersection of the sets of closure points of the four given functions, $\widetilde{C}$ is a club subset of κ^+.

Let $h(i) = h^*(A^i, \langle (T^i, g^{\eta,i}, D^i, \Delta^i, \mathrm{otp}(t^i), X^{\eta,i}) \mid \eta \in a^i \rangle, \overline{\kappa}^i)$ for $i \in \widetilde{C} \cap S_\kappa^{\kappa^+}$ and $h(i) = 0$ otherwise.

We have that $h^{*-1}(h(i)) \in H_i$ for all $i \in [\kappa, \kappa^+)$. If $i \in \widetilde{C} \cap S_\kappa^{\kappa^+}$, since $\overline{\overline{h^{*-1}(h(i))}} < \kappa$, there is some $i' < i$ such that $h^{*-1}(h(i)) \in H_{i'}$, and hence there is some $\widetilde{i} < i$ such that $h(i) < k(\widetilde{i})$.

Hence, as i is (amongst other things) a closure point of k, we have $h(i) < i$ for all nonzero $i < \kappa^+$.

Now suppose that $i, j \in \widetilde{C} \cap S_\kappa^{\kappa^+}$, $i < j$, and $h(i) = h(j)$. In particular we have that $A^i = A^j$ (and hence $a^i = a^j$) and $\overline{\kappa}^i = \overline{\kappa}^j$, that θ_0^i, θ_1^i, $\nu^i < j$, and finally for all $\eta \in a^i$

$$\langle T^i, g^{\eta,i}, D^i, \Delta^i, \mathrm{otp}(t^i), X^{\eta,i} \rangle = \langle T^j, g^{\eta,j}, D^j, \Delta^j, \mathrm{otp}(t^j), X^{\eta,j} \rangle.$$

We prove a series of lemmas which together describe the common parts of p^i and p^j. Let A be the common value of A^i and A^j and a that of a^i and a^j.

Lemma 5.6. (a) $(t^j \setminus t^i) \cap \{\alpha_\gamma \mid i \le \gamma < j\} = \emptyset$, and (b) $t^i \cap t^j \subseteq \{\alpha_\gamma \mid \gamma < i\}$.

Proof. Suppose $\alpha_\gamma \in t^j$. If $\gamma < j$ then $\gamma \in T^j$. But $T^j = T^i$, so $\gamma \in T^i$. Hence $\gamma < i$ and $\alpha_\gamma \in t^i$, proving (a). If $\alpha_\gamma \in t^i$ then $\gamma < \theta_0^i < j$. (For the definition of θ_0^i immediately gives that $\gamma < \theta_0^i$; and since $i, j \in \widetilde{C}$ and $i < j$ one has that $\theta_0^i < j$.) Thus if $\alpha_\gamma \in t^i \cap t^j$ we have $\gamma < i$ by (a). So (b) holds. $\blacktriangle_{5.6}$

Lemma 5.7. $d^i \cap d^j \subseteq i$.

Proof. If $\zeta \in d^i \cap d^j$ then $\zeta < \theta_1^i < j$. So $\zeta \in d^j \cap j = D^j = D^i = d^i \cap i$. $\blacktriangle_{5.7}$

Lemma 5.8. If $\eta \in a$ and $\alpha \in t^i \cap t^j$ then $f_\alpha^{\eta,i} \restriction d^i \cap d^j = f_\alpha^{\eta,j} \restriction d^i \cap d^j$.

Proof. Let $\eta \in a$ and $\alpha \in t^i \cap t^j$. By Lemma (5.6), if $\alpha = \alpha_\gamma$ then $\gamma < i$. By Lemma (5.7), $d^i \cap d^j \subseteq i$, and hence $d^i \cap d^j \subseteq i \cap d^i = D^i = D^j$, the first equality being the definition of D^i and the second holding since $h(i) = h(j)$.

As unpacked in the paragraph starting "Recall ..." immediately after the bullet-pointed table of definitions above, $D^i = \mathrm{dom}(g^{\eta,i}(.,\gamma))$ and $D^j = \mathrm{dom}(g^{\eta,j}(.,\gamma))$, and hence $\mathrm{dom}(g^{\eta,i}(.,\gamma)) = \mathrm{dom}(g^{\eta,j}(.,\gamma))$.

As $\alpha \in t^i \cap t^j$, if $\zeta \in d^i \cap d^j$ we have $e_\zeta(\pi_0(f_\alpha^{\eta,i}(\zeta))) = e_\zeta(\pi_0(f_\alpha^{\eta,j}(\zeta)))$ and, as e_ζ is a bijection, $\pi_0(f_\alpha^{\eta,i}(\zeta)) = \pi_0(f_\alpha^{\eta,j}(\zeta)) = b_\alpha \restriction \zeta$. Hence we have the claimed agreement. $\blacktriangle_{5.8}$

Recall that $f^i + f^j$ is the unique function f such that $\mathrm{dom}(f) = \mathrm{dom}(f^i) \cup \mathrm{dom}(f^j)$, $f \restriction \mathrm{dom}(f^i) = f^i$ and $f \restriction \mathrm{dom}(f^j) = f^j$.

Let $q = (A, B^i \cap B^j, t^i \cup t^j, f^i + f^j)$.

Lemma 5.9. *The quadruple q is a condition in Q^*.*

Proof. Recall that we defined $A = A^i = A^j$. Lemma (5.8) shows that f^i and f^j are compatible. $\blacktriangle_{5.9}$

Lemma 5.10. $\Gamma^i \cap \Gamma^j \subseteq i$.

Proof. If $\zeta \in \Gamma^i \cap \Gamma^j$ then $\zeta < \nu^i < j$ (the latter since $i, j \in \widetilde{C}$ and $i < j$). So $\zeta \in \Delta^j$. But $\Delta^j = \Delta^i$, so $\zeta < i$. $\blacktriangle_{5.10}$

Notation 5.11. For $\alpha \in t^i \cup t^j$ define $d^q_\alpha = d^i \cup d^j$ if $\alpha \in t^i \cap t^j$, $d^q_\alpha = d^i$ if $\alpha \in t^i \setminus t^j$, and $d^q_\alpha = d^j$ if $\alpha \in t^j \setminus t^i$.

Lemma 5.12. *For $\alpha \in t^i \cup t^j$ and $\eta \in a$ we have $d^q_\alpha = \mathrm{dom}(f^{\eta,q}_\alpha)$.*

Proof. Immediate from the definition of q, specifically the definition of f^q as $f^i + f^j$, the fact that $\mathrm{dom}(f^{\eta,i}_\alpha) = d^i$ for all $\eta \in a^i$ and $\alpha \in t^i$ – see the third paragraph of the proof of Proposition (5.5), and the definition by cases of the notation d^q_α just given. $\blacktriangle_{5.12}$

Lemma 5.13. *Suppose $\alpha, \beta \in t^i \cap t^j$, $\zeta, \zeta' \in d^q_\alpha \cap d^q_\beta$, $\zeta < \zeta'$, $\eta \in a$ and $f^{\eta,q}_\alpha(\zeta) = f^{\eta,q}_\beta(\zeta) \neq f^{\eta,q}_\alpha(\zeta') = f^{\eta,q}_\beta(\zeta')$. Then $\zeta, \zeta' \in d^i \cap d^j$.*

Proof. Let $\alpha = \alpha_\gamma$ and $\beta = \alpha_{\gamma'}$. Then $\gamma, \gamma' < i$ by Lemma (5.6) and $\Delta(\alpha, \beta) \in \Gamma^i \cap \Gamma^j \subseteq i$, by Lemma (5.10). As $b_\alpha \upharpoonright \zeta' = \pi_0(f^{\eta,q}_\alpha(\zeta')) = \pi_0(f^{\eta,q}_\beta(\zeta')) = b_\beta \upharpoonright \zeta'$ we have $\zeta' < \Delta(\alpha, \beta)$ and hence $\zeta, \zeta' < i$. Thus $\zeta' \in d^j \cap i \subseteq D^j = D^i \subseteq d^i$. Similarly $\zeta \in d^j$. $\blacktriangle_{5.13}$

Finally, we can now show that q satisfies (5) of the definition of Q.

Lemma 5.14. *Suppose $\alpha, \beta \in t^i \cup t^j$, $\zeta, \zeta' \in d^q_\alpha \cap d^q_\beta$, $\eta \in a$ and $f^{\eta,q}_\alpha(\zeta) = f^{\eta,q}_\beta(\zeta) \neq f^{\eta,q}_\alpha(\zeta') = f^{\eta,q}_\beta(\zeta')$ and y is harmonious with A^q past η. Then $y^\frown(w, B^q) \Vdash_{\mathbb{R}_w} " \zeta \dot{E}_\alpha \zeta' \iff \zeta \dot{E}_\beta \zeta' "$.*

Proof. If for some $k \in \{i, j\}$ we have $\alpha, \beta \in t^k$ then, using Lemma (5.13) if $\alpha, \beta \in t^i \cap t^j$, $\zeta, \zeta' \in d^k$. As $p^k \in Q$ we have $y^\frown(w, B^k) \Vdash$ " $\zeta \dot{E}_\alpha \zeta' \iff \zeta \dot{E}_\beta \zeta' "$. But $B^q = B^i \cap B^j \subseteq B^k$, so $y^\frown(w, B^q) \leq y^\frown(w, B^k)$ and $y^\frown(w, B^q) \Vdash$ " $\zeta \dot{E}_\alpha \zeta' \iff \zeta \dot{E}_\beta \zeta' "$.

Otherwise, either we have $\alpha \in t^i \setminus t^j$, $\beta \in t^j \setminus t^i$ and $\zeta, \zeta' \in d^i \cap d^j$, or we have the symmetric case with the roles of i and j exchanged. We treat the former; for the symmetric case exchange i and j throughout.

Suppose β is the ε-th element of t^j. As $\mathrm{otp}(t^i) = \mathrm{otp}(t^j)$, we can define β' to be the ε-th element of t^i. As $X^{\eta,i} = X^{\eta,j}$ we have that the 'F^η_ε' for the tuple that starts $\langle y, \varepsilon, \zeta, \zeta', \dots \rangle$ is the same for both i and j. Hence $e_{\zeta'}(\pi_0(f^{\eta,i}_{\beta'}(\zeta'))) = e_{\zeta'}(\pi_0(f^{\eta,j}_\beta(\zeta')))$. As $e_{\zeta'}$ is a bijection and π_0

is projection onto the first co-ordinate, this gives $b_\beta \restriction \zeta' = b_{\beta'} \restriction \zeta'$. Let $x = b_\beta \restriction \zeta' = b_{\beta'} \restriction \zeta'$, $f_{\beta'}^{\eta,i}(\zeta) = f_\beta^{\eta,j}(\zeta) = (x \restriction \zeta, \delta)$, $f_{\beta'}^{\eta,i}(\zeta') = f_\beta^{\eta,j}(\zeta') = (x, \delta')$, and, finally, $z = (\eta, y, x, \zeta, \zeta', \delta, \delta')$.

Then $z \in Z^{p^i} \cap Z^{p^j}$, and $\dot\sigma_z^i = \dot\sigma_z^j = \dot\sigma$ say, where $\dot\sigma$ is an $\mathbb{R}_y$-name for a truth value. Since $B^q = B^i \cap B^j$, $y^\frown(w, B^q)$ simultaneously reduces the truth values of the statements "$\zeta \dot{E}_\beta \zeta'$" and "$\zeta \dot{E}_{\beta'} \zeta'$" to $\dot\sigma$, so $y^\frown(w, B^q) \Vdash$ "$\zeta \dot{E}_{\beta'} \zeta' \Longleftrightarrow \zeta \dot{E}_\beta \zeta'$". If $\beta' = \alpha$ we are done. Otherwise, as $p^i \in Q$ we also have that $y^\frown(w, B^i) \Vdash$ "$\zeta \dot{E}_\alpha \zeta' \Longleftrightarrow \zeta \dot{E}_{\beta'} \zeta'$", and hence $y^\frown(w, B^q) \Vdash$ "$\zeta \dot{E}_\alpha \zeta' \Longleftrightarrow \zeta \dot{E}_\beta \zeta'$", as required.   ▲$_{5.14}$

Lemma (5.9) showed $q \in Q^*$ and Lemma (5.14) shows that in fact $q \in Q$ as well. Since by construction $q \leq p^i$, p^j, and both $p^i \leq p'^i$ and $p^j \leq p'^j$, we have shown that there is a suitable function h such that if $i, j \in \widetilde{C} \cap S_\kappa^{\kappa^+}$, $i < j$ and $h(i) = h(j)$ then p'^i and p'^j are compatible. Hence Q has the κ^+-stationary chain condition.

6. THE MAIN ITERATION

Let V be a model in which κ is supercompact, λ is a regular cardinal less than κ, and χ is a successor cardinal with predecessor χ^- such that $\mathrm{cf}(\chi^-) \geq \kappa^{++}$. Suppose also that GCH holds in V and hence, by [25], $\diamondsuit_\chi(S_{\kappa^+}^\chi)$ holds.

Let $\mathbb{L}$ be the Laver iteration ([16]), as defined in V, making the supercompactness of κ indestructible under $<\kappa$-directed closed forcing.

Let $\dot{e}$ be a canonical $\mathbb{L}$-name such that $\Vdash_\mathbb{L}$ "$\dot{e} : \kappa \longrightarrow V_\kappa$ is a bijection."

Using Theorem (1.2), Lemma (1.5), Proposition (2.3), Lemma (4.16) and Proposition (5.5), we will define an $\mathbb{L}$-name $\dot{\mathbb{P}}_\chi$ so that $\Vdash_\mathbb{L}$ "$\dot{\mathbb{P}}_\chi$ is an iteration of length χ consisting of $\dot{\mathbb{Q}}_0$, a name for the usual forcing to add a κ^+-Kurepa tree with χ^--many branches followed by a $<\kappa$-support iteration of $<\kappa$-directed closed, countably parallel-closed κ^+-stationary cc forcings each of size less than χ." Thus the iteration will add χ-many subsets of κ, but at each intermediate stage $2^\kappa = \chi^-$. The constituents of the iteration will depend on a fixed sequence $\langle \dot{x}_\alpha \mid \alpha < \chi^- \rangle$ of $\mathbb{L} * \dot{\mathbb{Q}}_0$-names for distinct branches through the κ^+-Kurepa tree. As we define the iteration we also build an enumeration of $\mathbb{L} * \dot{\mathbb{P}}_\chi$ as $\langle p_\xi \mid \xi < \chi \rangle$. As χ is a regular cardinal and since $\Vdash_\mathbb{L}$ "the iterands are of size less than χ" there will be a club set relative to $S_{\geq\kappa}^\chi$ of $\xi < \chi$ such that $\mathbb{L} * \dot{\mathbb{P}}_\xi = \{p_\varepsilon \mid \varepsilon < \xi\}$. (Formally, such that

for all $\varepsilon < \xi$ we have $\mathrm{supp}(p_\varepsilon) \subseteq \xi$ and $\mathbb{L} * \dot{\mathbb{P}}_\xi = \{p_\varepsilon \restriction \xi \mid \varepsilon < \xi\}$. See the proof of Proposition (2.3).) We simultaneously inductively define $\mathbb{L} * \dot{\mathbb{P}}_\xi$-names $\dot{S}_\xi$ as in §2 and derive $\mathbb{L} * \dot{\mathbb{P}}_\xi$-names $\dot{u}^\xi$.

As each $\dot{S}_\xi$ is a canonical name for a subset of ξ, when $\xi = \kappa\lambda\xi$ (ordinal multiplication) we can easily convert it into a name for a set of order type λ of sequences of ξ many subsets of κ. In order to do this, for each $\tau < \lambda$, set $\dot{u}^\xi_{1+\tau}$ to be the name derived from $\dot{S}_\xi$ for $\{\dot{e}\text{``}\{\eta < \kappa \mid \kappa\lambda\varepsilon + \kappa\tau + \eta \in \dot{S}_\xi \cap [\kappa\lambda\varepsilon + \kappa\tau, \widehat{\kappa\lambda\varepsilon + \kappa(\tau+1)})\}\mid \varepsilon < \xi\}$ and set $\dot{u}^\xi = \hat{\kappa}^\frown \langle \dot{u}^\xi_{1+\tau} \mid 1 + \tau < \lambda \rangle$.

We use the sequence $\dot{u}^\xi$ to help define the next stage in the iteration. Let $\dot{\mathcal{U}}^\xi$ be a $\mathbb{L} * \dot{\mathbb{P}}_\xi$-name for the class of all ultrafilter sequences. (As for each $\xi < \chi$ we have $\Vdash_{\mathbb{L}}$ " $\dot{\mathbb{P}}_\xi$ is $<\kappa$-directed closed ", we will have for all $\xi < \xi' \leq \chi$ that $\Vdash_{\mathbb{L}*\mathbb{P}_{\xi'}} \dot{\mathcal{U}}^\xi_\kappa = \dot{\mathcal{U}}^{\xi'}_\kappa$.)

If $\mathrm{cf}(\xi) = \kappa^+$, $\mathbb{L} * \dot{\mathbb{P}}_\xi = \{p_\varepsilon \mid \varepsilon < \xi\}$, $\xi = \kappa\lambda\xi$ and $\Vdash_{\mathbb{L}*\mathbb{P}_\xi}$ "$\dot{u}^\xi \in \dot{\mathcal{U}}^\xi$ and $\dot{\kappa}_{\dot{u}^\xi} = \kappa$" let $\langle \dot{E}^\xi_\alpha \mid \alpha < \chi^- \rangle$ enumerate the canonical $\mathbb{L} * \dot{\mathbb{P}}_\xi * \dot{\mathbb{R}}_{\dot{u}^\xi}$-names for graphs on κ^+ and let $\dot{\mathbb{Q}}_\xi = \dot{Q}(\dot{u}^\xi)$, where $\dot{Q}(\dot{u}^\xi)$ is an $\mathbb{L} * \dot{\mathbb{P}}_\xi$-name for the forcing defined from $\dot{u}^\xi$, $\langle \dot{x}_\alpha \mid \alpha < \chi^- \rangle$ and $\langle \dot{E}^\xi_\alpha \mid \alpha < \chi^- \rangle$ as in §4. Otherwise let $\dot{\mathbb{Q}}_\xi$ name trivial forcing.

Fix G which is $\mathbb{L}$-generic over V and H which is $\dot{\mathbb{P}}^G_\chi$-generic over $V[G]$. For $\xi < \chi$ let H_ξ be the restriction of H to $\dot{\mathbb{P}}^G_\xi$, let $\mathbb{Q}_\xi = \dot{\mathbb{Q}}^{G*H_\xi}_\xi$, and let $u^\xi = (\dot{u}^\xi)^{G*H_\xi}$.

If $V[G][H_\xi] \models u^\xi \in \mathcal{U}$, $\kappa_{u^\xi} = \kappa$ and $\mathrm{lh}(u^\xi) = \lambda$, let

- K_ξ be the $Q(u^\xi)$-generic over $V[G][H_\xi]$ induced by H,

- $A^\xi = \langle A_\rho \mid \rho < \kappa \ \& \ \exists p \in K_\xi \ A_\rho = (A_\rho)^p \rangle$,

- and $a^\xi = \bigcup \{a^p \mid p \in K_\xi\}$.

Fix an enumeration $\langle \dot{D}_\xi \mid \xi < \chi \rangle$ of the $\mathbb{L} * \dot{\mathbb{P}}_\chi$-names for subsets of V_κ such that each $\dot{D}_\xi$ is a $\mathbb{L} * \dot{\mathbb{P}}_\xi$-name.

Let $j : V \longrightarrow M$ witness that κ is 2^χ-supercompact, such that in the iteration $j(\mathbb{L})$ we force with $\dot{\mathbb{P}}_\chi$ at stage κ and then do trivial forcing at all stages between κ and $(2^\chi)^+$. Let $j(\mathbb{L}) = \mathbb{L} * \dot{\mathbb{P}}_\chi * \dot{L}$, where $\dot{L}$ is a $\mathbb{L} * \dot{\mathbb{P}}_\chi$-name and we note that by the choice of j we have $\Vdash_{\mathbb{L}*\dot{\mathbb{P}}_\chi}$ " $\dot{L}$ is $<(2^\chi)^+$-closed. "

We now carry out, in $V[G][H]$, an inductive construction of length χ in which we build a chain of conditions $(r_\xi, \dot{q}_\xi) \in \dot{\mathcal{L}}^{G*H} * j(\dot{\mathbb{P}}_\chi)$ for $\xi < \chi$. Note that forcing with $\dot{\mathcal{L}}^{G*H}$ over $V[G][H]$ always adds a generic embedding $j : V[G] \longrightarrow M[j(G)]$ – see [2, §9] – so that the name $j(\dot{\mathbb{P}}_\chi)$ is well defined. The construction includes arranging for each $\xi < \chi$ that $\Vdash_{\dot{\mathcal{L}}^{G*H}}$ " $\dot{q}_\xi \in j(\dot{\mathbb{P}}_\xi)$ " and that for all $p \in H_\xi$ we have that $r_\xi \Vdash_{\dot{\mathcal{L}}^{G*H}}$ " $\dot{q}_\xi \leq j(p)$. "

Define $\mathfrak{U} = (\{v \in \mathcal{U} \mid \kappa_v = \kappa \,\&\, \mathrm{lh}(v) \leq \lambda\})^{V[G][H]}$. As $V[G][H] \models \overline{\overline{2^\kappa}} = \chi$, χ is a successor cardinal and $\lambda < \kappa$, we have that $V[G][H] \models \overline{\overline{\mathfrak{U}}} \leq ((2^{2^\kappa})^\lambda) = 2^\chi$.

Inductive case: $\xi = 0$ *or limit* ξ. By the $<\chi$-closure of $\dot{\mathcal{L}}^{G*H} * j(\dot{\mathbb{P}}_\chi)$, just choose some $(r_\xi, \dot{q}_\xi)$ such that $\Vdash_{\dot{\mathcal{L}}^{G*H}}$ " $\dot{q}_\xi \in j(\dot{\mathbb{P}}_\xi)$ ", and for all $\xi' < \xi$ we have $(r_\xi, \dot{q}_\xi) \leq (r_{\xi'}, \dot{q}_{\xi'})$. Then for each $p \in H_\xi$ we have $r_\xi \Vdash_{\dot{\mathcal{L}}^{G*H}}$ " $\dot{q}_\xi \leq j(p)$. "

Inductive case: $\xi + 1$. If $\xi = 0$ then $\mathbb{Q}_0$ is the usual forcing to add a κ^+-Kurepa tree with χ^--many branches as computed in $V[G]$, and we set r_1 to be the trivial condition and $\dot{q}_1$ to be a name for the greatest lower bound of the union of the pointwise image of the $\mathbb{Q}_0$-generic filter. We now assume that $\xi > 0$.

If $V[G][H_\xi] \models u^\xi \notin \mathcal{U}$ or $u^\xi \in \mathcal{U}$ but $\kappa_{u^\xi} \neq \kappa$ or $u^\xi \in \mathcal{U}$ and $\kappa_{u^\xi} = \kappa$ but $\mathrm{lh}(u^\xi) \neq \lambda$ there is nothing to do and we can take $(r_{\xi+1}, \dot{q}_{\xi+1}) = (r_\xi, \dot{q}_\xi ^\frown \dot{\mathbb{1}}_{j(\mathbb{Q}_\xi)})$. So assume otherwise: $V[G][H_\xi] \models u^\xi \in \mathcal{U}$, $\kappa_{u^\xi} = \kappa$ and $\mathrm{lh}(u^\xi) = \lambda$.

By the construction so far $(r_\xi, \dot{q}_\xi) \Vdash_{\dot{\mathcal{L}}^{G*H} * j(\dot{\mathbb{P}}_\xi)}$ " there is a lifting $j : V[G][H_\xi] \longrightarrow M[j(G)][j(H_\xi)]$ of j."

As 2^χ is less than the closure of $\dot{\mathcal{L}}^{G*H} * j(\dot{\mathbb{P}}_\chi)$ we may, by shrinking if necessary, assume that there is some $(r'_\xi, \dot{q}'_\xi) \leq (r_\xi, \dot{q}_\xi)$ such that $(r'_\xi, \dot{q}'_\xi) \parallel$ "$v \in j(\dot{D}_\xi)$" for every $v \in \mathfrak{U}$.

Definition 6.1. If $v \in \mathcal{U}^{V[G][H]}$, $\mathrm{lh}(v) = \lambda$ and for all $\tau < \lambda$ we have $v_\tau \cap V[G][H_\xi] = u^\xi_\tau$ we say v *fills out* u^ξ.

Note that if v *fills out* u^ξ then, a priori, $v \in \mathfrak{U}$.

Lemma 6.2. *Suppose there is a lifting* $j : V[G][H_\xi] \longrightarrow M[j(G)][j(H_\xi)]$ *of* j. *Set* $\overline{B} = \bigcap \{j(B^q) \mid (A^q, B^q, t^q, f^q) \in K_\xi\}$. *Suppose there is some* $v \in \mathcal{U}^{V[G][H]}$ *which fills out* u^ξ *and for all* $\tau < \lambda$ *we have* $v \restriction \tau \in \overline{B}$.

Then there is a master condition for $j(\mathbb{Q}_\xi)$ such that on forcing below the master condition there is a lifting of j to a map $j : V[G][H_\xi][K_\xi] \longrightarrow M[j(G)][j(H_\xi)][j(K_\xi)]$ such that for every $v \in \mathcal{U} \cap \overline{B}$ filling out u^ξ and every $\tau < \lambda$ we have $v \upharpoonright \tau \in j(A^\xi)_\kappa$.

Proof. We construct a suitable master condition $p^* = (A^*, B^*, t^*, f^*)$.

- $A^* = A^\xi {}^\frown \{v \upharpoonright \tau \mid \tau < \lambda \ \& \ v \in \overline{B} \cap (\mathcal{U}_{\kappa^+} \setminus \mathcal{U}_\kappa) \ \& \ v \text{ fills out } u^\xi\}$.

- $B^* = \overline{B} \setminus \mathcal{U}_{\kappa^+}$.

- $t^* = \kappa \times j``\chi^-$.

- For each $\eta \in a^\xi$ and $\alpha < \chi^-$, $f^{*,\eta}_{j(\alpha)} = \bigcup\{j(f^{p,\eta}_\alpha) \mid p \in K_\xi\}$.

We note that since $f^{p,\eta}_\alpha$ is a partial function of size less than κ, and we have the density lemmas Lemma (4.8) and Corollary (4.13), it is easy to see that $d^{*,\eta}_{j(\alpha)} = \mathrm{dom}(f^{*,\eta}_{j(\alpha)}) = j``\kappa^+$ and $f^{*,\eta}_{j(\alpha)}(j(\zeta)) = j(f^{p,\eta}_\alpha(\zeta))$ for any $p \in K_\xi$ such that $(\eta, \alpha) \in t^p$. We also note that $a^{p^*} = a^\xi \cup \{\kappa\}$.

It is routine to verify that $p^* \in j(\mathbb{Q}^*_\xi)$ and that $p^* \leq_{j(\mathbb{Q}^*_\xi)} j(q)$ for $q \in K_\xi$. We must now show that p^* satisfies (5) in the definition of $j(\mathbb{Q}_\xi)$.

Let $\eta \in a^\xi$, and suppose that $f^{*,\eta}_{j(\alpha)}(j(\zeta)) = f^{*,\eta}_{j(\beta)}(j(\zeta)) \neq f^{*,\eta}_{j(\alpha)}(j(\zeta')) = f^{*,\eta}_{j(\beta)}(j(\zeta'))$ for some $\alpha, \beta \in \chi^-$ and $\zeta, \zeta' \in \kappa^+$. Let y be harmonious with A^* past η. If $\kappa_y < \kappa$ let $y' = y$. Otherwise let $y', \tau < \lambda$, $v \in B^*$ which fills out u^ξ, and $B \in \mathcal{F}(v \upharpoonright \tau)$ be such that $y = y'{}^\frown(v \upharpoonright \tau, B)$.

Using Corollary (4.10) and the definition of p^* we may find $p \in K_\xi$ with ρ^p a successor ordinal such that

- $\eta \in a^p$ with $\eta < \max(a^p)$,

- $\kappa_{y'} < \max(a^p)$,

- $(\eta, \alpha), (\eta, \beta) \in t^p$,

- $\zeta, \zeta' \in d^{p,\eta}_\alpha \cap d^{p,\eta}_\beta$,

- $f^{p,\eta}_\alpha(\zeta) = f^{p,\eta}_\beta(\zeta) \neq f^{p,\eta}_\alpha(\zeta') = f^{p,\eta}_\beta(\zeta')$,

- and y' is harmonious with A^p past η.

If $y = y'$ we have that y is harmonious with A^p past η and

$$y{}^\frown(j(u^\xi), B^*) \leq y{}^\frown(j(u^\xi), j(B^p)) = j(y{}^\frown(u^\xi, B^p)).$$

As p is a condition in $\mathbb{Q}_\xi$ we have $y{}^\frown(u^\xi, B^p) \Vdash `` \zeta \dot{E}^\xi_\alpha \zeta' \longleftrightarrow \zeta \dot{E}^\xi_\beta \zeta' "$, and we are done by elementarity.

So we may assume that we are in the other case, that is $y = y'^\frown(v \restriction \tau, B)$ for some $v \in \overline{B}$ which fills out u^ξ. We note that $v \restriction \tau \in \overline{B}$, since each of the sets B^q is closed under taking initial segments. Let $\rho^* < \rho^p$ be minimal such that $\max(\{\eta, \kappa_{y'}\}) < \kappa_{\rho^*}$.

Since y was chosen to be harmonious with A^* past η, $B^* \cap V_\kappa = \emptyset$, and $\eta < \kappa = \kappa_v$, we see that $B \subseteq \bigcup\{A^\xi_\rho \mid \eta < \kappa_\rho, \rho < \kappa\}$ and $\kappa_w > \eta$ for all $w \in B$. Since $y'^\frown(v \restriction \tau, B)$ is a condition, we also have that $\kappa_w > \kappa_{y'}$ for all $w \in B$. So by the choice of ρ^*, $B \subseteq \bigcup\{A^\xi_\rho \mid \rho^* \le \rho < \kappa\}$.

Since the condition $p \in K_\xi$, $A_\sigma = A^p_\sigma$ for $\rho^* \le \sigma < \rho^p$ and $A_\sigma \subseteq B^p$ for $\rho^p \le \sigma < \kappa$. It follows that $B \subseteq \bigcup\{A^p_\sigma \mid \rho^* \le \sigma < \rho^p\} \cup B^p$.

We now appeal to Lemma (4.20) to obtain the ρ^*-weakening of p, that is to say the condition $q \in \mathbb{Q}_\xi$ such that $\rho^q = \rho^*$, $p \le q$, $A^q = A^p \restriction \rho^*$, $B^q = \bigcup\{A_\sigma \mid \rho^* \le \sigma < \rho^p\} \cup B^p$, $t^q = \{(\eta', \alpha') \in t^p \mid \eta' \in a^q \cap \sup(a^q)\}$ and $f^q = \langle f^{p,\eta'}_{\alpha'} \mid (\eta', \alpha') \in t^q \rangle$.

Since $p \le q$ we have $q \in K_\xi$, and so $B^* \subseteq j(B^q)$. By the choice of ρ^*, since $\rho^q = \rho^*$, $A^q = A^p \restriction \rho^*$, $\max(\{\eta, \kappa_{y'}\}) < \kappa_{\rho^*}$ and y' is harmonious with A^p past η, we have that y' is harmonious with A^q past η.

By clause (3) of the conclusion of Lemma (4.20), the condition q enjoys a strengthened form of condition (5) in the definition of $Q(w)$, which implies in this case that $y'^\frown(u^\xi, B^q) \Vdash$ " $\zeta \dot{E}^\xi_\alpha \zeta' \longleftrightarrow \zeta \dot{E}^\xi_\beta \zeta'$. "

We claim that $y^\frown(j(u^\xi), B^*) \le y'^\frown(j(u^\xi), j(B^q)) = j(y'^\frown(u^\xi, B^q))$.

The only non-trivial point is that the pair $(v \restriction \tau, B)$ can be added to the condition $y'^\frown(j(u^\xi), j(B^q))$. This holds because $v \restriction \tau \in \overline{B} \subseteq j(B^q)$, and $B \subseteq \bigcup\{A^p_\sigma \mid \rho^* \le \sigma < \rho^p\} \cup B^p = B^q = j(B_q) \cap V_\kappa$. As in the case when $y = y'$, we are now done by elementarity.

Now that we have shown that p^* is a condition in $j(\mathbb{Q}_\xi)$ we have that $p^* \le_{j(\mathbb{Q}_\xi)} j(q)$ for $q \in K_\xi$, and so p^* is a master condition. The last thing to check in order to complete the proof of the lemma is that p^* is a master condition of the type required. However, this is almost immediate from its definition.

For as $j : V[G][H_\xi][K_\xi] \longrightarrow M[j(G)][j(H_\xi)][j(K_\xi)]$ is a lifting we have that $p^* \in j(K^\xi)$ (see [2], Proposition 9.1). We also have, by definition, that $A^\xi = \bigcap_{q \in K^\xi}[A^q, B^q]$, and so $j(A^\xi) = j(\bigcap_{q \in K^\xi}[A^q, B^q]) =$

$\bigcap_{q \in j(K^\xi)}[A^q, B^q]$. Hence if $q \in j(K^\xi)$ and $\rho^q \geq \kappa + 1$ then, by the definition of compatibility under $\leq_{\mathbb{Q}_\xi}$ and since K^ξ is a filter, we must have $A^q_\kappa = A^*_\kappa$.

Consequently, if v fills out u^ξ and $v \upharpoonright \tau \in \overline{B}$ then by the definition of p^* we have $v \upharpoonright \tau \in A^*_\kappa$ and hence, by the previous paragraph, $v \upharpoonright \tau \in j(A^\xi)$.   ▲$_{6.2}$

Having established the preceding lemma, we can now complete the inductive step.

Case (i). There is some $(r, \dot{q}) \leq (r'_\xi, \dot{q}'_\xi)$ with $\Vdash_{\dot{L}^{G*H}} \dot{q} \in j(\dot{\mathbb{P}}_\xi)$ such that

$$(r, \dot{q}) \Vdash_{\dot{L}^{G*H} * j(\dot{\mathbb{P}}_\xi)} \text{``}\exists v \in \mathcal{U}\,(\, v \text{ fills out } u^\xi \ \& \ \forall \tau < \lambda$$
$$v \upharpoonright \tau \in \overline{B} = \bigcap\{j(B^q) \,|\, (A^q, B^q, t^q, f^q) \in K_\xi\}\,)\text{''}.$$

Let $r_{\xi+1} = r$ and use Lemma (6.2) to choose $\dot{q}_{\xi+1}$ such that $\dot{q}_{\xi+1} \upharpoonright \xi = \dot{q}$, $(r_{\xi+1}, \dot{q}_{\xi+1} \upharpoonright \xi) \Vdash \text{``}\dot{q}_{\xi+1}(\xi)$ is a lower bound for $j\text{``}K_\xi\text{''}$ and $(r_{\xi+1}, \dot{q}_{\xi+1}) \Vdash \text{``}$ for every $v \in \mathcal{U} \cap \overline{B}$ which fills out u^ξ and every $\tau < \lambda$ we have $v \upharpoonright \tau \in j(A^\xi).\text{''}$

Case (ii). Otherwise. Again, by the closure of $\dot{L}^{G*H} * j(\dot{\mathbb{P}}_\chi)$, let $(r_{\xi+1}, \dot{q}_{\xi+1}) \leq (r'_\xi, \dot{q}'_\xi)$ be such that $(r_{\xi+1}, \dot{q}_{\xi+1} \upharpoonright \xi) \Vdash \text{``}\dot{q}_{\xi+1}(\xi)$ is a lower bound for $j\text{``}K_\xi\text{''}$ since Q_ξ is trivial.   ▲Inductive construction

When the construction is complete use the χ-closure of $\dot{L}^{G*H} * j(\dot{\mathbb{P}}_\chi)$ again and take a lower bound $(r^*, \dot{q}^*)$ for $\langle (r_\xi, \dot{q}_\xi) \,|\, \xi < \chi \rangle$ such that for all $q \in H$ we have $r^* \Vdash \dot{q}^* \leq j(q)$. Thus $(r^*, \dot{q}^*)$ forces that j can be lifted to some $j : V[G][H] \longrightarrow M[j(G)][j(H)]$ with $j(G) = G * H * j(G)/(G * H)$.

Claim 6.3. *If we generate $u = \langle u_\tau \,|\, \tau < \lambda \rangle$ from such a lifting of j to $V[G][H]$ in the usual inductive way, with $u_0 = \kappa$ and by setting $u_\tau = \{D \in V[G][H] \,|\, u \upharpoonright \tau \in j(D)\}$ for $0 < \tau < \lambda$, then in fact we have $u \in V[G][H]$ (and not merely $u \in V[G][H][j(G)/(G * H)][j(H)]$) and $u \in \mathcal{U}^{V[G][H]}$.*

Proof. The argument is an inductive repetition of a typical one in the context of Laver forcing.

By induction on $\tau < \lambda$ suppose that $u \upharpoonright \tau \in V[G][H]$. A priori, $u_\tau \in V[G][H][j(G)/(G * H)][j(H)]$. However $j(G)/(G * H) * j(H)$ is generic for a highly closed forcing – it is certainly $(2^\kappa)^+$-closed. So in fact $u_\tau \in V[G][H]$. As $V[G][H]$ is closed under sequences of length less

than or equal to λ we also have $u \restriction \tau + 1 \in V[G][H]$. Similarly, we inductively obtain that $u \restriction \tau \in V[G][H]$ for limit τ, and at the end of the induction that $u \in V[G][H]$.

We also need that normality and the two coherence conditions from Definition (3.1) hold for u_τ.

Suppose $f : \mathcal{U}_\kappa \longrightarrow V_\kappa$ is a function in $V[G][H]$. Observe that $V_\kappa^{V[G][H][j(G)/(G*H)][j(H)]} = V_\kappa^{V[G][H]}$. Then $\{w \in \mathcal{U}_\kappa \mid f(w) \in V_{\kappa_w}\} \in u_\tau$ if and only if $j(f)(u \restriction \tau) \in V_\kappa^{V[G][H]}$, if and only if there is some $x \in V_\kappa^{V[G][H]}$ such that $j(f)(u \restriction \tau) = x$, if and only if there is some $x \in V_\kappa^{V[G][H]}$ such that $\{w \in \mathcal{U}_\kappa \mid f(w) = x\} \in u_\tau$. Hence u satisfies normality in $V[G][H]$.

Similarly, suppose $f : \mathcal{U}_\kappa \longrightarrow \kappa$. Then $\{w \in \mathcal{U}_\kappa \mid f(w) < \mathrm{lh}(w)\} \in u_\tau$ if and only if $j(f)(u \restriction \tau) < \tau$, if and only if there is some $\sigma < \tau$ such that $j(f)(u \restriction \tau) = \sigma$. For this σ we then have for $X \subseteq V_\kappa$ that $X \in u_\sigma$ if and only if $u \restriction \sigma \in j(X)$ if and only if $j(X) \cap V_\kappa \in u_{j(f)(u \restriction \tau)}$ if and only if $\{w \in \mathcal{U}_\kappa \mid X \cap V_{\kappa_w} \in w_{f(w)}\} \in u_\tau$.

Finally, if $\sigma < \tau$ and $X \in u_\sigma$ then $u \restriction \sigma \in j(X)$ and so $\{w \in \mathcal{U}_\kappa \mid \exists \overline{\sigma} < \mathrm{lh}(w) \, X \cap V_{\kappa_w} \in w_{\overline{\sigma}}\} \in u_\tau$. So u also satisfies the two coherence conditions. $\blacktriangle_{6.3}$

Claim 6.4. *In $V[G][H]$ there is a stationary set $S \subseteq S_{\kappa^+}^\chi$ such that*

- *For every $\xi \in S$ one has $\mathrm{lh}(u^\xi) = \lambda$ and for every $\tau < \lambda$ that $u_\tau \cap V[G][H_\xi] = u_\tau^\xi$.*

- *For every ξ in the closure of S, $S \cap \xi \in V[G][H_\xi]$.*

- *For every $\xi \in S$, $u^\xi \in \mathcal{U}_{\kappa^+}^{V[G][H_\xi]}$.*

- *For every $\xi \in S$, $\{v \mid \exists \rho < \kappa \, v \in A_\rho^\xi\} \in \mathcal{F}(u)$.*

Proof. We start by outlining the motivation for the first step in the proof. As $S_{\kappa^+}^\chi$ is a stationary subset of $S_{\geq \kappa}^\chi$, Proposition (2.3) tells us that there is a $\diamondsuit_\chi(S_{\kappa^+}^\chi)$-sequence in $V[G][H]$, (with the properties given in Proposition (2.3 (1) and (2))), which predicts subsets of χ. The statement of the claim talks about predicting sequences of measures. So in order to apply Proposition (2.3) we must 'code' such sequences of measures by subsets of κ.

Working in $V[G][H]$, define a set $T \subseteq \chi$ 'coding' u by enumerating each u_τ in order type χ as $\langle y_{\tau\xi} \mid \xi < \chi \rangle$ and setting $T \cap [\kappa\tau\xi, \kappa\tau(\xi+1)) = \{\kappa\tau\xi + \eta \mid \eta \in y_{\tau\xi}\}$. Appealing to Proposition (2.3), we may find a

stationary set $S \subseteq S^\chi_{\kappa^+}$ such that S satisfies the first two clauses of the claim.

For $\xi \in S$ and $0 < \tau < \lambda$, we have that u^ξ_τ is a measure on V_κ in $V[G][H_\xi]$. Since $\mathcal{U}^{V[G]}_\kappa = \mathcal{U}^{V[G][H]}_\kappa$ it is clear that each u^ξ_τ concentrates on $\mathcal{U}_\kappa$.

Normality. Suppose $f : \mathcal{U}_\kappa \longrightarrow V^{V[G][H_\xi]}_\kappa$ and $\{w \in \mathcal{U}_\kappa \mid f(w) \in V_{\kappa_w}\} \in u^\xi_\tau$. As $u^\xi_\tau \subseteq u_\tau$ we can apply normality for u to get some $x \in V_\kappa$ such that $\{w \in \mathcal{U}_\kappa \mid f(w) = x\} \in u_\tau$. But $x, f \in V[G][H_\xi]$, hence $\{w \in \mathcal{U}_\kappa \mid f(w) = x\} \in u_\tau \cap V^{V[G][H_\xi]}_\kappa = u^\xi_\tau$.

Coherence (i). Suppose $f : \mathcal{U}_\kappa \longrightarrow \kappa$ with $f \in V[G][H_\xi]$ and $\{w \in \mathcal{U}_\kappa \mid f(w) < \mathrm{lh}(w)\} \in u^\xi_\tau$. Again as $u^\xi_\tau \subseteq u_\tau$ we can apply the first coherence condition for u to get some $\sigma < \tau$ with $x \in u_\sigma$ if and only if $\{w \in U_\kappa \mid x \cap V_{\kappa_w} \in w_{f(w)}\} \in u_\tau$. If $x \in V[G][H_\xi]$, so that $x \in u^\xi_\sigma$, then, recalling that $f \in V[G][H_\xi]$, we have $\{w \in U_\kappa \mid x \cap V_{\kappa_w} \in w_{f(w)}\} \in V[G][H_\xi]$, and hence $\{w \in U_\kappa \mid x \cap V_{\kappa_w} \in w_{f(w)}\} \in u^\xi_\tau$. Conversely, if $\{w \in U_\kappa \mid x \cap V_{\kappa_w} \in w_{f(w)}\} \in u^\xi_\tau$ then, as $u^\xi_\tau \subseteq u_\tau$, we have $x \in u_\sigma$.

Coherence (ii). If $\tau' < \tau$ and $x \in u^\xi_{\tau'}$, then $\{w \in \mathcal{U}_\kappa \mid \exists \sigma < \mathrm{lh}(w)\, x \cap V_{\kappa_w} \in w_\sigma\} \in u_\tau$ and the set is clearly in $V[G][H_\xi]$ as x is, and hence is an element of u^ξ_τ.

Finally, at each $\xi \in S$ when we did the inductive construction of $(r_{\xi+1}, \dot{q}_{\xi+1})$ we must have been in 'Case (i)' and used Lemma (6.2) because $(r^*, \dot{q}^* \upharpoonright j(\xi))$ would be an appropriate witness. Consequently for all $\tau < \lambda$ we have that $u \upharpoonright \tau \in j(A^\xi)$, and hence $A^\xi \in \mathcal{F}(u)$. $\blacktriangle_{6.4}$

Proposition 6.5. *Let $S \in V[G][H]$ be as given by Claim (6.4). Let ξ be a limit of elements of the set S of cofinality at least κ^+. For each $\tau < \lambda$ let $v_\tau = \bigcup\{u^\varepsilon_\tau \mid \varepsilon \in S \cap \xi\}$. Then $V[G][H_\xi] \models v$ is an ultrafilter sequence and $\forall \varepsilon \in S \cap \xi\ A^\varepsilon \in \mathcal{F}(v)$.*

Proof. By Claim (6.4) we have that $S \cap \xi \in V[G][H_\xi]$, thus $v \in V[G][H_\xi]$. It is clear that v is a sequence of measures concentrating on $\mathcal{U}^{V[G][H_\xi]}_\kappa$. In order to see that the normality and coherence conditions hold it is enough to observe that since $\mathrm{cf}(\xi) \geq \kappa^+$ we have that for each $z \in \mathcal{P}(\kappa)$ and $f : \mathcal{U}_\kappa \longrightarrow V_\kappa$ if $z, f \in V[G][H_\xi]$ there is some $\varepsilon \in S \cap \xi$ such that $z, f \in V[G][H_\varepsilon]$. Again by Claim (6.4) we have that $A^\varepsilon \in \mathcal{F}(v)$. $\blacktriangle_{6.5}$

7. Proof that we do get small universal families

Let $S \in V[G][H]$ be as given by Claim (6.4). As in Proposition (6.5), we choose ξ a limit point of S with $\mathrm{cf}(\xi) = \kappa^{++}$. Let $G * H_\xi$ be the $\mathbb{L} * \dot{\mathbb{P}}_\xi$-generic filter over V induced by $G * H$, and define an ultrafilter sequence $v \in V[G*H_\xi]$ by setting $v_\tau = \bigcup\{u_\tau^\varepsilon \mid \varepsilon \in S \cap \xi\}$ for $0 < \tau < \lambda$. Let g^* be $\mathbb{R}_v$-generic over $V[G * H_\xi]$. As we discussed in §3, by forcing below a suitable condition we may arrange that

- the generic object induced by g^* is a λ-sequence $\langle u_i \mid i < \lambda \rangle$ of ultrafilter sequences,

- defining $\kappa_i = \kappa_{u_i}$ for $i < \lambda$, the set $C = \{\kappa_i \mid i < \lambda\}$ is a club subset of κ,

- $\min(C) > \lambda$, and so $\mathrm{cf}(\kappa) = \lambda$ in $V[G * H_\xi * g^*]$.

Notation 7.1. For $\varepsilon \in S \cap \xi$ set H_ε to be the induced $\mathbb{P}_\varepsilon$-generic filter over $V[G]$, $Q_\varepsilon = \dot{Q}_\varepsilon^{G*H_\varepsilon}$, K_ε to be the Q_ε generic filter over $V[G][H_\varepsilon]$ induced by $G * H_\xi$, $A^{*\varepsilon} = \langle A_\rho \mid \rho < \kappa \ \& \ \exists p \in K_\varepsilon \, (\rho < \rho^p \ \& \ A_\rho = A_\rho^p) \rangle$, $\kappa_\rho^{*\varepsilon}$ to be the common value of κ_w for $w \in A_\rho^{*\varepsilon}$, and $a^{*\varepsilon} = \{\kappa_\rho^{*\varepsilon} \mid \rho < \kappa\}$.

As per Observation(3.24), the characterisation of genericity for Radin forcing implies that for every $\varepsilon \in S \cap \xi$, the sequence $\langle u_i \mid i < \lambda \rangle$ is $\mathbb{R}_{u^\varepsilon}$-generic over $V[G][H_\varepsilon]$. Let g^ε be the $\mathbb{R}_{u^\varepsilon}$-generic filter over $V[G][H_\varepsilon]$ induced by this sequence, so that easily $g^\varepsilon = g^* \cap \mathbb{R}_{u^\varepsilon}$.

Note that, by the characterisation of Radin-genericity from Theorem (3.22), for each $\varepsilon \in S \cap \xi$, since $A^\varepsilon \in \mathcal{F}(v)$ by Proposition (6.5), we have that $u_j \in \bigcup\{A_\rho^{*\varepsilon} \mid \rho < \kappa\}$ for all large $j < \lambda$. Let $i_\varepsilon < \lambda$ be the least successor ordinal such that $u_j \in \bigcup\{A_\rho^{*\varepsilon} \mid \rho < \kappa\}$ for $j \geq i_\varepsilon$, and let $\eta_\varepsilon = \kappa_{u_{i_\varepsilon}}$. Since η_ε is a successor point of the generic club C, the sequence $u_{i_\varepsilon} = \langle \eta_\varepsilon \rangle$.

We note that since $u_{i_\varepsilon} = \langle \eta_\varepsilon \rangle \in \bigcup\{A_\rho^{*\varepsilon} \mid \rho < \kappa\}$, we may define σ_ε as the unique ρ such that $\langle \eta_\varepsilon \rangle \in A_\rho^{*\varepsilon}$, and by definition we have $\kappa_{u_{i_\varepsilon}} = \eta_\varepsilon = \kappa_{\sigma_\varepsilon}^\varepsilon$. It follows that for any $q \in K_\varepsilon$ with $\sigma_\varepsilon < \rho^q$ we have that $\langle \eta_\varepsilon \rangle \in A_{\sigma_\varepsilon}^q$ and $\eta_\varepsilon \in a^q$.

Definition 7.2. For $\varepsilon \in S \cap \xi$ let $\mathfrak{C}_\alpha^\varepsilon = (\dot{E}_\alpha^\varepsilon)^{G*H_\varepsilon*g^\varepsilon}$, and, for $\alpha < \chi^-$, let $\mathfrak{f}_\alpha^\varepsilon = \bigcup\{(f_\alpha^{\eta_\varepsilon})^q \mid q \in K_\varepsilon \ \& \ \alpha \in t^{q,\eta_\varepsilon}\}$.

As we proved in Section 4 (see Corollary (4.13)), $\mathfrak{f}_\alpha^\varepsilon$ is a function with domain κ^+ such that $\mathfrak{f}_\alpha^\varepsilon(\zeta) \in \{b_\alpha \restriction \zeta\} \times \kappa$ for every $\zeta < \kappa^+$.

Definition 7.3. For each $\varepsilon \in S \cap \xi$ and $\alpha < \chi^-$ define $\mathcal{E}_\alpha^\varepsilon$ on $\mathrm{rge}(\mathfrak{f}_\alpha^\varepsilon)$ by $z \, \mathcal{E}_\alpha^\varepsilon \, z'$ if and only if $\mathrm{ht}(\pi_0(z)) \, \mathfrak{E}_\alpha^\varepsilon \, \mathrm{ht}(\pi_0(z'))$. That is, $\mathcal{E}_\alpha^\varepsilon = \mathfrak{f}_\alpha^\varepsilon \text{ "} \mathfrak{E}_\alpha^\varepsilon$. (For if $\zeta < \kappa^+$ and $\mathfrak{f}_\alpha^\varepsilon(\zeta) = z$ then $\mathrm{ht}(\pi_0(z)) = \zeta$.)

We now prove a short technical lemma which will allow us to give an equivalent characterization of $\mathcal{E}_\alpha^\varepsilon$, which in turn facilitates the proof that $\mathcal{E}_\alpha^\varepsilon$ and $\mathcal{E}_\beta^\varepsilon$ are coherent for $\alpha \neq \beta$.

Lemma 7.4. *Let $\varepsilon \in S \cap \xi$. Suppose $y^\frown(u^\varepsilon, D) \in g^\varepsilon$ and $q = (A^q, B^q, t^q, f^q) \in K_\varepsilon$ are such that*

- *ρ^q is a successor ordinal.*

- *$\eta_\varepsilon, \kappa_y \leq \sup(a^q)$.*

- *$B^q \subseteq D$.*

Then there is a lower part y' such that

- *$y'^\frown(u^\varepsilon, B^q) \leq y^\frown(u^\varepsilon, D)$*

- *$y'^\frown(u^\varepsilon, B^q) \in g^\varepsilon$.*

- *y' is harmonious with A^q past η_ε.*

Proof. Let $j_0 < \lambda$ be the largest ordinal such that $\kappa_{u_{j_0}} \leq \max(a^q)$. For all j with $j_0 < j < \lambda$ we have that $u_j \in \bigcup_{\sigma < \kappa} A_\sigma^{*\varepsilon}$ and $\kappa_{u_j} > \max(a^q)$, so that $u_j \in B^q$.

Appealing to the second part of Lemma (3.17) and to Lemma (3.26), we may now extend $y^\frown(u^\varepsilon, D)$ to a condition $\overline{y}^\frown(u^\varepsilon, B^q) \in g^\varepsilon$ by first adding in (as necessary) the pair $(\langle \eta_\varepsilon \rangle, \emptyset)$ and a pair with first entry u_{j_0}, then shrinking D to B^q. Note that $\kappa_{\overline{y}} = \max(\{\eta_\varepsilon, \kappa_y, \kappa_{u_{j_0}}\}) \leq \max(a^q)$.

Appealing to Lemma (3.29), there is y' directly extending $\overline{y}$ in $\mathbb{R}_{\overline{y}}$ such that y' conforms with g^ε, and y' is harmonious with A^q past η_ε. $\blacktriangle_{7.4}$

Proposition 7.5. *For each $\varepsilon \in S \cap \xi$, $\alpha < \chi^-$ and $z, z' \in \mathrm{rge}(\mathfrak{f}_\alpha^\varepsilon)$ we have $z \, \mathcal{E}_\alpha^\varepsilon \, z'$ if and only if there exist a condition $q \in K_\varepsilon$ and a lower part y harmonious with A^q past η_ε such that q and $y^\frown(u^\varepsilon, B^q)$ witness that $z \, \mathcal{E}_\alpha^\varepsilon \, z'$, i.e., letting $\zeta = \mathrm{ht}(\pi_0(z))$ and $\zeta' = \mathrm{ht}(\pi_0(z'))$, such that*

- *$\alpha \in t^{q, \eta_\varepsilon}$, $\zeta, \zeta' \in d_\alpha^{q, \eta_\varepsilon}$, $f_\alpha^{q, \eta_\varepsilon}(\zeta) = z$ and $f_\alpha^{q, \eta_\varepsilon}(\zeta') = z'$.*

- *y is harmonious with A^q past η_ε.*

- *$y^\frown(u^\varepsilon, B^q) \in g^\varepsilon$.*

- $y^\frown(u^\varepsilon, B^q) \Vdash^{V[G][H_\varepsilon]}_{\mathbb{R}_{u^\varepsilon}} \zeta \dot{E}^\varepsilon_\alpha \zeta'$.

Proof. If q and y are as in the equivalent, then $y^\frown(u^\varepsilon, B^q) \in g^\varepsilon$ and $y^\frown(u^\varepsilon, B^q) \Vdash^{V[G][H_\varepsilon]}_{\mathbb{R}_{u^\varepsilon}} \zeta \dot{E}^\varepsilon_\alpha \zeta'$, so that $\zeta \; \mathfrak{E}^\varepsilon_\alpha \; \zeta'$.

For the converse direction, suppose that $\zeta \; \mathfrak{E}^\varepsilon_\alpha \; \zeta'$. Choose a condition $y^\frown(u^\varepsilon, D) \in g^\varepsilon$ such that $y^\frown(u^\varepsilon, D) \Vdash^{V[G][H_\varepsilon]}_{\mathbb{R}_{u^\varepsilon}} \zeta \dot{E}^\varepsilon_\alpha \zeta'$ and $\langle \eta_\varepsilon \rangle$ appears in y. By the choice of η_ε, the sequence $\langle \eta_\varepsilon \rangle$ and all subsequent sequences appearing in y are members of $\bigcup_{\rho < \kappa} A^{*\varepsilon}_\rho$.

Choose a condition $q \in K^\varepsilon$ such that η_ε, $\kappa_y \in a^q$, ρ^q is a successor ordinal, and $B^q \subseteq D$. Appealing to Lemma (7.4) we find a condition $y'^\frown(u^\varepsilon, B^q)$ which is exactly of the kind needed to form a witness (together with q) that $z \; \mathfrak{E}^\varepsilon_\alpha \; z'$. $\blacktriangle_{7.5}$

Next we show that $\mathfrak{E}^\varepsilon_\alpha$ and $\mathfrak{E}^\varepsilon_\beta$ cohere for $\alpha \neq \beta$.

Lemma 7.6. *Let $\varepsilon \in S \cap \xi$, $\alpha \neq \beta$ and z and $z' \in \mathrm{rge}(\mathfrak{f}^\varepsilon_\alpha) \cap \mathrm{rge}(\mathfrak{f}^\varepsilon_\beta)$. Then $z \; \mathfrak{E}^\varepsilon_\alpha \; z'$ if and only $z \; \mathfrak{E}^\varepsilon_\beta \; z'$.*

Proof. Choose $r \in K_\varepsilon$ and y a lower part witnessing the equivalent conditions listed in Proposition (7.5) that $z \; \mathfrak{E}^\varepsilon_\alpha \; z'$. Choose $p \leq r$ such that $p \in K_\varepsilon$, ρ^p is a successor ordinal, and p contains enough information to verify that $z, z' \in \mathrm{rge}(\mathfrak{f}^\varepsilon_\beta)$, that is to say that $\beta \in t^{p,\eta_\varepsilon}$, $\zeta, \zeta' \in d^{p,\eta_\varepsilon}_\beta$, $f^{p,\eta_\varepsilon}_\beta(\zeta) = z$ and $f^{p,\eta_\varepsilon}_\beta(\zeta') = z'$.

We now recall the notion of ρ^*-weakening from Definition (4.19). Let q be the ρ^r-weakening of p, and note that by definition $A^q = A^p \restriction \rho^r = A^r$ and $a^q = a^r$. It follows from Lemma (4.20) and Corollary (4.21) that $p \leq q \leq r$, in particular $q \in K_\varepsilon$ and $B^q \subseteq B^r$.

We claim that y and B^q will serve as witnesses that $z \; \mathfrak{E}^\varepsilon_\beta \; z'$, that is that $y^\frown(u^\varepsilon, B^q) \in g^\varepsilon$ and that $y^\frown(u^\varepsilon, B^q) \Vdash \zeta \dot{E}^\varepsilon_\beta \zeta'$.

By Lemma (3.26), to check that $y^\frown(u^\varepsilon, B^q) \in g^\varepsilon$ we must show that y conforms with g^ε and $u \in B^q$ for all u appearing on the generic sequence with $\kappa_y < \kappa_u$. Since $y^\frown(u^\varepsilon, B^r) \in g^\varepsilon$, we see that the lower part y conforms with g^ε and also that $u \in B^r$ for all u appearing on the generic sequence with $\kappa_y < \kappa_u$. Fix such a sequence u. Since r is a condition and $u \in B^r$ it follows that $\mathrm{ssup}(a^q) = \mathrm{ssup}(a^r) \leq \kappa_u$. By the choice of η_ε and y, we have $u \in \bigcup\{A^{*\varepsilon}_\rho \mid \rho < \kappa\}$, and since $q \in K_\varepsilon$ we have that $u \in B^q$.

Since $B^q \subseteq B^r$, we see that $y^\frown (u^\epsilon, B^q)$ is a refinement of $y^\frown (u^\epsilon, B^r)$, and in particular $y^\frown (u^\epsilon, B^q) \Vdash \zeta \dot{E}^\varepsilon_\alpha \zeta'$. Since $A^q = A^r$ and q is a condition, it follows from Clause (5) in Definition (4.5) that $y^\frown (u^\epsilon, B^q) \Vdash \zeta \dot{E}^\varepsilon_\beta \zeta'$. $\blacktriangle_{7.6}$

Definition 7.7. Working in the model $V[G][H_\xi][g^*]$, we define for each $\varepsilon \in S \cap \xi$ a relation $\mathcal{E}^\varepsilon$ on the set $\bigcup \{\, \{b_\alpha \restriction \zeta \mid \zeta < \kappa^+\} \times \kappa \mid \alpha < \chi^- \}$ by

$$z \,\mathcal{E}^\varepsilon\, z' \longleftrightarrow \exists \alpha < \chi^- \; z \,\mathcal{E}^\varepsilon_\alpha\, z'.$$

Proposition 7.8. $V[G][H_\xi][g^*] \models \mathcal{E}^\varepsilon$ *is of size* κ^+ *and is universal for* $\{\, \mathfrak{E}^\varepsilon_\alpha \mid \alpha < \chi^- \}$.

Proof. $\mathcal{E}^\varepsilon$ has size κ^+ by the choice of $\{b_\alpha \mid \alpha < \chi^-\}$, as for $\alpha < \chi^-$ and $\zeta < \kappa^+$ we have that $b_\alpha \restriction \zeta \in \mathcal{T}$, where $\mathcal{T}$ is the κ^+-Kurepa tree added by $\mathbb{Q}_0$.

The universality follows from the definitions of $\mathfrak{E}^\varepsilon_\alpha$, $\mathcal{E}^\varepsilon_\alpha$ and $\mathcal{E}^\varepsilon$, and Lemma (7.6) which ensures that each $\mathfrak{f}^\varepsilon_\alpha$ is an embedding of $\mathfrak{E}^\varepsilon_\alpha$ into $\mathcal{E}^\varepsilon$: for Definition (7.3) ensures that each $\mathfrak{f}^\varepsilon_\alpha$ is an embedding of $\mathfrak{E}^\varepsilon_\alpha$ into $\mathcal{E}^\varepsilon_\alpha \subseteq \mathcal{E}^\varepsilon$, and Lemma (7.6) ensures the compatibility of these embeddings. $\blacktriangle_{7.8}$

Proposition 7.9. *In* $V[G][H_\xi][g^*]$ *suppose* Ξ ($\in V[G][H_\xi][g^*]$) *is cofinal in* $S \cap \xi$ *and for all* $\varepsilon \in \Xi$ *we have that* $\{\, \dot{E}^\varepsilon_\alpha \mid \alpha < \chi^- \}$ *is a list of canonical names for all graphs on* κ^+. *Then* $\{\, \mathcal{E}^\varepsilon \mid \varepsilon \in \Xi \}$ *is a universal family in the collection of graphs on* κ^+.

Proof. Let $\mathfrak{E}$ be a graph on κ^+ in $V[G][H_\xi][g^*]$ and let $\dot{E}$ be a canonical $\mathbb{L} * \dot{\mathbb{P}}_\xi * \dot{\mathbb{R}}_v$-name such that $\mathfrak{E} = \dot{E}^{G*H_\xi*g^*}$. As $\dot{E}$ is a canonical name it has size at most κ^+ and hence, as $\mathrm{cf}(\xi) > \kappa^+$, there is some $\varepsilon \in \Xi$ such that $\dot{E}$ is an $\mathbb{L} * \dot{\mathbb{P}}_\varepsilon * \dot{\mathbb{R}}_{u^\varepsilon}$-name and hence $\mathfrak{E} = \dot{E}^{G*H_\varepsilon*g^\varepsilon}$. By Proposition (7.8) we thus have that $\mathfrak{E}$ embeds into $\mathcal{E}^\varepsilon$. $\blacktriangle_{7.9}$

Theorem 7.10. *Suppose* κ *is a supercompact cardinal,* $\lambda < \kappa$ *is a regular cardinal and* Θ *is a cardinal with* $\mathrm{cf}(\Theta) \geq \kappa^{++}$ *and* $\kappa^{+3} \leq \Theta$. *There is a forcing extension in which cofinally many cardinals below* κ, κ *itself and all cardinals greater than* κ *are preserved,* $\mathrm{cf}(\kappa) = \lambda$, $2^\kappa = 2^{\kappa^+} = \Theta$ *and there is a universal family of graphs on* κ^+ *of size* κ^{++}.

Proof. Let $\chi = \Theta^+$. As mentioned in the first paragraph of §6, it is standard that if κ is supercompact there is a forcing extension in which κ remains supercompact and GCH and $\Diamond_\chi(S^\chi_{\kappa^+})$ hold. Now force with

$\mathbb{L} * \mathbb{P}_\xi * \mathbb{R}_v$ and work in $V[G][H_\xi][g^*]$. Cofinally many cardinals below κ, κ itself and all cardinals above κ are preserved from V to $V[G]$ and the remaining factors preserve all cardinals. The final factor Radin forcing makes λ the cofinality of the previously regular (indeed supercompact) cardinal κ. The second factor of the forcing makes $2^\kappa = 2^{\kappa^+} = \Theta$ and the final Radin factor does not increase these values as it has size 2^κ. By Proposition (7.9) $\{\mathcal{E}^\varepsilon \mid \varepsilon \in \Xi\}$ is a universal family in the collection of graphs on κ^+ and has size $\overline{\overline{\Xi}} = \kappa^{++}$. ▲7.10

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