CHALLENGING THE MATHEMATICIAN'S 'ULTIMATE SUBSTANTIATOR' ROLE IN A *LOW LECTURE* INNOVATION

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In this paper we draw on our experiences as member of the International Advisory Board and principal investigator of a research project on undergraduate mathematics teaching and learning to comment on the study of university mathematics as a process of enculturation into new mathematical practices and new ways of constructing and conveying mathematical meaning. We see this enculturation as the adaptation of different ways to act and communicate mathematically. We take a discursive perspective and we treat the changes to the mathematical and pedagogical perspectives of those who act – students and lecturers – as discursive shifts (Sfard, 2008). Our particular focus is on the shifts concerning the 'ultimate substantiator' role typically attributed to the lecturer.

UNIVERSITY MATHEMATICS: AN ENCULTURATION PERSPECTIVE

Mathematics undergraduates, and their lecturers, often describe university mathematics as a process of enculturation into new mathematical practices and new ways of constructing and conveying mathematical meaning (Nardi, 1996). As often described in the literature (e.g. Artigue, Kent & Batanero, 2007), what characterises the breadth and intensity of this enculturation varies according to factors that include: student background and preparedness for university level studies of mathematics; the aims and scope of each of the courses that the students take at university; how distant the pedagogical approaches taken in these courses are from those taken in the secondary schools that the students come from; the students' affective dispositions towards the subject and their expectations for what role mathematics is expected to play in their professional life. On their part, lecturers' views on their pedagogical role (e.g. Nardi, 2008) may also vary according to factors such as: length of teaching experience; type of courses (pure, applied, optional, compulsory etc.) they teach; perceptions of the goals of university mathematics teaching (such as to facilitate access to the widest possible population of participants or select those likely to push the frontiers of the discipline); and, crucially, institutional access to innovative practices (Skovsmose, Valero & Christensen, 2009).

Here we draw on our experiences, respectively, as member of the International Advisory Board (Nardi, 2014) and principal investigator of the LUMOS project (*Learning in Undergraduate Mathematics: Output Spectrum*; Barton & Paterson, 2013) to comment on aforementioned student enculturation, particularly with regard to how students and lecturers experience the innovations introduced in the project. We first outline the project.

LUMOS AND THE *LOW LECTURE* INNOVATION

LUMOS is a two-year project funded by Ako Aotearoa, the New Zealand government body that distributes national research grants for tertiary education research, as well as the New Zealand Teaching & Learning Research Initiative (TLRI). Its main aim is to understand how course delivery at class level can achieve a range of desired learning outcomes for undergraduate mathematics that includes content and skill related outcomes as well as outcomes related to the processes of mathematics, affect, and broader graduate issues. It is expected that the project will generate evidence that different types of courses contribute to student learning in different ways. Therefore developing a variety of pedagogical practices is part of the project. Three innovations are currently under trial: *team-based learning*, *intensive technology* and *low lecture*. The third of these, *low lecture*, is the focus of this paper.

There are three key assumptions behind the *low lecture* innovation. First, lectures are not necessarily the best means of imparting information or developing skills. They are however useful for material overviews, demonstrating model ways of communicating mathematical ideas and enthusing newcomers with the skill and fluency that can often be found in the communicational practices of old-timers – thus one per week is sufficient. Second, responsibility for learning content and acquiring skills is handed back to students using specific guides of what they are expected to learn and where to find print and online resources, and with regular self- and lecturer-monitoring of progress. Third, learning about, and induction into, the processes of being mathematical are absent from most undergraduate courses, hence the time saved from lecturing is spent in small group sessions of semi-authentic mathematical experiences free from content-learning requirements.

The *Low lecture* innovation was trialled for the first time in 2013, with 14 MATHS108 students. MATHS108 is a Year 1 course for non-mathematical majors that covers: linear functions, linear equations and matrices; functions, equations and inequalities; limits and continuity; differential calculus (one/two variables); and, integral calculus (one variable). Faculty members, as members of the LUMOS team, run the trial on an extra-to-load basis. The trial consists of one lecture per week for the duration of the semester and three 2-hour *engagement sessions* which students need to prepare for in advance, as well as write up a report for afterwards. These reports substitute assignments. The remaining parts of MATHS108 (tutorials, tests and final written examination) stay the same.

The discussion we present here was initiated by the first author's account (Nardi, 2014) of her experience of observing an *engagement session* and the discussions that followed this observation. Our account adopts the *commognitive* perspective (Sfard, 2008). *Commognitive* terms in it are in *italics* and used as defined in the abridged presentation of the framework in (Nardi, 2014, p. 5-6) and (Nardi, Ryve, Stadler & Viirman, 2014, p. 183-5). We conclude the paper with a consideration of the shifts in the lecturer's role as experienced by the observed lecturer (second author).

As outlined in (Nardi, 2014) the observed *engagement session* was part of the *low lecture* MATHS108 course. Five students (thereafter Students B, N, J, D and A) participated in the session which was their first *engagement session* and took place in the early weeks of the first semester. The session was run by the second author, leading member of the LUMOS team (thereafter Lecturer L). In the account that follows we outline what unfolded in the session and then present the discussions between the observer (first author) and L (the lecturer and second author) that followed.

In presenting this account we are driven by the following questions:

- What were these 'newcomers' to the practices of university mathematics to make of the open task set to them (see below)?
- What were their expectations of the 'old-timer' who led the session?
- In return, what were the 'old-timer''s expectations of the students?
- And, finally, what kind of bearing, if any, did the slightly unexpected nature of the task have on the session and its aftermath?

OBSERVING AN *ENGAGEMENT SESSION* OF A *LOW LECTURE* COURSE

The five MATHS108 students arrived in the small, cosy meeting room where their first experience of an *engagement session* was about to kick off. Their preparation for the session consisted of engaging with an open task, sent to them a week prior to the session: exploring functions from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$ - see an outline of the task in Figure 1 The students expected to be invited to share their explorations with the lecturer and the group. We note the deliberately unexpected nature of the task: these students were so far accustomed to working with functions from $\mathbb{R}$ to $\mathbb{R}$ and may have had a general awareness of functions from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$. L comments on the spirit of the task as follows:

> "Engagement Session situations are intended to be open-ended mathematically, both conceptually and procedurally. That is, they are intended to ask students questions about mathematical concepts that they have not encountered before, although they may be related to the work in the course. Additionally, there are not only many "take-off" points (places where students can start working), but also several, different ways of developing their work.
>
> Furthermore, there is no presumed "correct" process or result. […] What is important is what they then do, mathematically. […] Students are not given marks for "correctness". They are marked on "mathematical thinking" in whatever form it is exhibited."

We return to the two omitted ([…]) parts from the above L quotation later in the paper when we examine a little more closely some of the student productions in preparation, during and after the session.

Engagement Situation #1: Functions from ℝ to ℝ×ℝ

Most functions we have been using map a Real Number onto a Real Number.

We write $f : \mathbb{R} \rightarrow \mathbb{R}$ and we say "f maps ℝ onto ℝ."

But functions can be about any numbers, not necessarily the Real Numbers. That is why we have to specify the domain when we define a function. In fact, a function can map anything onto anything, vectors or matrices, for example.

Not only that, we can define functions that map TWO numbers onto one number. You will learn more about such functions later. An example of such a function is

$$f(x, y) = 3x - y^2.$$

We write $f : \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}$ and we say "f maps ℝ cross ℝ onto ℝ".

What about a function that works the other way? It starts with a Real Number but produces TWO Real Numbers. That is $f : \mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$.

Our first problem is to find a suitable notation. Let's take an example. We start with a function f and a variable x. Let the first number created by the function be x^2, and the second number be $(1/x)$. Thus $f(2)$ is 4 and ½.

1. Devise a suitable notation for this.
2. Devise a new function $h : \mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$. Make up your own rules for h. Explore some values of h. Check: is h a function? That is, will each separate input x give a unique output pair?
3. Can you find a way of graphing h ? This will need to be a new kind of graph.
4. What can you say about the values of h for different inputs? E.g. what happens to $h(x)$ when x is close to zero, when x gets very large, when x is negative? Find some other things to investigate about $h(x)$.
5. Can you find another function, $j(x)$, which behaves differently? Will your graphing and notation scheme work for $j(x)$?

Figure 1. An outline of the *Engagement Situation* task pre-distributed to the students

The students and the lecturer were seated around a rotund table, arranged in the middle of a small meeting room. The students arrived with their preparatory work in hand. One – N, the only female in the group – also had her laptop with online access, which she used often during the session. The ambience was convivial and highly respectful of all. The students granted permission to the observer (first author) to join the session and seemed comfortable with her presence. The account that follows – described in (Nardi, 2014) as a sequence of episodes that evidence a substantial shakeup of the *learning-teaching agreement* – is based on notes jotted down during and right after the session. The account is written from a *commognitive perspective* and aims broadly at addressing the questions listed at the end of the previous section.

Shakeup of the *learning-teaching agreement* in a *low lecture* session: evidence

At the very start of the session L reminds the students that its overall aim is set out in the preparation sheet (Figure 1). L had set two tasks for this exploration: first, propose a notation for this type of function; second, devise a relationship of this type and explore how we would secure that it is a function, what its range of values would be, what its graph would look like and what its behaviour would be for very small or very large values of *x*. The preparation sheet ends with a request to devise a second function of this type and repeat the exploration with a view to comparing with the first. The students are also reminded that they will be expected to communicate the outcomes of their exploration and that some aids to doing so will be available in the room for them to this purpose. As the session starts, L reminds them that they are ultimately expected to produce a four-page report consisting of: an account of their pre-session efforts (on the first page), their take on the exchanges during the session (on the second and third pages) and their further explorations soon after (on the fourth and final page).

The final words on the preparation sheet were 'happy mathematising and they encapsulate explicitly the *discursive object* of the activity that the students are invited to participate in. L's overall demeanour and utterances throughout the session also convey exactly that: this session is about engaging with the *routines* of a mathematician (he lists several of these in at least two occasions, including hypothesising, justifying, proving, visualising, extrapolating etc.). The students' responses to these meta-discursive utterances by L – particularly when L asks them to cease activity for a moment to heed what they are doing, and how – is rather mute: they seem keen and confident to act but perhaps less so to take up this invitation for reflective distancing from the action. In fact it takes no more than a few seconds for them to return to the vicarious discussion of their exploratory work.

On the grounds of this discussion – which we sample selectively in what follows – there was little doubt that the students' take on the purpose of the session was essentially congruent to that of L. Sfard (2008, p. 223 onwards) speaks of mathematical *routines* in terms of *deeds*, *rituals* and *explorations* and it would be hard to perceive what was happening in the session as anything other than evidence of *exploration*. L's recollection of the students' work substantiates this claim further:

> […] in this situation, while students may graph their functions as lines in 3-space, other alternatives are acceptable. For example, students have used the first element created to define a new (curved) axis, on which the second element is plotted; others have used the first element to define a line in 2-space as in a conventional graph, and the second element to determine the width of the line, hence creating a ribbon. […] For example, the ribbon is not a function, as it is not 1 to 1. However an attempt to redefine "1 to 1" for this context would be an entirely acceptable process.

Let us now consider two aspects of the students' activity that relate to the questions we listed earlier: first, some **features of the students' *exploratory* work**, particularly in relation to the slightly unexpected nature of the task (from '$\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$' to '$\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$' functions); then, some evidence of the students' – and L's – **perceptions of the *learning-teaching agreement*** that sessions such as this may bring to question.

With regard to the first (students' *exploratory* work), the session was marked by the high likelihood on several occasions of *commognitive conflict*, emerging from the students' *word use* and form of *visual mediation*. Throughout the session the students' standard approach to *substantiation* was to *endorse or reject a narrative* about the objects at stake through indications in favour of – or against – a claim as evident on a screen, or on roughly produced drawings on paper. Combined with their generally non-standard use of *symbolic realizations* (notation, graphs and related terms), the ingredients seemed to be there for *commognitive conflict*. According to the task set by L in the preparation sheet (Figure 1), the students were expected to consider how a graph of a function from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$ would look. However, on various occasions, their utterances, and scribbles produced during the session, seemed to concern functions that looked more like $f + g$, fg, $f \circ g$, rather than $f : \mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$. In this sense the question 'what does a function from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$ look like?' – central in the preparation sheet – was not pursued as directly as L might have expected.

L's recollects some of the student productions (not only the five observed in this session) as follows:

> "In their preparatory work most students defined two functions, e.g. $g(x) = x^2$; $h(x) = \sin(x)$, and then, to draw the graph, composed them in some way to graph the equivalent of $g+h$, gh, or $h\circ g$.
>
> When it was pointed out that they had essentially created a function $\mathbb{R}$ to $\mathbb{R}$, this led to other suggestions such as using the first element to define the axis for the second element, or the first element to create a conventional graph that then got altered by the second element to create some kind of ribbon (2D) or envelope (3D).

Throughout L's contribution was to point out anomalies in such a way that further mathematical invention or adjustments could be attempted, to ask for more exact formulations of what was intended, or formulations using known conventional terms. We remind the reader that the aim was to trigger mathematical actions from the students, not "correct" objects.

In the *commognitive* perspective, one way to evaluate whether the focus and object of the exchanges amongst interlocutors (here L and the five students) are well-coordinated is to examine the forms of *word use* evident in these exchanges. Sfard (2008, p.181-2) distinguishes between *passive*, *routine-driven*, *phrase-driven* and *object-driven word use* – and systematic scrutiny of the exchanges can reveal the type of *word use*. In sessions such as the one we are discussing here there is plenty of deictic language, aimed at screen or paper, and this renders such scrutiny more difficult. Audio or video recording of the sessions (not done for the session we discuss here) is then crucial and this is a methodological decision that the LUMOS team might consider (taking account of the intended non-intrusiveness of the innovations).

A similar observation to the one made above considering how the students' engagement met L's expectations applies to the students' loose, non-standard deployments of notation. During the session L seems also alarmed by this and on several occasions he draws on his *ultimate substantiator* (Sfard, 2008, p. 234) status to alert the students to the precariousness of such loose use of notation (see later in the paper one such occasion concerning the use of the expression '$\cos(10x)$ on x^2'). There was one occasion, initiated by Student A, who proposed the introduction of the notation $t \rightarrow (f(t), g(t))$, which came closest to a standard notational *realization* of the type of function that the preparation sheet invited the students to consider. We elaborate some repercussions of not pursuing this in the session towards the end of the paper.

Further, while the confidence with which the students deployed online software to generate complex and attractive *visual realizations* of their suggestions – often gazed at from all angles and bringing home the potentiality of speedy, intuition-friendly resources – was impressive, it was also notable that these visual awe-inspiring moments were hardly interpreted or explicitly connected to the task set by L in the preparation sheet.

With regard to the second aspect we wish to examine in this account (the students' and L's perceptions of the *learning-teaching agreement*), our account is far less hesitant: simply put, these 'newcomers'' expectations of the 'old-timer', L, who led the session, were very open. It is in fact this openness which brought about the use of 'shakeup' in the title of (Nardi, 2014), the first account of these observations.

Certainly the ethics requirement of the *learning-teaching agreement* for 'tolerance and solidarity' (Sfard, 2008, p.287) was amply met. One incident illustrates what we see as a substantial power-shifting observed in the session: the exchanges taking place in such a session will, in Sfard's terms, eventually result in conceding to one of the present discourses being ultimately accepted by the interlocutors as privileged and paradigmatic. In a more conventional setting this would most likely be L's discourse. In the observed session this conceding did occur – but on the discursive path proposed by one of the students, not L. This was Student D, who proposed an innovative elaboration of the graph of a function from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$: The student defined two functions, the first was drawn in the conventional manner, and then the second was drawn using the graph of the first as the independent axis with scales along this graph, and perpendicular to it, being the same as the originals. A short time after the session Student D had worked out how to use a computer graphing package to handle drawing such a function and offered the following (Figure 2):

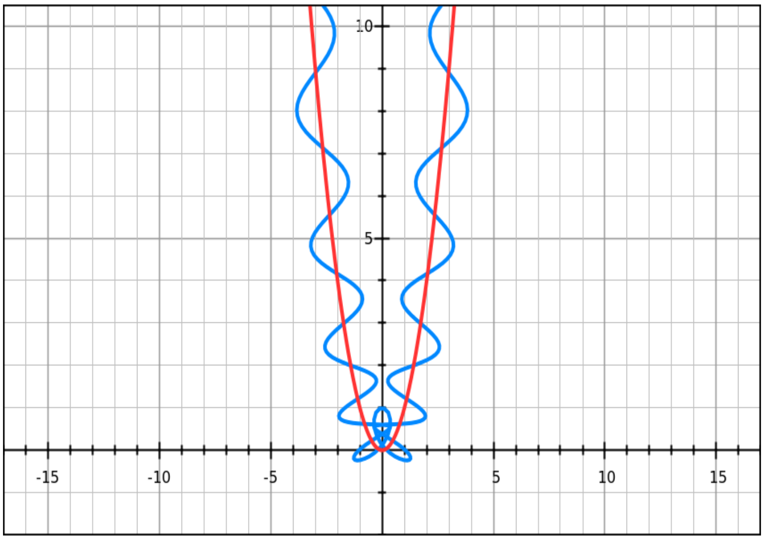

Figure 2. Student D's "$f(x)$ = cos(10x) on x^2" production. He defines f: $\mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$, $f(t) = (x,y)$, where $x=t-\cos(10t)\sin(\mathrm{atan}(10t))$ and $y= t^2+\cos(10t)\cos(\mathrm{atan}(10t))$.

In all this, L coordinated the intense exchanges with explicit and deliberate distancing, in fact with minimal use of his ultimate substantiator status.

It is in this ease with conceding this status that, in our view, the grandest element of the aforementioned 'shakeup' lies: L seemed uniformly open to the *narratives* proposed by the students; he seemed to actively hold back from encouraging their endorsement or rejection by the group. He seemed to sustain a mental list of proposed *narratives* that there had been no time to pursue, such as Student A's (see earlier in this section). In the pragmatic context of limited time – and Student D's more vocal presence attracting perhaps more attention than Student A's – this is not unlikely to happen during teaching. The observations of the session suggest that Student N appeared to experience the most obvious discovery moments. Student J's gestural language and body positioning also suggested so, particularly when 3D images started appearing on the screen of Student N's laptop. Only Student B appeared minimally participant, and quietly perplexed.

The session had buzz and warmth – but also left a slightly anxious sense of unfinished business about not having worked on Student A's proposed *narrative*. We note however that the events that followed on the same evening of the session to some extent appeased that anxiety: Student A wrote to L with an imaginative account of Student D's idea (omitted here due to limitations of space). He had nobly conceded to the temporary dominance of another student's proposed *narrative* in the session but made the most of it …afterwards. There is at least one implication of this turn of events (and we do say this in full awareness of the modesty of a claim based on evidence from a single observation of a LUMOS innovation session):

> "for at least the two hours of this *engagement session* these 'newcomers' slipped comfortably into the shoes of the 'old-timers', with all the fallibility and excitement that walking in these shoes entails. For that alone, surely this is an innovative path worth treading." (Nardi, 2014, p.10)

A *COMMOGNITIVE* TRACING OF DESIRED LEARNING OUTCOMES?

Several questions emerge from our account of the ‘shakeup’ of the *learning-teaching agreement* in the observed session: Is this ‘shakeup’ liberating, perplexing to the students, both? How does it sit alongside the rest of these students’ experiences at this university? They seem comfortable with it but will they stay so throughout? When, if at all, will they demand a reinstatement of L’s *ultimate substantiator* status in the form of a demand for (say) specific assessment of their proposed *narratives* (on functions from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$, and beyond)? We conclude with tentative responses to these questions, based on the written reflections of L (second author).

L notes that the ‘conceding of much of [his] status’ we evidence here does not refer to his ‘administrative status, nor [his] professional status’ but his ‘status as ‘old-timer’’. He prefers, however, the phrasing ‘controller of content of discussion’:

> ‘I still controlled the direction quite a lot, although I used their prompts, choosing between them for (hidden) pedagogical and mathematical reasons. I believe that I can remember making both pedagogical and mathematical decisions at such moments.’

While on that point he stresses the pressures of ‘running one of these sessions’:

> ‘[it] is exhausting for the lecturer because of the constant attending to the direction of the conversation and evaluating it for potential mathematical (content and process) and pedagogical value. It is why, when a mathematician first watched me run a session […], then tried it himself, he said that it was much, much harder than it looked.’

As to whether the experience is liberating or perplexing to students, he estimates that ‘about half’ ‘find it liberating’ and recalls students talking about ‘re-finding the creativity in mathematics’ and ‘expressing their pleasure at the sessions’. For ‘about a quarter’ though ‘it is perplexing – they just do not seem to get what it is about’ and for ‘another quarter it is a mix between the two – interesting but they feel a bit out of their depth’. These estimates are his ‘subjective judgement’ and he highlights that ‘these groups are not at all related to the students’ mathematical ability’.

In relation to how the *low lecture* experience sits alongside the rest of the students’ experiences L notes that his institution is ‘reasonably liberal’ and that it would not be unusual to find lecturers who are willing to ‘cede some of their status’. Also many of these students ‘will have had a similar sort of experience at times in their final year of school’ where they are likely to have been ‘treated quite respectfully as mature learners’. While ‘probably unusual at first year’ this respectful treatment in the *low lecture* innovation would then be ‘not so strange’. Other factors, such as the presence of mature students in the group, may also reduce the ‘strangeness’ of the experience and make the students’ commitment to this approach more resilient too:

> ‘I’ve not seen any students in any session get MORE perplexed or uncomfortable, I’ve seen some get less and some stay the same. For those who were comfortable with it, a few grow into it significantly quite quickly.’ [L’s emphasis]

As an example, L returns to Student A's 'radical' follow up (see earlier) of the discussion in the observed session: 'He was checking with me that [his ideas] were ok, but he had really taken on the idea that the mathematics was there to be played with'. And while the students 'do check things out' with him (L),

> 'they have never seen this as "assessment" in the formal way (is it right or wrong) but rather (do you think this is an ok track to follow). All the students who have done this seem to have caught on to the fact that this is exploration, and anything goes in some respects – it is what you do with it that counts, not what it is you are working with. I take this as a huge endorsement of the idea of the *engagement sessions* – that they are not about content but about process. I did not expect that most of the students would "get" this so quickly, although I did reiterate it often both in writing and verbally.'

Analyses from the implementation of the LUMOS innovations are ongoing. With this brief account we aim to make the most of the potency of a *commognitive* approach to this analysis. Our efforts here are a small step towards meeting this aim.

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