Arithmetic and dynamical systems

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By
Robert Royals
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Abstract

In this thesis we look at a number of topics in the area of the interaction between dynamical systems and number theory. We look at two diophantine approximation problems in local fields of positive characteristic, one a generalisation of the Khintchine–Groshev theorem, another a central limit theorem. We also prove a Pólya–Carlson dichotomy result for a large class of adelicly perturbed rational functions. In particular we prove that for a finite set of primes S that the power series $f(z)$ generated by the Fibonacci series with all primes in S removed has a natural boundary.

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Contents

Introduction

In this thesis we explore a number of topics in the area of the interaction between dynamical systems and number theory. These areas of work include the proof of a Khintchine–Groshev analogue in the setting of imaginary quadratic function fields, which the author and Anish Ghosh coauthored a paper on [11]. In Chapter 2 we look at the proof of a result by Deligero and Nakada [8] in detail and do the work required to correct a mistake. In Chapter 3 we look at certain power series and prove a Pólya–Carlson dichotomy theorem for a large class of sequences.

A Khintchine–Groshev theorem for imaginary quadratic function fields

The classic result of Khintchine of 1926 [14] is about ψ-approximable numbers where $\psi : \mathbb{N} \to \mathbb{R}$ is a real valued function. We say that a number $x \in \mathbb{R}$ is ψ-approximable if there exist $p/q \in \mathbb{Q}$ with $(p,q) = 1$ and arbitrarily large $|q|$ such that $|x - p/q| \leq \psi(q)/|q|$. Khintchine's result is as follows:

Theorem 1. *Let $\psi : \mathbb{N} \to (0,\infty)$ be continuous and non-increasing. Then almost all or almost no $x \in \mathbb{R}$ are ψ-approximable according as $\sum_{n=1}^{\infty} \psi(n)$ diverges or converges.*

The convergent case follows easily as a result of the Borel–Cantelli Lemma

Lemma 1. *Let X be a probability space with measure μ and (A_n) a sequence of measurable sets in X. Then if we let*

$$S = \{x \in X \mid x \in A_n \text{ for infinitely many } n\}$$

then if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then $\mu(S) = 0$. Conversely if for all $1 \leq m < n$ we have $\mu(A_m \cap A_n) = \mu(A_m)\mu(A_n)$ then one has $\mu(S) = 1$ whenever $\sum_{n=1}^{\infty} \mu(A_n) = \infty$.

We shall quickly demonstrate the convergent case. For each $q \in \mathbb{N}$ define $B_q(\psi)$ as the set of all $x \in [0,1)$ for which there exists some $p \in \mathbb{Z}$ such that $|x - p/q| \leq \psi(q)/|q|$. We may assume that $\psi(q) \leq 1/2$ for sufficiently large q, otherwise every $x \in \mathbb{R}$ is ψ-approximable. So for sufficiently large q we have $2\psi(q)/q \leq 1/q$ and for such q the set $B_q(\psi)$ consists of q intervals of length $2\psi(q)/q$ centered at the points $\{0, 1/q, 2/q, \ldots, (q-1)/q\}$ allowing for wrap-around at the ends. Thus $|B_q(\psi(q))| = 2\psi(q)$. Let $B'_q(\psi)$ be the set of all $x \in [0,1)$ for which there exists some $p \in \mathbb{Z}$ with $(p,q) = 1$ such that $|x - p/q| \leq \psi(q)/|q|$. Clearly we have that $B'_q(\psi) \subseteq B_q(\psi)$ so $|B'_q(\psi)| \leq |B_q(\psi)|$. Now $x \in [0,1)$ is ψ-approximable precisely when $x \in B'_q(\psi)$ for infinitely many $q \in \mathbb{N}$, which by the Borel–Cantelli Lemma holds for almost no $x \in [0,1)$ when $\sum_{n=1}^{\infty} |B'_n(\psi)| < \infty$. This follows immediately from $\sum_{n=1}^{\infty} B'_q(\psi) \leq \sum_{n=1}^{\infty} |B_n(\psi)| = \sum_{n=1}^{\infty} 2\psi(n) < \infty$. It is easily shown that x is ψ-approximable if and only if $x + n$ is ψ-approximable for all $n \in \mathbb{Z}$. Thus proving the result for $[0,1)$ is sufficient.

The divergence case is more difficult, so we won't repeat its proof here. However, it is worth noting that in the convergence case above we did not use the fact that ψ was non-increasing. Without this assumption the divergence case is false [9], but with a modification it becomes the Duffin–Schaeffer conjecture [9]

Conjecture 1 (Duffin–Schaeffer)**.** *Let $\psi : \mathbb{N} \to [0, \infty)$ be a real valued function, then almost no or almost all $x \in \mathbb{R}$ are ψ-approximable according as*

$$\sum_{n=1}^{\infty} \psi(n) \frac{\varphi(n)}{n}$$

converges or diverges.

Here $\varphi(n)$ is the Euler totient function, which is the number of integers $m \in [1, n] \cap \mathbb{N}$ such that $(m, n) = 1$. The factor $\varphi(n)/n$ arises from the fact that the measure of our sets $B'(n)$ defined above is $2\psi(n)\varphi(n)/n$. Without the assumption of ψ being non-increasing, we could choose $\psi(n)$ to be a non-zero constant only on a sequence (n_i) such that $\sum_{i=1}^{\infty} \varphi(n_i)/n_i < \infty$. Then we would have

$$\sum_{n=1}^{\infty} \psi(n) = \infty \text{ and } \sum_{n=1}^{\infty} \psi(n) \frac{\varphi(n)}{n} < \infty,$$

and the convergence case of the Borel-Cantelli lemma tells us that almost no $x \in \mathbb{R}$ are ψ-approximable, however $\sum_{n=1}^{\infty} \psi(n) = \infty$. Thus the assumption that ψ is non-increasing is necessary.

In fact if we assume that ψ is non-increasing, then the following holds [5]

$$\sum_{n=1}^{\infty} \psi(n) < \infty \text{ if and only if } \sum_{n=1}^{\infty} \psi(n) \frac{\varphi(n)}{n} < \infty,$$

and being able to eliminate the $\varphi(n)/n$ term makes the problem easier to work with.

The theorem of Khintchine was generalised to higher dimensions in 1938 by Groshev [12]. For this we define a matrix $x \in \mathrm{M}_{m \times n}(\mathbb{R})$ to be ψ-approximable if there exist infinitely many

$q \in \mathbb{Z}^n$ such that there exists a $p \in \mathbb{Z}^m$ with

$$\|xq + p\|^m \leq \psi(\|q\|^n),$$

where $\|\cdot\|$ indicates the sup-norm. Then Groshev's theorem is as follows:

Theorem 2. *Let $\psi : \mathbb{N} \to (0, \infty)$ be continuous and non-increasing. Then almost all or almost no $x \in \mathrm{M}_{m\times n}(\mathbb{R})$ are ψ-approximable according as $\sum_{n=1}^{\infty} \psi(n)$ diverges or converges.*

In 1990, A.D. Pollington and R.C. Vaughan proved a higher dimensional version of the Duffin–Schaeffer conjecture [20] which allows the non-increasing condition of ψ to be removed, however it requires dimension of at least 2, so the original Duffin–Schaeffer conjecture still remains an open problem.

Work has been done extending these results to fields of positive characteristic. For example using the field of Laurent polynomials $\mathbb{F}((T^{-1}))$ over a finite field $\mathbb{F}$ intead of $\mathbb{R}$ and approximating with polynomials in place of p and q, a precise introduction to the exact setup is given in Chapter 1. B. de Mathan [7] proved an analogue of Khintchine's theorem in this setting and Kristensen [16] generalised it to higher dimensions in an analogue of Groshev's theorem.

Work has also been done in extending these results to $\mathbb{C}$, approximating numbers $z \in \mathbb{C}$ by elements of some number field. Hermite [13] in 1854 proved the following result.

Theorem 3. *For every $z \in \mathbb{C} \setminus \mathbb{Q}(i)$, there exist infinitely many $p, q \in \mathbb{Z}[i]$ such that*

$$\left| z - \frac{p}{q} \right| \leq \frac{1}{\sqrt{2}|q|}.$$

Sullivan [24] proved a generalisation of Khintchine's theorem in this setting. If we let d be a positive non-square integer, denote by $\mathfrak{O}$ the ring of integers of the field $\mathbb{Q}(\sqrt{-d})$. Then if $\psi : \mathbb{R} \to [0, 1)$ is differentiable such that for some constant $c > 0$ we have $|\psi'(x)| < c\psi(x)$. Then

Theorem 4. *For almost every $z \in \mathbb{C}$, there exist infinitely many $p, q \in \mathfrak{O}$ with $(p, q) = \mathfrak{O}$ and*

$$\left| z - \frac{p}{q} \right| \leq \frac{\psi(|q|)}{|q|^2}.$$

Here (p, q) denotes the ideal generated by $p, q \in \mathfrak{O}$ and is a generalisation of coprimality.

In this thesis we look at extending the Khintchine–Groshev theorem to quadratic extensions of function fields. If K_∞ is a quadratic extension of a field of Laurent polynomials $\mathbb{F}((T^{-1}))$ by a quadratic element $\sqrt{f(T)}$ with $\sqrt{f(T)} \notin \mathbb{F}((T^{-1}))$, then we call $x \in \mathrm{M}_{m\times n}(K_\infty)$ ψ-approximable if there exist infinitely many $q \in \mathbb{F}[T]\big(\sqrt{f(T)}\big)^n$ such that for some $p \in \mathbb{F}[T]\big(\sqrt{f(T)}\big)^m$ we have

$$\|xq + p\|^{2m} \leq \psi(\|q\|^{2n}).$$

In particular our main theorem will be the following.

Theorem 5. *Let $\psi : \mathbb{N} \to (0, \infty)$ be continuous and non-increasing. Then almost all or almost no $x \in \mathrm{M}_{m\times n}(K_\infty)$ are ψ-approximable according as $\int_1^\infty \psi(x)dx$ diverges or converges.*

The measures and metrics that the above are given by are the Haar measures and sup-metrics where appropriate. Our proof uses results from ergodic theory and the theory of group actions on homogeneous spaces.

A positive characteristic central limit theorem

In Chapter 2 we will be looking in close detail at the proof of a result by Deligero and Nakada [8]. Again in this chapter we are looking at diophantine approximation in positive characteristic. Let $\{l_n\}$ be a non-decreasing sequence of numbers and let $\mathbb{F}$ be a finite field of q elements. Given some $f \in \mathbb{F}((T^{-1}))$, we will be looking at solutions to

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2 q^{l_n}}, \qquad (P, Q) = 1 \tag{1}$$

where $P, Q \in \mathbb{F}[T]$ and $n = \deg Q$. Let $\mathbb{L}$ be the unit ball in $\mathbb{F}((T^{-1}))$ and set

$$Z_N(f) = \#\{P/Q \in \mathbb{F}(T) \mid (P, Q) = 1, \deg Q \le N, \text{ and } P/Q \text{ satisfies (1)}\}.$$

The central limit theorem from [8] is as follows:

Theorem 6. *If $\sum_1^\infty q^{-l_n} = \infty$ then for any $\alpha \in \mathbb{R}$*

$$\lim_{N\to\infty} m\left\{ f \in \mathbb{L} : \frac{Z_N(f) - \sum_{n=1}^N q^{-l_n}(1 - \frac{1}{q})}{\sqrt{\sum_{n=1}^N q^{-l_n}(1 - \frac{1}{q})}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

The proof of the theorem involves using Lyapunov's condition, stated as Lemma 11, which requires for a certain collection of random variables U for us to calculate

$$\mathbb{E}\left[|U|^{2+\delta}\right]$$

for some $\delta > 0$. The random variable U is of the form $U = X_1 + \ldots + X_n$ with the X_i being indicator functions centralized to have zero expectation. That is to say that for each i we have $X_i = \mathbb{1}_{G_i} - m(G_i)$ for some measurable set G_i with measure $m(G_i)$. Now it turns out that this is only easy to compute by hand for δ an even integer. In [8], Deligero and Nakada attempt to calculate it for $\delta = 1$ and run into difficulties, so here we correct their proof by working with $\delta = 2$. The reason for choosing an even integer is that we have

$$\begin{aligned} |U|^4 &= U^4 \\ &= \left(\sum_{i=1}^{n} \mathbb{1}_{G_i} - \sum_{i=1}^{n} m(G_i) \right)^4 \end{aligned}$$

which we then expand out with the binomial theorem. In [8] working with an odd power they could not do the first step, and proceeded with the following steps

$$\begin{aligned} |U|^3 &= \left| \left(\sum_{i=1}^{n} \mathbb{1}_{G_i} - \sum_{i=1}^{n} m(G_i) \right) \right|^3 \\ &\leq \left(\sum_{i=1}^{n} \mathbb{1}_{G_i} + m(G_i) \right)^3 . \end{aligned}$$

It turns out that the loss of precision after applying the triangle inequality is too much to be able to prove what they require.

Adelic perturbations of certain power series

In this section we look at power series which have natural boundaries. A function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

with radius of convergence

$$r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f_n|}} > 0 \tag{2}$$

is said to have a natural boundary if there is no meromorphic extension of $f(z)$ to the whole complex plane $\mathbb{C}$. This can happen, for example, if on a dense subset S of the radius of convergence $\{z \in \mathbb{C} \ | \ |z| = r\}$ one has $\lim_{\lambda\to 1^-} |f(\lambda x)| \to \infty$ for all $x \in S$, which indeed is how all natural boundary functions appearing in this section occur. A classical example of such a function attributed to Weierstrass is $f(z) = \sum_{n=0}^{\infty} z^{2^n}$. The function $f(z)$ has radius of convergence 1 and $\lim_{\lambda\to 1^-} f(\lambda) = +\infty$. Noting that $f(z^2) = \sum_{n=0}^{\infty} (z^2)^{2^n} = \sum_{n=1}^{\infty} z^{2^n} = f(z) - z$, we see that $\lim_{\lambda\to 1^-} f(-\lambda) = \lim_{\lambda\to 1^-} f(\lambda^2) - 1 = +\infty$. Further, one has $\lim_{\lambda\to 1^-} |f(i\lambda)| = \lim_{\lambda\to 1^-} |f(-\lambda^2) + i| = \infty$. Continuing in this way recursively, we have that for all $x \in \{\mu \in \mathbb{C} \ | \ \mu^{2^n} = 1 \text{ for some } n \in \mathbb{N}\}$, $\lim_{\lambda\to 1^-} |f(\lambda x)| = \infty$. As the set S is dense on the unit circle, we have shown that $f(z)$ has a natural boundary at the unit circle. Throughout this chapter, the method of obtaining a functional equation of the form $f(z^k) = g(z) + h(z) f(z)$ for some rational functions $g, h \in \mathbb{C}(z)$ and some $k > 1$, to demonstrate $f(z)$ has a natural boundary is used many times. However, care has to be taken as the functions $g(z)$ and $h(z)$ can cause problems.

In the 2005 paper by Everest, Stangoe, and Ward, [10] they calculated explicitly that

$$f(z) = \sum_{n=1}^{\infty} |2^n - 1|_3 z^n$$

has a natural boundary on the unit circle. They achieved this by calculating

$$|2^n - 1|_3 = \begin{cases} \frac{1}{3}|n|_3 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases},$$

and then it becomes equivalent to showing that $g(z) = \sum_{n=1}^{\infty} |n|_3 z^n$ has a natural boundary. They obtain a functional equation by

$$\begin{aligned} g(z^3) &= \sum_{n=1}^{\infty} |n|_3 z^{3n} \\ &= 3 \sum_{n=1}^{\infty} |3n|_3 z^{3n} \\ &= 3 \left(\sum_{n=1}^{\infty} |n|_3 z^n - \sum_{3 \nmid n} |n|_3 z^n \right) \\ &= 3 \left(g(z) - \sum_{3 \nmid n} z^n \right) \\ &= 3 \left(g(z) + \frac{z^3}{1 - z^3} - \frac{z}{1 - z} \right) \end{aligned}$$

and then use this to show that $f(z)$ has a singularity at all 3^n-th roots of unity for $n \geq 0$, which forms a dense subset of the unit circle.

Later in 2014, Bell, Miles and Ward [2], suggest the possibility that if $\theta : G \to G$ is a group automorphism with the property that

$$F_\theta(n) = |\{g \in G \ \mid \ \theta^n x = x\}|$$

is finite for all $n \in \mathbb{N}$. Then the associated zeta function

$$\zeta_\theta(z) = \exp \sum_{n \geq 1} \frac{F_\theta(n)}{n} z^n$$

which is related to the function $\sum_{n \geq 1} F_\theta(n) z^n$ by

$$z\zeta_\theta'(z)/\zeta(z) = \sum_{n \geq 1} F_\theta(n) z^n$$

is either rational or has a natural boundary. That is to say that this class of functions satisfies a Pólya–Carlson dichotomy. The Pólya–Carlson Theorem [4], [21] is as follows:

Theorem 7. *A power series with integer coefficients and radius of convergence* 1 *is either rational or has the unit circle as a natural boundary.*

In [2] Bell, Miles and Ward go on to prove a Pólya–Carlson dichotomy for all functions of the form

$$\sum_{n=1}^{\infty} |r^n - 1|_S z^n$$

where $r \in \mathbb{Q}$, S is a finite set of primes such that for all $p \in S$, $|r|_p \leq 1$ and $|r^n - 1|_S = \prod_{p \in S} |r^n - 1|_p$. In this chapter we prove a Pólya–Carlson dichotomy for all functions of the form

$$\sum_{n=1}^{\infty} |n - r|_S z^n$$

where S is finite and $r \in \mathbb{Q}$. The main Theorem of this section is that we prove a Pólya–Carlson dichotomy for the class of functions $\sum_{n=1}^{\infty} a_n z^n$ where the rational sequence (a_n) satisfies Property 1, which is

Property. *For every prime* p *there exist constants* $n_p \in \mathbb{N}$, $c_{p,0}, c_{p,1}, c_{p,2}, \ldots, c_{p,n_p-1} \in \mathbb{Q}$ *and* $e_{p,0}, e_{p,1}, e_{p,2}, \ldots, e_{p,n_p-1} \in \{0, 1, 2, \ldots\}$ *such that for all* $k \in \{0, 1, \ldots, n_p - 1\}$

$$|a_n|_p = c_{p,k} |n|_p^{e_{p,k}} \text{ if } n \equiv k \mod n_p.$$

Chapter 1

A Khintchine theorem for quadratic function fields

1.1 Introduction

In this chapter we will be looking at the work written for a paper coauthored by myself and my adviser, Anish Ghosh. We will begin with some background information on the Khintchine-Groshev Theorem.

1.1.1 Basic notation

Let $\mathbb{F} = \mathbb{F}_s$ be a field of s elements, where s is a prime power such that $2 \nmid s$. Let $A = \mathbb{F}[T]$ be the ring of polynomials over $\mathbb{F}$ and let $k = \mathbb{F}(T)$ be its field of fractions. Denote by $|\cdot|$ the absolute value function on k generated by

$$|f(T)| = s^{\deg(f)}$$

for $f(T) \in A$. It is easy to see that this valuation is ultrametric. Let k_∞ be the completion of k with respect to this absolute value function. This gives us the field of Laurent polynomials in T^{-1} over $\mathbb{F}$. That is

$$k_\infty = \mathbb{F}((T^{-1})) = \left\{ \sum_{i=-\infty}^{N} x_i T^i : N \in \mathbb{Z}, x_i \in \mathbb{F} \right\}.$$

Let $\mathfrak{o} = \{x \in k_\infty : |x| \leq 1\}$ and denote by $\mathfrak{p}$ its unique prime ideal, i.e. $\mathfrak{p} = \{x \in k_\infty : |x| < 1\}$. For $x \in k_\infty$ denote by $[x]$ the polynomial part of x and by $\{x\}$, the tail of x, i.e.

$$[x] = \left[\sum_{i=-\infty}^{N} x_i T^i \right] = \sum_{i=0}^{N} x_i T^i \in A \text{ and } \{x\} = \left\{ \sum_{i=-\infty}^{N} x_i T^i \right\} = \sum_{i=-\infty}^{-1} x_i T^i \in \mathfrak{p}.$$

For $x \in k_\infty$ define $|\langle x\rangle|$ to be the distance of x to A given by

$$|\langle x\rangle| = \min\{|x-p| : p \in A\}.$$

1.1.2 ψ-approximable matrices and quadratic extensions

Let $\psi : [1,\infty) \to (0,\infty)$ be a non-increasing continuous function with $\psi(x) \to 0$ as $x \to \infty$. Let $f(T) \in A$ be squarefree non-unit so that $K = k(\sqrt{f(T)})$ is a quadratic extension of k. Depending on our choice of $f(T)$ it may or may not be true that $\sqrt{f(T)} \in k_\infty$. We are interested in the case when $k_\infty(\sqrt{f(T)})/k_\infty$ is an extension of degree 2. Let d be the degree of $f(T)$. It turns out that $\sqrt{f(T)} \in k_\infty$ occurs exactly when d is even and has square leading coefficient (see Proposition 14.6 in [22]). Following E. Artin, we call K *imaginary* if this is not the case. We fix an f such that K is imaginary, and set $K_\infty := k_\infty(\sqrt{f(T)})$. Let $B \subseteq K$ be the integers over A, that is

$$B = \{x \in K : x \text{ is a root of some monic } h(U) \in A[U]\}.$$

It is not difficult (See Lemma 2 below) to see that $B = A[\sqrt{f(T)}]$. The study of Diophantine approximation of Laurent series in K_∞ by ratios of polynomials in B is thus a function field analogue of Diophantine approximation of complex numbers discussed earlier. Let $\mathrm{M}_{m\times n}(K_\infty)$ denote the set of $m \times n$-matrices with K_∞ valued entries, $\|\ \|$ denote the L^∞ norm and I be the ball $\{z \in \mathrm{M}_{m\times n}(K_\infty) \ : \ \|z\| \le 1\}$. We denote by μ the Haar measure on $\mathrm{M}_{m\times n}(K_\infty)$ normalized so that $\mu(I) = 1$. We say that z is ψ-approximable if there exist infinitely many $q \in B^n$ and $p \in B^m$ such that

$$\|zq+p\|^{2m} \le \psi(\|q\|^{2n}). \tag{3}$$

Denote by $\mathscr{W}_{m\times n}$ the set of all ψ-approximable matrices in $\mathrm{M}_{m\times n}(K_\infty)$. It is not difficult to see (see Proposition 1 below), using Lemma 1 that

$$\mu(\mathscr{W}_{m\times n}) = 0 \text{ if } \int_1^\infty \psi(x)dx < \infty.$$

The main result of this chapter is the converse:

Theorem 8.

$$\mathscr{W}_{m\times n} \textit{ has full } \mu \textit{ measure if } \int_1^\infty \psi(x)dx = \infty.$$

where by full measure we mean that

$$\mu\left(M_{m\times n}(K_\infty) \setminus \mathscr{W}_{m\times n}\right) = 0.$$

1.2 Preliminaries and the convergence case

In this section, we record some preliminary lemmas and also prove the convergence case of Theorem 8. We begin with some facts about quadratic function fields.

Lemma 2. *If $2 \nmid s$ then $B = A[\sqrt{f(T)}]$.*

Proof. Let $x \in A[\sqrt{f(T)}]$. We can write x in the form $x = a + b\sqrt{f(T)}$ for some $a, b \in A$. Then x is a root of the monic polynomial

$$U^2 - 2aU + (a^2 - b^2 f(T)) \in A[U]$$

and so $A[\sqrt{f(T)}] \subseteq B$.

Conversely let $x \in K$ be integral over A. Write $x = (a + b\sqrt{f(T)})/c$ with $a, b, c \in A$ not all sharing a common factor, which we can do since $A = \mathbb{F}[T]$ is a unique factorization domain. It is easy to check that x is a root of the following quadratic in $k[U]$

$$U^2 - \frac{2a}{c}U + \frac{a^2 - b^2 f(T)}{c^2}.$$

If $x \in k$ then as $A = \mathbb{F}[T]$ is integrally closed in k, we have that $x \in A \subset A[\sqrt{f(T)}]$. On the other hand, if $x \notin k$ then the minimal monic polynomial of x must divide the above quadratic, and have degree of at least 2, and so is equal to it. Thus $c \mid 2a$ and $c^2 \mid a^2 - b^2 f(T)$. Now suppose $d \in A$ is an irreducible factor of c. If $d \mid 2a$ then $d \mid a$ as 2 is a unit. Thus $d^2 \mid a^2 - b^2 f(T) \Rightarrow d^2 \mid b^2 f(T)$, and as $f(T)$ is squarefree, we have $d \mid b$. This contradicts a, b, c sharing no common factor. Hence c has no irreducible factors and so is a unit. So $x \in A[\sqrt{f(T)}]$. $\square$

Lemma 3. *Let $f(T) \in A$ be such that $\sqrt{f(T)} \notin k_\infty$. Then for all $x, y \in k$,*

$$|x + y\sqrt{f(T)}|' = \max(|x|, |y||f|^{1/2})$$

for any extension $|\cdot|'$ of $|\cdot|$ to K.

Proof. Any extension $|\cdot|'$ of $|\cdot|$ to K must satisfy that $|\cdot|'$ restricts to $|\cdot|$ on k and $|\sqrt{f}|' = |f|'^{1/2} = |f|^{1/2}$. The theory of valuations tells us that if $|\cdot|_1, \ldots, |\cdot|_d$ are all extensions of $|\cdot|$ to K then

$$\sum_{i=1}^{i=d} [K_i : k_\infty] = [K : k]$$

where K_i is the completion of K with respect to the valuation $|\cdot|_i$. Let K_∞ be the completion of K with respect to $|\cdot|'$ and we have that $[K_\infty : k_\infty] = 2$ because $\sqrt{f} \notin k_\infty$ by assumption. Since $[K : k] = 2$ this means that there are no more extensions of $|\cdot|$ to K. Since the map $K \to K$ sending $\sqrt{f}$ to $-\sqrt{f}$ is a k-automorphism of K, we must have that

$|x + y\sqrt{f}|' = |x - y\sqrt{f}|'$ for all $x, y \in k$ or else we would generate another valuation lying over $|\cdot|$ for K.

Suppose there exists $x, y \in k$ such that $|x + y\sqrt{f}|' < \max(|x|', |y\sqrt{f}|')$. Then by the ultrametric property we have that $|x|' = |y\sqrt{f}|'$. Since there is only one valuation extension, we must have that $|x - y\sqrt{f}|' < \max(|x|', |y\sqrt{f}|')$ holds also. But then

$$|(x + y\sqrt{f}) + (x - y\sqrt{f})|' = |2x|' = |x|'$$

and

$$\begin{aligned}|(x + y\sqrt{f}) + (x - y\sqrt{f})|' &\leq \max(|x + y\sqrt{f})|', |x - y\sqrt{f}|') \\ &= |x + y\sqrt{(f)}|' \\ &< \max(|x|', |y\sqrt{f}|') = |x|'\end{aligned}$$

is a contradiction. □

Since the extension $|\cdot|'$ of $|\cdot|$ to K_∞ is unique we will simply write $|\cdot|$ in both cases.

1.2.1 The Haar measures on k_∞ and K_∞.

Measuring balls in k_∞

Let $v > 0$ and denote by $B_{k_\infty}(v)$ the ball of radius v centered at 0 in k_∞. That is

$$B_{k_\infty}(v) = \{x \in k_\infty \mid |x| < v\}.$$

Let μ_{k_∞} be the Haar measure on k_∞ normalized such that

$$\mu_{k_\infty}(B_{k_\infty}(1)) = 1.$$

The range of possible values of $|\cdot|$ on k_∞ is $\{s^n \mid n \in \mathbb{Z}\}$, so if we let $n_v \in \mathbb{Z}$ be that unique integer such that $s^{n_v - 1} < v \leq s^{n_v}$, namely $n_v = \lceil \log(v)/\log(s) \rceil$, we have that $B_{k_\infty}(v) = B_{k_\infty}(s^{n_v})$. Thus it suffices to compute the measure $\mu_{k_\infty}(B_{k_\infty}(v))$ where v is of the form s^n.

Lemma 4. *For all $m \in \mathbb{Z}$ we have $B_{k_\infty}(s^m) = T^m B_{k_\infty}(1)$.*

Proof. Let $x \in B_{k_\infty}(s^m)$, thus $|x| < s^m$ and so $|T^{-m}x| = |T^{-m}||x| = s^{-m}|x| < 1$, and $T^{-m}x \in B_{k_\infty}(1)$. So $x = T^m(T^{-m}x) \in T^m B_{k_\infty}(1)$. Hence $B_{k_\infty}(s^m) \subseteq T^m B_{k_\infty}(1)$. Conversely, if $x \in T^m B_{k_\infty}(1)$ then $x = T^m y$ for some $y \in B_{k_\infty}(1)$ and so $|x| = |T^m||y| < s^m$, thus $x \in B_{k_\infty}(s^m)$. Hence $T^m B_{k_\infty}(1) \subseteq B_{k_\infty}(s^m)$. □

Lemma 5. *Let $m, n \in \mathbb{Z}$ and $n \geq 0$. Then*

$$B_{k_\infty}(s^{m+n}) = \bigcup_{\substack{f \in \mathbb{F}_s[T] \\ \deg(f) < n}} (T^m f + B_{k_\infty}(s^m))$$

and the union is disjoint.

Proof. It is clear from the ultrametric property of $|\cdot|$ that the right hand side is included within the left, so let $x \in B_{k_\infty}(s^{m+n})$. We have that $T^{-m}x = [T^{-m}x] + \{T^{-m}x\}$ where $\{T^{-m}x\} \in B_{k_\infty}(1)$, so $T^{-m}x \in [T^{-m}x] + B_{k_\infty}(1)$. So $x \in T^m[T^{-m}x] + T^m B_{k_\infty}(1) = T^m[T^{-m}x] + B_{k_\infty}(s^m)$ (Lemma 4). Now $\deg([T^{-m}x]) \leq \deg(T^{-m}x) = \deg(x) - m < n$, and thus x is contained in the right-hand side of the proposed equality. To prove that the union is disjoint, suppose that $x \in T^m f + B_{k_\infty}(s^m)$ for some $f \in \mathbb{F}_s[T]$ with $\deg(f) < n$. Then $T^{-m}x \in f + B_{k_\infty}(1)$, and so $x \in (f + B_{k_\infty}(1)) \cap ([T^{-m}x] + B_{k_\infty}(1))$ which implies by the ultrametric property of $|\cdot|$ that $f - [T^{-m}x] \in B_{k_\infty}(1)$. Since $B_{k_\infty}(1) \cap \mathbb{F}_s[T] = \{0\}$, we have that $f = [T^{-m}x]$ is uniquely determined by x. □

Lemma 6. *For all $n \in \mathbb{Z}$*

$$\mu_{k_\infty}(B_{k_\infty}(s^n)) = s^n.$$

Proof. If $n \geq 0$ then by Lemma 5 we have that

$$B_{k_\infty}(s^n) = \bigcup_{\substack{f \in \mathbb{F}_s[T] \\ \deg(f) < n}} (f + B_{k_\infty}(1)).$$

Since the union is disjoint and the Haar measure μ_{k_∞} is translation invariant, it follows that

$$\mu_{k_\infty}(B_{k_\infty}(s^n)) = \mu_{k_\infty}(B_{k_\infty}(1)) \#\{f \in \mathbb{F}_s[T] \mid \deg(f) < n\} = s^n.$$

If $n < 0$ then again by Lemma 5 we have that

$$B_{k_\infty}(1) = B_{k_\infty}(s^{n-n}) = \bigcup_{\substack{f \in \mathbb{F}_s[T] \\ \deg(f) < -n}} (T^n f + B_{k_\infty}(s^n))$$

from which it follows that

$$1 = \mu_{k_\infty}(B_{k_\infty}(s^n)) \#\{f \in \mathbb{F}_s[T] \mid \deg(f) < -n\} = s^{-n} \mu_{k_\infty}(B_{k_\infty}(s^n))$$

and again $\mu_{k_\infty}(B_{k_\infty}(s^n)) = s^n$. □

Since we have been working with powers of s the following easy corollary will be useful when we are working with a general $v > 0$.

Corollary 1. *If $v > 0$ and $n \in \mathbb{Z}$ we have that*

$$\mu_{k_\infty}(B_{k_\infty}(s^n v)) = s^n \mu_{k_\infty}(B_{k_\infty}(v)).$$

and

$$v \leq \mu_{k_\infty}(B_{k_\infty}(v)) < sv.$$

Measuring balls in K_∞

Let $v > 0$. We denote by $B_{K_\infty}(v)$ the ball of radius v about 0 in K_∞. That is

$$B_{K_\infty}(v) = \{x \in K_\infty \mid |x| < v\}.$$

Let μ_{K_∞} be the Haar measure on K_∞ normalized such that

$$\mu_{K_\infty}(B_{K_\infty}(1)) = 1.$$

We know that every element of K_∞ is of the form $x + y\sqrt{f}$ for some $x, y \in k_\infty$ and that $|x + y\sqrt{f}| = \max(|x|, |y||f|^{1/2})$ by Lemma 3, so

$$B_{K_\infty}(v) = \{x + y\sqrt{f} \mid x, y \in k_\infty, |x| < v, |y| < v/|f|^{1/2}\}.$$

Treating K_∞ as the product k_∞^2, the product measure $\mu_{k_\infty^2}$ is a Haar measure for K_∞ so there exists some constant $c > 0$ such that

$$c\mu_{k_\infty^2} = \mu_{K_\infty}$$

and so

$$\mu_{K_\infty}(B_{K_\infty}(v)) = c\mu_{k_\infty^2}(B_{K_\infty}(v)) = c\mu_{k_\infty}(B_{k_\infty}(v))\mu_{k_\infty}(B_{k_\infty}(v/|f|^{1/2})).$$

The value c must be such that

$$c\mu_{k_\infty}(B_{k_\infty}(1))\mu_{k_\infty}(B_{k_\infty}(1/|f|^{1/2})) = c\mu_{k_\infty}(B_{k_\infty}(1/|f|^{1/2})) = 1$$

thus $c = s^{\lfloor(\deg f)/2\rfloor}$ and

$$\mu_{K_\infty}(B_{K_\infty}(v)) = \mu_{k_\infty}(B_{k_\infty}(v))\mu_{k_\infty}(B_{k_\infty}(v/|f|^{1/2})s^{\lfloor(\deg f)/2\rfloor}.$$

It then follows easily from Corollary 1 that

Corollary 2. *For all $v > 0$ and $n \in \mathbb{Z}$*

1. $\mu_{K_\infty}(B_{K_\infty}(s^n v)) = s^{2n}\mu_{K_\infty}(B_{K_\infty}(v))$,
2. $\mu_{K_\infty}(B_{K_\infty}(s^n)) = s^{2n}$,
3. $v^2/s^2 \le \mu_{K_\infty}(B_{K_\infty}(v)) < v^2 s^2$.

1.2.2 The convergence case

We now prove the convergence case of the Khintchine-Groshev theorem. While Theorem 8 requires ψ to be monotone non-increasing, the convergence part does not require this assumption. Firstly we note that $\mathcal{W}_{m\times n}(\psi)$ is invariant under translation by $M_{m\times n}(B)$. Indeed, if $z \in \mathcal{W}_{m\times n}(\psi)$ and $z' \in M_{m\times n}(B)$ then for each $q \in B^n$ and $p \in B^m$ such that $\|zq+p\|^{2m} \leq \psi(\|q\|^{2n})$ we have

$$\|(z+z')q + (p - z'q)\|^{2m} = \|zq+p\|^{2m} \leq \psi(\|q\|^{2n}),$$

and so $z + z' \in \mathcal{W}_{m\times n}(\psi)$. Consider the additive subgroup $P \subset K_\infty$ defined by

$$P = \{x + y\sqrt{f} \;\mid\; x, y \in k_\infty \text{ and } \max(|x|, |y|) < 1\}.$$

Clearly P is a fundamental domain for K_∞/B. That is to say, for every $x+y\sqrt{f} \in K_\infty$ there are unique $z \in P$ and $b \in B$ such that $x + y\sqrt{f} = z + b$. In particular, $b = [x] + [y]\sqrt{f}$ and $z = \{x\} + \{y\}\sqrt{f}$. So $M_{m\times n}(P)$ is a fundamental domain for $M_{m\times n}(K_\infty)/M_{m\times n}(B)$. Let $\mathscr{P} = M_{m\times n}(P)$. Since $\mathcal{W}_{m\times n}(\psi)$ is invariant under translation by $M_{m\times n}(B)$, this means that for any $z \in M_{m\times n}(K_\infty)$

$$\mu(\mathscr{P} \cap \mathcal{W}_{m\times n}(\psi)) = \mu((\mathscr{P} + z) \cap \mathcal{W}_{m\times n}(\psi)).$$

Since $M_{m\times n}(K_\infty)$ is the countable union of translations $\bigcup_{z\in M_{m\times n}(B)}(z + \mathscr{P})$ to prove that $\mu(\mathcal{W}_{m\times n}(\psi)) = 0$ we only need to prove that $\mu(\mathscr{P} \cap \mathcal{W}_{m\times n}(\psi)) = 0$. We now fix some nonzero $q \in B^n$ and consider the maps

$$M_{m\times n}(K_\infty) \xrightarrow{q} K_\infty^m \xrightarrow{\pi} P^m$$

where q represents right multiplication by q and π is the quotient map $K_\infty^m \to P^m$ given by reduction modulo B^m.

Given a measurable $\mathscr{B} \subseteq P^m$ define

$$\widetilde{\mu}(\mathscr{B}) = \mu(\mathscr{P} \cap q^{-1}\pi^{-1}(\mathscr{B})).$$

We will show that $\widetilde{\mu}$ is translation invariant. Let $y \in P^m$ and consider the set $q^{-1}\pi^{-1}(\mathscr{B} + y)$. As q is nonzero the map $M_{m\times n}(K_\infty) \xrightarrow{q} K_\infty^m$ is surjective so there exists some $Y \in M_{m\times n}(K_\infty)$ such that $\pi(Yq) = y$. It is easily verified that $q^{-1}\pi^{-1}(\mathscr{B}) + Y = q^{-1}\pi^{-1}(\mathscr{B} + y)$. Thus

$$\widetilde{\mu}(\mathscr{B} + y) = \mu(\mathscr{P} \cap (q^{-1}\pi^{-1}(\mathscr{B}) + Y)).$$

Now μ is a translation invariant so

$$\widetilde{\mu}(\mathscr{B} + y) = \mu((\mathscr{P} - Y) \cap (q^{-1}\pi^{-1}(\mathscr{B}))).$$

The set $q^{-1}\pi^{-1}(\mathscr{B})$ is invariant under translation by elements of $M_{m\times n}(B)$ so we have

$$\mu(\mathscr{P} \cap (q^{-1}\pi^{-1}(\mathscr{B}))) = \mu((\mathscr{P} - Y) \cap (q^{-1}\pi^{-1}(\mathscr{B})))$$

and thus $\widetilde{\mu}(\mathscr{B}) = \widetilde{\mu}(\mathscr{B}+y)$. It is also easy to see that $\widetilde{\mu}$ is a measure on P^m. Up to multiplication by positive constant, the only translation invariant measure of P^m is the Haar measure. Therefore $\widetilde{\mu}$ is the Haar measure on P^m and $\widetilde{\mu}(P^m) = \mu(\mathscr{P} \cap (q^{-1}\pi^{-1}(P^m)) = \mu(\mathscr{P}) = \mu_{K_\infty}(P)^{mn}$. Thus we can relate $\widetilde{\mu}$ with the Haar measure $\mu_{K_\infty^m}$ by

$$\widetilde{\mu}(\mathscr{B}) = \frac{\mu_{K_\infty}(P)^{mn}}{\mu_{K_\infty}(P)^m}\mu_{K_\infty^m}(\mathscr{B}).$$

Proposition 1. *If*

$$\int_1^\infty \psi(x)dx < \infty \text{ then } \mu(\mathcal{W}_{m\times n}(\psi)) = 0.$$

Proof. Fix some nonzero $q \in B^n$. Let S_q be the set

$$S_q = \{z \in M_{m\times n}(K_\infty) \ \mid\ \exists p \in B^m \text{ such that } \|qz+p\|^{2m} < \psi(\|q\|^{2n})\}.$$

Given some $x \in K_\infty^m$ and $v > 0$ the condition

$$\exists p \in B^m \text{ such that } \|x+p\| < v$$

is equivalent to

$$\exists p \in B^m \text{ such that } x+p \in B_{K_\infty^m}(v)$$

where $B_{K_\infty^m}(v)$ denotes the open box of radius v about some $0 \in K_\infty^m$ in the sup-metric. This is again equivalent to

$$x \in \bigcup_{p\in B^m}\left(B_{K_\infty^m}(v)+p\right).$$

Now the pre-image $\pi^{-1}(y)$ of some $y \in P^m$ is the coset $\{y+b \mid b \in B^m\}$. Also given some $z \in K_\infty^m$ the point $\pi(z) \in P^m$ is equivalent to z modulo B^m and so the cosets $\{z+b : b \in B^m\}$ and $\{\pi(z)+b : b \in B^m\}$ are equal. So $\{z+b : b \in B^m\} = \pi^{-1}\{\pi(z)\}$ and our condition is equivalent to

$$x \in \pi^{-1}(\pi(B_{K_\infty^m}(v))).$$

So the set S_q consists of elements $z \in M_{m\times n}(K_\infty)$ such that $zq \in \pi^{-1}(\pi(B_{K_\infty^m}(v_q)))$ where $v_q = \psi(\|q\|^{2n})^{1/2m}$, and so

$$S_q = q^{-1}\pi^{-1}(\pi(B_{K_\infty^m}(v_q))).$$

So now we can compute the measure

$$\begin{aligned}\mu(S_q \cap \mathscr{P}) &= \mu(q^{-1}\pi^{-1}(\pi(B_{K_\infty^m}(v_q)) \cap \mathscr{P})\\ &= \widetilde{\mu}(\pi(B_{K_\infty^m}(v_q))\\ &= c\mu_{K_\infty^m}(\pi(B_{K_\infty^m}(v_q))\end{aligned}$$

where $c = \mu_{K_\infty}(P)^{m(n-1)}$. Now suppose that $v_q \leq 1$. In this case $B_{K_\infty^m}(v_q) \subseteq \mathscr{P}$ and thus $\pi(B_{K_\infty^m}(v_q)) = B_{K_\infty^m}(v_q)$, so

$$\begin{aligned}
\mu(S_q \cap \mathscr{P}) &= c\mu_{K_\infty^m}(B_{K_\infty^m}(v_q)) \\
&= c\mu_{K_\infty}(B_{K_\infty}(v_q))^m \\
&< c(v_q^2 s^2)^m \\
&= cs^{2m}\psi(\|q\|^{2n}).
\end{aligned}$$

Next let $d > 0$ and define

$$\mathscr{S}_d = \bigcup_{\substack{q \in B^n \\ \|q\| = s^d}} S_q.$$

This union can only be non-empty for $d \in \frac{1}{2}\mathbb{N}$. If for some $d \in \frac{1}{2}\mathbb{N}$ the value $v_d = \psi(s^{2dn})^{1/2m} \leq 1$, then any $q \in B$ with $\|q\| = s^d$ satisfies $v_q \leq 1$ and we have

$$\begin{aligned}
\mu(\mathscr{S}_d \cap \mathscr{P}) &< \#\{q \in B^n \mid \|q\| = s^d\} cs^{2m}\psi(s^{2dn}) \\
&\leq s^{2(d+1)n} cs^{2m}\psi(s^{2dn}) \\
&= c' s^{2dn}\psi(s^{2dn}),
\end{aligned}$$

where $c' = cs^{2n+2m}$. We know that $z \in \mathcal{W}_{m\times n}(\psi)$ if and only if $z \in \limsup_{d \in \frac{1}{2}\mathbb{N}} \mathscr{S}_d$. There can only be finitely many d with $v_d > 1$ as $\lim_{x\to\infty} \psi(x) = 0$, thus the Borel–Cantelli lemma tells us that if

$$\sum_{d \in \frac{1}{2}\mathbb{N}} s^{2dn}\psi(s^{2dn}) = \sum_{e=1}^{\infty} s^{en}\psi(s^{en}) < \infty$$

then

$$\mu(\mathcal{W}_{m\times n}(\psi) \cap \mathscr{P}) = 0.$$

The convergence of the sum follows from the fact that ψ is non-increasing and that

$$\int_1^\infty \psi(x)dx < \infty.$$

□

1.3 The divergence case

The purpose of this section is to prove Theorem 8. Let $G = \mathrm{SL}_{m+n}(K_\infty)$ and $\Gamma = \mathrm{SL}_{m+n}(B)$. We then let $\Upsilon = G/\Gamma$. This space can be considered as the space of unimodular B-modules in K_∞^{m+n}. Since every such B-module can be represented in the form xB^{m+n} for some $x \in G$ and two lattices, xB^{m+n} and yB^{m+n}, are equal if and only if there is some transformation matrix $a \in \Gamma$ such that $xa = y$. Given a matrix $z \in M_{m\times n}(K_\infty)$, we associate to it the following module $\Lambda_z \in \Upsilon$ defined by

$$\Lambda_z = \begin{pmatrix} I_{m\times m} & z \\ 0_{n\times m} & I_{n\times n} \end{pmatrix} B^{m+n}.$$

A typical element of the B-module Λ_z is of the form

$$\begin{pmatrix} I_{m\times m} & z \\ 0_{n\times m} & I_{n\times n} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} zq+p \\ q \end{pmatrix}$$

where $p \in B^m$ and $q \in B^n$. For $t \in \mathbb{Z}$ we define

$$f_t = \mathrm{diag}(\underbrace{T^{nt},\ldots,T^{nt}}_{m \text{ times}},\underbrace{T^{-mt},\ldots,T^{-mt}}_{n \text{ times}}), \tag{4}$$

and on Υ, define the function,

$$\Delta(\Lambda) := \max_{v\in\Lambda\setminus\{0\}} \log_s \frac{1}{\|v\|}. \tag{5}$$

We will go on to show below that for a certain function $r(t) : \mathbb{N} \to \mathbb{R}$ that if there are infinitely many $t > 0$ such that

$$\Delta(f_t\Lambda_z) \geq r(t)$$

then $z \in \mathcal{W}_{m\times n}(\psi)$. First we will need the following preliminary lemma.

Lemma 7. *Fix $m, n \in \mathbb{N}$, $u > 1$ and $x_0 > 0$, and let $\psi : [x_0,\infty) \mapsto (0,\infty)$ be non-increasing and continuous. Then there exists a pair of continuous functions $\lambda, L : [t_0,\infty) \to \mathbb{R}$, where $t_0 = \frac{\log_u(x_0)}{n(m+n)} - \frac{\log_u \psi(x_0)}{m(m+n)}$, such that*

$$\begin{aligned} &\lambda(t) \textit{ is strictly increasing and} \to \infty \\ &L(t) \textit{ is non-decreasing } (\to \infty \textit{ if } \psi \to 0) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \psi(u^{\lambda(t)}) &= u^{-L(t)} \\ L(t) &= mt(m+n) - \tfrac{m}{n}\lambda(t) \qquad \forall t \geq t_0. \end{aligned} \tag{7}$$

Define the Dani function $r(t) : [t_0,\infty) \to \mathbb{R}$ by $r(t) = \frac{L(t)-\lambda(t)}{m+n}$. Then

$$\int_{x_0}^{\infty} \psi(x)dx < \infty \iff \int_{t_0}^{\infty} u^{-(m+n)r(t)}dt < \infty. \tag{8}$$

Proof. For each $t \in \mathbb{R}$ consider the functions $L_1(\lambda) = L_1^t(\lambda) = mt(m+n) - \frac{m}{n}\lambda$ and $L_2(\lambda) = -\log_u(\psi(u^\lambda))$. The first function L_1 is a decreasing line of gradient $-m/n$ and the second L_2 is a continuous non-decreasing function. Due to the definition of ψ, L_2 is only definined for $\lambda \geq \log_u(x_0)$. Notice that if we have functions L and λ as desired in the lemma, then we would have

$$L_1(\lambda(t)) = L_2(\lambda(t)) = L(t).$$

That is $(\lambda(t), L(t))$ is a point of intersection of L_1 and L_2. Now L_1 and L_2 have at most one point of intersection, so if there is one we define $(\lambda(t), L(t))$ as that point. There is an intersection if

$$L_1(\log_u(x_0)) \geq L_2(\log_u(x_0))$$

that is,

$$\begin{aligned} mt(m+n) - \tfrac{m}{n}\log_u(x_0) &\geq -\log_u \psi(u^{\log_u(x_0)}) \\ mt(m+n) &\geq \tfrac{m}{n}\log_u(x_0) - \log_u \psi(x_0) \\ t &\geq \tfrac{\log_u(x_0)}{n(m+n)} - \tfrac{\log_u \psi(x_0)}{m(m+n)} = t_0. \end{aligned}$$

Thus we have defined $(\lambda(t), L(t))$ for all $t \geq t_0$. Note that we have forced the equalities (7), now we must check that the other conditions hold.

Given a point (x, y) the unique value of t such that the line L_1^t intersects (x, y) is $t = \frac{x}{n(m+n)} + \frac{y}{m(m+n)}$. Now let $t_0 \leq t < t'$. If $\lambda(t') \leq \lambda(t)$ then

$$L(t') = L_2(\lambda(t')) \leq L_2(\lambda(t)) = L(t).$$

Since $L_1^{t'}$ intersects $(\lambda(t'), L(t'))$ we have

$$t' = \frac{\lambda(t')}{n(m+n)} + \frac{L(t')}{m(m+n)} \leq \frac{\lambda(t)}{n(m+n)} + \frac{L(t)}{m(m+n)} = t.$$

This is a contradiction. Thus $\lambda(t') > \lambda(t)$.

Now let $T \geq \log_u(x_0)$ and $t = \frac{T}{n(m+n)} + \frac{L_2(T)}{m(m+n)}$ so that L_1^t intersects the point $(T, L_2(T))$ and so $\lambda(t) = T$. We have that λ is strictly increasing and takes arbitrarily large values, and thus $\lambda \to \infty$. By the fact that $L(t) = -\log_u \psi(u^{\lambda(t)})$ we immediately get that L is non-decreasing and that $L \to \infty$ if $\psi \to 0$. Finally, (8) follows from a simple change of coordinates just as in [15]. We will repeat this here in more detail with our particular setting in mind, as there are some differences in the set up.

First, suppose that

$$\int_{x_0}^{\infty} \psi(x)dx = \infty.$$

Substituting $x = u^\lambda$ we obtain

$$\begin{aligned}\int_{\lambda_0}^{\infty} \psi(u^\lambda) \frac{d}{d\lambda} \left(u^\lambda\right) d\lambda &= \ln(u) \int_{\lambda_0}^{\infty} \psi(u^\lambda) u^\lambda d\lambda \\ &= \ln(u) \int_{\lambda_0}^{\infty} u^{\lambda - L_2(\lambda)} d\lambda\end{aligned}$$

where $\lambda_0 = \log_u(x_0)$. Now on the other hand the integral

$$\begin{aligned}\int_{t_0}^{\infty} u^{-(m+n)r(t)} dt &= \int_{t_0}^{\infty} u^{\lambda(t) - L(t)} dt \\ &= \int_{t_0}^{\infty} u^{\lambda(t) - L_2(\lambda(t))} dt. \qquad (9)\end{aligned}$$

Since we have that for all $t \geq t_0$

$$t = \frac{\lambda(t)}{n(m+n)} + \frac{L_2(\lambda(t))}{m(m+n)}$$

and hence

$$1 = \frac{\frac{d}{dt}\lambda(t)}{n(m+n)} + \frac{\frac{d}{dt}\left(L_2(\lambda(t))\right)}{m(m+n)}$$

then integral (9) is equal to

$$\begin{aligned}&\int_{t_0}^{\infty} u^{\lambda(t) - L_2(\lambda(t))} \left(\frac{\frac{d}{dt}\lambda(t)}{n(m+n)} + \frac{\frac{d}{dt}\left(L_2(\lambda(t))\right)}{m(m+n)} \right) dt \\ &= \frac{1}{n(m+n)} \int_{t_0}^{\infty} u^{\lambda(t) - L_2(\lambda(t))} \frac{d}{dt}\lambda(t) dt \qquad (10) \\ &\quad + \frac{1}{m(m+n)} \int_{t_0}^{\infty} u^{\lambda(t) - L_2(\lambda(t))} \frac{d}{dt}\Big(L_2(\lambda(t))\Big) dt. \qquad (11)\end{aligned}$$

Now the integral (10), by substitution of $\lambda = \lambda(t)$ gives us

$$\int_{t_0}^{\infty} u^{\lambda(t) - L_2(\lambda(t))} \frac{d}{dt}\lambda(t) dt = \int_{\lambda_0}^{\infty} u^{\lambda - L_2(\lambda)} d\lambda = \infty.$$

In integral (11) have that $\frac{d}{dt}\Big(L_2(\lambda(t))\Big) \geq 0$ as $L_2(\lambda(t))$ is non-decreasing. Also we have $u^{\lambda(t) - L_2(\lambda(t))} > 0$ and so we obtain that

$$\int_{t_0}^{\infty} u^{\lambda(t) - L_2(\lambda(t))} dt = \infty.$$

Now it remains for us to assume that

$$\int_{x_0}^{\infty} \psi(x) dx < \infty.$$

Using exactly the same decomposition as above, we need only show that

$$\int_{t_0}^{\infty} u^{\lambda(t) - L_2(\lambda(t))} \frac{d}{dt}\Big(L_2(\lambda(t))\Big) dt < \infty. \qquad (12)$$

Notice that

$$\frac{d}{dt}\left(u^{-L_2(\lambda(t))}\right) = -\ln(u)u^{-L_2(\lambda(t))}\frac{d}{dt}\Big(L_2(\lambda(t))\Big)$$

so inequality (12) is equivalent to

$$-\int_{t_0}^{\infty} u^{\lambda(t)}\frac{d}{dt}\left(u^{-L_2(\lambda(t))}\right)dt < \infty.$$

By integration by parts, this is equivalent to

$$-\left[u^{\lambda(t)-L_2(\lambda(t))}\right]_{t_0}^{\infty} + \int_{t_0}^{\infty} u^{-L_2(\lambda(t))}\frac{d}{dt}\left(u^{\lambda(t)}\right)dt < \infty.$$

The right-hand integral becomes

$$\begin{aligned}\int_{t_0}^{\infty} u^{-L_2(\lambda(t))}\frac{d}{dt}\left(u^{\lambda(t)}\right)dt &= \ln(u)\int_{t_0}^{\infty} u^{\lambda(t)-L_2(\lambda(t))}\frac{d}{dt}\lambda(t)dt\\ &= \ln(u)\int_{\lambda_0}^{\infty} u^{\lambda-L_2(\lambda)}d\lambda,\end{aligned}$$

which we have assumed to be finite. So we just need to consider the term

$$-\left[u^{\lambda(t)-L_2(\lambda(t))}\right]_{t_0}^{\infty}.$$

Looking back at where this term comes from, the function in the integral

$$\int_{t_0}^{\infty} u^{\lambda(t)}\frac{d}{dt}\left(-u^{-L_2(\lambda(t))}\right)dt$$

is non-negative for all $t \geq t_0$ because $-u^{-L_2(\lambda(t))}$ is monotone non-decreasing. So to prove that the integral is finite, we need only prove that

$$\int_{t_0}^{T} u^{\lambda(t)}\frac{d}{dt}\left(-u^{-L_2(\lambda(t))}\right)dt$$

is bounded above as $T \to \infty$. So this is equivalent to proving that

$$\begin{aligned}-\left[u^{\lambda(t)-L_2(\lambda(t))}\right]_{t_0}^{T} &= u^{\lambda_0-L_2(\lambda_0)} - u^{\lambda(T)-L_2(\lambda(T))}\\ &\leq u^{\lambda_0-L_2(\lambda_0)}\end{aligned}$$

is bounded above as $T \to \infty$. This is clearly true and concludes the proof. □

The following definition is a variation of a definition from section 8 of [15] modified for our own purposes.

Definition 1. *We say that a lattice $\Lambda \in \Upsilon$ is (ψ, n) approximable if there exist $v \in \Lambda$ with arbitrarily large $\|v_{(n)}\|$ such that*

$$\|v^{(m)}\|^{2m} \leq \psi(\|v_{(n)}\|^{2n})$$

where $v^{(m)}$ and $v_{(n)}$ are the first m and last n entries of v respectively.

Proposition 2. *Fix $m, n \in \mathbb{Z}$ let $u = s^2$ and let $\psi : [x_0, \infty) \to (0, \infty)$ be continuous, differentiable and non-increasing. Let $\Lambda \in \Upsilon$. Define $r : [t_0, \infty) \to \mathbb{R}$ using Lemma 7. Then if there exist arbitrarily large $t \in \mathbb{N}$ such that*

$$\Delta(f_t \Lambda) \geq r(t)$$

then Λ is (ψ, n)-approximable. Furthermore,

$$\int_{x_0}^{\infty} \psi(x) dx < \infty \text{ if and only if } \int_{t_0}^{\infty} s^{-2(m+n)r(t)} dt < \infty. \tag{13}$$

Proof. Lemma 7 immediately gives us (13). Suppose $\Lambda \in \Upsilon$ is such that for arbitrarily large t, $\Delta(f_t \Lambda) \geq r(t)$. Then for each pair $t > 0$ and $v \in \Lambda \setminus \{0\}$ with $\log_s(1/\|f_t v\|) \geq r(t)$, we have

$$\begin{aligned} & \frac{1}{\|f_t v\|} \geq s^{r(t)} \\ \Leftrightarrow \quad & \|f_t v\| \leq s^{-r(t)}. \end{aligned} \tag{14}$$

Applying f_t to v we have

$$f_t v = f_t \begin{pmatrix} v^{(m)} \\ v_{(n)} \end{pmatrix} = \begin{pmatrix} T^{nt} v^{(m)} \\ T^{-mt} v_{(n)} \end{pmatrix}.$$

So (14) becomes

$$\max(\|T^{nt} v^{(m)}\|, \|T^{-mt} v_{(n)}\|) \leq s^{-r(t)},$$

and so

$$s^{nt} \|v^{(m)}\| \leq s^{-r(t)} \text{ and } s^{-mt} \|v_{(n)}\| \leq s^{-r(t)}$$

Rearranging further, we get

$$\begin{cases} \|v^{(m)}\|^{2m} & \leq & s^{-2mnt-2mr(t)} & = & (s^2)^{-(mnt+mr(t))} & = & (s^2)^{-L(t)} \\ \|v_{(n)}\|^{2n} & \leq & s^{2mnt-2nr(t)} & = & (s^2)^{mnt-nr(t))} & = & (s^2)^{\lambda(t)} \end{cases}$$

where the last equality comes from the fact that

$$L(t) = mnt + mr(t) \text{ and } \lambda(t) = mnt - nr(t).$$

Now if $v^{(m)} = 0$ for some $v \in \Lambda$ then for all integer multiples w of v we have w satisfying $0 = \|w^{(m)}\|^{2m} \leq \psi(\|w_{(n)}\|^{2n})$ meaning that Λ is (ψ, n)-approximable. So we ignore this case. From the above we have

$$\|v^{(m)}\|^{2m} \leq (s^2)^{-L(t)} = \psi((s^2)^{\lambda(t)}).$$

By decreasing monotonicity of ψ we have that $\psi((s^2)^{\lambda(t)}) \leq \psi(\|v_n\|^{2n})$ and thus we have

$$\|v^{(m)}\|^{2m} \leq \psi(\|v_n\|^{2n}).$$

As $v^{(m)} \neq 0$ and the above inequalities hold for pairs (v,t) with arbitrarily large t, the fact that $\|v^{(m)}\|^m \leq s^{-L(t)}$ ($L(t) \to \infty$ holds if $\psi \to 0$) implies that $\|v^{(m)}\|$ is arbitrarily small and so $\|v_{(n)}\|$ is arbitrarily large. Thus we have shown that Λ is (ψ, n)-approximable. $\square$

Definition 2. *We call a function $f : [x_0, \infty) \to \mathbb{R}$* quasi-increasing *if there exists a constant $C > 0$ such that for all $t_1, t_2 \in [x_0, \infty)$, if $t_1 \leq t_2 < t_1 + 1$ then*

$$f(t_2) > f(t_1) - C.$$

Lemma 8. *Given $m, n \in \mathbb{N}$, $u > 1$, $x_0 \in \mathbb{R}$ and a non-increasing continuous function $\psi : [x_0, \infty) \to (0, \infty)$. Then the function $r(t) : [t_0, \infty) \to \mathbb{R}$ as given by Lemma 7 is quasi-increasing with $c = n$.*

Proof. We have that the function $L(t) = mnt + mr(t)$ is non-decreasing. So if $t_1, t_2 \in [t_0, \infty)$ and $t_1 \leq t_2 < t_1 + 1$ then

$$L(t_1) \leq L(t_2)$$

that is

$$\begin{aligned} & mnt_1 + mr(t_1) \leq mnt_2 + mr(t_2) \\ \Leftrightarrow \quad & r(t_1) + n(t_1 - t_2) \leq r(t_2). \end{aligned}$$

Now $t_1 - t_2 > -1$ so

$$r(t_2) > r(t_1) - n.$$

$\square$

Lemma 9. *If $r : [0, \infty) \to \mathbb{R}$ is a quasi-increasing function and $a : \mathbb{R} \to \mathbb{R}$ is non-increasing and $\exists c > 0$ such that $c^{-1} s^{-kx} \leq a(x) \leq cs^{-kx}$ for some $k > 0$ then*

$$\sum_{n=1}^{\infty} a(r(n)) < \infty \text{ if and only if } \int_1^{\infty} a(r(x))dx < \infty.$$

Proof. First suppose that $\sum_{n=1}^{\infty} a(r(n)) < \infty$ and let $n \in \mathbb{N}$. If $n \leq x < n+1$ then $r(x) > r(n) - C$ where C is the constant making r quasi-increasing, and $a(r(x)) \leq a(r(n) - C)$. Thus

$$\int_n^{n+1} a(r(x))dx \leq \int_n^{n+1} a(r(n) - C)dx = a(r(n) - C).$$

So

$$\int_1^{\infty} a(r(x)) \leq \sum_{n=1}^{\infty} a(r(n) - C) \leq c \sum_{n=1}^{\infty} s^{-k(r(n) - C)} \leq s^{kc} c^2 \sum_{n=1}^{\infty} a(r(n)) < \infty.$$

If instead we have $\sum_{n=1}^{\infty} a(r(n)) = \infty$ then for any $x \in (n, n+1]$ we also have $n+1 \in [x, x+1)$ and so $r(n+1) > r(x) - C$. Thus $a(r(n+1)+C) \leq a(r(x))$, and

$$\begin{aligned}\int_1^{\infty} a(r(x))dx &\geq \sum_{n=1}^{\infty} a(r(n+1)+C) \\ &= \sum_{n=2}^{\infty} a(r(n)+C) \geq c^{-1} \sum_{n=2}^{\infty} s^{-k(r(n)+C)} \geq c^{-2} s^{-kc} \sum_{n=2}^{\infty} a(r(n)) = \infty.\end{aligned}$$

□

We now introduce the notion of UDL, or "ultra distance like", as defined by Ghosh in [11], inspired by the notion of DL, or "distance like", as appearing in Kleinbock–Margulis [15]. A function $\Delta : \Upsilon \to \mathbb{R}$ is said to be smooth if there exists a compact open subgroup $U \subseteq G$ such that Δ is U-invariant. That is for every $g \in G$ and $u \in U$ we have $\Delta(ug\Gamma) = \Delta(g\Gamma)$.

Definition 3. *A function $\Delta : \Upsilon \to \mathbb{R}$ is called k-UDL if it is smooth and there exists a $k > 0$ such that the tail distribution function*

$$\Psi_{\Delta}(z) = \mu\left(\{\Lambda \in \Upsilon \;\mid\; \Lambda(\Delta) \geq z\}\right)$$

satisfies

$$\Psi_{\Delta}(z) \asymp s^{-kz}.$$

We call a function $\Delta : \Upsilon \to \mathbb{R}$ UDL if it is k-UDL for some $k > 0$.

Theorem 9. *The function $\Delta : \Upsilon \to \mathbb{R}$ as in* (5) *is $2(m+n)$-UDL.*

This result comes from a generalised version of Siegel's mean value theorem [23]. The original theorem relates the integral of a function $\psi : \mathbb{R}^k \to \mathbb{R}$ with $k > 1$ to a function $\widehat{\psi} : \mathrm{GL}_k(\mathbb{R})/\mathrm{SL}_k(\mathbb{Z}) \to \mathbb{R}$ on the space of all unimodular lattices in $\mathbb{R}^k$, defined by $\widehat{\psi}(\Lambda) = \sum_{v \in P(\Lambda)} \psi(v)$, where $P(\Lambda) \subset \Lambda$ is the set of all *primitive* vectors $v \in \Lambda$, that is those elements of Λ which do not occur as an integer multiple of another element of Λ, or equivalently, those $v \in \Lambda$ for which there exist $v_2, \ldots, v_k \in \Lambda$ such that $\{v, v_2, \ldots, v_k\}$ forms a $\mathbb{Z}$-basis for Λ. Siegel's original theorem then states:

Theorem 10. *For any $\psi \in L^1(\mathbb{R}^k)$, one has $\int_{X_k} \widehat{\psi} d\mu = c_k \int_{\mathbb{R}^k} \psi dv$ where $X_k = \mathrm{GL}_k(\mathbb{R})/\mathrm{SL}_k(\mathbb{Z})$ with Haar measure μ and $c_k = \frac{1}{\zeta(k)}$ where ζ is the Riemann zeta function.*

A direct generalisation of Siegel's mean value theorem appears in [15] where if we let $1 \leq d < k$, we say that an ordered d-tuple $(v_1, \ldots, v_d)$ of vectors in a lattice $\Lambda \subset \mathbb{R}^k$ is *primitive* if it is extendable to a basis $\{v_1, \ldots v_d, v_{d+1}, \ldots, v_k\}$ of Λ, and denote $P^d(\Lambda)$ the set of all such d-tuples. Then, given a function $\psi : \mathbb{R}^{kd} \to \mathbb{R}$, define the function $\widehat{\psi}^d(\Lambda) = \sum_{(v_1,\ldots,v_d) \in P^d(\Lambda)} \psi(v_1, \ldots, v_d)$, then we have the following theorem from [15]:

Theorem 11. *For $1 \leq d < k$ and $\psi \in L^1(\mathbb{R}^{kd})$,*

$$\int_{X_k} \widehat{\psi}^d d\mu = c_{k,d} \int_{\mathbb{R}^{kd}} \psi dv_1 \cdots dv_d,$$

where $c_{k,d} = \prod_{i=0}^{d-1} \frac{1}{\zeta(k-i)}$.

Now for our case we are not working in $\mathbb{R}$ or $\mathrm{GL}_k(\mathbb{R})/\mathrm{SL}_k(\mathbb{Z})$, but in K_∞ and Υ. So we now take $P^d(\Lambda)$ to be the d-primitive vectors of some $\Lambda \in \Upsilon$, where a d-tuple $(v_1, \ldots, v_d) \in K_\infty^k$ is d-primitive if it can be extended to a B-basis of Λ, redefine $\widehat{\psi}^d(\Lambda) = \sum_{(v_1,\ldots,v_d) \in P^d(\Lambda)} \psi(v_1, \ldots, v_d)$, for $\psi : K_\infty^{kd} \to \mathbb{R}$, and use the following theorem which is from a generalisation of Siegel's theorem due to Morishita [18] and explained in Section 4 of the paper of Athreya, Ghosh and Prasad [1]:

Theorem 12. *If $1 \leq d < k$ then there exists a $c_{d,k} > 0$ such that if $\psi \in L_1(K_\infty^{kd})$, then one has*

$$\int_\Upsilon \widehat{\psi}^d d\mu = c_{d,k} \int_{K_\infty^{kd}} \psi dv_1 \cdots dv_d$$

where μ is the Haar measure on Υ and for each i, dv_i is the standard product measure on K_∞^k.

We now have the tools to be able to prove Theorem 9. The following proof is based on Proposition 7.1 in [15] adapted for our positive characteristic setting making use of Theorem 12

Proof of Theorem 9. For each $z \in \mathbb{R}$ define $B_z = \{x \in K_\infty^{m+n} \mid \|x\| \leq s^{-z}\}$. Our aim is to prove that $\Psi_\Delta(z) = \mu(\Lambda \in \Upsilon \mid \Delta(\Lambda) \geq z) \asymp s^{-2(m+n)z}$. Now a lattice $\Lambda \in \Upsilon$ satisfies $\Delta(\Lambda) \geq z$ if it containts some $v \in \Lambda \setminus \{0\}$ such that $\log_s \frac{1}{\|v\|} \geq z$, or $\|v\| \leq s^{-z}$, or equivalently $v \in B_z$. If v is not primitive, then there is some primitive $v' \in \Lambda \setminus \{0\}$ with $\|v'\| < \|v\|$ so one has $\Delta(\Lambda) \geq z$ if and only if $P(\Lambda) \cap B_z \neq \emptyset$. Let $\psi : K_\infty^{m+n} \to \mathbb{R}$ be the characteristic function of B_z and applying Theorem 12 we have

$$\int_\Upsilon \widehat{\psi} d\mu = c_{1,m+n} \int_{K_\infty^{m+n}} \psi dv.$$

Now the integral $\int_{K_\infty^{m+n}} \psi dv$ is simply the volume of the box $B_{K_\infty^{m+n}}(s^{-z})$ which by Corollary 2 is $\asymp s^{-2z(m+n)}$. The function $\widehat{\psi}(\Lambda)$ counts how many primitive vectors in $P(\Lambda)$ lie in B_z, so $\widehat{\psi}(\Lambda) = \#P(\Lambda) \cap B_z$. Thus

$$\begin{aligned}
\int_\Upsilon \widehat{\psi} d\mu &= \int_{\substack{\Lambda \in \Upsilon \\ \text{such that} \\ \Delta(\Lambda) \geq z}} \widehat{\psi} d\mu \\
&\geq \int_{\substack{\Lambda \in \Upsilon \\ \text{such that} \\ \Delta(\Lambda) \geq z}} 1 d\mu \\
&= \mu\big(\Lambda \in \Upsilon \mid \Delta(\Lambda) \geq z\big) = \Phi_\Delta(z).
\end{aligned}$$

Therefore we obtain that there is some $c > 0$ such that for all z, $\Phi_\Delta(z) \leq cs^{-2z(m+n)}$. Now for the other bound, the proof involves demonstrating that those $\Lambda \in \Upsilon$ with $\#P(\Lambda) \cap B_z > U$, where $U = \#(B^\times)$ is the number of units in the ring B, form an insignificant portion of the total. The ring $B \subset K_\infty$ contains U units, so if $v_1, \ldots, v_{U+1}$ are primitive vectors of some lattice Λ, then they cannot all be unit multiples of each other. Therefore there will be two vectors, say $v_1, v_2 \in P(\Lambda) \cap B_z$, which are linearly independent over B. That is to say $(v_1, v_2) \in P^2(\Lambda)$. Now applying Theorem 12 again, this time with $d = 2$ and $\phi : K_\infty^{2(m+n)} \to \mathbb{R}$ the characteristic function of $B_z \times B_z$ we have

$$\int_\Upsilon \widehat{\phi}^2 d\mu = c_{2,m+n} \int_{K_\infty^{2(m+n)}} \phi(v_1, v_2) dv_1 dv_2$$

and $\int_{K_\infty^{2(m+n)}} \phi(v_1, v_2) dv_1 dv_2 \asymp s^{-4(m+n)z}$ by Corollary 2. So since if $v_1 \in P(\Lambda)$ and $\#P(\Lambda) \cap B_z > U$ then there exists another $v_2 \in P(\Lambda) \cap B_z$ with $(v_1, v_2) \in P^2(\Lambda) \cap B_z^2$. This gives us an injection $P(\Lambda) \cap B_z \to P^2(\Lambda) \cap B_z^2$ and so $\widehat{\psi}(\Lambda) \leq \widehat{\phi}^2(\Lambda)$. Furthermore if $1 \leq \#P(\Lambda) \cap B_z \leq U$ then $P^2(\Lambda) \cap B_z^2 = \emptyset$ as if $v \in P(\Lambda)$ then $uv \in P(\Lambda)$ for all U units of B, thus $\#P(\Lambda) \cap B_z = U$ consists of U linearly dependent primitive vectors. So $\widehat{\phi}^2(\Lambda) > 0$ only when $\#P(\Lambda) \cap B_z > U$. Therefore

$$\int_{\substack{\Lambda \in \Upsilon \\ \text{such that} \\ \#P(\Lambda) \cap B_z > U}} \widehat{\psi} d\mu \leq \int_\Upsilon \widehat{\phi}^2 d\mu \asymp s^{-4(m+n)z}.$$

Therefore, to finish the proof we have

$$\begin{aligned} \int_\Upsilon \widehat{\psi} d\mu &= \int_{\substack{\Lambda \in \Upsilon \\ \text{such that} \\ \#P(\Lambda) \cap B_z = U}} \widehat{\psi} d\mu + \int_{\substack{\Lambda \in \Upsilon \\ \text{such that} \\ \#P(\Lambda) \cap B_z > U}} \widehat{\psi} d\mu \\ &\leq U\mu\big(\{\Lambda \in \Upsilon \mid \Delta(\Lambda) \geq z\}\big) + \int_\Upsilon \widehat{\phi}^2 d\mu \\ &\leq U\Phi_\Delta(z) + c's^{-4(m+n)z} \end{aligned}$$

for some $c' > 0$, and so there is some $c'' > 0$ such that

$$\Phi_\Delta(z) \geq c''s^{-2(m+n)z} - c'/Us^{-4(m+n)z}.$$

So for sufficiently large z we have $\Phi_\Delta(z) \asymp s^{-2(m+n)z}$. $\square$

Next we will introduce the concept of Borel–Cantelli families. The traditional Borel-Cantelli theorem is as follows

Theorem 13. *Let X be a probability space with measure μ and let (A_n) be a sequence of measurable sets in X. Then*

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty \quad \Rightarrow \quad \mu\Big(\limsup_{n\to\infty} A_n\Big) = 0.$$

Conversely, if the sets (A_n) are pairwise independent, that is for all $m \neq n$ we have $\mu(A_m \cap\ A_n) = \mu(A_m)\mu(A_n)$, then

$$\sum_{n=1}^{\infty} \mu(A_n) = \infty \quad \Rightarrow \quad \mu\Big(\limsup_{n\to\infty} A_n\Big) = 1.$$

Clearly the converse direction is false if we drop the independence condition, for example take $X = [0,1]$ and $A_n = [0, 1/n]$. It is for this reason we introduce the concept of Borel–Cantelli families. The following terminology is taken from [15].

Definition 4. *Let $\mathcal{B}$ be a collection of measurable sets of Υ and let $\mathfrak{F} = \{f_t\}$ denote a sequence of μ-preserving transformations of Υ. We say that $\mathcal{B}$ is* Borel–Cantelli for $\mathfrak{F}$ *if for every sequence $\{A_n \mid n \in \mathbb{N}\}$ of sets from $\mathcal{B}$,*

$$\mu\Big(\limsup_{n\to\infty} f_t^{-1} A_n\Big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mu(A_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \mu(A_n) = \infty. \end{cases}$$

So for example if $\mathscr{B}$ was a collection of pairwise independent sets and $\mathfrak{F} = (id_v)$ then $\mathscr{B}$ would be a Borel–Cantelli family for $\mathfrak{F}$ by Theorem 13. The second main result which we need in the proof of Theorem 16 is the following:

Theorem 14. *If we take $\mathfrak{F} = (f_t)$ as in (4). Then the collection of sets*

$$\mathscr{B}(\Delta) = \{\{\Lambda \in \Upsilon \mid \Delta(\Lambda) \geq z\} \mid z \in \mathbb{R}\}$$

is a Borel–Cantelli family for $\mathfrak{F}$.

For this we use the following result of Athreya, Ghosh and Prasad, Theorem 1.6 from [1].

Theorem 15. *Let $\mathfrak{F} = (f_n)_{n\in\mathbb{N}}$ be a sequence of G such that for all $\beta > 0$*

$$\sup_{m\in\mathbb{N}} \sum_{n=1}^{\infty} \|f_n f_m^{-1}\|^{-\beta} < \infty,$$

and let Δ be a UDL function on Υ. Then

$$\mathscr{B}(\Delta) = \{\{\Lambda \in \Upsilon \mid \Delta(\Lambda) \geq z\} \mid z \in \mathbb{R}\}$$

is Borel–Cantelli for $\mathfrak{F}$.

Will we verify that $\mathfrak{F} = (f_t)$ as in (4) satisfies the condition of Theorem 15.

Proof of Theorem 14. Given $t_1, t_2 \in \mathbb{N}$ we have

$$f_{t_1} f_{t_2}^{-1} = f_{t_1 - t_2} = \mathrm{diag}(\underbrace{T^{n(t_1-t_2)}, \ldots, T^{n(t_1-t_2)}}_{m \text{ times}}, \underbrace{T^{-m(t_1-t_2)}, \ldots, T^{-m(t_1-t_2)}}_{n\text{times}}),$$

and so

$$\|f_{t_1} f_{t_2}^{-1}\| = \max(s^{n(t_1-t_2)}, s^{-n(t_1-t_2)}) = \begin{cases} s^{n(t_1-t_2)} & \text{if } t_1 \geq t_2 \\ s^{m(t_2-t_1)} & \text{if } t_1 < t_2. \end{cases}$$

Therefore, given t_2 we have

$$\begin{aligned}\sum_{t_1=1}^{\infty} \|f_{t_1} f_{t_2}^{-1}\|^{-\beta} &= \sum_{1 \leq t_1 < t_2} s^{-\beta m(t_2-t_1)} + \sum_{t_1=t_2}^{\infty} s^{-\beta n(t_1-t_2)} \\ &= \sum_{1 \leq t_1 < t_2} s^{-\beta m(t_2-t_1)} + \sum_{i=0}^{\infty} s^{-\beta n i} \\ &= \sum_{1 \leq t_1 < t_2} s^{-\beta m(t_2-t_1)} + \frac{1}{1 - s^{-\beta n}}.\end{aligned}$$

For the first sum we have

$$\begin{aligned}\sum_{1 \leq t_1 < t_2} s^{-\beta m(t_2-t_1)} &= \sum_{i=1}^{t_2-1} s^{-\beta m i} \\ &\leq \sum_{i=0}^{\infty} s^{-\beta m i} \\ &= \frac{1}{1 - s^{-\beta m}}.\end{aligned}$$

Thus

$$\sup_{t_2 \in \mathbb{N}} \sum_{t_1=0}^{\infty} \|f_{t_1} f_{t_2}^{-1}\|^{-\beta} \leq \frac{1}{1 - s^{-\beta n}} + \frac{1}{1 - s^{-\beta m}} < \infty$$

for all $\beta > 0$. Applying Theorem 15 we are done. $\square$

Theorem 16. *Given $m, n \in \mathbb{N}$, Δ as in (5), (f_t) as in (4) and a quasi-increasing function $r : [t_0, \infty) \to \mathbb{R}$, then for almost every or almost no $\Lambda \in \Upsilon$ there exist arbitrarily large $t \in \mathbb{N}$ such that*

$$\Delta(f_t \Lambda) \geq r(t)$$

according as the integral

$$\int_{t_0}^{\infty} s^{-2(m+n)r(t)}$$

diverges or converges.

Proof. The family $\mathcal{B}(\Delta)$ is Borel-Cantelli for (f_t), which means that if we let

$$U_t = \{\Lambda \in \Upsilon \ \mid \ \Delta(\Lambda) \geq r(t)\}$$

and take the sequence of sets $\{U_t \ \mid \ t \in \mathbb{N}\}$ of $\mathcal{B}(\Delta)$ then

$$\mu\Big(\limsup_{t \to \infty} f_t^{-1}(U_t)\Big) = \begin{cases} 0 & \text{if } \sum_{t=1}^{\infty} \mu\big(U_t\big) < \infty \\ 1 & \text{if } \sum_{t=1}^{\infty} \mu\big(U_t\big) = \infty. \end{cases}$$

Now $f_t^{-1}(U_t) = \{\Lambda \in \Upsilon \mid \Delta(f_t\Lambda) \geq r(t)\}$, so $\limsup_{t\to\infty} f_t^{-1}(U_t)$ is precisely the subset of $\Lambda \in \Upsilon$ for which there exist arbitrarily large t such that $\Delta(f_t\Lambda) \geq r(t)$. So we just need to prove that $\sum_{t=1}^{\infty} \mu\big(U_t\big) < \infty$ if and only if $\int_{t_0}^{\infty} s^{-2(m+n)r(t)} < \infty$. Now $\mu\big(U_t\big) = \Psi_\Delta(r(t))$, and $\Psi_\Delta(x)$ is non-increasing and by Theorem 9 satisfies that for some $c > 0$ and sufficiently large x we have $c^{-1}s^{-2(m+n)x} \leq \Psi_\Delta(x) \leq cs^{-2(m+n)x}$ since Δ is $2(m+n)$-UDL. Thus applying Lemma 9 to Φ_Δ and $r(t)$ which is quasi-increasing by Lemma 8 finishes the proof. □

Theorem 8 now follows as a Corollary of Theorem 16. Taking $r(t)$ as our quasi-increasing function from Lemma 7, and also from Proposition 2 we note that $\int_{x_0}^{\infty} \psi(x) < \infty$ if and only if $\int_{t_0}^{\infty} s^{-2(m+n)r(t)}dt$. The proof uses a Fubini-type argument as in 8.7 of [15] or Section 2 of [6] which we shall not repeat here.

Corollary 3. *Given $m, n \in \mathbb{N}$, Δ as in* (5)*, (f_t) as in* (4) *and a quasi-increasing function $r : [t_0, \infty) \to \mathbb{R}$, then for almost every or almost no $z \in \mathrm{M}_{m\times n}(K_\infty)$ there exist arbitrarily large $t \in \mathbb{N}$ such that*

$$\Delta(f_t\Lambda_z) \geq r(t)$$

according as the integral

$$\int_{t_0}^{\infty} s^{-2(m+n)r(t)}$$

diverges or converges.

Chapter 2

Correction to a paper by Deligero and Nakada

2.1 Background

Here we will be looking at writing out more rigorously and providing a correct argument to the proof of Theorem 2 in "On the Central Limit Theorem for Non-Archimedean Diophantine Approximations" by Eveyth Deligero and Hitoshi Nakada [8]. I will begin by briefly explaining what the notation is, and what the basic properties of the sets we describe are. All omitted details are supposed to either be adequately explained in the aforementioned paper, or amount to an easy exercise.

2.1.1 Definitions and notation

- $\mathbb{F} = \mathbb{F}_q$ is a finite field of q elements.
- $k = \mathbb{F}(T)$ is the field of rational polynomials over $\mathbb{F}$.
- $\overline{k} = \mathbb{F}((T^{-1}))$ is the field of formal Laurent series over $\mathbb{F}$.
- $\mathfrak{o} = \mathbb{F}[T]$ is the ring of polynomials over $\mathbb{F}$. We refer to $\mathfrak{o}$ as the integers of k.
- $|\cdot|$ is the normalized T^{-1}-adic valuation on $\overline{k}$. For a polynomial $f \in \mathfrak{o}$, it takes the value $|f| = q^{\deg f}$.
- $\mathbb{L} = \{x \in \overline{k} \, : |x| < 1\}$ is the unit ball in $\overline{k}$ with respect to $|\cdot|$. Equivalently, $\mathbb{L}$ consists of those $x \in \overline{k}$ such that the polynomial part of x is zero.
- Let m be the Haar measure on $\overline{k}$ normalized so that $m(\mathbb{L}) = 1$.

Throughout the sequence $\{l_n\}$ will be a non-decreasing sequence of non-negative integers, and we will be considering solutions to the diophantine equation

$$\left|f - \frac{P}{Q}\right| < \frac{1}{|Q|^2 q^{l_n}}, \quad (P, Q) = 1, \quad n = \deg Q \tag{15}$$

where $f \in \overline{k}$.

2.1.2 What we need to know

The sets F_n for $n \in \mathbb{N}$ are defined as follows.

$$F_n = \{f \in \mathbb{L} : \exists P/Q \in k \text{ with } \deg Q = n \text{ as a solution to (15)}\}.$$

We will proceed from the preliminary results from [8] that $m(F_n) = q^{-l_n}(1 - q^{-1})$ and furthermore, the sets F_n satisfy the following property.

Lemma 10 (Deligero–Nakada [8]). *For any increasing positive integers $i_1, i_2, \ldots, i_k$,*

$$m\left(\bigcap_{j=1}^{k} F_{i_j}\right) = \begin{cases} \prod_{j=1}^{k} m(F_{i_j}) & \text{if } (*) \text{ holds} \\ 0 & \text{otherwise,} \end{cases}$$

where the property $(*)$ means that for all $j = 1, \ldots, k-1$,

$$i_j + l_{i_j} < i_{j+1}.$$

In the case where the measure of the intersection is zero, the intersection is in fact the empty set. Interpreting Lemma 10 for $k = 2$, it says that if $i < j$ and $|i-j|$ is small, then F_i and F_j are disjoint, whereas if $|i - j|$ is big, then F_i and F_j are independent. Here "small" means $|i - j| < l_i$ and "big" means $|i - j| \geq l_i$. Using this idea we define the sets G_i, which are maximal disjoint unions of the F_n in the following sense:

1. Define $G_1 = \bigcup_{i=1}^{N} F_i$ where N is maximal so that $F_1, \ldots, F_N$ are pairwise disjoint. We define $\widetilde{l}(1) = 1$ and $\widetilde{l}(2) = N + 1$.

2. Suppose that G_m and $\widetilde{l}(m+1)$ are defined for some $m \geq 1$. Then we define $G_{m+1} = \bigcup_{i=\widetilde{l}(m+1)}^{N} F_i$ where N is again chosen to be maximal so that $F_{\widetilde{l}(m)}, \ldots, F_N$ are pairwise disjoint. We define $\widetilde{l}(m+2) = N + 1$.

Having defined the sets G_i as above we have

$$G_j = \bigcup_{i=\widetilde{l}(j)}^{\widetilde{l}(j+1)-1} F_i,$$

$$\widetilde{l}(1) = 1, \quad \text{and} \quad \widetilde{l}(m) = \widetilde{l}(m-1) + l_{\widetilde{l}(m-1)} \text{ for } m > 1.$$

We also have the property that if $|i-j| > 1$ then G_i and G_j are independent, and indeed for any $k \geq 1$ the σ-algebras generated by the sets $\{G_1, \ldots, G_{k-1}\}$ and $\{G_{k+1}, G_{k+2}, \ldots\}$ are independent. This is a result of Lemma 10. We will refer to this property by saying that the sequence of random variables $\{G_n\}$ is 1-dependent.

Remark 1. *This observation turns out to be very important in the calculations later. We define the centralised random variables*

$$X_i = \mathbb{1}_{G_i} - m(G_i) \tag{16}$$

for our later calculations. Say we have two polynomials $f(T_1, \ldots, T_m)$ and $g(T_1, \ldots, T_n)$ over $\mathbb{R}$ in m and n variables respectively. Then if $k > 0$, $(u_1, \ldots, u_m)$, $(v_1, \ldots, v_n)$ are all positive integers such that $u_j < k$, $v_j > k$ for all j, then the random variables

$$f(X_{u_1}, \ldots, X_{u_m}) \quad \text{and} \quad g(X_{v_1}, \ldots, X_{v_m})$$

are measureable over the independent σ-algebras generated by $\{G_1, \ldots, G_{k-1}\}$ and $\{G_{k+1}, G_{k+2}, \ldots\}$ respectively, and thus:

$$\mathbb{E}[f(X_{u_1}, \ldots, X_{u_m}) g(X_{v_1}, \ldots, X_{v_n})] = \mathbb{E}[f(X_{u_1}, \ldots, X_{u_m})]\mathbb{E}[g(X_{v_1}, \ldots, X_{v_n})].$$

In the paper, Deligero and Nakada seek to prove a central limit theorem for the random variable $Z_n : \mathbb{L} \to \mathbb{R}$ of F_n which is defined as

$$Z_N(f) = \#\{P/Q \in K : (P,Q) = 1, \deg Q \leq N, \text{ and } P, Q \text{ satisfy (15)}\},$$

and their theorem is

Theorem 2. *If $\sum_1^\infty q^{-l_n} = \infty$ then for any $\alpha \in \mathbb{R}$*

$$\lim_{N\to\infty} m\left\{ f \in \mathbb{L} : \frac{Z_N(f) - \sum_{n=1}^N q^{-l_n}(1 - \frac{1}{q})}{\sqrt{\sum_{n=1}^N q^{-l_n}(1 - \frac{1}{q})}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

If an element $f \in \overline{k}$ has P/Q with $\deg Q = n$ as a solution to (15), then this is the only such solution for Q of degree n. This is due to the fact that if

$$|f - P/Q| < 1/|Q|^2 = q^{-2n},$$

and

$$|f - P'/Q'| < 1/|Q'|^2 = q^{-2n},$$

then by the ultrametric property, we have

$$|P/Q - P'/Q'| < q^{-2n}$$

and if $P/Q \neq P'/Q'$ then

$$|P/Q - P'/Q'| \geq (|Q||Q'|)^{-1} = q^{-2n}.$$

Thus Z_n is related to the sets F_n in the following explicit manner:

$$Z_n = \sum_1^n \mathbb{1}_{F_i}$$

where $\mathbb{1}_{F_i}$ is the indicator function of F_i.

We can take this idea further and apply it to our sets G_i which are disjoint unions of the F_n. For all $n \geq 1$, if $N = \widetilde{l}(n+1) - 1$ then

$$Z_N = \sum_{i=1}^{N} \mathbb{1}_{F_i} = \sum_{i=1}^{n} \mathbb{1}_{G_i}.$$

Notice that since $m(F_n) = q^{-l_n}(1 - q^{-1})$ the condition that $\sum q^{-l_n} \to \infty$ implies that $\sum m(F_n) \to \infty$, and $\sum G_n \to \infty$.

The tactic of proving Theorem 2 employed by Deligero and Nakada is to prove a modified version involving sums of indicator functions of G_i instead of the Z_i. Proving the following turns out to be enough to prove Theorem 2 (see their paper for details):

Proposition 3. *For all real numbers $\alpha \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} m\left\{ f \in \mathbb{L} : \frac{\sum_1^n \mathbb{1}_{G_i}(f) - \sum_1^n m(G_i)}{\sqrt{\sum_1^n m(G_i)}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

That is to say, the sequence of random variables

$$\frac{\sum_1^n \mathbb{1}_{G_i}(f) - \sum_1^n m(G_i)}{\sqrt{\sum_1^n m(G_i)}}$$

converges in distribution to $\mathcal{N}(0, 1)$.

2.2 Proving Proposition 3

To prove Proposition 3, we use Lyapunov's condition for arrays of random variables.

Lemma 11 (Lyapunov's condition [3])**.** *Let $(\Omega, \mathscr{B}, m)$ be a probability space. For each $n \in \mathbb{N}$ let r_n be a positive integer, and let*

$$X_{n1}, \ldots, X_{nr_n}$$

be r_n independent random variables $\Omega \to \mathbb{R}$ such that

$$\mathbb{E}[X_{nk}] = 0 \quad \textit{for all } n \in \mathbb{N} \textit{ and } k \in \{1, \dots, r_n\}.$$

Furthermore, suppose that there exists some $\delta > 0$ such that for all $n \in \mathbb{N}$ and $k \in \{1, \dots, r_n\}$, $|X_{nk}|^{2+\delta}$ is integrable. Then if we define

$$\sigma_{nk}^2 = \mathbb{E}[X_{nk}^2], \quad s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2, \quad S_n = \sum_{k=1}^{r_n} X_{kn}$$

and if we have the following

$$\lim_{n\to\infty} \left(s_n^{-(2+\delta)} \sum_{k=1}^{r_n} \mathbb{E}[|X_{nk}|^{2+\delta}] \right) = 0$$

then for all $\alpha \in \mathbb{R}$

$$\lim_{n\to\infty} m \left\{ f \in \Omega : \frac{S_n(f)}{s_n} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

We cannot use Lyapunov's condition by setting $X_{nk} = \mathbb{1}_{G_k} - m(G_k) = X_k$ (16) because our G_k aren't independent. If we could, we would get our result immediately. However, the G_j are 1-dependent, which enables us to use the following trick.

For each n set $r_n = \left\lfloor (\sum_{i=1}^n m(G_i))^{1/3} \right\rfloor$. The only special thing about the choice of power 1/3 is that it is less than 1/2. For large enough n, since the sum $\sum_1^n m(G_i) \to \infty$, we can make sure that $r_n \geq 2$, ensuring that the following construction makes sense.

Since each $m(G_i) > 0$ we can define uniquely for each $k \in \{0, 1, \dots, r_n\}$ the values τ_{nk} by

$$\sum_1^{\tau_{nk}} m(G_i) \leq \frac{k}{r_n} \sum_1^n m(G_i) < \sum_1^{\tau_{nk}+1} m(G_i).$$

Note that this gives $\tau_{n0} = 0$ and $\tau_{nr_n} = n$. Since $\frac{1}{r_n} \sum_1^n m(G_i) > 1$ and $m(G_i) \leq 1$, we have that the τ_{nk} are distinct for each n. We then define for $k \in \{1, \dots, r_n\}$ the random variables

$$U_{nk} = \sum_{\tau_{n(k-1)}+1}^{\tau_{nk}-1} X_i,$$

so that

$$\sum_1^n X_i = \sum_{k=1}^{r_n} U_{nk} + \sum_{k=1}^{r_n} X_{\tau_{nk}}.$$

To make what follows less cumbersome we will write for $k \in \{1, \dots, r_n\}$

$$\alpha_{nk} = \tau_{n(k-1)} + 1 \quad \text{and} \quad \beta_{nk} = \tau_{nk} - 1$$

so that

$$U_{nk} = \sum_{\alpha_{nk}}^{\beta_{nk}} X_i.$$

The idea is that we have thrown away r_n of the terms in the sum $\sum_1^n X_i$ that we're interested in, and split it into r_n blocks $U_{n1}, \ldots, U_{nr_n}$ which are in fact independent (since the G_i are 1-dependent) and thus we can try to use Lyapunov's condition to get a result. We then hope to show that what we have thrown away is in fact negligible as $n \to \infty$ and that this gives us our result. So first we will show that if Lyapunov's condition holds for U_{nk} then Proposition 3 is true.

We will begin with a technical lemma and a corollary which will be used repeatedly throughout the calculations that follow.

Lemma 12. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of positive real numbers such that $\frac{x_n}{y_n} \to 0$ and $\sum_1^\infty y_n = \infty$, and let $\delta > 0$. Then there exists an $N \in \mathbb{N}$ such that for each $n \geq N$, for every $k \in \{1, \ldots, r_n\}$ we have*

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} x_i}{\sum_{\alpha_{nk}}^{\beta_{nk}} y_i} < \delta.$$

Proof. We start with the case $k = 1$. Let M' be large enough so that for all $m \geq M'$ we have $x_m/y_m < \frac{\delta}{2}$. Then let $M > M'$ be large enough so that

$$\frac{\sum_1^{M'} x_i}{\sum_1^M y_i} < \delta/2.$$

Then for all $m \geq M$

$$\begin{aligned}
\frac{\sum_1^m x_i}{\sum_1^m y_i} &= \frac{\sum_1^{M'} x_i + \sum_{M'+1}^m x_i}{\sum_1^m y_i} \\
&< \delta/2 + \frac{\sum_{M'+1}^m x_i}{\sum_1^m y_i} \\
&< \delta/2 + \frac{\sum_{M'+1}^m \frac{\delta}{2} y_i}{\sum_{M'+1}^m y_i} \\
&= \delta.
\end{aligned}$$

So for the $k = 1$ case, since $\alpha_{n1} = 1$ we choose N large enough so that for all $n \geq N$ we have $\beta_{n1} \geq M$.

With this same value N it then follows that the result holds for all $k > 1$ also. Indeed, for all $m \geq \alpha_{nk} > \beta_{n1} \geq M$ we have $x_m/y_m < \delta/2$, so

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} x_i}{\sum_{\alpha_{nk}}^{\beta_{nk}} y_i} < \frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \frac{\delta}{2} y_i}{\sum_{\alpha_{nk}}^{\beta_{nk}} y_i} = \delta/2 < \delta.$$

□

Remark 2. *We will repeatedly be considering an array of values $x_{nk} \in \mathbb{R}$ for $n \in \mathbb{N}$ and $k \in \{1, \ldots, r_n\}$ and trying to show that for some $\ell \in \mathbb{R}$, that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that $n \geq N$ implies that*

$$|x_{nk} - \ell| < \varepsilon$$

for every $k \in \{1, \ldots, r_n\}$. We will refer to this property by saying that x_{nk} converges uniformly to ℓ.

An example in the previous lemma is with

$$x_{nk} = \frac{\sum_{\alpha_{nk}}^{\beta_{nk}} x_i}{\sum_{\alpha_{nk}}^{\beta_{nk}} y_i}$$

and the claim was that x_{nk} converged uniformly to 0. We have the various nice properties of arithmetic associated with convergence applying here. For example if $x_{nk} \to \alpha$ uniformly and $y_{nk} \to \beta$ uniformly, then $x_{nk} + y_{nk} \to \alpha + \beta$ uniformly and $x_{nk} y_{nk} \to \alpha\beta$ uniformly.

Lemma 13. *If x_{nk} converges uniformly to ℓ then*

$$\lim_{n \to \infty} \frac{1}{r_n} \sum_{k=1}^{r_n} x_{nk} = \ell.$$

Proof. Let $\epsilon > 0$. Since $x_{nk} \to \ell$ uniformly, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|x_{nk} - \ell| < \epsilon$$

for each $k \in \{1, \ldots, r_n\}$. So take some $n \geq N$ and then by the triangle inequality

$$\left| \sum_{k=1}^{r_n} x_{nk} - r_n \ell \right| < r_n \epsilon.$$

Dividing through by r_n gives us the result. □

Corollary 4. *For all $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies that for every $k \in \{1, \ldots, r_n\}$ we have*

$$\left| \frac{\mathbb{E}[U_{nk}^2]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} - 1 \right| < \delta. \tag{17}$$

That is $\mathbb{E}[U_{nk}^2] / \sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)$ converges uniformly to 1.

Proof. We begin by examining $\mathbb{E}[U_{nk}^2]$ a little more closely.

$$\begin{aligned}
\mathbb{E}[U_{nk}^2] &= \mathbb{E}\left[\Big(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)\Big)^2\right] \\
&= \sum_{\alpha_{nk}}^{\beta_{nk}} \sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[m(G_i)m(G_j)] \qquad \text{by linearity of expectation} \\
&= \sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X_i]^2 + 2\sum_{\alpha_{nk}}^{\beta_{nk}-1} \mathbb{E}[X_i X_{i+1}] \qquad \text{by 1-dependence of the } G_i
\end{aligned}$$

We will look at these two sums separately. First we note that $\mathbb{E}[X_i^2] = m(G_i)(1 - m(G_i))$. So

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} E[X_i^2]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} = 1 - \frac{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)^2}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)}.$$

Now we can use Lemma 12 as $\sum_1^\infty m(G_i) = \infty$ and $m(G_i)^2/m(G_i) \to 0$, and so

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} E[X_i^2]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} \to 1 \text{ uniformly.}$$

Next we note that $\mathbb{E}[X_i X_{i+1}] = m(G_i \cap G_{i+1}) - m(G_i)m(G_{i+1})$, so

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}-1} E[X_i X_{i+1}]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} = \frac{\sum_{\alpha_{nk}}^{\beta_{nk}-1} m(G_i \cap G_{i+1})}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} - \frac{\sum_{\alpha_{nk}}^{\beta_{nk}-1} m(G_i)m(G_{i+1})}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)}.$$

Again we want to use Lemma 12. We have that $\sum m(G_i) \to \infty$, and $m(G_i)m(G_{i+1})/m(G_i) \to 0$ so the second term above converges to 0 uniformly. For the first term, note that the G_i are disjoint union of sets F_j which by Lemma 10 are either disjoint or independent, and thus $m(G_i \cap G_{i+1}) \le m(G_i)m(G_{i+1})$ and so Lemma 12 applies here too. Thus

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}-1} E[X_i X_{i+1}]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} \to 0 \text{ uniformly.}$$

□

We can now prove the following proposition.

Proposition 4. *If*

$$\frac{\sum_{k=1}^{r_n} U_{nk}}{\sqrt{\sum_{k=1}^{r_n} \mathbb{E}[U_{nk}^2]}}$$

converges in distribution to $\mathcal{N}(0,1)$ *as* $n \to \infty$ *then so does*

$$\frac{\sum_1^n X_i}{\sqrt{\sum_1^n m(G_i)}}.$$

Proof. First we will show the following:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{r_n} \mathbb{E}[U_{nk}^2]}{\sum_1^n m(G_i)} = 1. \tag{18}$$

If we can show that

$$r_n \frac{\mathbb{E}[U_{nk}^2]}{\sum_1^n m(G_i)} \tag{19}$$

converges uniformly to 1, then (18) follows from Lemma 13. From Corollary 4 we know that

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[U_{nk}^2]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)}$$

converges uniformly to 1. By the definition of α_{nk} and β_{nk} and the fact that $m(G_i) \to 0$ it is simple to check that

$$\frac{r_n \sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)}{\sum_1^n m(G_i)} \tag{20}$$

converges uniformly to 1 also. Multiplying these together gives us (19), and hence (18).

Pointwise convergence implies convergence in distribution so

$$\frac{\sum_{k=1}^{r_n} U_{nk}}{\sqrt{\sum_1^n m(G_i)}} = \frac{\sum_{k=1}^{r_n} U_{nk}}{\sqrt{\sum_{k=1}^{r_n} \mathbb{E}[U_{nk}^2]}} \sqrt{\frac{\sum_{k=1}^{r_n} \mathbb{E}[U_{nk}^2]}{\sum_1^n m(G_i)}}$$

also converges in distribution to $\mathcal{N}(0, 1)$.

So we are almost done, we just need to show that

$$\frac{\sum_{k=1}^{r_n} X_{\tau_{nk}}}{\sqrt{\sum_1^n m(G_i)}} = \frac{\sum_1^n X_i}{\sqrt{\sum_1^n m(G_i)}} - \frac{\sum_{k=1}^{r_n} U_{nk}}{\sqrt{\sum_1^n m(G_i)}}$$

converges pointwise to 0. This is simple as

$$\left| \frac{\sum_{k=1}^{r_n} X_{\tau_{nk}}}{\sqrt{\sum_1^n m(G_i)}} \right| < \frac{r_n}{\sqrt{\sum_1^n m(G_i)}} \leq \left(\sum_1^n m(G_i) \right)^{-\frac{1}{6}} \longrightarrow 0 \text{ as } n \to \infty.$$

□

So now we have seen that it is worthwhile trying to check Lyapunov's condition for U_{nk}, so this is our next task. We will prove Lyapunov's condition for $\delta = 2$.

Proposition 5. *The Lyapunov condition for U_{nk} holds for $\delta = 2$. That is to say*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{r_n} \mathbb{E}[U_{nk}^4]}{(\sum_{k=1}^{r_n} \mathbb{E}[U_{nk}^2])^2} = 0. \tag{21}$$

Proof. First we borrow some ideas and results from our proof of Proposition 4. We observe that by (18), (21) is true if and only if

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{r_n} \mathbb{E}[U_{nk}^4]}{(\sum_1^n m(G_i))^2} = 0. \tag{22}$$

We claim that

$$r_n^2 \frac{\mathbb{E}[U_{nk}^4]}{(\sum_1^n m(G_i))^2} \to 3 \text{ uniformly,} \tag{23}$$

which by Lemma 13 would give us that

$$\lim_{n\to\infty} r_n \frac{\sum_1^{r_n} \mathbb{E}[U_{nk}^4]}{(\sum_1^n m(G_i))^2} = 3.$$

Diving this though by r_n then would give us (22). By (20), to prove this it would suffice to show that

$$\frac{\mathbb{E}[U_{nk}^4]}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \to 3 \text{ uniformly,} \tag{24}$$

which is what we will go on to prove.

Let us now examine $\mathbb{E}[U_{nk}^4]$.

$$\mathbb{E}[U_{nk}^4] = \mathbb{E}\left[\Big(\sum_{\alpha_{nk}}^{\beta_{nk}} X_i\Big)^4\right] = \sum_{i,j,k,l=\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X_i X_j X_k X_l]$$

by linearity of expectation. So we have

$$\begin{aligned}\mathbb{E}[U_{nk}^4] = &\sum_i \mathbb{E}[X_i^4] + 3\sum_{i\neq j} \mathbb{E}[X_i^2 X_j^2] + 4\sum_{i\neq j} \mathbb{E}[X_i X_j^3] \\ &+ 6\sum_{i\neq j\neq k\neq i} \mathbb{E}[X_i X_j X_k^2] + \sum_{i,j,k,l \text{ distinct}} \mathbb{E}[X_i X_j X_k X_l] \\ =& S_1 + 3S_2 + 4S_3 + 6S_4 + S_5\end{aligned}$$

where i, j, k, l take values in $\{\alpha_{nk}, \ldots, \beta_{nk}\}$.

We will examine the uniform convergence of

$$\frac{S_i}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2}$$

in turn for each i and determine that they all converge to 0 uniformly, except for

$$\frac{S_2}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \to 1 \text{ uniformly,}$$

which will give us (24).

Case 1 ($S_1 = \sum \mathbb{E}[X_i^4]$).

Note that $|\mathbb{E}[X_i^4]| \leq \mathbb{E}[|X_i|^4] \leq \mathbb{E}[|X_i|] = 2\mathbb{E}[X_i^2]$. Now,

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X_i^2]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)}$$

converges uniformly to 1 as $n \to \infty$ as seen in the proof of Corollary 4. So,

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X^4]}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \leq 2\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X^2]}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2}$$

converges uniformly to 0 as $n \to \infty$.

Case 2 ($S_2 = \sum \mathbb{E}[X_i^2 X_j^2]$).

Following Remark 1 we write this in the form

$$\begin{aligned}
&\sum_{|i-j|>1} \mathbb{E}[X_i^2 X_j^2] + 2\sum_{\alpha_{nk}}^{\beta_{nk}-1} \mathbb{E}[X_i^2 X_{i+1}^2] \\
&= \sum_{|i-j|>1} \mathbb{E}[X_i^2] E[X_j^2] + 2\sum_{\alpha_{nk}}^{\beta_{nk}-1} \mathbb{E}[X_i^2 X_{i+1}^2] \\
&= \Big(\sum_{\alpha_{nk}}^{\beta_{nk}} E[X_i^2]\Big)^2 - \sum_{\alpha_{nk}}^{\beta_{nk}} E[X_i^2] - 2\sum_{\alpha_{nk}}^{\beta_{nk}-1} \mathbb{E}[X_i^2]\mathbb{E}[X_{i+1}^2] + 2\sum_{\alpha_{nk}}^{\beta_{nk}-1} \mathbb{E}[X_i^2 X_{i+1}^2].
\end{aligned}$$

From Corollary 4 we saw that

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X_i^2]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} \to 1 \text{ uniformly}$$

so

$$\frac{(\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X_i^2])^2}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \to 1 \text{ uniformly}$$

and

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X_i^2]}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \to 0 \text{ uniformly.}$$

For the last two terms, we just note that $\mathbb{E}[X_i^2]\mathbb{E}[X_{i+1}^2] \leq E[X_i^2]$ and $\mathbb{E}[X_i^2 X_{i+1}^2] \leq \mathbb{E}[X_i^2]$ and we get that

$$\frac{S_2}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \to 1 \text{ uniformly.}$$

Case 3 ($S_3 = \sum \mathbb{E}[X_i X_j^3]$).

Utilizing the 1-dependence of G_n we have that

$$\sum_{i\neq j}\mathbb{E}[X_iX_j^3] = \sum_{\alpha_{nk}}^{\beta_{nk}-1}\mathbb{E}[X_iX_{i+1}^3] + \sum_{\alpha_{nk}}^{\beta_{nk}-1}\mathbb{E}[X_i^3X_{i+1}].$$

Each of these terms gets dealt with swiftly by noticing that

$$\mathbb{E}[X_iX_{i+1}^3] \le \mathbb{E}[|X_i|] = 2\mathbb{E}[X_i^2]$$

and similarly for $\mathbb{E}[X_i^3X_{i+1}]$. So

$$\sum_{i\neq j}\mathbb{E}[X_iX_j^3] \le 4\sum_{\alpha_{nk}}^{\beta_{nk}}\mathbb{E}[X_i^2]$$

and we already saw that

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}}\mathbb{E}[X_i^2]}{(\sum_1^n m(G_i))^2} \to 0 \text{ uniformly.}$$

Case 4 ($S_4 = \sum\mathbb{E}[X_iX_jX_k^2]$)**.**

This case is a little more complicated, and needs to be broken down.

$$\sum_{i\neq j\neq k\neq i}\mathbb{E}[X_iX_jX_k^2] = 2\sum_{i<j<k}\mathbb{E}[X_iX_jX_k^2] + 2\sum_{i<j<k}\mathbb{E}[X_iX_j^2X_k] + 2\sum_{i<j<k}\mathbb{E}[X_i^2X_jX_k].$$

The middle of these three terms is the simplest to deal with as by 1-dependence of the G_i we have

$$\sum_{i<j<k}\mathbb{E}[X_iX_j^2X_k] = \sum_{\alpha_{nk}}^{\beta_{nk}-2}\mathbb{E}[X_iX_{i+1}^2X_{i+2}]$$

and we note that $|\mathbb{E}[X_iX_{i+1}^2X_{i+2}]| \le \mathbb{E}[|X_i|] = 2\mathbb{E}[X_i^2]$ as in previous cases. For the outer sums, 1-dependence gives us

$$\sum_{i<j<k}\mathbb{E}[X_i^2X_jX_k] + \sum_{i<j<k}\mathbb{E}[X_iX_jX_k^2] =$$

$$\sum_{\substack{i+2<j\\ \text{or}\\ j+1<i}}\mathbb{E}[X_iX_{i+1}]\mathbb{E}[X_j^2] + \sum_{\alpha_{nk}}^{\beta_{nk}-2}(\mathbb{E}[X_iX_{i+1}X_{i+2}^2] + \mathbb{E}[X_i^2X_{i+1}X_{i+2}]).$$

The second part is again dealt with easily as both $|\mathbb{E}[X_iX_{i+1}X_{i+2}^2]|, |\mathbb{E}[X_i^2X_{i+1}X_{i+2}]| \le \mathbb{E}[|X_i|]$. So we just need to examine the first part in more detail. Notice that

$$\left(\sum_{\alpha_{nk}}^{\beta_{nk}-1}E[X_iX_{i+1}]\right)\left(\sum_{\alpha_{nk}}^{\beta_{nk}}\mathbb{E}[X_j]^2\right) = \sum_{\substack{i+2<j\\ \text{or}\\ j+1<i}}\mathbb{E}[X_iX_{i+1}]\mathbb{E}[X_j^2] + \sum_{i-1\le j\le i+2}\mathbb{E}[X_iX_{i+1}]\mathbb{E}[X_j^2].$$

The last sum again can be ignored since $|\mathbb{E}[X_iX_{i+1}]E[X_j^2]| \leq \mathbb{E}[|X_i|]$. We know already that

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}} \mathbb{E}[X_j]^2}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} \to 1 \text{ uniformly}$$

and

$$\frac{\sum_{\alpha_{nk}}^{\beta_{nk}-1} E[X_iX_{i+1}]}{\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i)} \to 0 \text{ uniformly}$$

by the proof of Corollary 4, and thus

$$\frac{S_4}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \to 0 \text{ uniformly.}$$

Case 5 ($S_5 = \sum \mathbb{E}[X_iX_jX_kX_l]$)**.**

The 1-dependence property of the G_i immediately gives us that

$$\sum_{i,j,k,l \text{ distinct}} \mathbb{E}[X_iX_jX_kX_l] = 24 \sum_{\alpha_{nk}}^{\beta_{nk}-3} \mathbb{E}[X_iX_{i+1}X_{i+2}X_{i+3}]$$

and noting that $|E[X_iX_{i+1}X_{i+2}X_{i+3}]| \leq \mathbb{E}[|X_i|]$ gives us that

$$\frac{S_5}{(\sum_{\alpha_{nk}}^{\beta_{nk}} m(G_i))^2} \to 0 \text{ uniformly.}$$

$\square$

Chapter 3

Natural boundaries resulting from adelic perturbations of certain power series

3.1 Functions of the form $\sum_{n=1}^{\infty} |n|_S z^n$

Given a set of primes $S \subseteq \mathscr{P}$ we will consider the function

$$F_S(z) = \sum_{n=1}^{\infty} |n|_S z^n$$

where

$$|n|_S := \prod_{p \in S} |n|_p.$$

For all S, F_S has radius of convergence 1. This is due to the root test which states that we have radius of absolute convergence c^{-1} where

$$c = \limsup_n |n|_S^{1/n}.$$

For all integers $n \geq 1$ we have

$$1/n \leq |n|_S \leq 1$$

so

$$\limsup_n (1/n)^{1/n} \leq \limsup_n |n|_S^{1/n} \leq \limsup_n 1^{1/n} = 1,$$

and

$$\begin{aligned}\limsup_n (1/n)^{1/n} &= \lim_{n\to\infty} (1/n)^{1/n} \\ &= \lim_{n\to\infty} (1/e^n)^{1/e^n} \\ &= \lim_{n\to\infty} e^{-n/e^n} \\ &= e^0 = 1.\end{aligned}$$

Thus we have absolute convergence or divergence according as

$$|z| < 1 \text{ or } |z| > 1.$$

What is not clear is the behaviour when $|z| = 1$, which we will investigate for certain S. First we have this general result due to Theorem 7.

Proposition 6. *For any $S \subset \mathscr{P}$ with $S \neq \emptyset$ and $S \neq \mathscr{P}$ the function $F_S(z)$ has a natural boundary on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.*

Proof. Consider the derivative $F_S'(Z)$ of $F_S(z)$. We have

$$F_S'(z) = \sum_{n=1}^{\infty} n|n|_S z^{n-1}.$$

The coefficients $n|n|_S$ are integers and the function $F_S'(z)$ has radius of convergence 1 by the root test since

$$\begin{aligned}\limsup_n (n|n|_S)^{1/n} &= \lim_{n\to\infty} n^{1/n} \lim_{n\to\infty} (|n|_S^{1/n}) \\ &= \lim_{n\to\infty} n^{1/n} \cdot 1 \\ &= \lim_{n\to\infty} (e^n)^{1/e^n} \\ &= \lim_{n\to\infty} e^{n/e^n} \\ &= e^0 = 1.\end{aligned}$$

Thus $F_S'(z)$ is either rational or has a natural boundary by the Polya–Carlson dichotomy. A power series is rational if and only if its coefficients are eventually periodic. As $S \neq \mathscr{P}$, $n|n|_S = n$ for infinitely many n, and so $F_S'(z)$ has infinitely many coefficients, and thus its coefficients cannot be eventually periodic. Thus $F_S'(z)$ has a natural boundary. If the original function $F_S(z)$ had a meromorphic extention on any open set intersecting with the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ then so too would the derivative $F_S'(z)$. This cannot be the case as $F_S'(z)$ has a natural boundary on this circle, therefore $F_S(z)$ too has a natural boundary on the unit circle. □

Now that we know the existence of a natural boundary, the Polya–Carlson theorem tells us little about the nature of the poles themselves. To study the poles in more details a more hands-on approach is required. Suppose that $S \neq \emptyset$ and that $p \in S$. Let $e : S \to \mathbb{Z} \cap [1, \infty)$ and write $e_p = e(p)$. From now on we will let

$$F_{S,e}(z) = \sum_{n=1}^{\infty} |n|_{S,e}$$

where $|n|_{S,e} = \prod_{p \in S} |n|_p^{e_p}$. We will reorder the infinite summation for $F_{S,e}$ as follows

$$\begin{aligned}
F_{S,e}(z) &= \sum_{n=1}^{\infty} |n|_{S,e} z^n \\
&= \sum_{i=0}^{\infty} \sum_{p^i \| n} |n|_{S,e} z^n \\
&= \sum_{i=0}^{\infty} \sum_{p \nmid n} |p^i n|_{S,e} z^{p^i n} \\
&= \sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} \sum_{p \nmid n} |n|_{S,e} z^{p^i n}.
\end{aligned}$$

Let

$$G(z) = \sum_{p \nmid n} |n|_{S,e} z^n$$

so that

$$F_{S,e}(z) = \sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} G(z^{p^i}).$$

Since $G(z)$ is summed only over those n for which $p \nmid n$ we have

$$\begin{aligned}
G(z) &= \sum_{p \nmid n} |n|_{S \setminus \{p\},e} z^n \\
&= \sum_{n=1}^{\infty} |n|_{S \setminus \{p\},e} z^n - \sum_{n=1}^{\infty} |pn|_{S \setminus \{p\},e} z^{pn} \\
&= F_{S \setminus \{p\},e}(z) - F_{S \setminus \{p\},e}(z^p).
\end{aligned}$$

So

$$\begin{aligned}
F_{S,e}(z) &= \sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} \left(F_{S \setminus \{p\},e}(z^{p^i}) - F_{S \setminus \{p\},e}(z^{p^{i+1}}) \right) \\
&= F_{S \setminus \{p\},e}(z) - (p^{e_p} - 1) \sum_{i=1}^{\infty} \frac{1}{p^{ie_p}} F_{S \setminus \{p\},e}(z^{p^i}).
\end{aligned}$$

This rearrangement is valid since for example, the sums

$$\sum_{i=0} \frac{1}{p^i} F_{S \setminus \{p\},e}(z^{p^{ie_p}})$$

and

$$\sum_{i=0} \frac{1}{p^i} F_{S\setminus\{p\},e}(z^{p^{(i+1)e_p}})$$

are absolutely convergent inside the unit circle. Indeed, if we take some $z \in \mathbb{C}$ with $|z| = 1$, let

$$C = \sup\{|F_{S\setminus\{p\},e}(w)| \ \mid \ w \in \mathbb{C} \text{ and } |w| \leq |z|\}.$$

The value C is finite as $F_{S\setminus\{p\},e}(z)$ is continuous inside the whole compact set $\{w \in \mathbb{C} \ \mid \ |w| \leq |z|\}$. As all powers $|z^{p^i}| \leq |z|$ we have

$$\sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} |F_{S\setminus\{p\}}(z^{p^i})| \leq \sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} C < \infty.$$

Expressing $F_{S,e}(z)$ in terms of $F_{S\setminus\{p\},e}(z)$ will come in useful when inducting on the size of S.

Definition 5. *We say that $\mu \in \mathbb{C}$ is a* primitive *n-th root of unity if $\mu^n = 1$ and for all $k \in \{1, 2, \ldots, n-1\}$, $\mu^k \neq 1$.*

By basic group theory, this definition is equivalent to saying that $\mu \in \mathbb{C}$ is a primitive n-th root of unity if $\mu^n = 1$ and for all $k \mid n$ with $|k| < |n|$ we have $\mu^k \neq 1$.

Lemma 14. *If S is a finite set of primes, $e : S \to \mathbb{Z} \cap [1, \infty)$ and $n > 1$ is an integer divisible by some prime q with $q \notin S$ then there exists a constant $c_{n,S,e} > 0$ such that for all primitive n-th roots of unity $\mu \in \mathbb{C}$ and for all $\lambda \in [0, 1)$*

$$|F_{S,e}(\lambda\mu)| < c_{n,S,e}.$$

Proof. We will proceed by induction on the size of S. First suppose that S is empty and that $n > 1$ is any integer greater than 1. We have that

$$F_{S,e}(z) = \sum_{i=1}^{\infty} z^i = \frac{z}{1-z}$$

for all $z \in \mathbb{C}$ with $|z| < 1$. The closure of the set

$$Q = \{\lambda\mu \mid \lambda \in [0, 1) \text{ and } \mu \text{ is a primitive } n\text{-th root of unity}\}$$

does not contain 1 and so there exists a constant $c > 0$ such that for all $z \in Q$

$$|1 - z| > c.$$

So for all $z \in Q$

$$|F_{S,e}(z)| = \left|\frac{z}{1-z}\right| < \left|\frac{1}{1-z}\right| < c^{-1}.$$

Now suppose that $|S| \geq 1$ and that n is divisible by some prime q with $q \notin S$. Let

$$R = \{\mu \in \mathbb{C} \mid \mu \text{ is a primitive } r\text{-th root of unity for some } r \text{ with } q \mid r \mid n\}$$

and

$$T = \{\lambda\mu \mid \mu \in R \text{ and } \lambda \in [0,1)\}.$$

Note that R contains all primitive n-th roots of unity as $q \mid n \mid n$. Also note that the set R has the property that if $\mu \in R$ then $\mu^m = 1 \Rightarrow q \mid m$. Let $p \in S$. We have

$$F_{S,e}(z) = \sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} \left(F_{S\setminus\{p\},e}(z^{p^i}) - F_{S\setminus\{p\},e}(z^{p^{i+1}}) \right).$$

Since there is some prime q dividing n which does not belong to S the set T is closed under powers of p as $\lambda^p \in [0,1)$, $(\mu^p)^n = (\mu^n)^p = 1$ and $\mu^p \neq 1$ as $q \nmid p$. Thus for all i, $z^{p^i} \in T$ and $z^{p^i} = \lambda'\mu'$ where μ' is some primitive r-th root of unity with $q \mid r$. So by our inductive hypothesis as $q \notin S \setminus \{p\}$, take $c = \max\{c_{r,S\setminus\{p\},e}, q \mid r \mid n\}$, then for any $z \in T$

$$\begin{aligned}
|F_S(z)| &\leq \sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} \left| F_{S\setminus\{p\}}(z^{p^i}) - F_{S\setminus\{p\}}(z^{p^{i+1}}) \right| \\
&< \sum_{i=0}^{\infty} \frac{1}{p^{ie_p}} 2c \\
&= 2c \frac{p^{e_p}}{p^{e_p} - 1}.
\end{aligned}$$

As the set T contains all $\lambda\mu$ with μ a primitive n-th root of unity, and $\lambda \in [0,1)$, we are done. □

Proposition 7. *If $S \neq \emptyset$ is finite, then $F_{S,e}(z)$ has a pole at every n-th root of unity where n is divisible only by primes contained in S. Furthermore, if μ is a primitive n-th root of unity divisible by exactly k different primes in S then*

$$\Re(F_{S,e}(\lambda\mu)) \to (-1)^k \infty$$

as $\lambda \to 1^-$ and $|\Im(F_{S,e}(\lambda\mu))|$ is bounded for all $\lambda \in [0,1)$.

Proof. Clearly we have that

$$\lim_{\lambda \to 1^-} F_{S,e}(\lambda) = +\infty$$

as $|n|_S = 1$ infinitely often and all terms are positive, so $F_{S,e}(z)$ has a pole at 1. We will induct on the number of distinct prime factors of n. Suppose that $p_1, \ldots, p_k \in S$ are distinct, $n = p_1^{f_1} \cdots p_k^{f_k}$, $f_i \geq 1$, and $k \geq 1$. Let $p = p_k$ and for now we will assume that $f_k = 1$. Write

$$F_{S,e}(z) = F_{S\setminus\{p\},e}(z) - (p^{e_p} - 1) \sum_{i=1}^{\infty} \frac{1}{p^{ie_p}} F_{S\setminus\{p\},e}(z^{p^i}).$$

If $z = \lambda\mu$ where $\lambda \in [0, 1)$ and μ is a primitive n-th root of unity, by Lemma 14 as $p \mid n$ and $p \notin S \setminus \{p\}$ the first term $F_{S\setminus\{p\},e}(z)$ is bounded. For each $i \geq 1$ we can write $z^{p^i} = \lambda'\mu'$ where $\lambda' \in [0, 1)$ and μ' is a primitive $n/p_k = p_1^{e_1} \cdots p_{k-1}^{e_{k-1}}$-th root of unity, so by our inductive hypothesis, the real part of the term $\Re(F_{S\setminus\{p\},e}(z^p)) \to (-1)^{k-1}\infty$ as $\lambda \to 1^-$. Considering $F_{S\setminus\{p\},e}(\lambda\mu)$ as a function $\lambda : [0, 1) \to \mathbb{R}$, it is a continuous function going to $-(-1)^k\infty$ as $\lambda \to 1^-$ and so it is bounded in the opposite direction. That is to say that there exists a $c > 0$ such that for all $\lambda \in [0, \infty)$ and primitive n-th roots of unit μ, $(-1)^k\Re(F_{S\setminus\{p\},e}(\lambda\mu)) < c$. So

$$\begin{aligned}
(-1)^k\Re(F_{S,e}(z)) &= (-1)^k\Re(F_{S\setminus\{p\},e}(z)) - (-1)^k(p^{e_p} - 1)\sum_{i=1}^{\infty}\frac{1}{p^{ie_p}}\Re(F_{S\setminus\{p\},e}(z^{p^i})) \\
&= (-1)^k\Re(F_{S\setminus\{p\},e}(z)) \\
&\quad - (-1)^k(p^{e_p} - 1)\left(\frac{1}{p^{e_p}}\Re(F_{S\setminus\{p\},e}(z^p)) + \sum_{i=2}^{\infty}\frac{1}{p^{ie_p}}\Re(F_{S\setminus\{p\},e}(z^{p^i}))\right) \\
&> (-1)^k\Re(F_{S\setminus\{p\},e}(z)) - (p^{e_p} - 1)\left((-1)^k\frac{1}{p^{e_p}}\Re(F_{S\setminus\{p\},e}(z^p)) + \sum_{i=2}^{\infty}\frac{1}{p^{ie_p}}c\right) \\
&= (-1)^k\Re(F_{S\setminus\{p\},e}(z)) - c/p^{e_p} + \frac{p^{e_p} - 1}{p^{e_p}}(-1)^{k-1}\Re(F_{S\setminus\{p\},e}(z^p)).
\end{aligned}$$

So since $(-1)^k\Re(F_{S\setminus\{p\},e}(z)) - p^{e_p}c$ is bounded and $(-1)^{k-1}\Re(F_{S\setminus\{p\},e}(z^p)) \to +\infty$ as $\lambda \to 1^-$ we have that

$$\Re(F_{S,e}(z)) \to (-1)^k\infty$$

as $\lambda \to 1^-$. For the imaginary part we know that $|F_{S\setminus\{p\},e}(\lambda\mu)|$ and hence $\Im(F_{S\setminus\{p\},e}(\lambda\mu))$ is bounded by Lemma 14 and by our inductive hypothesis we know that there exists some $c > 0$ such that $|\Im(F_{S\setminus\{p\},e}(z^{p^i})| < c$ for all i and λ, so

$$\begin{aligned}
|\Im(F_{S,e}(z))| &\leq |\Im(F_{S\setminus\{p\},e}(z))| + (p^{e_p} - 1)\sum_{i=1}^{\infty}\frac{1}{p^{ie_p}}|\Im(F_{S\setminus\{p\},e}(z^{p^i}))| \\
&< |\Im(F_{S\setminus\{p\},e}(z))| + (p^{e_p} - 1)\sum_{i=1}^{\infty}\frac{1}{p^{ie_p}}c \\
&= |\Im(F_{S\setminus\{p\},e}(z))| + c
\end{aligned}$$

is bounded. Now we induct on f_k and assume $f_k > 1$. Writing

$$F_{S,e}(z) = F_{S\setminus\{p\},e}(z) - F_{S\setminus\{p\},e}(z^p) + \frac{1}{p^{e_p}}F_{S,e}(z^p),$$

Lemma 14 gives us that the first two terms $F_{S\setminus\{p\},e}(z) - F_{S\setminus\{p\},e}(z^p)$ are bounded and $z^p = \lambda'\mu'$ where μ is a primitive $p_1^{f_1} \cdots p_{k-1}^{f_{k-1}} \cdot p_k^{f_k - 1}$-th root of unity, so by the inductive hypothesis the term

$$\frac{1}{p^{e_p}}F_{S,e}(z^p) \to (-1)^k\infty$$

as $\lambda \to 1^-$ and thus so does $F_{S,e}(z)$. □

Proposition 8. *For every finite $S \subset \mathscr{P}$*

$$\lim_{\lambda\to 1^-}(1-\lambda)F_S(\lambda) = \prod_{p\in S}\frac{1}{\sum_{i=0}^{e_p} p^{-i}}.$$

Proof. We induct on the size of S. If $S = \emptyset$ then in the unit disk

$$\begin{aligned} F_{S,e}(z) &= \sum_{n=1}^{\infty}|n|_{S,e}z^n \\ &= \frac{z}{1-z} \end{aligned}$$

so

$$\begin{aligned} \lim_{\lambda\to 1^-}(1-\lambda)F_{S,e}(\lambda) &= \lim_{\lambda\to 1^-}(1-\lambda)\frac{\lambda}{1-\lambda} \\ &= \lim_{\lambda\to 1^-}\lambda = 1. \end{aligned}$$

If $S \geq 1$ then choose any $p \in S$ and as discussed earlier

$$F_S(z) = \sum_{i=0}^{\infty}\frac{1}{p^{ie_p}}(F_{S\setminus\{p\},e}(z^{p^i}) - F_{S\setminus\{p\},e}(z^{p^i+1}))$$

so

$$\lim_{\lambda\to 1^-}(1-\lambda)F_{S,e}(\lambda) = \sum_{i=0}^{\infty}\frac{1}{p^{ie_p}}\left(\lim_{\lambda\to 1^-}(1-\lambda)F_{S\setminus\{p\},e}(\lambda^{p^i}) - \lim_{\lambda\to 1^-}(1-\lambda)F_{S\setminus\{p\},e}(\lambda^{p^{i+1}})\right).$$

Now by our inductive hypothesis we know that

$$\lim_{\lambda\to 1^-}(1-\lambda)F_{S\setminus\{p\},e}(\lambda) = \prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}}$$

and so

$$\lim_{\lambda\to 1^-}(1-\lambda^{p^i})F_{S\setminus\{p\},e}(\lambda^{p^i}) = \prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}}.$$

Thus

$$\begin{aligned} \lim_{\lambda\to 1^-}(1-\lambda^{p^i})F_{S\setminus\{p\},e}(\lambda^{p^i}) &= \lim_{\lambda\to 1^-}(1-\lambda)(1+\lambda+\ldots+\lambda^{p^i-1})F_{S\setminus\{p\},e}(\lambda^{p^i}) \\ &= p^i\lim_{\lambda\to 1^-}(1-\lambda)F_{S\setminus\{p\}}(\lambda^{p^i}) = \prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}} \end{aligned}$$

and therefore

$$\lim_{\lambda\to 1^-}(1-\lambda)F_{S\setminus\{p\},e}(\lambda^{p^i}) = \frac{1}{p^i}\prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}}.$$

So we have

$$\begin{aligned}
\lim_{\lambda\to 1^-}(1-\lambda)F_{S,e}(\lambda) &= \sum_{i=0}^{\infty}\frac{1}{p^{ie_p}}(1/p^i - 1/p^{i+1})\prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}} \\
&= (1-1/p)\sum_{i=0}^{\infty}\frac{1}{p^{i(e_p+1)}}\prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}} \\
&= \frac{1-1/p}{1-1/p^{e_p+1}}\prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}} \\
&= \frac{1}{1+p^{-1}+\ldots+p^{-e_p}}\prod_{q\in S\setminus\{p\}}\frac{1}{\sum_{i=0}^{e_q} q^{-i}} \\
&= \prod_{q\in S}\frac{1}{\sum_{i=0}^{e_q} q^{-i}}.
\end{aligned}$$

□

3.2 Arithmetic sequences

To allow us to prove our main theorem in the next section, we need to examine functions of the form

$$F_{S,e,r}(z) = \sum_{n=1}^{\infty}|n-r|_{S,e}z^n$$

for some $r \in \mathbb{Q}$. Now if for some $p \in S$ we have $|r|_p > 1$ then for all $n \in \mathbb{N}$ we will have $|n-r|_p = |r|_p$. So

$$\begin{aligned}
F_{S,e,r}(z) &= |r|_p^{e_p}\sum_{n=1}^{\infty}|n-r|_{S\setminus\{p\},e}z^n \\
&= |r|_p^{e_p}F_{S\setminus\{p\},e,r}(z)
\end{aligned}$$

and therefore we know as much about the singularities on the unit circle for $F_{S,e,r}(z)$ as we do for $F_{S\setminus\{p\},e,r}(z)$. So without loss of generalisation, we will assume that for all $p \in S$, $|r|_p \leq 1$. We will proceed in a similar vein as to the previous section by reordering the sum of $F_{S,e,r}(z)$ as follows

$$\begin{aligned} F_{S,e,r}(z) &= \sum_{n=0}^{\infty} |n-r|_{S,e} z^n \\ &= \sum_{k=0}^{\infty} \sum_{\substack{n\geq 0 \\ \text{such that} \\ |n-r|_p = p^{-k}}} |n-r|_{S,e} z^n \\ &= \sum_{k=0}^{\infty} \frac{1}{p^{ke_p}} \sum_{\substack{n\geq 0 \\ \text{such that} \\ |n-r|_p = p^{-k}}} |n-r|_{S\setminus\{p\},e} z^n. \end{aligned}$$

Now as $|r|_p \leq 1$ we can write r in the following p-adic representation

$$r = r_0 + r_1 p + r_2 p^2 + \ldots$$

where for all i, $r_i \in \{0, 1, \ldots, p-1\}$. The $n \geq 0$ such that $|n-r|_p = p^{-k}$ are precisely those n such that

$$n \equiv r_0 + r_1 p + \ldots + r_{k-1} p^{k-1} \mod p^k$$

and

$$n \not\equiv r_0 + r_1 p + \ldots + r_{k-1} p^{k-1} + r_k p^k \mod p^{k+1}.$$

That is $n = mp^k + r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}$ for some $m \geq 0$ and $m \not\equiv r_k \mod p$. So in particular

$$\begin{aligned} &F_{S,e,r}(z) \\ &= \sum_{k=0}^{\infty} \frac{1}{p^{ke_p}} \sum_{m \not\equiv r_k(p)} |mp^k + r_0 + r_1 p + \ldots + r_{k-1} p^{k-1} - r|_{S\setminus\{p\},e} z^{mp^k + r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}} \\ &= \sum_{k=0}^{\infty} \frac{1}{p^{ke_p}} z^{r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}} \sum_{m \not\equiv r_k(p)} \left| m - \frac{r - (r_0 + r_1 p + \ldots + r_{k-1} p^{k-1})}{p^k} \right|_{S\setminus\{p\},e} z^{mp^k}. \end{aligned}$$

If for each k we write

$$t_k = \frac{r - (r_0 + r_1 p + \ldots + r_{k-1} p^{k-1})}{p^k}$$

then if we let

$$Q = \sum_{k=0}^{\infty} \frac{1}{p^{ke_p}} z^{r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}}$$

we have

$$
\begin{aligned}
F_{S,e,r}(z) &= Q\left(\sum_{m=0}^{\infty} |m-t_k|_{S\setminus\{p\},e} z^{mp^k} - \sum_{m\equiv r_k(p)} |m-t_k|_{S\setminus\{p\},e} z^{mp^k}\right) \\
&= Q\left(\sum_{m=0}^{\infty} |m-t_k|_{S\setminus\{p\},e} z^{mp^k} - \sum_{m=0}^{\infty} |pm+r_k-t_k|_{S\setminus\{p\},e} z^{(pm+r_k)p^k}\right) \\
&= Q\left(\sum_{m=0}^{\infty} |m-t_k|_{S\setminus\{p\},e} z^{mp^k} - z^{r_k p^k}\sum_{m=0}^{\infty} |pm-pt_{k+1}|_{S\setminus\{p\},e} z^{p^{k+1}m}\right)
\end{aligned}
$$

as $t_{k+1} = (t_k - r_k)/p$. As $|p|_{S\setminus\{p\},e} = 1$ we therefore have

$$
\begin{aligned}
F_{S,e,r}(z) &= Q\left(F_{S\setminus\{p\},e,t_k}(z^{p^k}) - z^{r_k p^k} F_{S\setminus\{p\},e,t_{k+1}}(z^{p^{k+1}})\right) && (25) \\
&= \sum_{k=0}^{\infty} \frac{1}{p^{ke_p}} z^{r_0+r_1p+\ldots+r_{k-1}p^{k-1}} F_{S\setminus\{p\},e,t_k}(z^{p^k}) - \\
&\qquad \sum_{k=0}^{\infty} \frac{1}{p^{ke_p}} z^{r_0+r_1p+\ldots+r_{k-1}p^{k-1}+r_kp^k} F_{S\setminus\{p\},e,t_{k+1}}(z^{p^{k+1}}) && (26) \\
&= F_{S\setminus\{p\},e,r}(z) - (p-1)\sum_{k=1}^{\infty} \frac{1}{p^{ke_p}} z^{r_0+r_1p+\ldots+r_{k-1}p^{k-1}} F_{S\setminus\{p\},e,t_k}(z^{p^k}). && (27)
\end{aligned}
$$

Lemma 15. *If S is a finite set of primes with corresponding exponents $e = \{e_p \mid p \in S\}$ and $n > 1$ is an integer divisible by some prime $q \notin S$ then there exists a constant $c_{n,e,S} > 0$ such that for all primitive n-th roots of unity $\mu \in \mathbb{C}$ and for all $\lambda \in [0,1)$*

$$|F_{S,e,r}(\lambda\mu)| < c_{n,e,S}.$$

Note that the constant $c_{n,e,S}$ does not depend on the value r providing that for all $p \in S$, $|r|_p \leq 1$.

Proof. We will proceed by induction on the size of S. First we suppose that $S = \emptyset$. We have

$$F_{S,e,r}(z) = \sum_{n=0}^{\infty} |n-r|_{\emptyset} z^n.$$

Now unless $r \in \mathbb{N}$ we have

$$F_{S,e,r}(z) = \frac{1}{1-z}$$

otherwise we have

$$F_{S,e,r}(z) = \frac{1}{1-z} - z^r.$$

Either way, as we noted in the proof of Lemma 14 it is clear that the constant $c_{n,e,S}$ exists.

Now we suppose that $|S| \geq 1$. Let $p \in S$ and write as we did in the earlier discussion

$$F_{S,e,r}(z) = F_{S\setminus\{p\},e,r}(z) - (p^{e_p}-1)\sum_{k=1}^{\infty} \frac{1}{p^{ke_p}} z^{r_0+r_1p+\ldots+r_{k-1}p^{k-1}} F_{S\setminus\{p\},e,t_k}(z^{p^k}).$$

So

$$|F_{S,e,r}(z)| \leq |F_{S\setminus\{p\},e,r}(z)| + (p^{e_p}-1)\sum_{k=1}^{\infty}\frac{1}{p^{ke_p}}|z^{r_0+r_1p+\ldots+r_{k-1}p^{k-1}}||F_{S\setminus\{p\},e_p,t_k}(z^{p^k})|$$

$$\leq (p^{e_p}-1)\sum_{k=0}^{\infty}\frac{1}{p^{ke_p}}|F_{S\setminus\{p\},e_p,t_k}(z^{p^k})|.$$

As in the proof of Lemma 14 we note that if $z = \lambda\mu$ for some $\lambda \in [0,1)$ and μ a primitive n-th root of unity with $q \mid n$ that $z^{p^k} = \lambda'\mu'$ where $\lambda' \in [0,1)$ and $\mu' \in \mathbb{C}$ is a primitive n'-th root of unity where $q \mid n'$ and n' is one of finitely many possible values. Since we plan to use our inductive hypothesis we need to check that for all $p' \in S \setminus \{p\}$ that $|t_k|_{p'} \leq 1$. This is clearly the case as

$$\begin{aligned}
|t_k|_{p'} &= |r-(r_0+\ldots+r_{k-1}p^{k-1})|_{p'}/|p^k|_{p'} \\
&= |r-(r_0+\ldots+r_{k-1}p^{k-1})|_{p'} \\
&\leq \max(|r|_{p'}, |r_0+\ldots+r_{k-1}p^{k-1}|_{p'} \\
&\leq 1.
\end{aligned}$$

Therefore by our inductive hypothesis we can find a constant c such that for all k

$$|F_{S\setminus\{p\},e,t_k}(z^{p^k})| < c$$

and hence

$$|F_{S,e,r}(z)| \leq (p^{e_p}-1)c\frac{p^{e^p}}{p^{e_p}-1}.$$

Taking this as our $c_{n,e,S}$ we are done. $\square$

Next we will obtain some functional equations relating the $F_{S,e,t_m}(z)$. For any $m \geq 1$, equation (25) gives us

$$\begin{aligned}
F_{S,e,t_m}(z) = \sum_{k=0}^{\infty}\frac{1}{p^{ke_p}}&z^{r_m+r_{m+1}p+\ldots+r_{m+k-1}p^{k-1}} \\
&\cdot\left(F_{S\setminus\{p\},e,t_{m+k}}(z^{p^k}) - z^{r_{m+k}p^k}F_{S\setminus\{p\},e,t_{m+k+1}}z^{p^{k+1}})\right).
\end{aligned}$$

So

$$\begin{aligned}
F_{S,e,t_m}(z^p) = \sum_{k=0}^{\infty}\frac{1}{p^{ke_p}}&z^{p(r_m+\ldots+r_{m+k-1}p^{k-1})} \\
&\cdot\left(F_{S\setminus\{p\},e,t_{m+k}}(z^{p^{k+1}}) - z^{r_{m+k}p^k}F_{S\setminus\{p\},e,t_{m+k+1}}z^{p^{k+2}})\right),
\end{aligned}$$

and

$$z^{r_{m-1}} F_{S,e,t_m}(z^p) = \sum_{k=0}^{\infty} \frac{1}{p^{ke_p}} z^{r_{m-1}+\ldots+r_{m-1+k}p^k}$$
$$\cdot \left(F_{S\setminus\{p\},e,t_{m+k}}(z^{p^{k+1}}) - z^{r_{m+k}p^k} F_{S\setminus\{p\},e,t_{m+k+1}} z^{p^{k+2}}) \right), \tag{28}$$
$$= \sum_{k=1}^{\infty} \frac{1}{p^{(k-1)e_p}} z^{r_{m-1}+\ldots+r_{m-1+k-1}p^{k-1}}$$
$$\cdot \left(F_{S\setminus\{p\},e,t_{m-1+k}}(z^{p^k}) - z^{r_{m-1+k}p^k} F_{S\setminus\{p\},e,t_{m-1+k+1}} z^{p^{k+1}}) \right) \tag{29}$$
$$= p^{e_p} F_{S,e,t_{m-1}}(z) - \left(F_{S\setminus\{p\},e,t_{m-1}}(z) - z^{r_{m-1}} F_{S\setminus\{p\},e,t_{m-1}}(z^p) \right). \tag{30}$$

In the following theorem, given some $r \in \mathbb{Q}$ we will use the notation

$$r \mod p^e$$

to indicate the positive integer

$$r_0 + r_1 p + \ldots + r_{e-1} p^{e-1}$$

as we will be working with more than one prime p. So in particular, $r \mod p^e$ is the smallest non-negative integer such that

$$|r - (r \mod p^e)|_p \leq p^{-e}.$$

If $n = p_1^{e_1} \cdots p_j^{e_j}$ for distinct primes p_i, then we will denote

$$r \mod n$$

to be the smallest non-negative integer simultaneously satisfying

$$|r - (r \mod n)|_p \leq p_i^{e_i}$$

for all $i = 1, \ldots, j$. The existence of $r \mod n$ is guaranteed by the Chinese Remainder Theorem (see Theorem 3.6 of [19]).

Theorem 17. *Let S be a finite set of primes and let $r \in \mathbb{Q}$ be such that $|r|_p \leq 1$ for all $p \in S$. Suppose that $n \geq 1$ is an integer divisible only by primes in S and that $\mu \in \mathbb{C}$ is a primitive n-th root of unity. Writing $n = p_1^{f_1} \cdots p_j^{f_j}$ where $p_1, \ldots, p_j \in S$ are distinct and $f_i \geq 1$ for all $i = 1, \ldots, j$, the value*

$$|F_{S,e,r}(\lambda\mu)| \to \infty$$

as $\lambda \to 1^-$. More precisely,

$$\Re\left((-1)^j \mu^{-(r \mod n)} F_{S,e,r}(\lambda\mu) \right) \to +\infty$$

and

$$\Im\left((-1)^j \mu^{-(r \mod n)} F_{S,e,r}(\lambda\mu)\right)$$

remains bounded as $\lambda \to 1^-$. The bounded constant does not depend on r.

Proof. We will induct on the number of distinct prime factors in n. For the base case we will take $n = 1$, a product of zero prime factors. In this case, the theorem tells us to expect that

$$\Re(F_{S,e,r}(\lambda)) \to +\infty$$

as $\lambda \to 1^-$, which is clearly true as $F_{S,e,r}(\lambda) = \sum_{n=0}^{\infty} |n-r|_{S,e} z^n$ and $|n-r|_{S,e} = 1$ infinitely often and for each n, $\lambda^n \to 1$ as $\lambda \to 1$. The imaginary part is clearly bounded as $F_{S,e,r}(\lambda)$ is real for all $\lambda \in [0,1)$. Let $p_1, \ldots, p_j \in S$ be distinct and let $n = \prod_{i=1}^{j} p_i^{f_i}$ where for all i, $f_i \geq 1$. Let $p = p_1$. We will use the variables $r_0, r_1, \ldots$ to indicate the p-adic coefficients of r and $t_0, t_1, \ldots$ to indicate the values

$$t_k = \frac{r - (r_0 + r_1 p + \ldots + r_{k-1} p^{k-1})}{p^k}.$$

If $f_1 = 1$ then equation (27) gives us

$$F_{S,e,r}(z) = F_{S\setminus\{p\},e,r}(z) - (p^{e_p} - 1) \sum_{k=1}^{\infty} \frac{1}{p^{ke_p}} z^{r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}} F_{S\setminus\{p\},e,t_k}(z^{p^k}). \tag{31}$$

Now, as for all $k \geq 1$, μ^{p^k} is a primitive $p_2^{f_2} \cdots p_j^{f_j}$-th root of unity, by our inductive hypothesis,

$$\Re\left((-1)^{j-1} (\mu^{p^k})^{-(t_k \mod n/p)} F_{S\setminus\{p\},e,t_k}(z^{p^k})\right) \to +\infty$$

as $\lambda \to 1^-$. Now since μ^{p^k} is a n/p-th root of unity, we need only express its exponent modulo n/p as adding to the exponent any factor of n/p leaves it unchanged, as $(\mu^{p^k})^{n/p} = 1$. On the other hand, μ is an n-th root of unity and so we must express its exponent modulo n. If

$$(t_k \mod n/p) \equiv t_k \mod n/p$$

then

$$\begin{aligned} p^k (t_k \mod n/p) &\equiv p^k t_k \mod n \\ &\equiv r - (r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}) \mod n. \end{aligned}$$

So

$$(\mu^{p^k})^{-(t_k \mod n/p)} = \mu^{r_0 + \ldots + r_{k-1} p^{k-1} - r \mod n},$$

so we have

$$\Re\left((-1)^{j-1} (\mu^{r_0 + \ldots + r_{k-1} p^{k-1} - r} \mod n) F_{S\setminus\{p\},e,t_k}(z^{p^k})\right) \to +\infty.$$

Now $z \to \mu$ as $\lambda \to 1^-$ so we have for every $k \geq 1$

$$\Re\left((-1)^{j-1}(\mu^{-(r \mod n)})z^{r_0+\ldots+r_{k-1}p^{k-1}}F_{S\backslash\{p\},e,t_k}(z^{p^k})\right) \to +\infty.$$

Hence it is clear from equation (31) and Lemma 15 that

$$\Re\left((-1)^j\mu^{-(r \mod n)}F_{S,e,r}(z)\right) \to +\infty$$

as $\lambda \to 1^-$. Now we have to show that

$$\Im\left((-1)^j\mu^{-(r \mod n)}F_{S,e,r}(z)\right)$$

remains bounded as $\lambda \to 1^-$. Our inductive hypothesis tells us that

$$\begin{aligned}&\Im\left((-1)^{j-1}(\mu^{p^k})^{-(t_k \mod n/p)}F_{S\backslash\{p\},e,t_k}(z^{p^k})\right)\\&=\Im\left((-1)^{j-1}(\mu^{(r_0+\ldots+r_{k-1}p^{k-1}-r) \mod n}F_{S\backslash\{p\},e,t_k}(z^{p^k})\right)\end{aligned}$$

is bounded as $\lambda \to 1^-$ by a constant not dependent on t_k. And hence so is

$$=\Im\left((-1)^{j-1}(\mu^{-(r \mod n)}z^{r_0+\ldots+r_{k-1}p^{k-1}}F_{S\backslash\{p\},e,t_k}(z^{p^k})\right).$$

So as these terms appear as factors within the geometric progression $\sum_{k=1}^{\infty}p^{-ke_p}$ and are uniformly bounded in the imaginary direction after a factor of $(-1)^{j-1}\mu^{-(r \mod n)}$, and Lemma 15 deals with the leading term $F_{S\backslash\{p\},e,r}(z)$, we have shown that

$$\Im\left((-1)^j\mu^{-(r \mod n)}F_{S,r}(z)\right)$$

is bounded by a constant not dependent on r. We have now proved the inductive step for the case $f_1 = 1$, we will now use this as a base case for a second inductive proof for $f_1 > 1$. Using the functional equation (30) we have

$$z^{r_0}F_{S,e,t_1}(z^p) = p^{e_p}F_{S,e,r}(z) - \left(F_{S\backslash\{p\},e,r}(z) - z^{r_0}F_{S\backslash\{p\},e,r}(z^p)\right). \tag{32}$$

Now as μ^p is a n/p-th root of unity, our new inductive hypothesis tells us that

$$\Re\left((-1)^j(\mu^p)^{-(t_1 \mod n/p)}F_{S,e,t_1}(z^p)\right) \to +\infty$$

as $\lambda \to 1^-$. Now

$$(\mu^p)^{-(t_1 \mod n/p)} = \mu^{-(pt_1 \mod n)}$$

and

$$pt_1 = r - r_0$$

so

$$\Re\left((-1)^j\mu^{(r_0-r) \mod n}F_{S,e,t_1}(z^p)\right) \to +\infty$$

and so

$$\Re\left((-1)^j \mu^{-(r \mod n)} z^{r_0} F_{S,e,t_1}(z^p)\right) \to +\infty.$$

Comparing this to equation (32), and noting that by Lemma 15 the term

$$F_{S\setminus\{p\},e,r}(z) - z^{r_0} F_{S\setminus\{p\},e,r}(z^p)$$

is bounded as $\lambda \to 1^-$, it follows that

$$\Re\left((-1)^j \mu^{-(r \mod n)} F_{S,e,r}(\lambda\mu)\right) \to +\infty$$

as $\lambda \to 1^-$. Similarly our inductive hypothesis also tells us that

$$\Im\left((-1)^j \mu^{-(r \mod n)} z^{r_0} F_{S,e,t_1}(z^p)\right)$$

is bounded as $\lambda \to 1^-$ in a way that does not depend on r, and again by equation (32) and Lemma 15 so is

$$\Im\left((-1)^j \mu^{-(r \mod n)} F_{S,e,r}(\lambda\mu)\right).$$

This concludes the inductive proof. □

3.3 Fibonacci numbers

We will now go on to look at the function

$$f(z) = \sum_{n=1}^{\infty} |F_n|_S z^n$$

where F_n denotes the Fibonacci sequence with $F_1 = F_2 = 1$ and S is a set of primes. Given a prime p let $v_p : \mathbb{Z} \to \mathbb{N}$ be defined by

$$v_p(n) = \sup\{e \geq 0 \mid p^e \mid n\}$$

allowing $v_p(0) = \infty$. From Lengyel's paper [17], we know that

$$v_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \mod 3 \\ 1, & \text{if } n \equiv 3 \mod 6 \\ v_2(n) + 2, & \text{if } n \equiv 0 \mod 6 \end{cases}$$

$$v_5(F_n) = v_5(n)$$

and for $p \neq 2, 5$

$$v_p(F_n) = \begin{cases} v_p(n) + e(p), & \text{if } n \equiv 0 \mod n(p) \\ 0, & \text{if } n \not\equiv 0 \mod n(p) \end{cases}$$

where $n(p)$ is the first positive index such that $p \mid F_{n(p)}$ and $e(p)$ is defined as being $v_p(F_{n(p)})$. It is a well known fact that for every m that for some n we have $m \mid F_n$, therefore ensuring that $n(p)$ and $e(p)$ are in fact defined.

For simplicity, let's first consider the case $S = \{2\}$. We have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} |F_n|_2 z^n \\ &= \sum_{n=1}^{\infty} 2^{-v_2(F_n)} z^n \\ &= \sum_{n \equiv 1,2(3)} z^n + \frac{1}{2} \sum_{n \equiv 3(6)} z^n + \frac{1}{4} \sum_{n \equiv 0(6)} |n|_2 z^n \\ &= \sum_{n \equiv 1(3)} z^n + \sum_{n \equiv 2(3)} z^n + \frac{1}{2} \sum_{n \equiv 3(6)} z^n + \frac{1}{4} \sum_{n \equiv 0(6)} |n|_2 z^n. \end{aligned}$$

Now as

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$$

we have

$$\begin{aligned} f(z) &= \frac{z}{1-z^3} + \frac{z^2}{1-z^3} + \frac{1}{2} \frac{z^3}{1-z^6} + \frac{1}{4} \sum_{n \equiv 0(6)} |n|_2 z^n \\ &= \frac{z}{1-z^3} + \frac{z^2}{1-z^3} + \frac{1}{2} \frac{z^3}{1-z^6} + \frac{1}{8} G(z^6). \end{aligned}$$

where

$$G(z) = \sum_{n=1}^{\infty} |n|_2 z^n.$$

We know $G(z)$ to have a natural boundary at the unit circle, and the first three terms are rational and thus there are at most finitely many singularities on the radius of convergence which can be effected by these terms. Therefore $f(z)$ has a natural boundary on the unit circle.

Next let us consider the case $S = \{2, 5\}$. This time we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} |F_n|_{\{2,5\}} z^n \\ &= \sum_{n=1}^{\infty} |F_n|_2 |F_n|_5 z^n \\ &= \sum_{n=1}^{\infty} |F_n|_2 |n|_5 z^n \\ &= \sum_{n \equiv 1,2(3)} |n|_5 z^n + \frac{1}{2} \sum_{n \equiv 3(6)} |n|_5 z^n + \frac{1}{4} \sum_{n \equiv 0(6)} |n|_2 |n|_5 z^n. \end{aligned}$$

If we write

$$G_5(z) = \sum_{n=1}^{\infty} |n|_5 z^n \qquad \text{and} \qquad G_{2,5}(z) = \sum_{n=1}^{\infty} |n|_{\{2,5\}} z^n$$

then we have

$$f(z) = z^{-2}G_5(z^3) + z^{-1}G_5(z^3) + \frac{1}{2}z^{-3}G_5(z^6) + \frac{1}{8}G_{2,5}(z^6).$$

By Proposition 7 if $\mu \in \mathbb{C}$ is a primitive 2^a5^b-th root of unity then

$$\lim_{\lambda \to 1^-} |G_{\{2,5\}}(\lambda\mu)| \to \infty.$$

By Lemma 14 if $a \geq 1$ then $|G_5(\lambda\mu)|$ is bounded for all $\lambda \in [0, 1)$. Therefore, any primitive 2^a5^b-th root of unity with $a \geq 2$, the first three terms in the above sum for $f(\lambda\mu)$ are bounded for all $\lambda \in [0, 1)$ and

$$\lim_{\lambda \to 1^-} |G_{\{2,5\}}(\lambda^6\mu^6)| = \infty.$$

Thus $|f(\lambda\mu)| \to \infty$ as $\lambda \to 1^-$ and $f(z)$ has a natural boundary.

Now for considering any finite S we are going to generalise to any sequence (a_n) of positive integers which satisfy the following property.

Property 1. *For every prime p there exist constants $n_p \in \mathbb{N}$, $c_{p,0}, c_{p,1}, c_{p,2}, \ldots, c_{p,n_p-1} \in \mathbb{Q}$ and $e_{p,0}, e_{p,1}, e_{p,2}, \ldots, e_{p,n_p-1} \in 0, 1, 2, \ldots$ such that for all $k \in \{0, 1, \ldots, n_p - 1\}$*

$$|a_n|_p = c_{p,k}|n|_p^{e_{p,k}} \text{ if } n \equiv k \mod n_p.$$

When we characterise a sequence of positive integers satisfying Property 1 like this we will adhere to the following conventions. If $c_{p,k} = 0$ for some p and k we will automatically take $e_{p,k} = 0$ as the power of $|n|_p$ clearly plays no role. Another case we wish to avoid is if for some p and $k \in \{0, 1, \ldots, n_p - 1\}$, for all $n \equiv k \mod n_p$ the value $|n|_p$ is constant. This happens exactly when $v_p(n_p) > v_p(k)$ and in this case $|n|_p = |k|_p$. If this is the case and $e_{p,k} \neq 0$, we will set $e_{p,k} = 0$ and substitute $c_{p,k}|k|_p$ for $c_{p,k}$.

Theorem 18. *For any sequence (a_n) satisfying Property 1 with constants n_p, $c_{p,0}, \ldots, c_{p,n_p-1}$, $e_{p,0}, \ldots, e_{p,n_p-1}$ for each prime p all adhering to the aforementioned conventions, then if S is a finite set of primes such that for some $p \in S$ and some $k \in \{0, 1, \ldots, n_p - 1\}$ we have $e_{p,k} \geq 1$ then the function*

$$f(z) = \sum_{n=1}^{\infty} |a_n|_S z^n$$

has a natural boundary at the unit circle. If S contains no such prime p then $f(z)$ is rational.

Proof. Let $N = \operatorname{lcm}\{n_p \mid p \in S\}$. For each $j \in \{0, 1, \ldots, N-1\}$ we consider the value of $|a_n|_S$ for when $n \equiv j \mod N$.

$$|a_n|_S = \prod_{p \in S} |a_n|_p.$$

Now for each p, we know that $n \equiv j \mod N$ and so we also have $n \equiv j \mod n_p$ as $n_p \mid N$. Let $k_{p,j}$ be that unique element of $\{0, 1, \ldots, n_p - 1\}$ such that $k_{p,j} \equiv j \mod n_p$. So

$$\begin{aligned}|a_n|_S &= \prod_{p \in S} |a_n|_p \\ &= \prod_{p \in S} c_{p,k_{p,j}} |n|_p^{e_{p,k_{p,j}}}\end{aligned}$$

as $n \equiv j \equiv k_p \mod n_p$ for all $p \in S$. Now if for any nonzero n with $n \equiv j \mod N$ should we have $|a_n|_S = 0$, or equivalently $a_n = 0$ we define the set

$$S_j = \emptyset$$

and the value

$$d_j = 0.$$

If this is the case then it follows that for this value n that

$$0 = \prod_{p \in S} c_{p,k_{p,j}} |n|_p^{e_{p,k_{p,j}}}$$

and $|n|_p^{e_{p,k_{p,j}}} \neq 0$ implies that $c_{p,k_{p,j}} = 0$ for some $p \in S$. This in turn implies that $|a_m|_S = 0$ and hence $a_m = 0$ for any $m \equiv j \mod N$. If, on the other hand, for some $n \equiv j \mod N$, we have that $|a_n|_S \neq 0$ then it follows that, for all $m \equiv j \mod N$ we have $|a_m|_S \neq 0$ and hence $c_{p,k_{p,j}} \neq 0$ for all $p \in S$. If for a prime $p \in S$ we have that $v_p(N) > v_p(j)$, then for all $n \equiv j \mod N$ we have

$$|n|_p = |j|_p.$$

We will split S into the disjoint union $S_j \cup S'_j \cup S''_j$ where

$$S_j = \{p \in S \mid v_p(N) \leq v_p(j) \text{ and } e_{p,k_{p,j}} \neq 0\}$$

$$S'_j = \{p \in S \mid v_p(N) > v_p(j) \text{ and } e_{p,k_{p,j}} \neq 0\}$$

and

$$S''_j = \{p \in S \mid v_p(N) > v_p(j) \text{ and } e_{p,k_{p,j}} = 0\}.$$

It then follows that for all $n \equiv j \mod N$ we have

$$|a_n|_S = \prod_{p \in S} c_{p,k_{p,j}} \cdot \prod_{p \in S'_j} |j|_p^{e_{p,k_{p,j}}} \cdot |n|_{S_j, e^{(j)}},$$

where $e^{(j)}$ denotes the collection of exponents $\{e_{p,k_j} \mid p \in S_j\}$. So we will set

$$d_j = \prod_{p \in S} c_{p,k_{p,j}} \cdot \prod_{p \in S'_j} |j|_p^{e_{p,k_{p,j}}}$$

and

$$|a_n|_S = d_j |n|_{S_j, e^{(j)}} \text{ for all } n \equiv j \mod N.$$

We will now prove the first part of the theorem. Let's assume that for some $p \in S$ and some $j \in \{0, 1, \ldots, n_p - 1\}$ that $e_{p,j} > 0$. We will first show that there exists some j for which S_j is non-empty. By our assumed conventions, $e_{p,j} > 0$ means that $c_{p,j} \neq 0$ and so for all $n \equiv j \mod n_p$ that $|a_n|_p = c_{p,j}|n|_p^{e_{p,j}}$. We also have the convention that in this case, $v_p(n_p) \leq v_p(j)$. If $v_p(n_p) = v_p(N)$ then we have $p \in S_j$. If $v_p(n_p) < v_p(N)$ let $p^d \| n_p$ and $p^e \| N$. The statement $n \equiv j \mod n_p$ is the same as saying that $n \equiv n_p k + j \mod N$ for some $k \in \{0, 1, \ldots, N/n_p - 1\}$. So we wish to choose k so that $p^e \mid n_p k + j$. Now

$$n_p k + j \equiv 0 \mod p^e \Leftrightarrow \frac{n_p}{p^d} k + \frac{j}{p^d} \equiv 0 \mod p^{e-d}.$$

Note that $p^d \mid j$ because $v_p(n_p) \leq v_p(j)$. The term n_p/p^d is coprime to p and so it has an inverse modulo p^{e-d}, and so

$$k \equiv -\frac{j}{p^d} \left(\frac{n_p}{p^d}\right)^{-1} \mod p^{e-d}.$$

So if we take $k_0 \in \{0, 1, \ldots, p^{e-d} - 1\}$ so that

$$k_0 \equiv -\frac{j}{p^d} \left(\frac{n_p}{p^d}\right)^{-1} \mod p^{e-d}$$

then $p^e \mid n_p k_0 + j$. We see that if $n \equiv n_p k_0 + j \mod N$ then $p^e \mid n_p k_0 + j$. So $p \in S_{n_p k_0 + j}$ since $v_p(N) \leq v_p(n_p k_0 + j)$.

Next we consider the set of sets $\{S_j \mid 0 \leq j < N\}$ partially ordered by inclusion. As it is a finite partially ordered set, there exists a maximal element S_{j_0} say such that if $S_{j_0} \subseteq S_{j_1}$ for some j_1, then $S_{j_0} = S_{j_1}$. As there exist non-empty S_j's, our maximal set S_{j_0} is also non-empty. We will rearrange the sum of $f(z)$ as follows

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} |a_n|_S z^n \\ &= \sum_{j=0}^{N-1} \sum_{n \equiv j(N)} |a_n|_S z^n \\ &= \sum_{j=0}^{N-1} f_j(z) \end{aligned}$$

where

$$\begin{aligned}
f_j(z) &= \sum_{n \equiv j(N)} |a_n|_S z^n \\
&= \sum_{n \equiv j(N)} d_j |n|_{S_j, e^{(j)}} z^n \\
&= \sum_{k=0}^{\infty} d_j |kN + j|_{S_j, e^{(j)}} z^{kN+j} \\
&= d_j |N|_{S_j, e^{(j)}} \sum_{k=0}^{\infty} |k + j/N|_{S_j, e^{(j)}} z^{kN+j}
\end{aligned}$$

Let

$$g_j(z) = \sum_{n=0}^{\infty} |n + j/N|_{S_j, e^{(j)}} z^n$$

so that

$$f_j(z) = d_j |N|_{S_j, e^{(j)}} z^j g_j(z^N).$$

We will split the sum

$$f(z) = \sum_{j=0}^{N-1} f_j(z)$$

into two parts $f(z) = h_1(z) + h_2(z)$ where

$$h_1(z) = \sum_{\substack{0 \le j < n \\ \text{such that} \\ S_j = S_{j_0}}} f_j(z)$$

and

$$h_2(z) = \sum_{\substack{0 \le j < n \\ \text{such that} \\ S_j \ne S_{j_0}}} f_j(z).$$

Let $n = \prod_{q \in S_{j_0}} q^{f_q}$ be an integer divisible by every prime in S_{j_0} and by no other primes such that for each $q \in S_{j_0}$, $f_q > v_q(N)$. Let $\mu \in \mathbb{C}$ be a primitive n-th root of unity. Now if $0 \le j < N$ is such that $S_j \ne S_{j_0}$ then

$$f_j(\lambda\mu) = d_j |N|_{S_j, e^{(j)}} (\lambda\mu)^j g_j(\lambda^N \mu^N)$$

is bounded as $\lambda \to 1^-$ by Lemma 15 as μ^N is an n/N-th root of unity and n/N is divisible by every prime in S_{j_0} and hence by some prime not in S_j by maximality of S_{j_0}. Thus

$$|h_2(\lambda\mu)|$$

is bounded as $\lambda \to 1^-$. Suppose instead that $S_j = S_{j_0}$. By Theorem 17 we have that

$$\Re\left((-1)^m (\mu^N)^{-(-j/N \mod n/N)} g_j(z^N)\right) \to +\infty$$

where $m = |S_{j_0}|$. This is equal to

$$\Re\left((-1)^m \mu^{(j \mod n)} g_j(z^N)\right) \to +\infty,$$

and thus

$$\Re\left((-1)^m z^j g_j(z^N)\right) \to +\infty.$$

Similarly by Theorem 17 we have that

$$|\Im\left((-1)^m z^j g_j(z^N)\right)|$$

is bounded as $\lambda \to 1^-$. So as every term in $h_1(z)$ goes to $+\infty$ and has bounded imaginary part, this means that

$$\Re\left((-1)^m f(\lambda\mu)\right) \to +\infty$$

as $\lambda \to 1^-$. Since this is true for any μ being some $\prod_{q\in S_{j_0}} q^{f_q}$-th root of unity with each $f_q > v_q(N)$, these singularities form a dense set on the unit circle. Therefore $f(z)$ has a natural boundary on the unit circle.

Now we finish off with the second part of the theorem, so we assume that for every $p \in S$ and every $k \in \{0, 1, \ldots, n_p - 1\}$ that $e_{p,k} = 0$. This means that for every $j \in \{0, 1, \ldots, N-1\}$ if $n \equiv j \mod N$ then

$$|a_n|_S = \prod_{p\in S} c_{p,k_{p,j}} = d_k.$$

So we have

$$\begin{aligned} f(z) &= \sum_{j=1}^{N} \sum_{n\equiv j(N)} d_k z^n \\ &= \sum_{j=1}^{N} d_k \sum_{m=0}^{\infty} z^{mN+j} \\ &= \sum_{j=1}^{N} d_k \frac{z^j}{1 - z^N}, \end{aligned}$$

which is rational and so this concludes the proof. □

The consequence of Theorem 18 is that we have provided a tool for discerning whether or not a large class of functions have a natural boundary, where previously authors had calculated on a case by case basis (see [10] [2]). What precisely this class of functions is or contains is not entirely clear, and requires further investigation.

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