Partially Ordered Sets and Their Invariants

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Francesco Piccoli

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Abstract

We investigate how much information cardinal invariants can give on the structure of the ordered set on which they are defined. We consider the basic definitions of an ordered set and see how they are related to one another. We generalize some results on cardinal invariants for ordered sets and state some useful characterizations. We investigate how cardinal invariants can influence the existence of some special suborderings. We generalize some results on the Dilworth and Sierpinski theorems and explore the conjecture of Miller and Sauer. We address some open problems on dominating numbers. We investigate Model Games to find some characterizations on the cardinality of a set.

Contents

Chapter 1

Some Order on Order

Introduction

We know that there are different ways to consider an ordered set. We will distinguish when a set is partially ordered, strictly partially ordered, quasi-partially ordered and linearly ordered. The aim of this section is to provide the various definitions needed for this thesis and to see how they are related to one another. We shall focus on the most important relationships for the purposes of this thesis.

Summary

In the first section of this chapter we revise old concepts. In the second section we introduce the concept of *induced relation*. In the third section we prove Theorem 1 on *double order systems*. In the forth section we establish the setting of the thesis and finally in the last two sections we revise some old concepts and their properties.

1.1 Basic definitions

In this section we are going to give our fundamentals definitions. We will try to make clear how the abstract concept of order is mathematically formalized.

All the definitions in this section are very simple and well-known in literature, see for example [9]. However the nomenclature is very erratic and concepts are often confused, so we will use this section to make things more precise.

Before formally introducing the various notions we recall the definition of a binary relation.

Definition 1. *We say that R is a binary relation if and only if R is a set of ordered pairs*

$$\forall u \in R \exists x, y (u = \langle x, y \rangle).$$

We abbreviate $\langle x, y \rangle \in R$ with xRy.

Note here $x = \langle x, y \rangle$ is the set-theoretical formula saying that u is an ordered couple of some set x and y.

The *field* of R is the set $field(R) = \{x : \exists y(\langle x, y \rangle \in R)\}$.

Finally, we say that two elements a and b are *comparable*, if aRb or bRa; otherwise we say that they are *incomparable*.

Let us recall here only the properties of binary relations which will be useful for our study.

Definition 2. *Let R be a binary relation and P be a set.*

The relation R is transitive *on P if and only if:*

$$\forall a, b, c \in P((aRb \wedge bRc) \to aRc);$$

The relation R is reflexive *on P if and only if:*

$$\forall a \in P(aRa);$$

The relation R is irreflexive *on P if and only if:*

$$\forall a \in P((a, a) \notin R);$$

The relation R is anti-symmetric *on P if and only if:*

$$\forall a, b \in P((aRb \wedge bRa) \rightarrow a = b);$$

The relation R is free *on P if and only if:*

$$\forall a, b \in P(a \neq b \rightarrow a \textit{ and } b \textit{ are incomparable});$$

The relation R is total *on P if and only if:*

$$\forall a, b \in P(a \neq b \rightarrow a \textit{ and } b \textit{ are comparable}).$$

If the relation R is transitive, (reflexive, total, ...) on it's own field then we will simply say that R is a transitive (reflexive, total, ...) binary relation.

Given the previously defined properties of a binary relation, let us see how they can define different orders on a set. All this definitions are well known and widely used in literature (with incoherent nomenclature), except the *pre-order*, a concept we isolate in order to better understand some connection between the various definitions.

Definition 3. *Let R be a binary relation and P be a set.*

We say that R is a pre-partial order *on P, if R is transitive on P;*

R is a strict partial order *on P, if R is an irreflexive pre-partial order on P; R is a* quasi-partial order *on P, if R is a reflexive pre-partial order on P; R is a* partial order *on P, if R is an anti-symmetric quasi-partial order on P; R is a* (strict) linear order *on P, if R is a total (strict) partial order on P and R is a* (strict) free order *on P, if R is (strict) free on P.*

If the relation R is a pre-partial order (strict partial order, partial order, ...) on it's own field, then we will simply say that R is a pre-partial order (strict partial order, partial order, ...).

Observe that transitivity is a necessary condition for a binary relation to become an order; this comes from the fact that an order must be coherent.

Intuitively for a binary relation R the properties of being freely, partially or linearly ordered is dependent on how much of the set is covered by the order. The property of being strict, quasi and "normally" ordered depends on how the notion of "equivalent" (that is the symmetry property) is handled by the binary relation.

An order relation is always intended to be applied to a specific set. Different order relations can be applied to the same set. To clarify this relation we need to consider the ordered couple (P, R_Q) where P is a set and R is a relation with some properties on some other set Q. Since usually we have that $P = Q$, we will often use the short notation $\mathbb{P}$ to denote the couple (P, R_P).

If the order relation R_P is a pre-partial order, $\mathbb{P}$ is called *pre-partially ordered set*; if it is a quasi-partial order, $\mathbb{P}$ is called *quasi-partially ordered set*, and so on for the other relations defined above.

The set P is called the *base* of $\mathbb{P}$.

Orders and ordered sets are two very close concepts and often confounded in literature, also because an order can be seen as an ordered set with the base equal to the field of the order and an ordered set can be seen as an order when its base coincides with the field of its binary relation.

Here we keep these two concepts distinct so that we can more freely interchange between them and we refer to the present section for clarification.

1.2 From order to order

In this section we will explore how to pass from one pre-order to another in a natural way. We will first try to define the process that allows this transfer and then investigate the way we can put in relation one order to another.

We will start next section by defining the concept of "induced" relation in order to give appropriate general setting to the study. The first and third part of the section's statements and definitions are mathematical folklore. In the second part we isolate and introduce some new concepts in order to complete the study.

Intuitively, we can pass from one type of pre-order to another by adding or removing certain couples of points of the base of the binary relation. These two opposite processes are referred to as augmentation and reduction as specified more precisely in the following formal definition.

Definition 4. *Let R_1 and R_2 be two pre-partially order; R_1 is an* augmentation *of R_2 if $R_2 \subseteq R_1$. In this case R_2 is called a* reduction *of R_1.*

Before we pass to the next subsection let us note that we will often use the symbols $<, \precsim, \leq$ to denote a strict, quasi-partial and partial order R.

1.2.1 Induced relation

Starting with a transitive binary relation R_1 we can define an augmented or reduced transitive binary relation R_2 with the same field, through a formula $\phi(x, y)$ with two variables ranging on P, defined in the language of set theory with the predicate R_1. The new relation R_2 is defined by

$$aR_2b \Longleftrightarrow \phi(a, b).$$

In this case, R_2 is termed *a transitive binary relation ϕ induced by R_1*.

From pre-partial order to quasi-partial order

A first case come into play when we want to define a quasi partial order from a pre-partial order in a 'minimal' fashion.[1]

Consider a pre-partial order R. Then define a quasi partial order from R in a natural way is to define a relation, denoted by $R_=$, through the formula:

$$aRb \vee a = b.$$

Usually, the relation $R_=$ is referred to as *the quasi-partial order naturally induced by R*. Notice that if R is a binary relation on a set P then

$$R_= = R \cup \{(a, a) : a \in P\}. \tag{1.1}$$

The following proposition is now clear.

Proposition 1. *Let R be a pre-partial order on P. Then $R_=$ is a quasi-partial order on P.*

[1] Here for minimal we mean that there does not exist any other quasi- partial order induced by the same transitive binary relation that is contained in it.

In the case the pre-partial order is irreflexive we have the following.

Proposition 2. *Let R be a strict-partial order on P. Then $R_=$ is a partial order on P.*

From pre-partial order to strict partial order

A second case arises when we want to pass from an arbitrary pre-partial order to a strict partial order which preserves some properties of the original relation. Let R be a pre-partial order; a natural way to define a strict partial order is to define a relation $<^R$ through the following formula:

$$a <^R b \Longleftrightarrow aRb \wedge a \neq b.$$

However in this way transitivity is lost even if irreflexivity of R is obtained.

For example, if we take the set $S = \{a, b, c\}$ and the pre-partial order $R = \{(a, b), (b, c), (c, a), (b, a), (c, b), (a, c), (a, a), (b, b), (c, c)\}$, then $<^R$ is irreflexive but not transitive since aRb and bRa hold true, but the relation $a <^R a$ is not verified.

Therefore in order to pass from a pre-partial order to a strict partial order, we need to be more careful and for this purpose, we need to introduce the concept of good equivalence relation.

Definition 5. *Let R be a transitive binary relation on P and let $\sim$ be an equivalence relation on P. Say that $\sim$ is a* good equivalence relation for R *if and only if:*

$$\forall a, b, c \in P((aRb \wedge a \not\sim b \wedge bRc \wedge b \not\sim c) \Rightarrow (a \not\sim c)). \tag{1.2}$$

Given a good equivalence relation $\sim$, define the strict partial order $R_{\not\sim}$

by:

$$aR_{\not\sim}b \iff aRb \wedge a \not\sim b.$$

The relation $R_{\not\sim}$ defined above is referred to as *the strict partial order induced by $\sim$ and R.* The definition is justify by the following proposition.

Proposition 3. *Let P be a pre-partial order on R and $\sim$ be a good equivalence relation on R. Then $P_{\not\sim}$ is a strict partial order on S.*

Proof. We need to prove that $P_{\not\sim}$ is an irreflexive transitive binary relation on S.

The transitivity is guaranteed by the (1.2) and the fact that R is a transitive binary relation.

The irreflexivity comes form the fact that is an equivalence relation. $\square$

If the transitive binary relation $\precsim$ is a quasi-partial order, the most natural equivalence relation to take is defined as follows.

$$a \sim b \iff a \precsim b \wedge b \precsim a.$$

Note that here is crucial that the transitive binary relation is reflexive. The induced equivalence relation described above is always a good equivalence relation for $\precsim$ and it is for this reason that when we define a strict partial order induced by $\precsim$. We will refer to the induced relation as *the strict partial order naturally induced by $\precsim$.*

From pre-partial order to linear order

A non-trivial consequence of the Axiom of Choice is that given a partial order P_1 one can always define an augmentation P_2 such that P_2 is a linear order. This is known as the *order-extension principle* and the first published proof was made by Edward Marczewski in [10].

Furthermore for a pre-partially order we have the following results.

Proposition 4. *Let R be a pre-partially order. Then there exists an augmentation L that is a quasi-partial order and L is total.*

Proof. We know from Proposition 1 that we can pass from a pre-partially order to a quasi-partial order and then the rest follows from Marczewski Theorem mentioned above. $\square$

1.3 Double order system

In this section we will focus on some relationships between different kinds of pre-partial orders on the same set. In particular we shall explore the relationship between a quasi-partial order and a strict partial order.

The definitions of this section are from [14] as the Proposition 5.

Definition 6. *[14] Let P be a set, let $\precsim$ be a quasi-partial order on P and let $<$ be a strict partial order on P. Then $(P, \precsim, <)$ is a double ordered system if the following condition are satisfied:*

1. *for all $p, q \in P$, if $p < q$ or $p = q$, then $p \precsim q$;*
2. *for all $p, q, r \in P$, if $p < q \precsim r$, then $p < r$;*
3. *for all $p, q, r \in P$, if $p \precsim q < r$ then $p < r$.*

Now we will see that if we move a quasi-partially order in the natural way as we did in the Section 1.2 to a strict partial order, then the two form a double order system.

Theorem 1. *Assume (P, R_1) is a quasi-partially ordered set. Let $\sim$ be a good equivalence relation for R_1 and let R_2 be the strict partial order induced by R_1 and $\sim$. Then (P, R_1, R_2) is a double order system.*

Proof. First of all we have to check that pR_2q or $p = q$ implies pR_1q, which follows immediately from the definition of the strict partial order R_2 and the reflexivity of R_1. Therefore we have to check that pR_2qR_1r or pR_1qR_2r implies pR_2r. Suppose pR_2qR_1r, then pR_1q and pR_1r from the transitivity of the binary relation R_1. Now it remains to prove that $p \nsim r$. If $q \nsim r$ then $p \nsim r$, since $\sim$ is a good equivalence relation for R_1. Otherwise, if $q \sim r$ and $p \sim r$, then $p \sim q$ by transitivity of the equivalence relation $\sim$, now a contradiction follows from the definition of the relation R_2. The case for pR_1qR_2r is analogous. $\square$

In particular we have the following.

Corollary 1. *Let $(P, \precsim)$ be a quasi-partially ordered set and let $\precnsim$ be the strict partial order naturally induced by $\precsim$. Then $(P, \precsim, \precnsim)$ is a double order system.*

Here we introduce a very common kind of double order system.

Definition 7. *[14] A double order system (P, R_1, R_2) is* simple *if and only if R_1 is the same as the quasi-partial order naturally induced by R_2.*

It follows from Corollary 1 that a strict partial order and its naturally induced quasi-partial order gives a simple double order system.

Introducing the definition of a simple double order system Monk also proved the following fact about that class.

Proposition 5. *[14] Let (P, R_1, R_2) be a double order system. If R_1 and R_2 are linear, then the double order system is simple.*

We could ask if any partially ordered and strict-partially ordered set that form a double order system is simple. This occurred in the case of the linear orders, but as demonstrate with the following example it is not true in the general case.

Example 1. *Consider the partial order ($\mathcal{P}(\omega)$, $\subseteq$) and the equivalence relation $\sim$ defined by*

$$\forall a, b \in \mathcal{P}(\omega)(a \sim b \leftrightarrow |a| = |b|).$$

It is not hard to show that $\sim$ is a good equivalence relation for the partial order $\subseteq$ so that we can define a strict partial order $\subset^*$ naturally induced by $\subseteq$ and $\sim$. Therefore ($\mathcal{P}(\omega), \subseteq, \subset^*$) is a double order system by Theorem 1, but we can easily find two different elements a and b belonging to $\mathcal{P}(\omega)$, of the same cardinality, such that $a \subseteq b$, and so this double order system is not simple.

1.4 The setting

Now that we have explored and compared the various structures related to the various concepts of order, we can make an explicit statement about the order system we refer. Later we will be allowed to be a little less informative since we will know how to find explicitly our order system of reference.

For a given set P we are interested in a quasi-partial order R defined on it. This structure will implicitly bring out two other structures: namely the strict partial order that is naturally induced by the quasi-partially order R and the partial order that is naturally induced by the strict partial order just defined. The first two structures will form a double order system. Sometimes we could refer to a strict order for R different than the one naturally induced; in this case we will explicitly define it.

As there will be no ambiguity, we can sometimes refer to such structures as an order system and talk more generally about order, keeping in mind all the underlying structures of such a system.

1.5 Ordering the orders

In this section the focus is on how we can compare orders. This will be done through the use of a special map that preserves certain properties, or equivalence relation that underline certain similarities.

1.5.1 Orderings

In order to build our class order we will need to define a map that preserves certain properties of the orders. Then we will use such map to define some class order on the class of quasi-partially ordered set.

Definition 8. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two quasi-partially ordered sets. A map f from P to Q is order preserving*[2] *if and only if:*

$$\forall p, q \in P(pR_Pq \Rightarrow f(p)R_Qf(p)).$$

If the map is injective, then $\mathbb{Q}$ is called an extension of $\mathbb{P}$. We often abbreviate this by $\mathbb{P} \leqslant_E \mathbb{Q}$.

If the map is bijective, then $\mathbb{Q}$ is called an augmentation of $\mathbb{P}$. We often abbreviate this by $\mathbb{P} \leqslant_A \mathbb{Q}$.

[2]To be more pedantic we should say quasi-partial order preserving but this time we abandon precision for readability.

It is important to observe that all the freely ordered sets will be at the bottom of the augmentation order, and linearly ordered sets will be at the top. Also, notice that if two ordered sets have different cardinalities, with respect to the base set, then they will be incomparable in $\leq_A$.

These particular maps don't preserve the complete structure of the orders involved since they do not take into account incomparability. In order to define a stricter order on orders we should define a stronger pre-order preserving map.

Definition 9. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two quasi-partially ordered sets. A map f from P to Q is an* embedding

$$\forall p, q \in P(pR_Pq \Leftrightarrow f(p)R_Qf(p)).$$

Then we say that $\mathbb{Q}$ is embedded on $\mathbb{P}$ *and we often abbreviate this by $\mathbb{P} \leqslant_e \mathbb{Q}$.*

Because of its frequency of use, we will omit the subscript so that if $\mathbb{Q}$ is embedded on $\mathbb{P}$ we just write $\mathbb{P} \leqslant \mathbb{Q}$.

Another very important order that we want define is the cofinal order.

Definition 10. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be quasi-partially ordered sets and let the map $f : P \to Q$ be an embedding map. We say that $\mathbb{P}$ is cofinally embedded in $\mathbb{Q}$ through f if and only if:*

$$\forall q \in Q\ \exists p \in P(qR_Qf(p)).$$

We will often abbreviate it by $\mathbb{P} \leqslant_C \mathbb{Q}$.

1.5.2 Equivalence relations

Now that we have defined some ordering on quasi-partial orders, we can introduce important equivalence relations that will be used continuously throughout this thesis. Note that an equivalence relation is a symmetric quasi-partial order that is a quasi-partial order such that for two elements a, b of the field both (a, b) and (b, a) are in the relation.

The first of such equivalence relations is the order type equivalence relation defined by the isomorphism between two pre-partially ordered set.

Definition 11. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ two pre-partially ordered set. An order isomorphism is an embedding $f : P \to Q$ which maps P onto Q.*

We say that $\mathbb{P}$ and $\mathbb{Q}$ have the same order type, and write $\mathbb{P} \simeq \mathbb{Q}$.

Thanks to the cofinal order introduced in the previous section we can define another important equivalence relation of cofinal type.

Definition 12. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two quasi-partially ordered sets. If $\mathbb{P} \leq_C \mathbb{Q}$ and $\mathbb{Q} \leq_C \mathbb{P}$, then we say that $\mathbb{P}$ and $\mathbb{Q}$ have the same cofinal type, and write $\mathbb{P} \simeq_C \mathbb{Q}$.*

1.6 Subordering

In order to understand the structure of a quasi-partial order, it is interesting to look at their most simple substructures. Thus we need to define formally what is meant by a substructure of a quasi-partially ordered set.

Definition 13. *Let $\mathbb{P} = (P, R_P)$ and $\mathbb{Q} = (Q, R_Q)$ be two quasi-partially ordered sets. $\mathbb{Q}$ is called a subordering of $\mathbb{P}$ if $Q \subseteq P$ and $R_Q = R_P \cap Q \times Q$.*

There are two sort of subordering we would like to explore in this section chains and antichains, specifically the ones that make the quasi-partially order relation total and the other that makes it free.

Definition 14. *Let $\mathbb{P}$ be a quasi-partially ordered set.*

$\mathbb{C}$ *is a* chain *of* $\mathbb{P}$ *if* $\mathbb{C}$ *is a subordering that is a linearly order set.*

$\mathbb{A}$ *is an* anti-chain *of* $\mathbb{P}$ *if* $\mathbb{A}$ *is a subordering that is a freely order set.*

The question is now which kind of chains and anti-chains a pre-partial order can have. In the case of anti-chains, this question reduces to the question of its size. On the other hand, in the case of chains, this question can have more complicated answers, since there are different kinds of linear order types for the same cardinality.

We will come back to this question in Chapter 3, where we will discuss these concepts and their generalizations.

Chapter 2

Bounding and Dominating Numbers

Introduction

We will present here a study of generalised bounding and dominating numbers. These two invariants in their original form have attracted the attention of set theorists in the last fifty years since they capture some properties of the continuum in different models of ZFC. They were introduced by Rothberger and Katetov respectively. A modern introduction is given by van Douwen in [22]. We will recall here the definition:

$$\mathfrak{d} = \min\{|C| : \ C \ \text{ cofinal in } (\omega^\omega, \leqslant^*)\}$$

$$\mathfrak{b} = \min\{|B| : \ B \ \text{ unbounded in } (\omega^\omega, \leqslant^*)\},$$

where $\leqslant^*$ here is the eventual dominance order.

In this chapter we are going to explore what can be said about these two invariants in the case of a more abstract context, namely we will consider quasi-partial ordered sets.

Summary

In the first section of this chapter first we recall some definitions on the generalized invariants, and then we prove Theorem 2 and Theorem 3. In the third section we revise old results on the bounding number. After introducing two new invariants in Definition 18, we state the main result of the chapter in Theorem 4 and finally, in the last section, we revise some known results on the relation between dominating and bounding numbers.

2.1 Basic definitions and theorems

In the following section we will use the vocabulary developed in the previous chapter and see its relation with the concepts of unboundedness and being a dominating subset.

The aim of this section is find the right context for the study of our cardinal invariant and to do this we will prove Theorem 2 and Theorem 3.

We will start by defining such concepts in the general setting of pre-ordered set. This definition and the one that will follow realize the definitions that can found in [3].

Definition 15. *Let $\mathbb{P} = (P, R)$ be a pre-ordered set;*

- *A subset $U \subseteq P$ is a R-unbounded subset if and only if*

$$\forall p \in P \; \exists q \in U (q \not R p \vee pRq;)$$

- *A subset* $D \subseteq P$ *is a* R*-dominating subset if and only if*

$$\forall p \in P\ \exists q \in D(pRq \vee p = q;)$$

Depending on the context we could just write that they are $\mathbb{P}$*-unbounded or* $\mathbb{P}$*-dominating subsets, or, if there is not risk of confusion, simply for the first case unbounded subsets and for the second dominating or cofinal subsets.*

Note that for all pre-ordered sets (P, R) there exists at least a R-dominating subset and a R-unbounded one, namely P. Furthermore we have that every R-dominating subset is also R-unbounded. Therefore we can define the following cardinal invariant for a pre-ordered set.

Definition 16. *Let* $\mathbb{P} = (P, R)$ *be a pre-ordered set. Then we define*

- $\mathfrak{b}_R(P) = \min\{|U| : U \text{ is a } R\text{-unbounded subset}\}$,
- $\mathfrak{d}_R(P) = \min\{|D| : D \text{ is a } R\text{-dominating subset}\}$.

Sometimes we will also use the notation $\mathfrak{b}(\mathbb{P})$ *and* $\mathfrak{d}(\mathbb{P})$*.*

The importance of the orders defined in the previous chapter is stated in the following theorem.

Theorem 2. *Let* $\mathbb{P} = (P, R_P)$ *and* $\mathbb{Q} = (Q, R_Q)$ *be pre-ordered sets. Then we have*

(1) if $\mathbb{P} \leq_A \mathbb{Q}$*, then* $\mathfrak{d}(\mathbb{P}) \geq \mathfrak{d}(\mathbb{Q})$ *and* $\mathfrak{b}(\mathbb{P}) \leq \mathfrak{b}(\mathbb{Q})$*;*

(2) if $\mathbb{P} \leq_C \mathbb{Q}$*, then* $\mathfrak{d}(\mathbb{P}) = \mathfrak{d}(\mathbb{Q})$ *and* $\mathfrak{b}(\mathbb{P}) = \mathfrak{b}(\mathbb{Q})$*.*

Proof. (1) The first implication follows almost directly from the definition. We will prove the statement just in the case of dominating numbers. To do that, it will be enough to show that if D is a R_P-dominating subset of P, and f is the map witnessing $\mathbb{P} \leq_A \mathbb{Q}$, then $f(D)$ is R_Q-dominating in Q. For the rest, since f is a bijection, every dominating subset of $\mathbb{P}$ will be mapped to a subset of $\mathbb{Q}$ of the same size.

Let $p \in Q$. Then $f^{-1}(p) \in P$ and therefore supposing D R_P-dominating there exists $r \in D$ such that $f^{-1}(p) R_P r$. Since f is a preserving pre-order map we have $f(f^{-1}(p)) R_Q f(r)$, which, since f is bijective, implies that $p R_Q f(r)$, where $f(r) \in f(D)$. Hence we get that $f(D)$ is a R_Q-dominating in Q.

(2) Let f be the cofinally embedding from $\mathbb{P}$ to $\mathbb{Q}$. We will denote by $D_1 = f[P]$ the image of the base P of $\mathbb{P}$ into the base Q of $\mathbb{Q}$, and by $D_2 = f[D]$ the image of a subset D cofinal in $\mathbb{P}$. We know that D_1 is cofinal in $\mathbb{Q}$ because f is a cofinal embedding, and, thanks to the order-preserving property of an embedding, we also have that D_2 dominates D_1. Hence we get that D_2 is cofinal in $\mathbb{Q}$, which yields that

$$\mathfrak{d}(\mathbb{P}) \geq \mathfrak{d}(\mathbb{Q}), \tag{2.1}$$

in fact for every cofinal subset of $\mathbb{P}$ we can find one subset of $\mathbb{Q}$ of smaller or equal size which is cofinal in $\mathbb{Q}$. Consider now a cofinal subset D of $\mathbb{Q}$. Then there exists a cofinal subset D' of $f[P]$ that dominates D with $|D'| \leq |D|$. The fact that f^{-1} is pre-order preserving implies that $f^{-1}[D']$ is cofinal in $\mathbb{P}$, and therefore we can conclude that the inequality in (2.1) should be an equality.

Let us now prove the bounding numbers equality. It easy to see that $\mathbb{P}$ cannot have an unbounded subset of size strictly less than the size of $\mathfrak{b}(\mathbb{Q})$.

In fact, if this would be the case, it would contradict the pre-order and inverse pre-order preserving property of f. Let B be an unbounded subset of $\mathbb{Q}$. Then clearly there exists a subset B_1 of $\mathbb{P}$ such that $f[B_1]$ dominates B and such that $|B_1| \leq |B|$. Indeed suppose that B_1 has a bound a, then $f(a)$ would be a bound for B which leads yet to a contradiction. We can therefore conclude that the equality $\mathfrak{b}(\mathbb{P}) = \mathfrak{b}(\mathbb{Q})$ must hold. $\square$

We say that a pre-ordered set (P, R) is *extensible* if for each $p \in P$ exists $r \neq p$ such that $pRr \wedge r\not{R}p$. In other words the pre-ordered set is extensible if does not have maximal point. Here a *maximal* point is a point $p \in P$ such that for all $r \in P$ we have that $p\not{R}r \vee rRp$.

A double order system (P, R_1, R_2) is extensible if the quasi-partial ordered set (P, R_1) and the strict partial ordered set (P, R_2) are extensible. We will see in the Section 2.2 and 2.3 that this is the only class of pre-ordered sets interesting for the dominating and unbounded numbers. In fact in all other cases the computation of these invariants can be deduced from the case of extensible pre-ordered sets.

We can finally state that in the presence of an extensible double order system the dominating and unbounded cardinals do not change if we are considering its quasi-partial order or its strict partial order.

Theorem 3. *Let $(P, \precsim, <)$ be an extensible double order system. Then*

$$\mathfrak{d}_{\precsim}(P) = \mathfrak{d}_{<}(P) \text{ and } \mathfrak{b}_{\precsim}(P) = \mathfrak{b}_{<}(P).$$

Proof. Let us begin by proving the equality for the dominating number case.

On the one hand, inequality $\mathfrak{d}_{\precsim}(P) \leq \mathfrak{d}_{<}(P)$ follows directly from the definition of a double order system. In fact the quasi-partial order is an augmentation of the strict partial order and so we can apply Theorem 2 to conclude.

On the other hand, to obtain the opposite inequality $\mathfrak{d}_{\precsim}(P) \geq \mathfrak{d}_{<}(P)$, we should show that every $\precsim$-dominating subset of P is $<$-dominating. Let B be a $\precsim$-dominating subset of P and let p be an arbitrary element of P, then there exists an element q of B such that $p \precsim q$. The fact that $<$ is an extendible order allows us to consider an $r \in P$ such that $q < r$, and, thanks to property 2 of Definition 6, we can then consider $p < r$. Moreover we can choose again an element $q_2 \in B$ such that $r \precsim q_2$ and, thanks to property 3 of Definition 6, we obtain that $p < q_2$. In conclusion, for every element of P we can find a $<$-bound in B i.e. B is $<$-dominating.

Let us proceed to the bounding number case. Inequality $\mathfrak{b}_{\precsim}(P) \geq \mathfrak{b}_{<}(P)$ holds for the same reason as in the case of the dominating number.

For the rest of the proof let U be a $\precsim$-unbounding subset of P and let p be an arbitrary element of P. Then there exists an element q of U such that $p \not\precsim q$; it follows directly from property 1 of Definition 6 that it is also true that $p \not< q$, so that U is also $<$-unbounding. $\square$

Therefore from now on we will mainly deal with extensible quasi-partial order, knowing that it possible to pass to a strict partial order preserving our cardinal invariants.

2.2 About the bounding number

In the following section we will focus on the bounding number.

All the results in this section where the authors are not cited are mathematical folklore.

In the case of non extensibility the quasi-partial order we will have a maximal point and that point will form an unbounded subset and in this case $\mathfrak{b}(\mathbb{P}) = 1$.

Thanks to remarks made in the previous section we can restrict our attention to the bounding number for quasi-partial orders. So, in the rest of the section, $\mathbb{P}$ will denote an extensible quasi-partial order, we will abbreviate it in *eqpo*.

For the bounding number there are special classes of quasi-partial orders whose cardinal invariants are easily calculable.

Definition 17. *Let $\mathbb{P}$ be a eqpo. We say that $\mathbb{P}$ is directed, if for every $a, b \in P$, there exists a $c \in P$ such that $a \leqslant c,\ b \leqslant c$.*

We say that $\mathbb{P}$ is $\lambda-$directed, if for each $A \subseteq P$ such that $|A| < \lambda$, there exists a $c \in P$ such that $\forall a \in A\ a \leqslant c$.

We can state the following:

Proposition 6. *Let $\mathbb{P}$ be a eqpo. Then if $\mathbb{P}$ is not directed, then $\mathfrak{b}(\mathbb{P}) = 2$.*

Therefore some important quasi-partial orders, such as trees and boolean algebras with or without maximum removed, are not interesting in the study of these cardinal invariants.

Note that if an infinite quasi-partial order is directed, then every finite subset is bounded. As a consequence, if an extensible quasi-partial order $\mathbb{P}$ is directed, then its bounding number is an infinite cardinal not larger than the size of $\mathbb{P}$.

Proposition 7. *Let $\mathbb{P}$ be a eqpo. Then if $\mathbb{P}$ is directed, then $\aleph_0 \leqslant \mathfrak{b}(\mathbb{P}) \leqslant |\mathbb{P}|$.*

The link between directedness and the bounding number is much stronger. Indeed it is expressed by the following easily verifiable fact.

Proposition 8. *Let $\mathbb{P}$ be a eqpo. Then $\mathfrak{b}(\mathbb{P}) = \min\{\lambda \mid \mathbb{P}$ is not $\lambda^+ -$ directed$\}$.*

Another important restraint for the bounding number for extensible quasi-partial orders is that it cannot be singular.

Proposition 9. *[3] Let $\mathbb{P}$ be a eqpo. Then $\mathfrak{b}(\mathbb{P})$ is a regular cardinal.*

Proof. Suppose the contrary. Then there exists a subset B of singular cardinality λ which is unbounded in $\mathbb{P}$. Then B is the union of smaller subsets of length $cof(\lambda) = \kappa$ i.e. $B = \bigcup_{\alpha<\kappa} B_\alpha$ with $|B_\alpha| < \lambda$. Therefore we can consider the bound of each subset, and note that the sequence of such elements is unbounded in $\mathbb{P}$. This implies that the sequence has cardinality less or equal than the cofinality of λ, which leads us to a contradiction. □

We cannot hope to have another restriction for the spectrum of the bounding numbers for the class of extensible directed quasi-partial orders because for each infinite regular cardinal κ the quasi-partial order $\mathbb{P} = (\kappa, \in)$ is such that $\mathfrak{b}(\mathbb{P}) = \kappa$.

We will now explore the connection between the unbounded chains (that are chains that are unbounded as subset) of an extensible quasi-partial order and its bounding number.

Given a quasi-partial order $\mathbb{P}$ it is not always true that for each unbounded subset of size λ we have an unbounded chain of the same size. This is the case when the unbounded subset has size $\mathfrak{b}(\mathbb{P})$, as shown in the following proposition, well known in the context of the continuum invariants.

Proposition 10. *Let $\mathbb{P}$ be a eqpo. Then if $\mathbb{P}$ is directed, then*

$$\mathfrak{b}(\mathbb{P}) = \min\{\lambda \mid \lambda \textit{ is the size of an unbounded chain}\}.$$

In particular $\mathbb{P}$ *has an unbounded chain of cardinality* $\mathfrak{b}(\mathbb{P})$*.*

Proof. It is easy to see that $\mathfrak{b}(\mathbb{P}) \leq \min\{\lambda : \lambda \text{ is the size of an unbounded chain}\}$. Thus we have to prove that for a given quasi-partial order $\mathbb{P}$ with $\mathfrak{b}(\mathbb{P}) = \lambda$ we can build an unbounded chain of size λ; we will do that by induction.

Let U be an unbounded subset of size λ. We will build a chain that dominates U, namely a chain which contains for each element of U an element greater or equal to it. Notice that such a chain has to be unbounded (otherwise U would be bounded).

Let the sequence $\langle x_\beta : \beta < \lambda \rangle$ be an enumeration of the subset U. We will call C_α the chain of order type α that dominates the firsts α elements of U, $\{x_\beta : \beta < \alpha\}$, and whose existence we shall prove by induction on α. To begin with, taken x_0, we can find an element x'_0 such that $x_0 \leq x'_0$, so that we can define $C_1 = \{x'_0\}$.

Suppose now we have defined C_α and we want to define the chain $C_{\alpha+1}$. The fact that $|\alpha| < \lambda$ implies that C_α is bounded by some element y, and remembering that $\mathbb{P}$ is directed, we can find an element $x'_\alpha \in \mathbb{P}$ such that $y, x_\alpha \leq x'_\alpha$; so that we can define $C_{\alpha+1} = C_\alpha \cup \{x'_\alpha\}$.

Let β be a limit ordinal and suppose we have defined the chain C_α for each $\alpha < \beta < \lambda$; we want to define the chain C_β. We define $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$. It is clear that C_β is a chain and that it dominates the set $\{x_\zeta : \zeta < \beta\}$. In fact, let x_ζ be such that $\zeta < \beta$. Since β is a limit, there exists β' such that $\zeta < \beta' < \beta$. Hence $\exists x \in C_{\beta'} \subset C_\beta$ such that $x_\zeta \leq x$.

Now we have a sequence of chains $\mathcal{C} = \langle C_\beta : \beta < \lambda \rangle$ and we can finally define the announced chain dominating U as:

$$C = \bigcup \mathcal{C}.$$

The arguments to show that $\mathcal{C}$ dominates $\langle x_\beta : \beta < \lambda \rangle$ are the same used in the limit case of the induction. $\square$

2.3 About the dominating number

The dominating number of a quasi-partial order in the case of partial orders is often called *cofinality*.

In the case that the quasi-partial order $\mathbb{P} = (P, \leq)$ is not extensible, we have a set M of maximal elements of $\mathbb{P}$. Consider the subset $N = \{x \in \mathbb{P} : \forall y \in M (x \not\leq y)\}$. We have two possibilities: either $N = \emptyset$ and then $\mathfrak{d}(\mathbb{P}) = |M|$, or $N \neq \emptyset$ and in this case $\mathfrak{d}(\mathbb{P}) = |M| \cdot \mathfrak{d}_{\leq \restriction N}(N)$, where $(N, \leq \restriction N)$ is an extensible quasi-partial ordered set. We can conclude that also in this section we can restrict our attention to the case of extensible quasi-partial orders. Thus, also in this section, $\mathbb{P}$ will denote such an order, if not stated otherwise.

It is easy to find a bound for $\mathfrak{d}(\mathbb{P})$, because we can always take the whole set as dominating subset. So we have that

$$\mathfrak{d}(\mathbb{P}) \leq |\mathbb{P}|.$$

In this section we will give a useful characterization of the dominating number of a quasi-partially ordered set as a product of two new invariants that depend on the configuration of its unbounded chains.

Here are the main results of the chapter. We first isolate two important concepts and we propose two new definitions to handle them. Then we prove the characterization in the Theorem 4. At the end of the section we also formally introduce a new order (the dominating order) that we will use in the rest of the thesis.

We call a *dominating chains base* of a quasi-partial order $\mathbb{P}$ a family of unbounded regular chains of $\mathbb{P}$, whose union is cofinal in $\mathbb{P}$; a chain is said to be *regular* if its cardinality is regular.

The cardinals invariant we want introduce are the following.

Definition 18. *The* chain base number $\mathfrak{bc}(\mathbb{P})$ *is defined as*

$$\mathfrak{bc}(\mathbb{P}) = \min\{|B| \ : \ B \textit{ is a dominating chains base of } \mathbb{P}\};$$

The liner height number $\mathfrak{lh}(\mathbb{P})$ *is defined as*

$$\mathfrak{lh}(\mathbb{P}) = \sup\{|C| \ : C \textit{ is an unbounded regular chain of } \mathbb{P}\}.$$

We can state the characterization of the dominating number that we announced.

Theorem 4. *Let $\mathbb{P} = (P, \leq)$ be an extensible quasi-partially ordered set. Then we have*

$$\mathfrak{d}(\mathbb{P}) = \mathfrak{bc}(\mathbb{P}) \cdot \mathfrak{lh}(\mathbb{P})$$

Proof. Let $\mathcal{D}$ be a dominating chains base of size $\mathfrak{bc}(\mathbb{P})$ and consider the quasi-partial order $(\bigcup \mathcal{D}, \leq \restriction \bigcup \mathcal{D})$. Thanks to Part (2) of Proposition 2 we have that $\mathfrak{d}(\mathbb{P}) = \mathfrak{d}(\bigcup \mathcal{D})$. We will split the proof into two claims.

Claim 1 If $\mathfrak{bc}(\mathbb{P}) \leq \mathfrak{lh}(\mathbb{P})$, then $\mathfrak{d}(\mathbb{P}) = \mathfrak{lh}(\mathbb{P})$.

On the one hand:

$$\mathfrak{d}(\mathbb{P}) = \mathfrak{d}(\bigcup \mathcal{D}) \leq |\bigcup \mathcal{D}| \leq \sum_{C \in \mathcal{D}} |C| \leq |\mathcal{D}| \cdot \sup\{|C| \mid C \in \mathcal{D}\} \leq |\mathcal{D}| \cdot \mathfrak{lh}(\mathbb{P})$$

Since we have chosen $\mathcal{D}$ such that $|\mathcal{D}| = \mathfrak{bc}(\mathbb{P}) \leq \mathfrak{lh}(\mathbb{P})$ we can conclude from our hypothesis that the dominatig number of $\mathbb{P}$ is less or equal to $\mathfrak{lh}(\mathbb{P})$.

On the other hand, if B is a dominating subset of $\mathbb{P}$ and C is an unbounded regular chain of $\mathbb{P}$ then B has to contain a subset dominating C. That subset has to be of cardinality $|C|$ because the chain C is unbounded and its size is a regular cardinal. To conclude, we just notice that if $|B| < \mathfrak{lh}(\mathbb{P})$ then we can find a chain of $\mathbb{P}$ which cannot be dominated by B. In this way we have proved that $\mathfrak{d}(\mathbb{P}) \geq \mathfrak{lh}(\mathbb{P})$ and so the claim.

Claim 2 if $\mathfrak{bc}(\mathbb{P}) \geq \mathfrak{lh}(\mathbb{P})$, then $\mathfrak{d}(\mathbb{P}) = \mathfrak{bc}(\mathbb{P})$.

From inequality (2.3) and the hypothesis we can conclude that the dominating number is less or equal to $|\mathcal{D}| = \mathfrak{bc}(\mathbb{P})$. To show that it cannot be strictly less than $|\mathcal{D}|$ we will build a dominating chains base from a dominating subset.

As our partial order is extensible, for each chain C we can consider its extension C^* defined as a chosen unbounded regular chain containing C. For each family of chains $\mathcal{D}$ let $\mathcal{D}^*$ denote the family of extended chains, $\mathcal{D}^* = \{C^* : C \in \mathcal{D}\}$. Striving for a contradiction we suppose that $\mathfrak{bc}(\mathbb{P}) > \mathfrak{d}(\mathbb{P})$. So we have a dominating subset B of $\mathbb{P}$ of size $\mathfrak{d}(\mathbb{P})$. Now we define $\mathcal{D} = \{ \{x\} \ : \ x \in B\}$, it is now enough to take $\mathcal{D}^*$ to get a dominating chains base, so $|\mathcal{D}^*| < \mathfrak{bc}(\mathbb{P})$ which is a contradiction.

Finally from Claims 1 and 2 we conclude that the dominating number of a quasi-partially ordered set is the maximum between the supremum of the size of its unbounded regular chain and the chains base number. □

Before we pass to the next section we will define for every quasi-partial ordered set a new quasi-partial order set that have as the power set of the previous base, more precisely we state the following definition.

Definition 19. *Let $\mathbb{P} = (P, \leq)$ be a quasi-partial order. We will define the quasi partial order $\mathbb{P}^* = (\mathcal{P}(P), \lhd)$ where $\mathcal{P}(P)$ is the power set of P and $\lhd$ is defined as:*

$$\forall x_1, x_2 \in \mathcal{P}(P)((x_1 \lhd x_2) \Leftrightarrow (\forall a \in x_1 \exists b \in x_2)\ (a \leq b)).$$

We will call $\lhd$ the dominating order.

Remark that from the quasi-partial order $\mathbb{P}^*$ we can define a new sub-ordering $\mathbb{P}^{\uparrow}$ whose base is the family of unbounded regular chains. It is not difficult to see that $\mathfrak{d}(\mathbb{P}^{\uparrow}) \geq \mathfrak{bc}(\mathbb{P})$.

2.4 Relationship between the bounding and dominating numbers

In the following section we will talk about the possible configurations of our invariants in the context of extensible quasi-partial orders. As usual $\mathbb{P}$ will be such an order.

All results here all well known and we report them here for completeness.

Since every dominating subset is also unbounded, the first relationship is $\mathfrak{b}(\mathbb{P}) \leqslant \mathfrak{d}(\mathbb{P})$.

With the next proposition we will show that we have an even stronger inequality.

Proposition 11. *[3] Let $\mathbb{P}$ be a eqpo. Then $\mathfrak{b}(\mathbb{P}) \leqslant cof(\mathfrak{d}(\mathbb{P}))$.*

Proof. Let D be a cofinal set of cardinality $\mathfrak{d}(\mathbb{P})$, that we can write as $D = \bigcup_{\alpha < cof(\mathfrak{d}(\mathbb{P}))} D_\alpha$ with $|D_\alpha| < \mathfrak{d}(\mathbb{P})$. For each α, we can choose an element p_α of $\mathbb{P}$ which is not dominated by an element of D_α. If $\mathfrak{b}(\mathbb{P}) > cof(\mathfrak{d}(\mathbb{P}))$ then we can bound the sequence $(p_\alpha)_{\alpha < cof(\mathfrak{d}(\mathbb{P}))}$ and state that the bound cannot be dominated by any element of D. □

We will show that under certain hypotheses we cannot hope to have a new constraint for these two invariants. To do that we will fix some well chosen cardinals and build a partial order that has the bounding and dominating numbers equals to the well chosen cardinals.

Proposition 12. *[3] Assume* GCH*, let β and δ be cardinals such that β is regular and δ is such that $\beta \leqslant cof(\delta)$. Consider the partial order $\mathbb{P} = \beta \times [\delta]^{<\beta}$ with the coordinate-wise order. Then*

$$\mathfrak{d}(\mathbb{P}) = \delta \text{ and } \mathfrak{b}(\mathbb{P}) = \beta.$$

Proof. Firstly we will prove that $\mathfrak{b}(\mathbb{P}) = \beta$. On the one hand it is clear that if $p \in [\delta]^{<\beta}$, then the sequence $(\alpha, p)_{\alpha<\beta}$ is an unbounded sequence in $\mathbb{P}$; on the other hand, if we suppose that there exists an unbounded subset B of cardinality less than β, we can consider the supremum of B on the first coordinate and call it γ. Since $|B| < \beta$ we have $\gamma < \beta$. Let p be the supremum of B on the second coordinate. Since β is regular, we can state that $p \in [\delta]^{<\beta}$. Hence we have that $(\gamma, p) \in \mathbb{P}$ is a bound of B, which leads to a contradiction.

Secondly we prove that $\mathfrak{d}(\mathbb{P}) = \delta$. If we consider a subset D of $\mathbb{P}$ with cardinality less than δ, then we know that it cannot dominate $\mathbb{P}$. In fact if it were the case then we could cover δ with the union of $< \delta$ many subsets of size less than β. Which is in contradiction with our hypotheses that $\beta \leqslant cof(\delta)$ and β is regular. Finally since we assume GCH and $\beta \leqslant cof(\delta)$, we have $\delta^{<\beta} = \delta$ and $\mathfrak{d}(\mathbb{P}) \leqslant |\mathbb{P}| = \delta$. $\square$

Chapter 3

Chains, Antichains and their Relatives

Introduction

To continue our exploration on ordered sets we will focus our attention in this chapter on some of their suborderings.

In the special case of linearly ordered sets we have a class of order type (namely the class of ordinals) that allow us to visualize and build more complicated examples.

In the first part of the chapter we will try to do the same thing and define for partially ordered sets a class of special order types that will help us to visualize and build more complicated ones. An example of these classes are the class of quasi-linearly ordered sets, an extension of the concept of being a linearly order set. As for linearly and freely ordered sets we will use these structures to look at special suborderings, namely, the quasi-chains and the quasi-antichains.

In the second part we will look at chains and their generalization in the context of cofinal suborderings. We will see how this kind of subordering are affected by cardinal invariants and how some of these special suborderings can influence the dominating and unbounding numbers' configuration.

Summary

In the first section of this chapter there are no new concepts or results introduced. In the second section we introduce the concept of quasi-linear orders in Definition 21. The main result of the section is Theorem 5. In this section we also prove Proposition 15 with Lemma 1 and Lemma 2 needed in the proof of Theorem 5. In the third section we prove Corollary 2 and Proposition 19 that are relatad to the well known concept of a scale. Then we introduce a new concept of (κ, λ)-scale in Definition 24 and, after proving Lemma 4, we arrive at the main result of this section, Theorem 6.

3.1 Preserving chains and antichains

In the first chapter we considered special mappings between ordered sets. The first of these mappings we introduced was the order-preserving function which as its name states preserves the order. The injective version of this map induces an order on the ordered sets. When such a mapping exists between two ordered set $\mathbb{P}$ and $\mathbb{Q}$ we called $\mathbb{Q}$ an extension of $\mathbb{P}$ and noted $\mathbb{P} \leq_E \mathbb{Q}$.

A similar map that can be defined between ordered sets is the one that maps incomparable pairs to incomparable pairs. As for the previous map we will note the existence of such an injective map between two ordered sets $\mathbb{P}$ and $\mathbb{Q}$ by $\mathbb{P} \leq_I \mathbb{Q}$.

To understand why we are interested in these orders when we are talking about chains and antichains we need to state the following.

Proposition 13. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two partially ordered sets such that $\mathbb{P} \leq_E \mathbb{Q}$. For every chain C of $\mathbb{P}$ there exists a chain C' of $\mathbb{Q}$ such that $|C| = |C'|$.*

Proof. We have that $f[C]$ is a chain because f is order preserving and $f[C]$ has the same size as C since f is injective. $\square$

Proposition 14. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two partially ordered sets such that $\mathbb{P} \leq_I \mathbb{Q}$. For every chain C of $\mathbb{P}$ there exists a chain C' of $\mathbb{Q}$ such that $|C| = |C'|$.*

Proof. We have that $f[A]$ is an anti-chain because f is incomparability preserving and $f[A]$ have same size sine f is injective. $\square$

3.2 Generalizing Chains and Antichains

To define a partially ordered set that looks like a linearly ordered set and a freely ordered set we will need a way to define an order on the power set of the base, so that our order will be linear and free over more complex subset rather that on single points.

Once the order is defined we will look at the more simple forms of generalizations that we will call quasi-linearly ordered set for their closeness to the linear ordered sets. Then we shall look at a simple example that will turn out to be cononical and we will state important properties about this example

The order we need for our generalization is called the dominating order, and was defined in Definition 19 of Chapter 2. The dominating order is an order on the subsets of the base of a partially ordered set.

Now we can define our quasi-linearly ordered set.

Definition 20. *Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $\{C_i\}_{i\in I}$ a family of unbounded chains of $\mathbb{P}$ such that $P = \bigcup_{i\in I} C_i$ and $(\{C_i\}_{i\in I}, \lhd)$ is a linearly ordered set, then $\mathbb{P}$ is a quasi-linearly ordered set.*

Suborderings that are quasi-linear order are called quasi-antichains ordered and quasi-chain ordered sets.

An example of quasi-linearly ordered set is the set $\omega \times \omega_1$ with the *product order* i.e. for all $(x_1, x_1), (y_1, y_2)$ we have that $(x_1, x_2) \leq_{\text{prod}} (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$.

In the next proposition we will prove that this partially ordered set is a quasi-linearly ordered set. For now notice that the $\aleph_1$ sized family of subsets $\{\omega \times \{\alpha\} : \alpha < \omega_1\}$ are chains that cover the partial order.

This example is a canonical one in the sense that all quasi-linear orders can be seen as the product order of some product of cardinals, as we will show later.

More generally we can state the following about product of two cardinals with the product order.

Proposition 15. *Let κ and λ be two cardinal numbers then $(\kappa \times \lambda, \leq_{prod})$ is a quasi-ordered chains.*

Proof. As for the special case $\omega \times \omega_1$ we need to define the family of chains:

$$C_\alpha = \kappa \times \{\alpha\}.$$

It follows that $P = \bigcup_{\alpha\in\lambda} C_\alpha$ and it is left to prove that

$$(\forall \alpha, \beta)(\alpha \leq \beta \to C_\alpha \lhd C_\beta).$$

Take $x \in C_\alpha$ so that there exists $k \in \kappa$ such that $x = (k, \alpha)$. We have that the point $t = (k+1, \beta)$ belongs to C_β . We have also that $x \leq_{\text{prod}} t$ since $k \leq k+1$ and $\alpha \leq \beta$. It follows that $C_\alpha \lhd C_\beta$ and so $(\kappa \times \lambda, \leq_{\text{prod}})$ is a quasi-linear order. $\square$

We can now introduce a special class of quasi-linear orders that will help us to characterize the more general class.

Definition 21. *Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $\{C_i\}_{i \in I}$ a family of unbounded regular chains of $\mathbb{P}$ such that $P = \bigcup_{i \in I} C_i$ and $(\{C_i\}_{i \in I}, \lhd)$ is a linear order set, then $\mathbb{P}$ is a simple quasi-linear order set.*

One of the interesting properties that simple quasi-linearly ordered sets have is that the size of the chains that cover the base must be of the same size.

Lemma 1. *Let $\mathbb{P} = (P, \leq)$ be a simple quasi-linearly ordered set such that $\{C_i\}_{i \in I}$ is the family of regular unbounded chains, then there exists κ such that for each $i \in I$ we have that $|C_i| = \kappa$.*

Proof. Suppose there exists $i, j \in I$ such that $|C_i| = \kappa_i$ and $|C_j| = \kappa_j$ where $\kappa_i \neq \kappa_j$. Since $\mathbb{P}$ is a quasi-linearly ordered set we have that $C_j \lhd C_i$ or $C_i \lhd C_j$. Without loss of generality we may assume that $C_j \lhd C_i$. Then we have that $\kappa_j \leq \kappa_i$ for otherwise the set C_i could dominate the set C_j. If $\kappa_j < \kappa_i$ then it would be possible to build a map $f : C_i \to C_j$ where $f(C_i)$ is a cofinal subset of C_j of size less than κ_j, and that would contradict the fact that κ_j is regular cardinal. So we can conclude that $\kappa_i = \kappa_j$ and so all the chains must have same size. $\square$

The importance of simple quasi-linearly ordered sets is that every quasi-linearly ordered set have a simple quasi-linearly ordered set cofinally embedded in it.

Lemma 2. *Let $\mathbb{P}$ be a quasi-linearly ordered set, then there exists a cofinal subordering $\mathbb{P}'$ that is a simple quasi-linearly ordered set.*

Proof. Since $\mathbb{P} = (P, \leq_P)$ is a quasi-linearly ordered set we have a family of chains $\{C_i\}_{i\in I}$ and from this family we can build a family of regular chains by taking for each $i \in I$ a dominating subset of C_i and name it C_i'.

Consider the new partial order $\mathbb{P}' = (P', \leq_P)$ such that $P' = \bigcup_{i\in I} C_i$. By construction it is clear that $\{C_i'\}_{i\in I}$ is a family of unbounded regular chains of $\mathbb{P}'$. We can conclude that $\mathbb{P}'$ is the simple quasi-linearly ordered set confinal subordering of $\mathbb{P}$ we were looking for. $\square$

The canonicity of the product of cardinals examples stated above comes from the fact that all quasi-linearly ordered set contain cofinally an extension of these partially ordered sets.

Theorem 5. *Let $\mathbb{P} = (P, \leq)$ be a quasi-linearly ordered set then there exist κ and λ cardinals numbers, such that $(\kappa \times \lambda, \leq_{prod}) \leq_E' \mathbb{P}$ and the image is cofinal in $\mathbb{P}$.*

Proof. By Lemma 2 we can assume that $\mathbb{P}$ is a simple quasi-linearly ordered set.

Let $\mathbb{P} = (P, \leq)$ be a quasi-linearly ordered set, then we know there exists a family of chains $\{C_i\}_{i\in I}$ of $\mathbb{P}$ such that $P = \bigcup_{i\in I} C_i$. By Proposition 1 there exists a cardinal κ such that $\forall i \in I$ we have that $|C_i| = \kappa$.

Furthermore $(\{C_i\}_{i\in I}, \lhd)$ is a linear order of size $|I| = \lambda$.

We want prove that $(\lambda \times \kappa, \leq_{\text{prod}})$ can be cofinally mapped in $\mathbb{P}$ with an order preserving function. To do it we will build a function that will map $(\lambda \times \kappa, \leq_{\text{prod}})$ in $\mathbb{P}$.

From the dominating order we can define the following order on I. $\forall x, y \in I$ we have that $x \leq_I y$ if and only if $C_x \lhd C_y$. So I will be linearly ordered by $\leq_I$ and we can define a map $f : \lambda \to I$ such that $f(\lambda)$ is cofinal in I.

Similarly for every $i \in I$ we want to define a map $g_i : \kappa \to C_i$ such that $g_i(\kappa)$ is cofinal in C_i and for every $i \leq_I j$ and $\alpha \leq \beta$ we have that $g_i(\alpha) \leq g_j(\beta)$. We will define such a family of functions recursively. We take $g_0 : \kappa \to C_0$ such that $g_0(\kappa)$ is cofinal in C_0.

Suppose we have already define g_α and we want to define the successor step $g_{\alpha+1}$. To do it we first note that since $C_\alpha \lhd C_{\alpha+1}$, we can build a function $r_{\alpha,\alpha+1} : C_\alpha \to C_{\alpha+1}$ such that $\forall x \in C_\alpha$ we have $x \leq r_{\alpha,\alpha+1}(x)$. So now we can define $g_{\alpha+1}(\gamma) = r_{\alpha,\alpha+1}(g_\alpha(\gamma))$.

For the limit step consider the function $r_\beta : \bigcup_{\alpha<\beta} C_\alpha \to C_\beta$ such that $\forall x \in \bigcup_{\alpha<\beta} C_\alpha$ we have that $x \leq r_\beta(x)$. So now we can define $g_\beta(\gamma) = r_\beta(\bigcup_{\alpha<\beta} g_\alpha(\gamma))$.

We now have all the elements needed to define our cofinal map $h : \lambda \times \kappa \to P$. We do that by defining

$$\forall(\alpha, \beta) \in \lambda \times \kappa \;\; h(\alpha, \beta) = g_{f(\alpha)}(\beta).$$

So we need to prove that h is order preserving and that $f(\kappa \times \lambda)$ is cofinal in $\mathbb{P}$.

First let's see that it is an order preserving map. $\forall(\alpha_1, \beta_2), (\alpha_2, \beta_2) \in \lambda \times \kappa$ such that $(\alpha_1, \beta_2) \leq_{\text{prod}} (\alpha_2, \beta_2)$ we have

$$g_{f(\alpha_1)}(\beta_1) \leq g_{f(\alpha_2)}(\beta_1) \leq g_{f(\alpha_2)}(\beta_2).$$

The first inequality comes from the construction of the family $\{g_\beta\}_{\beta<\kappa}$ and the second from the fact that the functions of the family are increasing.

It remains to prove that $h(\lambda \times \kappa)$ is cofinal in $\mathbb{P}$. Take a point p in P, that point will belong to some chain C_i with $i \in I$. Then we can chose $j \in J$ such that $C_i \lhd C_j$ and so we have $q \in C_j$ such that $p \leq q$. Finally since $g_j(\kappa)$ is cofinal in C_j we can find $r \in \kappa$ such that $q \leq h(j, r)$. □

3.3 Some special cofinal subsets

In this section, we are interested to know what kind of subsets could be cofinal in $\mathbb{P}$, an extensible quasi-partial order (eqpo), and which relationships can exist between the range of possible cofinal subsets and the bounding and dominating number's configuration.

We know, for example, that for every quasi-partial order, we can find a dominating subset of it such that as a suborder it is well founded.

Lemma 3. *[3] Let $\mathbb{P} = (P, \leq_P)$ be eqpo. Then we can find $Q \subseteq P$ such that Q is a $\leq_P$-dominating subset and $(Q, \leq_{P\restriction Q})$ is a well founded order.*

The natural question arising from the lemma will be if is it possible for a quasi-partial order to have a dominating subset that as a suborder is a chain. For the importance of such dominating subsets we will give them a name.

Definition 22. *Let $\mathbb{P}$ be an eqpo and S be a subset of $\mathbb{P}$. Then we say that S is a scale if S is a $\mathbb{P}$-dominating chain.*

In contrast to the well founded case, this time the answer will depend on the kind of quasi-partial order we are considering and on the axiomatic setting we have chosen.

We will prove that there exists a strict relationship with the bounding-dominating numbers configuration, i.e. this configuration will be decided by whether such a subset can dominate the full quasi-partial order.

Proposition 16. *Let $\mathbb{P}$ be an eqpo. Then $\mathfrak{d}(\mathbb{P}) = \mathfrak{b}(\mathbb{P})$ if and only if there exists a scale.*

Proof. If the partially ordered set has a well ordered dominating subset, evidently it is cofinally embedded. So by Proposition 2 our partial order has dominating and bounding numbers equal to those of the well ordered set, where all unbounded subsets are also cofinal, hence the two cardinal invariants coincide.

On the other hand, if $\mathfrak{d}(\mathbb{P}) = \mathfrak{b}(\mathbb{P})$, then we have a set $\{c_\alpha : \alpha < \mathfrak{b}(\mathbb{P})\}$ that dominates $\mathbb{P}$; now we substitute each c_α with a b_α which is a bound of $\{c_\delta : \delta < \alpha\}$. Since such a set has cardinality strictly less than the bounding number, it will always have a bound. Now we can consider the sequence $\{b_\delta : \delta < \mathfrak{b}(\mathbb{P})\}$. This sequence is cofinal in $\mathbb{P}$ since it dominates $\{c_\delta : \delta < \mathfrak{b}(\mathbb{P})\}$, and it is linearly ordered thanks to the careful choice of the b_α. $\square$

A consequence of this proposition is that if we have a quasi-partial order $\mathbb{P}$ with dominating number a singular cardinal then we cannot dominate the whole of $\mathbb{P}$, with only one chain, so that $\mathfrak{bc}(\mathbb{P}) > 1$. A corollary of Theorem 4 improves upon this result.

Corollary 2. *Let $\mathbb{P}$ be an eqpo and λ be a singular cardinal. Then if $\mathfrak{d}(\mathbb{P}) = \lambda$ then $\mathfrak{bc}(\mathbb{P}) \geq cof(\lambda)$.*

Proof. In Theorem 4 we proved that $\mathfrak{d}(\mathbb{P}) = \mathfrak{bc}(\mathbb{P}) \cdot \mathfrak{lh}(\mathbb{P})$. Clearly if $\mathfrak{bc}(\mathbb{P}) = \lambda$ then we have done. If otherwise $\mathfrak{bc}(\mathbb{P}) < \lambda$ then we have $\mathfrak{lh}(\mathbb{P}) = \lambda$ where λ is a singular cardinal. Suppose now that $\mathfrak{bc}(\mathbb{P}) < cof(\lambda)$, and let $\mathcal{D}$ be a family of chains witnessing this inequality. If we have $\sup\{|C| : C \in \mathcal{D}\} = \mathfrak{lh}(\mathbb{P}) = \lambda$ this leads to a contradiction since all chains have regular size less than λ and the size of $\mathcal{D}$ is strictly less than $cof(\lambda)$. So now it suffices to prove the equality $\sup\{|C| : C \in \mathcal{D}\} = \mathfrak{lh}(\mathbb{P})$ Recall that a dominating chains base of $\mathbb{P}$ is a subfamily of the family of the all unbounded regular chains of $\mathbb{P}$, so that one direction is obvious. For the other one we can suppose that $\sup\{|C| : C \in \mathcal{D}\} < \lambda = \mathfrak{lh}(\mathbb{P})$ and prove that in this case $\bigcup \mathcal{D}$ is not cofinal. We know that $\mathfrak{bc}(\mathbb{P}) = |\mathcal{D}| < \lambda$ and $\mathfrak{d}(\mathbb{P}) = \mathfrak{d}(\bigcup \mathcal{D}) \leq |\bigcup \mathcal{D}| \leq |\mathcal{D}| \cdot \sup\{|C| : C \in \mathcal{D}\} < \lambda \cdot \lambda = \lambda$, a contradiction. □

It is interesting to ask what happens with the order $\leq_C$. Is this an order preserving scale? It follows immediately from Proposition 2 and 16 that in the case of a cofinal order the answer is positive.

Proposition 17. *Let $\mathbb{P}$ be an eqpo. Then if $\mathbb{P} \leq_C \mathbb{Q}$ then $\mathbb{P}$ has a scale if and only if $\mathbb{Q}$ has a scale.*

In the case of the augmentation order we have a weaker result. Indeed in an augmentation map every dominating subset is mapped to a dominating subset but the converse is not always true.

Proposition 18. *Let $\mathbb{P}$ and $\mathbb{Q}$ be eqpo. Then if $\mathbb{P} \leq_A \mathbb{Q}$ and $\mathbb{P}$ has a scale, then $\mathbb{Q}$ has scale.*

Note that it is easy to see that the converse is not always true, take for example the partial order $\mathbb{Q} = (\omega^\omega, \leq^*)$ and $\mathbb{P} = (\omega^\omega, \leq)$. Then $\mathbb{P} \leq_A \mathbb{Q}$ and $\mathbb{P}$ does not have a scale even though $\mathbb{Q}$ has.

We say that a scale is minimal if there does not exist a scale of smaller size. So now we can ask if also the minimality of a scale is preserved by augmentation. Since this implies that dominating and bounding numbers are equal and augmentation maps do not preserve these cardinal invariants we may expect that this is not the case. Surprisingly, we have:

Proposition 19. *Let $\mathbb{P}$ and $\mathbb{Q}$ be eqpo. Then if $\mathbb{P} \leq_A \mathbb{Q}$ and $\mathbb{P}$ has minimal scale of size λ, then $\mathbb{Q}$ has a minimal scale of size λ.*

Proof. First note that by minimality, λ must be regular. Let S be a minimal scale of $\mathbb{P}$ of size λ. By Fact 18 we know that $\mathbb{Q}$ has a scale. Suppose that there exists a scale S' of $\mathbb{Q}$ of size $\mu < \lambda$. Let f be an augmentation map from $\mathbb{P}$ to $\mathbb{Q}$. It is easy to see that $f[S]$ is a cofinal subset of $\mathbb{Q}$ and, as S' is also cofinal, we find that S' dominates $f[S]$.

For $x \in \mathbb{Q}$ we denote by $\ulcorner x \urcorner$ the set $\{y : y \leq_Q x\}$. We know that $\ulcorner x \urcorner \cap f[S]$ has size strictly less than λ for all $x \in S'$, as otherwise since this is an initial segment of a cofinal chain in $\mathbb{Q}$ of size λ and every such initial segment of a cofinal chain of size λ is also cofinal, we will have x as maximum of $\mathbb{Q}$. We can then cover $f[S]$ with the union of fewer than λ subsets of size strictly less than λ that is $f[S] = \bigcup_{x \in S'} \ulcorner x \urcorner \cap f[S]$. This leads to a contradiction since λ is a regular cardinal. $\square$

We have seen that a quasi-partial order has a scale if and only if it has a well-ordered set cofinally embedded in it. We now introduce another special substructure of the partial order that can be seen, as will be evident in next lemma, as a generalization of a scale.

Definition 23. *Let λ and κ be two regular cardinals we will call a family of chains $\mathcal{C}$ a (κ, λ)-scale, if it is a dominating chains base, such that:*

1. *$\forall x \in \mathcal{C}$, the $|x| = \kappa$*

2. *there exists* $\langle c_\alpha : \alpha < \lambda \rangle$ *an enumeretion of* $\mathcal{C}$ *such that for all* $\alpha < \beta < \lambda$*:*

$$c_\alpha \lhd c_\beta \text{ and } c_\beta \not\lhd c_\beta.$$

It is clear that a scale of size κ is a (κ, 1)- scale, and that not all (κ, λ)-scales can be dominated by a scale. We can characterize the case where a (κ, λ)- scale is dominated by a scale with the following lemma.

Lemma 4. *Let* $\mathbb{P}$ *be an eqpo. Then* $\mathbb{P}$ *has a (*κ*,* λ*)- scale dominated by a scale if and only if* λ *is finite or* $\lambda = \kappa$*.*

Proof. If $\mathcal{C}$ is a (κ, λ)-scale of $\mathbb{P}$, we can suppose without loss of generality that $\mathbb{P} = \bigcup \mathcal{C}$.

Firstly we prove the implication from the right to the left. It easy to see that if there exists a (κ, n)- scale $\langle c_m : n < \omega \rangle$ with $n < \omega$ then c_n is a scale; so we will just prove that a (κ, κ)-scale is dominated by a scale. We will do this by induction. Let $\langle c_\alpha : \alpha < \kappa \rangle$ be our (κ, κ)-scale ordered by $\lhd$. By going to a cofinal well-founded subset, if necessary, we can assume that, for all $\alpha < \kappa$, we have $c_\alpha = \{c_{\alpha,\beta} : \beta < \kappa\}$ is a linearly $\leq$-ordered enumeration of the chain c_α. We suppose, without loss of generality, that if $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ then $c_{\alpha_1,\beta_1} \leq c_{\alpha_2,\beta_2}$.

With the previous assumptions we can claim that the chain $\langle c_{\alpha,\alpha} : \alpha < \kappa \rangle$ is cofinal for the quasi-partial order $\mathbb{P}$. To show the claim let $x \in \mathbb{P}$. Since we are taking $\mathbb{P} = \bigcup \mathcal{C}$, we assume that $x \in c_\alpha$ for some ordinal $\alpha < \kappa$. So we can denote it as $c_{\alpha,\beta}$ for some ordinal $\beta < \kappa$. It is enough now to take an ordinal $\kappa > \gamma > \max\{\alpha, \beta\}$ and note that $c_{\gamma,\gamma} \geq c_{\alpha,\beta}$.

The other direction follows directly from the Proposition 16 and the next proposition. $\square$

A natural consequence of Proposition 16 is that the singularity of the dominating number for a quasi-partial order prevents it from having a scale. We will see now that it prevents it from also having a (κ, λ)-scale for all κ and λ. This is because the existence of (κ, λ)-scale decides the bounding and dominating number for a quasi-partial order as shown in the following proposition.

Theorem 6. *Let $\mathbb{P}$ be a directed eqpo; If there exists a (κ, λ)-scale in $\mathbb{P}$, then $\mathfrak{b}(\mathbb{P}) = \min\{\kappa, \lambda\}$ and $\mathfrak{d}(\mathbb{P}) = \max\{\kappa, \lambda\}$.*

Proof. Let $\mathcal{C}$ be a (κ, λ)-scale of $\mathbb{P}$. We can suppose without loss of generality that $\mathbb{P} = \bigcup \mathcal{C}$. Thanks to Lemma 4 we already know that the proposition is true in the case $\kappa = \lambda$, so we now treat the cases $\kappa > \lambda$ and $\kappa < \lambda$.

Firstly we will prove that $\mathfrak{b}(\mathbb{P}) = \min\{\kappa, \lambda\}$. Remember that we are considering $\mathbb{P}$ to be directed so that we can use the characterisation from Proposition 10. Thus we need to prove that in our quasi-partial order there are no unbounded chains of size strictly less than $\min\{\kappa, \lambda\}$.

We first consider the case where $\kappa < \lambda$. In this case we should prove that there are no unbounded chains of size strictly less than κ. We suppose instead that C is an unbounded chain in $\mathbb{P}$ of size $\mu < \kappa$. For all $x \in C$ we will denote by $f(x)$ any chosen element among these that is in $\bigcup \mathcal{C}$ and that are greater or equal to x.

Now we have two possibilities. The first is that $f[C]$ is unbounded in some $X \in \mathcal{C}$, and this is impossible as C has size $\mu < \kappa$ and κ is a regular cardinal. The second is that $f[C] \cap X \neq \emptyset$ is bounded in every $X \in \mathcal{C}$ and in this case since $\mu < \kappa \leq \lambda$ and λ is regular, then we could $\lhd$-bound $f[C]$ by some disjoint $X \in \mathcal{C}$. Once we have such an X it easy to find an element $x \in X$ which is greater than every element of C. This is a contradiction with C being unbounded in $\mathbb{P}$.

Analogously in the case $\kappa > \lambda$ we cannot have an unbounded chain in $\mathbb{P}$ of size $\mu < \lambda$. In this case show that there exists an unbounded chain of size λ. We can build inductively such a chain for the enumeration of the (κ, λ)-scale $\langle c_\alpha : \alpha < \lambda \rangle$ by choosing carefully one element from each chain in a way so that the obtained sequence $\langle x_\alpha : \alpha < \lambda \rangle$ is linearly ordered by the strict linear order induced by $\leq$ and every element x_α can not be bounded from any element from $\bigcup_{\alpha<\beta} c_\beta$. Suppose that there exists an element y that bounds such a sequence; then y will belong, for some α, to the chain c_α and clearly this is in contradiction with how we constructed the sequence.

We now need to prove that the dominating number is equal to the maximum of the two cardinals.

Remember we are considering $\mathbb{P} = \bigcup \mathcal{C}$. From this it follows immediately that:

$$\mathfrak{d}(\mathbb{P}) \leq |\mathbb{P}| \leq |\bigcup \mathcal{C}| \leq \kappa \cdot \lambda.$$

So it remains to prove that

$$\mathfrak{d}(\mathbb{P}) \geq \kappa \cdot \lambda.$$

To do this we consider a dominating subset D of $\mathbb{P}$. It is clear that D should meet cofinally every chain of the (κ, λ)-scale, i.e. $\forall \alpha \exists \beta > \alpha$ such that $c_\beta \cap D \neq \emptyset$ and from the regularity of λ it follows that $|D| \geq \lambda$. We also know that D will dominate every chain of the (κ, λ)-scale, so that from the regularity of κ it will follow that $|D| \geq \kappa$. So we can conclude that $|D| \geq \kappa \cdot \lambda$. $\square$

Chapter 4

Dilworth, Sierpinski and the Sauer Miller Conjecture

Introduction

In this chapter we will explore two very important results of Order Theory, putting them in the context of our setting and finally in light of this exploration we will try to better understand a very important conjecture of the theory.

The first result we will talk about in Section 1 of this chapter is the Dilworth theorem. This result tries to calculate the size of an antichain in a partially ordered set given the number of chains needed to cover that set.

The Dilworth theorem is very useful for the case when the partial order is finite, but fails badly when we take infinite partially ordered sets. We will see a way to refine it using a more appropriate cardinal.

The point of doing that is to allow a better characterization of the partially ordered sets that do not satisfy the Dilworth theorem, and this is a characterization provided by Uri Abraham in [1].

In Section 2 we then look at how cardinality can effect these suborderings as chains, antichains, quasi-chains and quasi-antichains defined in the last section of Chapter 1 and the first section of Chapter 3. We will see how the counterexample of Sierpinski on chains and antichains does not apply to the more general concepts of quasi-chains and quasi-antichains.

Finally in the third section we will talk about an old conjecture of Sauer and Miller that given a partially ordered set with a singular dominating number asserts that there exists an antichain with cardinality equal to that of the cofinality of the original partial order.

All the results and conjectures we talk about in this chapter have an underling task in common, and that is to try to guess the length of antichains in a given partial order given a certain cardinal invariant of that partial order.

Summary

In the first section of this chapter, after recalling some old results on the subject, we introduce a new invariant in Definition 27. After proving two propositions on this invariant in Proposition 22 and Proposition 23, we have the main result of the section in Theorem 9. In the second section we look at some consequences of our generalization and prove Theorem 11. With the same approach, in the last section we prove Theorem 12.

4.1 Dilworth theorem and its generalizations

In this section we are going to look at a very important result in order theory regarding chains and antichains and see how this result can be extended thanks to some of the definitions introduced in the last chapter.

The theorem we are talking about is the Dilworth theorem and before we state it we need to introduce some new cardinal invariants.

4.1.1 Basic definitions

Definition 24. *If $\mathbb{P} = (P, \leq_P)$ is a partially ordered set and κ a cardinal number, we say that $\mathbb{P}$ cannot be covered by less than κ chains if there does not exist a set of κ chains from $\mathbb{P}$ such that the union of their bases is $\mathbb{P}$.*

We call $\mathrm{cov}(\mathbb{P})$ the minimum cardinal κ such that κ can be covered by κ chains.

Notice that this cardinal invariant is in strict connection with the base chains cardinal introduced in the last chapter. Actually we have that a family of chains that cover $\mathbb{P}$ can be a base of chains of $\mathbb{P}$ and so we have:

$$\mathfrak{bc}(\mathbb{P}) \leq \mathrm{cov}(\mathbb{P})$$

.

Proposition 20. *Let $\mathbb{P} = (P, \leq_P)$ be a partially ordered set defined as follow: $P = \omega \cup \{a, b, c\}$ and $\leq_P = <_\omega \cup \{(a, 0), (b, 0), (c, 0), (a, b), (a, c)\}$ where $\leq_\omega$ is the usual order on ω, then $\mathfrak{bc}(\mathbb{P}) = 1$ and $\mathrm{cov}(\mathbb{P})$ cannot be covered by less than two chains.*

The question that Dilworth tried to address asks if a partially ordered set that needs many chains to cover it necessarily contains large antichains. To better capture this statement we introduce another cardinal invariant for partially ordered sets.

Definition 25. *Let $\mathbb{P} = (P, \leq_P)$ be a partially ordered set then:*

$$\mathrm{width}(\mathbb{P}) = \sup\{|A| : A \textit{ is an antichain in } \mathbb{P}\}$$

We will call it the width of $\mathbb{P}$

4.1.2 Dilworth Problem

Now that we introduced some new cardinal invariants, we restate the problem mentioned above as a problem of how these invariants are related.

Since to cover a partially ordered set with an antichain of size κ we will need at least κ chains, we can already state the following fact.

Proposition 21. *Let $\mathbb{P} = (P, \leq_P)$ be a partially ordered set then* $\mathrm{width}(\mathbb{P}) \leq \mathrm{cov}(\mathbb{P})$

Our problem can now be stated as follows:

$$\text{Is } \mathrm{cov}(\mathbb{P}) = \mathrm{width}(\mathbb{P})?$$

In the case of finite partially ordered sets, Dilworth showed that the answer is positive.

Theorem 7. *[4] Let $\mathbb{P} = (P, \leq_P)$ be a finite partially ordered set, then:*

$$\mathrm{width}(\mathbb{P}) = \mathrm{cov}(\mathbb{P})$$

For the infinite case Perles showed that that Dilworth theorem cannot extend to the partially ordered set with base $\omega_1 \times \omega_1$ and order $\leq_{\mathrm{prod}_{\omega_1,\omega_1}}$ and we will call this structure in this chapter the *Perles partially ordered set*.

Theorem 8. *[16] Let $\mathbb{P} = (P, \leq_P)$ be the Perles partially ordered set, then*

$$\mathrm{width}(\mathbb{P}) < \mathrm{cov}(\mathbb{P}).$$

The Perles partially ordered set contradicts badly the Dilworth theorem since the partially ordered set does not have infinite antichains but still needs uncountable chains to cover it.

4.1.3 Dilworth generalizations

To try to extend the Dilworth theorem for infinite sets we will explore the same property in a cofinal context. We already have a cofinal version of the covering number, namely, the base chains number and so we now need a cofinal version of the width of the partial order.

Definition 26. *Let $\mathbb{P} = (P, \leq_P)$ be a partially ordered set, then*

$$\mathrm{cofwidth}(\mathbb{P}) := \min\{\mathrm{width}(\mathbb{A}) : \mathbb{A} \leq_C \mathbb{P}\}$$

And so we can state the following proposition.

Proposition 22. *Let $\mathbb{P} = (P, \leq_P)$ be a partially ordered set, then*

$$\mathrm{cofwidth}(\mathbb{P}) \leq \mathfrak{bc}(\mathbb{P}).$$

Proof. Let $\mathrm{cofwidth}(\mathbb{P}) = \kappa$ and for a proof by contradiction assume that $\mathfrak{bc}(\mathbb{P}) = \lambda < \kappa$. Since $\mathfrak{bc}(\mathbb{P}) = \lambda$, we can find a family of chains $\{C_i\}_{i \in I}$ such that $A := \bigcup_{i \in I} C_i$ where $|I| = \lambda$ where the restricted order is a cofinal subordering.

Let's then set $\mathbb{A} = (A, \leq\restriction A)$ so that $\mathbb{A} \leq_C \mathbb{P}$. We know by Fact 2 that

$\mathrm{width}(A) \leq \mathrm{cov}(A) \leq \lambda < \kappa$. This contradicts the definition of cofwidth since by minimality we cannot have a cofinal subordering smaller than κ that is we have that:

$$\mathrm{width}(\mathbb{A}) \geq \min\{\mathrm{width}(\mathbb{B}) : \mathbb{B} \leq_C \mathbb{P}\}$$

$\square$

We can pose the analogous question:

$$\text{Is } \mathrm{cofwidth}(\mathbb{P}) = \mathfrak{bc}(\mathbb{P})?$$

This time Perles's counter example will not work.

Proposition 23. *Let $\mathbb{P} = (P, \leq_P)$ be the Perles partially ordered set, then*

$$\mathrm{cofwidth}(\mathbb{P}) = \mathfrak{bc}(\mathbb{P})$$

Proof. When we take the cofinal version of these two cardinal invariants the invariants collapse to 1. We will see in fact that such partially ordered sets have a scale, and once we prove that, such cofinal linear subordering will need just 1 chain to be covered and so $\mathfrak{bc}(\mathbb{P}) = 1$, since this invariant cannot be less than 1 for non-empty partially ordered sets. In the same way the subordering will have width of $\mathbb{P}$ equal to 1 since all points are comparable and so $\mathrm{width}(\mathbb{P}) = 1$.

To see that $\mathbb{P}$ has a scale it's enough to recall from last chapter that $\mathbb{P}$ is a quasi-linear order and

$$\mathfrak{b}(\mathbb{P}) = \mathfrak{d}(\mathbb{P}) = \omega_1$$

The existence of a scale now follows from Proposition 1.2. □

As the last part of this proof suggests, if we want to build a counter example for this version of the conjecture, we need to look for a quasi-linearly ordered set without a scale and this means that we must see to it that the bounding and dominating numbers of this quasi-linear order are not the same.

The natural candidate would be the partially ordered set with base $P := \omega_2 \times \omega_1$ and $\leq_P := \leq_{\mathrm{prod}_{\omega_2,\omega_1}}$, which we shall call in this chapter the ω_2-*Perles partially ordered set.*

So now we can state the following proposition.

Theorem 9. *Let $\mathbb{P} = (P, \leq_P)$ be the ω_2-Perles partially ordered set, then*

$$\mathrm{cofwidth}(\mathbb{P}) < \mathfrak{bc}(\mathbb{P})$$

Proof. To prove the inequality we will split the proof into two parts.

On the one hand we want to prove that cofwidth($\mathbb{P}$) stays low. Since $\mathbb{P} \leq_C \mathbb{P}$, we have that $\mathrm{cofwidth}(\mathbb{P}) \leq \mathrm{width}(\mathbb{P})$, and so we need to prove that $\mathbb{P}$ contains no infinite antichains.

On the other hand we want prove that $\mathfrak{bc}$ stays high, and that means we must prove that we cannot cover a cofinal subordering of $\mathbb{P}$ with just countably many chains.

To prove the first part let's consider an infinite antichain A, and let's order the subset by the first coordinate as follows:

$$(\alpha, \beta) \leq (\alpha', \beta') \leftrightarrow \alpha \leq_{\omega_2} \alpha'$$

We can define a subordering $(\{(\alpha_n, \beta_n)\}_{n\in\omega}, \leq)$ of order type ω.

Now we show that the sequence $\{\beta_n\}_{n\in\omega}$ forms an infinite strictly decreasing chain in ω_1, and this is a contradiction.

Indeed, let $m, n \in \omega$ be such that $m < n$ so that by definition of $\leq$ we have that $\alpha_m \leq_{\omega_2} \alpha_n$. Since the two points (α_m, β_m) and (α_n, β_n) belong to the antichain A, and in particular are incomparable, we must have that $\beta_m < \beta_n$. The decreasing sequence $(\beta_n)_{n\in\omega}$ now contradicts the well-foundedness of $(\omega_1, \leq_{\omega_1})$.

Finally let's affirm by contradiction that there is a subordering $\mathbb{C}$ that can be covered by countably many chains.

Given the family of chains of the ω_2-Perles partially ordered set $\mathbb{P}$ defined in the following way:

$$C_\alpha = \{(\alpha, \beta) \in P : \beta \in \omega_2\}$$

Where α is a generic element of ω_2, so that this define a family of chains $\{C_\alpha\}_{\alpha\in\omega_2}$ all of size $\aleph_1$.

$$C'_\beta = \{(\alpha, \beta) \in P : \alpha \in \omega_1\}$$

Where β is a generic element of ω_2, so that this define a family of chains $\{C_\beta\}_{\alpha\in\omega_1}$ all of size $\aleph_2$.

This family of chains $\{C_\alpha\}_{\alpha\in\omega_2} \bigcup \{C'_\beta\}_{\alpha\in\omega_1}$ is dominating in the sense that for every chain $C \in \mathbb{P}$ there exist $\alpha \in \omega_2$ where $C \triangleleft C_\alpha$ or $C \triangleleft C'_\alpha$. To see this take the generic chain C and first notice that this chain cannot be cofinal for otherwise ω_2-Perles partially ordered set would have had a scale. That means that there exists a point (α, β) that cannot be dominated and since the chains are unbounded that means that either α will bound the projection of C on the first coordinate or β will bound the projection of C in the second coordinate. In the first case C_α will dominate C and in the second it will be C_β that dominates C.

Now if we have a countable family of chains with a cofinal union in the ω_2-Partially ordered set that means that there is a countable subfamily of $\{C_\alpha\}_{\alpha\in\omega_2} \bigcup \{C'_\beta\}_{\alpha\in\omega_1}$ with the same property.

Let's call α' the supremum of this sequence for the family $\{C_\alpha\}_{\alpha\in\omega_2}$ and β' the supremum of this sequence for the family of $\{C'_\beta\}_{\beta\in\omega_1}$. Then the subordering $(C_{\alpha'} \cup C_{\beta'}, \leq_{prod_{\omega_2,\omega_1}})$ should be cofinal in the ω_2-Perles partially ordered set. But that implies a contradiction since there are no points in either chain that is bigger than the point (α'', β'') such that $\alpha'' > \alpha'$ and $\beta'' > \beta'$. $\square$

4.2 Sierpinski Problem

One of the fundamentally properties of a partially ordered set is its cardinality. A first question that arises is what can we possibly infer on the structure of the partially ordered set just from its size. For example we have the following problem.

Probelem 1. *If $\mathbb{P} = (P, \leq)$ us a partially ordered set then does $\mathbb{P}$ have a chain or an antichain of size $|P|$?*

This question arises from the difficulty to imagine a subset of P of size κ that has neither a chain nor an antichain, and it is here that our definition of generalized chains and antichains will be useful.

Sierpinski indeed found a counter example of size $\aleph_1$ and we present Sierpinski's proof to see how this partial order is constructed.

Theorem 10 ([6])**.** *There exists a partially ordered set $\mathbb{P}$ of size $\kappa > \omega_1$ that does not contain any chains or antichains of size κ.*

Proof. Let $P := \omega_1$ and let $r : P \to \mathbb{R}$ be an injection. Define $<_P$ as follow:

$\forall x, y \in P$ we have that $x <_P y$ if and only if $x <_{\omega_1} y$ and $r(x) <_{\mathbb{R}} r(y)$

If C is a totally ordered subset of P with $<_C = <_P \restriction C$ then we can find a totally ordered subset of the reals which cannot happen, since C is uncountable and that we know reals cannot embed uncountable linear orders that are well ordered.

If C an antichain of $(P, <_P)$ then the reverse of $<_{\mathbb{R}}$ restricted to C gives another uncountable chain in $\mathbb{R}$, which is again a contradiction. □

The Sierpinski partially ordered set does not have chains or antichains of its size, but as we will see in the next theorem we cannot say the same when we talks about quasi-chains and quasi-antichains.

Theorem 11. *The Sierpinski partially ordered set $\mathbb{P} = (P, <)$ has a quasi-antichain or a quasi-chain of size ω_1.*

Proof. We consider the underling set ω_1 and we define recursively a family of chains. To build the first chain C_0 we take a point $x_0 = 0 \in \omega_1$ and define C_0 as the $<$-increasing sequence $\{c_i\}_{i<\lambda}$.

Now suppose we have a sequence of chains $\{C_i\}_{i\in\alpha}$ such that.

$$(\forall i \in \alpha C_i \exists j \in \alpha)((i < j \to C_i \lhd C_j) \vee (\forall i < j < \alpha C_i \perp C_j))$$

. We consider the cases when α is a successor ordinal and when it is a limit ordinal separately.

In the case when α is a successor we need to define a chain C_α that dominates $C_{\alpha-1}$, otherwise we need to ensure that successive chains will be incomparable.

Since $\{c_i\}_{i\in\lambda} \in C_{\alpha-1}$ are increasing also in $<_{\mathbb{R}}$ that mean the size of $C_{\alpha-1}$ is $\aleph_0$. Since $\{c_i\}_{i\in\lambda}$ are increasing in $<_{\omega_1}$ we have that there exists x_α such that $\forall i \in \omega c_i < x_i$.

Now from $x_\alpha \in \omega_1$ we will define another chain C_α and we have either that $C_{\alpha-1}$ can be dominated by some increasing sequence of points or that all the increasing sequences above x_α will be $\leq_{\omega_1}$ dominating and so incomparable with $C_{\alpha-1}$

For the limit case we have to consider the sequence of chains so far and since they are all of size $\aleph_0$ and the underling sets have size ω_1 we have that the union of all this chain will be bounded by same point x that will be the root of the chain for the limit case. And now we can apply the same reasoning that the successor case to define a chain C_α that will dominating the other chains otherwise the possible defined chains will be incomparables.

Now that we have our family of chains $\{C_i\}_{i\in\omega_1}$ and that we know that for each $\alpha, \beta < \omega_1$ such that $\alpha < \beta$ we have $C_\alpha \lhd C_\beta$ or that C_α and C_β are incomparables, we can look at the largest subsequence $A \subset \omega_1$ such that $C_\alpha \lhd C_\beta$ for all $\alpha, \beta \in A$.

We have in this situation two different cases, either the sequence have size $\aleph_1$ then we have found a quasi-chain of size $\aleph_1$ otherwise we can select the same number of chains but this time with the property that they are two by two incomparable, and so we have a quasi-antichain of size $\aleph_1$. $\square$

4.3 The Sauer and Miller Conjecture

In this section we will try to explore a conjecture of Sauer Miller and see how our setting can help to understand it.

The Sauer Miller Conjecture states the following.

Conjecture 1. *Let $\mathbb{P}$ be a partially ordered set such that $\mathfrak{d}(\mathbb{P}) = \kappa$ where κ is singular then*

$$\text{width}(\mathbb{P}) \geq \text{cof}(\kappa)$$

In the second chapter we stated the following characterization of the dominating number.

$$\mathfrak{d}(\mathbb{P}) = \mathfrak{hl}(\mathbb{P})\mathfrak{bc}(\mathbb{P})$$

The following result now restricts the class of the partially ordered sets where the conjecture is false.

Theorem 12. *Let $\mathbb{P}$ be a partially ordered set such that $\mathfrak{hl}(\mathbb{P}) = \kappa$, where κ is a singular cardinality, then $\mathbb{P}$ has an antichain of size* $\text{cof}(\kappa)$.

Proof. If $\mathfrak{hl}(\mathbb{P}) = \kappa$ and κ is a singular cardinal then for the definition of linear height we cannot have a chain of a singular cardinality so that there must exist a sequence of regular chains $\{C_i\}_{i \in \lambda}$ such that $sup\{|C_i| : i \in \lambda\} = \kappa$ and so $\lambda \leq \text{cof}(\kappa)$.

The sequence defined in the previous paragraph can be refined so that for all $i \neq j$ we have that $|C_i| \neq |C_j|$ and such that $|C_0| > \text{cof}(\kappa)$.

By Proposition 1 of the previous chapter we can see to it that for all $i \neq j$ the couple of chains C_i, C_j are incomparable for the dominating order, for otherwise the chains should have the same cardinality and that contradict our refinement.

We can conclude that we have a quasi-antichain of size $\text{cof}(\kappa)$ of chains of size greater the $\text{cof}(\kappa)$ and so we can apply lemma and obtain an antichain of size $\text{cof}(\kappa)$. $\square$

4.4 Conclusion

We have restricted the Sauer Miller conjecture under less general hypothesis and we could be tempted now to reformulate it as:

False Conjecture 1. *Let $\mathbb{P}$ be a partially ordered set such that $\mathfrak{bc}(\mathbb{P}) = \kappa$*

$$\text{width}(\mathbb{P}) \geq \text{cof}(\kappa)$$

We already know from Proposition 3 of the first section of this chapter that $\mathfrak{bc}(\mathbb{P}) = \kappa$ is not enough to find a big antichain. But that brings us to consider, as in the case of the original Dilworth theorem, the class of counter examples. That class looks like the class of generalized quasi-linear order and we know from Proposition 4 of the third chapter, that this class cannot be dominated by a subset of singular cardinality. So if that equality reveals itself to be true then the Sauer Miller Conjecture follow since the quasi-partially ordered sets where the false conjecture 1 fails is the one that cannot have a singular dominating number.

So now we have a new conjecture to explore.

Conjecture 2. *Let $\mathbb{P}$ be a partially ordered set such that*

$$\text{width}(\mathbb{P}) < \mathfrak{bc}(\mathbb{P})$$

then $\mathbb{P}$ is a generalized n-quasi-linear order for some finite number n.

Chapter 5

Cardinal invariants above the continuum

Introduction

In the first chapter we will pass from our general approach to a more specific and well-known one, putting the old definition in this new context. The results in this part will be motivated by two problems stated by Monk in [15] and by Shelah and Cummings in [3].

On the one hand we will "come full circle" for the invariant cardinals we started in the first chapter. We started that chapter with the definitions of bounding and dominating numbers for the continuum and from that definition we progressed to a more abstract one. Now we will return to the definition motivating that passage.

In the second part, on the other hand, we will concentrate on the dominating number and tackle two problems that address the generalization from another perspective, namely that of considering different orders defined by different ideals of the fields.

Summary

In the first section we prove Lemma 5. In the second section we solve a problem posed by Monk [1] in Theorem 13 and we give a partial solution of a question Shelah-Cummings in Theorem 14.

5.1 The reduced power of a partial orders

In this section we are going to fill in the gap between the general setting of a quasi-partial order and the special case of a cardinal invariant defined on and above the continuum that will be the object of interest in the first part of this chapter.

Before starting, we recall some basic definitions.

Definition 27. *Let I be a family of subsets of A. We will call I an* ideal *if and only if it satisfies the following properties:*

- *if $a, b \in I$, then $a \cup b \in I$*
- *if $b \subseteq a \subseteq A$ and $a \in I$, then $b \in I$*

If $A \notin I$ then we will call I a proper ideal.

We will consider now the power of partial orders on a set A. First taking an ideal I we will define the more general reduced power of a quasi-partial order.

[1]There should be a theorem of van Douwen published prior to Monk's paper that solve the problem but which reference we were unable to find again.

Definition 28. *Let $\mathbb{P} = (P, \leq_P)$ be a quasi-partial order and A a set. Consider I an ideal of A; We define the* reduced power *of $\mathbb{P}$ on A and I to be $\Pi^I_A \mathbb{P} = (\prod_{a \in A} P_a, <_{\Pi^I_A \mathbb{P}})$ where $P_a = P$ $\forall a \in A$ and if $p, q \in \Pi^I_A \mathbb{P}$ then*

$$p \leq_{\Pi^I_A \mathbb{P}} q \quad \textit{if and only if} \quad \{a \in A \ : \ p(a) \not\leq_{\Pi^I_A \mathbb{P}} q(a)\} \in I$$

When we take the ideal $\{\emptyset\}$ we will say that Π^I_A is the power *of $\mathbb{P}$.*

Sometimes we can define the reduced power by using the dual concept of a filter. When this is the case then we will write F^* for the dual ideal of the filter F.

At this point we will concentrate on powers of linear orders and we will show that we can reduce the investigation of this special case to the study of bounding and dominating numbers for larger cardinals.

To achieve this result we will need the following three lemmas. In the first, we will pass from the reduced power of a linear order to the reduced power of an ordinal. Then we will pass from the reduced power of an ordinal to the reduced power of a regular cardinal. Finally, we will pass from the reduced power on a general set to the reduced power on a cardinal.

Lemma 5. *Let $\mathbb{P}$ be a linear order, A a set and I an ideal on A. Then there exists an ordinal α such that we can embed cofinally $\Pi^I_A \alpha$ into the reduced power of $\mathbb{P}$ on A and I.*

Proof. We know by Lemma 3 that we can cofinally embed in $\mathbb{P}$ a wellordering that is isomorphic to an ordinal, that will be our α. So let f be a cofinal embedding of α in $\mathbb{P}$, and extend it to a cofinal embedding F of $\Pi^I_A \alpha$ in $\Pi^I_A \mathbb{P}$, taking $F(\{\beta_a\}_{a \in A}) = \{f(\beta_a)\}_{a \in A}$.

It easy to see that this is a cofinal embedding; We will just check here its cofinality. So let $g \in \Pi^I_A \mathbb{P}$ and consider $\{g(a)\}_{a \in A}$. We know that for each element $f(a)$ there exists an ordinal β_a less than α such that $\beta_a > f(a)$. So the function defined by $\{\beta_a\}_{a \in A}$ in $\Pi^I_A \alpha$ will be greater than f. □

Lemma 6. *[14] Let α be an ordinal, A a set and I an ideal on A. Then there exists a cofinal embedding between $\Pi^I_A \alpha$ and $\Pi^I_A \operatorname{cof}(\alpha)$.*

Lemma 7. *[14] Let $\mathbb{P}$ be a partial order, A a set and I a ideal on A. Let κ be the cardinality of A and f a bijection from A to κ. Finally, let the ideal E be defined on κ as: $B \subseteq \kappa$ is in E if and only if $f^{-1}(B)$ is in I. Then $\Pi^I_A \mathbb{P}$ is isomorphic to $\Pi^E_\kappa \mathbb{P}$.*

From the general setting of a quasi-partial order we can finally state the definitions which will be the object of the second part of this study namely the bounding number and the dominating number for larger cardinals.

Definition 29. *Let κ and λ be two cardinals with λ regular. Let I be an ideal on κ. We define*

$$\mathfrak{d}(\kappa, \lambda, I) = \mathfrak{d}(\Pi^I_\kappa \lambda)$$

and

$$\mathfrak{b}(\kappa, \lambda, I) = \mathfrak{b}(\Pi^I_\kappa \lambda).$$

We will denote $\mathfrak{d}(\kappa, \kappa, I)$ by $\mathfrak{d}_I(\kappa)$ and $\mathfrak{b}(\kappa, \kappa, \mathrm{Fin})$ by $\mathfrak{b}(\kappa)$. Where Fin is the ideal of κ of all the finite subset of κ

Now we can "come full circle" by properly defining the two invariants that historically motivated the generalizations we have taken into account in this thesis.

$$\mathfrak{d} = \mathfrak{d}(\omega, \omega, Fin)$$

$$\mathfrak{b} = \mathfrak{b}(\omega, \omega, Fin)$$

5.2 Problems on the dominating cardinal above the continuum

Now that we have defined the cardinal invariants for the well-known space of functions from one cardinal to another, with the order depending on the ideal we are quotienting, let's look at some problems on the dominating cardinal invariant.

5.2.1 Monk Problem

Most of the problems on this subject are concerned with how far the invariant $\mathfrak{d}(\kappa)$ is from the same invariant defined on a different ideal.

For example the problem that Monk states in [15] asks the following:

Probelem 2 (Monk)**.** *Let I be the ideal $\{\{\}\}$, can we have $\mathfrak{d}_I(\kappa) > \mathfrak{d}(\kappa)$ for some κ regular?*

The following proposition answers negatively the above question:

Theorem 13. *Let I be the ideal $\{\{\}\}$, $\mathfrak{d}_I(\kappa) = \mathfrak{d}(\kappa)$ for each κ regular.*

Proof. It is clear that $\mathfrak{d}_I(\kappa) \geqslant \mathfrak{d}(\kappa)$ as any dominating subset in $(\kappa^\kappa, \leqslant)$ is dominating in $(\kappa^\kappa, \leqslant^*)$.

So now we want prove that $\mathfrak{d}_I(\kappa) \leqslant \mathfrak{d}(\kappa)$. To prove this we need to introduce some notation. I will denote by γ^* the constant function from κ to κ taking each element of κ to the ordinal γ. I will denote by $\mathcal{C}$ the family of constant functions, so that with our notation $\mathcal{C} = \{\gamma^* \ : \ \gamma < \kappa\}$. Now let $f, g \in \kappa^\kappa$. We consider the pairwise sum of functions $f + g$ i.e.

$(f+g)(\beta) = f(\beta)+g(\beta)$ for each $\beta \in \kappa$. Hence $(f+g)(\beta) \geqslant \max\{f(\beta), g(\beta)\}$ for each $\beta \in \kappa$; we note also that $f+g$ is well-defined as the sum of any two ordinals less than κ is again less than κ.

Let D be a dominating subset of $(\kappa^\kappa, \leqslant^*)$ and let $D^1 = \{f+g \ : \ f \in D,\ g \in \mathcal{C}\}$. We note that the size of D is the same as the size of D^1. Indeed $|D| \leqslant |D^1| \leqslant |D| \cdot \kappa = |D|$, where the last inequality follows from $\mathfrak{d}(\kappa) > \kappa$ and the first by taking $g = 0^*$ for $f \in D$. So $|D^1| = \mathfrak{d}(\kappa)$.

It remains to prove that D^1 is a dominating subset of $(\kappa^\kappa, \leqslant)$. So now we take a $f \in \kappa^\kappa$. We know, as D is dominating in $(\kappa^\kappa, \leqslant^*)$, that there exists $g \in D$ such that $(\exists \beta^* < \kappa \forall \alpha > \beta^*)\ (g(\alpha) \geqslant f(\alpha))$ Now we consider the set $\{f(\alpha) \ : \ \alpha \leqslant \beta^*\}$. As κ is regular and the size of this set is strictly less than κ, this set has sup strictly less than κ. We denote this sup by δ. We now take an ordinal $\gamma > \delta$ and consider the function γ^*. We finally claim that $g + \gamma^* \geqslant f$. To see this, we need only remember that this function is such that $(g+\gamma^*)(\alpha) \geqslant \max\{g(\alpha), \gamma\}$ for each $\alpha < \kappa$. Then for $\alpha \geqslant \beta^*$ we have $(g+\gamma^*)(\alpha) \geqslant \max\{g(\alpha), \gamma\} \geqslant g(\alpha) \geqslant f(\alpha)$, while for $\alpha \leqslant \beta^*$ we have $(g+\gamma^*)(\alpha) \geqslant \max\{g(\alpha), \gamma\} \geqslant \gamma > \delta \geqslant f(\alpha)$. $\square$

5.2.2 Some important ideals

Other than Fin there are other very important ideals when we are considering reduced products. For example, the not stationary ideal, denoted NS. The not stationary ideal of a cardinal κ is defined from a special family of subsets of κ. To define this family, we need the notions of a subset being unbounded and closed. We already are familiar with the first. For the second, we have the following definition.

Definition 30. *Let κ be a cardinal. A subset C of κ is said to be* closed *if* *and only if for all limit ordinals $\alpha < \kappa$ we have that $C \cap \alpha$ is unbounded in α.*

So now we can define the family of subsets we are interested in.

Definition 31. *A subset of a cardinal κ is a* club *if it is closed and unbounded in κ.*

Finally the not stationary ideal can be defined as follows.

Definition 32. *Let κ be a cardinal with uncountable cofinality. A stationary subset of κ is a subset S having non-empty intersection with each club subset of κ. The* n*ot stationary ideal for κ is the family of all not stationary subsets of κ.*

We denote the not stationary ideal as NS_κ or just NS when the context is clear.

5.2.3 The Cummings–Shelah Problem

We will close this chapter and the part of this text dealing with cardinal invariants with this section in which we will state and try to address the question that Shelah and Cummings raise in [3].

Probelem 3 (Shelah, Cummings)**.** *Do we have $\mathfrak{d}_{NS}(\kappa) = \mathfrak{d}(\kappa)$ for every regular κ?*

Here $\mathfrak{d}_{NS}(\kappa) = \min\{|D| \ : \ D$ is dominating in $(\kappa^\kappa, \leqslant_{NS})\}$ and $f, g \in \kappa^\kappa$ satisfy $f \leqslant^* g$ if and only if $\{\alpha \in \kappa \ : \ g(\alpha) > f(\alpha)\} \in NS(\kappa)$.

Proposition 24 (Shelah, Cummings)**.** $\mathfrak{d}_{NS}(\kappa) \leqslant \mathfrak{d}(\kappa) \leqslant (\mathfrak{d}_{NS}(\kappa))^\omega$

Let $\mathfrak{CB}(\kappa) = \{|B| \ : B$ is a base for $\mathcal{C}(\kappa)\}$ where $\mathcal{C}(\kappa)$ is the family of the club set of κ.

Theorem 14. *If* $\mathfrak{CB}(\kappa) = \mu$, *then* $\mathfrak{d}_{NS}(\kappa) \leqslant \mathfrak{d}(\kappa) \leqslant \mu \cdot \mathfrak{d}_{NS}(\kappa)$

Proof. On the one hand, it is clear that a dominating subset of $(\kappa^\kappa, \leqslant^*)$ is a dominating subset of $(\kappa^\kappa, \leqslant_{NS})$, so that the first inequality follows: $\mathfrak{d}_{NS}(\kappa) \leqslant \mathfrak{d}(\kappa)$.

On the other hand, let $f \in \kappa^\kappa$ and let $C \in \mathcal{C}(\kappa)$. Enumerate C as $C = \{c_\alpha : \alpha < \kappa\}$. We can now define $f_C \in \kappa^\kappa$ where $\forall c \notin C$ $f_C(x) = f(x)$ and $\forall c \in C \exists \alpha < \kappa$ such that $c = c_\alpha$, then $f_C(c_\alpha) = f(c_{\alpha+1})$.

If $A \in [\kappa]^\kappa$ and A is enumerated as $A = \{a_\alpha : \alpha\}$. We can define the order preserving function $f^A \in \kappa^\kappa$ where $f^A(\alpha) = a_\alpha$. Given a function $g \in \kappa^\kappa$ we denote $g^* = f^{g''\kappa}$, so that g^* is order preserving and $\forall \alpha \in \kappa$ $g(\alpha) \leqslant g(\alpha)$.

We consider D a dominating subset of $(\kappa^\kappa, \leqslant_{NS})$ and B a base of $\mathcal{C}(\kappa)$ of cardinality $\mathfrak{CB}(\kappa) = \mu$. We define now the following family of functions $D' = \{f^*_C \ : \ f \in D$ and $C \in B\}$. We claim that D' is a dominating subset of $(\kappa^\kappa, \leqslant^*)$.

To see this, take $g \in \kappa^\kappa$. Then $\exists C \in \mathcal{C}(\kappa)$ and $\{\alpha \ : \ f(\alpha) \geqslant g(\alpha)\} = C$. We can write the enumeration of the club as $C = \{c_\alpha : \alpha \in \kappa\}$. Let now $\gamma > c_1$ and define $c_\beta = \min(C \setminus \gamma + 1)$. Then

$$g(\gamma) \leqslant g^*(\gamma) \leqslant g(c_\beta) \leqslant f(c_\beta).$$

As C is a club, c_β must be a successor. Hence we can write $c_\beta = c_{\alpha+1}$ where $c_\alpha \leqslant \gamma \leqslant c_{\alpha+1}$ and continue our inequality with

$$f(c_{\alpha+1}) = f_C(c_\alpha) \leqslant f^*_C(c_\alpha) \leqslant f^*_C(\gamma).$$

We can now conclude by considering that the cardinality of this dominating subset is $|D'| = \mathfrak{CB}(\kappa) \cdot |D| = \mu \cdot \mathfrak{d}_{NS}$. We obtain the second inequality:

$\mathfrak{d}(\kappa) \leqslant \mu \cdot \mathfrak{d}_{NS}(\kappa)$. $\square$

Chapter 6

Exploring Cardinals with Games

Introduction

In this last chapter we want to use a different approach in studying the relationship between the cardinality of a set and the structures that can be built on it.

We will look at some theoretical combinatorial properties of the game that can be strongly influenced by the cardinality and that can even, how we shall see in the last section, determine the cardinality.

We will start the chapter by defining the rules of our game. Then we will give a new simple proof of a result of Abraham and Schipperus in [2] and finally state a characterisation of the continuum through the definition on Model Games.

In the rest of the chapter we use the following notation: with the symbol $\mathfrak{c}$ we indicate the cardinal arithmetical result of $2^{\aleph_0}$ and with the abbreviation

CH we indicate the continuum hypothesis $\mathfrak{c} = \aleph_1$

Summary

After introducing the basic properties and definitions in the first section, we prove the main result of the chapter in Theorem 16, and give a new proof of old results in Theorem 15 and Corollary 3.

6.1 An Introduction to Model Games

Model games are set theoretical games on infinite sets introduced by Boris Model in [13].

The aim of this section is to define the rules of the games and state some simple fact about them.

6.1.1 The Rules

To define the games we need an arbitrary set X of cardinality bigger than n and a subset of X of n elements that we denote by B_0.

We begin a game $MG(X, n)$ with two players in the following way: at the stage 0 the first player, who we refer to as Remover, removes an element b_0 from the base B_0, the second player, who we refer to as Adder, adds an element from $X \setminus (B_0 \setminus \{b_0\})$ to form the new base B_1.

The play continues in this way: at the stage $k < \omega$, Remover removes b_k from the base B_k and Adder adds an element from $X \setminus (B_k \setminus \{b_k\})$ to form the new base B_{k+1}.

A fixed point in this game is an element $x \in X$ such that there exists m such that for all $k > m$ we have that $x \in B_k$.

For $t \leq n$ a t-winning strategy for Remover is a strategy that guarantees at most t fixed points.

If $t_1 < t_2$ and Remover has a t_1-winning strategy than he has a t_2-winning strategy, so that we can define

$$c(X,n) = \min\{t : \text{ Remover has a } t\text{-winning strategy for } MG(X,n)\}.$$

If Remover has all the informations of the game before his turn then he has a 0-strategy and the game does not have any interest. So we will consider games where the only information that Remover has is at which stage he is.

Note that the definition is justified by the fact that Remover have always a n-winning strategy which implies the following fact.

Proposition 25. *For every set X and natural number n we have that*

$$c(X,n) < n.$$

6.2 Model Games On Countable Sets

To understand how the games work, we prove the following simple fact about Model Games and countable sets.

Proposition 26. *If X is a countable set, then $c(X,n) = 0$ for all $n < \omega$.*

Proof. If X is countable set, then there exist a bijection between any infinite subset of $\mathbb{N}$ to X. We can choose an infinite partition of $\mathbb{N}$ such that by means of the bijection we can define a function f with the properties that for all $x \in X$ and $k < \omega$ there exist an $m > k$ such that $f(m) = x$. Now that we have this function, we can define a strategy in the following way: at stage m Remover looks at the value of $f(m)$. If it is in B_m then he removes $f(m)$. Otherwise, he picks whatever he wants among the points of B_m. It follows from the properties of f that such a strategy is a 0-winning strategy. $\square$

A first interesting question is if there exists an uncountable set such that for some $n < \omega$ we have that $c(X, n) = 0$.

Before answering that we state another fact about Model games which follows directly from the definitions

Proposition 27. *Let $n_1 < n_2$, then $c(X, n_1) \leq c(X, n_2)$.*

Proof. Consider a t-winning strategy for $MG(X, n_2)$. We want to find a t-winning strategy for $MG(X, n_1)$.
First of all we fix a linear order $\leq$ for the set X. Remover at any stage m looks at the n_1 points of the games, adds artificially $(n_1 - n_2)$ points and looks at the moves he would have done in the case of the t-winning strategy for $MG(X, n_2)$.

If this strategy brings him to remove one of the artificially added points, then Remover picks the smallest point among the firsts n_1 following the linear order $\leq$. Otherwise he does the same move.

This strategy must been winning for the $MG(X, n_2)$ otherwise we could build a counterexample for the case $MG(X, n_1)$. Indeed if we consider the sequence of the n_2 point just build and apply the original strategy, then the strategy will have the same point fixed since the choice of Remover for such points in strategy remains the same. $\square$

Thus we can first of all answer the following question: Are there an uncountable set X such that $c(X, 2) = 0$? The answer is negative.

Proposition 28. *If* $c(X, 2) = 0$ *then* X *is a countable set.*

Proof. Consider x a point of X. If Remover has a 0-winning strategy, then at some stage k_0 he removes x, a whatever other point y in the base set, as otherwise Adder could fix x adding at each stage such a y. For the same reason the same will happen at stage $k_1 > k_0$ and so on for an infinite subset K_x of $\mathbb{N}$.

Consider the family of subset K_x for $x \in X$. Since the strategy must choice a unique point of the same base at some fixed stage, then each of these subsets must be pairwise disjoint. Thus we can conclude that X must be countable. □

After these proposition we can note that the function c evaluated on two point games give some information on the countability and uncountability of the set X.

In fact we have the following result.

Proposition 29. *Let* X *be an arbitrary set, then*

- $c(X, 2) = 0$ *if and only if* $|X|$ *is countable;*
- $c(X, 2) = 1$ *if and only if* $|X|$ *is uncountable.*

Proof. It will be enough to prove the first point, since $c(X, 2) \leq 1$ thanks to Fact 25. One direction come from Theorem 26 since is a special case of theorem. The other direction is equivalent to Theorem 28. □

6.3 Model games for three point on uncountable sets

Since for a games $MG(X, n)$ there is always an $(n-1)$-winning strategy, the first interesting case for uncountable set is the case $MG(X, 3)$.

The following result can be found in Abraham and Schipperus [2], but we report it with a different proof.

Theorem 15. *If $\aleph_0 < |X| \leq 2^{\aleph_0}$, then $c(X, 3) = 1$.*

Proof. Since $|X|$ is uncountable, we need to prove that $c(X, 3) < 2$, that is, Remover have a 1-winning strategy.

Since $|X| \leq 2^{\aleph_0}$ there exists an injective function f from X to $\mathbb{R}$. Let g be an enumeration with repetitions of $\mathbb{Q}$. The strategy will be the following: at the stage m with $B_m = \{x, y, z\}$, look at $g(m)$ and at the ordered set $(\mathbb{R}, <)$. If all of $f(x), f(y), f(z)$ are above or under $g(m)$, then remove the inverse image of the greater of the three, otherwise $g(m)$ isolates one of the three points, and Remover removes the inverse image of the isolated point under g.

Consider now two points x_1, x_2 at the stage m . We want to prove that at some stage one of these two points will be removed by Remover. Following the strategy we consider the image of the points $f(x_1)$ and $f(x_2)$. Since these are two points of the real line there will be a rational point q between them.

We can consider now the play at the stage $\min\{g^{-1}(q)\} \setminus m + 1$ so that we are sure that one of the points $f(x_1)$ and $f(x_2)$ is isolated, and hence one of the points among x_1 an x_2 will be removed. □

We are left with the case where the set X has cardinality above the continuum. In this case as shown by Abraham and Shipperus [2], we have the following result. We report the proof for the reader convenience.

Proposition 30 ([2])**.** *If* $|X| > 2^{\aleph_0}$*, then* $c(X, 3) = 2$*.*

Proof. We recall a classical result by Erdös: if we colour pairs of uncountable many elements with countably many different colours, we can extract an infinite subset such that every pair has the same colour; in symbols

$$(2^{\aleph_0 +}) \to (\omega_1)^2.$$

Now suppose for contradiction that $c(X, 3) = 1$. Then we can define a colouring as follows since in this case it is not possible that a couple of points remains fixed at all stages, that means that for all couples (a, b) there exists $n < \omega$ such that for all $x \in X$ at the stage n, from the set $B_n = \{a, b, x\}$ Remover picks a or b. Therefore for each couple we can give the color. This coloring cannot have a homogeneous triplet, contradicting Erdös Theorem. $\square$

After these proposition we can note that the function c evaluated on tree point games gives some information on the countability or, in the case of uncountability, tells us if the set is above or under the continuum of the set X.

In fact we prove the following results.

In the case of Model games on tree points we have that:

Theorem 16. *Let* X *be an arbitrary set, then*

- $c(X, 3) = 0$ *if and only if* $|X|$ *is countable;*
- $c(X, 3) = 1$ *if and only if* $\aleph_0 < |X| \leq \mathfrak{c}$*;*

- $c(X, 3) = 2$ *if and only if* $|X| > \mathfrak{c}$.

Proof. The first assertion comes from Proposition 26 and from Proposition 28 and Proposition 27.

To prove the second one for one direction we will use straightforward the Proposition 15. For the other we will assume for absurd that the cardinality is strictly greater than the continuum and we will use Proposition 30 to spot the contradiction.

The third point will follows from the first two. □

As Corollary we have one of the surprising results of Abraham and Shipperus [2].

Corollary 3. $c(\omega_2, 3) = 2$ *if and only if* CH

Proof. If $c(\omega_2, 3) = 2$ then by Theorem 16 we have that $\omega_2 > \mathfrak{c}$ and since we have that $\omega_2 > \omega_1$, for the uncountability of $\mathfrak{c}$ we get that $\mathfrak{c} = \omega_1$ that is CH.

On the other hand the continuum hypothesis $\mathfrak{c} = \omega_1$ implies that $\omega_2 > \mathfrak{c}$ and so $c(\omega_2, 3) = 2$ by Theorem 16. □

This latter fact, which concludes this chapter, gives a sense of the interest in exploring this kind of games.

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