Leaving Mathematics As It Is: Wittgenstein's Later Philosophy of Mathematics

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Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.

For it cannot give it any foundation either.

It leaves everything as it is.

It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other. (*PI*, §124)

Abstract

Wittgenstein's later philosophy of mathematics has been widely interpreted to involve Wittgenstein's making dogmatic requirements of what can and cannot be mathematics, as well as involving Wittgenstein dismissing whole areas (e.g. set theory) as not legitimate mathematics. Given that Wittgenstein promised to 'leave mathematics as it is', Wittgenstein is left looking either hypocritical or confused.

This thesis will argue that Wittgenstein can be read as true to his promise to 'leave mathematics as it is' and that Wittgenstein can be seen to present coherent, careful and non-dogmatic treatments of philosophical problems in relation to mathematics. If Wittgenstein's conception of philosophy is understood in sufficient detail, then it is possible to lift the appearance of confusion and contradiction in his work on mathematics. Whilst apparently dogmatic and sweeping claims figure in Wittgenstein's writing, they figure only as pictures to be compared against language-use and not as definitive accounts (which would claim exclusive right to correctness).

Wittgenstein emphasises the importance of the applications of mathematics and he feels that our inclination to overlook the connections of mathematics with its applications is a key source of a number of philosophical problems in relation to mathematics. Wittgenstein does not emphasise applications to the exclusion of all else or insist that nothing is mathematics unless it has direct applications. Wittgenstein does question the alleged importance of certain non-applied mathematical systems such as set theory and the logicist systems of Frege and Russell. But his criticism is confined to the aspirations towards philosophical insight that has been attributed to those systems. This is consonant with Wittgenstein's promises in (*PI*, §124) to 'leave mathematics as it is' and to see 'leading problems of mathematical logic' as 'mathematical problems like any other.' It is the aim of this thesis to see precisely what Wittgenstein means by these promises and how he goes about keeping them.

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Introduction

> The philosopher easily gets into the position of a ham-fisted director, who, instead of doing his own work and merely supervising his employees to see they do their work well, takes over their jobs until one day he finds himself overburdened with other people's work while his employees watch and criticize him. (*PG*, p.369)

Wittgenstein was particularly keen to stress that he did not want his later philosophy of mathematics to get involved in pronouncing upon which mathematics was valid and which was not – making those decisions is a job for mathematicians. One of the key themes of his remarks upon his own approach to the philosophy of mathematics is that mathematics should be seen to be left untouched by his investigations. Even stronger than this, he felt that it was a serious mistake by other philosophers to get involved in either criticising or justifying mathematics (*PI*, §124). Despite Wittgenstein's intentions, many interpretations of Wittgenstein's work on mathematics paint a picture of him as the 'ham-fisted director' that he warned against. The tone was set for this by early reviews of *RFM*[1] by the likes of Kreisel (1958, p.143–144) and Dummett (1964, p.491), which criticised Wittgenstein for not knowing enough about mathematics.[2] But even some of the most sympathetic of contemporary interpreters of Wittgenstein have painted a similar picture. Amongst them Monk (1995) puts the criticism especially poignantly[3]:

> ...he [Wittgenstein] hardly seems to believe it [pure mathematics] exists. The only activity that might deserve the name 'pure mathematics' that emerges from his 'description' is the construction of calculi for either use or amusement; that is, an activity that is either indistinguishable from applied mathematics or else is a frivolous pastime that has nothing to do with science. (1995[4])

The principle of charity surely weighs heavily against interpreting any philosopher as doing exactly what they said they would not do. And yet interpreters have been prepared to bite this particular bullet. The feeling has perhaps been that Wittgenstein's comments leave them little choice – Putnam puts this particularly clearly when he says that "the philosopher who famously said "Take your time" failed to take his time" (2007, p.246).[5] Despite, or perhaps because of, Putnam's strong sympathies for Wittgenstein, at points one can feel Putnam's exasperation leaping off the page:

> There is no system of irrational numbers- but also no super-system, no "set of irrational numbers" of higher-order infinity. This from a philosopher who doesn't put forward philosophical "theses"?! (2007, p.239)

This thesis will advance a case that the exasperation of Wittgenstein's interpreters can be spared and Wittgenstein himself can be spared from being painted as a 'ham-fisted director.' My suggestion will be that many of the difficulties in interpreting Wittgenstein's philosophy of mathematics arise from a failure to be sufficiently clear about Wittgenstein's approach to philosophy.

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[1] References to the 3rd edition. The usual abbreviations for Wittgenstein's works are followed, as given in the Bibliography.
[2] Monk (1995) covers the history of how Wittgenstein's philosophy of mathematics has been received.
[3] It should be noted that (Monk, 2007) sounds a much more sympathetic tone.
[4] The source is online and has no page references.
[5] Putnam is presumably thinking of "This is how philosophers should salute each other: 'Take your time!'" (*CV*, p.91).

It is perhaps understandable if interpreters have struggled to place the intentions of Wittgenstein's project in relation to his approach to philosophy in general. Whilst Wittgenstein gives numerous asides like the remark about the 'ham-fisted director', neither of the key texts with regard to his later philosophy of mathematics (*RFM* and *LFM*) provides a clear statement or extended discussion of his approach to philosophy. I take the first clear statement of Wittgenstein's later methodology to be the 'chapter on philosophy' in the *Big Typescript*.[6] This was composed in 1930-31[7] and Wittgenstein does not appear at any later point to renounce or significantly modify what he says there, instead including those remarks later in the *Philosophical Investigations* as (*PI*, §89–133). There is hence reason to think, as I intend to assume, that Wittgenstein thereafter does relatively little in the way of revisiting his approach to philosophy because he does not see any benefit in revisiting what he had already formulated as well as he could.[8]

Following Kuusela (2008), I will argue that Wittgenstein's approach to philosophy can be seen as aimed to give us a set of techniques with which to avoid dogmatism with regard to philosophical claims. Instead of claiming to capture the necessary characteristics of mathematics, Wittgenstein instead aims to bring out the particular aspects of mathematics relevant to particular questions. Since the aim is not to bring out a single essence that characterises all of mathematics, philosophical claims need not (for Wittgenstein) be construed as competing with one another for the exclusive right of correctness. Wittgenstein can thus be seen as avoiding commitments to sweeping claims of a form like 'all mathematics must be X', where 'X' expresses some philosophical requirement that X be seen as a necessary property of mathematics. Whilst statements like this might sometimes appear in Wittgenstein's work, it will be argued that they appear only for the purpose of comparing different models to bring out different aspects of mathematics (or whatever is at issue in the particular context). Rather than simply rejecting dogmatic claims, Wittgenstein instead tries to show what is dogmatic about them by exploring the ways in which they misrepresent (as well as what truth they do contain).[9] In order to do so, Wittgenstein needs to look at a wide range of cases and sometimes also to see each case from a variety of angles. If Wittgenstein is viewed as aspiring to make claims concerning necessary characteristics (e.g. the essential characteristics of mathematics) then this variety can seem deeply confusing. When struggling to find a foothold with the text it can be tempting to read Wittgenstein as dogmatically advocating one of the models (claiming for it exclusive right of correctness across a wide field) which he is in fact exploring (i.e. using

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[6]In this I am siding with Kuusela (2008, p.313-314) against Diamond (2004, section 5) and Pichler (2004, section 4.3). Both Diamond and Pichler read the *Big Typescript* as prior to Wittgenstein's arriving at an answer to the problem of dogmatism. Whilst the transition in Wittgenstein's philosophy may have been gradual, I will assume that *Big Typescript* expresses an important part of the transition.

[7]See (Stern, 2005, p.222). This 1930 date justifies my using any post-1930 text in this thesis. The composition of *PR* and discussions in *WVC* date to just before the 'chapter on philosophy' (though there may have been overlap). But this does not necessarily mean that Wittgenstein did not have a methodology in mind at that point – he may simply not have written it down yet. For this reason those texts will also be drawn upon, though with some caution.

[8]If *LFM* perhaps provides more in the way of methodological asides than *RFM*, this is most likely because not all of Wittgenstein's lecture audience was familiar with his approach.

[9]Wittgenstein comments that "one cannot take too much care in handling philosophical mistakes, they contain so much truth" (*Z*, §460).

to make a comparison which brings out a particular limited point). Identifying and resisting the temptation to misread Wittgenstein in this way will be central to the aims of this thesis. It will be argued that the appearance of inconsistency or incompetence surrounding Wittgenstein's philosophy of mathematics can be lifted by showing that Wittgenstein does not dogmatically advocate claims which would make him inconsistent or incompetent.

In order to read Wittgenstein as true to his conception of philosophy, it will be especially important to understand how Wittgenstein's avoidance of dogmatism relates to his aspiration to avoid being a 'ham-fisted director', overburdened by questions which are more properly left to mathematicians. One remark in particular within (*PI*, §89–133) expresses a number of dimensions of this aspiration and how it relates to Wittgenstein's conception of philosophy. For this reason, this thesis will be structured around that remark:

> Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.
>
> For it cannot give it any foundation either.
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> It leaves everything as it is.
>
> It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other. (*PI*, §124)

Part 1 will give an explanation (following Kuusela (2008)) of the conception of philosophy expressed in (*PI*, §89–133), together with explanation of how this relates to Wittgenstein's work on mathematics in particular. Part 1 will therefore develop an initial understanding of (*PI*, §124) and what Wittgenstein meant by it.

In order to appreciate the significance of (*PI*, §124) more fully, it is necessary to see how Wittgenstein's methodology is employed. This will be discussed in the rest of the thesis (including parts 2, 3 and 4) with the intention of bringing out the significance of (*PI*, §124) in relation to Wittgenstein's later philosophy of mathematics. Part 2 will focus on Wittgenstein's treatment of questions related to the 'peculiar inexorability of mathematics' (*RFM*, p.37), with a view to showing how it is that philosophy 'cannot give any foundation' to mathematics. This will begin to undermine the picture of Wittgenstein as a 'ham-fisted director' by suggesting that some of the theses which have been attributed to Wittgenstein are better seen as models which Wittgenstein *makes use of* but does not dogmatically advocate.[10]

Part 2 looks at Wittgenstein's view of the role of mathematical propositions and his arguments that it is misleading to view mathematical propositions as descriptions. If one subscribes to a Platonist view of mathematics, then one will take mathematical propositions to be descriptions of mathematical entities in a pre-existing mathematical reality. In part 2 we see that this model tends to give a distorted picture of how we perform basic mathematical inferences like expanding a series, as well as making mathematical

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[10]Maddy (1993) and Steiner (2009) are explicitly argued against in part 2, with more left to be said against their readings in part 3. Whilst not central targets, conventionalist readings like those of Wright (1980) and Kripke (1982) are implicitly criticised (for more explicit criticism see Diamond (1989, p.14 & p.28-29)).

propositions appear mysterious by positing objects which then seem like they must somehow 'exist necessarily.' Positing mathematical objects may be intended to allow mathematics to be seen as non-arbitrary and as sensitive to some kind of reality, thereby giving mathematical practice a kind of philosophical foundation. But the Platonist's model tends to give a distorted view of the way we use mathematical propositions and as such it makes basic mathematical inferences and even mathematical propositions themselves (insofar as they are portrayed as having a mysterious subject-matter) appear problematic. The Platonist can therefore be said to be guilty of contributing to philosophical confusion by applying a model (the model of descriptive propositions) to cases which it does not fit. In part 2 we will see how Wittgenstein employs his clarificatory approach (discussed in part 1) to resolve the problem which concerns the Platonist, acknowledging that mathematics is non-arbitrary without needing to posit mysterious objects.

Part 3 will take up some questions closely related to those raised in part 2 with a view to showing that Wittgenstein does not adopt any dogmatic theses (accounts claiming to have discovered necessary truths) and as such his philosophy 'leaves mathematics as it is.' This will require addressing topics which have been thought by interpreters to do the opposite. When seen in the light of a precise reading of Wittgenstein's method, it will be possible to see that some of these discussions are some of the most illustrative of Wittgenstein's non-dogmatic approach. Whilst I do not think that Wittgenstein held any particular topics to be central to his work in the sense of having other treatments dependent upon those discussions, nonetheless some of the topics to be covered are central in a different sense. They are topics where the temptation to say something dogmatic is particularly strong and hence it can be hard to see how Wittgenstein's approach philosophy of mathematics is to be credible unless one can understand Wittgenstein as avoiding dogmatism in relation to these topics.

Part 4 will bring out how Wittgenstein's discussions treat 'leading problems of mathematical logic' as 'problems of mathematics like any other.' This comment within (*PI*, §124) can be seen to relate to interpretations of particular mathematical results which have been thought to be of philosophical significance. Wittgenstein borrows the phrase from Ramsey (1987, p.2), which Ramsey had used to refer to the problem of finding a procedure to determine the truth or falsity of any given logical formula (the famous *Entscheidunsproblem* problem). Whilst such problems might appear philosophically important, Wittgenstein cautions against supposing that pursuing them would "bring to light essential truths about mathematics" (*PG*, p.196). For Wittgenstein such pursuits should not be interpreted in a way which gives them an inflated philosophical significance and he comments that "there can't be any 'leading problems' of mathematical logic if those are supposed to be problems whose solution would at long last give us the right to do arithmetic as we do" (*PG*, p.196).

Wittgenstein's rejection that we could "at long last" earn the "right to do arithmetic as we do" stands in contrast to the views of Frege and Russell. Frege and Russell interpreted their logical systems as providing a definition of the notion of 'natural number.' Whilst Frege and Russell thought that their systems would provide an insight into the nature of natural numbers and provide arithmetic with a secure foundation, Wittgenstein argues that insisting on seeing the natural numbers as Frege and Russell define them would rob arithmetic of much of its power. The significance which Frege and Russell attribute to their systems is misplaced since they fail to see that their interpretations of the significance

of their logical systems are not as compelling as they take them to be. This issue embodies a theme which re-emerges within a number of Wittgenstein's discussions in *RFM* in particular. Wittgenstein discusses cases of philosophical significance being attributed to mathematical results and then critically suggests alternative interpretations and scenarios that put pressure on any claim to exclusivity that the initial interpretation might have seemed to have had.

The contention of this thesis is that, despite widespread belief to the contrary, Wittgenstein's philosophy of mathematics is consistent with (*PI*, §124). In order to argue for this, close attention to Wittgenstein's conception of philosophy is needed – closer attention than has been given by many commentators on Wittgenstein's philosophy of mathematics. My insistence on close attention to Wittgenstein's approach to philosophy in relation to his philosophy of mathematics is not entirely new in the literature as this sort of insistence can be found in, for example, the work of Floyd (1991, p.145), Mühlhölzer (Gréve and Mühlhölzer 2014, p.172) and Diamond (1991, p.179). What is distinctive about this thesis is the detailed reading of (*PI*, §124) and the application of this reading to the wide variety of topics necessary to see (*PI*, §124) and Wittgenstein's philosophy of mathematics as consistent with one another. Each topic will be discussed with a view to seeing Wittgenstein's treatment in the light of the reading of Wittgenstein's view of philosophy elaborated in chapter 1. My contention is that this enables Wittgenstein's philosophy of mathematics to be seen as much stronger and more sophisticated than many of his interpreters have been in a position to see. Contrary to Putnam (2007, p.246), Wittgenstein really did 'take his time' but we can only see this by looking at his work in the light of his view of philosophy.

Wittgenstein's work on mathematics can be seen as part of a wider effort to "demonstrate a method, by examples" (*PI*, §133). The method is a particular approach to doing philosophy which "leaves everything as it is" (*PI*, §124). The examples are applications of the method to particular problems. The method itself is not immediately easy to grasp and the challenge is particularly difficult because the individual examples are not entirely unrelated to one another. A particular problem is treated and it sometimes leads naturally to another question, where one again might struggle to see how to apply the method. This is part of why this thesis needs to cover the wide range of topics that it does – it is very difficult to see what Wittgenstein is trying to show unless one follows him far enough (by seeing the method demonstrated in enough cases).

The problem is exacerbated by the number of accusations of dogmatism that have been levelled at Wittgenstein. Each demonstration of Wittgenstein applying his approach tends to prompt a question of the form 'but what about...?', where another example of Wittgenstein allegedly being dogmatic in some related way is brought up. This thesis covers a very wide range of topics so that enough of the web of accusations are answered that the promises of (*PI*, §124) can be seen as genuine and not empty. In this way (*PI*, §124) can be seen as a careful articulation of the approach to philosophy that is exhibited in Wittgenstein's work.

Part 1 – Wittgenstein's Method and Mathematics

Chapter 1 - Wittgenstein's Methodology

1.1. Chapter Introduction

I wish to suggest that Wittgenstein's work on the philosophy of mathematics be seen in the light of a particular reading of Wittgenstein's philosophical methodology. I will argue that much of the appearance of inconsistency and incompetence that has surrounded Wittgenstein's philosophy of mathematics can be removed when his work is seen in the light of his philosophical methodology. This chapter will focus on elaborating a reading of Wittgenstein's approach to philosophy – namely that advanced by Kuusela (2008). In order to argue that Wittgenstein's philosophy of mathematics can be seen as consistent, it needs to first be shown to be plausible that Wittgenstein's conception of (and approach to) philosophy in general is consistent. Whilst a full defence of Wittgenstein's methodology is beyond my scope (for this the reader should consult Kuusela (2008)), it will be important to be able to see what it is that makes Wittgenstein's conception of philosophy so distinctive (when compared with more traditional conceptions of philosophy) and so easy to misrepresent.

1.2. Wittgenstein's Conception of Philosophical Problems

An important theme of Wittgenstein's early and later work is his view that philosophical problems arise from misunderstandings of language (Kuusela 2008, p.17). Attempts to directly answer the questions that express these problems therefore do not address the source of the problem, namely the misunderstanding that leads to it. Wittgenstein's explanation of this in his earlier work is short on examples (Kuusela 2008, p.30) but in his later work the idea is developed and seen in application to detailed examples. For this reason I wish to begin with an example from Wittgenstein's later work and consider in the next section how this relates to Wittgenstein's early work. In illustration of the point that philosophical problems are misunderstandings of language, Kuusela (2008, p.30-31) cites the following remark from 1933:

> Let us consider a particular philosophical problem, such as 'How is it possible to measure a period of time, since the past and the future aren't present and the present is only a point?' The characteristic feature of this is that a confusion is expressed in the form of a question that doesn't acknowledge the confusion, and that what *releases* the questioner from his problem is a particular alteration of his mode of expression. (*PG*, p.193)

The problem is posed as a question but what drives us to pose the question is a confusion. The problem can be understood as arising through a mixing up of different senses of "to

measure", or different types of measurement. The asker of the question is thinking of time on the model of measuring length, such as with measuring tape or a ruler. A ruler or tape is typically laid against what is to be measured in proximity to the object in its entirety. But time is not present or extended in a physical sense. If the present moment is conceived of as a point between what is to happen and what has happened then the present appears to have no extension at all. The problem thus "arises from a failure to notice that what is called 'measuring' is not just one kind of activity, but a variety of activities, different types of measuring corresponding to different types of objects" (Kuusela 2008, p.31). The word 'measure' might be said to have different uses corresponding to our different ways of measuring. Accordingly, the problem can be resolved by pointing out the differences in these uses, especially the difference between measuring lengths and measuring time. Thus the resolution to the problem is a change in the questioner's 'mode of expression', since the questioner is brought to speak of measuring in a way which recognises the different uses without running them together.

The questioner's mistake is not resolved by pointing out something which is unknown to the questioner. It is not the case that the questioner fails to see that time is often measured by means of some periodic motion (such as that of a clock). The questioner is rather led to the problem by taking up a particular view of measuring. As Wittgenstein says "the phenomena that now strike us as so strange are the very familiar phenomena .... They don't strike us as strange until we put them in a peculiar light by philosophising" (*PG*, p.169).[11] Unlike with empirical problems, philosophical problems do not arise because we lack factual knowledge. Instead the problem arises because we see what we are already familiar with through "the medium of a misleading form of expression" (*BB*, p.31). Thus a philosophical problem is characteristically a problem in making sense of what one already knows rather than a problem that requires the acquisition of new knowledge (Kuusela 2008, p.30).

The difficulty with resolving philosophical problems is compounded, Wittgenstein thinks, because philosophical problems do not immediately present themselves, especially not to the questioners, as problems concerning forms of expression. Instead it can seem to the questioner that the problem is connected with or leads to a peculiarly deep insight. It will help to explore this in relation to an example. Consider that a solipsist might take the question 'how do we know that others feel pain?' to lead to a deep insight in the claim that 'only my pain is real pain.' The significance of this alleged insight proves difficult to pin down specifically. The claim that 'only my pain is real pain' might be taken to mean that other people are merely pretending, or it might be taken to mean that one can never say of others that they are in pain (Kuusela 2008, p.32). If the claim were that others are merely pretending then this would be an empirical claim. The solipsist does not, however, intend to claim that others are merely pretending and is rather trying to say something concerning the nature of pain. Whilst the solipsist takes himself to have arrived at an insight concerning pain, Wittgenstein takes the solipsist to be expressing a convention concerning the use of the word 'pain' – namely that it should not be applied to others:

> The man who says 'only my pain is real' doesn't mean to say that he has found out by the common criteria ... that others who said they had pains were cheating....
> [He] objects to using this word in the particular way in which it is commonly

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[11]Cited by Kuusela (2008, p.30).

used. On the other hand, he is not aware that he is objecting to a convention. (*BB*, p.56–57)[12]

The solipsist confusedly takes himself to be expressing a peculiar kind of truth – a necessary truth. The solipsist's claim that 'only my pain is real pain' is supposed to reveal a characteristic of pain which is part of the essence of pain, the very nature of the concept. The solipsist's claim is thus a case of a metaphysical claim – a purported necessary truth that pertains to reality (Kuusela 2008, p.1, p.97). The claim that 'only my pain is real pain' can seem to be at once so general as to hold for all possible instances of pain and a claim about the way things are (or 'really are' (Kuusela 2008, p.98)). Wittgenstein calls it "the essential thing about metaphysics" that "the difference between factual and conceptual investigation is not clear to it. A metaphysical question is always in appearance a factual one, although the problem is a conceptual one" (Ms134, 1531).[13] Wittgenstein says:

> We feel that we have said something about the nature of pain when we say that one person can't have another person's pain.... as though it would be not false but nonsense to say 'I feel his pains,' but as though this were because of the nature of pain, of the person etc. as though, therefore, this statement were ultimately a statement about the nature of things. (Ms148, 32r)

Of course the solipsist is not making an empirical claim but his claim takes on that appearance. It is not being presented as contingently true that "I cannot feel another person's pain" but rather that it is inconceivable that I should be able to feel another person's pain (Kuusela 2008, p.103).

The metaphysical claim seems like it points out a necessary feature of pain so that all instances of pain must possess the characteristics which the solipsist ascribes to them (i.e. they must be his own). Whilst this might appear like a generalisation over a range of phenomena (instances of pain), Wittgenstein urges against seeing this as analogous to an empirical generalisation (such as the claim that everyone is pretending to be in pain). As Wittgenstein sees it, the 'cannot' (of "I cannot feel another person's pain") reveals that the metaphysical "proposition hides a grammatical rule" (*BB*, p.55). The metaphysical claim appears as though it were both a claim about the world and also such that we cannot conceive of its opposite – we cannot imagine circumstances which would count against the claim. As Wittgenstein says:

> The avowal of adherence to a form of expression, if it is formulated in the guise of a proposition dealing with *objects* (instead of signs) must be *'a priori.'* For its opposite will really be unthinkable, inasmuch as there corresponds to it a form of thought, a form of expression that we have excluded. (*Z*, §442[14])

If the claim is to be about the world (about 'objects') then, like other empirical claims, one would expect it to be such that we could imagine testing it by experience. But the metaphysician's claim does not admit of testing by experience, since its opposite is being excluded as unintelligible (according to the solipsist, I cannot be in a position to ascribe pain to others):

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[12]Cited by Kuusela (2008, p.32).
[13]Cited by Kuusela (2008, p.103).
[14]Cited by Kuusela (2008, p.104).

> When one wants to show the senselessness of metaphysical turns of phrase, one often says "I couldn't imagine the opposite of that", or "What would it be like if it were otherwise?" (When, for instance, someone has said that my images are private, that only I alone can know if I am feeling pain, etc.) Well, if I can't imagine how it might be otherwise, I equally can't imagine that it is *so.* For here "I can't imagine" doesn't indicate a lack of imaginative power. I can't even *try* to imagine it; it makes no sense to say "I imagine it". (*PG*, p.129)

If the question concerns what we can imagine and not imagine, then the question is not a question about the world so much as about our concepts or ways of talking. The metaphysician expresses a commitment to a particular way of talking (an 'adherence to a form of expression' (*Z*, §442)) but in a way which disguises the claim as a proposition about objects (in the solipsist case, instances of pain). Since the proposition is not arrived at by experience, one may be inclined to characterise it as "a priori" – as known to be true independently of empirical facts. But this only serves to further the conflation of empirical proposition and rule – the metaphysical "a priori" statement is not a statement about anything and rather expresses a commitment to a form of expression.

As Kuusela puts it, "one might sum up the metaphysician's confusion by saying that she projects a way of using language onto reality, as exemplified by the 'a priori' proposition" (2008, p.104). The necessary characteristics that the metaphysician points to (e.g. in the solipsist case, the impossibility of knowing of another's pain) are projected onto reality, so that the convention which the metaphysician expresses (one might say, 'I must/can only apply the word pain to myself') takes on the appearance of expressing necessary features of reality. The later Wittgenstein thus sees it as part of his role as a philosopher to unmask metaphysical uses of expressions so that the problem can be seen to be a problem concerning the use of language. This needs to be achieved so that the confusion/s that leads to the adherence to a particular form of expression can be brought out (*PI*, §116).

The question needs to be addressed as to how Wittgenstein thinks that one can go about revealing the sources of philosophical confusions and how this is to be done without requiring some kind of claim concerning essences or necessities. Following Kuusela (2008) it will be explained that Wittgenstein lays out in (*PI*, §89–133) a method for addressing philosophical questions which is intended as "a strategy for avoiding metaphysical projections of forms of presentation onto the objects of investigation in the guise of statements about necessary truths" (Kuusela 2008, p.111). Whilst this strategy is intended to be a general one to be applied to philosophical problems (2008, p.111), Wittgenstein introduces it by comparison with the approach to philosophy which he subscribed to at the time of the *Tractatus.* The later Wittgenstein uses the *Tractatus* as his "primary example of the confusion of metaphysics" (2008, p.105) and since it is his approach for dealing with this confusion that we need to understand, we also need to understand that example.

1.3. Wittgenstein's Turn Away from a *Tractarian* Conception of Philosophy

The comparison between Wittgenstein's later approach to philosophy and that of the *Tractatus* seems for Wittgenstein to be an especially illuminating one in part because the *Tractatus* attempted to make a key move which his later approach was meant to follow through upon (and therefore succeed where the *Tractatus* had failed). The *Tractatus* attempted to move away from statements of essences (Kuusela 2008, p.98) but on

Wittgenstein's later understanding the *Tractatus* committed itself to metaphysical theses nonetheless (2008, p.99-100). The way in which Wittgenstein saw this as coming about is revealing both as an example of metaphysics and with regard to how Wittgenstein later came to think that metaphysics could be avoided.

Whilst the purpose here is not to defend any particular reading of the *Tractatus*, nonetheless some characterisation of the *Tractatus* approach to philosophical problems has to be given in order to understand what Wittgenstein says about his later approach to philosophy in (*PI*, §89–133). In order to make sense of the contrast, I will follow Kuusela in seeing the *Tractatus* as putting "forward a program for philosophy as logical analysis that is universally applicable to all philosophical problems" (2008, p.99). As I have mentioned, Wittgenstein at the time of the *Tractatus* (as well as later) understood philosophical problems as arising from misunderstandings of language (2008, p.98). For the *Tractatus* the key problems relate to using the same expression (sign) in different ways so that one is inclined to mix up the different uses (the *Tractatus* speaks of 'symbols', which are signs with a logico-syntactic use (Kuusela 2008, p.19)). For the *Tractatus* this kind of error can be avoided by using a notation that excluded logical errors:

> In order to escape such errors we must make use of a sign-language that excludes them by not using the same sign for different symbols and by not using in a superficially similar way signs that have different modes of signification: that is to say, a sign-language that is governed by *logical* grammar-by logical syntax.
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> (The conceptual notation [Begriffsschrift] of Frege and Russell is such a language, which, however, does still not exclude all errors.) (*TLP* 3.325)[15]

The concept-script is meant to make logico-syntactical distinctions easy to see so that we are not misled by using signs in multiple ways that can easily be confused (2008, p.56). The concept-script would exclude logical errors and therefore the confusions of metaphysicians would not arise. Philosophical problems would be handled by logically analysing the expressions in question by means of the concept-script (2008, p.59).

For the *Tractatus* then the way to resolve philosophical problems would be to clarify expressions of our vague ordinary language by analysis into a concept-script which would be such as to eliminate ambiguities by using a different sign for each logically distinct symbol (2008, p.59). The concept-script might in this sense be said to be more in tune with logic than ordinary language, since each symbol in it would correspond to a logically distinct role (2008, p.57). Whilst it happens that languages typically do employ the same symbol as different signs, this is an accidental feature of particular languages and not an essential or logical feature (2008, p.67). A complete analysis of an expression would bring to light all of its logical features, removing any possibility of confusion due to ambiguities.

One might ask how Wittgenstein thought that such a concept-script would be created and in what way the *Tractatus* indicates how to undertake this. Whilst Kuusela has much to say on this (2008, p.59-63), the key point for understanding Wittgenstein's later remarks on philosophy is that he did previously think that a complete analysis of expressions was possible by means of which language could be made completely exact, in that all possible

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[15]Cited by Kuusela (2008, p.56).

misunderstandings would be excluded.[16] When Wittgenstein looks back on his old views later, this idea of a complete analysis is part of what he finds problematic:

> Formerly I myself spoke about the 'complete analysis,' the idea being that philosophy should decompose all propositions once and for all, thus laying down clearly every connection and removing every possibility of misunderstanding. As if there were a calculus in which this decomposition were possible.... All of this was based on a mistakenly idealised picture of language and its use. (*PG*, p.211)[17]

The idea of making entirely clear all of the logico-syntactical rules for language-use can start to look problematic if one tries to picture in detail how all of these rules would be laid out in a way that would prevent all possible misunderstandings. One might plausibly tabulate all of the rules and employments of a simple language with clear and simple rules (a 'calculus') but it is not clear that this could be done for a natural language. Moreover, even for a simple calculus with an explicitly laid-out notation, it is not clear that misunderstandings would not be possible (2008, p.69). If one wanted to formulate the rules in a way which prevented *particular* misunderstandings, then in that case one would be able to specify criteria for having achieved a complete and exact notation (namely that the particular misunderstandings were removed). But if the idea were, as was the case with the *Tractatus* conception of analysis, to eliminate all possible misunderstandings then the criteria for completion have not yet been made clear (2008, p.68). As Wittgenstein says:

> One might say: an explanation serves to remove or to avert a misunderstanding-one, that is, that would occur if but for the explanation; not every one that I can imagine. (*PI*, §87)[18]

Likewise an explanation can only be said to be exact in specific respects (2008, p.69). A concept-script might be said to be exact in the sense of removing particular ambiguities but it is not clear how all possible ambiguities could be specified. The ideals of absolute exactness and absolute clarity both appear, from Wittgenstein's later perspective, to be confused.

The *Tractatus* was meant to present a programme by which metaphysics could be excluded as nonsense (2008, p.98) since it was thought that metaphysics arose from misunderstandings of language which a complete analysis would exclude. And yet the idea of a complete analysis actually commits the *Tractatus* to a metaphysics of language. The problem is that the *Tractatus* requires that all propositions (here meaning sentences which

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[16]Kuusela (2008, p.298) notes that the key point concerns the possibility of complete analysis and so his reading is not committed to a view (such as that criticized by Floyd (2007, p.194-6)) of the *Tractatus* as holding that there is a logical order of language which a correct concept-script should capture, at least not explicitly so. The necessary/logical features of language were to be elucidated in terms of a concept-script (2008, p.98) but whilst this resulted in a metaphysical picture of logic it was not intended to be metaphysical (Kuusela calls it a "relapse" (2008, p.105-106)). See also Kuusela (2008, p.61-62).

[17]Cited by Kuusela (2008, p.67). In order to understand (*PI*, §89–133), Kuusela makes extensive use of Nachlass material (which he explains at (2008, p.13)). I have here followed him in using Nachlass quotations where these are more illuminating. Nonetheless the reading is meant to fit (*PI*, §89–133). On this particular point see (*PI*, §88) on the problem with an ideal of absolute exactness.

[18]Cited by Kuusela (2008, p.69).

can be true or false) admit of a single complete analysis and that this complete analysis will remove all possible misunderstandings (2008, p.99-100). Thus the *Tractatus* is committed to a thesis about propositions – the thesis that every proposition will admit of a single analysis by which it is absolutely clarified (2008, p.100). Whilst the *Tractatus* may aim to exclude statements of necessary truths from what can be said using the concept-script (2008, p.98), nonetheless a necessary truth concerning propositions is "built into its conception of the method of logical analysis" (2008, p.100). As long as a single complete analysis is thought to be possible, then "language continues to be seen as possessing an essence that can be captured once and for all in some logical notation or another that shows what meaningful expressions *must* be" (2008, p.100). Thus Wittgenstein became committed to a thesis concerning language, though he did not at the time see it as such:

> We have a *theory* ... of the proposition; of language, but it does not seem to us a theory. For it is characteristic of such a theory that it looks at a special, clearly intuitive case and says: *'That* shows how things are in every case. This case is the exemplar of *all* cases.'-'Of course! It has to be like that' we say, and are satisfied. We have arrived at a form of expression that *enlightens us.* ... (*Z*, §444)[19]

As we have seen from the characterisation of metaphysics in the previous section, the metaphysician does not take himself/herself to be laying down any requirements or conventions for the use of an expression (2008, p.106). Rather, they take themselves to have arrived at an insight or necessary truth. When impressed by this apparent insight, it can seem as though every proposition must be such as to admit of a complete analysis that would exclude all misunderstandings. As with the solipsist's apparent insight concerning pain, the view that propositions should admit of a complete analysis appears as an a priori truth rather than a convention concerning the term 'proposition' (2008, p.104).

The *Tractatus* may not have aimed to lay down any requirements concerning what language must be but in making methodological assumptions concerning the possibility of a single complete analysis of propositions it was thus committed to such requirements. As a more specific example, the *Tractatus* treated every proposition (true/false sentence, including everyday assertions) as a true/false (re)presentation of a state of affairs (2008, p.110). If everyday propositions are sometimes ambiguous and might not have a single definite sense, then it would seem that they might not have a single truth-value. Since this would not fit the mould of a true/false (re)presentation of a state of affairs, one is led to postulate that it is only at surface level that everyday propositions appear to be indefinite and analysis would reveal a definite sense (2008, p.111). Thus the fully-analysed proposition appears to be at a deeper level than the everyday proposition (2008, p.110). Similarly, not all everyday propositions appear to be representations of states of affairs, but at the deeper level they must turn out to be such if they are to be propositions (2008, p.110). Thus Wittgenstein was led to think in terms that led to the "infamous philosophical postulation of a realm of the "really real" behind the veil of appearances that everyday thought (wrongly) assumes to be real" (2008, p.110). Looking back on his thinking later, Wittgenstein speaks of this as the postulation of a 'real sign' that is to be found behind everyday expressions:

> When we believe that we must find that order, the ideal in actual language, we are easily led to speak of the *'real' sign* (sentence or word), to look for the real sign, so to speak, behind what is called that in customary language use. For we aspire after

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[19]Cited by Kuusela (2008, p.106). His modified translation. Cf (*PI*, §104-108).

> something more pure than the sign in the sense of a written or printed word etc. We are in search of a sublime essence. (Ms142, 88)[20]

From the point of view of Wittgenstein's earlier work, it may have looked like an incidental detail that everyday language did not on the surface fit how he came to think that language must work. He could see himself as being concerned with language in the abstract rather than the incidental features of particular instances of language-use. But from the point of view of Wittgenstein's later philosophy, the move to speaking of language in an abstract way put Wittgenstein's *Tractatus* project on 'thin ice':

> "The 'sublime conception' forces me to move away from the concrete case since what I say doesn't fit it. I now move into the ethereal region, talk of the *real* sign, of rules that must exist (even though I can't say where & how) – and find myself on 'thin ice'." (*PPO*, p.173[21])

We are on thin ice because we are starting to lose contact with concrete cases of language-use (Kuusela 2013, p.99) and it was just such cases that philosophical analysis was meant to be able to clarify. In order to apply it to particular cases, one would seemingly need to find the real signs behind specific instances of language-use. But the nature of the real sign proves difficult to be clear about – it might be seen as a kind of Platonic entity or as a thought in the speaker's mind at the time of utterance (Kuusela 2013, p.99). We struggle to say what this real sign is ("I can't say where & how" it exists). Instead of being in a position to clarify particular uses of language by means of insights into language in the abstract, we find ourselves struggling to characterise the existence of 'real signs' or to explain how they can relate to real language-use. The hunt for the nature of these real signs becomes a "pursuit of chimeras" (*PI*, §94; Kuusela 2013, p.100).

The remark (*Z*, §444) cited previously both shows that Wittgenstein did not realise at the time that he was committed to a thesis ("it does not seem to us a theory") and it can also be taken to reveal "the anatomy of the *Tractatus's* mistake" (Kuusela 2008, p.106). Wittgenstein diagnoses the error as fastening on to a particular kind of case and seeing all others by means of it. The early Wittgenstein did not notice that the 'intuitive case' (for example, the case where a proposition serves clearly as a representation of a state of affairs) was for him taking on the role of a mode of presentation (2008, p.106), functioning as a model for all of the cases of which it is an exemplar. It seemed rather as though the example brought out what had to be characteristic to all of the cases. This projection is, for the later Wittgenstein, characteristic of metaphysics. He elsewhere elaborates:

> "Every proposition says: This is how things stand." Here we have the kind of form that can mislead us. (Misled me.) ...
>
> This is the kind of proposition that one repeats to oneself countless times. One thinks that one is tracing the outline of the thing's nature over and over again, and one is merely tracing round the frame through which we look at it...
>
> Again and again we trace out the form of expression and think we have depicted the thing.-Due to an optical illusion we appear to see within the thing what is marked on our spectacles.... Only when this illusion has been removed can we simply see language, as it is.

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[20]Cited by Kuusela (2008, p.110). Cf (*PI*, §105).
[21]Cited by Kuusela (2013, p.98). Cf (*PI*, §107).

> The expression of this confusion is the metaphysical use of our words. For now we predicate of the thing what lies in our mode of presenting it. Impressed by the possibility of a comparison, we think we are perceiving a state of affairs of the highest generality. (Ts220, §110)[22]

When 'impressed by the possibility of a comparison,' one does not see that it is a comparison that one is making. It can be easy to miss that the comparison is functioning as a mode of presentation if one assumes that there are common features that all exemplars of a concept must share (2008, p.107). This kind of assumption may be an "unexamined presupposition of thinking" (Kuusela 2008, p.107). If all examples of propositions must share the features common to the concept, then just a few examples or even a single example would be sufficient to see this feature. As Wittgenstein puts it:

> The tendency to generalize the case seems to have a strict justification in logic: here one seems completely justified in inferring: 'If one proposition is a picture, then any proposition must be a picture, for they must all be of the same nature.' For we are under the illusion that what is sublime, essential about our investigation consists in grasping one comprehensive essence. (Ts220, §93)[23]

So the key to what led to the *Tractatus's* project being left unable to do clarificatory work (by losing contact with concrete cases) was an idealisation concerning language. It was assumed that the investigation was to be an investigation of concepts and concepts were assumed to be "unified through universally shared characteristics" (2008, p.107). This assumption masks that any metaphysical projection is going on, making it seem as though what is projected must be what is really in the examples (2008, p.108).

This understanding of Wittgenstein's later view of the failings of the *Tractatus* gives us a background against which to understand Wittgenstein's later method for clarifying philosophical problems. Through the lens of a metaphysical projection it looks as though concepts are governed by strict rules, each one being characterised by a definite set of universally shared characteristics. If we are to get back to talking about actual instances of language, then we need to accept our comparisons for what they are. Instead of seeing the characteristics of our comparison to be characteristics of the objects satisfying a concept/expression, we need to see these characteristics as part of our mode of representation.[24] As Kuusela (2013, p.106) puts it, the "problem can be avoided by putting forward the strict and precise rules, not as a claim about language, but as a particular way in which logic, for its purposes, seeks to describe the uses of language."[25]

Philosophical clarification can use models of expressions as being governed in strict and precise ways without falling into the projection on to particular cases that characterises metaphysics. The method by which it can do so is the subject of the next section.[26]

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[22]Cited by Kuusela (2008, p.105). Cf (*PI*, §103, §113, §116).

[23]Cited by Kuusela (2008, p.107)

[24]This methodological shift is what Kuusela (2008, p.111) understands Wittgenstein to mean by the move of "turning our whole examination round" (*PI*, §108).

[25]Wittgenstein makes the same point, differently phrased, in *PG* (p.77). See also (MS 140, 33).

[26]There is of course more that can be said with regard to Wittgenstein's motivations for his later method. For example, I have not mentioned Kuusela's view that the *Tractatus* made the development of a method into the 'fundamental problem' and so adopted a hierarchical

1.4. Philosophy and Objects of Comparison

Wittgenstein speaks in the preface to the *Investigations* in a way that contrasts his "new thoughts" with his "old way of thinking" (*PI*, viii) and this contrast seems to be picked up within the discussion of *PI* when Wittgenstein speaks of "turning the whole examination around" (*PI*, §108). Kuusela points to similar remarks in Wittgenstein's notebooks from 1936-37 and argues that this talk of turning around is a way of expressing and clarifying a shift in Wittgenstein's approach to philosophy that had taken place around 1930-1932 (Kuusela 2008, p.121). Wittgenstein's remarks on the shift concern "how to avoid dogmatism, prejudices, and metaphysical projections in philosophy" (2008, p.121). This section will explain the core ideas of that shift and summarise the perspective on philosophy that results. This conception of philosophy will be applied in the next chapter in order to arrive at an understanding of the claims of (*PI*, §124) concerning how philosophy 'leaves everything as it is' and also 'leaves mathematics as it is.'

To summarise what was said in the last section, we have seen that Wittgenstein speaks of his previously taking the idea that a proposition is a representation of a state of affairs in a way that led him to see all propositions through the lens (Wittgenstein makes a comparison with wearing spectacles) of that comparison (Ts220, §110). The idea was taken in a metaphysical way, so that it was seen not to apply to actual instances of propositions (sentences of actual languages) but propositions in the abstract or 'real signs' which were meant to stand behind actual instances (Ms142, 88). Thus the features of the model were imposed upon the particular cases in a way Wittgenstein takes to be characteristic of metaphysics (*Z*, §444).[27] The metaphysical projection results in a kind of dogmatism since the particular cases are not considered in their own right and are only seen through the lens of the projection. We also saw that the projection does not appear to be a projection to the metaphysician, since if one assumes that concepts have a unified essence (or set of essential features) then application of the model seems like it would be perfectly justified (Ts220, §93).

In order for philosophy to do clarificatory work without falling into the dogmatism that is characteristic of metaphysical projection, comparisons need to be identified as comparisons rather than metaphysical insights (*PI*, §131). In order to remain clear that a comparison is just that, we need to be clear about the cases to which it fits and the cases which it does not:

> When you are tempted to make general metaphysical statements, ask yourself (always): What cases am I actually thinking of?-What sort of case, which conception do I have in mind here? Now something in us resists this question for we seem to jeopardize the ideal through it: whereas we are doing it only in order to put it in the place where it belongs. For it is supposed to be a picture with which

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approach to philosophy (2008, p.11). My focus is only on understanding the *Tractatus* as an example of metaphysics so that Wittgenstein's later method can be understood in a way which can help resolve apparent inconsistencies in Wittgenstein's philosophy of mathematics. Most central are the notions of metaphysics as involving a projection of a model of representation from a particular comparison and the idea that this can be avoided by recognizing the comparison for what it is.

[27]Wittgenstein calls this kind of thinking "the origin of a kind of dogmatism" since "everything which holds of the model will be asserted of the object of the examination; & asserted: it *must always* be" (Ms115, 56, 57). Cited by Kuusela (2008, p.109).

we compare reality, through which we represent how things are. Not a picture by which we falsify reality. (*PPO*, p.97)[28]

One way to put this point that a model naturally fits some cases and not others is to say that particular cases might be considered 'prototypes' for the model in question. To return to an example from section 1.2, we might fasten on to a particular case of the use of the word 'measure', such as a measurement of length. We then come to see other examples of the use of the term 'measure' in the light of the original example (Kuusela 2008, p.30-31). So we might then think that the entirety of the thing to be measured always has to be present at the point of measuring, as is the case when measuring with a ruler or tape-measure. This requirement that the whole object be present at the time of measuring can seem much more compelling if one is not clear about its origin. It can seem especially compelling when it is not announced as a requirement and is only expressed through a question such as 'How is it possible to measure a period of time, since the past and the future aren't present and the present is only a point?' (*PG*, p.193). The key move, then, is to see which example/s are the origin of the prototype:

> The object of comparison [insertion: model], the object from which a way of conceiving things is derived, should be announced so that the examination does not become unjust. For now everything which holds of the model will be asserted of the object of the examination; & asserted: it *must always* be ... This is the origin of a kind of dogmatism. One forgets the role of the prototype in the examination: it is as it were the unit of measurement with which we measure the object of examination. Dogmatism, however, claims that every measured object must be a whole number of the units of measurement. (Ms115, 56-57)[29]

A prototype is an exemplary case which stands as a representative for other cases and thus shapes the way in which they are represented. When a prototype becomes the basis for a metaphysical projection, then the prototype is seen as something which the various cases *must* match, since the prototype is seen as bringing to light a necessary truth – a truth which must concern all of the cases falling under the concept (Kuusela 2008, p.123). Asking the question "what cases am I really thinking of?" forces one to come down to a level of particular cases and thus to "bring a statement down to earth from the heights of metaphysical abstraction" (Kuusela 2008, p.123).

This is not to say that philosophy is therefore limited only to dealing with particular cases. If one were limited to discussing only particular cases then one might worry that philosophy would lose generality and would collapse into becoming an empirical investigation. Using prototypes to model a range of cases is perfectly valid, so long as it does not give rise to metaphysical projection:

> ...since we confuse prototype & object we find ourselves dogmatically conferring on the object properties which only the prototype necessarily possesses. On the other hand we think the examination will lack the generality we want to give it if it really holds of the one case. But the prototype must be presented for what it is; as characterizing the whole examination and determining its form. In this way it

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[28]Cited by Kuusela (2008, p.122). Wittgenstein is here discussing a particular view of naming but for my purposes what matters is that the view is, as Kuusela says, "an example of the metaphysical projection" (2008, p.122).
[29]Cited by Kuusela (2008, p.122).

> stands at the head & is generally valid by virtue of determining the form of examination, not by virtue of a claim that everything which is true only of it holds for all the objects to which the examination is applied. (Ms111, 119-120)[30]

Immediately prior to this remark, Wittgenstein refers to "the object from which this way of conceiving things is derived" as the "object of comparison". The prototype may bring to light a point that holds of a range of cases without necessarily leading to dogmatism. The problem of dogmatism arises when the applicability of the prototype is claimed to be greater than it is, such as when it is taken as a reflection of a necessary truth (and thus as applicable to all cases). In order to explicitly treat the prototype as an object of comparison, we need to be clear that it does not come with the claim that it *must* hold of all cases. Rather, one uses it to compare ranges of cases against it and one notes both the similarities and the differences of the cases to the object of comparison (Kuusela 2008, p.124; *PI*, §130).

The model when used as an object of comparison is thus a vehicle for philosophical clarification as it gives a way of presenting the object of investigation that brings to light relevant features that are being overlooked or muddled by the asker of the philosophical question (such as the differences between 'to measure' in relation to times and in relation to lengths (*PG*, p.193)). As Wittgenstein says in the passage quoted above (Ms111, 119-120), the model is part of the *form* of the investigation. Presenting such models is part of a strategy to present the object of investigation in a way that resolves the problem. This strategy of presenting our usage of relevant expressions in an enlightening way is referred to by Wittgenstein as a 'perspicuous representation' (sometimes also called by interpreters a 'perspicuous overview' or 'overview'):

> A main source of our failure to understand is that we do not *command a clear view* of the use of our words.—Our grammar is lacking in this sort of perspicuity. A perspicuous representation produces just that understanding which consists in 'seeing connexions'. Hence the importance of finding and inventing *intermediate cases*.
>
> The concept of a perspicuous representation is of fundamental significance for us. It earmarks the form of account we give, the way we look at things. (*PI*, §122)

I will return to the relevance alluded to here of 'intermediate cases' but first it is worth noting that Wittgenstein refers to the idea of a perspicuous representation as being of 'fundamental importance'. The perspicuous representation is said here to bring about a 'clear view of the use of our words', indicating that when we fall into a philosophical misunderstanding we misunderstand or lose sight of key features of how our words are used. The perspicuous representation is thus also referred to as a "rearrangement" which makes our use of words more easily "surveyable" (*PI*, §92). Wittgenstein contrasts this manner of investigation with the search for unified essences to our concepts (*PI*, §92). The idea of searching for essences was identified in the last section as leading to the metaphysical projections that the *Tractatus* fell into (see especially (Ts220, §110) cited in the last section), with the *Tractatus's* fall into dogmatism being emblematic for the later Wittgenstein of philosophical dogmatism in general (Kuusela 2008, p.105). By saying that the approach of providing "perspicuous representations" is "of fundamental significance for us" and that it "earmarks the form of account we give", Wittgenstein is putting forward a

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[30]Cited by Kuusela (2008, p.124). Kuusela's translation modified from (*CV*, p.21-22).

contrast between his particular conception of philosophy and the conception of philosophy as a search for essences (Kuusela 2008, p.237).

Whilst Wittgenstein is presenting a conception of philosophy as offering perspicuous representations, he is not committing himself to a single particular form that such overviews must take. The "seeing connections" that he refers to might be brought about in more than one way. Wittgenstein emphasises that "*one* means of perspicuously framing the grammar" is "the introduction of a novel mode of expression particularly suited for this purpose" (Ms152, 91).[31] The introduction of a novel mode of expression is roughly the presentation of a particular picture[32] — for example, offering a definition of a word which might well be known to be a simplification of the word but which is still enlightening for a particular purpose (Kuusela 2008, p.144, p.234). Perhaps such a definition might be inspired by an extrapolation from a prototypical case. So long as one is clear that the definition is not meant such that it must apply neatly to all cases of the word's use then such a simplified definition need not lead to dogmatism. It might play a key role in perspicuous presentation, if it helps to bring to light the misunderstanding in question.

Another form that an overview might take is a series of prototypical examples. This way of presenting overviews is alluded to in (*PI*, §122) when Wittgenstein mentions "finding and inventing *intermediate cases.*" It may be that an overview needs to bring out not just a series of particular ways of using a word but also the possibility of further cases between the examples given. Using an example in this way is referred to by Wittgenstein as using it as a "centre of variation":

> This is the case when we are asked: what is the essence of punishment and now one says that every punishment is really society's revenge, and another, its essence is deterrence etc. But are there not typical cases of society's revenge and typical cases of deterrent measures and others of punishment as reform, and countless mixtures and intermediate cases? If we, therefore, were asked about the essence of punishment, essence of revolution, of knowledge, of cultural decline or refined sense for music-we should not try to give something *common* to all cases, not what they all *really* are, that is, an ideal which is *contained* in them all; but instead of this examples, as it were centres of variation. (Ms152, 16-17[33])

We can easily think of cases where a particular punishment seems to play the role of a deterrent and others where society's revenge might be said to be the key role of the punishment. Such prototypical cases might be put forward as exemplifying a range of cases, helping us to see the range of roles that punishment serves. But it need not be that the roles are always clearly individuated and that the word 'punishment' is only used in a definite number of ways. It can also be that there are "countless mixtures and intermediate cases" and highlighting a few such cases would allow the examples presented to convey the possibility of intermediate cases. One way to put the point that the term 'punishment' has various uses with intermediate cases in between is to say that the unity of the concept is complex and is not constituted by any single feature or essence (Kuusela 2008, p.173). The concept instead comprises various meanings with a possibility of mixtures and variation in between.

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[31] Cited by Kuusela (2008, p.234)

[32] A picture for Wittgenstein is a mode of presenting facts or things (Kuusela 2008, p.36).

[33] This remark is cited by Kuusela (2008, p.173).

The use of examples as "centres of variation" and the proposal of new modes of expression (particular models such as the model of punishment as society's revenge or of punishment as deterrent) are thus examples of techniques that can be used to bring out differences in ways of using words for the purpose of resolving a particular philosophical problem. Wittgenstein does not identify a single method to be used in all cases and this is perhaps a reflection that philosophical problems are not all alike. Rather than trying to give a definitive account of philosophical problems and the exclusively correct method for handling them, Wittgenstein can rather be seen to be offering, in (*PI*, §89–133), a characterisation of a method which he aims to demonstrate by examples. Wittgenstein signals this when he says in (*PI*, §133) that "we now demonstrate a method, by examples" and emphasises that there are methods and not a single method.[34] The demonstration by examples that Wittgenstein refers to appears to be rest of *PI*.

1.5. Chapter Conclusion

The examples that I have given so far in demonstration of Wittgenstein's method/s might be said to be simplistic or chosen to suit the method but this is because such simple examples are most illustrative. It will be argued in later chapters that Wittgenstein's conception of philosophy can be seen as applicable to much more sophisticated examples of philosophical problems. The central aim of this thesis is to read pertinent topics from Wittgenstein's philosophy of mathematics as treatments of philosophical questions in line with the conception of philosophy discussed in (*PI*, §89–133). It will be argued that the appearance of inconsistency surrounding Wittgenstein's philosophy of mathematics can be removed by seeing Wittgenstein's discussions in the light of the conception of philosophy summarised here. Much of the appearance of inconsistency is related to the promises that philosophy, as Wittgenstein sees it, 'leaves mathematics as it is' and 'cannot give it any foundation either' (*PI*, §124). This chapter has sufficiently elaborated Wittgenstein's conception of philosophy to be able to begin an initial interpretation of these promises, which will be the aim of the next chapter.

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[34]See also Kuusela (2008, p.271).

Chapter 2 - Wittgenstein's Methodology and PI §124

2.1. Chapter Introduction

In this chapter it will be argued that Wittgenstein's later understanding of philosophy can be used to make sense of Wittgenstein's more controversial claims concerning his approach to philosophy in general and to the philosophy of mathematics in particular. This chapter will apply the understanding of Wittgenstein's methodology discussed in the last chapter with a view to arriving at an initial understanding of the claims that Wittgenstein makes in (*PI*, §124), such as the claims to 'leave everything as it is' and to 'leave mathematics as it is'. Achieving this will require being able to apply the distinction between a metaphysical account and a perspicuous representation/overview, with a view to avoiding readings of Wittgenstein which would attribute to him some form of metaphysical claim.

2.2. Wittgenstein's Conception of Philosophy and 'Leaving Everything As It Is'

Having summarised Wittgenstein's conception of philosophy, we are now in a position to begin to understand the significance of (*PI*, §124):

> Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.
>
> For it cannot give it any foundation either.
>
> It leaves everything as it is.
>
> It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other. (*PI*, §124)

This section will look at the first three sentences in (*PI*, §124) before moving on to the last paragraph about mathematics in particular in subsequent sections.

In the light of Wittgenstein's conception of philosophy, we can say that philosophy does not interfere with the actual use of language because philosophy simply clarifies misunderstandings of the use of words by means of perspicuous presentations. This does not alter the actual use of language because the perspicuous presentation just draws our attention to uses of words. It is not a case of changing or replacing the uses of words but seeing the connections between existing uses of words in a way that enables the misunderstanding to be addressed. This kind of perspicuous presentation might be characterised as a description in the sense that it does not lay down new rules, as metaphysical accounts (surreptitiously) do, concerning how words should or should not be used and instead only draws attention to existing uses.

The contrast between Wittgenstein's idea of 'leaving actual use of language as it is' and how he conceives of a metaphysical approach can be illustrated by returning in more detail to the solipsism example from section 1.2. The solipsist proposes that 'only my pain is real pain' and in doing so is laying down a rule concerning the use of the expression pain (that 'pain' must only be applied in the first-person). However, this rule is presented as though it were a factual insight (a necessary truth) concerning the essence of the term 'pain'. Thus

the solipsist claims to have arrived at a kind of discovery by means of analysing the concept pain and by this discovery they think they have shown that pain cannot be used in the third-person. For Wittgenstein a metaphysical account such as that of the solipsist does not arrive at an insight into the nature of pain and instead it simply latches on to one way of using words and then projects this model onto other cases. In the case of solipsism, the solipsist might be said to have a picture of a pain as kind of object so that reporting a pain were like reporting that there is a sofa in the next room. If reports of pain are seen through the lens of the model of empirical descriptions then third-person ascriptions look problematic since if the pain-object is in the mind of another person then one cannot see it.

In contrast to the metaphysical account of the solipsist, a Wittgensteinian approach to the problem that the solipsist is concerned with would aim to address the solipsist's misunderstanding by presenting our uses of relevant expressions in a perspicuous way. Whilst we cannot go into the full details of such a resolution here, the beginnings of such a perspicuous presentation can be seen by pointing to Wittgenstein's comparison between expressions of pain ("I am in pain") and primitive expressions of pain behaviour such as rubbing one's knee or saying "ouch" (*PI*, §244; Kuusela 2008, p.88). When we see first-person reports of pain in this way then there is no need for an inner pain-object, since one can then resist the temptation to see the expressions on the analogy of empirical descriptions. If first-person reports can be seen as primitive expressions, then the problem with respect to third-person ascriptions of pain can also be resolved. We could then be said to ascribe pain to others through observations concerning their expressions of pain-behaviour. An ascription such as "X is in pain" could then be seen as being logically related to our views about X, such as our taking X to need to stop and rest his knee. But for present purposes the key point is not so much how this Wittgensteinian approach could be followed through in detail so much as how it contrasts with the solipsist's.

A metaphysical position such as the solipsist's can give an appearance of exposing the underlying foundations of actual language use. The solipsist's model of empirical reports (such as "there is a sofa in the next room") requires that there be an object to report upon and this feature of the model is projected on to all of the cases so that expressions of pain look like reports upon inner pain-objects. This postulation of an entity in order to satisfy the requirements of the model parallels the postulation of 'real signs' (explored in section 1.3) standing behind expressions in order to satisfy requirements that the *Tractatus* laid down for propositions (Ts220, §110; *PI*, §104). If one stays within the terms of the model that the metaphysician subscribes to, then doing away with the postulated entities looks like an admission that we use language invalidly – that our apparent reports of pain are empty utterances (after all, how can one have a report without something to report upon?) or our propositions are not really propositions at all. The postulated entities (pain-object or real signs) thus look to the metaphysician like they must be the foundation of our being able to use expressions as we do.

For Wittgenstein the appearance that the postulated entities of the metaphysician are needed in order for us to use our expressions as we do is to be seen as a result of the metaphysician dogmatically applying an inappropriate model. It is because the model is inappropriate that entities need to be postulated in order to fit it. Resolution of the misunderstanding by means of a perspicuous presentation would show the metaphysician's model to be inappropriate and would thus also show that the postulated entities are not necessary. The perspicuous presentation does not provide the foundation that the

metaphysician offers and nor does it dismissively reject the postulated existence of these entities. The perspicuous presentation does need to remove the apparent need for the postulated entities since what these entities are is unclear. The postulation leads to troubling questions such as 'where are these pain-objects and how am I made aware of them?' or 'what are these real signs? Can we give examples of them?' But the postulations are not to be simply dismissed and instead need to be removed in a way that shows that we don't need them. This is achieved by presenting an alternative picture of the use of the expression/s, which resolves our philosophical unease:

> Obviously what calms us is that we see a system which (systematically) excludes those constructions that always made us uneasy, those we were unable to do anything with, and which we still thought we had to respect. (Ms112, 119v / Ts213, 416[35])

In showing that the metaphysician's model is inappropriate we also remove the need for the "constructions" that were postulated in order to fit it[36] and thus removes the appearance of a need for a foundation. Actual uses of language are not questioned as the need for talk of entities is removed along with the talk itself.

2.3. Wittgenstein's Conception of Philosophy and 'Leaving Mathematics As It Is'

Wittgenstein takes philosophy to 'leave mathematics as it is' in much the same way that he takes it to leave actual uses of language as it is (of which mathematical language is a subset), albeit with some minor special considerations incorporated into the approach. As he says at the beginning of *LFM*:

> I can as a philosopher talk about mathematics because I will only deal with puzzles which arise from the words of our ordinary everyday language, such as "proof", "number", "series", "order", etc. (*LFM*, p.14)

The suggestion of this remark is that questions related to these terms are to be treated in much the same way as philosophical questions relating to terms like 'pain.' This may make it seem like he won't need to talk about any detailed mathematics and will instead focus on everyday words. However, the remark continues:

> Knowing our everyday language- this is one reason why I can talk about them [the puzzles concerning "proof", etc.]. Another reason is that all the puzzles I will discuss can be exemplified by the most elementary mathematics – in calculations which we learn from ages six to fifteen, or in what we easily might have learned, for example, Cantor's proof. (*LFM*, p.14)

So Wittgenstein will need to talk about mathematics but it will be mathematics that it should be within his and his audience's power to grasp. One might wonder why Wittgenstein should need to talk about mathematics at all given that the puzzles arise from ordinary language. But to look at the various uses of a term like "number", one has to look

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[35]Cited by Kuusela (2008, p.255).
[36]I take this to be Cavell's point when he says that "scepticism for Wittgenstein is the intellectual twin of metaphysics" (Cavell 2005, p.195) – both are cleared away by an overview.

at some mathematics - at the least one will have to be sensitive to differences like that between complex numbers and natural numbers (*LFM*, p.15).

Evidently confusions arising from ordinary everyday language can relate to mathematics. Wittgenstein says that these confusions are particularly "tenacious" and describes them as arising in a way in line with the characterisation of philosophical problems from section 1.2:

> What kind of misunderstandings am I talking about? They arise from a tendency to assimilate to each other expressions which have very different functions in the language. We use the word "number" in all sorts of different cases, guided by a certain analogy. We try to talk of very different things by means of the same schema... Hence I will have to stress the differences between things. (*LFM*, p.15)

A philosophical problem in connection with the term "number" is thus not a mathematical problem but it is nonetheless connected with mathematics. Whilst the term "number" is perfectly familiar and might be said to be a term of our "ordinary everyday language", it is used in particular ways in relation to different areas of mathematics (cardinal numbers, imaginary numbers, real numbers, transfinite numbers etc.). Wittgenstein says that "Mathematicians only go astray, when they want to talk about calculi in general" rather than a "particular calculus" (*PG*, p.369) and this is presumably because wanting to talk about, say, "number" in general terms gives rise to the temptation to see "number" as a concept with a single essence. One might then be tempted towards a particular model of "number" and lose sight of some of the various different senses of "number".

If one is in the grip of a metaphysical conception of "number" (or "proof" or "series") then all of the particular instances of ways of using the term "number" will be seen on the same model. When seen through the lens of such a metaphysical projection, particular uses of "number" will seem problematic insofar as they can only be made to fit the model with awkwardness. We might then be tempted to give explanations concerning how the particular instances of the use of the term should be seen in terms of the model. Metaphysical philosophers may regard this as an important task but Wittgenstein regards such explanations as 'vapour'. They are an expression or continuation of the philosophical problem/confusion:

> The philosophy of mathematics consists in an exact scrutiny of mathematical proofs - not in surrounding mathematics with a vapour. (*PG*, p.367)

Whilst philosophical confusions may arise in relation to everyday terms such as "number" or "proof", noting the differences in their uses can require a detailed examination of how they are used in particular mathematical contexts. Exploring sufficient range and depth of cases can be crucial to resolving the problems.

Despite noting that mathematicians only get into philosophical confusions when they try to talk about "calculi in general" (*PG*, p.369), it is not the case that mathematical works are therefore entirely free of philosophical influence. Wittgenstein says that mathematicians are sometimes tempted to talk about calculi in general within otherwise mathematical works and emphasises that the presence of such talk does not compromise the mathematical value of the works or the validity of the mathematics:

> In mathematics there can only be mathematical troubles, there can't be philosophical ones.
>
> The philosopher only marks what the mathematician casually throws off about his activities. (*PG*, p.369)

He speaks of philosophical expressions which relate to mathematics (that which is 'casually thrown off') as "prose" and distinguishes them from the mathematics itself. He thought that being clear about separating prose from mathematics was an important part of resolving at least some of the puzzles he was interested in:

> It is a strange mistake of some mathematicians to believe that something *inside* mathematics might be dropped because of a critique of the foundations. Some mathematicians have the right instinct: once we have calculated something it cannot drop out and disappear! And in fact, what is caused to disappear by such a critique are names and allusions that occur in the calculus, hence what I wish to call *prose*. It is very important to distinguish as strictly as possible between the calculus and this kind of prose. Once people have become clear about this distinction, all these questions, such as those about consistency, independence, etc., will be removed. (*WVC*, p.149)

This remark reveals that at least some of the puzzles that interested Wittgenstein were also of concern to mathematicians. Wittgenstein is prepared to question pictures to which mathematicians subscribe but the pictures that he will question are not mathematical pictures. Wittgenstein will question some of what mathematicians say but he will not question their mathematics.

This *WVC* remark also reveals that Wittgenstein's idea that philosophy 'cannot give any foundation' to ordinary language (*PI*, §124) also applies to mathematics. Wittgenstein is making the contention that some mathematicians had fallen into thinking that a 'foundation' were needed for mathematics because they had fallen into dogmatically applying certain specific pictures of mathematics. Even if the sort of 'foundation' being sought were mathematical, as was the case with the search for consistency proofs, Wittgenstein suggests that the motivation was at least in part philosophical. Wittgenstein takes the motivation to have been sceptical concerns about mathematics arising from metaphysical pictures. Wittgenstein is especially concerned with the quasi-empirical leaning that these pictures exhibit as in this connection he mentions in particular that "analysis and set theory are always taken to be theories describing something, not calculi" (*WVC*, p.141). Whilst this remark on its own is presumably not meant as anything more than a first indication of the kinds of pictures involved, it suggests that the pictures are derived from other forms of language (e.g. physics). Much as expressions of pain can seem mysterious and problematic when seen on the model of descriptions of entities, likewise statements of mathematics can seem mysterious and problematic when seen on the same model. Wittgenstein draws exactly this parallel between problems related to psychological expressions and problems related to mathematical expressions in the last remark of what is published as *PI*:

> An investigation is possible in connexion with mathematics which is entirely analogous to our investigation of psychology. It is just as little a *mathematical* investigation as the other is a psychological one. It will *not* contain calculations, so

> it is not for example logistic. It might deserve the name of an investigation of the 'foundations of mathematics.' (*PI*, p.232[37])

So Wittgenstein is interested in what might be called the 'foundations of mathematics' but he is not interested in providing a foundation *for* mathematics. The idea that mathematics should need a foundation to protect it against scepticism is a confusion, to be addressed by philosophical clarification rather than the production of more mathematics or by making changes to mathematics.

The appearance that mathematics should be secured from scepticism by mathematical means is connected with the appearance of 'leading problems of mathematical logic.' Wittgenstein borrows this phrase from Ramsey (1987, p.2) and uses it to refer the decision problem and other mathematical problems which might be thought to be of special philosophical significance (*PG*, p.196). In pointing out what might be problematic about this, Wittgenstein cautions against thinking that solving a mathematical problem could "at long last give us the right to do arithmetic as we do" (*PG*, p.196). When in the grip of certain pictures, particular mathematical problems such as the establishing of a logical model of arithmetic might seem to have foundational significance. Through the lens of certain pictures it might seem as though arithmetic were problematic as it stood and in need of being put on a new footing in order to secure it against scepticism. Whilst Wittgenstein has no need to question the mathematical validity of systems produced with this kind of foundational motivation, he does question whether the systems have the significance attributed to them. He encourages us (through his philosophical questioning) to see them as "mathematics like any other."

In saying that "leading problems of mathematical logic" are "mathematics like any other", Wittgenstein is suggesting that certain mathematical systems have a different significance from that which is sometimes ascribed to them. If certain systems were, for example, seen by mathematicians as providing a foundation for everyday arithmetic then Wittgenstein would be disagreeing with mathematicians concerning the significance of these systems. As we shall see in chapter 8, Wittgenstein is critical of Frege and Russell's interpretations of their logical systems as providing such an alleged foundation for arithmetic. He is likewise critical of certain interpretations of the significance of other parts of mathematics and his remarks on such topics, especially set theory and the real numbers, have proven highly controversial.[38]

Part of the reason why Wittgenstein's remarks have been so controversial is that readers can find it hard to see why Wittgenstein should want to separate 'prose' (non-mathematical or philosophical comment) from 'mathematics' unless he were to have some philosophical conception of what mathematics is.[39] It can be hard to see *how* Wittgenstein can hope to distinguish prose from mathematics in particular cases unless he were to have an account (perhaps some kind of definition) of what these notions are. If one looks at Wittgenstein in this way then it can be easy to read Wittgenstein as presupposing a certain philosophical

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[37]The preceding remark concerns seeing statements of memory on the model of descriptions. The investigation he mentions here is presumably the investigation of 'foundation' in the phrase 'foundations of mathematics mentioned at the start of *LFM* (p.14). This also seems to square well with Wittgenstein says of purposes in *RFM* (p.376-383).

[38]Putnam (2007), for example, is especially critical.

[39]This topic will be explored further in the next section.

picture of mathematics (say, a broadly constructive or conventionalist one) and applying this particular picture to various pieces of mathematics. If the mathematics in question does not fit with the picture, then Wittgenstein allegedly criticises it (supposedly suggesting the mathematics to be not *bona fide* in some way). As we shall see in chapters 4 and 5, this is how Maddy (1993, p. 67) and Steiner (2009, p.23) read Wittgenstein on set theory.

Pressure can be put on the temptation to read Wittgenstein as using some particular philosophical picture of mathematics to decide what is mathematics and what not by considering Wittgenstein's criticism of the 'dogmatism' that metaphysics gives rise to. If Wittgenstein were to use a particular philosophical picture of mathematics to decide what is mathematics and what not, Wittgenstein would be just as dogmatic as any metaphysician. Not only would he be imposing a particular picture on all cases, he would be dismissing particular cases as not legitimate on the basis of failing to fit the model. In deciding what does and does not count as mathematics, Wittgenstein would surely be making decisions that one would expect a mathematician to be making (*PG*, p.369).

Before taking up arguments concerning Wittgenstein's remarks on particular topics (such as set theory), the question first needs to be addressed of how the role of Wittgenstein's notion of 'prose' should be understood and why interpreters have been so tempted to read Wittgenstein as employing the notion dogmatically.

2.4. Reading and Misreading Wittgenstein on 'prose'

Wittgenstein's notion of 'prose' seems to be intended in part to contrast exact mathematical language with ambiguous ordinary (non-mathematical) language. Writing concerning certain work of Skolem's, he comments:

> An explanation in word-language of the proof (of what it proves) only translates the proof into another form of expression: because of this we can drop the explanation altogether. And if we do so, the mathematical relationships become much clearer, no longer obscured by the equivocal expressions of word-language. (*PG*, p.422)

One might be tempted to therefore take Wittgenstein's notion of 'prose' to be based upon a prior theory or conception of what mathematics is, this theory being such as to bring out the superior precision of mathematical language. This is how Shanker (1987, p.209) takes the notion and it is worth considering Shanker's view in some detail since Shanker's view risks making it impossible to take Wittgenstein seriously in his promise not to disagree with mathematicians about mathematics (*PG*, p.369). If Wittgenstein were to be read, as Shanker reads him, as having a theory of what mathematics is, then it would have to be answered how that theory could be understood in terms of Wittgenstein's conception of philosophy and how it could avoid being dogmatic. We shall explore how Shanker goes about reading Wittgenstein as having a theory of mathematics and this will provide a useful way to approach the question of what role the notion of 'prose' has for Wittgenstein. The discussion will naturally lead us on to the question of how the role of 'prose' can be understood as non-dogmatic.

As Shanker describes him, Wittgenstein holds that mathematics is made up of a large number of systems, each of which has the meaning of its symbols set by the rules of the system. The most important rules of a system are the axioms and any further rules are

derived from there. Whilst this view needs to be explained at more length[40], the part which is most important for the purpose of how Shanker distinguishes 'prose' from 'proof' is that the meanings of mathematical terms are taken to be fixed by the axioms of the system in question. In order to put across this picture, Shanker lays a particular stress upon remarks like:

> Mathematics consists entirely of calculations. In mathematics everything is algorithm and nothing is meaning: even when it doesn't look like that because we seem to be using words to talk about mathematical things. Even these words are used to construct an algorithm. (*PG*, p.208)

In remarks like this Wittgenstein can be read as articulating a conception of mathematical propositions as rules that fix the ways that terms are to be used within a mathematical system. Under this view mathematical systems can be seen to be 'autonomous', in that each system is not reliant upon anything other than the propositions of the system itself for its validity (*PR*, §111; Shanker 1987, p.305-306).

The pictures of mathematical propositions as rules and of mathematical systems as autonomous are important parts of Wittgenstein's thinking. Before directly considering how these ideas relate to the role of the notion of 'prose', it is worth considering the relationship to some other key themes, especially Wittgenstein's thoughts on the significance of metamathematics and the possibility of scepticism in relation to mathematics. This will help us to better understand what motivates Shanker's view and allow us to do justice to these motivations without following Shanker in ascribing a thesis concerning the nature of mathematics to Wittgenstein.

If mathematics is not seen as a set of autonomous systems and is instead seen as a single global system (perhaps unified by a single set of axioms) then the edifice of mathematics might seem to be open to the possibility of global doubts – problems that could bring down the entire edifice. This kind of picture might seem tempting when talking about 'mathematics' in very general terms, perhaps looking upon 'the body of mathematics' as akin to 'the body of history.' If it were to turn out that some crucial detail of history had been gotten wrong, say that Julius Caesar was not a real person, then this would force us to revise vast amounts of history. But if mathematical systems are autonomous then no analogous relationship holds with problems in mathematical systems (especially contradictions) since then at worst only the system in question could be affected.

This kind of system-specific thinking can be found in Hilbert's writing as well and Friederich (2011, p.5, p.8) suggests that Hilbert may have been an influence on Wittgenstein's development of the idea. But the picture of autonomy that Wittgenstein articulates goes further, since Wittgenstein also stresses that mathematical systems are only related to one another by relationships of analogy or by transformation of one system into another. Hilbert, by contrast, wanted to develop mathematics which would be 'about' mathematical systems. His idea was that mathematical techniques could be used to show whether certain important mathematical systems were consistent. These metamathematical techniques were intended to be part of a foundational programme of putting mathematics on a solid footing by showing mathematical systems to be consistent.

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[40]The view of Wittgenstein as taking mathematics to consist exclusively of rules which explicitly fix the meanings of terms will be discussed in chapters 5 and 6. I wish to agree with Floyd's point that it "would be an overstatement to hold that for Wittgenstein all mathematics is, as such, algorithmic or that only conjectures for which a method of resolution is in hand count as mathematical propositions" (2000, p.248-249).

If metamathematical expressions were mathematics 'about' mathematical systems, at least in the referential sense, then metamathematical propositions would be descriptive propositions and not rules. The metamathematical expressions would not be parts of an autonomous system and would instead be dependent upon other systems for their meaning (namely the systems which they are 'about'). This would run contrary to the picture expressed in (*PG*, p.208) and hence one might wonder whether Wittgenstein would therefore have to reject metamathematics or whether he might instead philosophically interpret metamathematics in a different way from Hilbert.

Whilst Wittgenstein acknowledges the validity of metamathematics, he can be seen as disputing its alleged significance. Rather than interpreting metamathematics to be mathematics 'about' mathematics, he presses a picture in which metamathematical techniques just appear as more techniques. The picture is that when one employs a metamathematical technique, one is introducing a new technique and thereby adding something that enables one to do things in the system that one could not do before. In this sense one is creating a new system in which the old system might be seen to figure as a part – one now has a larger system in which a simulation of the old system can be seen. So the metamathematical technique allows one to prove results in the expanded system but it is a matter of prose to say that they are results 'about' the original system, at least if 'about' is meant referentially. One may well say this but saying it would not be to give an interpretation of the mathematical result rather than to simply state the result. One could put the point by saying that metamathematics for Wittgenstein would be mathematics 'about' mathematics in a very different way from how a description is a statement about its subject-matter.[41] So it seems that Wittgenstein could have maintained a view of mathematical systems as autonomous (Shanker 1987, p.305-306) and still acknowledged the validity of metamathematics.

Wittgenstein's thoughts related to metamathematics and its use can be hard to follow, especially for contemporary mathematicians, because of the way in which techniques related to metamathematics have become widespread. Wittgenstein's thinking might even appear confused to a contemporary mathematician but it is worth noting that Wittgenstein's thinking does not have to be seen as confused. The development that produces the most confusion in relation to understanding Wittgenstein is that it has become common to distinguish between the syntax and the semantics of a mathematical system. In loose terms the idea is that a system is syntactically a set of expressions the system is associated with the possible structures (described set-theoretically) which would satisfy the expressions – these are the models of the system. A sentence of the system is then described as true if it is satisfied in all consistent models of the system. This mathematical approach post-dates Wittgenstein's work but it is plausible that if Wittgenstein had encountered the approach then he would have acknowledged the validity of these techniques and denied that they offer a definitive analysis of 'truth.'[42] He would perhaps say that the move to considering models for the system effectively moves us to an expanded system, allowing us to show things in the expanded system that relate by means of analogy to the original system. This sort of question will become particularly important in chapter 10, where we will see that some of the objections to Wittgenstein's remarks on Gödel's incompleteness theorems have originated from a failure to see how Wittgenstein understands such metamathematical techniques.[43]

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[41]For more on this topic see Mühlhölzer (2012), Shanker (1988) and Floyd (2001).
[42]On this see Floyd (2001, p.304).
[43]Floyd and Putnam's defence of Wittgenstein on this has had a difficult reception for this same reason. See in particular their reply to Bays and Steiner (2006). I note it at this

If one looks at the picture of mathematical propositions as rules, of mathematical systems as autonomous and the criticism of interpretations of metamathematics as mathematics 'about' mathematics then a picture might seem to emerge of Wittgenstein as advocating a conception of mathematical systems as self-defining. According to such a picture, the axioms of a system might be said to be rules which fix certain aspects of the way that the terms should be used and then the propositions are further rules which are derived from the initial rules. Wittgenstein might then be seen as, as he is portrayed by Shanker (1987, p.305-306) as advocating an account of mathematics as the totality of all such systems. Any expressions that are not part of these systems would be non-mathematical, even if they appeared to have some connection to the systems. Such expressions which appeared to be connected to the systems but were not themselves expressions of the systems could be distinguished by being referred to as 'prose.'

Whilst the notion of mathematical systems as autonomous systems of rules has various advantages (such as, as we shall see in chapter 7, undermining the notion that contradictions might pose a threat to all of mathematics) and it undoubtedly *figures* in Wittgenstein's thought, the key question is whether this conception is the *basis* of Wittgenstein's notion of 'prose' in the way that Shanker suggests (1987, p.209). Revealing of the difficulty with this claim is when Shanker says that Wittgenstein's idea of mathematical statements as rules is not an "alternative picture" (p.64) but "a precise philosophical clarification of mathematical syntax" (p.65). The trouble with Shanker's claim is that it if the idea of mathematical propositions as rules is part of a clarification then it surely *is* just one possible picture and cannot be said to be simply part of "mathematical syntax." Otherwise comments like the following would ring hollow:

> ...the whole point is that I must not have an opinion... I have no right to want you to say that mathematical propositions are rules of grammar. I only have the right to say to you, "Investigate whether mathematical propositions are not rules of expression..." (*LFM*, p.55)

If Wittgenstein has to assume that mathematical systems *are* bodies of rules in order to say what is prose and what is not, then any of his claims concerning what is prose and what is not are just matters of opinion that he would have 'no right' to present as clarifications. The danger here is the danger of supposing that mathematical statements *are* rules rather than proposing the picture as an object of comparison with which to *model* mathematical statements *as* rules. Taking mathematical propositions to *be* rules would be problematic both because of Wittgenstein's promises not to be dogmatic, and also because Wittgenstein at points seems to point out the limitations of this picture. As Floyd (2000, p.251) notes (in criticism of Shanker), if mathematical propositions were *simply* rules that set up a system then the contrary of a mathematical proposition would not be a part of a system at all and would thus be meaningless. She cites the following remark by Wittgenstein:

> My explanation mustn't wipe out the existence of mathematical problems.
>
> That is to say, it isn't as if it were only certain that a mathematical proposition made sense when it (or its opposite) had been proved. (This would mean that its opposite would never have a sense (Weyl).) On the other hand, it could be that

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relatively early stage because the assumption is so widespread and it can be difficult to understand Wittgenstein on other matters (e.g. accounts of mathematical propositions or the roles of proof) if one is committed to this assumption.

certain apparent problems lose their character as problems—the question as to Yes or No. (*PR*, p. 170)

This kind of self-cautioning remark makes much more sense if Wittgenstein is saying that mathematical propositions are 'like' rules or that it can help us see past certain problems if we see mathematical propositions as akin to rules. This strongly suggests that Shanker's reading is attributing a thesis to Wittgenstein which Wittgenstein is keen to avoid. But if mathematical propositions are only akin to rules (rather than actually being rules) then we are left with the problem of how Wittgenstein does go about using the term 'prose.' Contrary to Shanker, I want to suggest that Wittgenstein's use of this notion can be understood without any need to invoke a prior conception of mathematics.

2.5.Wittgenstein notion of 'prose' and 'leading problems of mathematical logic'

Wittgenstein comments at the beginning of *LFM* concerning how he intends to question the significance of mathematical systems and his comments suggest that his method in relation to the significance of mathematical systems is closely connected with his method of presenting alternative pictures to resolve philosophical problems by means of perspicuous overviews:

> Mathematicians tend to think that interpretations of mathematical symbols are a lot of jaw – some kind of gas which surrounds the real process, the essential mathematical kernel. A philosopher provides gas, or decoration – like squiggles on the wall of a room.
>
> I may occasionally produce new interpretations, not in order to suggest they are right, but in order to show that the old interpretation and the new are equally arbitrary. I will only invent a new interpretation to put side by side with an old one and say, "Here, choose, take your pick." I will only make gas to expel old gas. (*LFM*, p.13-14)

The remark seems to concern prose since he is talking about 'interpretations of mathematical symbols.' The suggestion here is that prose can be revealed as such by showing it to be 'arbitrary' because we would give it up if we were to adopt an alternative interpretation. The idea seems to be that each of the prose pictures simplifies the mathematics in its own way and so we can call the choice between them 'arbitrary', since we would be prepared to give up any or every such picture without giving up the mathematics. A particular interpretation of a mathematical system/expression is thus much like an object of comparison. The interpretation may be enlightening with regard to the system/expression in some particular respect but the interpretation is not put forward in order to be advocated as *the* correct interpretation. Like an object of comparison, the interpretation serves to remind us of particular aspects which we were in some sense already familiar with. The interpretation can be useful without needing to be put forward as the definitive account and is only meant to be useful insofar as it resolves the particular problem/s.

On this reading 'prose' is not an idea presented as part of a definitive account of what mathematics is. Contrary to Shanker, Wittgenstein is not committed to a thesis whereby any given statement would always have to be definitively classifiable as either mathematics or prose. Instead Wittgenstein is introducing a distinction that can be used to resolve particular confusions in particular situations. Marking out particular expressions as prose rather than mathematics in particular cases can help us to see that other pictures are possible with respect to the mathematics and this can allow us to overcome the grip of a particular picture (this attachment to the particular picture being the root of the particular confusion).

A key part of the test of this reading of course lies in whether Wittgenstein's discussions can be seen to fit with it. Particular topics in relation to the alleged significance of mathematical systems will be covered in parts 3 and 4. It will there be argued that Wittgenstein's method in treating those topics does fit with the promise to "only make gas to expel old gas" (*LFM*, p.13-14). He typically introduces a particular interpretation of a mathematical system (say, a Platonist interpretation), proposes alternative interpretations (say in terms of a picture of mathematical propositions as rules) and points to ways that the interpretations start to look problematic by considering ways in which we would want to use the propositions of the system in different scenarios and what we might say of applications that the system could have.

Wittgenstein is aware that certain philosophical perspectives on mathematics do recommend that we give up certain mathematical propositions, asking us to treat as prose any purported mathematics which, for example, uses the law of excluded middle (this claim being part of how he, on my reading, views both Intuitionism and Constructivism).[44] Wittgenstein is generally disparaging about such positions, saying of Intuitionism that it is "all bosh – entirely" (*LFM*, p. 237). He makes it quite clear that he does not want to be identified with revisionist positions:

> Turing doesn't object to anything I say. He agrees with every word. He objects to the idea he thinks underlies it. He thinks we're introducing Bolshevism[45] into mathematics. But not at all. (*LFM*, p. 67)

I will argue in sections 6.3 and 6.4 that a key part of what is distinctive about Wittgenstein's approach (when contrasted with Intuitionism and Constructivism) is that, contrary to Shanker's reading, he does not draw any principled line between prose and mathematics apart from all of the particular cases. This is a key criticism that he voices of Intuitionist and Constructivist approaches. Commenting upon the restriction that they place on what can be counted as an existential theorem, he remarks that:

> When the intuitionists and others talk about this they say: 'This state of affairs, existence, can be proved only thus and thus.' And they don't see that by saying that they have simply defined what they call existence. (*PG*, p.374)

As I will argue, Wittgenstein can be seen to be criticising the Intuitionists and Constructivists for dogmatically claiming to have an account of what can and cannot be accepted as a valid piece of mathematics in advance of the individual cases. As he puts it, "What is an existential theorem? The answer is this, and this, and this . . . " (*AWL*, p.116).

Rather than starting from a particular account of what mathematics must be or some metaphysical consideration of the limits of mathematical reasoning, Wittgenstein looks at the role played by particular statements in particular contexts. This in itself does not guarantee that Wittgenstein won't identify a statement as prose that we might otherwise take to be mathematics but when he does so he will go about it by showing that the appeal and plausibility of the statement in question rests upon a particular picture and that the picture can be given up without giving up the mathematics.

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[44]Wittgenstein mentions Weyl in this vein in the remark cited above - (*PR*, p. 170). More will be said on this in chapter 6.

[45]This term very likely comes from Ramsey's reference to "the Bolshevik menace of Brouwer and Weyl" (1931, p. 56).

Rather than taking the classical mathematician to be wrong in failing to see some intuitionist or constructivist insight, Wittgenstein wishes to consider particular cases on their respective merits. This is important because Wittgenstein is aware that mathematicians are unlikely to give up a mathematical proposition or argument unless they can be given a mathematical reason for doing so. The fact that some mathematics does not conform to some particular philosophical picture will not influence mathematicians – it is simply not the type of concern that they (*qua* mathematicians) will be influenced by. Wittgenstein hence does not want to be dictating on what is and is not mathematics in the way that Intuitionists and Constructivists arguably do. The Intuitionists and Constructivists risk being left looking dogmatic as they try to decide what mathematics is admissible by appeal to criteria that classical mathematicians consider to be philosophical rather than mathematical.

The questions of whether Wittgenstein thought that mathematical propositions really are rules and of whether he advocated changes to mathematical practice on this basis are, as I have suggested, key questions for the interpretation of Wittgenstein's philosophy of mathematics. I have here begun to answer both questions in the negative and will later make both cases in more detail. However, these are not the questions that I will discuss in the next chapter. It is necessary to first establish a clearer sense of why these interpretative problems arise with the poignancy that they do. Wittgenstein's criticisms of certain metaphysical approaches to the philosophy of mathematics, especially Platonism, can easily be read (or, as I will argue, misread) in a way which makes it seem as though Wittgenstein were dogmatically advocating a particular picture of mathematics. For this reason the next two chapters will focus on some aspects of how Wittgenstein distances himself from more traditional philosophical positions and how he provides alternative ways of addressing certain problems in the philosophy of mathematics.

2.6. Chapter Conclusion

This chapter has covered some of the special challenges that Wittgenstein's use of the notion of 'prose' can give rise to for interpreters. It can easily seem as though Wittgenstein has a philosophical theory by which he aims to define what 'mathematics' is and thereby separate mathematics from prose. This is how Shanker reads Wittgenstein and Shanker bases this in part on attributing to Wittgenstein a view of mathematics as comprised of self-defining rule-systems or 'calculi'. However, this chapter has argued that reading Wittgenstein in this way attributes to Wittgenstein an inconsistent point of view (with respect to his promises in (*PI*, §124)). It should be acknowledged that calculi play a role in Wittgenstein's thinking but not the role of an exhaustive and final account of what mathematics is. Prose can instead be understood as a dispensable accompaniment of mathematics. Prose is an attempted translation of mathematics into ordinary language and is marked out by being unnecessary for the validity of the mathematics.

Part 2 – Not Needing Philosophical Foundations for Mathematical Practice: 'for it cannot give it any foundation either'

Chapter 3 – Mathematical Inference without foundations

3.1. Chapter Introduction

The purpose of this chapter and the next is to show how Wittgenstein applies his approach (as described in part 1) to questions which best exhibit the application of certain pictures in a way which gives rise to philosophical problems which can make mathematical practice appear as though it were problematic and in need of philosophical justification. These are questions concerning how we use the related terms 'mathematical inference' (to be covered in this chapter) and 'mathematical proposition' (to be covered in the next). These are problems which connect with certain traditional philosophical positions such as Platonism. The discussions also help to illustrate Wittgenstein's claim, made in (*PI*, §124), that philosophy 'cannot give any foundation' to how we use expressions. They also go to the heart of the web of problems that Wittgenstein wants to address in his philosophy of mathematics (meaning that themes explored in this chapter will re-emerge in later discussions, especially in part 3).

Wittgenstein's discussion of inference is rich in connections and not all connections can be explored here. This chapter will follow Wittgenstein's suggestion that inferring 'consists in the transition from one assertion to another' by means of a rule (*RFM*, p.39) and his exploration of certain problems that one can get into when attempting to give a definitive account which aims to justify inference in general, especially problems with regard to logical and mathematical inference. Following a certain tempting line of thinking can make such an account seem to be needed, as though mathematical inferences would be doubtful without such an account. But, if we follow through on this line of thinking, anything we appeal to in justification of an inference would itself be open to similar doubts and would be similarly in need of support. Using (Floyd 1991), I will explore how Wittgenstein both shows us a path into this position of apparently inescapable doubt and also shows us what mistakes lead us there, so that we can see a way out again.

3.2. Mathematical Inference

The material that this chapter focuses on is now widely known as the beginning sections of *RFM*. As Floyd (1991, p.50) explains, this material was at one point intended to form part of *PI* and was planned to link with the discussion in *PI* of rule-following. It deals with issues which have become iconic of Wittgenstein's philosophy and therefore gives a pertinent place to start a detailed examination of Wittgenstein's work on mathematics

(although not an easy place as the material is some of the most difficult to get to grips with). The question with which *RFM* begins is the following:

> We use the expression: "The steps are determined by the formula..." How is it used? (*RFM*, p.35)

This question, though general, is not typical of philosophical questions. But in asking it Wittgenstein does intend to say something that will help to address certain philosophical questions. This becomes clear when in the course of discussion Wittgenstein's interlocutor asks "what does the peculiar inexorability of mathematics consist in?" (*RFM*, p.37). This question is a more typical philosophical question and as such it is not obviously a question concerning the way words are used. The interlocutor's thought seems to be that when we calculate using a formula we are doing something which is guided by the 'peculiar inexorability of mathematics' (as though this inexorability were itself a thing). Wittgenstein aims to address this line of thinking by considering a specific case-study or example which the interlocutor will agree should show the 'peculiar inexorability of mathematics,' if indeed anything will. The example is "the inexorability with which two follows one and three two" – the example of counting or expanding an arithmetic series according to a formula.

The aim of the discussion is said to be to "get clear what inferring really consists in" (*RFM*, p.39). Wittgenstein wants to consider particular inferences (such as the inference from one step in a sequence to the next) as examples of what we call 'inferring', so as to see in what sense the 'peculiar inexorability of mathematics' might be present. Wittgenstein's interlocutor is tempted in the course of the discussion towards certain accounts of how we are able to make inferences – accounts which might explain what justification I have for inferring that three follows two in a particular sequence. To preview where the discussion will go, we will come to see that the model of pointing to a thing which justifies an inference does not fit certain basic cases of mathematical and logical inference (such as following a basic series). In other cases there may be something we can appeal to which we should say were the basis for our inference - for example, if the appropriate steps were already presented beforehand in some other form (*RFM*, p.45-46) or if the transition were made via another proposition as part of a chain (*RFM*, p.39). But in certain key cases our picture of the inference will be distorted and the inference will appear to us problematic if we see it as having a basis or grounding analogous to the other cases.

An early move in this dialectic is Wittgenstein's suggestion that any explanation of how we infer from one term to the next in a simple series (such as counting out the natural numbers) would be 'superfluous' (*RFM*, p.37). To the interlocutor this suggestion itself (despite the anti-sceptical intentions behind it) looks like a kind of scepticism – as though it were a denial that we really do infer at all. Floyd notes that in an earlier version of the manuscript intended for *Philosophical Investigations* the interlocutor asks "are the steps then not determined by the algebraic formula?", to which Wittgenstein responds that "the question contains a mistake" (Floyd 1991, p.151). The point, then, of refocusing the discussion on how we use the expression "the steps are determined by the formula..." is to show the interlocutor that what is at issue is not whether the steps in expanding a series are determined by a formula but whether anything is added by the suggestion that there might be something, "some truth corresponding to this sequence" (*RFM*, p.37), which explains the steps being determined by the formula. The point is to show that seeking this 'truth corresponding to the sequence' is not a way of achieving greater clarity about certain key cases of inference in accordance with a formula or series.

Wittgenstein's interlocutor is tempted to say that the formula itself (or some abstract entity corresponding to it) determines the steps, that it is the justification of our particular

inferences. But there is something strange about this way of talking. It seems plausible that if I were challenged on my expanding a series as I did I would point to the formula that I was using but it's not clear that my pointing to the formula would be any justification of my particular inferences – more a reminder of the fact that I'm trying to expand this series (as opposed to some quite different series). There is something odd about offering the formula itself as justification and Wittgenstein puts pressure on this idea by exploring how we might use the expression "the steps are determined by the formula." If determination by a formula were to be a matter of the steps being such as to correspond to the formula/sequence as an abstract entity then we would expect to be able to see this in relation to the particular cases.

Wittgenstein hence presents various models/scenarios for particular ways we might use the expression "the steps are determined by the formula." The first model is as a description (though naturally not a complete description, to the exclusion of others) of how people behave when using a formula:

> We may perhaps refer to the fact that people are brought by their education (training) so to use the formula $y = x^2$, that they all work out the same value for y when they substitute the same number for x. (*RFM*, p.35)

Since they all get the same result, we might use the expression 'the steps are determined by the formula' to point to this training-induced correlation. The model is quite behaviouristic – Floyd notes that the German term for 'training' here is meant to convey training by rote (Floyd 1991, p.152) – but Wittgenstein does not dwell on any behaviouristic suggestions (he is not advocating any behaviourist doctrine). Instead he immediately moves on to propose another model suggesting that the people are given an explicit instruction to 'add 3' and they all behave likewise (or one might call it a variation on the first model, since it just replaces the operation of squaring with adding). In this circumstance we might again say that "the steps are determined by the formula" describes the observed correlation, or rather by the instruction to 'add 3.'

The interlocutor might think these cases irrelevant and unrelated to the essence of 'determination by a formula' (since the interlocutor's sought-after truth corresponding to the formula is not seen to be in play in them). So Wittgenstein continues his survey and proposes that we might understand "the steps are determined by the formula" in a very different way – we might use it as a way of distinguishing different kinds of formula:

> Then we *call* formulae of a particular kind (with the appropriate methods of use) "formulae which determine a number y for a given value of x", and formulae of another kind, ones which "do not determine the number y for a given value of x." ($y = x^2 + 1$ would be of the first kind, $y > x^2 + 1$, $y = x^2 \pm 1$, $y = x^2 + z$ of the second.) (*RFM*, p.35)

This would be another very specific way of using the expression "the steps are determined by the formula", namely to distinguish formulae that determine y from those that don't.

Wittgenstein draws the classificatory model out in some detail, noting that it is easy to imagine the classifications being applied for, say, a list of formulae that are all written down but not given in the question. Wittgenstein notes that it is not so easy to make sense of attempting to apply the classification by asking "is $y = x^2$ a formula which determines y for a given value of x?" (*RFM*, p.35). The answer seems too obvious for the question to need asking, to the extent that we could give no better explanation of what it means for a formula to determine y for a value of x than to point to a formula like "$y=x^2$". This is

perhaps why Wittgenstein wants to consider this – he wants to draw attention to how the classificatory scheme is itself understood by reference to certain kinds of paradigmatic examples. He perhaps wants to suggest the more general point that when we call something an inference we understand it to be such by reference to certain paradigmatic cases. We can give examples of what we mean by 'inference' but there may be no general account available to us which will explain the most basic examples. Perhaps there is no single thing for us to find in which all inferring might be said to consist (except perhaps in the transition from one proposition to another by means of a rule, which would appear to be an answer that says too little to satisfy the interlocutor's aspirations). There is a hint of this suggestion in the way that Wittgenstein does suggest a use for the too-obvious question – he says we "might address this question to a pupil in order to test whether he understands the use of the word 'to determine'" (*RFM*, p.35-36). One might likewise imagine putting the question "can you infer the value of y if $y=x^2$ and x is 3?" in order to check that a non-native English speaker were familiar with the English word 'infer.'

The question "is $y=x^2$ a formula which determines y for a given value of x?" is also illuminating as it gives a scenario where it is not clear what is wanted. Whilst the other proposed models are various and their variety does suggest that there is not a single thing that we can call 'determination by a formula', they do share a common theme in that they all point to contexts where it is clear what is wanted. When a question such as "is $y=x^2$ a formula which determines y for a given value of x?" or "what sort of determination is the determination of steps by a formula?" is put then determination can be different things and in these particular cases it is not clear what is wanted (Floyd 1991, p.157). And if determination can be different things in different cases, even for determination by an algebraic formula, then the idea that there is a quality of 'correspondence to the truth/formula' which explains this determination is then challenged. It is starting to look like this notion of such a quality cannot be brought to bear on the case to which it was supposed to be best exemplified (namely determination by a formula).

The interlocutor tries to rescue the idea that the steps are determined by something corresponding to the formula by proposing another picture. They propose that the "way the formula is meant determines which steps are to be taken" (*RFM*, p.36). Wittgenstein's response is to suggest that it will be equally problematic to attempt a general account of 'the way the formula is meant.' There is no reason to think that there is something common to all instances of meaning a formula in a particular way. In order to make this suggestion, he asks us to consider how we would see what were meant by a new symbol such as '$x!2$' (*RFM*, p.36). We would have to be shown how to use it to determine particular steps (perhaps using the symbol to mean 'x^2'). In order to learn the use of the symbol we have to be made to see it as determining the particular steps and in this first instance we have no prior grasp of the meaning. If, as the interlocutor proposes, using the symbol correctly is a matter of following the meaning then we have to know the meaning in order to use the symbol. But we have no way of learning the meaning without seeing what constitutes a correct use and so the interlocutor's picture leads us into an apparently vicious circle. The question of how the formula can determine the steps and how it can have a certain meaning now appear as two sides of the same question.

In order to make the interlocutor feel more acutely the problem with attempting a general account of inference as following something (such as a 'meaning') corresponding to the formula, Wittgenstein suggests a new question:

> *How do I know* that in working out the series + 2 I must write
>
> "20004, 20006"
>
> and not
>
> "20004, 20008"? (*RFM*, p.36)

A general account would tell us what it is in virtue of that I know what to write. But finding such an account looks problematic since the use of 'how do I know how to write' here seems to be empty. There are other circumstances where there might be something that we could point to, some justification that we could give but here no justification looks like it could be any more certain than my certainty in continuing the series as I do. As Floyd puts it, any justification would look "superfluous, ad hoc – hardly informative about this particular step" (Floyd 1991, p.161). The interlocutor's account (the proposal that I know what to write in virtue of what is meant) is supposed to explain how we make the step from one term in the series to the next in a way which is independent of our actually making the step. Whatever answer the interlocutor might give to the question of what this justification is, whatever principle he invokes, one might then ask whether that principle has been applied correctly, thus invoking the same question of in virtue of what does he know that. If, for example, he says that we know what to write in virtue of knowing what is meant, then the question becomes how do we know what is meant. The picture leads us into a regress of interpretations of what is meant, requiring further interpretations indefinitely.

The interlocutor fails to understand Wittgenstein's suggestion that any justification would be superfluous and instead asks whether Wittgenstein means "to say that the expression '+2' leaves you in doubt what you are to do e.g. after 2004?" (*RFM*, p.37).[46] Wittgenstein replies that:

> No; I answer "2006" without hesitation. But just for that reason it is superfluous to suppose that this was determined earlier on. My having no doubt in face of the question does not mean that it has been answered in advance. (*RFM*, p.37)

Wittgenstein does not deny that there is certainty here. Wittgenstein's point is that it misrepresents this certainty to characterise it as knowledge of something in advance of taking the step. It would be less misleading to characterise it instead simply as a kind of certainty in a way of acting -a certainty in acting appropriately rather than a certainty concerning something known.[47] The certainty can then be understood as indicating an absence of any doubts (or indeed grounds for doubts) rather than the possession of a

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[46]Kripke has attributed to Wittgenstein a position whereby there is a "paradox" here concerning how to follow a rule whose sceptical results have to be in part accepted (Kripke 1982, p.20, p.89). I follow Floyd (1992, fn 55) in rejecting such a reading implicitly by following a line of interpretation whereby the sceptical view is attributable only to the interlocutor and not to Wittgenstein. Whilst it is not my intention to cover this point in the same detail as the authors which Floyd references (1992, fn 55), I will return to the point in the next section.

[47]On this Floyd (1991, p.165) cites *PI* section 211. , *OC* sections 448-9 and (*PI*, p. 211).

justification.[48] But the interlocutor is unlikely to be fully persuaded of this whilst the picture of something corresponding to the sequence continues to hold some appeal.

3.3. Something Behind Mathematical Inference?

The interlocutor still wants to say that the determinations of the next step in the series are in some sense already made before we make them. Perhaps the interlocutor thinks, in Platonic fashion, that I know what to write in continuation of the series because the correct thing to write is a 'mathematical fact' (Floyd 1991, p.165). Perhaps the interlocutor thinks of the rule itself as kind of pre-existing entity which somehow determines what we are to do. Whilst there might be cases where it would make sense to say that the determinations are made 'in advance', it is not clear what this can mean for this case. Wittgenstein comments:

> In his fundamental law Russell seems to be saying of a proposition: "It already follows-- all I still have to do is, to infer it." Thus Frege somewhere says that the straight line which connects any two points is really already there before we draw it; and it is the same when we say that the transitions, say in the series + 2, have really already been made before we make them orally or in writing--as it were tracing them. (*RFM*, p.45[49])

This idea that the transition has already been made in advance might well be applicable to some cases but not for the case where we are expanding the series. For this case it looks like the idea of the determinations being made 'in advance' is only a metaphor that it is not clear how to cash out. Wittgenstein elaborates:

> One might reply to someone who said this: Here you are using a picture. One *can determine* the transitions which someone is to make in a series, by doing them for him first. E.g. by writing down in another notation the series which he is to write, so that all that remains for him to do is to translate it; or by actually writing it down very faint, and he has to trace it. In the first case we can also say that we don't write down *the* series that he has to write, and so that we do not ourselves make the transitions of that series; but in the second case we shall certainly say that the series which he is to write is already there. We should also say this if we *dictate* what he has to write down, although then we are producing a series of sounds and he a series of written signs. It is at any rate a sure way of *determining* the transitions that someone has to make, if we in some sense make them first. (*RFM*, p.45-46[50])

If the next thing to write down were already traced out for us then we could point to the traced-out picture and say that we were following it – we could then appeal to the tracing

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[48]As we shall see in the ensuing chapters, counting is an activity that Wittgenstein thinks might better be understood as an alignment in practices to be applied in other ways (much like an agreement in a way of measuring) rather than as knowledge of some content (*RFM*, p.356).

[49]Also cited by Floyd (1991, p.166).
[50]Again cited by Floyd (1991, p.166-167).

to provide a justification for our writing what we do. In this case, unlike the others, there is something (a thing) which determines the steps in advance – namely the tracing.

One might try to use the expression 'the steps are determined in advance' in a more general way. One might say that the steps are already determined in advance as a way of expressing that we are certain that somebody has now learned how to continue the series. But in this the Platonic idea of responding to a pre-existing entity (or fact concerning pre-existing entities) only seems to figure as a metaphor, if at all. For the tracing case there is a specific way of answering the question 'how do you know' as the pupil can point to the tracing. When we ask the 'how do you know' question in relation to expanding the series, the question tempts us to carry the expression 'determined in advance' over to cases where there is nothing analogous to the tracing to point to. And the Platonic answer is equally misleading in that it likewise carries the idea of knowledge of something (knowledge of a shadowy fact) over into the cases where it is not appropriate (Floyd 1991, p.167).

Even if one thinks that there is some mathematical fact to appeal to which is somehow independent of my making the step that I do in the series, it is hard to see how I could appeal to my perception of that fact as justification for making the step. In order to do so, my appeal to the mathematical fact would have to be independent of and more certain than my making the judgement that I do. If the idea is that the rule or the series were a kind of entity to be perceived, then this entity would 'stand there like a sign-post' (*PI*, §85) leaving me still having to make a judgement. Positing the rule or series as a kind of entity leads to a regress of interpretations problem much like we encountered in the last section with the appeal to the meaning of the formula (or one might say another side of the same problem). Whatever principle or entity that the interlocutor invokes as justification, we could still ask how we know that the principle or entity's significance were interpreted correctly (Floyd 1991, p.162). In order to avoid the regress, the principle or entity would somehow need to be such that its interpretation was beyond question. But it is difficult to see how there could be anything in regard to which individual interpretation were less of an issue than the continuation of a simple series by the rule 'add 2' and this is the very case at issue. No matter how much we may want to give a justification for continuing the series as we do, no appropriate candidate for a justification is available. In *PI* Wittgenstein is emphatic that we reach a point where we can give no further justification:

> "How am I able to obey a rule?"—if this is not a question about causes, then it is about the justification for my following the rule in the way I do.

> If I have exhausted the justifications I have reached bedrock, and my spade is turned. Then I am inclined to say: "This is simply what I do." (*PI*, §217)

If we feel a demand for a justification particularly strongly, then it can seem as though not giving a justification would leave the whole practice of expanding the series in doubt (*RFM*, p.37). But the demand itself is a manifestation of a confusion – (*PI*, §217) continues with the parenthetical remark:

> (Remember that we sometimes demand definitions for the sake not of their content, but of their form. Our requirement is an architectural one; the definition a kind of ornamental coping that supports nothing.)(*PI*, §217)

Wittgenstein elsewhere comments that a philosophical question often manifests as a demand for a definition but the demand is an expression of a confusion (*BB*, p.26). The root problem is that we don't see our way around the uses of the relevant expressions and in this case the key expression is 'the steps are determined by the formula.' We carry a picture over from a case where the expression fits to other cases where the expression does not fit and so we see certain cases in a distorted light. In particular we carry over a model from cases like where the steps are traced out, perhaps because this kind of case is more concrete and more easily pictured (on which see the quotation below). We apply the model to all cases of determination and this gives rise to an 'architectural requirement' that we should have something to point to as a justification. In order for the requirement to be met, we think we have to speak of a 'meaning' as an object or a 'mathematical fact' as a configuration of pre-existing mathematical entities. Otherwise the requirement would go unsatisfied and our making inferences as we do would then look unjustified, as though we could infer whatever we wanted. When the interlocutor objects that "I must only infer what really *follows*" Wittgenstein replies by pointing to the presupposition of the inappropriate model:

> Is this supposed to mean: only what follows, going by the rules of inference; or is it supposed to mean: only what follows, going by *such* rules of inference as somehow agree with some (sort of) reality? Here what is before our minds in a vague way is that this reality is something very abstract, very general, and very rigid. Logic is a kind of ultra-physics, the description of the 'logical structure' of the world, which we perceive through a kind of ultra-experience (with the understanding e.g.). Here perhaps inferences like the following come to mind: "The stove is smoking, so the chimney is out of order again." (*RFM*, p.40)

The model of inferences being grounded by an agreement with reality is perfectly appropriate to empirical inferences (it does not lead to confusion in relation to them) and with the inference that the chimney is out of order we can point to the fact that leads us to it (namely that the stove is smoking). The model thus requires that there be something that can be pointed to as a justification and so we impose this requirement of the model upon the mathematical inferences in question (the expansion of a series in accordance with a formula).[51] We posit meanings or mathematical entities to satisfy this requirement by giving us things to point to as justification. The nature of these posited entities is not entirely clear – we struggle to say how or where they exist or give specific examples of them.[52] Instead we treat the postulated entities as standing behind our concrete examples of inferences, perhaps as part of a 'logical structure' that cannot easily be seen at the surface but which a complete analysis would bring out. The point of these postulated entities is supposed to be to make sense of the practice of inferring – to show us 'what inferring

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[51]This parallels how the *Tractatus* requirements for language were imposed upon language itself (*PI*, §104). See section 1.2.

[52]This parallels the *Tractatus* picture of 'real signs' as a hidden depth (*PI*, §92). See sections 1.3 and 1.4. The following remark, discussed in section 1.3, connects especially closely:

> "The 'sublime conception' forces me to move away from the concrete case since what I say doesn't fit it. I now move into the ethereal region, talk of the *real* sign, of rules that must exist (even though I can't say where & how) – and find myself on 'thin ice'." (*PPO*, p.173[52])

consists in' and thereby explain the 'peculiar inexorability' of mathematical and logical inference. But every attempt to understand the expansion of the series in the light of them just leaves us in doubt as to what mathematical inference could be:

> The difficult thing here is not, to dig down to the ground; no, it is to recognize the ground that lies before us as the ground.

> For the ground keeps on giving us the illusory image of a greater depth, and when we seek to reach this, we keep on finding ourselves on the old level. (*RFM*, p.333)

The 'ground' appears to be our ordinary practice (e.g. of expanding a series). We keep looking at our ordinary practice through various pictures and seeing an illusion of greater depth. Each time we see that a posited entity would only 'stand there like a sign-post', we seek to posit a new entity behind it or restructure the picture somehow. We therefore try to develop our accounts further in order to justify our practice but the misguided attempts at justification only lead to a need for yet further accounts.

Wittgenstein is suggesting that the whole dilemma between positing a problematic abstract object or seeing mathematical inferences as unjustified is a manifestation of a misunderstanding. The misunderstanding arises from the carrying over of the inappropriate model of inference as grounded by agreement with reality. If the model simply does not apply to key cases of mathematical and logical inference (such as expansion of a basic series) then there is no question of whether the inference is justified or not. It is simply not the sort of case in which we would speak of something that we would call 'the justification.' This is the shift in perspective that is required in order to "recognise the ground that lies before us as the ground" (*RFM*, p.333). [53]

Wittgenstein's aim can thus be understood as giving a clarification whereby we see how the problem (the problem of "what inferring really consists in" (*RFM*, p.39)) emerges and how to resist the temptation towards the thinking that leads to it. Resolving the problem and closing out the sceptical concerns associated with it are thus deeply connected with one another.

3.4. Chapter Conclusion

The next chapter will further explore the theme that we can avoid certain misunderstandings about mathematics if we try to see certain kinds of mathematical inference as a kind of practice or skill, rather than seeing them through the lens of an analogy derived from empirical reasoning. Before doing so, let us first summarise the results of this chapter.

This chapter began by making use of Floyd's (1991) treatment of Wittgenstein's discussion at the beginning of *RFM* of the expression "The steps are determined by the formula..." This discussion revealed that Wittgenstein aims to release the grip of the pictures that incline us towards giving an account of how we make mathematical inferences which require a justification to be given for how the steps are arrived at. Whilst it may be

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[53]Cf (*PI*, §198) cited below.

appropriate to point to a justification for empirical inferences (for example, one can point to the smoking of the stove as justification for inferring that the chimney is out of order (*RFM*, p.40)), the requirement for a justification is not appropriate in the case of, for example, expanding a series by a rule (such as the rule 'add 2'). The model can make it tempting to postulate a meaning of the formula as an abstract object and the abstract object might then appear as a foundation for our practice. But any such account seems superfluous when applied to the case of expanding the series as the meaning-object would still stand in need of interpretation and the original problem would recur (since we could then ask what our justification was for that interpretation). Wittgenstein thus aims to show that the model of inference which imposes this requirement for a justification is the source of the sceptical problem and the meaning-objects which it seems to require are a manifestation of this problem rather than a 'foundation.'

Chapter 4 - Mathematical Propositions without foundations

4.1. Chapter Introduction

Much as (as we saw in the last chapter) attempts to see all inferences under a single model (a model to fit all empirical and logico-mathematical cases) give rise to scepticism concerning inference, likewise attempts to understand our assent to mathematical propositions under a single unified model also feeds sceptical concerns in relation to mathematics. Scepticism arises if mathematical propositions are understood as descriptions of mathematical facts, since mathematical propositions then appear to be dependent upon a strange subject-matter whose nature is unclear. Whilst it may be tempting to avoid this scepticism by treating mathematical propositions as conventions, this would neglect the importance of the use of mathematical propositions outside of mathematics, thus giving rise to the sceptical suggestion that mathematics is made up arbitrarily rather than in relationship with certain applications.

Wittgenstein suggests an alternative model of mathematical propositions as 'rules of description', designed to bring out the important aspects of how we use mathematical propositions (especially their application outside of mathematics) and to release the grip of the other pictures just mentioned. In line with the approach to philosophy discussed in section 1.4, Wittgenstein can be seen to emphasise mathematical propositions as neither entirely conventional nor entirely driven by sensitivity to application.[54] This incorporation of both aspects is important since otherwise Wittgenstein would be seen to be himself giving a dogmatic account, under which any mathematical propositions lacking in extra-mathematical applications (and therefore not being a neat fit for the account) would be the subject of the kind of sceptical doubt that Wittgenstein's method is intended to overcome. Wittgenstein should instead be seen as offering an object of comparison with which to examine (by similarity and dissimilarity of fit with examples) the ways that mathematical propositions are used and so to resolve the sceptical problems connected with dogmatic accounts.

4.2. Mathematical Propositions

A central theme that Wittgenstein pursues in his writing on mathematics is that we are inclined to view mathematics as having a subject-matter like physics does. This inclination can be understood as part of what drives the temptation, discussed in the last chapter, to see mathematical inferences as being grounded in such a way that there should be some demonstrable thing we can point to as their justification.[55] We carry over a model that better suits empirical inferences and apply it to mathematical inferences, leading us to need to posit a parallel to the physical structures of the world in some kind of 'logical structure' (*RFM*, p.40). This determination to see mathematics as like physics is a key theme of the pictures that tend to be applied dogmatically (with this fixation upon a particular kind of picture becoming especially prominent even among mathematicians during the foundations

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[54]I take this to be what is called the 'arbitrariness and non-arbitrariness of grammar' in chapter 5 of (Kuusela, 2008).

[55]Such a model is applicable in cases where there is a physical object to point to, such as the case where one expands a formula by following a traced pattern (*RFM*, p.45-46).

crisis[56]). Wittgenstein aims in various ways in his work on mathematics to undermine the attraction of this picture by highlighting the differences between the ways that empirical descriptions are used and the ways that mathematical statements are used.

Wittgenstein puts the point quite forcefully when responding in *LFM* to what he takes to be a suggestion by Godfrey Hardy[57] that "a reality corresponds to" mathematical theorems:

> We have here a thing which constantly happens. The words in our language have all sorts of uses; some very ordinary uses which come into one's mind immediately, and then again they have uses which are more and more remote... A word has one or more nuclei of uses which come into everyone's mind first.
>
> So if you forget where the expression "a reality corresponds to" is really at home-
>
> What is "reality"? We think of "reality" as something we can *point* to. It is to *this, that.*
>
> Professor Hardy is comparing mathematical propositions to propositions of physics. This is extremely misleading. (*LFM*, p. 239-40)

The expression "a reality corresponds to" is one which we associate most clearly with cases of being able to point to objects in physical reality in comparison with our empirical descriptions – cases like using an expression "there is a green sofa" and pointing to the green sofa in order to justify the assertion. Wittgenstein accepts that it is possible to talk of a reality corresponding to mathematical propositions but thinks that it misleads us about mathematical propositions to think of mathematical propositions as made true or false by mathematical facts. What he seems to be especially concerned about is that when we regard mathematical propositions as justified by a mathematical reality then it is natural to start to ask questions about this alleged 'reality' - questions such as where it is, how it is that we are able to talk about it and 'see' it and why statements about it (i.e. mathematical statements) should be so useful in science and in everyday life. The mathematical reality starts to look 'fishy'[58] (*LFM*, p. 145) and this 'fishiness' gets transposed onto mathematics itself, making it appear that we need to determine the true nature of mathematical entities in order to secure confidence in mathematics.[59] Wittgenstein wants to show that these problems arise because of our attachment to the picture of mathematical statements as descriptions and are thus problems with our picture rather than with mathematics (Kuusela 2008, p.198).

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[56]For example, (*LFM*, p. 240). Set theory is seen as a particular point of attraction for this picture as Wittgenstein comments that "analysis and set theory are always taken to be theories describing something, not calculi" (*WVC*, p.141). At the opposite extreme from (and perhaps a reaction against the ontological issues of) the physics analogy seems to be an analogy to chess (*PG*, p.30, p.40 and p.192), which has the defect of making mathematics look arbitrary.

[57] Hardy (1929, p.4) – Hardy actually speaks of theorems as 'concerning' reality.

[58]Wittgenstein also says 'shadowy' (*RFM*, p.202), though the term 'shadowy' is also used in other ways in his work.

[59]Wittgenstein saw this as a key motivation that drove, for example, Frege and Russell to undertake their foundation projects (*LFM*, p.273; *AWL*, p.150-152) with the idea being that the projects "would at long last give us the right to do arithmetic as we do" (*PG*, p.296).

A central theme of Wittgenstein's criticism of looking at mathematical statements as descriptions is that mathematical entities then look like they are part of a kind of 'shadow reality.' This shadow quality is in part because the picture fails to illuminate, and rather tends to distort, key notions like 'determination by a formula' (as we saw in the last chapter) and also because the picture undermines the variety and creativity that we see in mathematics (as Wittgenstein suggests at (*LFM*, p. 145)). Another problem arises when we attempt to take account of the fact that we tend to say that true mathematical propositions are not just true but 'necessarily' true, whereas we don't usually say this about empirical descriptions.[60] If we construe mathematical propositions as descriptions then it looks as though this 'necessity' were a feature of mathematical objects, as though mathematical objects were special necessarily-existing objects that were "not subject to wind and weather" (*RFM*, p.74). The difference between necessary and non-necessary propositions is better characterised (though not explained or justified), he proposes, by saying that necessary statements are such that we do not allow any empirical fact to count for or against them (*RFM*, p.47-50) or by saying that there is no empirical observation which would confirm or disconfirm '2+2=4' (*RFM*, p.96-97).[61]

Whilst we can explain arithmetical expressions like '2+2=4' by way of illustrations like putting two apples together with another two apples, if one day we were to do this and find that we had three apples then we would say that we had lost an apple somehow (*RFM*, p.97). We would appeal to some physical explanation first and if we found that it kept happening then we would simply say that apples were not good things to count with (as we do say with regard to objects that easily fall apart or mix into one another like blobs of jelly). The mathematical proposition ('2+2=4') would not itself come into question as a result of these observations. Whereas if I say that there is a green sofa in the next room and then we go into the next room and see no sofa or a red sofa then I would have to concede that I was mistaken. Empirical descriptions are called true or false in the light of empirical observations and we can say in advance what sort of observations would verify or falsify them, whereas all that can count for or against a mathematical proposition is a proof (*RFM*, p.98-99).[62]

When the immunity of mathematical propositions from empirical revision is presented as arising from a feature of mathematical entities, then both mathematical entities and mathematical propositions start to look fishy. To repeat the point, mathematical entities look fishy because properties are ascribed to them that we do not have any clear model of, since it is not clear what it means for an object to exist necessarily (*RFM*, p.64). Further, mathematical expressions also start to look fishy, since then it seems mysterious that we should be able to describe or refer to these entities in a 'shadowy' realm (*RFM*, p.202).

In order to release the grip of the picture of mathematical propositions as descriptions, Wittgenstein offers an alternative picture (as an object of comparison rather than a

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[60]We might say that some statements with the form of an empirical description were 'necessary' but that would not be to treat them as empirical descriptions. See (*OC* 96).

[61]He sometimes puts this by saying that mathematical statements, as grammatical statements, are non-temporal (Kuusela 2008, p.195-196) or that they are 'deposited in the archives' (*LFM*, p. 104) or that the proposition serves as a paradigm (*RFM*, p.50).

[62]Although this need not imply that unproven mathematical propositions are meaningless – see chapter 6.

replacement account[63]) which is intended to avoid the ontological difficulties and present mathematical language in a more down to earth fashion.[64] Wittgenstein's suggestion is that we consider seeing mathematical propositions on the model of statements that he calls 'preparations for description' or 'rules of description', of which he gives examples such as 'red is' (*RFM*, p.64) and 'there is no reddishgreen' (*LFM*, p.245).

Whilst 'there is no reddishgreen' might initially look like an empirical description, there is no single empirical fact that clearly corresponds to it:

> The correspondence is between this rule and such facts as that we do not normally make a black by mixing a red and a green; that if you mix red and green you get a colour which is "dirty" and dirty colours are difficult to remember. All sorts of facts, psychological and otherwise. (*LFM*, p.245)

When Wittgenstein says that the expression might be said to correspond to the fact that 'if you mix red and green you get a colour which is "dirty"', he does not mean that this is what the proposition says in the way that 'there is a sofa in the next room' says that there is a sofa in the next room. He is suggesting rather that the expression shows us something concerning how we use the terms 'red' and 'green' and other colour terms. We tend to give names to clearly identifiable colours and use those colours as points of reference. Whilst mixing samples of red and green will naturally result in something that might be called a colour, it is not a colour that is useful to us as a point of reference and so we don't give it any name. The statement 'there is no reddishgreen' might be said to be part of our system of colour terms, which we use for describing objects as having colours.[65]

Carrying this idea over to mathematics, we might say that mathematical propositions like '2+2=4' set up a system which shows how '2' and '4' are to be used and we apply this system when we describe situations in the world in terms of numbers – statements such as 'there are 2 apples over there.' The system of arithmetic being what it is allows us to make the transition (barring any apples going missing or any miscounting) from 'there are 2 apples over there' and 'there are 2 apples here' to 'there are 4 apples altogether' (*AWL*, p.154). Arithmetic is set up such that it is applicable in this way, and if we want to talk of a reality corresponding to arithmetic then we should say that this reality is to be seen in our using arithmetic in the way that we do (*LFM*, pp. 248-9).[66] This is naturally connected with many facts about us and the world, much as our using the colour terminology that we do is connected with many facts about the world, our visual systems and our brains. But the rule is not itself a summary of these facts and nor is it justified by them. So long as we are comfortable with using a rule then the rule does not stand in any need of justification. On this model we don't need to be able to postulate entities to stand behind and justify

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[63](*LFM*, p.55), quoted later, is particular forceful on this.

[64]I believe this is what Floyd means by saying that "a distinction "between 'ordinary' and 'mathematical' (or 'scientific') language is utterly alien to [Wittgenstein's] philosophy" (2005, p.233). Wittgenstein seems to put this point thus – "the words 'world', 'experience', 'language', 'proposition', 'calculus', 'mathematics' can stand only for trivial demarcations, similar to 'eat', 'rest', etc." (*BT*, 54e). Certain terms and/or areas of language can acquire a metaphysical complexion when seen through the lens of a reductive picture.

[65]Likewise 'a sofa is longer than a chair' tells us something about how we use the terms 'sofa' and 'chair' (*LFM*, p.250). See (Diamond 1991, p.233-234).

[66]See (Diamond 1991, p.233).

'2+2=4' any more than we would need to postulate entities to stand behind 'there is no reddishgreen'.

4.3. Are Mathematical Propositions Really Rules of Description?

Whilst arithmetic is a favourite source of comparisons, Wittgenstein wants the idea of mathematical statements as rules of description to be seen to be illuminating with regard to more than just arithmetic and seems to have regarded kinematics as another key case. As Diamond puts it:

> Wittgenstein's idea that in mathematics we are developing our means of description should be seen with his view that there are many different kinds of description, in which a variety of techniques are used. One technique of description, for example, is the formulation of some kind of "ideal case," which enables us to describe actual cases as departures of one or another sort from the ideal. Wittgenstein thinks of kinematics, for example, as providing such a means of description... Mathematics is integrated into the body of standards for carrying out methods of arriving at descriptive propositions, for locating miscounts (for example), or mistakes or inaccuracies of measurement. (Diamond 1991, p.234)

Wittgenstein does not pin down mathematical propositions to a particular kind of rule of description (for example, in terms of 'ideal cases' as with kinematics, or propositions that license one term to be substituted for another as seems to fit better with arithmetic) and this is likely because he wants to use the notion of a rule of description as an object of comparison – the point is that it is illuminating to see mathematical propositions as rules of description, not that they really are rules of description or that this model will perfectly fit every case (this latter being a point to be returned to later in this section). Despite having often been read as advocating a dogmatic account of mathematical propositions as rules of description (for example, Maddy (1993, p.67) and Steiner (2009, p.23)), Wittgenstein seems to disavows such an aim:

> ...the whole point is that I must not have an opinion... I have no right to want you to say that mathematical propositions are rules of grammar. I only have the right to say to you, "Investigate whether mathematical propositions are not rules of expression, paradigms – propositions dependent on experience but made independent of it. Ask whether mathematical propositions are not made paradigms or objects of comparison in this way." Paradigms and objects of comparison can only be called useful or useless. (*LFM*, p.55)

The use of 'paradigms and objects of comparison' here is confusing, since the term 'paradigm' is used both for the model of mathematical expressions as rules of description (a paradigm being a type of rule of description) and also as a synonym for 'object of comparison.' Thus the term 'paradigm' seems to be used both for the specific model being offered and for the role that the model is to play. Confusing as the expression might be, it is likely that Wittgenstein intends the model of mathematical propositions as rules of description to be used as an object of comparison since this would be consonant with his claim that he 'must not have an opinion.' He can claim not to have an opinion because an opinion would be true or false in itself, whereas objects of comparison are neither simply

true nor simply false (at least not in an empirical way) and are better understood as being 'useful or useless' insofar as they bring out features in a way which helps or does not help in the resolution of the philosophical problem.[67] (The question of the truth or falsity of mathematical propositions is to be returned to below and also in chapter 6 on questions related to the role of proof.)

In a similarly cautioning spirit, Wittgenstein also notes that there may be no sharp line between using an expression as a description and using an expression as a rule of description (*RFM*, p. 363), pointing in particular to cases where we might follow rules as a kind of experiment to see what comes out (*RFM*, p.359-360). As Mühlhölzer (2012, p.104) suggests such a case might be "if you add the first four powers of 7, you will get 400." This could well be surprising, since one might not expect such a round result. Insofar as the statement tells you what you will get when you perform the operation, then it functions as a prediction. But insofar as the statement tells us what we must get if we perform the operation correctly, then it functions as a rule. Wittgenstein acknowledges that there could be cases where it is not clear which role an expression is playing (*LFM*, p.103-104) but he takes the view that the two pictures are incompatible – if an expression is used as a rule then it is not also (at the same time) a prediction (*LFM*, p. 95; *RFM*, p.49-52; *OC*, §97-98). If we see that the result is what *have* to get, then we cannot also see the result as what we *happen* to get.

Wittgenstein clearly takes it to be useful to compare the use of mathematical expressions to that of rules of description, since this picture helps to dislodge the idea that mathematical statements function as descriptions. If mathematical propositions are seen as rules then the truth or falsity of mathematical statements is then not to be understood as being determined by a shadowy reality. But this is not to say that truth and falsity do not apply (even though it is odd to call rules true or false) – as Wittgenstein would say in *OC* of expressions that play the role of rules:

> The reason why the use of the expression "true or false" has something misleading about it is that it is like saying "it tallies with the facts or it doesn't", and the very thing that is in question is what "tallying" is here. (*OC*, §199)

Much as Wittgenstein wants to say that there is no single way of accounting for our making mathematical inferences, he likewise wants to say that there is no single way of accounting for what makes mathematical propositions true. The point he makes with regard to mathematical inference (in the discussion of 'determination by a formula' at the beginning of *RFM*) seems to apply also for propositions:

> "But isn't there a truth corresponding to logical inference? Isn't it true that this follows from that?"—The proposition: "It is true that this follows from that" means simply: this follows from that. (*RFM*, p. 38)

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[67]This reading of the passage fits with the reading of Wittgenstein's methodology advocated in sections 1.3 and 1.4. Others may read the claim not to have an opinion as the claim that the points being made are beyond question – see Kuusela (2008, p.247) on the readings of Glock (1991) and Hacker (2001). If so then it would need to be explained why Wittgenstein should have 'no right to want to say' that mathematical propositions are rules of description and why Wittgenstein seems to think that the outcome of the investigation is not presupposed.

Wittgenstein is aware that his pressing a model of mathematical propositions as rules of description might make it look like mathematical statements are called true because they are useful to us in describing the world (making predictions, measurements and so on). But he cautions himself against trying to account for the truth of mathematical propositions in this way:

> A rule *qua* rule is detached, it stands as it were alone in its glory; although what gives it importance is the facts of daily experience.

> What I have to do is something like describing the office of a king;--in doing which I must never fall into the error of explaining the kingly dignity by the king's usefulness, but I must leave neither his usefulness nor his dignity out of account. (*RFM*, p.357)

The suggestion is that whilst the usefulness of our rules may be relevant to our using them as rules, it is not a justification or explanation – to treat usefulness as the grounds of mathematical propositions would be to "fall into the error of explaining the kingly dignity by the king's usefulness."[68] Again this point is raised in relation to 'determination by a formula':

> "Then do you want to say that 'being true' means: being usable (or useful)?"--No, not that; but that it can't be said of the series of natural numbers--any more than of our language--that it is true, but: that it is usable, and, above all, *it is used.* (*RFM*, p.37-38)

There is reason to think that in these last two remarks Wittgenstein is voicing another denial that the *only* appropriate way to understand the use of mathematical propositions is as rules of description. He can be taken as indicating that there are cases of mathematical systems which do not appear to be used in this way (the object of comparison is for these cases not a good fit). This is because sometimes mathematical propositions are not intended to be used outside of mathematics at all:

> But then why doesn't it need a sanction for this? Can it extend the network *arbitrarily*? Well, I could say: a mathematician is always inventing new forms of description. Some, stimulated by practical needs, others, from aesthetic needs, -and yet others in a variety of ways. And here imagine a landscape gardener designing paths for the layout of a garden; it may well be that he draws them on a drawing-board merely as ornamental strips without the slightest thought of someone's sometime walking on them. (*RFM*, p.99[69])

Whilst arithmetic and kinematics might be best understood as 'rules for description', other mathematical systems have uses only inside mathematics and not outside (*LFM*, p.254) and the landscape gardener remark suggests that some mathematical systems may be ends in themselves. This variety of uses is part of what Wittgenstein wants to point to in calling mathematics a family (*RFM*, p.399). This is an important point but detailed discussion of this shall be left for the next chapter.[70] For now the point I wish to press is that, much as it

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[68]The 'dignity' here appears to be immunity from empirical revision.

[69]See also (*LFM*, p.16) on the interest of some mathematics being aesthetic.

[70]Detailed discussion is certainly required as Wittgenstein has been read as requiring that all mathematical propositions have empirical applications – for example by Maddy (1993) and Steiner (2009).

would be dogmatic to give a single account of what licenses all inferences, likewise it would be dogmatic to give a single account of what leads us to assent to all mathematical propositions. In order to see why, we need to consider the range of the different types of case (the kinds of considerations that might lead us to call something a true mathematical proposition).

We have seen that if we wanted to point to facts that relate to our assent to 'there is no reddishgreen' (*LFM*, p.245) then we would point to facts about the human visual system and about the world (such as the reflective properties of the surfaces of objects). Pointing to these sorts of things would be relevant to the fact that the expression 'there is no reddishgreen' has a use for us – they are empirical points which relate to what makes the rule useful. The usefulness of the rule also comes out in our being able to apply it and its connection with empirical expressions such as 'that ball is red' and 'that ball is green.' But there are other cases such as the expression of the rules of chess, where there do not appear to be any relevant facts about the world to point to (Kuusela 2008, p.208-209) and there are no empirical applications. Rather than pointing to facts about the world, we would be more inclined to say that the rules of chess simply create the game of chess. The rules may have been devised to make the game enjoyable but they do not reflect anything about the world as the game is rather an end in itself. These two cases ('there is no reddishgreen' and a rule of chess) might be considered as at opposite ends (or in the case of chess perhaps near to the end, since it is at least connected with facts about what we find entertaining[71]) of a spectrum in regard to whether our assent to (or use of) the rule depends upon facts about the world or only on conventions we have laid down.[72] When compared against a rule of chess, "2+2=4" can be seen to be connected to empirical applications and thus Wittgenstein calls it an "instrument" rather than a "convention" (*AWL*, p.156-157).

Wittgenstein does not think that chess is a good model for arithmetic or kinematics or indeed any mathematics with applications in science or everyday life, since seeing mathematics in these terms tends to rob mathematics of its importance and make it appear as a 'mere game.'[73] Nonetheless, his acknowledgement that some mathematical systems are devised with a mind to 'aesthetic needs' suggests that there is a continuum between on the one hand mathematical systems with diverse and important extra-mathematical applications such as arithmetic and on the other hand systems that have no uses outside of mathematics itself. Wittgenstein seems to have thought of set theory as having no applications outside of mathematics and it will be argued in the next chapter that Wittgenstein regards set theory as a kind of limiting case of mathematics for this reason. Nonetheless, Wittgenstein, as I will read him, did not deny that the propositions of set theory are mathematical propositions.

Since there are parts of mathematics which have applications outside of mathematics itself, it would be misleading to see all of mathematics on the model of chess. This would reduce

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[71] To consider the end of the spectrum, one needs to consider cases of conventions which have little use except that one needs a convention. One might perhaps point to conventions that one says persist for historical reasons such as features of the way that cutlery is arranged at a table. Even these are connected with some facts (e.g. historical facts), though the facts do not make the convention useful so much as bring it about that we use it.

[72] I draw this point from the discussion of the 'arbitrariness and non-arbitrariness of grammar' in chapter 5 of Kuusela (2008).

[73] See (*RFM*, p.257) – where 'mere game' is used', (*PG*, p.30, p.40 and p.192) or (*AWL*, p.151-152) on formalism.

all of mathematics to the model of chess, thereby neglecting the important extra-mathematical uses of mathematical systems like arithmetic. Likewise it would be both dogmatic and misleading to say that all mathematical propositions have applications directly to empirical description. This would be to reduce all of mathematics to the model of 'there is no reddishgreen' (which is connected directly to empirical applications), thereby neglecting or rejecting areas such as set theory (on which much more will be said in the next chapter).

Wittgenstein therefore presents a more nuanced perspective by stressing that mathematical systems come in different varieties and limiting himself to the claim that the more characteristic cases of mathematical propositions are connected with empirical applications. The rules of description picture serves as a reminder of the important connections to applications exhibited by key mathematical propositions and serves to dislodge the picture of mathematical propositions as descriptions (and therefore also the problems that go along with fixation upon that picture).

4.4. Chapter Conclusion

Points concerning worries of dogmatism arising from Wittgenstein's clarification (including this worry concerning non-applied systems) will be taken up further in the next few chapters. First I will summarise the results of this chapter.

Mathematical propositions appear as subject to sceptical doubts if seen through the lens of a model more appropriate for empirical cases. The model of mathematical propositions as descriptions leads to the postulation of mathematical entities as the subject-matter of mathematical propositions, so that mathematical propositions are thought to be true of these entities. But the nature of these entities is unclear and it becomes confusing to see in what way mathematical propositions are necessarily true rather than contingently true. The mathematical entities are then posited as existing necessarily, adding further to their unclear nature. Wittgenstein can be seen as suggesting that a way out of this sceptical problem is to see the necessity of mathematical propositions as a feature of the way mathematical propositions are used – specifically in a way which is immune from empirical revision. In order to undermine the description model, Wittgenstein emphasises a model of mathematical propositions as similar to rules of description such as 'there is no reddishgreen.' This model is revealing with regard to the connection between mathematical propositions and empirical applications, suggesting that some mathematical systems are by their nature connected with these applications (rather than being descriptions of mathematical objects, a model which leaves the connection to empirical descriptions unclear). Whilst the rules of description picture can be seen to be of value for resolving the sceptical problems connected with the description model, not all mathematical propositions are employed in empirical propositions and thus not all mathematical systems are a good fit for the model.

Part 3 – Leaving Mathematics as it is

Chapter 5 – Pure and Applied Mathematics: philosophy 'leaves mathematics as it is'

5.1. Chapter Introduction

The purpose of part 3 is to further illustrate how Wittgenstein applies his philosophical method in relation to the philosophy of mathematics, focusing in particular on how he applies his method in a way that 'leaves mathematics as it is.' In giving these illustrations, I will aim to defend the view that Wittgenstein *can be coherently read as* 'leaving mathematics as it is.' Despite widespread belief to the contrary, Wittgenstein's philosophy of mathematics really is true to his methodological promises and does not advance 'theses' or make prescriptions about what mathematics must and must not do.

As was noted in the previous chapter, Wittgenstein has been read as dismissing set theory as not mathematics because of its lack of direct empirical applications, with Maddy (1993, p.67)[74] and Steiner (2009, p.23)[75] advocating readings whereby Wittgenstein is only prepared to use the term 'mathematics' where the connection of a mathematical system to empirical applications is direct. I will argue that Wittgenstein instead regards mathematics as a family, with some parts of mathematics being applied directly to experience and others not. Whilst Wittgenstein regards directly-applied mathematics like arithmetic and kinematics as most exemplary of mathematics, nonetheless he acknowledges set theory as a limiting case of mathematics.

5.2. Maddy's View of Wittgenstein on Pure and Applied Mathematics

If Wittgenstein were to dismiss set theory as not mathematics, then this would seem to contradict his promise to 'leave mathematics as it is.' Perhaps more importantly, if Wittgenstein were to claim that no system can be mathematics unless the system in question has direct empirical applications, then this would be a case of Wittgenstein laying down a stipulation about what mathematics must be. It would indicate that Wittgenstein were committing himself to a particular picture of what mathematics is and what mathematical propositions do.

The picture that Wittgenstein has been most widely read as committing himself to is the picture of mathematical propositions as rules of description which was discussed in the last chapter. According to this picture, mathematical propositions lay down rules for the usage of expressions in empirical contexts much as 'there is no reddishgreen' lays down a rule for

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[74]Maddy (1993, p.67) says that "the parts of mathematics without application are just empty games with meaningless signs" and cites set theory as an exemplary case of this (p.69).

[75]Steiner says that "canonical applications" are "the empirical regularities upon which mathematical theorems are based" and claims that Wittgenstein thus criticizes set theory for "having no empirical applications" (2009, p.23).

the use of 'red' and 'green' (*LFM*, p.245). Analogously, a role of a proposition such as '2+2=4' is to allow us to make the transition (barring any apples going missing or any miscounting) from 'there are 2 apples over there' and 'there are 2 apples here' to 'there are 4 apples altogether' (*AWL*, p.154). It was noted in the last chapter that Wittgenstein presents this picture in contrast to a Platonist picture and that he suggests that if we want to talk of a reality corresponding to arithmetic then we should say that this reality is to be seen in our using arithmetic in the way that we do (*LFM*, pp. 248-9).

It can be tempting to read Wittgenstein as advocating an account of mathematical propositions as rules of description as a superior account to Platonist accounts and with the same definitive aspirations as a Platonist has. If Wittgenstein is read in this way then any mathematical propositions which do not fit the rules of description model become problematic, making it appear mysterious as to how they can be legitimate cases of mathematics. Such a reading of Wittgenstein is advocated by Maddy (1993, p.67) and by Steiner (2009, p.23), both of whom take Wittgenstein to have rejected set theory as not mathematics for this reason. Since set theory has no empirical applications, its propositions cannot serve as rules of description (since there are no descriptions for which we could use these rules) and so, according to this picture, the propositions of set theory cannot be mathematical propositions.

As was noted in the last chapter, Wittgenstein seems to disavow any aim to present a definitive account of mathematical propositions and says, in the following important remark, that he is presenting the rules of description model as an object of comparison:

> ...the whole point is that I must not have an opinion... I have no right to want you to say that mathematical propositions are rules of grammar. I only have the right to say to you, "Investigate whether mathematical propositions are not rules of expression, paradigms – propositions dependent on experience but made independent of it. Ask whether mathematical propositions are not made paradigms or objects of comparison in this way." Paradigms and objects of comparison can only be called useful or useless. (*LFM*, p.55)

Given the reading of Wittgenstein's use of objects of comparison summarised in chapter 1, it would hence be inconsistent of Wittgenstein to then use the picture of mathematical propositions as rules of description for the purpose of rejecting certain cases of mathematical propositions as not legitimate. This provides some reason to question whether Wittgenstein really did use the picture of mathematical propositions as rules of description in order to reject set theory. However, by itself this remark is not decisive evidence. The remark might with some strain be read as saying that Wittgenstein's conclusion is definitive but has to be arrived at by discussion and not by Wittgenstein simply presenting it.[76]

It has to be considered whether Wittgenstein really did dismiss set theory, and other parts of pure mathematics, as not true mathematics in virtue of a lack of empirical applications. If Wittgenstein does take such a view, as Maddy (1993, p.67) and Steiner (2009, p.23) argue that he does, then it would be necessary to say that Wittgenstein in this respect commits

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[76]Such a reading is strained as Wittgenstein seems to say that the outcome of the discussion is not meant to be determined in advance. See Kuusela (2008, p.247) on the readings of Glock (1991) and Hacker (2001).

himself to a definitive account of mathematical propositions as rules of description and so he fails at this point to follow his own methodology. If Wittgenstein really does want to see pure mathematics "pruned away" (Maddy 1993, p.67) then his view will look "draconian" (Maddy 1993, p.70). Wittgenstein will look like a "ham-fisted director" (*PG*, p.369) telling mathematicians what they can and cannot do – an approach to philosophy of mathematics which he repeatedly cautioned against (*LFM*, p. 13; *PI*, §124). I will present and argue against Maddy's reading (taking hers over Steiner's since she gives most direct focus to the question of how pure and applied mathematics are to be distinguished and their respective roles for Wittgenstein). In doing so I will address the question of whether set theory is dismissed by Wittgenstein as not mathematics on the grounds of not having applications but I will leave aside the question of whether set theory might be dismissed by Wittgenstein as intrinsically metaphysical, deferring until part 4 my argument that Wittgenstein does not see set theory itself as metaphysical and only criticises a particular metaphysical interpretation of set theory.

Maddy gives focus to what she sees as a stipulation that the applications of mathematical statements (whether conceived as rules or not) have to be extra-mathematical in order for the statements to count as legitimately mathematical – a claim that "the parts of mathematics without application are just empty games with meaningless signs" (1993, p.67). Her primary motivation (1993, p.67-68) for reading Wittgenstein as requiring an extra-mathematical use for mathematical statements is that she cannot see room for any other kind of use. She cites (1993, p.68) Wittgenstein's rejection of Platonism (*LFM*, p.239-40) and takes it that since there is no mathematical reality for mathematical statements to apply to the only way in which they can have use is by applying indirectly to empirical reality – mathematical propositions need to have demonstrable "effectiveness in science" (Maddy 1993, p.68). For this reason she takes Wittgenstein's remarks rejecting Platonism's conception of mathematics as the study of mathematical objects not just as a rejection of Platonism but also as part of a rejection of pure mathematics itself.

Wittgenstein says that we must distinguish the work that a mathematician does from what a mathematician is inclined to say about the objectivity of mathematics (*PI*, §254), the latter being a subject for philosophical treatment:

> What we 'are tempted to say' in such a case is, of course, not philosophy; but it is its raw material. Thus, for example, what a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical *treatment*.

Maddy takes the treatment to reveal that applied mathematics is mathematics because it "can be made clear" and shown to have a "real use", whereas pure mathematics only has a prose-constructed illusion of use and so philosophical treatment will reveal that it can be "pruned away" (Maddy 1993, p. 67). She ascribes to Wittgenstein the view that pure mathematical statements, with their associated prose-constructed illusions, are aimed to be 'about' mathematical objects (their illusion of justification is an imagined "application in a purely mathematical realm") and since these objects don't exist we must take the statements to be meaningless (Maddy 1993, p.68).

Maddy's reading of (*PI*, §254) is difficult to square with the anti-sceptical way in which Wittgenstein speaks of the prose-mathematics distinction (*WVC*, p.149). Prose is a subject for philosophical examination because what a mathematician is inclined to say about the objectivity of mathematics is typically suggestive of some form of metaphysical thesis

(commonly some form of Platonism). Prose is not part of mathematics itself and can be done away with without detriment to the mathematics:

> An explanation in word-language of the proof (of what it proves) only translates the proof into another form of expression: because of this we can drop the explanation altogether. And if we do so, the mathematical relationships become much clearer, no longer obscured by the equivocal expressions of word-language. (*PG*, p.422)

Contrary to Maddy's reading, it is surely against what Wittgenstein says of his conception of 'prose' to say that any mathematics should depend upon prose for its acceptance. If mathematics were dependent upon prose, it would make no sense for the prose to be eliminable without detriment to the mathematics. Furthermore, Wittgenstein talks about the need to separate philosophical interpretation from mathematics so as to show that the mathematics is unaffected by philosophical scepticism or justification (*WVC*, p.149). Maddy reads Wittgenstein as using the prose-mathematics distinction to subject pure mathematics to a sceptical challenge in that she takes Wittgenstein's analysis to reveal that what mathematics had thought to be mathematics is not mathematics after all. But Wittgenstein is explicit that distinguishing between mathematics and prose should help us see that "once we have calculated something it cannot drop out and disappear" (*WVC*, p.149). It is not clear that Maddy's interpretation can make sense of Wittgenstein's professed anti-sceptical aspirations.[77]

Maddy's reading is also in conflict with the interpretation given in chapter 4 of Wittgenstein's criticism of the Platonist image of mathematical entities as analogous to physical entities. Contrary to how Maddy would have it, when Wittgenstein rejects the Platonist view of the existence of mathematical entities he also rejects the view that mathematical statements *aim* to be descriptive of a mathematical reality. Maddy takes Wittgenstein to criticise the Platonist ontology but leave the Platonist model of mathematical statements (as descriptions) untouched in the case of pure mathematics (1993, p.68).[78] But it is exactly this image of the functioning of mathematical statements that is Wittgenstein's focus. As was argued in the last chapter, a key part of Wittgenstein's aim in pressing a picture of mathematical propositions as rules is to dislodge the picture of mathematical propositions as descriptions. Rather than targeting only Platonism's ontology, Wittgenstein targets the picture which gives rise to the ontology. As I read him, Wittgenstein aims to move away from seeing mathematics as *dependent* upon a special mathematical realm, to make us see that this is a philosophical misconception. Maddy takes him to be saying that pure mathematics really is so-dependent. Because she sees this dependency upon a "mathematical realm" (1993, p.68), she thinks that a critique of Platonist ontology has to result in a rejection of pure mathematics.[79]

It could be argued that this response to Maddy does not get to the heart of the matter. Maddy might say that even if we follow the conception of mathematical statements as rules in the case of applied mathematics, it is less obvious where pure mathematics fits into matters. Maddy might suggest that we have no reason to speak of a statement as functioning as a rule for forming descriptions unless we form descriptive statements using that rule. Many of Wittgenstein's examples appear to be drawn from arithmetic and geometry so it is easy to form descriptive statements which show the mathematical statements being applied in descriptions. How then do we make sense of statements in

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[77] See chapter 2 for more on the notion of 'prose.'

[78] Maddy takes Wittgenstein to read set theory as aiming at the "description of a fantastic world of transfinite numbers" (1993, p.69), whereas I take Wittgenstein to be claiming that the description model leads to a misleading interpretation of set theory. This point will be pursued in much more detail in chapter 9.

[79] On this see Conant (1997, p.215-220).

higher mathematics? Perhaps it is because Maddy cannot see how such statements can have application that she takes Wittgenstein to read such statements as meaningless attempts to form descriptive statements about some fanciful Platonistic reality. She is influenced in this view by some of Wittgenstein's remarks related to higher-order set theory (1993, p. 69), this being a favourite example of a branch of mathematics limited on applications outside of mathematics.[80]

5.3. Pure and Applied Mathematics and Set Theory as a Limiting Case

Separating set theory from metaphysical (especially Platonist) interpretations of set theory is a significant problem in itself, one which I will defer until part 4. For now it is of note that Maddy takes Wittgenstein not to draw any such distinction, as though set theory were itself metaphysical. In part 4, I will argue that Wittgenstein does draw such a distinction and that some of Wittgenstein's remarks criticise a certain metaphysical interpretation of set theory rather than set theory itself. But the remarks that I will discuss in part 4 are not the remarks that Maddy points to. Rather, Maddy points to a sequence of remarks in *RFM* in which Wittgenstein is considering the significance of set theory's lack of empirical applications. Whilst these remarks have widely been read as highly revisionary, Wittgenstein says that set theory is "evidently mathematics" (*RFM*, p. 264). His question is not whether set theory is mathematical since its applications are not all clear, his question is how best to understand that it is that it is mathematical *even though* its applications are not all clear:

> And why is it evidently mathematics?--Because it is a game with signs according to rules?
> But isn't it evident that there are concepts formed here--even if we are not clear about their application?
>
> But how is it possible to have a concept and not be clear about its application? (*RFM*, p. 265)

We may not be clear about the application, or better "intended application" (*RFM*, p. 259), of certain concepts in set theory but this does not disqualify set theory from being mathematics. When Wittgenstein asks whether it makes sense to speak of concepts for which we are unclear about their application, he is asking whether set theory really fits the model of rules of description. If the propositions of set theory are not employed as rules of description, then perhaps what we have in set theory is not naturally described as 'concepts', if 'concepts' means notions to be employed in describing the world. Wittgenstein is holding up two models for set theory – the model of an arbitrary sign-game and the model of a system of rules of description. He clearly takes the model of sign-game to be a better fit for set theory but he does not dismiss the alternative model altogether. If set theory were only an arbitrary sign-game with no relation to empirical application or other mathematical systems, then we might not want to call set theory mathematics (it would then be more analogous to chess). But Wittgenstein is not saying this – rather, he is saying that it seems to be a fringe case of a system of rules of

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[80]I don't want to address the question of whether set theory really does have extra-mathematical applications. It seems reasonable to suppose that at least parts of set theory lack extra-mathematical applications and if this is granted then we can instead discuss those parts of set theory.

description. We can see this by considering another remark to which Maddy (1993, p.67) gives prominence (as does Rodych (1997, p.217)):

> Imagine that a calculating machine had come into existence by accident; now someone accidentally presses its knobs (or an animal walks over it) and it calculates the product 25×20.

> I want to say: it is essential to mathematics that its signs are also employed *in mufti.*
> It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics. (*RFM*, p. 257)

On a first reading this statement (particularly the 'it is essential to mathematics') appears to have an air of finality to it, appearing to proclaim that all mathematical symbols must be employed in descriptive statements about the world if the symbols are to count as mathematical. But we can begin to put pressure on this idea by noting the 'I want to say.' Morris (1994, p.295) has argued that Wittgenstein often uses expressions like this when he is drawing our attention to one possible point of view. Perhaps in this case, as with a number of others, this particular picture is possible because it is one of the senses of a family-resemblance term.[81] Wittgenstein can be seen as noting the attraction of the particular picture without thereby committing himself to it. Wittgenstein is perhaps suggesting that we can take it is a characteristic of mathematics that its signs are employed *in mufti,* not because this holds of all mathematics but because this holds of certain key exemplars of the family-resemblance term 'mathematics.'[82]

Of all the exemplars of the term 'mathematics', arithmetic is what comes most immediately to mind. And we can argue that it is arithmetic which Wittgenstein has in mind by noticing that the context of the cited remark relates only to arithmetic. The calculating machine (the one which has 'come into existence by accident') is not said to be a general theorem-proving machine, just a machine capable of producing the symbols corresponding to arithmetical operations. That this is the scope of Wittgenstein's remark becomes even more likely when we see that Wittgenstein is using the point to make a criticism of Russell and Whitehead's attempted reduction of arithmetic to logic in *Principia Mathematica*:

> But is it not true that someone with no idea of the meaning of Russell's symbols could *work over* Russell's proofs? And so could in an important sense test whether they were right or wrong?

> A human calculating machine might be trained so that when the rules of inference were shewn it and perhaps exemplified, it read through the proofs of a mathematical system (say that of Russell), and nodded its head after every correctly drawn conclusion, but shook its head at a mistake and stopped calculating. One could imagine this creature as otherwise perfectly imbecile.

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[81]Strictly Morris notes that the uses of these expressions are typically "modal" (1994, p.295) and relate express possible pictures. It seems a natural step, given the methodology described in chapter 1, for a possible picture to appeal because it captures one strand of a family-resemblance expression.

[82]Wittgenstein makes this same point at (*RFM*, p.400), quoted later in this section. I do not mean to claim that Wittgenstein *always* uses 'I want to say' to preface one sense of a family-resemblance concept but he does appear to be doing so in this case. He uses the same locution in this way at (*PI*, §494) but the usage appears subtly different, though still related, at (*PI*, §100) and (*PI*, §141).

> We call a proof something that can be worked over, but can also be copied. (*RFM*, p. 258)

Whilst the point in this section is not to discuss Wittgenstein's views on the *Principia* project of reducing arithmetic to logic at any length (that discussion is to be taken up in part 4), it is worth giving some explanation in order to follow through on the point that it is primarily arithmetic in play in the previously-cited remark. The attempted reduction of arithmetic to logic in *Principia* relied on being able to construct proofs which would relate statements of logic to corresponding statements of arithmetic. Whilst these imagined proofs would be impossibly long, Russell imagines that the process of constructing them would be a mechanical one and so it does not matter that we cannot actually carry it out. Wittgenstein's objection to Russell, to be explored at length in part 4, is that we need to be able to understand and apply these proofs if they are to count as providing a reduction of arithmetic to logic, which means that they would need to be 'surveyable' in Wittgenstein's sense. Wittgenstein's point with the calculating machine is that proof is not a matter of just constructing certain symbols. The symbols have to be used in a way which gives them significance, which is why surveyability is necessary in proofs. The calculating machine is an extension of the mechanistic view of proof that Wittgenstein finds in Russell's thought.

The claim that it "is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics" is a reasonable point with regard to arithmetic (as well as geometry, kinematics and other applied parts of mathematics). One might fairly say that our being able to use arithmetic for the purpose of counting is part of what makes arithmetic what it is and if Russell's logical equivalents of arithmetical statements were not usable for this purpose then it would be reasonable to question whether he had successfully captured the essence of arithmetic. Wittgenstein's objection is that Russell's logical equivalents of arithmetical statements would not be usable as arithmetical statements (since they would be far too long) and hence they cannot be taken to capture the meaning (use) of arithmetical statements. Wittgenstein's remark that it "is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics" can thus be seen, in the context in which he makes the remark, to be limited to arithmetic.

Wittgenstein furnishes us with an argument that it is application which makes systems like arithmetic into mathematics. In a discussion with the Vienna Circle (*WVC*, p.170), Wittgenstein suggests that we might imagine wars being fought using chess. The suggestion is that if this were to happen then chess would no longer be just a game. Mathematics is not just a sign game (as, for example, Sudoku is) because we do not use it as just a game. It does not matter what the intentions were behind chess as regards whether it is a game or not, it matters how we actually use it in our lives. To put the idea another way (not Wittgenstein's own way), we might say that a religious text is marked out as a religious text by the way that it is consulted and used rather than by its content. Wittgenstein's suggestion is that mathematical systems are not just games because we do not use them as just games.[83]

My interpretation of Wittgenstein's 'in mufti' remark (*RFM*, p. 257) might seem to have the downside that Wittgenstein said 'mathematics' when he should have said 'arithmetic' or

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[83]Shanker (1987, p.80) seems to agree on this point. Severin Schroeder has suggested to me that we could instead read Wittgenstein as saying that a system must have an 'intended application' in order to be mathematical. I think the *WVC* chess argument counts against Schroeder's suggestion, since the point seems to concern actual usage and not intended usage. Further, Wittgenstein sometimes uses 'intended application' to refer to something like a prose expression connected with some mathematics – for example, (*RFM*, p. 262) or (*RFM*, p.266). It strikes me as out of accord with Wittgenstein's view for any prose expression to be essential to a piece of mathematics, so these cases of Wittgenstein's using the term 'intended application' would have to be explained as somehow special cases.

'certain core parts of mathematics such as arithmetic.' But it is important to stress the point drawn from Morris (1994) that Wittgenstein tends to use expressions like 'I want to say' when he's pointing to one way of using a multi-faceted term (such as a family-resemblance term). There is nothing unclear or confused about Wittgenstein's expression if we see it as in line with a typical way of using expressions like 'I want to say.' In another remark Wittgenstein explicitly considers what he should say about mathematics which lacks applications and he is there explicit that it is important to remember that mathematics is "a family" (*RFM*, p.399). He wishes to stress that the connection to application is the "essential thing about a great part of mathematics (of what is called 'mathematics') and yet say that it plays no part in other regions" (*RFM*, p.399). I wish to return to this remark towards the end of this section. For now I will only note that if there were something awkward about reading Wittgenstein as expressing his point in the 'in mufti' (*RFM*, p. 257) remark in this way (and I don't think there is), then such awkwardness would surely be preferable to reading the remark in a way which attributes to Wittgenstein a rejection of pure mathematics – such a reading would be uncharitable in itself and also deeply out of step with Wittgenstein's methodological promises to do philosophy of mathematics in a non-revisionary way (*PI*, §124).

There is a second statement in *RFM* which can appear problematic for my reading of the 'in mufti' remark (*RFM*, p. 257) as saying that it is only characteristic of certain core members of the family 'mathematics' for which it is essential to their nature that they are applied. The potentially problematic remark reads:

> Now how about this--ought I to say that the same sense can only have *one* proof? Or that when a proof is found the sense alters?
>
> Of course some people would oppose this and say: "Then the proof of a proposition cannot ever be found, for, if it has been found, it is no longer the proof of *this* proposition." But to say this is so far to say nothing at all.-- It all depends *what* settles the sense of a proposition, what we choose to say settles its sense. The use of the signs must settle it; but what do we count as the use?--
>
> That these proofs prove the same proposition means, e.g.: both demonstrate it as a suitable instrument for the same purpose.
>
> And the purpose is an allusion to something outside mathematics. (*RFM*, p. 366-7)

It is the last sentence of the remark which is of importance but we have to understand the rest of the remark first. Here Wittgenstein's notion that a proof demonstrates how to use a statement (*RFM*, p. 305-307) is in play. He notes that this conception can lead us to ask how it is that we can have multiple proofs of the same statement, since each proof must bring out some different use for the statement. His answer (elaborated more after the quoted passage) is that it all depends upon the other connections that the statement has, the various roles that are ascribed to it. One proof may be geometrical and another algebraic but this need not prevent us from taking them as proofs of the same statement, depending upon the setting (i.e. depending upon the rest of the mathematical system in which the statement/s play a part[84]).

So why does Wittgenstein say that the purpose revealed by a proof is an allusion to something outside mathematics? I think that Wittgenstein is saying that mathematical statements play roles in systems and it is characteristic of proofs to establish the roles of the statements that they prove. When a mathematical system is looked at in a more general way then the system itself will have some kind of purpose/s behind it (the system itself will have roles to play) since mathematicians don't invent mathematical systems arbitrarily. It

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[84]For more on proof assigning sense to expressions, see Säätelä (2011).

may be tempting to take Wittgenstein as saying that mathematical systems have to have direct applications to empirical descriptions or else they are arbitrary but the following remark (also cited in a previous chapter) suggests that, if the point is meant to cover all cases of proofs, then Wittgenstein must have had a very wide conception of 'outside mathematics':

> But then why doesn't it need a sanction for this? Can it extend the network *arbitrarily*? Well, I could say: a mathematician is always inventing new forms of description. Some, stimulated by practical needs, others, from aesthetic needs, -and yet others in a variety of ways. And here imagine a landscape gardener designing paths for the layout of a garden; it may well be that he draws them on a drawing-board merely as ornamental strips without the slightest thought of someone's sometime walking on them. (*RFM*, p.99)

Aesthetic purposes are not the sorts of purposes that come to mind when Wittgenstein says 'outside mathematics' and yet the landscape gardener remark reveals that aesthetic purposes are part of his picture. Perhaps Wittgenstein had in mind 'aesthetic' ends like the parsimony that can be achieved by setting up concepts to link previously-unrelated mathematical systems.[85] It may be that Wittgenstein thought of this kind of aesthetic end as something distinct from the mathematics itself. The most natural alternative reading of the remark would be to say that the claim that "the purpose is an allusion to something outside mathematics" (*RFM*, p. 367) is not meant as a picture which should fit *all* cases of proofs and is limited only to applied systems such as arithmetic and geometry.

That Wittgenstein was not dismissive of pure mathematics is also suggested by his ways of talking about 'pure mathematics' in *RFM*. At one point he suggests that pure mathematics can be characterised as the practice of deriving mathematical rules from other mathematical rules:

> Any proof in applied mathematics may be conceived as a proof in pure mathematics which proves that *this* proposition follows from *these* propositions, or can be got from them by means of such and such operations; etc. (*RFM*, p. 436)

Rather than regarding pure mathematics as incidental to the core of mathematics, this conception seems to be an immensely wide one, suggesting that most branches of mathematics are treated by Wittgenstein as pure mathematics. The idea that statements of applied mathematics can be transformed into statements of pure mathematics is not an obvious one and requires some examination to understand. It immediately suggests that Wittgenstein means something very different by 'applied mathematics' than what we might naturally assume him to mean.

Light is shed on Wittgenstein's distinction between pure and applied mathematics when he considers whether we could imagine a people who have only an applied mathematics and not pure mathematics (*RFM*, p. 232). He imagines this people using a co-ordinate system to express physical predictions but he takes it that the theorems we associate with co-ordinate geometry would not be part of their mathematics, implying that these theorems are not part of applied mathematics. Wittgenstein asks whether the people would arrive at the commutative law of multiplication and suggests that they would not formulate it as a

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[85]When I say 'link previously-unrelated mathematical systems', I do not mean this to suggest that one system might be 'about' another. Wittgenstein regards mathematical systems as content-less and autonomous so one system cannot be 'about' another - see Mühlhölzer (2012). But this does not mean that there cannot be links between systems in the sense that one system is set up in a way analogous to another. For example, we can set up an analogue of Peano Arithmetic using Zermelo-Frankel set theory. In this sense the systems might be said to be linked.

rule since it is "not a rule of their notation" nor "a proposition of their physics" and would hence have to be pure mathematics.[86]

What Wittgenstein seems to be driving at is the question of whether it is characteristic of mathematical practices that we see the point in using the symbols we do in the way that we do, as opposed to following through the steps of a practice blindly (albeit a practice that yields desirable results, like always betting on the winning horse without knowing why it wins). Wittgenstein seems to be considering whether our being able to see the point of certain propositions and derive others from them is something crucial to what we call mathematics. Regarding the commutative law, he says that the imagined people "do not need to obtain any such proposition--even if they allow the shift of factors" (*RFM*, p.232). He is trying to imagine that their "mathematics were done entirely in the form of orders" (*RFM*, p.232). These orders are used to make predictions but he says that it "does not matter at all how these people have arrived at this method of prediction" (*RFM*, p.232). The idea of expressions in the form of orders comes up again later and an illustration is then given – Wittgenstein wonders whether we could do mathematics just by formulating instructions like "let 10 x 10 be 100" (*RFM*, p.276). Wittgenstein says that the "centre of gravity of their mathematics lies for these people entirely in doing" (*RFM*, p.232). The 'doing' here is presumably to be contrasted with activities like formulating, deriving, abstracting and seeing – as Wittgenstein says, "these people are not supposed to arrive at the conception of making mathematical discoveries" (*RFM*, p.233).

We can now understand how it is that a "proof in applied mathematics may be conceived as a proof in pure mathematics which proves that *this* proposition follows from *these* propositions"(*RFM*, p. 436). The transformation from applied to pure mathematics is being understood as a transformation from application of instructions formulated along the lines of "do this" and "do that", to rules formulated so as to allow derivations i.e. propositions. The people who lack pure mathematics do mathematics by moving from empirical statement to empirical statement without ever formulating the rules by which they make the transitions as propositions (formulating them instead only as instructions). Wittgenstein suggests that some techniques employed by physicists and engineers are like those employed by the tribe (in that the rule is never formulated as a proposition), citing specifically the determination of resultant force on an object by means of drawing a polygon to represent the individual forces:

> Take the construction of the polygon of forces: isn't that a bit of applied mathematics?
>
> And where is the proposition of *pure* mathematics which is invoked in connexion with this graphical calculation? Is this case not like that of the tribe which has a technique of calculating in order to make certain predictions, but no propositions of pure mathematics? (*RFM*, p.265)

The tribe might lack co-ordinate geometry and the commutative law but presumably they might have the technique of drawing force polygons.

Given that the theorems of co-ordinate geometry and the commutative law are considered to be pure mathematics by Wittgenstein, we have to wonder whether the sorts of examples that Wittgenstein uses in his illustrations of his idea of mathematical statements

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[86]Wittgenstein's point in considering this is to say that the people might well allow that factors can be switched without ever formulating the law and hence could get by without formulating this particular rule of pure mathematics.

functioning as rules really are, as Maddy seems to assume[87], seen by him as statements of applied mathematics. Perhaps surprisingly, it appears not:

> …the question "are there a hundred times as many marbles here as there?" is surely not a mathematical question. And the answer to it is not a mathematical proposition. A mathematical question would be: "are 170 marbles a hundred times as many as 3 marbles?" (And this is a question of pure, not of applied mathematics.)

> Now ought I to say that whoever teaches us to count etc. gives us new concepts; and *also* whoever uses such concepts to teach us pure mathematics? (*RFM*, p. 412)

Clearly Wittgenstein regards at least some of pure mathematics as quite easy to relate to applications in empirical propositions.

It is interesting that Wittgenstein does not seem to come to a firm view about whether we can imagine a people who lack a pure mathematics. He seems to be inclined towards the idea that it is essential to what we call mathematics that we formulate at least some of it as propositions. He says that it "is clear that mathematics as a technique for transforming signs for the purpose of prediction has nothing to do with grammar" (*RFM*, p.234). If we are simply working with instructions and predictions then there is no element of necessity:

> What is the transition that I make from "It will be like this" to "it *must* be like this"? I form a different concept. One involving something that was not there before. When I say: "If these derivations are the same, then it *must* be that...", I am making something into a criterion of identity. (*RFM*, p.237)

It seems that if we lacked a pure mathematics, at least as Wittgenstein uses the term in *RFM*, then what we would have would only be mathematics in a lesser sense, if it were mathematics at all. For us the technique of drawing force polygons is connected with other techniques which do involve formulating propositions and making derivations from them. For the tribe there can be no such connections as they do not practice pure mathematics. Without these connections it is not clear that their technique of using force polygons could be called a mathematical technique.

The interpretation of Wittgenstein's view of pure mathematics being offered here is a long way from Maddy's suggestion (1993, p. 68) that Wittgenstein regards pure mathematics as devoted to 'fanciful applications.' Maddy might well object that how we use the term 'pure mathematics' is not important. What is important, she might suggest, is that Wittgenstein regards some of mathematics to be devoted to 'fanciful applications' and that Wittgenstein is scornful of this. But this is a misidentification of Wittgenstein's target – Wittgenstein may be mindful to point out cases where we attribute 'fanciful applications' to a piece of mathematics and remind us that they are no more than fanciful (being as they are cases where our prose is a poor translation of the mathematics) but such misguided prose need not affect the validity of the mathematics. Even if the mathematical system in question lacks extra-mathematical application, it is enough that the mathematical system have connections[88] to other mathematical systems and that those systems have applications:

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[87] The question quoted below seems like it could easily figure in Wittgenstein's illustrations of his own conception of mathematical statements as rules. We surely can't take Wittgenstein to be rejecting as not bona fide mathematics the same sorts of statements with which he illustrates his own view.

[88] Again, these connections need not be referential. I think it would be enough that the connections be connections of analogy and parallel. As I suggested before, set theory is not a mere sign-game like Sudoku because (among other reasons) we can construct an analogue

> I have asked myself: if mathematics has a purely fanciful application, isn't it still mathematics?--But the question arises: don't we call it 'mathematics' only because e.g. there are transitions, bridges from the fanciful to non-fanciful applications? That is to say: should we say that people possessed a mathematics if they used calculating, operating with signs, *merely* for occult purposes?
>
> But in that case isn't it incorrect to say: the *essential* thing about mathematics is that it forms concepts?--For mathematics is after all an anthropological phenomenon. Thus we can recognize it as the essential thing about a great part of mathematics (of what is called 'mathematics') and yet say that it plays no part in other regions. This insight by itself will of course have some influence on people once they learn to see mathematics in this way. Mathematics is, then, a family; but that is not to say that we shall not mind what is incorporated into it. (*RFM*, p.399)

We should be careful not to misread this last remark (that mathematicians of the future might be more careful about what new mathematics is developed) as a prediction that mathematicians of the future will not count set theory (Wittgenstein refers to the axiom of choice shortly after this passage) as mathematics. Wittgenstein himself counts set theory as mathematics so there is no reason for him to make such a prediction. But Wittgenstein does clearly hope that mathematicians of the future will see mathematics with direct extra-mathematical applications to be core to mathematics and he hopes that this will lead them to direct mathematical enquiry differently.

Why might mathematicians want to direct mathematical enquiry differently if they saw systems with extra-mathematical applications to be core to mathematics? In part because mathematicians are likely to want to do work which is most central in the sense of being most characteristic of mathematics. But in part this is most likely also because if mathematicians did see it as characteristic of mathematics to be linked to extra-mathematical applications then this would put pressure on the dogmatic employment of Platonist and anti-Platonist pictures which were discussed in chapter 4 and which Wittgenstein takes to be at the heart of the foundations crisis. Mathematicians would become less interested in building foundational systems if they did not feel sceptical concerns about whether mathematics was well-founded. And if mathematicians were to consider mathematics as a family with applied mathematics at the core then they would have available a picture which can lead them to reminders of the variety of mathematics and so enable them to resist the scepticism that goes along with the dogmatic use of the pictures. Wittgenstein is suggesting, in line with the view of mathematical propositions discussed in chapter 4, that the motivation to unify all of mathematics under foundational systems would be reduced if it could be seen that the quest for unity that goes along with such projects is itself a source of the scepticism that motivates the projects (by making certain parts of mathematics look problematic because they do not fit a particular picture). Understanding mathematics as a *family* would mean accepting the variety of mathematics, which would take away much of the motivation for attempting to see mathematics through the reductive lens of a particular foundational picture.

Whilst Wittgenstein might not think that set theory has any direct applications to empirical description, his remark concerning 'transitions' and 'bridges' suggests that set theory is still more than a mere sign game. The suggestion we can read from this is that

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of Peano Arithmetic in set theory and argument-forms used in set theory are found in other parts of mathematics.

set theory, and likewise other parts of mathematics which lack empirical applications, is not completely unrelated to the parts of mathematics which do have applications. Set theory is still part of the family of mathematics because it has similarities and analogies that connect it with other areas of mathematics. At a superficial level, one might point to the fact that the reasoning employed in set theory is axiomatic and proceeds by means of proof. But more important is that models (one might say 'simulations'[89]) of arithmetic and other core mathematical systems can be constructed within set theory. These kinds of connections make it clear that set theory is at least analogous to other core parts of mathematics, even if it lacks the extra-mathematical applicability that those parts of mathematics possess.[90]

Rather than dogmatically dismissing pure mathematics and set theory as not *bona fide* mathematics, Wittgenstein can be seen to be using set theory as an illuminating limiting case of mathematics. Wittgenstein's interest in set theory is in part a manifestation of Wittgenstein's probing of the boundaries of what we call 'mathematics.' Far from neglecting his own philosophical promises, Wittgenstein can be seen to be employing his philosophical methodology by presenting philosophical pictures of mathematical propositions and using them as objects of comparison to be held up against particular examples. The propositions of set theory offer illuminating examples in part because they are only an awkward fit for the model of mathematical propositions as rules of description and in part because they are fringe cases of mathematical propositions. The overview which Wittgenstein elaborates reveals systems such as arithmetic and geometry to be core cases of mathematics because they have important applications. Chess would not be a mere game if it were used to fight battles (*WVC*, p.170) and so mathematical systems would be mere games if they were not connected somehow to applications. Whilst set theory has no direct applications, it can be seen to be mathematical nonetheless insofar as it is connected to core systems (*RFM*, p.399). It may be that the relationships needed to connect set theory to arithmetic and geometry need not themselves be mathematical – they might only be relationships of analogies in the kinds of symbols used and ways that they are employed (much as there are only loose connections between the various things which we call 'games'). However, one could easily make the point that simulations of arithmetic and geometry can be constructed using set theory so the "transitions" and "bridges" (*RFM*, p.399) that Wittgenstein mentions need not be understood to be particularly loose.

Clearly Wittgenstein takes set theory as a fringe case of mathematics because it lacks empirical applications and so is only an awkward fit for the model of mathematical propositions as rules of description. In this way Wittgenstein is giving the model of mathematical propositions as rules of description preferential treatment over the others. This preferential treatment is not ungrounded – as we have seen, Wittgenstein argues that mathematics would be a game if it were not applied, much as chess would no longer be a game if we used it to fight battles. Moreover, it is perfectly consistent with Wittgenstein's methodology for him to give preferential treatment to a particular picture – Wittgenstein needs to emphasise the picture/s that he takes to be most illuminating, the picture/s which can do most work in resolving the problem/s at hand. In this case the problem is the web

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[89]See Mühlhölzer (2005, p.78). The term 'simulation' avoids the suggestion that the one system is 'about' the other.

[90]One might wonder whether the set-theoretic simulation of arithmetic could be applied as arithmetic is. But the set-theoretic simulations would be too cumbersome for that purpose – see Mühlhölzer (2005, p.73). We shall see in part 4 (chapter 8) that Frege and Russell's simulations of arithmetic would be of limited applicability for the same reason.

of confusions that can make mathematics appear problematic. As we saw in chapter 4, the picture of mathematical propositions as rules of description can do a great deal of work in this regard by presenting a way of avoiding the problematic positing of a metaphysical Platonic reality and the related problem of reducing mathematics to a mere game. The picture avoids both such problems by allowing us to speak of mathematics as grounded in a reality of empirical applications.

Wittgenstein emphasises the model of mathematical propositions as rules of description not because he has a personal preference for it or because he takes it as a definitive account. Rather, he thinks that it can help us out of philosophical problems concerning mathematics and the picture he presents of set theory as a fringe member of the family of mathematics is part of how he hopes to allow us to work past these problems.

5.4. Chapter Conclusion

In relation to the contrast between pure and applied mathematics, Wittgenstein's discussions reinforce the approach (discussed in part 2) that he takes towards mathematical inference and mathematical propositions. Rather than pressing an account of mathematical propositions as essentially rules of description, Wittgenstein argues that this picture is illuminating with regard to how mathematical propositions are used (using it to dislodge dogmatic pictures such as the picture of mathematical propositions as descriptions or the picture of mathematical propositions as arbitrary stipulations). Rather than rejecting as not mathematical any instances of systems which do not naturally fit the rules of description model (especially systems which lack empirical applications), Wittgenstein insists that the model reveals what is most characteristic of mathematics and so he only suggests that systems without extra-mathematical applications should be regarded as peripheral cases of mathematics.

Chapter 6 – Proof, Constructive Proof and Leaving Mathematics as it is

6.1. Chapter Introduction

This chapter will address the difficult and complex topic of Wittgenstein's remarks upon proof, with a view to bringing out how Wittgenstein's exploration of the connection between a mathematical proposition and its proof is connected with his view that it distorts the grammar of mathematical propositions to see them as descriptive propositions.

Whilst Wittgenstein thinks that it is illuminating to regard mathematical propositions as rules whose use is fixed by proofs, he does not apply this model dogmatically. If he were to apply the model dogmatically, then we would expect him to deny that mathematical conjectures were meaningful, since a conjecture has no proof and so its use would not yet be fixed. Likewise, we might expect him to deny that mathematical propositions could be true, since rules are not true or false. Whilst such a view is attributed to Wittgenstein by Marion (1998, p.159-165), Wittgenstein in fact uses the picture of mathematical propositions as rules in order to explore the grammar of conjectures and mathematical truth. He suggests that mathematical conjectures are not meaningful in advance of proof in the way that empirical descriptions are typically meaningful in advance of verification but nonetheless we often have things we can say with regard to how a conjecture might be approached and to that extent we are not entirely in the dark with regard to conjectures. Likewise, Wittgenstein acknowledges that mathematical propositions are true and false, whilst stressing that mathematical propositions don't 'turn out to be' true or false in the way that empirical propositions do.

Wittgenstein's use of a picture of mathematical propositions as rules of description can easily be misread as part of a definitive account of mathematics as the construction of rules. Especially if Wittgenstein is taken to deny that mathematical conjectures or falsehoods are meaningful, as then deduction principles that invoke the law of excluded middle (the law "p or not p") start to look suspect (since it would not be clear on such a view that a conjecture could be called true or false). Marion (1998, p.175) interprets Wittgenstein as aligning himself to Brouwer and Weyl in rejecting these principles and as criticising only their "sales pitch" (1998, p.167). But Wittgenstein is critical of Brouwer and Weyl in a way that suggests that he takes their rejection to be dogmatic and is based upon extrapolating from particular cases to a rule of what can and cannot be a mathematical proof (*PG*, p.374). Rather than rejecting these proof-techniques, Wittgenstein can be seen as insisting that the way in which a mathematical proposition is proven makes a difference to its meaning (*PG*, p.373). As I will argue, Wittgenstein can thus be seen as distancing himself from Brouwer and Weyl's dogmatism and instead using their views as objects of comparison with which to consider the differences between kinds of proof.

6.2. The Role of Proof

It was emphasised in the last chapter that Wittgenstein takes it to be especially illuminating to regard mathematical propositions as rules of description rather than as

arbitrary stipulations or as descriptive propositions. As we have seen, regarding mathematical propositions as arbitrary stipulations neglects the connection of mathematics with empirical applications and makes mathematics appear as a mere game. Regarding mathematical propositions as descriptive propositions, however, leads to the positing of mathematical entities as the subject-matter for mathematical propositions to describe. Features of mathematical propositions are then projected onto the mathematical entities, with the necessity of mathematical propositions leading to a view of mathematical entities as existing necessarily. As we saw in chapters 4 and 5, it is better to understand this necessity as a feature of how the propositions are used, in that mathematical propositions are immune from empirical revision. In this section the focus will be on the significance of the parallel point that mathematical propositions are immune from empirical *verification*. This section will follow Wittgenstein's examination of the ways in which mathematical propositions are verified with a view to showing how this examination "leaves mathematics as it is" (*PI*, §124). Rather than attempting to cover Wittgenstein's discussions in full detail[91], the focus will be on what overview Wittgenstein gives and what dogmatic accounts it is meant to undermine.

It will be argued that Wittgenstein contrasts the verification of a mathematical proposition by a proof to verification of a descriptive proposition by experiment. It is *characteristic* of mathematical propositions that a proof defines or partially-defines[92] the mathematical proposition whereas the descriptive proposition is meaningful even before the experiment takes place. I make the qualification that it is *characteristic* of mathematical propositions to be determined by their proofs because there are boundary cases. The key boundary cases are mathematical conjectures, which are not entirely meaningless because they lack a proof (Floyd 1995, p.385) and Wittgenstein also makes suggestions that there are empirical propositions (or apparently empirical propositions) for which the meaning is not necessarily clear in advance of verification (*PI* §462-463; *LFM*, p.185-186). The argument for this qualification will be delayed until the next section on conjectures. If the qualification is granted for the time being then it makes it all the more pressing to answer the question 'what did Wittgenstein want to say about proof?' Given that he is not presenting a dogmatic account and his contrast rather plays a role of an overview, what attractions towards dogmatism does the overview protect us against?

In order to answer this question, it is best to consider Wittgenstein's discussion of proof in relation to the models of mathematical propositions discussed in chapter 4. That mathematical propositions are in some sense necessary (they exhibit a 'peculiar inexorability') is something that Wittgenstein wants to recognise as a characteristic feature of mathematics (*RFM*, p.170; *RFM*, p.37; *RFM*, p.84) which Platonist and conventionalist models both present in a misleading way. The Platonist model, as discussed in chapter 4, misleadingly makes it seem as though this necessity arises from mathematical entities rather than from the way we use the expressions. Aside from the sceptical problems connected with postulating mathematical entities, one might object to this model that it is "not something behind the proof, but the proof, that proves" (*RFM*, p.173). The model of

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[91]More detailed discussion can be found in Mühlhölzer (2005) and Floyd (2010).

[92]The proof may only partially-define because there may be more than one proof of the same proposition. It may be that two proofs assign different meanings to the proposition or it may be that the use assigned is the same (*RFM*, p.189; *RFM*, p.308). Sometimes we may need another proof to show that what two proofs prove is the same proposition (*RFM*, p.191-192; *RFM*, p.368).

mathematical propositions as descriptions can be taken to lead to or presuppose a model of mathematical proofs as experiments. In showing how this model misleads with regard to proof, Wittgenstein is further undermining the Platonist model.

This is not to say that the model of mathematical propositions as arbitrary stipulations, also discussed in chapter 4, fares any better. If taken in a simplistic way, the model makes it seem as though we could easily stipulate mathematical systems however we wish. As argued in chapter 4, this is misleading in part because it misses the importance of the purposes that mathematical systems serve. It is also misleading with regard to our experience of doing mathematics. For much of what we do in mathematics, we follow set rules which we do not think of ourselves as having control over. We may sometimes simply step through certain rules to see what result comes out for our calculation and we may then be surprised by the result – this can feel very much like an experiment. For this reason Wittgenstein acknowledges that doing mathematics (performing calculations) can sometimes feel like performing an experiment. This is part of the motivation for Wittgenstein's idea of proof as a picture – we may initially arrive at proofs by a semi-experimental process but we come to see the procedure of the proof as something necessary. With this idea Wittgenstein illustrates that it is possible to do justice to the conceptual nature and to the necessity of mathematical proof without having to posit a special mathematical subject-matter or treat all mathematical propositions as arbitrary stipulations. Let us first consider the contrast between mathematical proof and empirical verification.

When Wittgenstein says that a calculation is not an experiment (*RFM*, p.51, p.170, p.202-203), one way to put his point is to say that when we perform a calculation, say 13x13, it doesn't make sense to say that there's a possibility that we may get any result other than 169. The expression "13x13=169" is internally related to the meaning of the symbols in the expression – it partially defines what we mean by "13" and "169". For an experiment, the procedure does not determine what result we will get. There is not an internal relation between the conditions that define the experiment and the experiment's result. When I let go of a ball, it is not of the nature of the ball that it will bounce (some balls do not bounce). If we know the conditions of an experiment and the procedure carried out then we might be able to predict the outcome but the outcome does not follow of necessity.

Philosophers typically say that a statement such as "7 is prime" is both true and necessarily true.[93] For Wittgenstein, this necessity shows itself not in the object/s involved but in our not ascribing any mathematical sense to the idea that this could be otherwise. The proof shows 7 to be prime and gives no alternative scenarios (indeed, the idea of 'scenarios' is hard to make sense of) and for Wittgenstein the proof is part of what we mean by "7 is prime." The closest thing to an analogue of proof for an empirical description would be a proposition's being verified by checking (say, checking that the ball bounces when I drop it). This is quite different as the verification does not bear the same relation to what we mean by "the ball bounces when dropped" as a proof does to "7 is prime". The proof partly defines what we mean mathematically by "7 is prime", whereas we might understand "the ball bounces when dropped" in the same way irrespective of in which way we test it.[94] The

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[93]Wittgenstein might put this point by talking of the "peculiar inexorability" (*RFM*, p.37) of mathematics.

[94] This is not to say that the meaning of an empirical proposition is never related to our means of testing it. As I shall go on to argue in the next two sections, the point is that the

model of mathematical propositions as descriptions distorts this grammatical difference, since on that model the mathematical proposition's meaning is unchanged by the proof.

Another way in which to put the point that an experiment is not internally related to its result is in terms of the criteria for reproducing an experiment. With an experiment we set up the experimental conditions and then when we carry out the experiment Nature decides for us what the result will be. The criteria for reproducing an experiment include the experimental conditions but not the result. By contrast, the statement proved is part of what we must reproduce in reproducing a proof. If the proof we 'reproduce' were of a different statement then it would be a different proof (whereas the same experiment might perhaps give different results on different occasions). Whatever we are to say about proof from the point of view of the model of mathematical propositions as descriptions, it would thus be highly misleading to take proofs to be analogous to empirical experiments.

That the criteria for reproducing a proof are different from the criteria for reproducing an experiment is an important difference because it is connected with the different roles played by proof and experiment. An experiment is a kind of test, whereby we check an empirical prediction in a situation meeting the conditions of the prediction to see whether the prediction holds. The result of the experiment is that we are able to confirm or disconfirm the prediction in that case and in this way it tells us something about the world. What the proof serves to do, from the perspective that Wittgenstein presents, is to fix our way of using expressions.[95] The proof is able to fix our way of using certain expressions because it does not just tell us that the statement at the end of the proof follows, it shows us how it follows and thereby shows us how to use certain expressions. As Wittgenstein puts it, the proof functions as a picture:

> When I say "a proof is a picture"--it can be thought of as a cinematographic picture.
>
> We construct the proof once for all. A proof must of course have the character of a model.
>
> The proof (the pattern of the proof) shews us the result of a procedure (the construction); and we are convinced that a procedure regulated in *this* way always leads to this configuration.
>
> (The proof exhibits a fact of synthesis to us.) (*RFM*, p.159)

The idea is that when we are led through a particular procedure then we see that the result of that procedure must always be the same.

If we see that the model of a mathematical proof as an experiment is misleading then it might be tempting to turn instead to a model of mathematical proof as laying down stipulations. Given that the necessity of proof is something that arises in our way of using the proof, should we therefore say that the proof is simply a way of laying down a convention?

One reason why the conventionalist model of proof is misleading is that it depends upon a model of mathematical propositions as conventions which, as we saw in section *2.3* and

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meanings of empirical propositions are not typically so closely linked to particular means of verification.

[95]Or, more generically, symbols. I don't mean to exclude diagrammatic forms.

chapter 4, neglects the importance of the connection of mathematical propositions with applications. More importantly for present purposes, the model is also out of accord with much of our experience of doing mathematics. To reiterate the point, when we are trying to arrive at a result of a proof we don't feel like we are trying to lay down a rule. We are very much constrained by the relevant system as it stands and it can sometimes feel like we are simply running through a procedure by following rules to see what comes out.[96] When we first derive new rules then the experience can be very similar to performing an experiment, since we don't always see what we will get out.

Consider what happens when we first run through the procedure of adding the first four powers of seven and are led by this procedure to take "$7^0+7^1+7^2+7^3=400$" as a rule of the system. As we first run through the procedure, we may simply be taking it as a procedure (in the manner of "if you add the first four powers of 7 then 400 will be the result"). As Wittgenstein says:

> Is it experimentally settled whether one proposition can be derived from another? — It looks as if it were! For I write down certain sequences of signs, am guided by doing so by certain paradigms [...] and of what I get in this procedure I say: it follows. (*RFM*, p. 96)

When we first compose a proof we might not be aware of the result that we are working towards (although on many occasions we might be). Nonetheless, as we take in a proof we see necessity in the procedure's giving the result. A picture of the procedure which we carried out is encapsulated within the rule, insofar as this picture fixes the way in which we will use the proven statement. When we take the procedure as a picture (as a proof) then we also take the proven statement as a rule (as part of the mathematical system). Wittgenstein's idea of a proof as a picture of an experiment thus incorporates this experimental side to constructing proofs but without slipping into treating proofs themselves as experiments.

Wittgenstein gives examples of proofs in order to show this and it is worth considering at least one in order to make the idea clearer. Wittgenstein gives a drawing of a pentacle and draws lines on this to show how many points it has, calling this figure (c) (*RFM*, p. 47):

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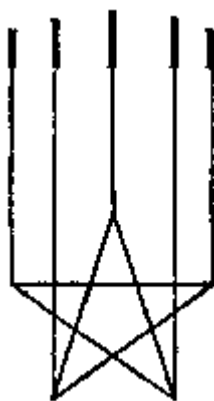
[96]It should be noted that Wittgenstein's interest is not driven by a commitment to a metaphysical position. As Floyd puts it, he "aims to do justice, instead, to ordinary experiences in mathematics and logic" (2010, p.316).

He calls the lines at the top of the figure (a) and the pentacle itself (b). He says that this can be seen as a proof that there are as many points on the pentacle as there are fingers on a (idealised) hand:

> Let us give names to the shapes of the patterns (a) and (b): let (a) be called a "hand", H, and (b) a "pentacle", P. I have proved that H has as many strokes as P has angles. And this proposition is once more non-temporal.[97]

In order to see the connection, we first need to trace the lines and see how the procedure of correlation runs for this particular pentacle. Once we see the connection and see that it generalises, we can then see the procedure as a picture. In doing so we use the image "as a new prescription for ascertaining numerical equality: if one set of objects has been arranged in the form of a hand and another as the angles of a pentacle, we say the two sets are equal in number." We don't need to check how many objects there are in the two sets, just by their arrangement we can then tell that they are equal in number.

By calling a proof a picture Wittgenstein is aiming to do justice to the necessity of proof. The Platonist model, as discussed in chapter 4, misleadingly makes it seem as though this necessity arises from mathematical entities rather than from the way we use the expressions. Aside from the sceptical problems connected with postulating mathematical entities, one might object to this model that it is "not something behind the proof, but the proof, that proves" (*RFM*, p.173).The model of mathematical propositions as arbitrary stipulations, at least if taken in a simplistic way, makes it seem as though we could easily stipulate mathematical systems however we wish. A more sophisticated version of the stipulation model might be sensitive to how proofs are developed in accordance with the rules already laid down in the system. But this would at the least leave a sense of arbitrariness still remaining concerning the choice of axioms. Whilst sometimes the choice of axioms for a mathematical system might be described as arbitrary, on other occasions the system serves a particular purpose and the axioms might then seem to choose themselves. Of axioms such as this Wittgenstein says that "when we say that it is evident, this means that we have already chosen a definite kind of employment for the proposition without realizing it" (*RFM*, p.224).One might say this concerning, for example, the axioms of Peano Arithmetic or the axioms of Euclid. We have seen in the last section that Wittgenstein regards cases of applied mathematical systems such as this as the most characteristic cases of mathematical systems. For cases such as this, the model of mathematical propositions as rules of description helps to bring out that the system has a point to it and its having this point is connected with our sense that the system *has* to be as it is. These systems are set up such as to serve this point and for the most part we cannot see how to change such systems (giving them alternative axioms) without sacrificing their point. Insofar as coming to see a procedure as a picture involves seeing the point in that procedure (seeing that it can be used to a particular end) then Wittgenstein's model of a proof as a picture is incompatible with a conventionalist approach. Commenting on how two proofs can prove the same proposition, Wittgenstein says:

> That these proofs prove the same proposition means, e.g.: both demonstrate it as a suitable instrument for the same purpose.
>
> And the purpose is an allusion to something outside mathematics. (*RFM*, p.367)

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[97]By saying that the proposition is 'non-temporal', Wittgenstein is saying that it expresses a rule. The key point is that a non-temporal expression has no exceptions because anything that looks like an exception won't be counted. So it may look like a generalization (description) but in fact it is a rule.

The point in a procedure lies in a use for the proposition and, as we have seen in the last chapter, for the most characteristic mathematical systems the propositions will have uses outside of the system itself. For these cases the conventionalist model is thus misleading with regard to proof, since in neglecting the applications of mathematical systems it also neglects a key aspect which enables us to take a procedure as a picture. But, as was also argued in the last section, for some systems and for some proofs it may be that the proof serves an aesthetic end or an end related to the organisation of the system itself. So long as a conventionalist model of proof was sensitive to such ends then the model need not be misleading with respect to such cases.

There is another strand to Wittgenstein's idea of proofs as pictures which should also be noted here. Part of the point of calling proofs 'pictures' is also that a proof is not just text on a page but is something conceptual that is not reducible to any one of its physical reproductions, at least if these reproductions are looked at only as physical configurations (*RFM*, p.143). Each reproduction both shows us a procedure and shows us the procedure in a 'perspicuous' (*RFM*, p.143) way that allows us to take it as a picture. Wittgenstein thus agrees with the Platonist that it would be misleading to take a view, as Wittgenstein attributes to formalists, that mathematics is about 'mere' marks on a page (*AWL*, p.151-152). It is not the symbols themselves but the way that we use them (the procedures that we embody in them) that makes them mathematics (*RFM*, p.257; *RFM*, p.224). However, as we have seen, the Platonist conception of mathematics as about mathematical objects is also misleading since mathematical propositions are not verified in the same ways as empirical propositions and are immune from empirical revision (and these distortions, as seen in chapter 4, are connected with sceptical problems). We can do greater justice to the use of mathematical propositions but saying that mathematical signs embody operations and procedures in a way that shows us how to use the expressions. In this way the picture of mathematical propositions as rules of grammar - rules which fix the use of expressions - can be seen to avoid the Platonist and anti-Platonist distortions (since it avoids speaking of mysterious objects referred to by mathematical propositions and also avoids taking mathematics to be statements about 'marks' (*AWL*, p.153)).

Wittgenstein's discussion of proof is thus highly nuanced and the nuances can be seen to play the role of acknowledging the valuable points within certain tempting pictures whilst blocking the attraction of applying those pictures dogmatically.[98] The key question for the purposes of how Wittgenstein is true to his promise to 'leave mathematics as it is' (*PI*, §124) is whether this kind of balancing act is entirely sustainable. Does Wittgenstein fall into employing a picture dogmatically? Are there philosophical problems that emerge as a result of dogmatic employment of philosophical pictures by Wittgenstein? Whilst I have argued that Wittgenstein does avoid taking up a dogmatic stance[99] with regard to proof in a number of respects, I have delayed the question of whether Wittgenstein makes it a dogmatic thesis that a mathematical proposition is not meaningful in advance of its proof. Wittgenstein has certainly been read by some[100] as claiming that mathematical conjectures

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[98]For further discussion of the nuances, see especially Mühlhölzer (2005) and Floyd (2010).
[99]A dogmatic stance here would be one which fastens on certain cases/aspects of proof and claims that all proof must fit a model which is only applicable to the selected cases/aspects.
[100]For example, Shanker says that for Wittgenstein a conjecture is a "meaningless expression albeit one which may exercise a heuristic influence on the construction of some new proof-system" (1987, p. 230). Though I will focus instead on Marion's claim that Wittgenstein held a "robust form of verificationism…where the method of verification—to be found in the proof—determines the sense of the statement" (Marion 1998, p.159).

are meaningless until they are proven. The next section will take up the question of whether Wittgenstein can be understood as avoiding this particular dogmatic claim.

6.3. The Status of Conjectures

If we accept that it is characteristic of a mathematical proposition to be internally related to its proof, then it can start to look like mathematical propositions are not 'truths' at all. If a mathematical proposition plays the role of a rule, then this would suggest that mathematical propositions are not so much true or false (in the manner of correct or incorrect descriptions) but rules of the system or not. A conjecture, then, would be a statement which is not yet proven or disproven with respect to the system and thus not really a proposition of the system at all. Marion attributes such a view to Wittgenstein, claiming that Wittgenstein held a "robust form of verificationism...where the method of verification—to be found in the proof—determines the sense of the statement" (Marion 1998, p.159). Even more radically, Marion cites Wittgenstein's remark that "it is part of the nature of what we call propositions that they must be capable of being negated" (*PG*, p.376) and argues that since we cannot imagine under what circumstances we would assert the negation of a mathematical truth then it must be that mathematical propositions are not really propositions at all (Marion 1998, p.168).

Clearly the view being attributed to Wittgenstein by Marion risks not 'leaving mathematics as it is.' Marion's view is primarily focused on the early 1930s and some of the remarks he cites pre-date the first statement of an aspiration to 'leave mathematics as it is.'[101] But with regard to the law of excluded middle he says that he believes that later remarks represent "no significant departure from this line of thought" (1998, p.170). I will suggest that the radical reading that Marion advocates is questionable even with regard to the early 1930s remarks. We do speak of truth and falsity with regard to mathematical propositions and it is not clear that this talk can easily be reinterpreted without changes to mathematical practice. For example, negation plays an important role in mathematics and the negation of a falsehood is taken to be a truth.[102] If mathematical 'truths' were really rules, then it is not clear what a mathematical 'falsehood' would be. If the negation of a rule is to be its opposite then presumably the negation would be a rule that were not part of the system in question (since the system would be defined by its rules). So falsehoods would then seem to have no role to play. Wittgenstein himself urges this point:

> My explanation mustn't wipe out the existence of mathematical problems.
>
> That is to say, it isn't as if it were only certain that a mathematical proposition made sense when it (or its opposite) had been proved. (This would mean that its opposite would never have a sense (Weyl).) On the other hand, it could be that certain apparent problems lose their character as problems—the question as to Yes or No. (*PR*[103], p. 170)

As was noted in chapter 2, if we are to read Wittgenstein as true to his methodology then we should not read him as saying that mathematical propositions really are rules of description. We should instead read him as presenting a comparison between mathematical

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[101]In (*BT*, 308e) composed 1930-31 – see (Stern, 2005, p.222).

[102]Embodied in the rule of double-negation elimination, to be discussed in the next section.

[103]*PR* was composed before the Big Typescript remarks that I take to be the first expression of Wittgenstein's later methodology. However, it is accepted that Wittgenstein became more concerned with a need to 'leave everything as it is' over time so there is no reason to think that he would have gone back on a remark which urges against revisions to mathematical practice, as I am suggesting that this one does.

propositions and rules of description and it is worth being clear on this point before considering Marion's reading in detail. Wittgenstein does not say that mathematical propositions are rules of description in every respect. When he talks about this idea in his lectures from 1934-1935 that the comparison is useful for understanding the "application of a mathematical sentence occurring in our language" (*AWL*, p.152) i.e. the sentence's application outside of mathematics. More fully:

> The arithmetic sentence in which "3" occurs is a rule about the use of the word "3". The relation of this sentence to a sentence such as "There are 3 men here" is that between a rule of grammar about the word "3" and a sentence in which the word "3" is used. The *application of a mathematical sentence occurring in our language* is not to show us what is true or false but what is sense and nonsense. (AWL, p.152[104])

He makes it clear in the next lecture what he means by an "application of a mathematical sentence occurring in our language." He says:

> I emphasise the word "rule" *when I wish to oppose rules to something else* e.g. when I wish to emphasise the difference between "2+2=4" and "If A gives me 2 apples and B gives me 2, then I have 4 apples in all." (*AWL*[105], p.154)

So mathematical expressions play the role of rules in relation to non-mathematical expressions e.g. licensing certain inferences concerning numbers of apples in particular situations. The remark concerning negation above (*PR*, p. 170) suggests that mathematical sentences do not play a rule-like role in all respects and that this comparison can be misleading when it comes to understanding the relation between mathematical sentences and each other. The point seems to be that mathematical sentences are rule-like in some respects and not rule-like in others. Wittgenstein stresses that the "primitive classification" (*AWL*, p.155) of sentences into different categories can itself be misleading and that some cases will defy easy classification (*PI*, §23). Certain cases "do not fall within any of the divisions of the classification, any more than spintheriscopes[106] belong to the classification dress, food, furniture" (*AWL*, p.154).

Wittgenstein is quite aware that if he "told a mathematician that 2+2=4 was a rule for the use of signs, he would feel uncomfortable" (*AWL*, p.156). Wittgenstein acknowledges that saying this seems to undermine the sense in which mathematical sentences are true and false (*AWL*, p.156), which is presumably the mathematician's main concern. The mathematician is less concerned with the relationship of mathematical sentences to their applications and so the comparison might seem strange to him. To dispel this concern Wittgenstein makes the point that "to say that something is a rule of grammar is not to say that it is always so used" (*AWL*, p.156) and insists that what the comparison brings out is the relationship of such sentences to their applications i.e. that mathematical propositions play a role of rules in relation to their empirical applications but need not always play a rule-like role.[107]

Here we can see Wittgenstein applying his methodological point that comparison can be illuminating through dissimilarity as well as similarity (*PI*, §130). Mathematical sentences defy "primitive classification" (*AWL*, p.155) and might be called rules of description for

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[104]My italics.

[105]Italics from Ambrose's notes.

[106]A device for measuring nuclear decay.

[107]Floyd makes the similar suggestion that Wittgenstein be seen as separating between the "applied role" of an equation and its role within a calculus (2005, p.107) but Floyd does not apply the point to questions of truth and falsity.

some purposes (*AWL*, p.156) and might be called propositions or 'statements' (*AWL*, p.153) for other purposes. The crucial thing for Wittgenstein is to bring out the respects in which each such comparison is illuminating and not, so that we are not tempted to apply a particular comparison dogmatically.

As I read Wittgenstein, undecided mathematical propositions (the "mathematical problems" that Wittgenstein "must not wipe out") are less rule-like than, say, axioms. If the axioms tell us to combine signs in certain ways, then certain configurations of signs are licensed by the axioms and the axioms play the rule-like role of licensing configurations. These configurations might allow us to make further transitions, allowing us to derive further rules from our existing rules. But what do we make of a configuration of signs which has not yet been derived from any other? The role of that proposition is not (at least not yet) especially rule-like and rather plays the role of a kind of target. Our interest in an unproven proposition is primarily in whether we can derive it or not rather than how the proposition itself can be used. When we apply a proposition in a derivation then we are using it in a rule-like way but the proposition that we are aiming to derive is not thereby treated as a rule.[108] When we use a proposition to make a derivation then we use it as a rule of grammar but we don't use a proposition as a rule of grammar when the proposition is itself being derived.

Marion is quite aware that to attribute to Wittgenstein the view that mathematical sentences are rules and that they have meaning only insofar as a proof assigns them meaning is to attribute to him "an unpopular stance on a controversial thesis" (Marion 1998, p.164). But Marion seems to think that the stance is "insightful" (p.164) and the reading is in any case unavoidable because of the remarks that Wittgenstein makes. Marion is especially focused on remarks relating to conjectures such as that 'three consecutive 7s' occur in the expansion of π.[109] Marion takes Wittgenstein to be saying that such conjectures are not meaningful unless we have a decision procedure for them (1998, p.162). The thesis that Marion ascribes to Wittgenstein is that mathematical propositions are expressions of rules or algorithms and so not properly described as true or false, which he captions with the phrase that 'there are no genuine alternatives in mathematics' (1998, p.168). Marion takes this as a reason behind remarks of Wittgenstein's which he thinks express the view that mathematical propositions are not properly subjected to truth-functional logical laws and thus that "the lack of validity of the Law of Excluded Middle in mathematics is a distinguishing feature of **all** mathematical propositions" (1998, p.168). Thus Marion takes Wittgenstein to reject the law of excluded middle (the law by which all propositions are true or false) and in this respect to be closely aligned to Intuitionists like Brouwer but actually more extreme than them in arguing not just "against the universal applicability of the Law of Excluded Middle" but also "arguing for its universal inapplicability" (1998, p.168).

Marion takes Wittgenstein to be in agreement with Brouwer with regard to questions like whether the law of excluded middle is applicable to "there are three consecutive 7s in the expansion of π" (1998, p.165). Marion points out Wittgenstein agrees with Brouwer's basic claim that one cannot say that the three 7s must occur or not without providing a method for determining this. On this reasoning, it is not enough to say that the expansion of π will

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[108] These targets for future derivations (i.e. conjectures) are of course of more interest to professional mathematicians than they are to non-mathematicians.

[109] It has turned out that this is no longer a conjecture – there is such a sequence of sevens (Shanker 1988, p.159). But for our purposes let us pretend that it is still a conjecture, as it was in the days of Wittgenstein and Brouwer.

settle the matter since the expansion is something which we need to derive. Marion cites (1998, p.165) the following:

> Will three consecutive sevens ever occur in an evaluation of π? People have an idea that this is a problem because they think that if we knew the whole evaluation we should know, and the fact that we don't know is merely a human weakness. This is a subterfuge. The mistake lies in the misuse of the word infinite, which is not the name of a numeral.

> 'If we find that three consecutive sevens occur, then we have proved that they do; but if we don't find them we still have not proved that they do not.' This gives us no criterion for falsehood, but only for truth. (*AWL*, p. 107)

Marion is certainly right that Wittgenstein is not outright rejecting Brouwer's view but it risks distorting Wittgenstein to take him to be closely aligned to Brouwer. It is typical of Wittgenstein to not outright reject philosophical theses since, as he says, "they contain so much truth" (Ms112, 99r). Instead he thinks that "we must only point out and resolve the injustices of philosophy, and not posit new parties-and creeds" (Ts213, 420). The question is how to understand the extent of the agreement and disagreement between Wittgenstein and Brouwer without seeing Wittgenstein as committed to a dogmatic position.

Wittgenstein acknowledges Brouwer's point that having a procedure for expanding π is not sufficient as a procedure for deciding whether three consecutive 7s appear in the expansion. But he takes this not as an indication that the law of excluded middle breaks down for mathematical propositions but as an indication that "there are three consecutive 7s in the expansion of π" is not a mathematical proposition in the same sense that a proven theorem is a proposition. Marion's focus is on the period 1929-1933 (1998, viii) but some of Wittgenstein's later remarks on this suggest that Wittgenstein is considering whether "there are three consecutive 7s in the expansion of π" is better seen as a prose expression until it is proven. Calling the pattern of 7s 'φ', Wittgenstein comments:

> But does this mean that there is no such problem as: "Does the pattern φ occur in this expansion?" −To ask this is to ask for a rule regarding the occurrence of φ. And the alternative of the existence or non-existence of such a rule is at any rate not a mathematical one.

> Only within a mathematical structure which has yet to be erected does the question allow of a *mathematical* decision, and at the same time become a demand for such a decision. (*RFM*, p.279)

If "the pattern φ occurs in the expansion of π" is not a proposition, then it is not a genuine question to ask "Does the pattern φ occur in this expansion?" But Wittgenstein does not want to deny that this is a genuine question, nor that the expression is in some sense a proposition. As he says:

> Besides, the question is not so much whether the prediction makes some kind of sense, as: what kind of sense it makes. (That is, in what language-game it occurs.) (*RFM*, p.281)

Wittgenstein is rather probing the role of an expression like "the pattern φ occurs in the expansion of π." He thinks that Brouwer has hit upon an unusual and illuminating case of a mathematical proposition and wants to do justice to what is unusual about it. Contrary to Marion's reading, Wittgenstein seems to be interested in the proposition because it is somehow different from paradigm cases of mathematical propositions like expressions of theorems. Rather than claiming that the law of excluded middle does not apply in

mathematics, Wittgenstein thinks that there is something wrong with insisting upon applying the law of excluded middle to *this particular case.*

The problem with the case is that we don't see why the law of excluded middle should apply here - we cannot see why it should be that the pattern should either occur or not occur - since we have no adequate means of deciding. Wittgenstein considers whether we even need to see why, suggesting that the law of excluded middle might be best understood as a 'commandment' - an instruction to follow without necessarily seeing the point of (*RFM*, p.271). He considers whether this might be a way to see mathematical propositions in general:

> Suppose we look at mathematical propositions as commandments, and even utter them as such? "Let 25^2 be 625." (*RFM*, p.271)

But Wittgenstein is uncomfortable with this suggestion – he seems to think that in normal cases the application of excluded middle is natural and not something we take on faith:

> But is this really a way out of the difficulty? For how about all the other mathematical propositions, say '$25^2=625$'; isn't the law of excluded middle valid for these *inside* mathematics?[110] (*RFM*, p.276-277)

Wittgenstein does not want to treat "the pattern φ occurs in the expansion of π" as typical of mathematical propositions. The expression better fits the model of a conjecture (a special kind, or limiting case, of proposition) since it admits of a decision only "within a mathematical structure yet to be erected" (*RFM*, p.279). A conjecture is a kind of special case of a proposition which expresses an open question. Wittgenstein's interest here appears to be in part in pointing out different cases that our language can tempt us to overlook. On this point Säätelä (2011, p.2) cites the following remark:

> Unfortunately, our language uses each of the words "question", "problem", "investigation", "discovery", to refer to such fundamentally different things. It's the same with the words "inference", "proposition", "proof". (*BT*, p.616)

As we have seen in the last section, mathematical propositions are not like empirical propositions in that a mathematical proposition is internally related to its proof, whereas empirical propositions are typically externally related to their verification. But this is not to say that Wittgenstein would insist that the way of verifying an empirical proposition has no bearing on its meaning or that we cannot find cases where we would want to say that it does (Diamond (1999) gives such an example). This is part of the contrast between the grammar of a rule of description and the grammar of a proposition and Wittgenstein insists that this contrast 'shades off' in particular cases (*RFM*, p. 363). Wittgenstein is pointing to differences which are not always clear-cut but which are important nonetheless (since we can be led to philosophical confusions if we are not attentive to them).

Likewise, the way we use the term "question" in relation to empirical questions is different from mathematical questions. An open empirical question might be clearly-defined in advance of our having a method for answering it. But mathematical questions are conceptual questions and this means that with a mathematical question the question itself is not entirely clear prior to our having a method for answering it. He says that "problem of

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[110] Wittgenstein's italics.

finding a mathematical decision of a theorem might with some justice be called the problem of giving mathematical sense to a formula" (*RFM*, p.296)[111] and he notes:

> A proposition with its proof belongs to a different category than a proposition without a proof. (Unproved mathematical propositions—signposts for mathematical research, stimuli for mathematical constructions.) (*BT*, p.630-631[112])

The example of "there are three consecutive 7s in the expansion of π" is interesting and worthy of special attention because it initially looks so much like a proposition which we should think that we have a decision-procedure for. But what we have is actually just something that *might* be a decision-procedure if we pursued it far enough. We are able to expand π up to an arbitrary n digits but if the sequence does not occur in the first n digits then we cannot guarantee that it won't occur later and the question will remain undecided. But this is not to say that we are entirely in the dark with regard to the proposition:

> It seems clear that we understand the meaning of the question: "Does the sequence 777 occur in the development of π?" It is an English sentence; it can be shown what it means for 415 to occur in the development of π; and similar things. Well, our understanding of that question reaches just so far, one may say, as such explanations reach. (*PI*, §516)

We can point to considerations that are relevant to checking the truth of the proposition and which might well yield a decision. Insofar as we can speak of such things then an analogy between this proposition and more paradigmatic case of propositions holds.[113] Insofar as we have approaches that are relevant to deciding the sentence, it is akin to a proposition. Insofar as we do not have a method of deciding the sentence, it is not akin to a proposition. Much as there are "typical cases of society's revenge and typical cases of deterrent measures and others of punishment as reform, and countless mixtures and intermediate cases" (Ms152, 16-17), so there are "mixtures and intermediate cases" with regard to the "primitive classification" (*AWL*, p.155) of sentences. The sentence "there are three consecutive 7s in the expansion of π" is an intermediate case (it is both proposition-like and rule-like – perhaps 'boundary case' might be a better term).

Marion (1998, p.165) takes Wittgenstein to be agreeing with Brouwer that the law of excluded middle is not applicable to the case of "there are three consecutive 7s in the expansion of π" without some further specification of a decision-procedure. Marion is not wrong about this but it does an injustice to Wittgenstein to say, as Marion goes on to say, that Wittgenstein was happy to reject the law of excluded middle and other logical laws across all of mathematics (1998, p.168). Wittgenstein can be seen to be criticising Brouwer for hitting upon an unusual case and extrapolating from this case to all of mathematics. As I will go on to argue, extrapolating from particular cases to claim insight into all of mathematics can be seen to be key to Wittgenstein's criticisms of intuitionism, such as his remark - "What is an existential theorem? The answer is this, and this, and this . . . "

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[111]Cited by Säätelä (2011, p.16).
[112]Again cited by Säätelä (2011, p.18).
[113]Wittgenstein speaks of an analogy between mathematical propositions and other propositions at (*PG*, p. 366).

(*AWL*, p.116).[114] This part of the argument will be deferred until the next section, on constructive and non-constructive proof-techniques. First I should make my points of agreement and disagreement with Marion with regard to "there are three consecutive 7s in the expansion of π" clearer.

Marion is right that Wittgenstein does want to emphasise differences between mathematical propositions and empirical propositions and he does indeed emphasise that mathematical truths could not have turned out to be false – there are no "genuine alternatives in mathematics" (1998, p.167). Mathematical propositions don't turn out to be true – insofar as they are mathematical they must be true. However, I wish to disagree with Marion in that I want to stress Wittgenstein's point that the negation of a mathematical truth is not nonsense (*PR*, p. 170). The negation of a mathematical truth is excluded by the calculus in which the truth is proven, though a corresponding expression might be a truth of a different calculus. In this way a mathematical falsehood is not something about which we can say nothing – it has a specific role in a mathematical system (*PG*, p.376). The point is not that mathematical propositions are not genuine propositions. The point is that mathematical propositions are not paradigmatic cases of propositions because they have a different grammar from empirical propositions.

I also want to agree with a characterisation which Marion gives early in his discussion (1998 p.161-162) when he suggests that Wittgenstein would have no qualms about applying the law of excluded middle in whole areas of mathematics, such as for solving arithmetic problems or algebraic equations. But Marion seems to me to weaken this point by saying that it is because of a decidability theorem that we are entitled to the law of excluded middle in elementary algebra (1998, p.161). It seems to me that Wittgenstein is not saying that we need a general theorem in order to be sure we have a decision procedure. Wittgenstein seems to resist laying down any criteria as he restricts himself to saying that "when the method has been evolved, then questions in that system become very like ordinary empirical questions" (*AWL*, p.199). How evolved the method needs to be and in what ways, he does not say. And nor would one expect him to, since to do so would be to draw a limit to the ways that mathematicians can work. There is no saying in advance where and when we will accept the law of excluded middle – certainly our acceptance of the applicability of the law of excluded middle is linked to our seeing an expression as decidable but this is not to say that we need a theorem in order to be sure of decidability.[115]

I am sympathetic to Marion when he initially suggests an interpretation which allows of qualified use of the law of excluded middle:

> Wittgenstein is ready to admit decidability (and hence the applicability of the Law of Excluded Middle) at the level of assertions (mathematical propositions) when there is a procedure of decision available (therefore a 'calculus') but not at the level of the expression of the procedures of decision themselves. (Marion 1998, p.162)

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[114]Another example is "When the intuitionists and others talk about this they say: 'This state of affairs, existence, can be proved only thus and thus.' And they don't see that by saying that they have simply defined what they call existence." (*PG*, p.374).

[115]Nor does Marion say that we would need a theorem in every case but his presentation seems to make this suggestion. I am not sure that Wittgenstein is even citing a proof in relation to algebra. Wittgenstein does mention a proof that every algebraic equation has a root (AWL, p.198) that seems to be an unrelated example – an example of how a conjecture can be given sense.

Marion is perhaps somewhat speculative in his characterisation of what kinds of propositions Wittgenstein might and might not allow excluded middle to be applied to. I am not sure that Wittgenstein would want to say very much at all in regard to when we will be happy to apply excluded middle and when not. Nonetheless, the reading seems to me to fit Wittgenstein. My key complaint is that when Marion goes on to "provide further arguments in support" (1998, p.164) of his reading then he ends up painting Wittgenstein as much closer to Brouwer than the above-quoted characterisation would suggest. Marion's elaboration of the arguments actually ends up resulting in a much more contestable reading with Wittgenstein "not so much doubting the reason set forth by Brouwer against the universal applicability of the Law of Excluded Middle as arguing for its universal inapplicability" (1998, p.168). As I have been arguing, it is this latter claim which distorts Wittgenstein by making it seem as though mathematical propositions really were rules rather than merely rule-like in their applications outside of mathematics.

I disagree with Marion insofar as I want to suggest that it is not only because of a broad difference in grammar between mathematical and empirical propositions that the application of the law of excluded middle to "there are three consecutive 7s in the expansion of π" looks doubtful. The problem with the conjecture is that it is not clear to what extent it, the particular statement, is decidable and so it is not clear to what extent it either fits the broader mould of a proposition or the more particular mould of a mathematical proposition.[116] One can actually put much the same point by means of empirical (or rather seemingly empirical) propositions alone [117] – "Fred went to see yesterday's match" clearly fits the mould of a proposition (since envisaging methods of checking is quite easy) whereas it is much less clear what to say about "Fred went to see tomorrow's match." Whether the latter is a proposition or not, I am not sure. I know how to ask people at a match whether Fred was there and check whether Fred knows things about what happened but I don't see how to apply these checks if the match was supposed to have happened tomorrow. This might look like a confusion but it need not necessarily be so – it could turn out, for example, that Fred went to the equivalent of tomorrow's fixture in the previous season and in that case I would know what to check. Wittgenstein makes a similar point when he compares mathematical conjecture to riddles – expressions for which we may or may not be able to find a definite meaning. [118] My suggestion is that Wittgenstein is making much the same point with regard to "there are three consecutive 7s in the expansion of π." It is similar to other expressions that we could decide and we can point to things relevant to deciding it but it is not clear whether these things entitle us to call it a proposition.

From this point of view it looks like a misrepresentation on Marion's part to say that Wittgenstein's criticisms of Brouwer are directed only against Brouwer's "sales pitch"

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[116]Wittgenstein's point with regard to Brouwer's pendulum number is similar – it is not clear that Brouwer's pendulum number is really a number (although Brouwer thinks that it is) – see Marion (1998, p.166).

[117]Diamond (1999) makes the point that there are cases of empirical expressions which we struggle to say whether they are meaningful because it is not clear whether they could be verified. She compares a case of pronouncing upon the age of ancient bones now that we have carbon dating to the case of making such a pronouncement hundreds of years ago.

[118]Wittgenstein compares a mathematical conjecture to the instruction to arrive 'neither naked nor dressed' (*LFM*, p.185-186) – it is not immediately clear whether this even makes sense i.e. it is not immediately clear whether it is even an instruction.

(1998, p.167) rather than any point of substance. Wittgenstein takes it as a substantial point that Brouwer fails to see any problem with calling an expression like "there are three consecutive 7s in the expansion of π" a proposition:

> I need hardly say that where the law of the excluded middle doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics. (Against Weyl and Brouwer.) (*PR*[119], p.176)

If Marion were right, it would have been clearer for Wittgenstein to say here 'in that case we aren't dealing with propositions because mathematics does not consist of propositions.' But this is not what Wittgenstein says – he does speak of mathematical propositions and he is suggesting that Brouwer's cases do not neatly fit that category.

Furthermore, it is a substantial criticism of Brouwer when Wittgenstein says that his example "doesn't reveal a peculiarity of propositions about infinite aggregates" (*PR*, § 173). Marion reads this as just a reaction on Wittgenstein's part against phrases like "infinite aggregate", which tend to suggest a Platonistic conception of mathematical objects (Marion 1998, p.167). But this could instead be read as a substantial criticism because Wittgenstein can be read as saying that Brouwer's position claims an insight into the nature of mathematics, or at least part of mathematics, and attempts to generalise a criterion that future proofs must meet. We can see Wittgenstein as criticising Brouwer for taking himself to have arrived at an insight of relevance to mathematicians, whereas Wittgenstein thinks that Brouwer has simply found an interesting example which initially looks meaningful but turns out not to be. If Brouwer were to have revealed "a peculiarity of propositions about infinite aggregates" (*PR*, § 173) then mathematicians would need to henceforth avoid the law of excluded middle with regard to *any* proposition involving infinite aggregates. But if the point concerns the availability of a decision-procedure then there is no reason to avoid the law of excluded middle when a decision-procedure is available, even if the expression does involve infinite aggregates.

On this reading we can also see Wittgenstein's diagnosis of the reason for Brouwer's error as making a more substantial point than Marion can allow for. Brouwer can be read as making a brash generalisation because he fails to see that we can have expressions which look at first glance like mathematical propositions but which turn out not to fit this mould. Brouwer's assumption that his case is mathematical suggests that he takes it that mathematical propositions are distinguished from non-mathematical propositions by their form or subject-matter, rather than by their role. But it is the way that we use expressions and their relations to other expressions that counts.[120] Whilst "there are three consecutive 7s in the expansion of π" may look like a generalisation over a peculiar mathematical subject-matter (namely the as-yet-unwritten expansion of π), if it were to be a mathematical proposition it would be such that we could use it as a rule:

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[119]Again this remark is from 1929-1930 but Marion follows the view that any doubts about the law of excluded middle and sympathies with the intuitionists would have been strongest in the late 1920s and early 1930s (Marion 1998, p.170). So the remark being an early one just makes it even more relevant.

[120] On this see sections 2.3 and 3.2, especially the argument that chess would not be just a game if it were used to fight battles.

> If someone says (as Brouwer does) that for $(x) \bullet f_1 x = f_2 x$, there is, as well as yes and no, also the case of undecidability, this implies that '$(x)...$' is meant extensionally and we may talk of the case in which all x happen to have a property. In truth, however, it's impossible to talk of such a case at all and the '$(x)...$' in arithmetic cannot be taken extensionally.[121] (*PR*, p.212)

Brouwer's dogmatism – very likely one of the strands of 'Bolshevism' that looms as a danger in *LFM*[122] - thus arises because Brouwer fails to adequately distinguish mathematical from empirical propositions.

It should again be noted that Marion's interest is primarily in the years 1929-33 and he excludes later remarks (Marion 1998, p.170), some of which I have made use of. Nonetheless, Marion thinks that the later Wittgenstein makes "no significant departure" (1998, p.170) from the line of thought that Marion attributes to the 1929-33 Wittgenstein. As I have argued, Marion's picture makes Wittgenstein's viewpoint seem simpler than it is, attributing to Wittgenstein dogmatic theses which are out of accord with Wittgenstein's conception of the aims of his philosophy and his criticisms of Brouwer. Marion is forced to treat it as misleading that Wittgenstein "nearly always puts the emphasis on differences between his views" (1998, p.164) and Brouwer's and suggest that Wittgenstein's admonitions covered a deeper agreement. As I see it, Wittgenstein admonishes Brouwer for extrapolating too far from a particular case. Where Marion attributes to Wittgenstein a view which he admits to be received as an "unpopular stance on a controversial thesis" (1998, p.164), instead Wittgenstein's examination can be seen to involve no controvertial theses and to be limited to evaluating the significance of Brouwer's case by comparing it similar and disimilar models (especially models of empirical proposition and models of decideable mathematical propositions). If my reading is to stand up then Wittgenstein needs to be seen as rejecting as dogmatic Brouwer and Weyl's rejection of proof-techniques involving the law of excluded middle and also rejecting as dogmatic the avocation of exclusively constructive methods. This is the topic of the next section.

6.4. The Validity of Non-constructive Proof

As was noted in the last section on conjectures and the law of excluded middle, Marion does not to take Wittgenstein's criticisms of Brouwer as the substantial points that they appear to be (1998, p.164; 1998, p.167), in part because Marion cannot see room for genuine criticism from the standpoint that he ascribes to Wittgenstein, whereby mathematical propositions really are not true or false and are not properly called propositions (1998, p.168). Marion's playing down of these criticisms is also in part motivated by Marion's view that Wittgenstein aligned himself with Brouwer and Weyl in adopting a broadly constructive view of proof in mathematics. As Marion puts it:

> Wittgenstein's strong words against the language of Brouwer and Weyl are likely to confuse some into thinking that his position is radically opposed to that of the

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[121]The idea that the quantifier is taken by Brouwer 'extensionally' here seems to mean roughly that the proposition is taken as a descriptive generalisation ranging over a particular domain.

[122]E.g. (*LFM*, p. 67). The term very likely comes from Ramsey's reference to "the Bolshevik menace of Brouwer and Weyl" (1931, p. 56).

> intuitionists. But this is not the case: his criticisms of their language in no way implies the rejection of their critical standpoint. If it did, one would have to explain away, on pain of contradiction, Wittgenstein's later remarks on existence, with their clear constructivist slant and the underlying distrust of Russellian mathematical logic. (Marion 1998, p.175)

It is not a simple question as to what 'constructivism' entails, as apart from an acceptance of certain proofs as 'constructive' and a rejection of others as 'non-constructive.' Constructivity might, as Marion (1998, p.171) cites Beeson as saying, be "possessed in a greater or lesser degree" (Beeson 1993, p.139). Nonetheless, there are certain distinguishing features – non-constructive proofs are taken to show the existence of an object or entity without demonstrating its form. This might be achieved by showing that the non-existence of the entity would lead to a contradiction. Accordingly, constructivists are typically taken to reject the use of *reductio ad absurdum* and double-negation elimination in mathematics. Both principles seem to rely upon a use of excluded middle, insofar as objects are taken to either exist or not exist, and so constructivists are sceptical of excluded middle in relation to the existence of mathematical entities.

The attribution to Wittgenstein of a constructive attitude to existence proofs is related, at least for Marion, to what he takes to be Wittgenstein's view that "the essential aspect of a proof is the recipe or algorithms that it provides" (Marion 1998, p.173) – a view which Marion calls an "'intensional' standpoint" (1998, p.175). A similar view is taken by Lampert (2008) and both take this to be at the heart of Wittgenstein's critical remarks, which we will now discuss, on 'extensional' ways of expressing mathematics (which is what Marion calls "Russellian mathematical logic" in the above). Wittgenstein is seen as regarding mathematics as essentially 'intensional' in that mathematics is taken to consist exclusively of algorithmic techniques, even though mathematics is sometimes misleadingly expressed in an extensional form, in terms of the existence of objects (as is the case with expressions using quantifiers).

The contrast between 'intension' and 'extension' in Wittgenstein is not spelled out very explicitly, especially in the period 1929-33, with which Marion and Lampert are primarily concerned. The contrast connects with the idea of giving an explanation in different ways – either by example or by definition. To illustrate, I might define countries of North America intensionally by specifying them as those countries on the North American continent (perhaps giving co-ordinates), or I might give an extensional explanation by simply listing the countries by name. It might be tempting to think that Wittgenstein thinks of mathematics as exclusively intensional, with the idea of the extensional figuring only as a Platonist misreading. There are certainly remarks of Wittgenstein's, especially in the 1929-1933 period, that are suggestive of this:

> Mathematics consists entirely of calculations.

> In mathematics everything is algorithm and nothing is meaning; even when it doesn't look like that because we seem to be using words to talk about mathematical things. Even these words are used to construct an algorithm. (*PG*, p. 468)

However, remarks like this can just be read as expressions of Wittgenstein's view (explored in chapter 4) that it misrepresents mathematical propositions to regard them as descriptions. If mathematical propositions were really intensional, even when they look extensional, then this would entail something much more radical. This would seem to suggest that pieces of mathematics have to be rewritten in order to put them into a more adequate form. The extent of the rewriting would depend upon whether any general

principles were thought to be rejected because of a failure to fit the intensional mould – Marion (1998, p.168) and Lampert (2008, p.63) both take Wittgenstein to reject excluded middle, *reductio ad absurdum* and double negation elimination. Given the prevalence of these principles in mathematics, rejection of them would require that significant amounts of mathematics be rewritten. This is exactly what Lampert (2008, p.71) and Mancousu and Marion (2003, p.173) ascribe to Wittgenstein.

Mancousu and Marion are well aware that Wittgenstein claimed that his investigation would "leave mathematics as it is" (*PI* §124) but they claim that his ambitions were only "limited" (Mancousu and Marion 2003, p.181) and suggest that the promise to leave mathematics as it is just amounts to the suggestion that nothing of value would be lost in reconstructing mathematics along intensional/constructive lines (Mancousu and Marion 2003, p.173). Aside from involving a highly optimistic thesis concerning how much of classical mathematics can be constructivised, their reading clearly does not square with some of Wittgenstein's later promises concerning his philosophy. Most especially Wittgenstein's promise that his philosophy would not require delving deep into technical matters of mathematics:

> I can as a philosopher talk about mathematics because I will only deal with puzzles which arise from the words of our ordinary everyday language, such as "proof", "number", "series", "order", etc... Knowing our everyday language- this is one reason why I can talk about them. Another reason is that all the puzzles I will discuss can be exemplified by the most elementary mathematics – in calculations which we learn from ages six to fifteen, or in what we easily might have learned, for example, Cantor's proof. (*LFM*, p.14)

Constructivising mathematics is not a simple task and the adequate constructivisation of proofs has been much debated by mathematicians. It is difficult to interpret this without great strain as consistent with the promise not to "interfere with the actual use of language" (*PI*, §124) or the promise "not to interfere with the mathematicians" (*LFM*, p. 13).

A reading which is more congruous with Wittgenstein's promises can be arrived at by looking at what Wittgenstein says about how the 'intensional' and 'extensional' figure in mathematics during the period outside of the main interest of Marion and Lampert's, as Wittgenstein's clearest statements on this are from 1942-1944. Marion claims that these remarks exhibit a "clear constructivist slant" (Marion 1998, p.175) but he does not cite the following remark:

> The range of certain extensions casts a sidelight on the algebraic property of the function. In this sense, then, the drawing of a hyperbola could be said to cast a sidelight on the equation of a hyperbola.

> It is no contradiction of this for those extensions to be the most important application of the rule; for it is *one* thing to draw an ellipse, and another to construct it *by means of its equation.*-

> Suppose I were to say: extensional considerations (for example the Heine-Borel theorem) show: *This* is how to deal with intensions... To give the illustrations here will in fact be to give a procedure. (*RFM*, p.293)

The suggestion here appears to be that extensions in mathematics can be used to reveal intensions in the same way that examples of the application of a rule can be used to reveal the rule itself. The remark does not seem to question whether extensions should be allowed in mathematics. Here it seems that extensional expressions lay down mathematical rules in a different way from intensional ones, with extensional considerations in mathematics

being more indirect. For example, a non-constructive existence proof might show us the existence of an object but not show us how to construct it and so the technique in question might therefore be called 'indirect.'

It might be thought that Wittgenstein's view here is that extensional mathematical techniques alone will not suffice and that any extensional constraints have to be filled in later by intensional developments. Then perhaps one might think that Wittgenstein were saying that an extensional proof would not be valid until it were filled in later by an intensional one. But Wittgenstein explicitly rejects such a line of thinking:

> I don't need to *assert* that it must be possible to construct the n roots of equations of the n-th degree; I merely say that the proposition "this equation has n roots" hasn't *the same* meaning if I've proved it by enumerating the constructed roots as if I've proved it in a different way. If I find a formula for the roots of an equation, I've constructed a new calculus; I haven't filled in a gap in an old one.
>
> Hence it is nonsense to say that the proposition isn't proved until such a construction is produced. (*PG*, p.373)

Here the point that a proof by enumeration wouldn't have the "same meaning" as a non-constructive proof seems to also be the point that an extensional demonstration of a result is different from an intensional one. More importantly, the different approaches are said to confer a different meaning upon the proved expression – they give it a different role in the calculus. Thus an expression like "there are three consecutive 7s in the expansion of π" might be said to have a different sense depending upon whether we have a proof showing *where* the 7s occur or whether we have a proof simply showing *that* the consecutives 7s occur.[123] This sort of observation would be entirely in line with Wittgenstein's methodological aspiration to remind us of the variety of uses that might be masked by the use of a single form of expression.

Marion cites the following remark as demonstrating a constructivist slant in Wittgenstein but again the remark can be read as simply pointing to how the role played by a proposition depends upon how it is proven:

> Hence the issue whether an existence-proof which is not a construction is a real proof of existence. That is, the question arises: Do I understand the proposition 'There is . . .' when I have no possibility of finding where it exists? And here are two points of view: as an English sentence for example, I understand it, so far, that is, as I can explain it (and note how far my explanation goes). But what can I do with it? Well, not what I can do with a constructive proof. And insofar as what I can do with the proposition is the criterion of understanding it, thus far it is not clear in advance whether and to what extent I understand it. (*RFM*, p.299)

Marion ends the quotation at this point, making it tempting to read the "not clear in advance whether and to what extend I understand it" as applying to an imagined non-constructive proof. But the question-mark about our understanding is only raised in relation to the prose expression 'There is...' This is made clear by the continuation of the remark, which raises a danger in interpreting expressions using quantifiers such as the existential quantifier (which might naturally be translated 'There is...'):

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[123]Though we need not *necessarily* say that the expression would have two senses if we had both proofs as the different senses *might* amount to the same rule by having the same relations to other expressions in the calculus (*RFM*, p.189; *RFM*, p.308). Sometimes we may need another proof to show that what two proofs prove is the same proposition (*RFM*, p.191-192; *RFM*, p.368).

> The curse of the invasion of mathematics by mathematical logic is that now any proposition can be represented in a mathematical symbolism, and this makes us feel obliged to understand it.

> Although of course this method of writing is nothing but the translation of vague ordinary prose. (*RFM*, p.299)

We can readily imagine somebody rewriting "there are three consecutive 7s in the expansion of π" in logical symbolism using the existential quantifier and in that case it would be all the more tempting to assume off-hand that the expression were a mathematical expression. Wittgenstein may even have exactly this expression in mind here, since his remark that "I understand it, so far, that is, as I can explain it (and note how far my explanation goes)" is remarkably similar to (*PI*, §516).[124] If we imagine that a non-constructive proof has been given showing that a contradiction would arise if there were not three consecutive 7s in the expansion of π, then would this confer a mathematical meaning upon the expression "there are three consecutive 7s in the expansion of π"? Wittgenstein seems to be expressing a doubt that the prose translation of the non-constructive result might be misleading. The prose expression might tempt us to think that the non-constructive result is equivalent in all respects to a similar constructive result. This is what Wittgenstein urges against when he says that what we can do with the expression is "not what I can do with a constructive proof" (*RFM*, p.299). Wittgenstein revisits the same point later, making it somewhat clearer:

> A proof that shews that the pattern '777' occurs in the expansion of π, but does not shew where. Well, proved in this way this 'existential proposition' would, for certain purposes, not be a rule. But might it not serve e.g. as a means of classifying expansion rules? It would perhaps be proved in an analogous way that '777' does not occur in π^2 but it does occur in $\pi \times e$ etc. The question would simply be: is it reasonable to say of the proof concerned: it proves the existence of '777' in this expansion? This can be simply misleading. It is in fact the curse of prose, and particularly of Russell's prose, in mathematics. (*RFM*, p.407-408)

Again the point concerns only the prose translation and whether it is 'misleading', not whether the proof should be accepted as valid mathematics. That Wittgenstein was only critical of prose translations making constructive and non-constructive results appear equivalent is suggested by the following remark:

> 'Every existence proof must contain a construction of what it proves the existence of.' You can only say 'I won't call anything an "existence proof" unless it contains such a construction.' The mistake lies in pretending to possess a clear concept of existence.

> We think we can prove a something, existence, in such a way that we are then convinced of it independently of the proof. (The idea of proofs independent of each other—and so presumably independent of what is proved.)

> Really, existence is what is proved by the procedures we call 'existence proofs.' When the intuitionists and others talk about this they say: 'This state of affairs, existence, can be proved only thus and thus.' And they don't see that by saying that they have simply defined what they call existence. For it isn't at all like saying

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[124]Cited previously in this chapter.

> 'that a man is in the room can only be proved by looking inside, not by listening at the door.'

> We have no concept of existence independent of our concept of an existence proof. (*PG*, p. 374)

The parenthetical remark that warns against thinking of proofs as 'independent of what is proved' is particularly illustrative. This is exactly the point that the way in which an expression is proved is linked to the role that the expression plays and so a constructive and a non-constructive proof confer different meanings upon an expression. The mistake of "pretending to possess a clear concept of existence" is clearly an accusation against constructivists. Wittgenstein is suggesting that constructivists unnecessarily recommend that all of mathematics be redeveloped along constructive lines, when all that their insight really warrants is that attention is paid to the difference between the invocation of 'existence' in expressions demonstrated constructively and expressions demonstrated non-constructively. Constructivists are led to this mistake because they think they have discovered the true nature of mathematical 'existence' and dogmatically insist that the term can only be used as they wish to use it. But Marion misreads this remark as a mere surface criticism of the language of constructivists (such as Weyl – identified as a likely target in a similar remark (*AWL*, pp. 116–17)) and intuitionists (Brouwer being the prime example):

> Wittgenstein shared broadly the intuitionist lack of satisfaction with existence proofs. What is left for him to criticize is the language in which they couched their remarks. For him there is no concept of existence independent of particular proofs, and 'Weyl talks as though he has a clear idea of existence independent of proof.' It looks, according to Wittgenstein, as if Weyl and Brouwer are making statements 'about the natural history of proofs'—something he strongly disagrees with: 'Confusion in these matters are entirely the result of treating mathematics as a kind of natural science' (*PG*, p. 375). So the presumed intuitionist prescription 'Every existence proof must contain a construction of what it proves the existence of' must be replaced by the more appropriate statement 'I won't call anything an "existence proof" unless it contains such a construction.' (Marion 1998, p.175)

It is perhaps true that Wittgenstein considered intensional techniques to be more *characteristic* of mathematics than extensional ones. But this does not amount to a "lack of satisfaction" with non-constructive existence proofs – at least not if that is supposed to mean a rejection of non-constructive proofs as inadequate or waiting to be filled in by intensional proofs. Brouwer and Weyl's mistake is to fall into dogmatism concerning methods of proof because they take themselves to have seen the true nature of mathematical existence. Removing any claim to see into the true nature of mathematical existence does not correct Weyl and Brouwer's language. Rather, it unmasks their claim as a dogmatic one. They are expressing a rule concerning how they think mathematical proofs should be done but they are expressing it as a factual expression – a kind of insight into the nature of mathematics.

The claim which Wittgenstein attributes to Weyl and Brouwer, the claim that 'Every existence proof must contain a construction of what it proves the existence of', is a case of a metaphysical claim. The claim is presented as though it were an observation, as though the nature of existence proofs had been studied and some observations of their features had led to spotting that this pattern would hold. But thinking of existence proofs in these terms seems like it could be problematic since an empirical investigation would not tell us whether some new construction will be something that we will count as an existence proof. The claim attributed to Brouwer and Weyl thus fits the mould of what Wittgenstein calls

metaphysics – he calls it the "essential thing about metaphysics" that "it obliterates the distinction between factual and conceptual investigations" (*Z*, §458).

Rather than arriving at an insight into the nature of existence proofs, Weyl and Brouwer are to be seen as laying down a rule concerning how they will use the term 'existence proof.' They are indirectly saying 'I won't call anything an "existence proof" unless it contains such a construction.' In making this point, Wittgenstein is doing what he calls "words back from their metaphysical to their everyday use" (*PI*, §116). He is undermining the apparent air of discovery about their claims and revealing them as presenting a disguised stipulation of how they think things should be done.[125] Some of the temptation towards Weyl and Brouwer's view is undermined once it is seen a proposed rule for using an expression. We can then start to see the proposal as an object of comparison, looking at ways in which the rule fits and does not fit our usage. Wittgenstein's point that there are a family of things that we might call mathematical existence proofs thus helps to release the temptation to see all existence proofs as fitting in a single mould (and in particular they need not all fit the constructivist mould).

As discussed in chapter 1, a dogmatic thesis tries to impose a model of the use of an expression upon all of that expressions various uses. Some of those uses will not fit the model and so one either has to introduce constructions and distortions by which to make the cases fit, or else dismiss the cases as not legitimate. In giving an overview we come to see that there is no need to impose a single model and so we see that we can do away with the constructions and distortions that "always made us uneasy, those we were unable to do anything with, and which we still thought we had to respect" (Ms112, 119v)[126]. The constructivist denial of non-constructive existence proofs is such a distortion, since it arises from a dogmatic imposition of a single model. When Wittgenstein says that that "it is nonsense to say that the proposition isn't proved until such a construction is produced" (*PG*, p.373), we can see him as identifying this claim as a distortion arising from the dogmatic imposition of the model. Wittgenstein's unmasking of the metaphysics in Weyl and Brouwer's claim can thus be seen as part of a move "from a piece of disguised nonsense to something that is patent nonsense" (*PI*, §464). Whilst 'Every existence proof must contain a construction of what it proves the existence of' is seen as an expression of an insight into the 'natural history of proofs', then it looks like we have to be prepared to reject non-constructive proofs. But when it is rephrased as 'I won't call anything an "existence proof" unless it contains such a construction', then the rejection of non-constructive proofs is seen to be dogmatic and to serve no clear purpose.

Whilst Wittgenstein may have characterised the dogmatic prescriptions of Brouwer and Weyl as 'nonsense', this does not mean that he simply dismissed their positions. Rather, he takes them to have arrived at a real insight in showing that there are different senses of 'existence' in play when an expression is demonstrated constructively from non-constructively. (In this respect Wittgenstein might be taken to use their views as 'objects of comparison' by which to clarify the grammar of 'existence' and 'existence proof.') Wittgenstein also acknowledges that one can 'do more' with a constructive than a non-constructive proof (*RFM*, p.299), perhaps suggesting that Wittgenstein may have taken constructive proofs to be more important to mathematics than non-constructive ones.

It is certainly the case that Wittgenstein was interested in Constructivism, especially so in 1930 when Wittgenstein undertook a constructivisation of Euler's proof of the infinity of primes. Mancousu and Marion (2003) explore this constructivisation and use it to question the widespread ascription to Wittgenstein of a "lack of proficiency in mathematical

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[125]For more on the metaphysical use of words see Kuusela (2008, p.103-104)

[126]Cited by Kuusela (2008, p.255). His translation modified from (*PO*, p.173-175).

matters” (Mancousu and Marion 2003, p.171). But it is unclear how far their discussion can go as “evidence in support of the claim that, after all, he held a constructivist stance, at least during the transitional period of his thought (1929-1933)” (Mancousu and Marion 2003, p.171). It is perfectly possible that Wittgenstein could have been interested in developing a constructivised proof for the purpose of exploring how the meaning that the constructivised version confers on “infinity of primes” is different from the non-constructivised version. The fact that Wittgenstein constructivised Euler’s proof is only evidence of a constructivist leaning on Wittgenstein’s part if Wittgenstein could be said to have been undertaking this because he felt that Euler’s proof was not valid.

There is some evidence that Wittgenstein was unhappy with Euler’s proof in 1930, at least if Waismann recorded him correctly. In a very brief comment from the Vienna Circle discussions Euler’s proof is noted as “immediately in error” (*WVC*, p.108). But Wittgenstein’s own comments can be read as criticising only the prose translation of Euler’s proof as demonstrating that “there are infinitely many prime numbers.” Wittgenstein asks “Euler’s proof that “there are infinitely many prime numbers” is supposed to be an existence proof, and how is such a proof possible without a construction?” (BT, 434e[127]). Certainly he is questioning (one might better say ‘investigating’) whether the proof deserves to be called an existence proof but he is not saying definitively that it does not.

Wittgenstein does use the very critical-sounding expression “proof by circumstantial evidence”, which he says is something that is inappropriate to mathematics and “absolutely never permitted” (*PG*, p.384). But here he may just be talking about the interpretation of the proof as establishing the prose expression, since he says that the “connection between the symptom and what we would like to have proved is a *loose* connection” (*PG*, p.384). If Euler’s proof is misleadingly interpreted as a proof of the prose expression, then it becomes equated with a constructive proof since both are seen as establishing exactly the same result. Then the proof does look like a “proof by circumstantial evidence”, since it then looks like an inferior version of the constructive proof. But this appearance is a result of taking two different proofs to establish the same proposition, as though the proposition had a determinate meaning prior to the proofs. Wittgenstein’s parenthetical warning that “mathematics is dressed up in false interpretations” (*PG*, p.385) seems to point to the prose as the source of the problem.

The following remark strongly suggests that Wittgenstein’s concern with Euler’s proof was a question of how its result is expressed in prose and the danger of taking a proposition to be independent of its proof (i.e. taking propositions to have the same meaning independently of whether they are proven constructively or non-constructively):

> A mathematical question must be no less exact than a mathematical proposition. You can see the misleading way in which the mode of expression of word-language represents the sense of mathematical propositions if you call to mind the multiplicity of a mathematical proof and consider that the proof belongs to the *sense* of the proved proposition, i.e. determines that sense. It isn't something that brings it about that we believe a particular proposition, but something that shows us what

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[127] The quotation-marks around the prose expression are omitted from the shortened *PG* version (PG, p.383), presumably removed by the editor. The quotation-marks seem to me to be important as they identify the expression as prose.

we believe - if we can talk of believing here at all. In mathematics there are concept words: cardinal number, prime number, etc. That is why it seems to make sense straight off if we ask "how many prime numbers are there?" (Human beings believe, if only they hear words...) In reality this combination of words is so far nonsense; until it's given a special syntax. Look at the proof "that there are infinitely many primes," and then at the question that it appears to answer. The result of an intricate proof can have a simple verbal expression only if the system of expressions to which this expression belongs has a multiplicity corresponding to a system of such proofs. (*PG*, p.375)

This passage can be difficult to understand if we read Wittgenstein as suggesting that Euler's proof might a proof of anything other than 'the infinity of primes.' Yet Wittgenstein does not have to be read as recommending an alternative prose translation so much as urging that we are not misled by the translation of the result as "there are infinitely many prime numbers." If we want to see what the proof establishes, he might have said that the proof sufficed to make us give up looking for a largest prime number (*WVC*, p.136). Just because a particular prose translation is misleading does not necessarily mean that there is a better prose translation available. Sometimes we need to be shown that "we can drop the explanation altogether" (*PG*, p.422). Insofar as Euler's proof shows that "that there are infinitely many primes" then this prose expression does not have the same meaning as it would if it were used in reference to a constructive proof. Wittgenstein can be seen to be making the point that that distinctions involved in the mathematics are finer than this prose translation reflects.

Brouwer and Weyl fail to "leave mathematics as it is" (*PI*, §124) insofar as they dogmatically impose a single model upon the term 'existence proof.' Whilst their rejection of non-constructive proofs may be dogmatic, it can be used to point the way to an insight concerning the varieties of mathematical proofs and how we treat mathematical results in prose. Whilst non-constructive proofs are valid proofs in their own right, even without constructive support (*PG*, p.373), they are different from constructive proofs and thus a non-constructive result is not necessarily the same result as a constructive one. Though we may sometimes translate a constructive and a non-constructive result by the same prose expression, this translation masks a mathematical difference in that what we can do with a non-constructive result is not what we can do with a constructive one (*RFM*, p.299). In this way Wittgenstein does not simply dismiss Brouwer and Weyl's dogmatic position and he instead uses their point of view as an object of comparison[128] with which to further explore the uses of the term 'existence proof.'

6.5. Chapter Conclusion

Following on from the last chapter, distortions resulting from seeing mathematical propositions as descriptions were further explored in relation to proof with Wittgenstein suggesting that it gives a distorted picture of the role of proofs to see mathematical propositions as verified in the manner of empirical descriptions. If that were the case then mathematical propositions would have a meaning independent of their proof, whereas a proof assigns a meaning to a mathematical proposition. Wittgenstein is careful not to

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[128]For more on objects of comparision see section 1.3.

overplay this observation on the roles of mathematical propositions, stressing that conjectures are not meaningless because we lack proofs for them. Rather, we typically have some idea of how to approach a conjecture and its relation to other propositions and in this regard we are not in the dark concerning conjectures. The notion that mathematical proofs are pictures of experiments is employed as a further object of comparison, enabling us to see the procedures involved in mathematics as being followed in an experimental fashion (i.e. without knowledge of the result) up until the procedure is seen to yield its result necessarily and so become a proof.

Whilst proofs may assign meanings to mathematical propositions, Wittgenstein is careful not to be dogmatic about how meanings are assigned. Neither conjectures nor false propositions are said to be meaningless and Wittgenstein criticises Weyl for suggesting that false propositions lack meaning (*PR*, p. 170). Wittgenstein draws an insight from Weyl in that he takes different proof-techniques to assign meanings in different ways and acknowledges that one cannot do with a non-constructive proof what one can do with a constructive proof. But Wittgenstein refuses to take this to indicate a deficiency in non-constructive proofs and instead takes the value of this insight to lie in pointing to the ways that apparently the same expression can actually have quite different uses, depending upon how it is proven.

Chapter 7 - The Status of Contradictions and Leaving Mathematics as it is

7.1. Chapter Introduction

When Wittgenstein in *PI* says that philosophy "leaves mathematics as it is" (*PI*, §124), the very next remark that he gives relates to the significance of contradictions and it seems that Wittgenstein may have taken this topic as offering an especially important application of his method (*WVC*, p.149). Contrary to widely-accepted interpretations, Wittgenstein did not dismiss sceptical concerns about the appearance of contradictions in mathematics by saying that we could deal with *all* contradictions "simply by refusing to draw any conclusions from a contradiction" (Potter 2011, p.131). Nor was Wittgenstein's view that *all* contradictions could or even should be easy to deal with by the addition of new rules (Potter 2011, p.129-130; Chihara 1977, p.370). Such an approach might work in certain cases and emphasising the variety of different cases can be seen to be Wittgenstein's point. Wittgenstein unmasks a mechanical picture of mathematical systems and suggests that this mechanical picture hides the variety of contradictions. Many contradictions arise in mathematical systems in ways which are not at the core of the applications of those systems and contradictions like this can typically be easily dealt with by a change to the system's axioms. Some contradictions can arise at the core and then we may not know how to correct the system and so it then becomes useless to us. This variety in the status of contradictions is masked by the mechanical picture, making it appear as though any contradiction could bring the whole system down and so creating a sceptical concern about the emergence of contradictions.

7.2. Wittgenstein's thinking on Contradictions

When Wittgenstein in *PI* says that philosophy "leaves mathematics as it is" (*PI*, §124), the very next remark that he gives relates to the significance of contradictions:

> It is the business of philosophy, not to resolve a contradiction by means of a mathematical or logico-mathematical discovery, but to make it possible for us to get a clear view of the state of mathematics that troubles us: the state of affairs *before* the contradiction is resolved.
>
> (And this does not mean that one is sidestepping a difficulty.)
>
> The fundamental fact here is that we lay down rules, a technique, for a game, and that then when we follow the rules, things do not turn out as we had assumed. That we are therefore as it were entangled in our own rules.
>
> This entanglement in our rules is what we want to understand (i.e. get a clear view of).
>
> It throws light on our concept of *meaning* something. For in those cases things turn out otherwise than we had meant, foreseen. That is just what we say when, for example, a contradiction appears: "I didn't mean it like that."

> The civil status of a contradiction, or its status in civil life: there is the philosophical problem. (*PI*, §125)

Wittgenstein perhaps took the problem relating to contradictions as offering an important illustration of what his method can achieve – namely, the disappearance of sceptical concerns by means of philosophical clarification (*WVC*, p.149). The problem might also be seen by Wittgenstein as an important one in the sense of being a meeting-place for a number of different confusions. Several lectures in *LFM* are devoted to it (*LFM*, p.185-230) and at one point Wittgenstein declares that the discussion has gotten them into a "mess" which getting out of will be like "unravelling a ball of wool" (*LFM*, p.220). At least one of the strands of this ball of wool runs into the temptation to give an account of mathematical inference, which we discussed in chapter 3.

We saw in chapter 3 that it can be tempting to try to give an account of mathematical inference as a kind of receptivity to mathematical facts (Floyd 1991, p.165), where a mathematical fact is construed as "some truth corresponding to" (*RFM*, p.37) and separate from the use of a mathematical expression. When one subscribes to a picture such as this then one will be inclined to say that "it already follows-- all I still have to do is, to infer it" (*RFM*, p.45). This model of mathematical inference is "superfluous" (Floyd 1991, p.161) in that any entity standing apart from the expression itself would 'stand there like a sign-post' (*PI*, §85) and still leave me requiring to make an inference.

The idea of a meaning as an object that stands apart from my words figures importantly in Wittgenstein's discussion of contradictions. If meanings are treated as entities separate from our use of expressions then a contradictory expression seems as though it describes a thing – a strange kind of entity or fact:

> The idea is that when I give you an order, there are the words- then something else, the sense of the words- then your action. And so with "sit and don't sit", it is supposed that besides the words and what he does, there is also the sense of the contradiction – that something which he can't obey. (*LFM*, p.185)

If the contradiction is seen as a thing which is expressed by a contradictory form of expression, then one is inclined to say that the contradiction 'jams.' We get a picture of a contradiction as a flaw in a mechanism and Wittgenstein wants to urge that we resist this picture:

> We most naturally compare a contradiction to something which jams. I would say that anything which we give and conceive to be an explanation of why a contradiction does not work is always just another way of saying that we do not want it to work. (*LFM*, p.187)

Explanations of 'why a contradiction does not work' are superfluous in much the way that explanations of why inference works are superfluous – they commit us to a picture which does not enlighten us with regard to particular cases and they provide us with no reassurance with regard to our ordinary practice. The picture of a contradiction as connected with a logical "mechanism" is said to be an "extremely misleading one" (*LFM*, p.190) and one misleading aspect of this picture is that it distorts the grammar of necessary propositions. It makes it seem like the necessity of necessary propositions were a feature of the proposition or its subject-matter rather than the way that the proposition is used:

> When we think of a logical machinery explaining logical necessity, then we have a peculiar idea of the parts of the logical machinery – an idea which makes logical necessity much more necessary than other kinds of necessity. If we were comparing the logical machinery with the machinery of a watch, one might say that the logical

> machinery is made of parts which cannot be bent. They are made of infinitely hard material – and so one gets an infinitely hard necessity. (*LFM*, p.196)

This is presumably why Wittgenstein says that the problem of getting clear about the status of contradictions "throws light on our concept of *meaning* something" (*PI*, §125) – the problem is connected with the misleading ideas that meanings are things and that inferences are made for us (by the meanings) before we actually make them.[129] If we subscribe to this picture then we will be inclined to think that contradictions are things that might be hidden away, either in the mathematical realm or in the super-rigid mechanism, waiting for us to discover them.

With regard to contradictions in particular (as opposed to mathematical propositions in general) this is a misleading picture because it makes it seem as though the expression of a contradiction were a meaningful proposition like any other but with a peculiar subject-matter. This misleading picture is connected with a picture of contradictory orders as orders that we cannot carry out (*LFM*, p.206). We say that we cannot carry out a contradictory order but the meaning of this is not the same as saying that we cannot carry out the order to, say, move a mountain. We cannot carry out the order to 'both sit and not sit' but this is because the order itself is not clear. It is an important part of the grammar of contradictions[130] that they have not been assigned a meaning in the way that more paradigmatic cases of propositions have. Expressions of contradictions are expressions that we "do away with" (*LFM*, p.206) because their connection to other techniques is such that the expression itself "is of no use" (*LFM*, p.207). We are inclined to form contradictory expressions like "I am lying" because we also have related expressions like "I am eating" (*LFM*, p.208) – the grammar that gives us the useful expression "I am eating" also suggests the form "I am lying", but the contradictory expression is not one that we can make use of in the same way as "I am eating." Contradictions in themselves are not useful but they are not entirely meaningless as they are connected with other propositions and as such they are limiting cases of propositions - Wittgenstein says "is this a statement or isn't it? I'd say: I don't know; call it what you like" (*LFM*, p.208).[131]

The picture of the contradiction as a thing denoted by a contradictory expression can lead us to overlook this connection between contradictions and the useful expressions that they indirectly relate to (the relationship between 'I am lying' and 'I am eating'). The picture also leads to another misrepresentation of contradictions as it makes it seem as though all contradictions were equally damaging. If a mathematical calculus were a kind of machine (*LFM*, p.196), a machine whose workings were described by our expressions, then perhaps a single jam anywhere could bring the whole thing to a halt. Wittgenstein acknowledges that some contradictions are such that their discovery renders the calculus of "no use to calculate with" (*LFM*, p.228) but he denies that this is the case with all contradictions. It depends upon the way that the contradiction arises and its position in the calculus.

If the contradiction is intimately connected with the way in which the calculus is used, then the contradiction might well be difficult to deal with and would render the calculus of no use. But this is only the worst kind of case – the case where the contradiction arises on "a thoroughfare" for the calculus (*LFM*, p.227). But if the contradiction is relatively isolated then its discovery need not lead us to doubt the whole calculus – in that case we can typically prevent any problems arising by adding axioms to prevent the unwanted consequences. In order to be able to add axioms that prevent the system from becoming unusable, we have to be able to retain a view of the system *as usable*. When we find the contradiction we are "entangled in our own rules" (*PI*, §125) and disentangling ourselves

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[129]For more on this see chapter 3.

[130]By which I mean that it is a central characteristic of their role or 'civil status' (*PI*, §125).

[131]See also (*PG*, p.317).

means finding a way to change the calculus to make it a calculus with which we could do the things that we thought we could do with it before we found the contradiction:

> Can we say: 'Contradiction is harmless if it can be sealed off'? But what prevents us from sealing it off? That we do not know our way about in the calculus. Then that is the harm. And this is what one means when one says: the contradiction indicates that there is something wrong about our calculus. It is merely the (local) symptom of a sickness of the whole body. But the body is only sick when we do not know our way about.
>
> The calculus has a secret sickness, means: What we have got is, as it is, not a calculus, and we do not know our way about - i.e., cannot give a calculus which corresponds 'in essentials' to this simulacrum of a calculus, and only excludes what is wrong in it. (*RFM*, p.209)

Here Wittgenstein compares the contradiction to a "symptom of a sickness" but his urging that the "body is only sick when we do not know our way about" is important. If we think that all contradictions might lead to this kind of sickness, then we might well become concerned not just about actual contradictions encountered but with the mere possibility of contradictions out there in the calculus somewhere, as though we had to rule out any possibility of contradictions arising before we could be sure of using the calculus.

Wittgenstein makes the point that it only makes sense to speak of something as 'hidden' if one has a conception of what it would mean to uncover what is hidden. For example, we can speak of somebody as being hidden behind a chair but we would not know what was meant if somebody were said to be hidden in a small room with no furniture such that we could see the whole room. In that case the worry about somebody being 'hidden' might be called "hysterical" (*LFM*, p.225). If one is driven to speak of the possibility of contradictions hidden in a simple calculus that we can survey entirely then that would be using the term 'hidden' in a 'hysterical' sense. Wittgenstein makes this point in the following remark:

> In a system with a clearly set out grammar there are no hidden contradictions, because such a system must include the rule which makes the contradiction discernible. A contradiction can only be hidden in the sense that it is in the higgledy-piggledy zone of the rules, in the unorganized part of the grammar; and there it doesn't matter since it can be removed by organizing the grammar. (*PG*, p.305)

Wittgenstein's unusual phrasing can make this remark difficult to follow. The expression "higgledy-piggledy" is oddly conversational and risks making mathematics sound like a more casual business than it is. Indeed, this is perhaps the most serious objection raised by Wittgenstein's critics – namely the claim that Wittgenstein trivialises serious mathematics. As I have already indicated already, I want to argue that Wittgenstein's target of criticism is not serious mathematics (or any kind of mathematics) and instead his target is a certain philosophical picture of contradictions as flaws in a mechanism. In order to make this case I need to address certain key criticisms.

7.3. Defending Wittgenstein on Contradictions

To start to see how some interpreters have seen a trivialising of mathematics in Wittgenstein's remarks, it is instructive to note what Michael Potter says of "higgledy-piggledy" remark quoted above. Potter calls the remark "very strange indeed" and he comments that "Wittgenstein seems to have imagined that if it is uncertain whether a

system is consistent, that can only be because the system has not been set out with sufficient clarity" and he says that this is "simply false" (Potter 2011, p.129-130). Under Potter's reading (which we shall now consider), Wittgenstein trivialises mathematical concerns to the point of not understanding how difficult serious mathematics is.

Potter thinks that Wittgenstein divides contradictions only into the categories "hidden" and "obvious" (Potter 2011, p.129) and Potter thinks that Wittgenstein does admit that "we should check for obvious inconsistencies" (2011, p.129). But obvious inconsistencies are presumably easy to deal with. So it is only hidden contradictions which might present a mathematician with a serious issue and Potter reads (*PG*, p.305) as saying that hidden contradictions can only arise because of an "ambiguity in the rules" (Potter 2011, p.129) arising "because the system has not been set out with sufficient clarity" (2011, p.130). Potter rightly says that this is "simply false" (p.130) and points to a formal system defined by these rules as a counter-example (2011, p.130):

> 1. You may write down any formula which it would be legitimate to write down in Peano Arithmetic.

> 2. You may not write down any formula which contradicts what you have already written down.

We have no "mechanical means" of deciding whether Peano Arithmetic is consistent[132] and so we do not know whether this system is consistent either. Thus we do not know whether the system contains any contradictions. This makes it seem as though Wittgenstein is guilty of underestimating the complexity of mathematical systems. It makes it seem as though Wittgenstein assumed that we would always be able to do a run-through of the rules and immediately see whether any of them were in conflict. If we can't easily do a run-through of the rules then, according to the view being attributed to Wittgenstein, we should just restructure the rules to make a run-through possible. If we can't do this then we've created a "higgledy-piggledy" and the possibility of contradictions arising is then our own fault (2011, p.130).

Wittgenstein's remark is certainly rather curt and it is understandable that Potter should find it "very strange" (2011, p.129). Nonetheless, I want to suggest that a more reasonable point can be seen in it than Potter allows for. When Wittgenstein says that in "a system with a clearly set out grammar there are no hidden contradictions" (*PG*, p.305), he is undoubtedly thinking of a very simple kind of system which most real mathematical systems will not match up to. But is the latter part of the remark really saying that it will always be easy to restructure the grammar of a system to make it such that we could easily check its rules? If so then he would seem to be in conflict with what he says later to his lecture audience:

> I don't say that a contradiction may not get you into trouble. Of course it may. (*LFM*, p.219)

It is quite possible that Wittgenstein's views changed on this but it need not necessarily be so. The relevance of the 'higgledy-piggledy' can be better understood in relation to the idea from the lectures of a calculus having certain "thoroughfares" (*LFM*, p.227). In both the lectures (*LFM*, p.227) and the *PG* remarks (*PG*, p.304) Wittgenstein emphasises the question of whether the calculus can be used to perform the particular purposes that we have in mind for it. A contradiction is only of the kind that get us into trouble if we can no

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[132]Potter appears to be invoking Gödel's second incompleteness theorem as showing that a proof of the consistency of Peano Arithmetic cannot be formalised within Peano Arithmetic itself. Though for the principled point it does not really matter whether a proof is possible so much as whether the consistency of the system is evident.

longer see how to restructure the calculus to allow us to use it for what we want to use it for. When Wittgenstein talks of the 'higgledy-piggledy', we can take him to be gesturing at how all of the implications of the rules and how they relate to one another might not have been worked out yet. Wittgenstein is presumably thinking of a mathematical system still under development, for which we are still deriving new rules. The 'higgledy-piggledy' is then a name for a space of derivations which have not yet been explicitly made. The 'higgledy-piggledy' space is still fluid, in that we have not yet worked out how the rules relate to one another in the higgledy-piggledy. One would expect all of the key applications of a calculus to arise within the space that is worked out at its conception, so the fluidity of the higgledy-piggledy suggests that a contradiction found there will not necessitate a radical reworking of the calculus. Given that the calculus is already in use and working as intended, a contradiction arising it he 'higgledy-piggledy' is unlikely to affect a 'thoroughfare' and so it is likely that it will be sufficiently isolated that we should be able to seal it off.

Whilst Wittgenstein does emphasise a difference between contradictions which are 'hidden' and those which are in view, the point is not that any contradiction is always easy to deal with as soon as it is out in the open. If the contradiction is on a "thoroughfare" (*LFM*, p.227) and is directly connected with the way that the calculus is used then eliminating it may be very difficult to do. We may then be in a situation in which we no longer "know our way about in the calculus" and we cannot see how to seal off the contradiction (*RFM*, p.209). Wittgenstein is sensitive to there being a whole spectrum of kinds of contradictions in terms of the difficulty that might be involved in salvaging the system (if indeed it is salvageable). Rather than subscribing to a simple division of contradictions into the "hidden" and the "obvious" (Potter 2011, p.129), Wittgenstein can be read as trying to remind us of the immense variety of contradictions. Rather than every contradiction being a 'jam' (*LFM*, p.187) which might bring the system to a halt, each contradiction arises in a particular way and needs to be dealt with in a particular way.

Wittgenstein was aware that his discussions of contradiction ran the risk of "meddling with the mathematicians" (*LFM*, p.223). In discussing the "civil status" of contradictions (*PI*, §125), he had to be careful to avoid making false generalisations about how contradictions arise or how they can be dealt with. If Wittgenstein were saying that all contradictions were either "hidden" or "obvious" (Potter 2011, p.129) then Wittgenstein would be guilty of meddling with the mathematicians – he would be effectively saying that no genuinely troubling contradictions should ever emerge. The delicacy and the novelty of the point that Wittgenstein is trying to make can mean that it is very difficult for interpreters to avoid ascribing a claim to Wittgenstein which would have him "meddling with the mathematicians" (*LFM*, p.223). Perhaps the most influential one has been the claim that Wittgenstein subscribed to a "misconception that we can repair a contradictory system simply by refusing to draw any conclusions from a contradiction" (Potter 2011, p.131), which is a charge suggested by a remark of Turing's and which is taken up by Chihara (1977, p.371).

Wittgenstein does at points suggest that a contradiction might be dealt with by making it a rule that we won't draw any conclusions from the contradiction (*LFM*, p.220). Turing makes the point that one wouldn't need to go through the contradiction to get "any conclusion which one liked" (*LFM*, p.220). But it is not clear that Wittgenstein meant this as a generalisation – the generalisation that one could *always* deal with a contradiction by making it a rule not to draw any conclusions from it. The remark can instead be taken to only relate to *particular* contradictions of the kind that he takes to be least threatening.[133] If

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[133]The example he mentions is a contradiction of the form "p and not-p" arising from Frege's system (*LFM*, p.220). Returning to the question later, Wittgenstein says that one could use Frege's calculus to do some basic counting (though not as a foundation for

Wittgenstein did think that all contradictions could be dealt with in this way then it would be rather odd for him to bother stressing a difference between contradictions which rendered a calculus useless and those which do not, since on such a view no contradiction would be problematic. To emphasise a point already made previously, Wittgenstein explicitly disavows such a simplification of his view[134]:

> I don't say that a contradiction may not get you into trouble. Of course it may. (*LFM*, p.219)

It is not clear that (as Chihara (1977, p.372) and Potter (2011, p.131) allege) Wittgenstein misses Turing's point that one need not go through the contradiction in order to derive falsehoods. Shanker (1987, p.140-145) points out that Waismann had already raised exactly this point to Wittgenstein years before and Wittgenstein's response is that one does not simply ignore the contradiction[135] but introduce axioms to exclude the contradiction explicitly.[136] By explicitly excluding the contradiction, one has restructured the calculus so that the contradiction does not arise. For more difficult cases we might not be able to see any way to do this and then we are thoroughly "entangled in our own rules" (*PI*, §125). We find that "we just don't know which things to eliminate and which not- then the calculus is no use to calculate with" (*LFM*, p.228).

Potter thinks that Wittgenstein must have missed Turing's point that one wouldn't need to go through the contradiction to get "any conclusion which one liked" because Wittgenstein responds that "we must continue the discussion next time" (*LFM*, p.220) and of the next lecture Potter thinks that it is "hard to see any of what he said as really answering Turing's objection" (2011, p.131). What Wittgenstein does in the next lecture is present by means of analogies the idea that a calculus has key "thoroughfares" (*LFM*, p.227) and lesser trodden paths, with greater and lesser connections with the rest of the calculus and its applications. Whilst this point does not directly address Turing's criticism it can be seen to put Wittgenstein in a position to respond to Turing. Depending upon the way that the contradiction arises and the way that we want to use the system, it may be that we can deal with contradiction very simply. If the system still allows us to do what we want to do with it and the contradiction emerges in an isolated way, then it may be viable to simply avoid the contradiction, physicists sometimes do (Ramharter 2010, p.294). But we will most likely want to restructure the system so that the contradiction does not emerge. This may be a simple task that just requires a small modification to an axiom, or it may require significant changes or it may even be that we end up having to give up on the system. Wittgenstein's point is not that one won't ever get into trouble with contradictions but to question the idea that "with contradictions one *must* get into trouble" (*LFM*, p.219).

Turing's point is that one can't simply seal off a contradiction by refusing to use it. One does not need to go directly through the contradiction in order to get into trouble as one might get into trouble with it indirectly. Wittgenstein's response indicates that what matters is the kind of trouble that one gets in. Physicists are able to avoid getting into

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mathematics (*LFM*, p.228)) and "it would not necessarily be detrimental if there were a contradiction in it" (*LFM*, p.227).

[134]This is perhaps what he means in (*PI*, §125) about not "sidestepping a difficulty".

[135]But note that physicists really do sometimes ignore contradictions (Ramharter 2010, p.294).

[136]I omit the details of Shanker's exploration of how a system can be restructured to prevent a contradiction arising (1987, p.140-145). The idea that such a restructuring can be done should be familiar enough.

trouble as a result of letting contradictions stand since the contradictions that they let stand are isolated and do not prevent them from using the systems in question in the ways that they want to use them. So long as the system can be used for what it was intended for, the possibility of a contradiction need not be troubling for them. Turing is quite right that Wittgenstein's suggestion that one "don't draw any conclusions from a contradiction" (*LFM*, p.220) won't fully seal off the contradiction in the sense of preventing falsehoods being derived from it indirectly. But the criticism does not touch Wittgenstein's point since Wittgenstein is only saying that there are cases where the possibility of deriving falsehoods need not trouble us. In the cases where it does trouble us, we should of course want to restructure the system to prevent the contradiction.

Chihara thinks that Wittgenstein "failed to grasp" (1977, p.372) Turing's "main point" (1977, p.373) that a contradiction can be such that a simple and isolated stipulation is not enough to remove it. But when Turing gives an example of how the discovery of one contradiction can lead to others (*LFM*, p.227) Wittgenstein does accept that there are cases like this and only denies that this was the case with the Russell contradiction found in Frege's system.[137] Wittgenstein says that Russell's contradiction was isolated enough that it could be and was sealed off by Russell (*LFM*, p.229) but one could use it to perform basic counting even with the contradiction in it (*LFM*, p.227). Turing's response does indicate that the two disagreed on how to interpret the Russell contradiction as Turing indicates that he would want to see the contradiction removed even before using Frege's system for some basic counting:

> *Turing*: If one eliminated the contradiction, then it would be all right. But if one simply avoids what feels fishy, then I would say that the contradiction did vitiate it. (*LFM*, p.229)

Thus it seems that Turing and Wittgenstein do not reach agreement about how to interpret this particular case. But Turing does not say that contradictions will *always* need to be eliminated and nor does he object to talking about 'thoroughfares' and 'circuses' in a calculus - actually he engages with this way of talking (*LFM*, p.227). Nor does Wittgenstein dispute that there are cases like Turing's where one contradiction leads to others (*LFM*, p.227). The two seem to be broadly agreeing on the wider question of whether there are a range of cases of kinds of contradiction. Turing seems to be prepared to follow Wittgenstein's broader point that contradictions are not all equally damaging and the key question for how damaging they are is how they are bound up with applications of the system. As Wittgenstein says, the two "agree that the point of avoiding a contradiction is not to avoid a peculiar untruth about logical matters – to avoid getting to that place from which you can go in every direction" since then we "might forfeit the point of our calculus" (*LFM*, p.224).

It makes perfect sense that the discussion of contradictions ends with the two disagreeing about the interpretation of Frege's system, since by that point the question of how to interpret the contradiction in Frege's system is not especially important. The main purpose for Wittgenstein appears to have been to undermine the picture of a contradiction as "a peculiar untruth about logical matters" (*LFM*, p.224) because this picture leads seeing all contradictions as 'jams' (*LFM*, p.187) that might bring down the whole system. Wittgenstein wants to remind us that a contradiction has a "civil status" (*PI*, §125) − a

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[137]He says "Yes; but it does not apply to Frege's logic" (*LFM*, p.227).

position within the calculus, whether that be on a "thoroughfare" (*LFM*, p.227) or a side-street. If Wittgenstein were saying that all contradictions could be easily dealt with or worked around (Potter 2011, p.131; Chihara 1977, p.370) then he would be making a dogmatic claim which many cases of contradictions would easily be seen to not fit. But instead Wittgenstein can be seen, as I have argued for seeing him, as providing an overview[138] of sorts by reminding us of the variety of possible significance that a contradiction might have.

7.4. Chapter Conclusion

A picture of mathematical propositions as descriptions of a mathematical reality (*LFM*, p.185) is linked to a picture of mathematical systems as mechanical systems (*LFM*, p.196). Under the mechanical picture, a contradiction looks like a flaw in the mechanism – a malfunctioning part which causes the mechanism to 'jam.' Any contradiction would then appear to be able to bring the whole system down and so the picture naturally leads to a sceptical concern about the possibility of contradictions. Wittgenstein emphasises that the concern can be alleviated by releasing the grip of the picture by seeing mathematical systems as intimately connected with applications. By looking at the variety of ways in which a contradiction may be connected with a calculus and its applications, we can come to see that not all contradictions are equally threatening. Only some contradictions arise in relation to the central parts of a calculus, the 'thoroughfares' on which the applications of the calculus most crucially hang. If a contradiction arises on such a thoroughfare then we might well have to significantly alter or even discard the calculus but not all contradictions are like this. Many contradictions will arise in the fringes of a mathematical system as the system is further developed and in these cases it is more likely that new axioms can be added to prevent contradictions from arising, since the fringes have fewer connections within the calculus than the thoroughfares and thus the problems are easier to isolate.

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[138]For more discussion of overviews see section 1.3.

Part 4 – Separating Mathematics from Philosophy

Chapter 8 - Frege and Russell's foundations for arithmetic: 'for us a problem of mathematics like any other'

8.1. Chapter Introduction

Part 4 concerns Wittgenstein's idea that "a 'leading problem of mathematical logic' is for us a problem of mathematics like any other" (*PI*, §124). Wittgenstein wanted to carefully distinguish philosophical interpretations of mathematical systems from statements of mathematics and he felt that doing so would dissolve the apparent significance that certain mathematical systems might otherwise seem to have (*WVC*, p.149).[139] He felt that an attachment to particular philosophical pictures could cause certain mathematical pursuits to take on a significance that arose not from any possible application of the system but rather from a philosophical concern.

Wittgenstein's determination to undermine philosophical interpretations of mathematical systems has struck some commentators as bizarre. Commentators have perhaps struggled to see why Wittgenstein would even be interested in showing pieces of mathematics to be less significant than commonly thought. It can be tempting to take this, as Putnam does, as evidence of an inadequate respect for science (2007, p. 246). But this is a strange thing to attribute to a man who once remarked that "great work of the modern mathematical logicians" had "brought about an advance in Logic comparable only to that which made Astronomy out of Astrology, and Chemistry out of Alchemy" (Wittgenstein 1913, p.1).[140] It can be seen that Wittgenstein's criticisms of certain views of the significance of mathematical systems are evidence of quite the opposite of an inadequate respect for science. Rather, Wittgenstein wants us to see mathematics for what it is and not to see it through the distorted lens of some particular philosophy. Given the criticism that Wittgenstein has received, this is a controversial claim. In order to defend this claim it is necessary to cover some of the most severely criticised topics from his writing. So the selection of topics is intended both to illustrate Wittgenstein's manner of undermining philosophical interpretations of the significance of mathematical systems and also to provide sufficient defence that a reading of Wittgenstein as true to his methodological promises can be seen to be credible.

I will begin with Wittgenstein's treatment of Frege and Russell's view of their logical systems as offering suitable analyses of the notion of 'natural number', thus taking their

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[139] Whilst the origin of the term 'leading problem of mathematical logic' is Ramsey's example of the decision problem (1987, p.2) Wittgenstein uses the term to include problems that might be thought to "bring to light essential truths about mathematics" or "give us the right to do arithmetic as we do" (*PG*, p.196).

[140] Cited by Floyd (2001, p.289).

systems to resolve a 'leading problem of mathematical logic' (i.e. the problem of defining the natural numbers). Wittgenstein aims to undermine their interpretations of the significance of their systems by showing that the notion of natural number arising from their systems would not be one which we could work with in practice without presupposing our everyday notion of natural number. Whilst Wittgenstein's remarks have sometimes been read as dismissive, I will argue that Wittgenstein is sensitive to the nuanced interpretations that Frege and Russell have of their projects. Instead of reading Wittgenstein as trying to offer a definitive refutation of Frege and Russell, I will read him as holding their interpretations of their systems up against possible uses of their systems and as putting forward an alternative interpretation of their systems as "frills tacked on to the arithmetical calculus" (*RFM*, p.146), useful for certain limited purposes but much less useful than arithmetic itself (*LFM*, p.228-229). Wittgenstein's remarks can thus be seen as criticising Frege and Russell's interpretations because those interpretations undervalue ordinary arithmetic.

8.2. Wittgenstein and Logicist Foundational Projects

When Wittgenstein criticises the logicist programme he employs his characteristic approach by not criticising the logicist systems themselves and instead assessing certain views of the *interest* or *philosophical significance* of those systems (*RFM*, p.143). This is part of the broader theme within Wittgenstein's philosophy of separating the mathematical (or in this case logico-mathematical) out from the philosophical (*PI* §124; *WVC*, p.149), a task which is particularly difficult to do with regard to logicism.[141] Wittgenstein takes on two pictures of the significance of the logicist systems, pictures which we can find in Russell and in Frege. One such picture is the picture of logicist systems as offering an analysis of the meaning of the terms of ordinary arithmetic (as they are used in arithmetic propositions and within empirical ascriptions of number). Another picture is of the logicist calculus as a justificatory foundation, upon which the validity of the propositions of pure arithmetic (such as "2+5=7") might be said to rest. Wittgenstein undermines both of these interpretations by arguing that the logicist systems cannot successfully account for either pure or applied arithmetic (an example of an applied statement being "there are 5 red apples"). The logicists presume a certain outline for how accounts of pure and applied mathematics can be provided and they fail to see that such an approach would trade off a tacit use of our ordinary number concepts. Once this tacit use is exposed then it is revealed that the proposed analysis is not adequate to replace our ordinary arithmetic concepts, nor can it plausibly act as a justificatory foundation of the propositions of pure arithmetic.[142] The logicist analyses of arithmetical expressions may be useful for certain mathematical purposes but any claim to philosophical significance is severely undermined by the failure to account for pure or applied arithmetic.

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[141] A task which Wittgenstein takes on by subjecting the Logicist systems themselves to close scrutiny (*PG*, p.367) so as to compare the claims made of the significance of these systems with what the systems actually do.

[142] One might say that they were foundational only in the non-justificatory sense of MacLane (1986, p.406), where a foundational system is only a way of organising different branches of mathematics. We might say on Wittgenstein's behalf that such an organization could be achieved by using the foundational system to create simulations of other systems.

It has sometimes been assumed that Wittgenstein's central argument against the logicist interpretation of their systems is constructivist or contains constructive assumptions about mathematics.[143] I will instead explain Wittgenstein's argument as embodying a view that our ordinary number concepts as we use them in pure and applied mathematics are richer than can plausibly be accounted for in terms of a notion of correspondence alone. The logicist programmes attempt to do more with less – they attempt to improve our ordinary arithmetic practices (insofar as they attempt to clarify the concept of number) and to do so with a smaller range of symbols and techniques than arithmetic uses. In an important sense, the logicist programmes underestimate arithmetic, both as it figures in pure mathematics and in empirical contexts. The overriding theme of Wittgenstein's criticism of the logicists' interpretations, then, is that he takes it as reductive with respect to arithmetic.

Rather than tarring Frege and Russell with the same brush, I will suggest that Wittgenstein is sensitive to the differences in their views. I will argue that Frege's interpretation of the significance of his system can be seen to be consistent even in spite of Wittgenstein's argument (whereas Russell's interpretation is made to look inconsistent). I will suggest that Wittgenstein is sensitive to this, though not especially concerned by it. Frege is only seen as consistent at the expense of adopting a philosophical thesis concerning 'mediate' proofs which requires taking mathematical structures to exist prior to our formulation of the relevant mathematical propositions. It is not Wittgenstein's primary purpose to show Frege or Russell to be inconsistent, although he will point out inconsistency if it serves his end. Rather, his primary purpose is to show that their interpretations of their logical systems presuppose misleading pictures of arithmetic and its uses.

8.3. Logicist Systems as an 'Analysis' of Arithmetic

The logicist programmes of Frege and Russell were in part logico-mathematical and in part philosophical. Each presents a logico-mathematical system and attributes a philosophical significance to that system. Perhaps the most obvious and well-known understanding of logicism is as providing an analysis (or rather a logical system which is interpreted as providing an analysis) of the terms of arithmetic which exposes the meaning of those terms as they figure in statements like "2+5=7" or "there are 5 red apples." This way of interpreting the logicist project might lead to a worry – how are we to know whether a concept has been captured correctly? What if the original concepts were not entirely clear in the first place?[144] This concern is expressed by Russell and he proposes a different line. Russell thinks that the analysis can at least replace ordinary number concepts, even if the original concepts might remain in some sense clouded:

> So far we have not suggested anything in the slightest degree paradoxical. But when we come to the actual definitions of the numbers we cannot avoid what must at first sight seem a paradox, though this perception will soon wear off. We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples. It is indubitable and

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[143] For example, (De Bruin, 2008). Marion (1998, p.221-223) has also noted that Wright takes Wittgenstein's view to be finitist, which may account for why Wright has not treated Wittgenstein's critique as of importance.

[144] For more on different conceptions of analysis involved in logicism, see (Reck, 2007).

> not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematic number 2, which must always remain elusive.

> Accordingly we set up the following definitions: ... At the expense of a little oddity, this definition secures definiteness and indubitability; and it is not difficult to prove that numbers so defined have all the properties that we expect numbers to have. (1919, p. 14)

Rather than claiming to clarify the original number concepts, Russell's remark suggests that he accepts that doubts may remain about them and instead aims to avoid these doubts by producing a new analysed concept. An analysis might normally be expected to expose the core meaning of the analysed term but some flexibility can be allowed on how much of the original usage needs to be retained in the analysis. It is not entirely clear how much of the original usage of mathematical terms Russell intended to retain in his analysis but he does say that he wants his definitions to retain the usage of ordinary arithmetical terms in regard to counting objects:

> (…) we want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties. This definite meaning is defined by the logical theory of arithmetic. (1919, p.10)

Carnap's conception of explication takes the flexibility afforded to analyses even further, allowing that the original concepts may be greatly changed so long as the change is fruitful enough. But even he stipulates that the "explicatum is to be similar to the explicandum in such a way that in most cases in which the explicandum has so far been used, the explicatum can be used" (1962, p.7). Carnap wishes to read Frege and Russell as providing an explication (1947, p. 7-8) and it has been argued that Carnap too saw accounting for applied arithmetic as crucial to the explication (Marion[145], 2006, p.12).

Frege's interpretation of the significance of his system appears to be less reliant upon accounting for the applicability of arithmetic and I will suggest later that we can read Frege's view as coming from a different perspective. Nonetheless, Frege does take accounting for applied arithmetic as one of his aims.[146] He says:

> I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one. To apply arithmetic in the physical sciences is to bring logic to bear on observed facts. (*Gl*, p.99)

And if this is not revealing enough then more so is his is his criticism of Newton's definitions:

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[145] Marion's paper (2006) also defends the view that Wittgenstein's surveyability criticism undermines Russell and Carnap's aspirations for the logicist undertaking.

[146] See (Blanchette, 1994) for more on this.

> Even so, we should still remain in doubt as to how the number defined geometrically in this way is related to the number of ordinary life, which would then be entirely cut off from science. Yet surely we are entitled to demand of arithmetic that its numbers should be adapted for use in every application of number, even although that application itself is not the business of arithmetic. (*Gl*, p.26)

The reference to 'ordinary life' strongly suggests that accommodating applied arithmetic is one of Frege's aims.

It will be important to be clear on a distinction between 'pure arithmetic' and 'applied arithmetic' and what it means for logicism to account for them. By pure arithmetic, I mean the use of arithmetical propositions in pure mathematics where no empirical terms feature. An example expression of pure arithmetic is "2+5=7" and in order for a logicist system to be taken to account for pure arithmetic, that system needs to be able to decide the truth or falsity of pure arithmetical expressions (since this is what we would otherwise do by employing the system of arithmetic i.e. by performing arithmetical calculations). Applied arithmetic is the use of arithmetical expressions in empirical expressions such as "there are 5 red apples." For a logicist system to be taken to account for applied arithmetic, the system needs to be able to show us how to decide these empirical expressions. This is not to say that all such expressions necessarily need to be decidable using the logicist system but that the logicist system should allow us to decide as much as, or at least most of, what we are able to decide using (non-logicist) arithmetic.

Whilst I have so far focused upon a logicist aspiration to account for applied arithmetic, the aspiration to account for pure arithmetic is even more central. Especially so for Frege, who says that "definitions must show their worth by their fruitfulness: it must be possible to use them for constructing proofs" (*Gl*, p.9). We will come back to why this especially important for Frege.[147] For now it is only necessary that the distinction between pure and applied arithmetic is clear so that it can be seen that Wittgenstein's criticism targets logicist claims to account for both pure and applied arithmetic.

8.4. Wittgenstein on *Principia Mathematica* and Logicism

Wittgenstein's approach, as I have mentioned, is to go closely over what the logicist systems do in order to see whether the interpretation that logicists have put on their systems fits well with what their systems do. Much of Wittgenstein's discussion focuses upon Russell but his criticism can be seen (and is seen by him) to generalise to Frege. In this section I will focus mostly upon Wittgenstein's scrutiny of *Principia Mathematica* and Russell's interpretation of it as an analysis of ordinary arithmetical propositions which accounts for pure and applied arithmetic. More will be said about Frege specifically in the last section.

How then, might we see *Principa* as analysing ordinary arithmetical expressions? Take an expression such as 'If there are 2 things here and 2 things there, there are 4 things

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[147] Ahead of time, it will turn out to be important because Frege takes proofs in his system to be the 'true' proofs underlying calculations in pure arithmetic. I will suggest that the idea of capturing an underlying (and Platonically pre-existing) mathematical structure plays an important role for Frege's view of the significance of his system.

altogether' (*AWL*, p.146). The ideas of Frege and Russell present us with a way in which such a conditional statement can be taken as a tautology (i.e. as analysable into an expression of their systems). The basic idea is to analyse out the concepts '2' and '4' in the expression by correlating objects (we might say correlating instead of counting but Frege and Russell will take the correlating to amount to the same thing as counting). This has become known as Hume's principle, which Frege states thus:

> …the number of objects falling under concept *F* is identical to the number of objects falling under concept *G* iff there is a one-one correlation between *F* and *G*. (*Gl*, p.63)

Frege and Russell construct systems based upon this principle, with the intention being that the system could be used to define and clarify the concept of natural number. Whilst these systems were complex and technical, the core idea behind each was to define the concept of natural number by defining the number 'three' as 'the class of all triplets' where the class of all triplets was to be determined by correlations, to define 'four' as the 'class of all quartets' and so on. As Wittgenstein says, Russell attempts to do this with expressions like this one:

((x=a) and (y=c)) or ((y=b) and (x=d))[148]

This expression would only be satisfied by a (for x) and c (for y) or by b and d. Russell's definition of the class of all couples could then be expressed as:

((x=a) and (y=c)) or ((y=b) and (x=d)) …

Wittgenstein takes each of Frege and Russell to identify '2+2=4' with a formal analogue of 'If there are 2 things here and 2 things there, there are 4 things altogether.' Their idea is to have a formal analogue of every arithmetical expression so as to present a logical analysis of arithmetic.

As Wittgenstein understands Frege and Russell's approach, each logical expression is meant as an analysis (Wittgenstein says 'theory' (*AWL*, p.148)) of the mathematical expression. The idea is that the mathematical expression is really a logical expression and is revealed as such by Frege and Russell's analysis. We might say that the Frege-Russell definition is meant to show us the essence of our arithmetical expressions:

> "By means of suitable definitions, we can prove '25x25=625' in Russell's logic."–
>
> And can I define the ordinary technique of proof by means of Russell's? But how can one technique of proof be *defined* by means of another? How can one explain the *essence* of another? For if the one is an 'abbreviation' of the other, it must surely be a *systematic* abbreviation. (*RFM*, p.175-176)

Here Wittgenstein talks of revealing the 'essence' of one proof-technique by another. The suggestion of Wittgenstein's interlocutor in the cited remark is that Russell's logical system yields the arithmetical expression '25x25=625' and further that the expression can be proven in Russell's system. The interlocutor's suggestion here looks like a strong one – the suggestion is not just that an analogue of '25x25=625' can be obtained but that '25x25=625', the very arithmetical expression itself, can be obtained. We would want to say this if what were provided were a strong mathematical reduction of one system to

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[148] This is taken from Ambrose's lecture notes (*AWL*, p.149) - it appears there are meant to be implicit second-order quantifiers for x and y.

another, whereby the system of arithmetic could be shown to be fully reducible to the Russellian system. But we might also say this if we were able to obtain a Russellian analogue of '25x25=625' and we were prepared to interpret that analogue as an acceptable analysis of '25x25=625.' And if we were prepared to interpret the Russellian analogue of '25x25=625' as an acceptable analysis of the arithmetical expression, then we would also be prepared to interpret the arithmetical expression as an abbreviation of the Russellian one.

Wittgenstein's point regarding the 'systematic abbreviation' is that, in order to accept such an interpretation, we need to be able to translate expressions of Russell's system into arithmetical expressions and vice versa. We need to be able to see that anything we could prove in arithmetic can also be proven in Russell's system. For this it would not be enough to pair up statements of Russell's system with statements of arithmetic one by one without having rules by which to do so. If we are to take '25x25=625' as an abbreviated expression of Russell's system, then Russell's system needs to give us systematic techniques for constructing analogues of arithmetical expressions from within Russell's system alone. Russell's system needs to show us how to decide whether a given arithmetical expression is a proposition of arithmetic or not by showing us whether the expression has a Russellian proposition as its analogue. If Russell's system is to be seen as an analysis of arithmetic, then it must "teach us to add" (*RFM*, p.146).

Wittgenstein has a particular kind of definition in mind when his interlocutor says 'by means of suitable definitions.' The definitions in question are presumably the ones that Wittgenstein uses in his lectures to bring out Russellian expressions as analogues of arithmetical ones such as 1+1=2:

> In my notation this is: $(E1x)fx.(E1x)gx.\neg(\exists x)fx.gx.\rightarrow.(E2x)fx \cup gx.$ (Recall that $(E2x)fx$ is short for $(\exists x,y)fx.fy:\neg(\exists x,y,z)fx.fy.fz.$) (*AWL*, p.147)

So 'E2x' is meant to play a role of 'there are exactly 2 xs.' It should abbreviate the longer Russellian expression. This has to be introduced because Russell's expressions are too long to be able to work with for an expression involving large numbers. For smaller numbers we can immediately see the relationship between the Russellian expression and a corresponding arithmetical expression since we can simply see in a glance all of the terms are involved in the expression – we don't have any need to count them (*RFM*, p.146). But for expressions involving larger numbers, then we cannot take in the number of terms in a glance. If we wanted to use Russell's system to decide whether 15+27=56 is an arithmetical theorem, then we would have to decide whether the Russellian expression corresponding to 'E15x+E27x=E56x' was a proposition (theorem) of Russell's system.

The problem which Wittgenstein raises is that the abbreviation 'E15x' is not systematic as far as Russell's system goes. The only way to use these definitions is by using an arithmetical technique i.e. we have to count the terms. If the technique of abbreviation that gives us 'E15x' is made part of Russell's system then an arithmetical technique is made part of Russell's system and any ambitions to analyse/explicate arithmetic are undermined (one is then using arithmetic to analyse/explicate arithmetic). And if the abbreviation technique is not made part of Russell's system then we have to somehow determine whether there is a Russellian analogue of '15+27=56' without having to count the terms.

The 'abbreviations' in question can seem so natural that we can fail to spot that there is even a technique in play here. We might think that the technique of using 'E15x' instead of the expression for 'there are exactly 15 xs' is simply justified by the logic of Russell's system, given that it is so natural. But Wittgenstein has a right to press for the nature of the justification. We might consider, as an analogy, the technique of writing exponents for

factored multiplications.[149] It seems so natural to write '16^{15}' instead of '16 x 16 x ... x 16' (with 15 factors) that we might think that exponentiation is simply contained in the technique of multiplication. This is a point which Wittgenstein considers (*RFM*, p.180).

Exponentiation is a good analogy because it is a technique which allows us to handle expressions which might otherwise be unmanageable. To use Wittgenstein's phrase, it is a technique which gives us 'surveyability.' A new technique or concept is introduced by introducing exponentiation into the symbolism – it is not just an abbreviation (although it may look like one). We can see that '16^{15}' is conceptually different from (though mathematically equal to) '16 x 16 x ... x 16' (with 15 factors) because we have to compute them differently. In the latter case we can, if we wish, blindly tick off 16s by doing multiplications until we run out. But with '16^{15}' we have no way to perform the calculation without being aware of the value of the exponent. The two expressions have different uses.

For Wittgenstein, if two expressions have different uses then they are conceptually different. But this is not definitive since perhaps the logicist will deny that meaning is a function of use. More important is perhaps that the technique of exponentiation enables us to arrive at results that we would not have been able to otherwise. Being forced to look at the number of factors (the exponent) leads to the seeing of new connections (such as "$a^{m+n}=a^m \times a^n$"). Some of these connections might have been possible even without the new notation but others would not have been. In particular Mühlhölzer (2005, p.84) cites an example which Wittgenstein gives in the Nachlass[150] of an inductive proof where the inductive step from n to n+1 refers to the exponent.

If we can arrive at results using exponentiation that we could not arrive at otherwise then exponentiation is doing mathematical work. It then has the character of a rule or posit of the system. Wittgenstein's contention is that it is the same with the 'E15x' technique.[151]

Wittgenstein is not denying that there are expressions possible in Russell's system which we might call analogues of ordinary arithmetical expressions. Rather, if Russell's system is to be taken as an analysis of arithmetic (and we are to interpret those Russellian analogues as equivalent to the arithmetical expressions) then we would need to be able to systematically construct analogues of arithmetical propositions within Russell's system and using the resources of Russell's system alone. As Wittgenstein says, he is "not trying to shew that it is impossible that, for every mathematical proof, a Russellian proof can be constructed which (somehow) 'corresponds' to it, but rather that the acceptance of such a correspondence does not lean on logic" (*RFM*, p.185). And by 'logic' Wittgenstein means by Russell's system alone.

We can now see how Wittgenstein goes about challenging the claim that "by means of suitable definitions, we can prove '$25 \times 25=625$' in Russell's logic" (*RFM*, p.175). Either the definitions are part of Russell's system or they are not. If they are then Russell's system makes use of arithmetic. If they are not then we cannot prove in Russell's system the expressions that we are easily be able to evaluate in arithmetic. So a logicist interpretation of Russell's system as accounting for pure arithmetic looks to have been severely undermined.

Wittgenstein's argument that pure arithmetic (the theorems of pure arithmetic) cannot be derived from Russell's system alone has a parallel with regards to applied arithmetic. The question with regard to applied arithmetic is whether we can use a system like Russell's (based upon a principle of correlation) to compare quantities of objects. Again Wittgenstein

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[149]Mühlhölzer (2005, p.83-84) has a revealing discussion of this, and of Wittgenstein's whole point of view with regards to surveyability.

[150](MS 122, 103r–104v).

[151]Marion points to instances of other surveyability arguments in relation to mathematics (2009, p.425-431; 2006).

makes the point that for small numbers of objects we can see whether they are correlated but for larger numbers the correlation can only be seen by counting:

> Can I know there are as many apples as pears on this plate, without *knowing* how many? And what is meant by not knowing how many? And how can I find out how many? (*PR*, p.140)

It is not enough for Frege and Russell to simply point to the idea of correlation and say that this is in some sense the basis of counting. We need to be shown how to perform the activities that we currently do with counting by means of correlation alone:

> When presented with thousands of dots we do not know when some have vanished...We can say that classes are equal in number when they *can be* correlated provided we give instructions for telling how we find whether they can be. (*AWL*, p.149)

So, as with the case of accounting for pure mathematics, the problem is that the approach does not adequately 'teach us to add' (*RFM*, p.146). Perhaps if the Russellian system were able to account for pure mathematics then it might also be possible to use it to account for applied mathematics, since then it would give us access to the propositions of arithmetic. But Wittgenstein's criticism indicates that it gives us neither.

There is a suggestion in Wittgenstein that this was a blindspot in Russell's view of the logicist programme – that the need to be taught how to add had somehow been forgotten:

> I should like to say: Russell's foundation of mathematics postpones the introduction of new techniques--until finally you believe that this is no longer necessary at all.

> (It would perhaps be as if I were to philosophize about the concept of measurement of length for so long that people forgot that the actual fixing of a unit of length is necessary before you can measure length.) (*RFM*, p.179)

8.5. Can Russell's View be Saved?

Perhaps the logicist could object that although the necessary proofs to systematically establish theorems in the Russellian system corresponding to ordinary arithmetical truths are not practically possible, nonetheless they are in some sense 'theoretically' possible (Russell might perhaps say that they are only 'medically impossible' (1936)). Russell's system may not give a practicable way of evaluating propositions of pure and applied arithmetic, nonetheless it does outline a general approach and the approach has a certain intuitive appeal (as demonstrated by our ability to work with it for small cases).

The main trouble with this line of response is that it is not clear what sense of 'theoretical' might be relevant. If the idea is that the Russellian system really could (with enough time and effort) be implemented to do pure and applied arithmetic, then this looks like it could be a misunderstanding. In that case Wittgenstein could simply point again at the need for the introduction of the arithmetical concepts by means of the definitions in question and reassert that the system lacks the conceptual resources to do the necessary work. It simply is not plausible that humans could do arithmetic in this way, no matter the time and effort. One would have to alter the human brain so that we could see correspondences in larger and larger classes. It would surely be against any accepted sense of 'analysis' (or, for that matter, 'explication') to present for our analysed term the outline of a concept as we think

that a hypothetical superman might be able to use it. If we are to analyse our concepts, then the analysis must sure be intended for us.

It might be better to ask whether there is a way to simply change Russell's system (or supply one like it) so that Wittgenstein's criticism would not apply. One such approach is to develop a system in which the axioms of Peano Arithmetic can be derived. Peano Arithmetic is the accepted basis of arithmetic so once we have that then it might be said that the aim has been achieved.[152]

Wittgenstein's reply is going to be to question the sense in which Peano Arithmetic is the 'basis' of arithmetic. Whilst Peano Arithmetic certainly expresses the familiar arithmetic operations, this is not to say that it can 'teach us to add' any more than Russell can. As Mühlhölzer points out (2005, p.74) the number 100,000 would have to be represented in Peano Arithmetic by the expression "1+1+...+1" (with a hundred-thousand 1s). Unless definitions are introduced into Peano Arithmetic to give us decimal notation then Peano Arithmetic will also be unworkable for the purposes of doing pure and applied arithmetic. Just as with Russell, the introduction of such definitions would be an introduction of our ordinary arithmetic concepts:

> I want to say: if you have a proof-pattern that cannot be taken in, and by a change in notation you turn it into one that can, then you are producing a proof, where there was none before. (*RFM*, p.143)

We may well understand Peano Arithmetic as casting light upon ordinary arithmetic. But it is not the same thing as ordinary arithmetic. We cannot use Peano Arithmetic alone to evaluate expressions like "125+73=198". Insofar as wish to use Peano Arithmetic to do this then we have to introduce a technique which trades off ordinary arithmetic.

There is a general point here about foundational systems. Foundational systems typically try to reduce a system with a comparatively larger number of symbols to a system with a smaller number of symbols. *Principia Mathematica*, Peano Arithmetic and (at least parts of) Set Theory are all examples. In order to achieve this, statements in the system to be reduced are modelled in the foundational system (i.e. analogues of statements of the system to be reduced are required in the foundational system). Since the foundational system has (by design) fewer symbols available, the statements in the foundational system are much longer than the corresponding statements in the system to be reduced. Actually, sometimes they are spectacularly long –Mühlhölzer (2005) mentions (p.73) that in "the foundational system of Nicholas Bourbaki, e.g., the expression for the number 1 already consists of 4 523 659 424 929 primitive symbols"![153]

In the case of *Principia Mathematica*, statements of ordinary arithmetic are simulated by means of expressions involving a small number of logical relations. The symbolism is not as powerful as that of ordinary arithmetic (since, in its aspiration to be more basic, it

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[152] Neologicists take such an approach. Note that neologicists take their systems as accounting for pure and applied mathematics. See the introduction to Wright (1983) and his reply to Boolos (2001,p.322). Also see MacBride's survey (2003,p.109) – he cites Demopoulos as an outlier with regards to his notion of 'analysis' but he also (2000, p.220-221) shares an aspiration to account for pure and applied arithmetic. I will later argue that Frege himself can be seen as having a response to Wittgenstein's criticism but only at the expense of a particularly strong form of Platonism. It is not clear (and it is not my purpose to determine) whether this approach could be seen to fit with the more modest Platonism to which neologicists aspire (see section 3 of McBride (2003)) or whether there is any alternative philosophical move which might protect neologicists.
[153]This number he takes from Mathias (2002).

contains fewer symbols) and so it is not surprising that an expression for 'there are exactly 5672 objects satisfying P' would have to be exceptionally large. Given that this is in line with what we typically see with foundational systems, it does not look likely that tinkering with logicist systems is going to get around Wittgenstein's objection. This is not to suggest that no mathematical system can ever be reduced to a simpler one without any surveyability issues arising, just that the prospects for such a system as a logicist reduction of arithmetic do not look good.

One might feel that there is still room to say that a system such as Russell's might be seen as a basis for arithmetic in a foundational sense, even if Wittgenstein's surveyability argument is left to stand. In my view Wittgenstein would accept that such an interpretation of a logicist system is possible but not appealing due to the metaphysical assumptions which it would require. We can see how such a perspective would go by looking at some of Frege's remarks.

8.6. Frege and Foundations

Russell's articulation of his system may postpone "the introduction of new techniques-- until finally you believe that this is no longer necessary at all" (*RFM*, p.179) but Frege seems to have been conscious that definitions would need to play an important role for him. In the preface to Begriffschrift, he says that the use of definitions in order to shorten expressions would be "advisable in the case of eventual application" (p.8). Yet the full quotation calls into question whether 'advisable' means optional here:

> ...transitions that this one mode of inference would not allow us to carry out except mediately, will be abbreviated into immediate ones. In fact this would be advisable in the case of eventual application. In this way, then, further modes of inference would be created (*Bs*, p.8)

Blanchette (1994, footnote 12) helpfully collects remarks that throw some light upon Frege's view of definition:

> The first definition of the Begriffsschrift is accompanied with the explanation that "we can do without the notation introduced by this proposition and hence without the proposition itself as its definition; nothing follows from the proposition that could not also be inferred without it. Our sole purpose in introducing such definitions is to bring about an extrinsic simplification by stipulating an abbreviation" (Bs §24). In the Grundlagen, a definition "only lays down the meaning of a symbol" (*Gl*, p.78), and in the Grundgesetze: "The definitions ... merely introduce abbreviated notations (names), which could be dispensed with were it not that lengthiness would then make for insuperable external difficulties" (*Gg*, p.2)

So Frege accepts that the necessary proofs could not be formulated without the definitions and yet he treats this as an 'external difficulty' rather than a limitation of his system. Frege's attitude seems to be bound up with his picture of mathematical proof in general. He thinks that the proofs of his system would be the true proofs, even though we would need definitions external to his system in order to 'mediately' formulate expressions of them. Frege seems to think it is a sign of weakness on the part of mathematicians that they do

not give adequate attention to how the proofs that they offer would relate to the true (longer and only 'mediately' accessible) proofs:

> ...in proofs as we know them, progress is by jumps, which is why the variety of types of inference in mathematics appears to be so excessively rich; for the bigger the jump, the more diverse are the combinations it can represent of simple inferences with axioms derived from intuition. Often, nevertheless, the correctness of such a transition is immediately self-evident to us, without our ever becoming conscious of the subordinate steps condensed within it. (*Gl*, p.102)

Frege seems to hold a view of his system as a justificatory foundational system. Whilst, as we have seen, Frege does think that his system gives an account of applied arithmetic, perhaps he thinks that it only does so indirectly by providing access to proofs which are the true basis of arithmetical propositions. An arithmetical calculation such as "2+5=7", for Frege, is only a shorthand for Fregean proof. We are in a certain sense lazy in that we do not give consideration to the Fregean proof when performing the calculation, although the Fregean proof grounds our intuition nonetheless.

There is something rather mysterious about Frege's perspective on 'mediate accessibility' and one might be tempted to take it as a general caution to mathematicians rather than an important part of his view. One might wonder how the mathematician is supposed to become conscious of the 'subordinate steps' within his proofs, given that those steps would only ever be available 'mediately' by means of certain definitions. But Frege's remarks here can be seen as giving a way to avoid the challenge of vicious circularity that he would otherwise be susceptible to (and which he to some extent anticipated). Frege seems to give no *direct* answer to Wittgenstein's challenge as Frege would have to accept that the definitions which he needs (for accounting for pure and applied arithmetic) are arithmetic-involving. But Frege could (and it seems would) maintain that our ability to do arithmetic is in any case a matter of our having intuitions which we could not have were it not for Frege's (only 'mediately' accessible) proofs. The circularity is then not vicious since both the 'mediately accessible' proof and the ordinary arithmetic calculation are (for Frege) different ways of grasping the same underlying mathematical structure.

This perhaps explains why Wittgenstein's consideration of this line of thinking is so cautious:

> Suppose someone were to say: "The only real proof of 1000 + 1000 = 2000 is after all the Russellian one, which shews that the expression... is a tautology"? For can I not prove that a tautology results if I have 1000 members in each of the two first pairs of brackets and 2000 in the third? And if I can prove that, then I can look at it as a proof of the arithmetical proposition.
>
> In philosophy it is always good to put a *question* instead of an answer to a question.
>
> For an answer to the philosophical question may easily be unfair; disposing of it by means of another question is not.
>
> Then should I put a *question* here, for example, instead of the answer that that arithmetical proposition cannot be proved by Russell's method? (*RFM*, p.147)

Wittgenstein's caution need not be seen as a hedging of bets. Whilst Wittgenstein's criticism may very much undermine the appeal of Frege's interpretation of his system by showing that his system cannot of its own accord (without the aid of a philosophical thesis

on 'mediate' proof) account for pure arithmetic, Frege's interpretation is not necessarily inconsistent.[154]

Wittgenstein would like to urge that the calculations of ordinary arithmetic should be seen as the correct procedures (rather than the unsurveyable procedures of a logicist system) and Frege's system only as a new system inspired by it (and in some respects a simulation of it):

> I can find out that 100×100 equals 10,000 by means of a 'shortened' procedure.
>
> Then why should I not regard *that* as the original proof procedure?
>
> A shortened procedure tells me what *ought* to come out with the unshortened one. (Instead of the other way round.) (*RFM*, p.157)

One might expect Wittgenstein to be more definitive here. Given that the unshortened procedure is not surveyable, the name 'proof' might not be appropriate for it, let alone 'only real proof.' But Wittgenstein only presses the idea cautiously. Whilst the shortened procedure does tell us what ought to come out with the unshortened one, Frege can say that this is because the shortened procedure and unshortened one are both insights into a single underlying mathematical reality.

Frege's point of view is consistent so long as it rests (as it seems it does) upon a commitment to a Platonist thesis concerning his 'mediately accessible' proofs. Frege thinks that there is a mathematical reality out there to be discerned and our mathematical techniques (including ordinary arithmetic) are just devices for helping us see that reality. Again the preface to the Begriffsschrift is enlightening:

> The most immediate point of contact between my formula language and that of arithmetic is the way in which letters are employed.
>
> I believe that I can best make the relation of my ideography to ordinary language clear if I compare it with that which the microscope has to the eye. Because of the range of the possible uses and the versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope. Considered as an optical instrument, to be sure, it has many imperfections, which ordinarily remain unnoticed only on account of its intimate connection with our mental life. But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others.
>
> This ideography, likewise, is a device invented for certain scientific purposes, and one must not condemn it because it is not suited to others. If it answers to these purposes in some degree, one should not mind that there are no new truths in my work. I would console myself on this point with a realisation that a development of method, too, furthers science. (*Bs*, p.6)

Frege's viewpoint looks difficult to argue with directly. One might press him on why he takes his system (his unshortened proofs) to provide a better view into the structure underlying arithmetic than another system (say Russell's or a set-theoretic system of arithmetic). Frege could not point in response to his system's accounting for pure and

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[154] Frege would have to accept that in a sense our acceptance of a correspondence between his 'mediately accessible' proofs and the ordinary calculations "does not lean on logic" (*RFM*, p.185), at least not logic alone. It leans also on a philosophical thesis which takes both proofs and calculations as means of grasping the same underlying mathematical reality.

applied arithmetic, since we have seen that Frege can only claim to account for pure and applied arithmetic on the basis of his view of his system as capturing an underlying reality (and so that line of response would be circular). Instead he would have to point to particular features of his system which make it especially intuitive or fruitful or which somehow suggest that the underlying reality is likely to follow the lines of his system.

Wittgenstein is then wise not to take issue with Frege's interpretation of his system on its own terms. To do so he would have to dispute Frege's Platonism and that would be to move on to another topic entirely.

From the point of view of a metaphysical interpretation like that which I have attributed to Frege, Wittgenstein's criticism can at worst only serve to point out the reductive character of Frege's point of view. This may or may not concern the Fregean – Frege himself is intentionally reductive, as can be seen in his remark that it is because mathematicians progress by jumps that the "variety of types of inference in mathematics appears to be so excessively rich" (*Gl*, p.102). For Frege, this richness is just a result of 'jumping' around the underlying mathematical structure. Wittgenstein would urge Frege to look again at the variety and inventiveness (as he says, 'colourfulness' (*RFM*, p.176[155])) to be found in mathematics and in arithmetic in particular, with a mind to suggesting that perhaps arithmetic has more to it than the simple components of the system to which it is supposed to be reduced. Wittgenstein does indeed make such remarks (*RFM*, p.176) but he does not press them very forcefully. Wittgenstein is aware that such a broad question of interpretation is unlikely to be easily settled and that a change in perspective from Frege's to his is too radical a change to come easily.[156]

What Wittgenstein can do is to put forward his own interpretation of the significance of the logicist systems. Whilst he may be disparaging about the logicists aspirations to see their systems as a ground or analysis of arithmetic, he can nonetheless acknowledge that the logicist systems are in some sense revealing with regard to arithmetic:

> But still for small numbers Russell does teach us to add; for then we take the groups of signs in the brackets in at a glance and we can take *them* as numerals; for example '*xy*', '*xyz*', '*xyzuv*.' Thus Russell teaches us a new calculus for reaching 5 from 2 and 3; and that is true even if we say that a logical calculus is only--frills tacked on to the arithmetical calculus. (*RFM*, p.146)

Russell's system clearly has structural similarities with regard to the system of arithmetic and in that regard it might be said to be revealing with regard to arithmetic. A system like Russell's will therefore stimulate new mathematical work, as well as being "a new bit of mathematics" (*RFM*, p.176) in itself. In this respect Wittgenstein can certainly see Frege's aspiration to 'further science' by the 'development of method' (*Bs*, p.6) as having been realised, though not with the same significance as Frege takes it to have.

In these ways Wittgenstein can interpret the systems of Frege and Russell in a way which is respectful of their mathematical contributions. Wittgenstein can even see their systems as 'foundational' systems if 'foundational' is understood in a non-justificatory way. I have in mind a sense of 'foundation' much like that of MacLane's (1986, p.406) idea of foundation-as-organisation, whereby other areas of mathematics are reconstructed (one might say 'simulated') and related to one another in the foundational system. If these systems can

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[155]I follow Mühlhölzer (2005, p.66) in using 'colourfulness' in place of Anscombe's 'motley.'
[156]Perhaps the best thing for Wittgenstein to do would be to point to the problems concerning taking every inference to admit of some further justification discussed in chapter 3, since Frege might well be accused of falling in with the "illusory image of greater depth" discussed in that connection (*RFM*, p.333).

play an organising role and bring forth revealing analogies between systems then this is not something which Wittgenstein would necessarily want to 'run down', much as he did not want to 'run down' (*LFM*, p.156) Frege and Russell's systems since those systems would be good simulations of arithmetic for cases limited to small numbers. What Wittgenstein does feel the need to criticise is Frege and Russell's views of the alleged philosophical significance of their projects. Their interpretations of their systems might seem very appealing if one misses the need for definitions which smuggle in arithmetical concepts but once this need is made clear then their interpretations look questionable and an interpretation like Wittgenstein's then looks like it might be preferable.

8.7. Chapter Conclusion

When Wittgenstein cautions against seeing parts of mathematics as revealing the "mysteries of the mathematical world" (*RFM*, p.137), it can be tempting to take this, as Putnam does, as evidence of an inadequate respect for science (2007, p. 246). But it can be seen as just the opposite – Wittgenstein wants us to see mathematics for what it is and not to see it through the distorted lens of some particular philosophy. For Wittgenstein, at least in respect of these kinds of questions of the significance of mathematics, "the philosophy of mathematics consists in an exact scrutiny of mathematical proofs - not in surrounding mathematics with a vapour" (*PG*, p.367).

It is an exact scrutiny that Wittgenstein subjects Frege and Russell's proofs to, in order to show that their logico-mathematical systems cannot substitute for ordinary arithmetic without presupposing the very ordinary arithmetic that it would allegedly be replacing. Wittgenstein shows every respect for their logico-mathematical achievements and even acknowledges that their systems could take the place of some very limited and basic counting practices. He takes issue with their interpretations of their systems as providing an analysis of arithmetic because those interpretations are not fair to arithmetic. Wittgenstein shows much respect for mathematics here in that he is determined to protect arithmetic from its devaluation by Frege and Russell.

Chapter 9 - The problems of Set Theory as 'problems of mathematics like any other'

9.1. Chapter Introduction

There is a strand of Wittgenstein's thinking on set theory which is very much in line with this thought from the mathematician Saunders MacLane:

> The idea that there is really an actual world of sets is a myth, perhaps convenient but nevertheless mythical... The effectiveness of the set-theoretic formalism, if indeed it is effective, is by no means evidence for the Platonic reality. (MacLane 1986a, p.8)

What MacLane and Wittgenstein have most in common is an insistence on drawing a dividing line between mathematics and philosophy. Many of Wittgenstein's remarks on set theory can be seen as attempting to insist on drawing this line right down to points of detail by testing what will count as a mathematical idea and what will count as a philosophical one. Set theory, for Wittgenstein, is not a theory about sets, not if sets are conceived as existing prior to the theory. Nor is it a theory of anything external to itself, or an exploration of any concepts apart from those elaborated within the mathematics and only as they are elaborated in the mathematics (and so it is not an exploration of the infinite either, except insofar as that term is given a meaning within the mathematics). Wittgenstein objects to the metaphysical speculation that sometimes surrounds set theory and in this some practically-minded mathematics may well find some appeal in Wittgenstein.[157]

The reception of this strand of Wittgenstein's thinking is likely to have been clouded by the widely-held view that Wittgenstein subscribed to a form of finitism and this led him to reject set theory as not legitimate mathematics.[158] This is particularly unfortunate given the stress that Wittgenstein laid upon 'leaving mathematics as it is' (*PI*, §124) and avoiding the 'Bolshevik' willingness to revise mathematics that he saw in certain constructive philosophies of mathematics (see chapter 6). I will argue that Wittgenstein's perspective on the infinite in mathematics is compatible with the notion of infinity as it figures in set theory and that at least some of the more dismissive-looking remarks by Wittgenstein on set theory are much more reasonable than they have been taken to be.

The idea that the notion of 'set' is implicitly defined is not new and nor is the relevance of Wittgenstein's thought to this idea.[159] But Wittgenstein's remarks on set theory, in their relation to his view of infinity, have not been fully appreciated. Wittgenstein's remarks on infinity might be seen, as Moore sees them (Moore 1990, p.206-208), as a connecting with a tradition of the 'potential' infinite going back to Aristotle. Seen in this light, many of

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[157] Of note is a sentiment expressed by (Hamming 1998, p.644) when he said that he saw no reason for walking into 'Cantor's paradise' – this is very close to a thought of Wittgenstein's (*LFM*, p.103). Also of note, as mentioned, is MacLane's critique of the interpretation of set theory as a theory about sets in (1986a) and (1986b, p.449). MacLane expresses a similar anti-metaphysical spirit to the critique that will be extracted from Wittgenstein here.

[158] For example, Marion's claim that Wittgenstein "never really accepted" the axiom of choice (1998, p.72) or Putnam's claim that Wittgenstein's view of infinity amounts to a "flat-unbelievable assertion" (2007, p.241).

[159] See (Muller 2004).

Wittgenstein's remarks on set theory can then be seen as responding to a possible objection that the 'actual' infinite figures within set theory as a mathematical notion (rather than as a philosophical image). Thus Wittgenstein can be understood to be taking on two philosophical interpretations with regard to set theory – the interpretation of set theory as a realm of sets (conceived as pre-existing objects) and an interpretation of set theory as an exploration of the nature of higher-order infinities (again conceived as somehow pre-existing).

9.2. Wittgenstein and Conceptions of the Infinite

Marion, in his discussion of Wittgenstein's view of infinity, helpfully distinguishes what a Platonist, an intuitionist and a strict finitist would say about the Fibonacci sequence (Marion 1998, p.184-5). A brief characterisation of these views will help us to locate Wittgenstein's perspective. The Platonist would say that the whole infinite extension of the sequence is given when the rule of the sequence is given – the extension is simply available as an object. The intuitionist would hold that the whole extension cannot be coherently thought of as an object but that we can progressively generate part of the extension by following the rule. The advocate of strict finitism (which is thought to arise from sceptical arguments concerning the idea of following a rule) would hold that the rule does not determine the sequence on its own, since the community would have to agree as to how the rule should be interpreted for each case. As Marion puts it:

> In contrast with the intuitionist, the strict finitist would make the further claim that there is no already defined a_{n-1} for a step n which has not been already computed. At each new step n, the community will decide what is the right a_{n-1}, and this still leaves a_n undecided. (Marion 1998, p.185)

Whilst Wittgenstein has famously been ready by Dummett (1978, p.249) as a strict finitist, Marion points to reasons for thinking that he should instead be read as resisting each of these three positions (strict finitism, intuitionism and Platonism).

Wittgenstein says that a technique is infinite if we don't prescribe any end to its repeated application:

> To say that a technique is unlimited does *not* mean that it goes on without ever stopping--that it increases immeasurably; but that it lacks the institution of the end, that it is not finished off. As one may say of a sentence that it is not finished off if it has no period. Or of a playing-field that is unlimited, when the rules of the game do not prescribe any boundaries (*RFM*, p.138)

This idea in itself is not especially new and Marion notes the similarity to Aristotle (1998, p.182). Wittgenstein thinks that we are tempted towards a mistaken view of infinity which is connected with a picture of mathematics as having a subject-matter (see chapter 4). If one thinks of mathematical propositions as describing a subject-matter then it can be very tempting to think of an infinite sequence as analogous to a finite sequence but bigger:

> There are two ways of using the expression 'and so on.' If I say, 'The alphabet is A, B, C, D, and so on', then 'and so on' is an abbreviation. But if I say, 'the cardinals are 1, 2, 3, 4, and so on', then it is not.—Hardy speaks as though it were always an abbreviation. As if a superman would write a huge series on a huge board—which is alright, but has nothing to do with the series of cardinals. (*LFM*, p. 255)

Wittgenstein regards the Platonist as having been seduced by an image of the infinite as analogous to 'the enormously big' and in this Wittgenstein agrees with the intuitionist (Brouwer, 1923). But Wittgenstein does not agree with the intuitionist's approach to resolving the problem. The disagreement between the Platonist and the intuitionist concerns whether conceiving of a completed extension is truly impossible (as the intuitionist holds) or as a Platonist like Russell would say only 'medically impossible'[160], meaning that the extension exists in its totality and is in some sense accessible even if we cannot fully enumerate it. For the Platonist, the completed infinite sequence exists even if our human limitations prevent us from getting all of the way to the end. For the intuitionist, we construct the sequence as we go along and so there is no sequence prior to our constructing it. Both Platonist and Intuitionist see the question to relate to the status of the extension of the mathematical expression, rather than to how the expression itself is used (Marion 1998, p.186). Likewise the strict finitist's claim, added on to the intuitionist's view, that "there is no already defined a_{n-1} for a step n which has not been already computed" (Marion 1998, p.185) is a claim concerning an extension (Marion 1998, p.186).

For Wittgenstein this debate is unresolvable because it misses the point – instead of looking at what humans are capable of conceiving, we should be looking to the role that the infinite plays in our mathematical systems. For Wittgenstein it is not that we hit a barrier in trying to imagine completing an infinite sequence, rather there is some confusion which leads us to even try to imagine this. The idea of a completed infinity is an image to which we have ascribed no meaning. It is part of our notion of an infinite sequence that the idea of completing it does not make sense. If we focus on the way in which infinite techniques work then it will be shown to be unnecessary to conceive of the extension as an object (which is what leads us to wonder whether we can grasp it) and instead see the extension as an aspect of a technique. Wittgenstein comments:

> Ought the word 'infinite' to be avoided in mathematics? Yes; where it appears to confer a meaning upon the calculus; instead of getting one from it. (*RFM*, p.141)

We should not take the infinite to be a thing of which we have a prior grasp even before it figures in a calculus. Nonetheless, there are techniques which we can call 'infinite' and which we do so without calling upon an explicit definition of 'infinite.' For example, we say that the natural numbers are infinite. With regards to this, Wittgenstein would say that our concept of the natural numbers is such that they cannot all be enumerated (since each in turn will have a successor, so there is no end-point). And this is what is characteristic of cases where we ordinarily use the term 'infinite' – the technique in question can be carried on without end. This is why the idea of a result or object being given in totality can seem perplexing for these techniques. If we try to think of an infinite technique as completed then we are left with an air of paradox:

> Wittgenstein in a lecture once asked his audience to imagine coming across a man who is saying, '…5, 1, 4, 1, 3—finished!', and, when asked what he has been doing, replies that he has just finished reciting the complete decimal expansion of pi backwards—something that he has been doing at a steady rate for all of past eternity. (Moore 1990, p.44)

Wittgenstein thinks that the confusion rests in our interpretation of mathematics and not in the mathematics itself. Once the mathematics is reinterpreted so as to relieve the temptation to talk of a completed totality then what had seemed perplexing will disappear.

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[160] See (Russell 1936).

To say that infinite techniques are techniques which can be applied repeatedly without end need not mean that we have to make new stipulations as we go, as the strict finitist says. Once the technique is set up it need only be a matter of following the technique from there. To say that new stipulations are required at each step is to doubt the objectivity of following a rule. As we have seen in chapter 3, Wittgenstein was not sceptical towards the objectivity of following a rule (such as expansion of a series). For Wittgenstein the phenomenon of rule-following is not properly seen as reducible to stipulations being made at the point of each application.

Wittgenstein's key idea, then, is that when we use an infinite technique then we are employing a technique that can be employed repeatedly without end. Wittgenstein treats the attempt to think of a completed infinity as leading to paradox and therefore to be avoided. This is not a decisive argument against Platonism with regard to infinity and I don't think Wittgenstein took it to be one. A Platonist might avoid speaking of a 'completed' infinity and confine himself to speaking of a 'complete' infinity, thereby avoiding the suggestion of process that seems to give rise to the paradoxes. Whatever the merits of demerits of such a move, from Wittgenstein's perspective the Platonist's postulation of a complete infinity looks unnecessary. There is nothing in the mathematics to require it and so it is just a philosophical picture.

There are cases where we might think that a 'complete' or 'completed' infinity really is necessary to at least some mathematics – specifically the set-theoretic mathematics of high-order infinities. We might feel inclined to say that there the infinite figures as a measure of a quantity of a mathematical entity, rather than as a form of a mathematical technique. But this temptation is a philosophical one and Wittgenstein can be seen as presenting an alternative interpretation whereby the infinite figures as a technique rather than a quantity in regard to Cantor's famous diagonal proof and in regard to set theory in general.

9.3. Cantor's Diagonal Proof

We can now attempt to make sense of Wittgenstein's criticisms in relation to Cantor's Diagonal Proof as not directed at the mathematics of Cantor's proof and instead as directed at a certain misleading prose interpretation of the proof. Given that my purpose is to focus on the prose interpretation of the diagonal proof, I shall not try to offer an accurate discussion of Cantor's original formulation of the diagonal proof and nor shall I offer a fully rigorous mathematical exposition. Rather, I will give an informal exposition of the proof with a mind to bringing out the misleading suggestions that promote the prose interpretation that Wittgenstein takes issue with.

Cantor's diagonal argument has been taken to show that the set of all real numbers is in some sense larger than the set of all natural numbers, with Cantor himself saying that the set of all real numbers is "genuinely infinite" (this being associated with completed infinity) rather than "non-genuinely infinite" (Cantor 1932, p.165). The argument proceeds by showing that the set of natural numbers cannot be put into a one to one mapping (commonly called a 'bijection') with the set of real numbers, whereas a subset of the real numbers can be put into a one to one mapping with the set of all natural numbers. Putting the subset of the reals into a one to one mapping with the natural numbers is easy to do, since all natural numbers are also real numbers so the subset could simply be the natural numbers themselves. But showing that the real numbers cannot be put into a bijection with the natural numbers requires the ingenuity of Cantor's diagonal method.

If one wanted to try to create a bijection from the reals to the natural numbers then one intuitive approach would be to think of each real number as an infinite expansion of digits and to try to find a way of ordering the real numbers so that one arrives at a sequence that covers all real numbers. Such an ordering would have to have a form like the below:

Place in Order	Real Expansion
1	$0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}d_{17}d_{18}d_{19}$ ....
2	$0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}d_{27}d_{28}d_{29}$ ....
3	$0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}d_{37}d_{38}d_{39}$ ....
4	$0.d_{41}d_{42}d_{43}d_{44}d_{45}d_{46}d_{47}d_{48}d_{49}$ ....
5	$0.d_{51}d_{52}d_{53}d_{54}d_{55}d_{56}d_{57}d_{58}d_{59}$ ....
$\vdots$	
n	$0.d_{n1}d_{n2}d_{n3}d_{n4}d_{n5}d_{n6}d_{n7}d_{n8}d_{n9}$ ...

But Cantor shows that such an approach cannot give us what we are looking for. No matter how sophisticated we make the ordering, it will always be possible to find a real number that is not included in the ordering. This is shown by a 'diagonal' procedure. For each real in the ordering, one can add one to first digit in the first real, then add one to the second digit in the second real and so on, using these digits to construct a number.[161] The constructed number will not be included in the ordering, since it differs in at least one place from each number in the ordering. So it must be impossible to put the real numbers into a one to one mapping with the natural numbers – in Cantor's terms, the real numbers are not 'denumerable.'

Given the elegance of Cantor's argument, it is understandable that many of Wittgenstein's readers have been surprised to find Wittgenstein referring to it as a "puffed-up proof" (*RFM*, p.132). Whilst Wittgenstein certainly takes issue with Cantor's understanding of the *significance*[162] of the diagonal proof, it is important to see that Wittgenstein does not take himself to be disagreeing with Cantor about the mathematics of the proof. This is why Wittgenstein cautions that the "result of a calculation expressed verbally is to be regarded with suspicion" (*RFM*, p.127). Seeing this requires being clear about what is determined by the proof and what seems to be added by prose.

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[161] This image is a useful illustration:

$0.[1]00000000…$

$0.0[1]0000000…$

$0.00[1]000000…$

$0.000[1]00000…$

Wittgenstein gives a similar image but without the bracketing (*RFM*, p.131).

[162] (*RFM*, p.142).

What has been shown is that there can be no bijection between **N** and all decimals expansion in the interval **(0,1)**[163]. One might say that this is analogous to showing that 5 objects cannot be put into bijection with 4 objects. But if we lay too much stress on this analogy then we might come to think of the proof as showing there to be more natural numbers than the real numbers in the *same* way that 5 objects are more than 4. Clearly we can tell that five things cannot be put into bijection with 4 things by counting them, whereas this is not possible when comparing **N** with **(0,1)**. The comparison between **N** and **(0,1)** might be called a conceptual comparison, since we require a new form of argument (specifically the diagonal argument) in order to make the comparison. Deciding whether 5 objects can be put into bijection with 4 could be called a quantitative determination, since arithmetic alone is sufficient to decide the issue. Of course we can decide to call the comparison a comparison of 'cardinality' in each case but doing so can easily create a misleading impression.[164] As Wittgenstein says, a "difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension" (*RFM*, p.132). What Wittgenstein takes to be a matter of prose, then, is a reading of Cantor's proof as showing that the real numbers are 'larger' than the natural numbers in the same way in which a collection of five things is larger than a collection of four things.

It might be objected that the comparison of cardinality in the case of the objects really is the same as the case of comparing **N** with **(0,1)**. It might be said that our determination that 5 objects cannot be put into bijection with 4 is not based upon a prior method of counting. It might be said that our determination that 5 objects cannot be put into bijection with 4 is somehow primitive, or that our notion of one-one correspondence underlies (or is the essence of) our method of counting. If that were the case then defining cardinality in terms of bijection would not just be a decision and would follow naturally from the nature of arithmetic. This contention about the nature of our concept of natural number is essentially the Logicist contention of Frege and Russell.[165] Given that this is treated as a philosophical thesis, it is not unreasonable to regard this as a thesis within the domain of prose.

If Wittgenstein is right and we want to avoid being misleading then we should say that the notion of constructing bijections between sets and subsets introduces a new concept of comparing collections. This is then applied in the diagonal proof to introduce a new concept for comparing notions of number (specifically the notion of natural number with a conception of real numbers as expansions):

> It means nothing to say: "*Therefore* the X numbers are not denumerable." One might say something like this: I call number-concept X non-denumerable if it has been stipulated that, whatever numbers falling under this concept you arrange in a series, the diagonal number of this series is also to fall under that concept. (*RFM*, p.128)

What Wittgenstein wants to resist is an impression of Cantor's proof as discovering a pre-existing difference in magnitude between the natural numbers and the real numbers. Instead he wants to see the proof as providing a new way of comparing number-concepts. Part of what contributes to the misleading impression of discovery is an image of the numbers themselves as objects that exist prior to the proof. Wittgenstein undermines this idea of the numbers as pre-existing by stressing that the idea of real numbers as decimal

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[163] Henceforth simply **(0,1)**.

[164] (Fogelin 2009, p.123-125) is useful on this. Unfortunately he goes on to suggest that Wittgenstein may have thought transfinite cardinals unintelligible (2009, p.127). I don't take Wittgenstein to be questioning the intelligibility of any *mathematical* ideas.

[165] See for example (*Gl, §63*).

expansions is just one conception of real numbers. We are only entitled to take the diagonal number as a legitimate real number if we assume that real numbers can be identified by decimal expansions.[166] As Han has argued (2010), the conception of real numbers as expansions is just one conception.[167] There is a separate and, Han argues, historically primary, conception whereby a real number is identified with a particular rule for generation of an expansion of digits. If real numbers are considered as rules for generating expansions then it has not been shown that there is a real number corresponding to the diagonal number unless the rule of the real number can be given or it be shown to be specified by a rule in a way analogous to the rules[168] that we are familiar with (Han gives the method of determination by Taylor series as the most relevant case) for real numbers.[169]

Wittgenstein can acknowledge that Cantor's proof is perfectly legitimate, so long as the diagonal procedure is only taken to be giving sense to a way of comparing our concept of natural numbers with the conception of real numbers as expansions. Wittgenstein acknowledges this as a perfectly 'sober' conclusion (*RFM*, p.131). What he cautions against is being tempted to overlook the difference between the conceptions of real numbers as rules and as expansions and the difference between our ordinary conceptions of magnitude (as bound up with counting) and the conception of comparison by bijections. If we overlook these differences then the result that the 'real numbers' are 'larger' than the natural numbers can look like a "fact of nature" (*RFM*, p.131). This is the sort of picture which Wittgenstein intends to caution against.[170] If we are taken in by this picture then we might be inclined to think of Cantor's work as exploring the nature of the infinite, as though the infinite were an independently-given object of study. Wittgenstein stresses his notion of infinite techniques as techniques stipulated to have no end to them in order to caution against the picture of the infinite as the 'enormously big' that dovetails with a reading of Cantor as making a discovery about the infinite:

> "Ought the word 'infinite' to be avoided in mathematics?" Yes; where it appears to confer a meaning upon the calculus; instead of getting one from it.
>
> This way of talking: "But when one examines the calculus there is nothing infinite there" is of course clumsy--but it means: is it really necessary here to conjure up the picture of the infinite (of the enormously big)?
>
> And how is this picture connected with the *calculus*? For its connexion is not that of the picture | | | | with 4. (*RFM*, p.141-142)

One might think that the infinite was already out there to be studied (either as a Platonic object or in infinitely-sized things) prior to Cantor's work. If the infinite is thought of as being available as an object of study ready for Cantor to explore then the picture adopted seems to be the Platonist conception of the infinite, whereby the infinitely-sized collection is conceived of as being present and available to us in its entirety. It is this picture of the infinite as the 'enormously big' that Wittgenstein is most at pains to reveal as a picture at

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[166] "Cantor shews that if we have a system of expansions it makes sense to speak of an expansion that is different from them all.--But that is not enough to determine the grammar of the word "expansion" (*RFM*, p.134)

[167] Even Cantor says that the notion of real number he worked with was not the accepted one and he expresses "the firm conviction that in due time this extension will come to be regarded as a thoroughly simple, appropriate, and natural one" (Cantor 1932, p.165). This extension of the concept has come to be so widely used that we now need reminding that it is only an extension and not in all respects a natural one.

[168] Taylor series and techniques with their basis in the method of interpolation.

[169] (*RFM*, p.130, p.136).

[170] As we have seen in chapters 3 and 4.

the level of prose and not to be found within Cantor's mathematics. As Wittgenstein sees it, there is nothing in Cantor's mathematics which requires a commitment to the Platonic infinite (and so there is nothing to invalidate Wittgenstein's own conception of infinity). When Cantor gives us a way of comparing our conception of the natural numbers with the real numbers to say that the real numbers are of higher-order infinity, he thereby develops a new conception of 'infinity.' Rather than showing us a pre-existing realm of different types of infinity, Cantor has shown us a new way of comparing number-concepts. It is not necessary to speak of a 'complete' or 'actual' infinity in order to understand Cantor's mathematics and if that talk encourages us to think of the infinite as a kind of object or encourages us to exaggerate the analogy between categorisation of concepts under types of infinity and quantities of collections in terms of counting then it would be better to avoid talk of 'complete' or 'actual' infinity.

Wittgenstein is not only concerned to show a Platonic conception of infinity to be unnecessary for understanding Cantor in order to defend the validity of his own notion of infinity. Wittgenstein also cautions against falling into an elevated idea of set theory's importance because of the 'dizziness' that one feels when in the pull of an illusory image of the infinite (*RFM*, p.137). When one thinks that one finds this image in mathematics then one feels a certain excitement, as though one were being introduced to the "mysteries of the mathematical world" (*RFM*, p.137). But to be drawn to Cantor's work with the intention of discovering the nature of the infinite would, in Wittgenstein's terms, be to be bewitched by language.[171]

9.4. Metaphysical Interpretations of Set Theory

Given that Wittgenstein argues that it is especially misleading to regard mathematical propositions as descriptions (see chapter 4), it is not surprising that Wittgenstein rejects a perspective whereby set theory is seen as a theory 'about sets' or an exploration of the nature of the infinity. As with his comments on Cantor's diagonal proof, Wittgenstein is concerned to separate the mathematics of set theory from a certain prose reading which makes it seem as though complete infinities must figure in set theory.

This image of a complete infinity is likely to be what Wittgenstein has in mind when he says that certain theorems in set theory inspire a certain 'dizziness' (*PI*, §412). There is something paradoxical about taking set theory to be descriptive of completed infinite sets, since the image of a complete infinite leads us into confusion if we try to apply it (we are led to cases like the man counting out the digits of pi backwards). Rather than taking this air of paradox to be indicative of probing into the "mysteries of the mathematical world" (*RFM*, p.137), Wittgenstein want us to see that this air of paradox arises only from a certain prose interpretation in relation to infinite sets. He calls this interpretation 'the theory of infinite aggregates':

> The theory of aggregates attempts to grasp the infinite at a more general level than a theory of rules. It says that you can't grasp the actual infinite by means of arithmetical symbolism at all and that therefore it can only be described and not represented. The description would encompass it in something like the way in which you carry a number of things that you can't hold in your hands by packing them in a box. They are then invisible but we still know we are carrying them (so to speak, indirectly). The theory of aggregates buys a pig in a poke. Let the infinite accommodate itself in this box as best it can. (*PR*, p.206)

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[171] Famously, "philosophy is a battle against the bewitchment of our intelligence by means of language" (*PI*, §109).

If we think that sets exist even before we formulate rules concerning them then we will likely take it that the rules of set theory are formulated so as to capture the features of sets. But this idea begins to look rather strange if one accepts, as Wittgenstein and the intuitionists do, that a complete infinity is an image which does no mathematical work and so any rules which we formulate could not capture it. One might instead follow a Platonic thought and suggest that any rules were inessential and we could get by with just describing the sets. But even this is confusing since how then are we to understand how to use the descriptions?

> If an amorphous theory of infinite aggregates is possible, it can describe and represent only what is amorphous about these aggregates.
>
> It would then really have to construe the laws as merely inessential devices for representing an aggregate. And abstract from this inessential feature and attend only to what is essential. But to what?
>
> Is it possible within the law to abstract from the law and see the extension presented as what is essential?
>
> This much at least is clear: that there isn't a dualism: the law and the infinite series obeying it; that is to say, there isn't something in logic like description and reality. (*PR*, p.221)

The idea of mathematical statements as descriptions, and our difficulty in resisting this idea, is a key part of what misleads us. Wittgenstein says that the inappropriateness of the idea of description to mathematics "of itself abolishes every 'set theory'" (*PR*, p.188). Here I take it as important that he puts 'set theory' in inverted commas − he does not mean the mathematics of set theory but the interpretation of set theory as a theory about sets. And this prose interpretation gets a particularly strong grip upon us because some of the symbolism of set theory is suggestive of it:

> Set theory builds on a fictitious symbolism, therefore on nonsense. As if there were something in Logic that could be known, but not by us. If someone says (as Brouwer does) that for $(x) \cdot f_1 x = f_2 x$ there is, as well as yes and no, also the case of undecidability, this implies that '$(x)...$' is meant extensionally and that we may talk of all x happening to have a property. (*PR*, p.33)

Note that Wittgenstein is not here saying that set theory is nonsense or that all extensional mathematics is nonsense. Preceding this remark we find Wittgenstein saying that the "expression '$(n)...$' has a sense if nothing more than the unlimited possibility of going on is presupposed."[172] Rather, the point is that we should not be misled by the symbolism '(x)' (or terms like 'first member' and 'last member'[173]) to take a statement employing this symbolism to be a descriptive statement *about* xs, as though the xs were there to be referred to prior to the system's being formulated.

Whilst this kind of symbolism may tempt us towards a misleading interpretation, Wittgenstein does not take it to undermine the mathematics of set theory:

> When set theory appeals to the human impossibility of a direct symbolization of the infinite it brings in the crudest imaginable misinterpretation of its own calculus. It is of course this very misinterpretation that is responsible for the invention of the calculus. But of course that doesn't show the calculus in itself to be something incorrect (it would be at worst uninteresting) and it is odd to believe

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[172] See also (Therrien 2012, p.55-56)
[173] (*PR*, p.33)

that this part of mathematics is imperilled by any kind of philosophical (or mathematical) investigations. (*PG*, p.469)

If we are to be clear about this distinction between prose and mathematics for set theory then we will need to see how set theory can be understood in a way which releases us from the pull of the interpretation of set theory as about completed infinite sets. We need to see how we can interpret set theory as compatible with a less misleading conception of the infinite i.e. as a calculus in which there are infinite sets in the sense that the set is specified to have no limit on its elements (not as a completed infinity). To this end, it is worth discussing cases of axioms of set theory for which it is particularly difficult to resist the pull of the image of the statement as a description of (completed infinite) sets, beginning with the power set axiom.

The axiom states that every set has a power set, given by the set of all of its possible subsets. For a finite set, $\{x, y, z\}$, its power set would be:

$$\{\{\}, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}^{174}$$

The power set of a set 'S' is typically denoted by 'P(S).' The image that tempts us with this axiom, and especially with illustrations such as that given here, is an image of laying out the set and dividing its elements in every way permissible. Whilst this image is one that we can get a clear grip upon for a finite set, it becomes rather confusing if we attempt to see the same idea as simply extended to an infinite set. For an infinite set, this laying out and dividing would seem to require that all of the infinite collection of elements were somehow accessible for being divided up. How are we to understand the power-set axiom as a rule which fixes meanings rather than as an operation to be performed?

Instead of seeing the power set axiom as describing an operation to be performed, we can see it as licensing a certain kind of inference. The role (and hence the meaning) of the power set axiom is that we can make certain inferences within the system (or calculus) which we would not be able to make without the axiom. We can therefore understand the power set axiom not as an operation or descriptive proposition but as the rule 'if T=P(S) then any subset of S is a member of T.' This rule can be particularly useful when we want to make use of some property in relation to T and S, perhaps using a particular subset with a special property or just taking a general subset. This can be made clearer by example of the use of the power set axiom.

Here is a textbook proof that a set X cannot be put into one-one correspondence with its power set:

> Assume that f is a one-to-one mapping from *X* onto P(X); our purpose is to show that this assumption leads to a contradiction. Write $A = \{x \in X : x \notin f(x)\}$; in words, *A* consists of those elements of *X* that are not contained in the corresponding set. **Since A $\in$ P(X)** and since f maps *X* onto P(X), there exists an element *a* in *X* such that $f(a) = A$. The element a either belongs to the set *A* or it does not. If $a \in A$, then, by the definition of A, we must have $a \in f(a)$, and since $f(a) = A$ this is impossible. If $a \notin A$, then, again by the definition of A, we must have $a \in f(a)$, and this too is impossible. The contradiction has arrived... (Halmos 1974, p.93)

The use of the power set axiom is highlighted. A is stipulated to be a subset of X and therefore it is taken to be a member of P(X). I include this here merely to show that there

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[174]The example is from (Puntambekar 2007, p.1-2). The {} is the empty set.

need not be anything particularly remarkable about saying that an axiom is used as a rule or that a proof can be seen as a derivation of a result by application of rules – a conception which makes no appeal to an actual infinity as an object.

The power set axiom need not be construed as an instruction (in a constructivist sense) about what to do with sets, nor need it be viewed as a description of the nature of sets. Instead, the axiom can be viewed as a rule within a system of rules. Whilst the image of elements being combined can be a difficult one to resist for set theory, it is possible to see set theory as a calculus if we focus upon the rules by which sets are defined. From this perspective it is better to resist saying that the power set axiom tells us something of what we can do with sets or of how the set hierarchy can be constructed. Rather, it would be better to say that the axiom partially defines the term 'set' by playing a key role in the system in which the term 'set' has meaning.

The most controversial of the axioms in set theory has been the axiom of choice, with various constructivist and even classical mathematicians expressing qualms about the way in which the infinite figures in the axiom. But Wittgenstein is willing to acknowledge the axiom of choice as mathematics (*RFM*, p.400). As with the power set axiom, if we try to see the axiom of choice as a kind of license or permission then we will be able to separate the axiom of choice from the prose interpretation which suggests itself and so see why Wittgenstein is willing to acknowledge the axiom of choice as mathematics.

Informally put, the axiom of choice states that for any collection of non-empty sets it is possible to select one element from each set. More formally:

> If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function f on S such that
>
> $(5.1)\ f(X) \in X$
>
> for every $X \in S$.

> The Axiom of Choice postulates that for every S such that $\emptyset \notin S$ there exists a function f on S that satisfies (5.1). (Jech 2006, p.47)

This applies in infinite as well as the finite cases and the axiom allows that there is no need for a method of selection to be specified. Wherever a definite choice function can be specified there is no need to use the axiom of choice. Rather, the axiom is of use when a choice function cannot be specified. Because the function cannot be specified, the axiom of choice allows that an object (a choice function or choice set) can exist without being fully specified.

One might think that Wittgenstein would object to the idea of selecting elements from an infinite collection of infinite sets on the basis that the idea of such a selection implies or presupposes a completed infinity. However, this line of thinking would make Wittgenstein's ideas inconsistent and in my view should not be attributed to him. For Wittgenstein a completed infinity is only a metaphysical image and cannot be a part of a consistent mathematical system or indeed any mathematical system. Whilst we may associate an image of a completed infinity with the axiom of choice, this is not stated in the axiom itself. The informal statement of the axiom that I gave above perhaps adds to the enticement of the image by speaking of 'selection' but the formal statement is less enticing.

The axiom of choice can be seen, in Wittgensteinian fashion, as licensing inferences involving choice functions without requiring such choice functions to be fully specified – where we are not able to provide a choice function then we use the axiom of choice to

postulate one.[175] Wittgenstein has no reason to object to the allowing of these inferences as he can simply take this license to be part of what defines the system in question.

To repeat, Wittgenstein is certainly critical, at least in earlier writing, of the image associated with the axiom of choice. He says:

> I can make something like a random selection from a finite class of classes. But is that conceivable in the case of an infinite class of classes? It seems to me to be nonsense. (*PR*, § 146)

Marion cites this remark and notes that Wittgenstein "simply rejected in the strongest possible terms the idea of an infinite number of selections for which no rule could be stated" (1998, p.76). Certainly Wittgenstein is questioning whether we really have a clear picture of this 'selection' and Wittgenstein's references to the axiom of choice in *RFM* suggest that he felt that something is not entirely clear when it comes to this axiom (*RFM*, p.283; *RFM*, p.400) and it is likely that what he felt to be unclear was the *picture* of an infinite number of selections without a rule. On this I agree with Marion but I disagree with Marion's claim that it puts Wittgenstein in the "camp of Kroneckerian finitists" (Marion 1998, p.76). The axiom of choice can be seen to be treated by Wittgenstein as a limiting case of a mathematical proposition since it is a proposition which gives license to certain inferences but which we don't fully understand apart from its applications. Wittgenstein's view can hence be seen to centre on the confused picture/s attached to the axiom of choice.[176]

It may well be objected that it is reductive to strip the axiom of choice of its associated picture and see it instead merely as a rule that a choice function need not be specified. Perhaps with the various formulations and uses of the axiom of choice it might turn out that the role played by the axiom is more sophisticated. This in itself need not be telling against Wittgenstein, so long as the axiom can be seen as operating as a rule (as was illustrated with the user of the power set axiom).

9.5. Chapter Conclusion

Whilst Wittgenstein acknowledges the axiom of choice as mathematics, he also says that it is not a core example of what we mean by mathematics. Specifically, he says:

> Mathematics is, then, a family; but that is not to say that we shall not mind what is incorporated into it.
>
> We might say: if you did not understand any mathematical proposition better than you understand the Multiplicative Axiom[177], then you would not understand mathematics. (*RFM*, p.399-400)

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[175]Friederich (2011, p.15-16) similarly talks of an axiom as "(normatively) licensing a certain conceptual connection". He discusses proofs using Zorn's lemma either as an axiom or as a theorem derived from the axiom of choice and notes that the step in the proof at which Zorn's lemma is used is the same in each case. I feel my point fits well with Friederich's – I wish to argue that axioms and theorems in set theory conjure up images of completed infinities when considered in isolation but these images can be dispersed by focusing on the employment of the axioms in the relevant steps of particular theorems.

[176]A point which Marion acknowledges would be a reasonable and non-revisionary view (1998, p.73).

[177]The axiom of choice.

This is one of many of Wittgenstein's statements which can easily look revisionary but which can be seen as non-revisionary after careful reading. His point is that our understanding of the axiom of choice is not a paradigm case of mathematical understanding. As we have seen, mathematics, for Wittgenstein, is best seen as being all about specifying rules and the axiom of choice is a rule that lets us get by when we don't have the kind of more specific rule that we would otherwise need. So it is natural for Wittgenstein to use a core-periphery contrast[178], with the axiom of choice at the periphery and more paradigmatic cases of mathematical techniques at the core.

I believe it is Wittgenstein's main criticism of set theory in general that it is more appropriately seen as peripheral to mathematics than as the foundation of mathematics. This would be a judgement concerning set theory's importance, not its validity. Wittgenstein is admittedly rather brief in his explanation of how to read set theory without the need to invoke the idea of a complete infinity. Nonetheless, there is enough in his remarks (together with his remarks on Cantor's diagonal proof) to see his reading in outline. If this reading is consistent then there is no need to see some of the more dismissive-looking remarks of Wittgenstein's remarks as rejecting part or all of set theory as invalid.

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[178] As Wittgenstein uses the term 'family', it is normal for there to be a core and periphery to the family.

Chapter 10 - The Significance of Gödel's First Incompleteness Theorem

10.1. Chapter Introduction

Wittgenstein's remarks on Gödel's first incompleteness theorem have proved highly controversial. Having received an initial reception of "nearly unanimous condemnation" (Floyd 2005, p.116), more recently there have been commentators who have tried to mount a defence.[179] The polarisation of the views to be found in the secondary literature is in part due to the difficulty of interpreting Wittgenstein's remarks. Part of what makes the remarks on Gödel's first incompleteness theorem so difficult is that so many issues can be taken to be involved in them and many of these issues are particularly subtle. Does Gödel's theorem represent an argument for mathematical Platonism?[180] Does it show that truth cannot be equated with provability? In what way is Gödel's theorem mathematics about mathematics? How can an expression be demonstrated as unprovable without showing it to be false? With issues like these being involved, questions are naturally raised about how such issues might figure in other parts of Wittgenstein's philosophy and especially whether Wittgenstein's comments might have been motivated by some agenda with regard to one of these issues. There are also questions concerning what Wittgenstein was even talking about in his remarks. Whilst Gödel's first incompleteness theorem undoubtedly figures, it is not clear that he meant to comment on the proof itself. Wittgenstein comments that his "task is, not to talk about (e.g.) Gödel's proof, but to by-pass it" (*RFM*, p.383). A remark such as this suggests that Wittgenstein's broader philosophical methodology also something that cannot be ignored in order to understand the remarks on Gödel.

Not every strand of the relevant issues can be discussed here. Floyd comments that "a detailed line-by-line treatment of Wittgenstein's writings on Gödel lies beyond the scope" of her 2001 paper (2001, p.286) and my scope is much more limited still. My purpose is to use work by Floyd (1995; 2001), Shanker (1988) and Floyd and Putnam (2000; 2006) to show that Wittgenstein's remarks on Gödel can be read in a way which is consistent with (*PI*, §124). This requires showing that Wittgenstein can be read as not criticising Gödel's proof itself. Instead Wittgenstein will be read as trying to put pressure upon or block certain misleading interpretations of the proof's significance and as doing so without himself advocating or presupposing a dogmatic thesis (and so his remarks are not motivated by e.g. a thesis concerning truth or proof).

Wittgenstein's remarks on Gödel's theorem can be taken as an interesting study in the way in which a prose translation of mathematical symbolism can be highly misleading. As Floyd puts it:

> Wittgenstein viewed Gödel's result as a striking example of a theorem that invites uncritical metaphysical speculation, and uncritical metaphysics was always for Wittgenstein an intellectual tendency to be investigated and unmasked. (Floyd 2001, p.286)

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[179] Floyd (1995; 2001), Shanker (1988) and Floyd and Putnam (2000; 2006).
[180] Gödel himself later claimed that it did. See the discussion in Gödel's unpublished lecture (Gödel 1951).

As an example of the metaphysical speculation that the theorem might invite, Gödel's first incompleteness theorem is sometimes thought to be in support of a Platonist view. The claim is that the theorem establishes the existence of propositions which are true and not provable. Since these propositions are not provable, the argument goes, they must be true in virtue of something other than the axioms i.e. some mathematical reality which is prior to the axioms.[181]

Whilst I do think that Wittgenstein would be critical of such an argument, it is important to note that Wittgenstein was not a verificationist (see the interpretation presented in chapter 6) and did not claim that mathematical propositions always need a proof in order to be meaningful (Floyd 1995, p.385). Rather, Wittgenstein would be critical of this kind of argument because it is not clear in what sense the theorem establishes the existence of 'propositions which are true and not provable.' An English expression like this can easily simplify and distort the mathematical purport of a proof. Wittgenstein aims to present the significance of Gödel first incompleteness theorem in a way that can reveal what is misleading about this kind of prose.

Wittgenstein's point is not an easy one to see at first and so it will help to clearly establish sufficient background before approaching it directly.

10.2. Wittgenstein and Proofs of Impossibility

We have a (philosophical) tendency to try to assimilate mathematical statements to empirical descriptions and in relation to truth this tendency can encourage us to think that statements within a system can be made true or false by considerations that are external to the system itself, in something like the way that we might be inclined to say that "Everest is the tallest mountain" is made true by physical relationships. We might then think that we can do 'mathematics about mathematics' in much the way that we can say that "Everest is the tallest mountain" is about a mountain. Wittgenstein can be taken as saying that if we do make such a move in mathematics (i.e. we do show a statement to be true or false by considerations that are not part of the techniques of the system at issue) then we have subtly shifted the subject and we are then talking about a different system (since a new technique has been added to it) and hence a different sentence (since it is part of a different system).

Wittgenstein's line of thinking can be hard to recognise because of the prevalence in parts of contemporary mathematics (and philosophy of mathematics) of a model-theoretic approach to truth. In loose terms such an approach would take a system as a set of syntactic rules and then consider the possible structures (described set-theoretically) which would satisfy the rules – these are the models of the system. A sentence of the system is then described as true if it satisfied in all consistent models of the system.

Rather than accepting a model-theoretic approach to truth as an analysis of the essence of truth, Wittgenstein would presumably have regarded a model-theoretic approach as just another proof-technique. He might have said that there is a sense in which applying a model-theoretic technique to a system that was previously only being treated syntactically then changes the system that we are talking about. The point is that introducing a technique for demonstrating propositions as true into a system must, for Wittgenstein, change that system. It is worth being clear about why this is and why it matters, since the question relates so closely to the interpretation of Wittgenstein's remarks on Gödel.

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[181]In this vein Penrose claims that there must be "something absolute and 'God-given' about mathematical truth" (Penrose 1989, p. 112). Cited by Floyd (2001, p.298).

The point concerns what it is to look at a system model-theoretically. One understanding (which we have reason to think Wittgenstein would object to[182]) is that when we look at a system syntactically then we are looking at what propositions follow from the roles given to the symbols alone and when we move to the semantic level (the model-theoretic level) then we are considering the ways that the objects of the system can be configured so as to make the possible formulas of the system true or false. On this understanding, it will seem like both levels of analysis will tell us which propositions are true or false and the propositions were true or false even before we came up with a way to do the analysis (since the propositions seem to have a subject-matter of mathematical objects, or possible objects). Wittgenstein would say that what is called a 'model-theoretic analysis' is a kind of proof-technique, since it gives us a way to show which propositions are true and which false. The consideration of possible models should not be regarded as a venture into the realm of possible mathematical objects, but rather a technique based upon the consideration of possible truth-values for well-formed formulas of the system. Regarded in this way, it seems justifiable to say that a system considered syntactically (i.e. in which model-theoretic proof is not allowed) is not the same system as when model-theoretic techniques are allowed, since we can count propositions as true model-theoretically which we could not count as true otherwise. To say this is not to object to a model-theoretic conception of truth. Rather, it need be no more than a caution to be careful about when we are using the model-theoretic conception and when we are employing a syntactic conception of truth. We should not assume that a truth is a truth and that it doesn't matter how we arrived at it – for mathematical propositions it does matter how we arrive at the truth since this is bound up with what system the proposition belongs to.

Whilst these questions related to model-theoretic approaches to mathematics are relevant, the version of Gödel's theorem that Wittgenstein remarked upon was not a model-theoretic proof. As Floyd and Putnam note (2006, p.107), Wittgenstein only saw a version of Gödel's proof which does not use model-theoretic methods. A model-theoretic version was subsequently developed by Rosser (1936) and this has led to a perception that Wittgenstein was commenting on the Rosser version.[183] For now I wish to be clear that I am taking Wittgenstein only to have commented on the Gödel version and I will return to the relationship of the comments to the Rosser version later.

In relation to Gödel's results, Wittgenstein aims to show us that if the claim that a sentence is 'true but unprovable' is to be a theorem (i.e. a claim for which we have a proof) then it must be a claim in a different system from the system at issue since if we do not have a proof in the system then the sentence is not true in that system (remembering that we are excluding model-theoretic considerations). The question naturally arises of what system such a claim would be located in. If a sentence is constructed which we are tempted to call 'true but unprovable', then in what way can that sentence tell us something about the system at issue? More generally, how can we have a proof that a proposition is not provable?

In order to better understand this question with regard to Gödel's theorem, Wittgenstein compares it to the question "Can there be true propositions in the language of Euclid, which are not provable in his system, but are true?" (*RFM*, I Appendix III §7[184]). Wittgenstein mentions (*RFM*, I Appendix III §14) a particular an example of such a proof

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[182] The reason for thinking this is Wittgenstein's reading of what metamathematics is. See Shanker (1988, p.162, p.185-195), Floyd (2001), Floyd and Putnam (2006, p.104-106) and also Mühlhölzer (2012).

[183] (Floyd and Putnam 2006, p.105) point to Bays (2004) and Steiner (2001) as taking Wittgenstein's comments to relate to the Rosser version. The situation is complicated by a brief sketch given but not advocated in Gödel's 1931 paper (1986, p.149-151).

[184] I use remark numbers rather than page numbers for this section since the remarks in Appendix III are so few and brief. They span p.116-123.

in this connection – a famous proof concerning the trisection of an angle using ruler and compasses.[185]

The ancient Greek approach to doing geometry was strongly visual (as opposed to algebraic) and relied heavily on producing diagrams and techniques of manipulating diagrams (Shanker 1988, p.163). It was widely known that an angle (drawn on a diagram) could be bisected using such techniques and so there was interest in whether an angle could be divided in other ways using the same (or similar) techniques. Nobody was able to find a way to trisect an angle and nor was anybody able to arrive at an explanation as to why, until Wantzel published an impossibility proof in the 19th century.[186] The details of Wantzel's proof are not of importance to this discussion but the form of his proof is. Whilst Wantzel's proof may offer explanation of why the Greeks were unable to find a proof, it is not a proof framed in their language – it is not a geometrical proof. Rather, Wantzel begins his proof by explaining a way to translate problems of constructing shapes with straight lines and circles into algebraic terms which allow analysis in terms of quadratic equations (Shanker 1988, p.164). The proof is then an application of this technique to the case of the trisection.

The Wantzel proof convinces us that we should not look for a trisection of an angle with ruler and compasses within Euclid's system. It does not tell us anything mathematical about the expression "trisection by ruler and compasses" *within Euclid's system* as we had no proof or disproof of that expression within Euclid's system (Shanker 1988, p.186, p.196). But the proof does convince us that we should not look for a way of trisecting the angle via ruler and compasses – it shows us that we will not be able to ascribe the kind of meaning that we would want to the expression "trisection by ruler and compasses" within Euclid's system (i.e. we could not give it a meaning which would be faithful to the system).

Wantzel's proof convinces us that we can't ascribe the kind of meaning that we would want to "trisection by ruler and compasses" within Euclid's system by setting up an extended system in which all of the techniques of Euclid's system figure (or we might say that they are presupposed) but in which we also have some new algebraic techniques (Shanker 1988, p.196). Using the algebraic techniques, Wantzel finds an algebraic way to specify what would count as a trisection by ruler and compasses (Floyd 1995, p.391) and proves that such a trisection is not part of the (extended) system (i.e. he finds a way to specifically exclude it).

So Wantzel's proof shows us that a trisection of the angle by ruler and compasses is excluded in a system which is in significant ways similar to the system of Euclid. This convinces us that there is no point looking for something similar to a trisection by ruler and compasses in Euclid's system. Wantzel's proof is undoubtedly of relevance to how we view certain expressions connected with Euclid's system (in particular the expression "trisection by ruler and compasses") but it does not reveal to us any new rules for expressions within Euclid's system (Floyd 1995, p.392). Seeing this distinction is important for being clear about the significance of another impossibility proof – namely Gödel's First Incompleteness Theorem.

Whilst it might seem natural to discuss some summary statement of what the theorem proves, this is problematic for the case at issue. Typical statements of the theorem in English would be likely to prejudice the discussion since the shortcomings of these prose statements as statements of the theorem is part of what is at issue. As Wang says, "apart from the matter of proving Gödel's theorem, just to interpret the statements of it …

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[185]Discussed at some length by Shanker (1988, p.163) and Floyd (1995).

[186]See Cajori (1918, p. 346–7).

becomes a complex task from Wittgenstein's perspective" (1991, section 6.2[187]). Instead we should first explore some of the key mathematical ideas involved in the proof of the first incompleteness theorem.

10.3. The Gödel Numbering Technique

The proof of Gödel's first incompleteness theorem involves some key mathematical notions developed in order to model features of a formal system (such as the system of *Principia Mathematica*). These rely on a device called Gödel numbering, which is a system whereby each symbol of the system is given a Gödel number and each expression which is possible in the system is then given a Gödel number derived from the Gödel numbers of the symbols from which it is composed. Using this device, we can speak of each statement or sequence of statements which is expressible in the system as having a Gödel number.

The power of Gödel numbering becomes clearer when the idea is applied to deductions and proofs. Deduction rules in the system can also be encoded using Gödel numbering (using the observation that a deduction rule is a transition from one arrangement of symbols to another). Proofs can be thought of as applications of deduction rules to statements which are either axioms or already proven. In this way it is possible to formalise a relation which expresses whether a statement is a proof of another statement within the system. This can be expressed as the relation Proof(x,t), where x is the Gödel number of a proof and t is the Gödel number of an expression which (if the relation holds) is established by the proof. This relation makes it possible to make deductions about what can and cannot be proven in the system.

Key to Gödel's first incompleteness theorem is the Gödel sentence. The Gödel sentence is a particular sentence of a formal theory which is constructed so that the expression Proof(x,t) is used with t as the Gödel number of the expression itself. The Gödel sentence is so constructed that it can be read as saying that no natural number encodes a proof of the Gödel sentence:

$$\neg(\exists x)(\text{NaturalNo.}(x).\text{Proof}(x,\mathbf{t}))$$

Here 'NaturalNo(x)' is another expression expressible in the formal system. It is crucial to keep in mind that these terms stand in for arithmetical expressions (which are being omitted because of their complexity). We can read this statement as "there does not exist a proof numbered by the natural number x and culminating in the expression numbered by t" but we need to keep in mind that this is a rendering in English that may not map perfectly (at least not in all circumstances) to the corresponding arithmetical expression. Using this expression, the proof goes on to show that if the formal system in question is consistent then this formula is unprovable (i.e. proving it would violate the consistency of the system). And if the system is ω-consistent[188], then the negation of the expression is also not provable.

Putting aside the points about the limitations of an English rendering of an arithmetical expression, we are now in a position to see how tempting it might be (especially for a Platonist) to read Gödel's theorem as establishing "true but unprovable sentences." Wittgenstein's remarks do not meet this head on and deny that there can be situations

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[187]Cited by Floyd (1995, p.374).

[188] A system is ω-inconsistent if it proves P(n) for all natural numbers n (and predicate P) but the general case of P(n) fails (i.e. the predicate is satisfied by something other than a natural number).

where we might want to say that there are true but unprovable sentences. Rather, Wittgenstein wants to stress "true but unprovable" as an interpretation in prose by showing that there can be different ways to take this expression. That there should be different ways to take this expression is not immediately obvious, not least because we ordinarily take it that we know what it means to call a sentence 'true' and that all meaningful sentences are simply either true or false. But Wittgenstein stresses that a particular string of symbols may be a truth in one mathematical system and not a truth in another. Understanding this point helps greatly in understanding Wittgenstein's remarks on Gödel's First Incompleteness Theorem.

10.4. Prose Translations and Gödel's First Incompleteness Theorem

Just as we need to be careful about which system we're talking about when we use a prose expression such as "trisection by ruler and compasses" in relation to Wantzel's proof, so we also need to be careful which system we're talking about when we use a prose expression such as "true but unprovable" in relation to Gödel's. Wittgenstein aims to bring out the subtleties of the uses of the prose expression by considering a particular case that might arise from application of Gödel's theorem. Wittgenstein imagines an interlocutor who says the following:

> "I have constructed a proposition (I will use 'P' to designate it) in Russell's symbolism, and by means of certain definitions and transformations it can be so interpreted that it says: 'P is not provable in Russell's system.' Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable." (*RFM*, I Appendix III §8)

So the interlocutor is imagining that a sentence is constructed in Russell's system (surely the system of *Principia Mathematica*[189]) using Gödel's techniques – in essence a Gödel sentence. Then the interlocutor frames an informal argument to the effect that the sentence is true but unprovable. Wittgenstein's response is to prompt us to ask the questions "provable in what system?" and "true in what system?"[190] The point seems to be to show us that the moves which the informal argument makes are not moves available in Russell's system:

> "True in Russell's system" means, as was said, proved in Russell's system, and "false in Russell's system" means the opposite has been proved in Russell's system.--Now what does your "suppose it is false" mean? *In the Russell sense* it means, "suppose the opposite is proved in Russell's system"; *if that is your assumption* you will now presumably give up the interpretation that it is unprovable. And by "this interpretation" I understand the translation into this English sentence. (*RFM*, I Appendix III §8 continued)

When the informal argument asks us to suppose the sentence to be true or false, it is not clear what this supposition is meant to amount to. Either the sentence is true/false in the Russell sense (the sense of having a proof of the sentence or its contrary in Russell's system) or it is true/false in some other sense. If it is the Russell sense of true/false that is intended then the translation of the Gödel sentence that the informal argument presses is one which can be shown on inspection to be at odds with the workings of Gödel's theorem.

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[189]It is *Principia* referred to in the title of Gödel's 1931 paper (1986).
[190] Cited from the same remark as the above.

If it is the Russell sense of 'false' which is intended by "suppose it is false", then the informal argument seems to take it for granted that the sentence P will be interpreted as 'P is not provable in Russell's system' in all situations. But we are only entitled to translate the sentence in this way if the conditions of Gödel's theorem are met. This means that if the negation of P has been proven then we have to assume that either Principia is inconsistent or it is ω-inconsistent.[191] As Floyd and Putnam (2000, p.2-3) argue, if Principia is ω-inconsistent then the mechanism of Gödel numbering no longer enables us to interpret the sentence P as asserting of itself that it has no proof. It must be remembered that the Gödel numbering mechanism only serves to construct an arithmetic sentence which we designate by the form "¬(∃x)(NaturalNo.(x).Proof(x,t))". If the system is ω-inconsistent then the arithmetical predicate designated by 'NaturalNo.(x)' will be satisfied by arguments which are not natural numbers. Likewise the predicate 'Proof(x,t)' will be satisfied by arguments which are not natural numbers (and so cannot serve as Gödel numbers of expressions). So the translation 'P is not provable' has to be given up. The other alternative is that the system is inconsistent and an inconsistent system can prove anything so the correlation of P with the English sentence 'P is not provable' then looks like we would want to give it up.[192]

It is also not clear how the argument can be entitled to translate P as 'P is not provable' in the case where P is supposed to be true in the Russell sense:

> If you assume that the proposition is provable in Russell's system, that means it is true *in the Russell sense*, and the interpretation "P is not provable" again has to be given up. If you assume that the proposition is true in the Russell sense, *the same* thing follows. (*RFM*, I Appendix III §8 continued)

If a proof has been found of P in Russell's system then Russell's system is inconsistent (there is no condition of ω-consistency for the Gödel sentence in the way that there is for the negation of the Gödel sentence – consistency is the only condition). So again we would presumably give up the interpretation.

Perhaps the informal argument intends for P to be true in some sense other than the Russell sense. Wittgenstein questions:

> "But may there not be true propositions which are written in this symbolism, but are not provable in Russell's system?"--'True propositions', hence propositions which are true in *another* system, i.e. can rightly be asserted in another game. Certainly; why should there not be such propositions; or rather: why should not propositions--of physics, e.g.--be written in Russell's symbolism? The question is quite analogous to: Can there be true propositions in the language of Euclid, which are not provable in his system, but are true? (*RFM*, I Appendix III §7)

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[191]Gödel's theorem does not just concern the Gödel sentence – it also shows that the negation of the Gödel sentence has no proof, provided that the system is ω-consistent. *Principia* would be ω-inconsistent if *Principia* proved P(n) for all natural numbers n but the general case of P(n) fails (i.e. the predicate is satisfied by something other than a natural number).

[192]Floyd and Putnam note (2000, p.3) that this is "is not to deny that in various contexts, and for various reasons, we may want to correlate its sentences with sentences in English." It is just that we would want to give up the interpretation of P as 'P is not provable.'

Perhaps what the informal argument intended was to call P true in some unspecified and system-independent sense.[193] Whatever such a sense of 'true' might be, Wittgenstein puts pressure on the idea that there is any need to invoke it by comparing the significance of P to a proof such as Wantzel's.

Just as there is nothing within Euclid's system which says (without invoking non-Euclidean algebraic techniques) that the trisection of the angle is impossible, so Russell's system does not say that the Gödel sentence is unprovable. An analysis of the Gödel sentence will convince us not to attempt to find a proof of it within Russell's system (Floyd 1995, p.406) but this would not yield any result which decides the truth or falsity of the sentence within Russell's system. Our being convinced that we should not pursue a proof of the sentence does not reveal that we see that it is true in some deeper way that goes beyond Russell's formalism. It simply shows that the sentence has been so-constructed (using the Gödel number technique) so as not to admit of a proof. Much as Wantzel manages to show us that there is nothing we would be prepared to call a trisection within Euclidean geometry, Gödel shows us that there is nothing we would call a proof of P (provided the conditions of his theorem are met). This point can be accepted without asking whether P is true or false:

> One could put the point this way. One often hears statements about 'true' and 'false'-for example that there are true mathematical statements which can't be proved in *Principia Mathematica*, etc. In such cases the thing is to avoid the words 'true' and 'false' altogether, and to get clear that to say that P is true is simply to assert P and to say that P is false is simply to deny P or to assert ¬P. It is not a question of whether p is 'true in a different sense.' It is a question of whether we assert p. (*LFM* p. 188[194])

As Floyd puts it, using a phrase of Wittgenstein's student Watson, "to construe Gödel's theorem in terms of the general notions of *true* and *unprovable* 'obscures rather than illuminates' the point of Gödel's proof" (2001, p.283).

The question "why should not propositions--of physics, e.g.--be written in Russell's symbolism?" (*RFM*, I Appendix III §7) is especially illuminating. One might want to say that P is simply true in the way that "Everest is the tallest mountain" is true. But it is not clear what it would mean to say that the string of symbols constructed to form P is true or false in virtue of something in the world. If it is to be called true or false within a particular system, then there surely must be some technique we employ by which to call it true or false.

If P were to be replaced by a quite different kind of sentence then there might be a clear sense in which we would want to call it true. As I mentioned previously, a different version of Gödel's proof was later developed by Gödel and Rosser and in this version of the proof the sentence in question is true in the Tarskian model-theoretic sense. Some of the criticism of Wittgenstein's remarks in relation to Gödel's theorem have taken Wittgenstein to have been using the informal argument to propound and then reject the *Rosser* version of the proof (Steiner 2001, p.259; Bays 2004, p.207). As I also noted previously, this attribution is unfair since Wittgenstein would have been unaware of the Rosser version beyond seeing a brief sketch given but not advocated in Gödel's 1931 paper (1986, p.149-

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[193]This is what Floyd and Putnam call the metaphysical claim that "there is a well defined notion of 'mathematical truth' applicable to every formula of *PM*" (2006, p.9).

[194]Cited by Floyd (2001, p.300). She also notes that Wittgenstein need not be read as holding a redundancy theory of truth (1995, p.401) and rather as making a particular point regarding what use it is to call the sentence at issue true or false.

151). Following Floyd and Putnam (2006, p.105-106), it is worth stressing the ways in which this reading seems to be quite unfair to Wittgenstein.[195]

Firstly, Wittgenstein's points are made in regard to the sentence P constructed in terms of the original Gödel proof. If it were a sentence constructed in terms of the Rosser proof, then the point regarding giving up the interpretation of P in the case of a proof of the negation of P because of ω-inconsistency becomes irrelevant since ω-inconsistency is not a condition of the Rosser proof (Floyd and Putnam 2006, p.105). Floyd and Putnam (2000, p.3-5) cite historical evidence[196] to support the attribution of the ω-inconsistency point to Wittgenstein.

Secondly, Wittgenstein is discussing *Principia Mathematica* ("Russell's system") and the Rosser proof concerns Peano Arithmetic. The soundness of Peano Arithmetic can be safely assumed (Floyd and Putnam 2006, p.105-106) and thus the idea of supposing a Rosser-style sentence to have a proof or disproof in Peano Arithmetic has a very different significance (2006, p.105) and would naturally look like a misunderstanding on Wittgenstein's part (if we do indeed interpret him to have been discussing a Rosser sentence).

Thirdly, the Rosser proof is a model-theoretic proof which shows the Rosser sentence to be true in the specific Tarskian sense. There would surely be no need (unless Wittgenstein really were either deeply suspicious or incompetent) to ask in what sense the sentence were true if it were a Rosser proof at issue (Floyd and Putnam 2006, p.105). [197]

The question therefore arises, if Wittgenstein was discussing Gödel's 1931 version of the proof and not the Rosser version then is what Wittgenstein says now of only historical relevance? If it is granted that we can translate the Rosser sentence as 'true but unprovable in Peano Arithmetic' then can we not again formulate a metaphysical argument such as the argument for Platonism given before?

Whilst this question goes outside the realms of pure Wittgenstein interpretation, it seems to me that the appropriate Wittgenteinian thing to say would be, following Floyd (2001, p.304) that the Rosser sentence is 'true' only in the specific Tarskian mathematical sense of 'true.' The metaphysical questions only arise if 'Tarski-true' is taken to mean 'true' in an absolute way that is supposed to apply across all systems and contexts. That is a whole different issue and beyond the scope of this thesis.

10.5. Chapter Conclusion

Despite appearances to contrary, Wittgenstein's remarks on Gödel's first incompleteness theorem do not have to be read in a way which commits Wittgenstein to any particular philosophical thesis or to a misunderstanding of the theorem. Wittgenstein can instead be read as pointing out that the proof does not support a certain tempting informal argument which seems to show the existence of a 'true but unprovable' sentence of *PM*. The prose

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[195]This is not to deny that Bays and Steiner may raise some good points regarding what can be said in interpretation of the Rosser version but my interest is in what can be fairly attributed to Wittgenstein.

[196]Specifically the testimony which they reference of Goodstein and Watson.

[197]Floyd notes (2001, p.300) that Turing was present during the lecture when he made the remark given above - (*LFM* p. 188). Watson's testimony suggests that Turing and Wittgenstein had discussed the matter before, which would explain why Turing did not accuse Wittgenstein of a mistake as he might have done if Wittgenstein were talking of the Rosser version.

translation 'true but unprovable' is actually a translation of the mathematical sentence which we would give up in certain key circumstances. Rather than being forced upon us, the prose translation is misleading in that it obscures rather than enlightens us with respect to the details of how Gödel's proof works.

Conclusion

At points I have suggested that readings of Wittgenstein on mathematics have not always been entirely charitable. My point is not that anybody has read Wittgenstein with the intention of finding sweeping and questionable assertions in his work. Though it is quite possible that some have done this, it is of no interest to me if they have. My point is rather that if we are to be charitable to Wittgenstein then we have to understand what he takes himself to be doing - what he takes his project to be. Many sensitive readers who have done much to further the reception of Wittgenstein have failed to understand Wittgenstein on mathematics for this very reason.

Few have done more to further the reception of Wittgenstein than Hilary Putnam, and yet Putnam is led to attribute "flat-unbelievable assertions" (2007, p.241) to Wittgenstein, whilst expressing his surprise and frustration at finding these assertions in the work of a philosopher who "doesn't put forward 'theses'" (2007, p. 239). It is perhaps telling that Putnam gives little indication of what would and would not be 'theses.' One is left to surmise that Putnam reads Wittgenstein looking for a traditional philosophical account and if one reads Wittgenstein in this way then one will have no trouble finding what look like many such theses. This is because a thesis is a dogmatic philosophical account which, though it may capture many aspects of our language-use, nonetheless distorts other aspects. Many such pictures appear within Wittgenstein's work and Wittgenstein is careful not to outright reject any of them. It can be very tempting to think that some such picture must be the one that Wittgenstein recommends but to read Wittgenstein in this way is not in accordance with his conception of philosophy. Under his conception, these various dogmatic pictures are employed as objects of comparison. The various pictures are not pitted against one another in a winner-takes-all battle, but employed as lenses through which to see how various and complex our language is.

It is perhaps unfortunate that *RFM* on its own is often taken as the key expression of Wittgenstein's philosophy of mathematics, since *RFM* does not contain any discussion of Wittgenstein's conception of philosophy analogous to (*PI*, §89–133). Without any explanation of what is going on, *RFM* can appear as series of loosely connected philosophical wonderings, seeming to wander from one bold assertion to another without the purpose of any of them being made especially clear. *LFM* has more in the way of indication of what Wittgenstein understands himself to be doing and in this regard gives the reader new to Wittgenstein more of a foothold, though perhaps not all that much more. It can be very tempting to ask what the point of all of Wittgenstein's wanderings is, as though there must be a single unifying aim to Wittgenstein's philosophy of mathematics. If this thesis is correct then the aim of Wittgenstein's philosophy of mathematics is also that of the *Investigations*, namely that:

> ...we now demonstrate a method, by examples; and the series of examples can be broken off. -Problems are solved (difficulties eliminated), not a *single* problem. There is not *a* philosophical method, though there are indeed methods, like different therapies. (*PI*, §133)

It is appropriate then that the last remark of the *Investigations* draws such a parallel:

> An investigation is possible in connexion with mathematics which is entirely analogous to our investigation of psychology. It is just as little a *mathematical* investigation as the other is a psychological one. It will *not* contain calculations, so it is not for example logistic. It might deserve the name of an investigation of the 'foundations of mathematics.' (*PI*, p.232)

The appearance that Wittgenstein was confused or inconsistent in his philosophy of mathematics can be dissolved if his work is seen in this light – as a series of demonstrations of his method of resolving philosophical difficulties. Given that the method is itself demonstrated by examples, understanding Wittgenstein in this way also helps to better understand his conception of philosophy. Hence we can now see (*PI*, §124) with greater clarity:

> Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.

> For it cannot give it any foundation either.

> It leaves everything as it is.

> It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other. (*PI*, §124)

Philosophy cannot interfere in the actual use of language because philosophy simply proposes models to be used as lenses through which to see our use of language – objects of comparison to be held up against and to cast light upon the actual use of language. The actual use of language is exhibited by examples, chosen carefully to be sensitive to points of detail. The point is not to raise every possible detail in regard to language-use but to bring into view enough of its variety that we give up our attachment to the particular picture/s that leads to the problem in question.

Philosophers have not always taken themselves to be proposing models for how particular expressions are used, often thinking that they have seen some insight into reality or into a world of concepts. When taken in this way, aspects of language-use which do not fit with the philosophical account begin to look problematic and scepticism arises. One thinks that one has to adapt the philosophical account to either account for the problematic areas or find a way to dismiss them as not relevant. Wittgenstein aims to avoid doing either by addressing the scepticism at its root, namely the dogmatic picture/s that give rise to it. As Wittgenstein says:

> For we can avoid ineptness or emptiness in our assertions only by presenting the model as what it is, as an object of comparison— as, so to speak, a measuring-rod; not as a preconceived idea to which reality *must* correspond. (The dogmatism into which we fall so easily in doing philosophy.) (*PI*, §131)

This, for Wittgenstein, is a key characteristic of philosophical questions – they relate to ways of using language which are so various and subtle that attempting to see them under a single account is unrealistic and the reason why ordinary expressions can appear problematic to us.

As we saw in chapter 3, even some of our simplest and most natural practices with mathematical language can appear problematic when seen through the lens of a philosophical account. Wittgenstein's discussion of "The steps are determined by the formula..." illustrates that apparently simple expressions can relate to subtle and sophisticated patterns of use. Even "working out the series + 2" (*RFM*, p.36) can appear problematic from the vantage-point of a philosophical account. When we perform such basic calculations we act with a certainty that we cannot hope to give a philosophical foundation for, since no foundation could be more solid than the certainty with which we act.

Nor can we understand mathematical propositions in a satisfactory way by means of a single unified account. Seeing mathematical propositions as descriptions makes necessity look like a feature of mathematical objects (chapter 4) and misrepresents the relationship between a mathematical proposition and its proof (as we saw in chapter 6, the meaning of a mathematical proposition is internally related to its proof). Nor are mathematical propositions arbitrary stipulations, since then mathematics would be just a game (chapter 4). The appearance of a problematic character to mathematical propositions can be released if we see mathematical propositions as rules of description, since then we can see mathematics as more than just a game and as anchored in practical applications but without the need to posit necessarily-existing objects.

Whilst the rules of description picture can be used to resolve certain problems (such as, as we saw in chapter 7, dissipating a sceptical fear of contradictions), it should be taken as an object of comparison and not a definitive account to which all cases must neatly conform. If the picture is relied upon dogmatically, then aspects of mathematics which lack empirical applications (set theory being a key example for Wittgenstein) then appear problematic. Rather than dogmatically rejecting such aspects as not mathematics, Wittgenstein acknowledges their validity and treats them as limiting cases (chapter 5).

A dogmatic attachment to the rules of description picture can also appear problematic when it comes to mathematical truth, since rules are not properly speaking true or false. (This is covered in chapter 5.) But Wittgenstein's point is not that mathematical propositions *are* rules of description. Wittgenstein's point is that mathematical propositions can be illuminating seen as rules of description for certain purposes. Mathematical propositions are alike to rules of description and this picture helps illuminate that mathematical propositions are not true or false indepedently of verification in the way that empirical descriptions are. Nonetheless, mathematical propositions are true or false. Whilst a mathematical proposition is internally related to its proof, a conjecture is not therefore to be treated as meaningless. We most often are not entirely in the dark with regards to how the verification of a conjecture might be approached. Insofar as we are able to talk about deciding mathematical propositions, the law of excluded middle is (as we saw in chapter 5) appropriately applied to them. Whilst there might well be cases where the application of the law of excluded middle looks questionable, these are also cases where it is not clear that we are dealing with a mathematical proposition and so it is not reasonable to dogmatically reject the law of excluded middle for all of mathematics.

When seen through the lens of a particular philosophical picture certain mathematical systems can become invested with a metaphysical significance. The system then seems as though it tells us about something that goes beyond the mathematics itself. In the case of Frege and Russell's logic-mathematical systems, these can take on the appearance of

providing a definition of the natural numbers and so as resolving a 'leading problem of mathematical logic.' But this picture is reductive with respect to the natural numbers and this comes out in the failure of Frege and Russell's systems to enable us to do what arithmetic does – we cannot use their systems to count using large numbers unless we rely on ordinary arithmetic (see chapter 8).

In other cases the metaphysical significance that a mathematical system can take on is a matter of our reading the system as revealing some reality – as probing the "mysteries of the mathematical world" (*RFM*, p.137). It can be tempting to read set theory as a description of a realm of sets or as an exploration of a realm of higher-order infinities (chapter 9). But such a reading is not forced upon us by the mathematics and it is better understood as a prose interpretation. The picture of set-theoretic propositions as descriptions can be avoided if set theory is read as a body of mathematical rules, for which we can talk about objects only insofar as they are defined by the rules.

Gödel's first incompleteness theorem (discussed in chapter 10) might easily be read as showing that the truth of mathematical propositions is independent of whether we have proofs of them. This line of thinking naturally arises from the translation of a true Gödel sentence as "this sentence is unprovable." In my view it was convincingly argued some time ago (Floyd 1995; Floyd and Putnam 2000; Floyd 2001) that Wittgenstein dissolves the appearance of a metaphysical insight by showing that this prose translation misrepresents a Gödel sentence in that there are circumstances in which we should wish to give up the translation. Yet the view persists that Wittgenstein's discussion of Gödel's first incompleteness theorem is a key open interpretative question. Whilst interpretative questions are never entirely closed, I feel that Floyd and Putnam especially have done much more than they have so far been given credit for. The resistance to fully accept this is perhaps a resistance to accept Floyd's point that "there is no new thesis argued for in Wittgenstein's remarks on Gödel" (Floyd 1995, p.410).

My feeling is that much of the best interpretative work on Wittgenstein's philosophy of mathematics has not received the credit that it deserves. The reason for this, I believe, is that philosophical readers are inclined to approach it looking for an answer as to what Wittgenstein really thinks, as though Wittgenstein must have a unified and definitive account that interpreters have not yet been able to discern. This approach to Wittgenstein robs both his work and that of his interpreters of much of its value. Wittgenstein's distinctive approach to philosophy is a key part of his work. This has not been fully accepted with regard to Wittgenstein's work on mathematics and there persists a "common opinion, that Wittgenstein's discussions of mathematics and logic are inferior to, and separable from, the rest of his later philosophy" (Floyd 1995, p.375). I am certainly in sympathy with Diamond when she writes that "there is almost nothing in Wittgenstein which is of value and which can be grasped if it is pulled away from that view of philosophy" (1991, p.179). This thesis has been devoted to exploring how that view of philosophy bears upon and comes out within Wittgenstein's philosophy of mathematics.

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