What makes a claim an acceptable mathematical argument in the secondary classroom? A preliminary analysis of teachers' warrants in the context of an Algebra Task

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The study we report builds on previous research conducted by Nardi, Biza and colleagues, which examined mathematics teachers' considerations of what makes a claim an acceptable mathematical argument in the secondary classroom. We identify teachers' considerations in their written responses to tasks and then in semi-structured interviews that probe these written responses. Here we present data from six teachers and one such task, a (GCSE-level) Algebra Task. The tasks we invite the teachers to engage with, of which the Algebra Task is one, are structured as follows: a mathematical problem that students are likely to encounter in typical secondary mathematics lessons; fictional student responses to the problem (grounded on student responses found by relevant research as typical); and, an invitation to teachers to solve the problem, consider the purposes of its use in the lesson, reflect on the student responses and describe the feedback they would provide to the students. So far, we have proposed a theoretical tool for analysing mathematics teachers' warrants for the preferences they express in their written responses and the interviews. The tool is based on our adaptation of Toulmin's model of argumentation in which we classify teachers' warrants according to pedagogical, epistemological and institutional considerations.

Keywords: teacher argumentation; warrants; Algebra; proof; Toulmin's model for argumentation.

Studying the practical rationality of mathematics teaching with Toulmin's model

The study we draw on here aims to refine typologies that describe teachers' knowledge and beliefs – such as Shulman's (1986; 1987) constructs of pedagogical content knowledge and Hill and Ball's (2004) mathematical knowledge for teaching – and explore how these knowledge and beliefs transform into pedagogical practice. Our aims resonate with those in works – such as Herbst and colleagues' (Herbst and Chazan, 2003; Miyakawa and Herbst, 2007) – that address the complex set of considerations that teachers take into account when they determine their actions. Of particular relevance to our analyses is what Herbst and Chazan (2007) call the practical rationality of teaching, "a network of dispositions activated in specific situations" (p. 13). To explore this, in the study we draw on here, we invite teachers' comments on classroom scenarios (Nardi, Biza and Zachariades, 2012; Biza, Nardi and Zachariades, 2007) that they are likely to experience in their lessons. We invite these comments first in writing and then in interview. We then analyse the arguments that teachers put forward using an adaptation of Toulmin's model of argumentation (1958) and Freeman's (2005) refinement of parts of the Toulmin model.

In what follows we describe this adaptation, and how it came to be, and illustrate its employment in a sample of recently collected data. We conclude with a brief discussion of how these recent analyses fit with other works and how they relate to where our subsequent research in this area is heading.

A classification of warrants in the arguments of mathematics teachers

Toulmin's (1958) model describes the structure and semantic content of an informal argument in terms of six basic types of statement, each of which plays a particular role but which are not necessarily all present in the utterance of an argument: the conclusion (C) is the statement of which the arguer wishes to convince; the data (D) are the foundations on which the argument is based; the warrant (W) is what justifies the connection between data and conclusion and is supported by the backing (B), which presents further evidence and justifications; the modal qualifier (Q) expresses degrees of confidence; and, finally, the rebuttal (R) consists of potential refutations of the conclusion.

Toulmin's model has been employed by researchers in mathematics education across educational levels and mainly to analyse student arguments. Most use reduced versions of Toulmin's model (CDWB or CDW) but recently researchers have argued in favour of employing the full model – see Nardi et al. (2012: 159-160) for examples of studies of both kinds. Of particular interest to us are works that elaborate the model by offering a classification of warrants, such as Freeman's (2005) and, within mathematics education, Inglis et al. (2007)'s classification of inductive, structural–intuitive and deductive warrants that underlie the *mathematical* arguments of their participants. Our analysis aims to discern, differentiate and discuss the range of influences (epistemological, pedagogical, curricular, professional and personal) on the arguments that *teachers* put forward when they elaborate the decisions they make in the course of a mathematics lesson.

Our adaptation of Freeman's (2005) classification of warrants is as follows:

- *a priori* warrant: resorting to a mathematical theorem or definition (*a priori–epistemological*) or to a pedagogical principle (*a priori–pedagogical*);
- *institutional* warrant*:* a justification of a pedagogical choice on the grounds of it being recommended or required by institutional policy, such as a national curriculum or a textbook (*institutional–curricular*) or that it reflects standard practices of the mathematics community (*institutional–epistemological*);
- *empirical* warrant: the citation of a frequent occurrence in the classroom (according to teaching experiences, *empirical–professional*) or resorting to personal learning experiences in mathematics (*empirical–personal*);
- *evaluative* warrant: a justification of a pedagogical choice on the grounds of a personally held view, value or belief.

The purpose of such an elaboration of the types of arguments teachers use is to demonstrate that the decisions teachers make do not have exclusively mathematical (epistemological) grounding. Their grounding is broader and includes a variety of other influences, most notably of a pedagogical, curricular, professional and personal nature. Acknowledging the breadth and scope of teachers' warrants implies the need to re-define our criteria for evaluating teachers' arguments in a pedagogical context and for exploring aforementioned practical rationality of teaching (Herbst and Chazan, 2007). We demonstrate this breadth and scope with samples of our preliminary analysis of recently collected data. We start with presenting the aims, methods and participants of our study.

Methodology

The six teachers who participated in this study were staff members in a single school, located in the East Midlands of England. The school was approached with an invitation to take part in the study because it had previously been involved in research projects with the University of Nottingham and Loughborough University. All members of the mathematics department were invited and the six teachers were self-selected. The school is an average-sized, mixed-gender, state-funded, secondary school serving the 14–19 age group. Students' results in GCSE examinations are higher than the national average. The proportion of students that could be described as disadvantaged – based on the number of students eligible for free school meals – was much lower than the national average. The school population is predominantly white British.

The mathematics department consisted of a newly appointed head of department, an assistant head of department, an advanced skills teacher and six mathematics teachers who were part- and full-time. Two of the teachers held senior leadership roles in the school, one of whom was the head of mathematics in the previous year and had been promoted recently.

The teachers who participated in the study were all female, and included: the new head of mathematics, Teacher P; Teacher S, an experienced teacher but also new to the school; Teacher R, a PGCE student; Teacher T, an Advanced Skills Teacher who had been teaching for five years; Teacher A, who was in her second year of teaching; and, Teacher M, the previous head of mathematics and now assistant principal who had twenty-five years' teaching experience. Each teacher was given £40 as a reward for participating in the project.

Anonymity, confidentiality and the right to withdraw were guaranteed to all participants who had to provide their consent in writing. Ethical approval for the study was awarded by the Research Ethics Committee of the School of Education and Lifelong Learning at UEA, which supported the study – see Acknowledgement.

Data-collection involved teachers completing a questionnaire in which they engaged with a Task – see next Section. The questionnaires were distributed and collected by the third author, Watson, who also carried out interviews with the teachers soon after. The interview protocol was designed to elaborate the teachers' written responses to the Task, with a particular focus on the warrants underlying the feedback they provided for each of the fictional student responses.

The interviews were audio recorded and transcribed verbatim. Initial analysis of the six datasets (each consisting of the Task script and interview transcript from each of the six teachers) was completed collectively by the team. During this first scrutiny of the data key issues and themes were identified. Each team member then carried out further independent analysis of two datasets steered by this initial identification of themes and deploying the theoretical approach developed in earlier studies by Nardi, Biza and colleagues (see Biza et al., 2012; Nardi et al., 2012). The preliminary analyses we present here are a first attempt to weave together these independent analyses.

A first observation that emerged from our initial scrutiny of the data concerned the relatively strong presence of pedagogical and institutional considerations in the teachers' justifications for what they appear to prioritise in their written and interview accounts. In the following, we sample from the six datasets in order to substantiate and elaborate this observation. First, however, we introduce the Task.

The Algebra Task

The Task that the six teachers engaged with consists of five questions (Figure 1). In question 1, teachers were asked to solve a mathematical problem from GCSE level algebra. Then, in the three parts of question 2 (2a, 2b and 2c), three fictional students' responses were offered to the same problem and teachers were asked to provide feedback to these responses. Finally, the teachers were asked to reflect on the aims of this problem (question 3); to comment on whether these fictional responses are likely to occur in their lessons (question 4); and, to offer any other comment on the Task (question 5).

Question 1
Write down your solution to the following problem.
Is the expression $x^2<x$ always true, sometimes true or never true? Justify your response.
Question 2
The teacher of a higher ability of year 10 class gave students the above question. Please provide feedback to the students who gave the following responses:

> **Question 2a**
> I tried several positive and negative figures and the outcome was never true:
> For $x= 0$, $0 < 0$ which is not true.
> For $x = 1$, $1 < 1$ which is not true.
> For $x = -1$, $1 < -1$ which is not true.
> For $x = 1.5$, $2.25 < 1.5$ which is not true.
> For $x = -1.5$, $2.25 < -1.5$ which is not true.
> So, this expression is never true.
> **Question 2b**
> This expression is *never* true because the square of a number is *always* bigger than the number.
> **Question 2c**
> If x is negative, x^2 is positive. So, $x^2>x$ is true, therefore, $x^2<x$ is not true.
> If $x = 0$, $0^2 = 0$. So, $x^2 <x$ is not true.
> If x is positive, x^2 equals x times x and this is always bigger than x.
> So, $x^2> x$ is true, therefore, $x^2<x$ is not true.
> As a result the expression is never true.

Question 3
In your view what is the aim of the problem featured?
Question 4
Are the student responses likely to be observed in your lessons? If so, which ones? If not, what responses would you expect?
Question 5
If you have any comments on the task, in relation to the main study we are intending to carry out, please write them here.

Figure 4: The Algebra Task

The idea for this Task originated in a popular activity for students on the *evaluation of validity of statements and generalisations* (Swan, 2006) in which students have to decide whether a statement is "'always', 'sometimes' or 'never' true [and] then justify their decisions with examples, counter-examples and explanations" (pp. 146-147). In resonance with this type of activities, in the Algebra Task teachers were given the algebraic expression $x^2<x$ and were asked to consider whether this expression is always, sometimes or never true. This expression is *sometimes true*: it is

true for any value of x between 0 and 1, and false for any other value of x. There is a range of approaches to justifying that the expression is *sometimes true*:

- Trial numbers for which the expression is not true (e.g. 0, 1, 1.5, etc.) and others in which it is true (e.g. $\frac{1}{4}, \frac{1}{2}$ etc.). A trial of one number for each case is sufficient evidence.
- Make the graphs of $y=x^2$ and $y=x$ and observe that the graph of $y=x^2$ is always above the graph of $y=x$, except for the interval (0,1).
- Solve the algebraic inequality: $x^2<x$ or $x^2-x<0$ or $x(x-1)<0$ which is true only when x and x-1 have different signs. This is true only when $x>0$ and $x<1$.

The expression and the fact that it is not true for all integer values of x was a deliberate choice. Students often try specific numbers to validate the truth of an expression and ignore fractions and decimal numbers, especially those which are between zero and one. This imprecise practice is reflected in the fictional response 2a. Additionally, the expression $x^2<x$ challenges a common misconception that the square of a number is always greater than the number, reflected in the response 2b and in the third step of the response 2c. Response 2c, in comparison to 2b, offers a more elaborate explanation with a distinction of cases for different numbers (i.e. negative, zero and positive). We included this type of response to examine if teachers' feedback would be affected by the format of the fictional student's response. Would teachers, for example, prefer a more apparently formal response that distinguished cases for different types of number? We sample from the six teachers' responses in order to explore what the teachers prioritised and how they justified these priorities.

Data and analysis: the six teachers' pedagogical and institutional considerations

Here we focus on the six teachers' pedagogical and institutional considerations as evident in their intended feedback to the students in their written responses to the Task and their comments in the interviews.

Regarding the teachers' ***pedagogical considerations***, the teachers appear willing to engage the students, ask probing questions that feedback directly to the students and generally move in accordance with where the students are. For example, Teacher A asks the student in her response to Question 2a: "Are there any other types of numbers that you haven't considered?." When, in the interview, she was asked to elaborate her choice, she responded:

> I try and get the students to work out for themselves what to do, kind of leave them an open question so I've asked if there are any other types of numbers that you haven't considered. Now, I'm not sure if they would get that from that, but I couldn't think of another way. […] so I don't want to give them the answer because at first I was going to write "have you thought about numbers between zero and one?" But then that's just giving them the answer and I want them to think more deeply about...

Teacher P also acknowledges that the teacher needs to start from the student's point of view: "I think it's depending on the…because of the answer and how it is and how the students approach the problem, it's getting them to basically sort of move on from where they are." We see Teacher A's intention to avoid 'just giving them the answer' and prioritising the opportunity to 'think more deeply' – as well as Teacher P's willingness to 'move on from where they are'– as *a priori pedagogical* warrants for their prioritising of certain approaches to the mathematical problem in question.

In the interview Teacher S starts off too with the statement of a pedagogical principle, the value of assessment for learning:

> When you're giving feedback obviously you're not just going to say, "no, that's wrong," so this is what you should have done. So if you are giving feedback and using assessment for learning then you are going to ask questions that hopefully will make them think further.

In response to Question 2b she writes: "Could you please explain how you know this? Could you give me some examples to justify this statement?" Invited to consider her response in this question she proclaims that her plea for more explanation is, "Because I've got nothing to work on there, I can guess at what is going on in their head but I have nothing, almost nothing to feedback with." Transparency of the student's mathematical thinking (for pedagogical reasons) is her priority, "So at least I can start seeing what their thinking [is] and then move them on a bit further."

A sharp contrast between the teachers' pedagogical and epistemological priorities emerges on the occasion of discussing what Teacher S describes as the illustrative power of examples (similar to Teacher R who wants to help the students see how different examples can lead to different answers). It seems that identifying and demonstrating a range of examples is crucial in some teachers' feedback (pedagogical priority), even though (for mathematical reasons) they recommend coverage of all real numbers in their responses to Questions 2a and 2c.

However, this explicit prioritising of pedagogy is not the case for all participants. Teacher T, for example, – who, incidentally, had misunderstood the mathematical problem in the Task and realised the misunderstanding towards the end of the interview – writes in response to question 1: "never true because squaring a number is always bigger than the value of x." Under the influence of this common misconception she writes in response to question 2a: "Well done for trying positive and negative values of x and also decimals. Could you provide any further reason for your answer? Any proofs?" Algebraic misconception notwithstanding, Teacher T, overall, wants to see the students involved in particular mathematical practices such as exploring and generalising:

> I would want them to think about it, then generalise from that, like we were just saying generalise whenever you square a number it always ends up bigger, I want them to put a bit more meat into it instead of just trying a few examples for it.

In question 2b she requests a trial of specific cases of numbers: "Well done for providing a reason for your answer would you write this as proof. But is this true for positive and negative values of x? What about decimals?" We note her use of words such as 'reason', 'proof' and 'generalise'. But we also note that she wishes to see more exploration of the problem and is interested in encouraging students to investigate the problem through specific types of numbers in order to gain access to their way of thinking (reverting in this way to a prioritising of pedagogy, which is similar to Teacher A, Teacher R and Teacher S):

> No you see I don't know whether they have thought about it in their head and that's where they've come from, we as teachers always need to see things written down don't we but they might have thought about things in their head.

Regarding the teachers' ***institutional considerations***, we noted that constructs such as student Ability Level and Strength/Target policy – both terms reflecting a use of language that is common in curriculum and policy documents – are part of the discourse of the teachers quite strongly (explicitly or implicitly) and appear to influence feedback to the students.

Teacher M, the most senior participant in the study, often responds in ways that demonstrate solid compliance with the school's marking policy: "identifying the 'strength' in a student response and making the 'target' the student should aim for explicit." She tailors her pedagogical approach to this frame:

> […] well I would I do now follow the school marking policy, which is s stands for strength, t stands for target, and so I'm just in that mind-set now when I'm marking work and so try to pick out the strength whatever that might be and then obviously for me the more interesting thing is the target in what direction to, you know, lead them into as a responsible piece of homework. This particular one yeah it's sometimes difficult to pick out the strengths you know, like I've just already said we're more interested in the targets about you know about using that information to stretch them and to push them further forward and so.

Analogously, the level of guidance Teacher A aims to offer students is tailored to their perceived ability:

> …depends on your group doesn't it? I think like with your middle ability and lower ability they might need a bit more direction. But I wouldn't…I don't think I would feel confident just going in and giving them something to do straight away. I like to let them feel uncomfortable because I think that's good and then when they start to kind of lose the will to carry on it's kind of then step in and…

Concluding remarks

Across the six datasets a strong observation is that the teacher responses are more intensely characterised by a pedagogical discourse, and less a mathematical one. For example, although some of the teachers mentioned 'proof', we have little evidence of what they actually mean by this term. Some stress the importance of 'justification' in students' responses but it is not clear whether they mean proof / verification of the statement, being convinced of the truth of the statement or some explanation accompanying the answer to the mathematical problem.

Furthermore, we noted that the teachers appraise the fictional student response in question 2c because it is written in "quite a mathematical way" (Teacher A); or, it is a "more general statement" (Teacher P); or, because students "managed to express what they're thinking in a more clear way […] they put it in logical order" (Teacher R); or, they made a "really brave attempt [to] use inequality signs" (Teacher S). These were occasions where a glimpse into their epistemological priorities was possible.

Generally, however, the teachers' pedagogical aims (and their *a priori pedagogical* warrants thereof) are explicit and detailed in their responses, while their mathematical aims (and *a priori epistemological* warrants thereof) are not always clearly discernible. Even when teachers speak about the trial of *other numbers* in questions 2a and 2c – Teacher S, for example, seems to prioritise rigour when she writes "to prove it for all possibilities, not just a selection" – they do not cite a fully-fledged, mathematically acceptable process for *checking* all numbers. It seems that, for most, what is missing is simply a trial of a number between 0 and 1. And, in most cases, priorities ultimately revert to pedagogy: when, for example, Teacher R expresses her willingness to "link back" to "concrete mathematics", her previously stated epistemological priorities (for the need for clarity and transparency) emerge as rather dimmer than her clear and explicit pedagogical priorities.

The claim that emerges from this preliminary analysis of the six datasets and which we put forward in this paper is as follows: explicitly, in at least four of the six datasets, the teachers propose the use of examples in question 2b, even though they recommend coverage of all real numbers in questions 2a and 2c. There is a pattern in the teachers' stated pedagogical priority that relies on this perceived power of exemplification. Teacher P says that she does this because she wants to start from the point students are. Teacher A says that she will start with examples and then she will ask the students to try other numbers, if they still believe that the expression is never true. Teacher R wants to help the students see how different examples give different answers and Teacher S talks about the illustrative power of examples.

This pedagogical prioritising – and the *strength and explicitness of the a priori pedagogical warrants* on which it stands – is in some contrast with the *less explicit, briefer and less strongly warranted epistemological prioritising*. We credit the classification of warrants that underpins the analysis of these data with allowing us this type of insight and we aim to expand and deepen this insight further in subsequent phases of our analysis.

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