Better-Quasi-Orders: Extensions and Abstractions

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For my mother Angela McKay, my father Stuart McKay, and especially for my late grandfather, Prof. Kenneth Burton (FRS).

"Logic is a little tweeting bird, chirping in a meadow. Logic is wreath of pretty flowers that smell bad. Are you sure your circuits are registering correctly? Your ears are green!"

— Mr Spock (2268)

Abstract

We generalise the notion of σ-scattered to partial orders and prove that some large classes of σ-scattered partial orders are better-quasi-ordered under embeddability. This generalises theorems of Laver, Corominas and Thomassé regarding σ-scattered linear orders, σ-scattered trees, countable pseudo-trees and N-free partial orders. In particular, a class of *countable* partial orders is better-quasi-ordered whenever the class of indecomposable subsets of its members satisfies a natural strengthening of better-quasi-order.

We prove that some natural classes of structured σ-scattered pseudo-trees are better-quasi-ordered, strengthening similar results of Kříž, Corominas and Laver. We then use this theorem to prove that some large classes of graphs are better-quasi-ordered under the induced subgraph relation, thus generalising results of Damaschke and Thomassé.

We investigate abstract better-quasi-orders by modifying the normal definition of better-quasi-order to use an alternative Ramsey space rather than exclusively the Ellentuck space as is usual. We classify the possible notions of well-quasi-order that can arise by generalising in this way, before proving that the corresponding notion of better-quasi-order is closed under taking iterated power sets, as happens in the usual case.

We consider Shelah's notion of better-quasi-orders for uncountable cardinals, and prove that the corresponding modification of his definition using fronts instead of barriers is equivalent. This gives rise to a natural version of Simpson's definition of better-quasi-order for uncountable cardinals, even in the absence of any Ramsey-theoretic results. We give a classification of the fronts on $[\kappa]^\omega$, providing a description of how far away a front is from being a barrier.

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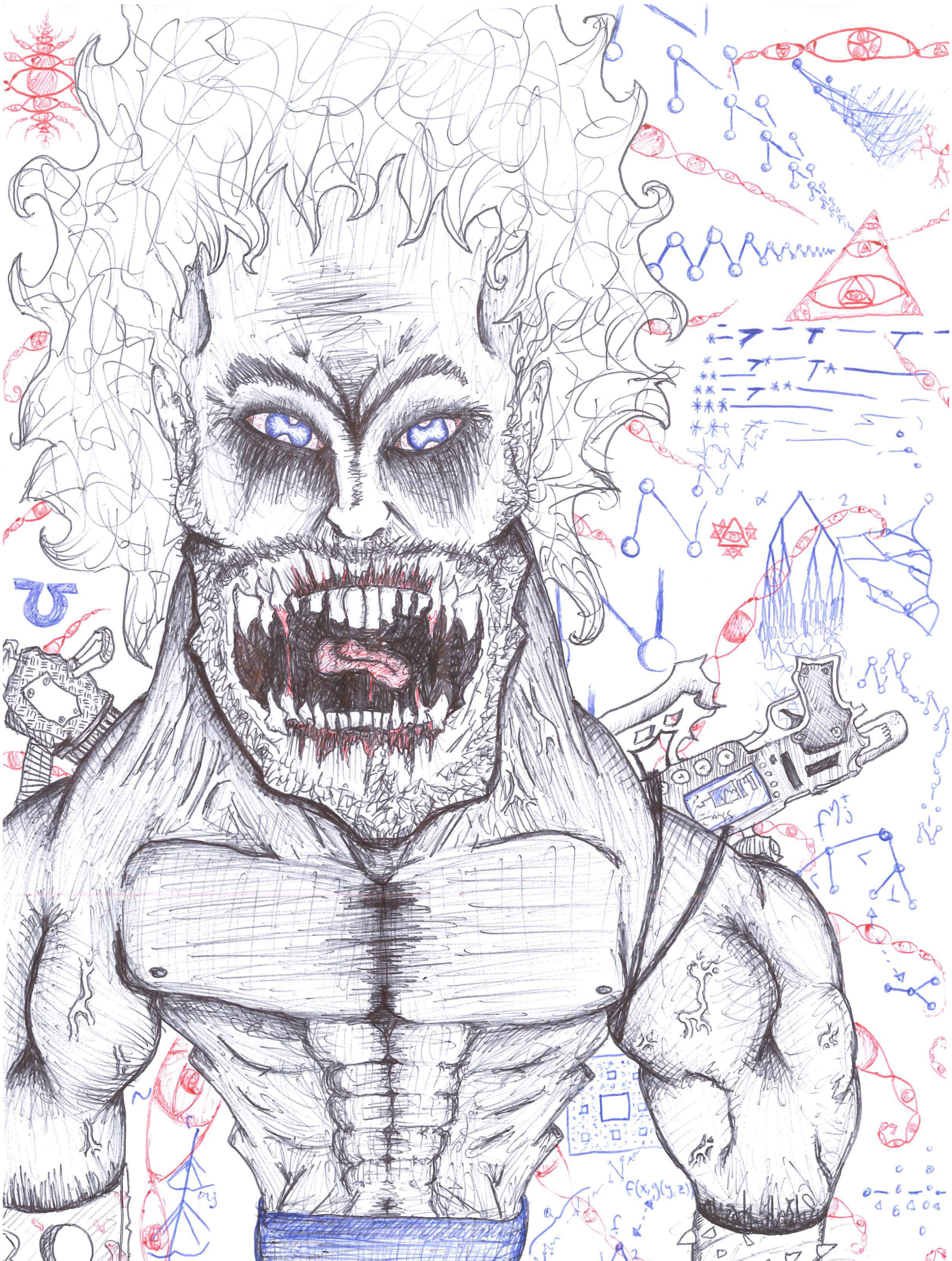

Figure 0: An appropriate beginning to a thesis.

Chapter 1

Introduction

Mathematical objects come in all shapes and sizes. There is such a variety that in order to make sense of these objects, it can often be useful to try to compare them according to their relative complexity. This essentially amounts to putting a *quasi-order* (i.e. a transitive reflexive relation) on a class of mathematical objects; one object is below another if it is of less or equal complexity.

When ranking objects by complexity, the notion of well-order is both natural and fundamental. Indeed, mathematicians will often give an ordinal ranking for the complexity of objects in a given class. However for some types of object (for example finite graphs or partial orders) an ordinal ranking may not be so natural; it may not in fact make sense for any two objects to be comparable with respect to an intuitive notion of complexity.

So in order to compare complexity of objects, we wish to generalise the notion of well-order to non-linear quasi-orders. One way to do this could be to simply consider *well-founded* quasi-orders, i.e. those with no infinite descending sequences. However this notion has a slightly different flavour, as we lose the property that there are at most finitely many minimal elements of any subset of the order. Indeed we could have infinitely many (non-comparable) least-complex elements.

Another natural way to generalise the notion of well-order is to not only forbid infinite descending sequences, but also infinite pairwise incomparable subsets or *antichains*. This is the definition of a *well-quasi-order* (wqo). Generalising in this way preserves some desirable properties of well-ordering, for example: any subset of a wqo has finitely many minimal elements; any infinite sequence of distinct elements of a wqo has an infinite strictly ascending subsequence; and the power set of a wqo is well-founded by the order $A \leqslant B$ if there is a function $f : A \to B$ with $a \leqslant f(a)$ for all $a \in A$. For these reasons and more, well-quasi-order is often a desirable property for a relational notion of complexity. The well-quasi-order concept was discovered multiple times independently by many different authors, for a detailed history and motivation of wqo theory see [27].

One useful and interesting property of wqos is that they can be used to construct other wqos. For example, if Q is a quasi-order, then consider the new quasi-order consisting of the class of finite sequences of elements of Q,

$$Q^{<\omega} = \{\langle q_i : i < n\rangle : n \in \omega, q_i \in Q\},$$

ordered by $\langle p_i : i < n\rangle \leqslant \langle q_i : i < m\rangle$ iff there is some injective increasing $f : n \to m$ such that for all $i < n$, $p_i \leqslant q_{f(i)}$. A theorem of Higman [20] states that if Q is wqo, then so is $Q^{<\omega}$ (see also [26]).

Well-quasi-order theory alone however is not sufficient for transfinite constructions. If Q is a quasi-order, then consider the class of transfinite sequences of elements of Q,

$$\tilde{Q} = \{\langle q_i : i \in \gamma\rangle : \gamma \in \mathrm{On}, q_i \in Q\},$$

ordered by $\langle p_i : i \in \gamma\rangle \leqslant \langle q_i : i \in \delta\rangle$ iff there is some injective increasing $f : \gamma \to \delta$ such that for all $i \in \gamma$, $p_i \leqslant q_{f(i)}$. If Q is well-quasi-ordered, then the same does not always hold for $\tilde{Q}$. Thus for transfinite constructions such as this one to be wqo, we require a stronger condition on the quasi-order Q.

The required stronger notion is that of a *better-quasi-order* (bqo). Developed by Nash-Williams in the 1960s [39], this is a strengthening of well-quasi-order for which many desired infinite constructions are wqo, when the assumption of wqo alone is insufficient. As alluded to, a theorem of Nash-Williams [40] states when Q is bqo, so is $\tilde{Q}$ (see also [38, 53]). In fact, Pouzet gives a characterisation of bqo in [45], that Q is bqo iff $\tilde{Q}$ is wqo. By a theorem of Rado [47], if Q is wqo and $Q^{<\omega}$ is not wqo, then Q contains an isomorphic copy of *Rado's poset* $\mathfrak{R}$ (see Definition 6.2.37 and Figure 6.4). Indeed the definition of bqo strives to forbid this from happening, as well as similar transfinite versions (see [38, 15, 42]).

Fortunately the majority of 'natural' wqo classes turn out to also be bqo (as noted by Kruskal in [27]), although the notions are not equivalent (see [47, 31, 38]). At first glance the definition of bqo is not necessarily intrinsically beautiful, but it can be seen as an infinitary strengthening of wqo and is an invaluable tool which can be used to show that certain classes are wqo. Indeed, it is often much easier to prove that a class is bqo, than it is to prove the weaker property of wqo directly. The theory of bqos may also be of interest to the reverse mathematician, with theorems such as Nash-Williams' on transfinite sequences [40] requiring strong subsystems of second order arithmetic for their proof [35, 36, 50]. For some introductory background reading on bqo theory, see [38, 44, 53].

1.1 Constructing better-quasi-orders.

In his 1948 paper [16], Fraïssé conjectured that the set of countable linear orders is wqo under *embeddability*. That is, for two linear orders L and L', we have $L \leqslant L'$ iff there is a function $f : L \to L'$ such that for all $a, b \in L$, $a < b$ iff $f(a) < f(b)$. Laver famously proved this conjecture in [30]. A relatively simple account of one proof of Fraïssé's conjecture is given by Simpson in [53] and another is given by Pouzet in [44]. Pouzet's version of this proof is a prototype example of a more general method of proving that a class is bqo, that we generally refer to as *constructing* bqos.

The rough idea behind this proof of Fraïssé's conjecture is as follows. The set of countable linear orders can be split into two subsets: the *scattered* orders (i.e. those linear orders that do not embed the rational numbers $\mathbb{Q}$) and the linear orders into which $\mathbb{Q}$ embeds. So since every countable linear order embeds into $\mathbb{Q}$, the quasi-order of countable linear order types consists of points for every scattered linear order, and infinitely many points larger than every scattered order which are all order-equivalent[1] under the embeddability

[1]That is $a \leqslant b$ and $b \leqslant a$.

ordering. Thus it only remains to prove that the scattered linear orders are wqo, because a descending sequence or antichain can only contain at most one point order-equivalent to $\mathbb{Q}$.

Now a famous theorem of Hausdorff [19] is used. This theorem states that the class of scattered linear order types, as well as having their *external* definition (of not embedding $\mathbb{Q}$) can also be defined *internally*. Let $\mathscr{S}_0$ consist only of the singleton linear order type and for $\alpha > 0$, let $\mathscr{S}_\alpha$ be the class of all well-founded lexicographic sums

$$L_0 + L_1 + ... + L_\beta + ... \qquad (\beta < \delta)$$

and converse well-ordered lexicographic sums

$$... + L_\beta + ... + L_1 + L_0 \qquad (\beta < \delta)$$

where every L_β is a member of $\bigcup_{\gamma<\alpha} \mathscr{S}_\gamma$. Then Hausdorff's theorem states that $\mathscr{S} = \bigcup_\alpha \mathscr{S}_\alpha$ is precisely the class of scattered linear order types. Thus each element $L \in \mathscr{S}$ can be represented by a well-founded tree, labelled by ordinals and reversed ordinals, which describes how L is built in this hierarchy, by recording the ordinals and reversed ordinals used in its construction. Furthermore, if the trees embed, then so will the linear orders. Thus the statement of Fraïssé's conjecture reduces to knowing that these well-founded trees are wqo. At this point we can invoke a well-known theorem of Kruskal stating that these trees are indeed wqo [26]. This idea is explained in more detail by Pouzet in [44].

Similar methods were used to expand Fraïssé's conjecture even further. Laver, in his original proof in [30], not only proved the conjecture for countable order types, but he also extended it into the transfinite. The full form of his theorem implies that in fact all σ*-scattered* linear orders form a bqo, these are the countable unions of the scattered orders defined above.

However, Fraïssé's conjecture has also been expanded in a different direction. Firstly by Corominas, who proved that the set of all countable pseudo-trees[2] forms a bqo under

[2] A partial order T is a countable pseudo-tree if for each $t \in T$, $\{u \in T : u \leqslant t\}$ is a countable linear order.

embeddability [6]. This was then further expanded by Thomassé, who showed that the class of countable N-free partial orders[3] is bqo under embeddability [55].

We will use a similar method to prove that some new classes of partial orders, pseudo-trees and graphs are bqo, by constructing them internally, using this construction to show that they are bqo and then characterising them externally in a similar manner to Hausdorff's theorem. For our construction, in place of Kruskal's tree theorem we will use a more complex bqo theorem on a larger class of trees, due to Kříž [25].

In particular, the theorem on partial orders (Theorem 3.5.12) will extend Fraïssé's conjecture even further, giving for each $n \in \omega$, a transfinite version. When $n = 1$ we have Laver's theorem, when $n = 2$ we have a transfinite version of Thomassé's theorem, and in general as $n \in \omega$ increases we obtain much larger transfinite bqo classes of partial orders. This would appear to be the ultimate version of Fraïssé's conjecture.

1.2 Colourings, partial orders and structured trees

Some of the most striking theorems in bqo theory are that certain classes of partial orders, often with colourings, are bqo under embeddability. Indeed, the notion of bqo was first used by Nash-Williams in order to prove that the class $\mathscr{R}$ of rooted trees of height at most ω is wqo (and bqo) under the embeddability quasi-order [39]. (See also [28] for a proof that uses more modern terminology.) Laver explored the coloured versions of such trees in [30], expanding Nash-Williams' method, he proved that $\mathscr{R}$ *preserves bqo*. (That is to say that if Q is bqo, then the class of trees of $\mathscr{R}$ coloured by Q is also bqo under a natural embeddability ordering, see Definition 2.2.7.)

We mention again the contribution from Nash-Williams [40], that if Q is bqo, then the class $\tilde{Q}$ of transfinite sequences of members of Q is bqo. This theorem can be viewed as an embeddability result on a class of coloured partial orders; since it is equivalent to saying that the ordinals preserve bqo. In fact Nash-Williams proved a stronger but more technical version of this statement, equivalent to saying that the ordinals are *well-behaved*

[3] A partial order is N-free if it does not embed the N partial order, see Definition 2.3.5.

(see Definition 2.2.8). This is an important strengthening of bqo that will be crucial for the results of chapters 3, 4 and 5.

Perhaps Laver's is the most famous result of this type. The full form of his theorem that proved Fraïssé's conjecture is that the class of σ-scattered linear orders preserves bqo [30]. A few years after his paper on σ-scattered linear orders, Laver also showed that the class $\mathscr{T}$ of σ-scattered[4] trees preserves bqo [32]. Initially we notice that there should be some connection between the theorems of σ-scattered linear orders and σ-scattered trees. In each, we first take all partial orders of some particular type (linear orders, trees) that do not embed some particular order (namely $\mathbb{Q}$ and $2^{<\omega}$). In both cases, the class of countable unions of these objects turn out to preserve bqo. Notice that these are the minimal elements of the increasing unions of smaller structures.

We prove a general theorem of this type (Theorem 3.5.3) which states that given well-behaved classes $\mathbb{L}$ and $\mathbb{P}$ of linear orders and partial orders respectively, the class of 'generalised σ-scattered partial orders' $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ will be well-behaved (see definitions 3.2.14 and 3.4.2). Letting $\overline{\mathbb{L}}$ be least class containing $\mathbb{L}$ and closed under lexicographic L-sums for all $L \in \overline{\mathbb{L}}$; we define our 'scattered' partial orders to be those orders X such that:

- Every *indecomposable* subset of X is isomorphic to a member of $\mathbb{P}$. (See Definition 3.2.10.)
- Every *linear* subset of X is isomorphic to a member of $\overline{\mathbb{L}}$ (See Definition 2.3.10).[5]
- The partial orders $2^{<\omega}$ and $-2^{<\omega}$ do not embed into X. (See Definition 3.2.13).[6]

Our class $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ is essentially then the class of *countable unions* of such X (see Definition 3.4.2). We note that in particular, if the class of countable linear order types $\mathscr{C}$ is a subset

[4]A tree T is σ-scattered if there are trees T_n $(n \in \omega)$ that do not embed $2^{<\omega}$ with $T_n \subseteq T_{n+1}$ for each $n \in \omega$, and $(\forall a \in T_{n+1} \setminus T_n), (\forall b \in T_n)$, $a \not< b$, satisfying $T = \bigcup_{n\in\omega} T_n$.

[5]In fact our scattered orders will have a more complex but more workable definition which potentially gives rise to a larger class. We state it in this simpler way for now, since the corresponding class is still shown to be well-behaved under some modest assumptions on $\mathbb{L}$. See Remark 3.5.5.

[6]In the initial definition of scattered partial orders we will also forbid embeddings of the partial order $2^{<\omega}_{\perp}$ (see Definition 3.2.13). However this assumption can be removed using Corollary 3.5.2.

of $\overline{\mathbb{L}}$, then $\mathscr{M}^{\mathbb{L}}_{\mathbb{P}}$ contains all countable partial orders satisfying the first point (Theorem 3.5.8).

Applying this theorem with classes known to be well-behaved yields generalisations of many other known results in this area. Van Engelen, Miller and Steel proved that the class $\mathscr{S}$ of scattered linear orders is well-behaved [14], which in turn was generalised to $\mathscr{M}$ by Kříž [25]. Corominas showed that the class of countable pseudo-trees preserves bqo [6] and Thomassé showed that the class of countable N-free partial orders preserves bqo [55]. We summarise known results as applications of Theorem 3.5.3 in Table 1.1. In each case Theorem 3.5.3 tells us that the given class is well-behaved.[7] We mark the *limits* column positively if the σ-scattered partial orders are necessary and negatively when the scattered orders will suffice.

Class	Description	$\mathbb{P}$	$\mathbb{L}$	Limits
$\mathscr{S}$	Scattered linear orders [30]	1, C_2	$\mathrm{On} \cup \mathrm{On}^*$	✗
$\mathscr{M}$	σ-scattered linear orders [30]	1, C_2	$\mathrm{On} \cup \mathrm{On}^*$	✓
$\mathscr{U}^{\mathrm{On}}$	Scattered trees [32]	1, C_2, A_2	On	✗
$\mathscr{T}^{\mathrm{On}}$	σ-scattered trees [32]	1, C_2, A_2	On	✓
$\mathscr{T}^{\mathscr{C}}$	Countable pseudo-trees [6]	1, C_2, A_2	$\mathscr{C}$	✓
$\mathscr{C}_{\{1,A_2,C_2\}}$	Countable N-free partial orders [55]	1, C_2, A_2	$\mathscr{C}$	✓

Table 1.1: Known results as applications of Theorem 3.4.12.

Here A_2 and C_2 are the antichain and chain of cardinality 2 respectively, On is the class of ordinals, On^* is the class of reversed ordinals and $\mathscr{C}$ is the class of countable linear orders.

Applying Theorem 3.4.12 with the largest known well-behaved classes $\mathbb{L}$ and $\mathbb{P}$ gives that some very large classes of partial orders are well-behaved (Theorem 3.5.12). For example, let $\mathbb{P}$ be the set of indecomposable partial orders of cardinality less than some $n \in \omega$, and $\mathbb{L} = \mathscr{M}$. Then for $n > 2$ the well-behaved class $\mathscr{M}^{\mathbb{L}}_{\mathbb{P}}$ contains the σ-scattered

[7] In the cases of $\mathscr{U}^{\mathrm{On}}$, $\mathscr{T}^{\mathrm{On}}$ and $\mathscr{T}^{\mathscr{C}}$ the constructed $\mathscr{M}^{\mathbb{L}}_{\mathbb{P}}$ is actually a larger class of partial orders.

linear orders, σ-scattered trees, countable pseudo-trees, countable N-free partial orders, and generalisations of such objects.

Crucial to the ideas in sections 3, 4 and 5 are those of constructing objects with so called 'structured trees'. Put simply, these are trees with some extra structure (usually a partial order) given to the set of successors of each element. Embeddings between structured trees are then required to induce embeddings of this extra structure.

Theorems on structured trees also appear throughout the literature on bqo theory (cf. [55, 6, 25, 44, 27, 33]). The rationale for their usefulness is described in the previous section, and explained in more detail by Pouzet in [44]. As mentioned in Pouzet's paper, and as touched on before, Kruskal proved that the set of finite trees structured with ω is bqo [27]. Pouzet's method is to take a 'simple' class of objects (e.g. partial orders) and a bqo class of multivariate functions sending a list of objects to a new object (so called 'operator algebras'). Closing the class under these functions then yields a new class, which one can prove to be bqo. The crucial step is to show that this construction can be encoded as a structured tree, contained inside a class which is known to preserve bqo. Pouzet's method however was limited in that the structured trees that he used were only 'chain-finite' (i.e. well-founded, or those trees for which every chain is finite).

More recently, larger classes of structured trees have been shown to preserve bqo. In particular, using a modification of the Minimal Bad Array Lemma (see [53]), Kříž managed to prove that if Q is well-behaved then $\mathscr{R}_Q$ (the class of Q-structured trees of $\mathscr{R}$, see Definition 2.4.5) is well-behaved [25]. This is the structured tree theorem that we will use to prove our results of sections 3 and 4.

It is worth noting, that once we have this correspondence between partial orders and structured trees, we can see why $\mathbb{Q}$ and $2^{<\omega}$ appear in the definition of σ-scattered linear orders and σ-scattered trees respectively; these are respectively the minimal partial orders that cannot be represented internally by a well-founded tree. Thus a partial order is externally scattered iff it has a well-founded internal tree representation.

Motivated by the fact that structured trees are extremely useful for constructing bqo classes, the aim of Section 4 is to expand the class of structured trees known to be well-behaved. To this end, we use a similar construction as with the partial orders of Section 3, in order to prove that some large classes of structured pseudo-trees are well-behaved (Theorem 4.3.4). This expands Kříž's structured tree theorem of [25] to certain classes of σ-scattered pseudo-trees, which incorporate both Laver's σ-scattered trees [32] and Corominas' countable pseudo-trees [6], generalising all three theorems simultaneously.[8]

We demonstrate the utility of our new structured tree theorem in Section 5, by proving Theorem 5.2.6. This is an analogue to graphs of our main theorem on partial orders (Theorem 3.5.12). We prove that a large transfinite class of 'generalised σ-scattered graphs' is well-behaved under the induced subgraph relation. This expands results of Damaschke [8] and Thomassé [55] and furthers the study of the wqo properties of classes of graphs, examples of which can be found in for example [54, 24, 11, 43, 48, 41].

1.3 Ramsey spaces

A quasi-order Q is defined to be bqo iff there is no 'bad' function $f : [\omega]^\omega \to Q$. Thus the very definition of better-quasi-order makes use of the space of infinite subsets of the natural numbers $[\omega]^\omega$. Many of the Ramsey properties of this space, such as the Galvin and Prikry Theorem [18] are heavily used throughout bqo theory (cf. [46, 53] among many others). This space is known as the Ellentuck space and is a prototypical example of a topological Ramsey space, originally described by Silver, Galvin and Prikry and Ellentuck [52, 18, 13]. In this context it is usually denoted $\mathbb{N}^{[\infty]}$.

Ramsey spaces $(\mathcal{R}, \leqslant, r)$ are abstract mathematical objects, which generalise the infinite dimensional Ramsey theory that can be done on the space $\mathbb{N}^{[\infty]}$. For an excellent and comprehensive text on this subject, see [56]. Objects such as *fronts*[9] and *barriers*, that will be familiar to anyone well-versed in bqo theory, have already been studied in this more abstract setting. Indeed in [56], Todorčević mentions that the study of infinite

[8] We consider only the *well-branched* trees T from Laver and Corominas' classes, i.e those such that $\forall x, y \in T$ the set $\{t \in T : t < x, y\}$ has a maximal element. This is necessary for the definition of structured pseudo-trees.

[9] Fronts are also sometimes called *blocks*, for example in [39].

dimensional Ramsey theory all began with Nash-Williams' initiation of bqo theory in [39].

So what happens to the definitions of wqo and bqo when we change this Ramsey space to a general Ramsey space $\mathcal{R}$? What does it mean to be wqo with respect to an unusual Ramsey space? Indeed, if bqos are a useful tool for showing that certain classes are wqo, then can we also generalise bqos and use them to show that classes are $\mathcal{R}$-wqo?

One contentious point here is how to define a *shift*, which gives some extra structure to a Ramsey space which is required for an analogous definition of bqo. Indeed, in the usual case, a continuous[10] function $f : [\omega]^\omega \to Q$ is defined to be 'bad' iff for all $X \in [\omega]^\omega$,

$$f(X) \not\leq f(X \setminus \{\min X\}).$$

So, implicit in the definition is a way of *shifting* a member X of our Ramsey space, to another member. For the Ellentuck space, the *usual* shift maps X to $X^+ = X \setminus \{\min X\}$. The Ramsey theory surrounding the usual shift on the Ellentuck space has been studied by Di Prisco and Todorčević in [10]. In general we must define what we mean by our shift function $\cdot^+$. We concede that there could be some debate here about which is the 'correct' definition of shift to take, so we try to make our definition as general as possible.

Interestingly, although Ramsey spaces themselves vary massively, the range of possible notions of $\mathcal{R}$-wqo seems relatively narrow. We classify the possibilities into seven types, and have found examples for three (we note that the majority of 'natural' shifts on well-known examples of Ramsey spaces have their $\mathcal{R}$-wqo equivalent to the usual notion of wqo). Since more examples appear to be difficult to find, it seems very likely that this classification can be refined further in the future.

Many of the theorems around the definition of bqo can be lifted to the new corresponding notion of $\mathcal{R}$-bqo. In particular, with one extra assumption relating to the shift map, we obtain that Q is $\mathcal{R}$-bqo iff $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-wqo for every $\alpha \in \mathrm{On}$.[11] This theorem allows us to prove that when two Ramsey spaces have shifts satisfying this added condition, and have equivalent corresponding notions of wqo, then their corresponding equivalent notions

[10] Giving Q the discrete topology and $[\omega]^\omega$ the product topology.

[11] Here $\mathcal{P}_0(Q) = Q$, $\mathcal{P}_{\alpha+1}(Q) = \mathcal{P}(\mathcal{P}_\alpha(Q)) \cup \mathcal{P}_\alpha(Q)$ and for limit $\lambda \in \mathrm{On}$, $\mathcal{P}_\lambda(Q) = \bigcup_{\alpha<\lambda} \mathcal{P}_\alpha(Q)$. See Definition 2.1.11

of bqo are also equivalent.

1.4 Fronts and barriers on an uncountable cardinal

One of the main variations of better-quasi-orders that can be found in the literature is Shelah's notion of κ-bqo for an uncountable cardinal κ. In his paper [49], he generalises the notion of bqo in such a way that the corresponding notion of wqo is equivalent to the statement that there are no length κ descending sequences and no antichains of cardinality κ. He avoids the difficulties that arise in the absence of any Ramsey-like properties by further considering more complex bqo notions that have some Ramsey-like properties built into their definition. By doing this, he ultimately goes on to prove a certain generalisation of Laver's theorem on σ-scattered linear orders to λ-scattered orders[12], as well as showing that the class $\mathscr{R}$ preserves his generalised notions of κ-bqo, in a similar way to the way that $\mathscr{R}$ preserves bqo.

We note that there are two equivalent notions of bqo that could potentially be generalised to an uncountable cardinal. The first, originally given by Simpson [53], is of a topological nature. Q is bqo iff there is no continuous 'bad' function $f : [\omega]^\omega \rightarrow Q$, where Q has the discrete topology and $[\omega]^\omega$ has the product topology.[13]

The second is more combinatorial. First define a *front* $\mathcal{F}$ to be a set of finite subsets of ω, such that:

- for all $a, b \in \mathcal{F}$, a is not an initial segment of b,
- and for all $X \in [\omega]^\omega$, there is an initial segment of X in $\mathcal{F}$.

Then Q is bqo iff there is no 'bad' function $f : \mathcal{F} \rightarrow Q$ for any front $\mathcal{F}$. These can be seen to be equivalent since any continuous function is constant on a set with a fixed initial segment and the shortest such elements form a front. For a more detailed analogous proof see Theorem 7.2.11.

[12] i.e. those linear orders that are unions of $\leqslant \lambda$ many scattered linear orders.

[13] In fact, Simpson's original definition has Borel measurable in place of continuous here but as in [53] this is still equivalent.

In the usual (countable) case, instead of a front we can use a *barrier*, i.e. a front with the extra property known as the *Sperner property*:

$$(\forall a, b \in \mathcal{B}), a \not\subset b.$$

This is because, using the Ramsey properties available in the countable case, we can find a restriction of $\mathcal{F}$ that is a barrier (see [38]). This result of Nash-Willaims [40] is often known as 'every block contains a barrier'.

In [49], Shelah uses an unusual property to define the barriers which he uses analogously in his definition of κ-bqo. He defines a front on some $A \subseteq \kappa$ whose order type is κ analogously to the countable case. He then defines a *κ-barrier* $\mathcal{B}$ to be a front on $[A]^\omega$ with the extra property:

$$(\forall a, b \in \mathcal{B}), b \text{ is not a strict initial segment of } a \setminus \min a.$$

We refer to this property as the *barrier property.* It is worth mentioning that this property is implied by the Sperner property, which is more usually seen in the definition of barriers. Shelah then defines: Q is κ-bqo iff there is no 'bad' $f : \mathcal{B} \to Q$ for a κ-barrier $\mathcal{B}$.

We ask if it is possible to give a version of Simpson's definition for κ-bqo, equivalent to Shelah's notion. In order to do so, we need to replace κ-barriers with fronts in Shelah's definition. So in Chapter 7, we prove that in fact this is possible and the notions are equivalent, even in the absence of 'every block contains a barrier'.

Once we have this equivalence, given a bad function $f : \mathcal{F} \to Q$ we obtain a bad function $g : \mathcal{B} \to Q$ for $\mathcal{F}$ a front on $[\kappa]^\omega$ and $\mathcal{B}$ a κ-barrier. However the relationship between f and g is indirect in the sense that we have no direct method to define g from f. We would like to ask the question - how 'close' is $\mathcal{F}$ to being a κ-barrier? In order to attempt to answer this, we investigate the method of *extending* the elements of a front $\mathcal{F}$ by adding final segments, resulting in a new front $\mathcal{F}'$. Then from any bad function $f : \mathcal{F} \to Q$ we can define a bad function $f' : \mathcal{F}' \to Q$ just by letting $f'(a) = f(b)$ where b is the initial segment of a contained in $\mathcal{F}$. Thus extending fronts is invariant for bad functions in this sense.

In Chapter 8, we investigate the interplay between this method of extending and the method of restricting. We classify the many possible types of front, depending on whether or not there is a process of extending and restricting to find a κ-barrier. We then prove some relationships between existence of some such types of front and some negative partition relations involving κ, in order to try to describe the cardinals at which these types of front exist.

1.5 The stucture of this thesis

The structure of this thesis is as follows:

- In Chapter 2 we give some basic bqo theory, notation, definitions and preliminaries.
- In Chapter 3 we aim to prove our expanded version of Fraïssé's conjecture. We begin by giving a modified version of Pouzet's operator algebra construction from [44], that can be used to show that a constructed class of partial orders is well-behaved. We then give a specific operator algebra that will construct the desired scattered partial orders internally. We define intervals and indecomposable orders, allowing us to give an external definition of the scattered partial orders. We prove an extension of Hausdorff's theorem on scattered linear order types [19], which amounts to saying that the class of internally defined scattered partial orders is the same as the class of externally defined scattered partial orders. We then externally define our σ-scattered partial orders. Then for each order, we define a structured tree which describes how this order is built internally. Using a theorem of Kříž from [25] we see that these trees are in a well-behaved class, which allows us to prove that our given class of σ-scattered partial orders is well-behaved. We then mention some implications, showing that some more simply defined classes of countable partial orders are well-behaved, before finally posing some related questions relating to future applications.
- In Chapter 4 we aim to expand Kříž's structured tree theorem to a large class of

pseudo-trees. The method of the proof is essentially similar to that of Chapter 3. We give both internal and external definitions, prove their equivalence and use the internal definition along with Křížʼs structured tree theorem to prove that our class of trees is well-behaved. This result also generalises Corominas' theorem on pseudo-trees [6] into the transfinite and Laver's theorem on σ-scattered trees [32] to pseudo-trees.

- In Chapter 5 we apply the new structured pseudo-tree theorem of Chapter 4 to graphs. We give an external definition of a class of scattered graphs, analogous to that of the partial orders of Chapter 3. For each externally defined scattered graph, we define a pseudo-tree that describes an internal construction. We then define a class of σ-scattered graphs. Given a σ-scattered graph G, we then construct a pseudo-tree describing an internal construction G by using the pseudo-trees which internally characterise the scattered graphs used to define G. This pseudo-tree is a member of the class constructed in Chapter 4, which allows us to prove that our class of σ-scattered graphs is well-behaved.

- In Chapter 6 we take a different direction. We begin be reviewing some basic definitions and properties related to topological Ramsey spaces, as well as giving a few examples. We then define precisely what we mean by a *shift map* in as general a way as possible, before defining the notion of $\mathcal{R}$-wqo for a general Ramsey space $\mathcal{R}$. We attempt to classify the possible types of $\mathcal{R}$-wqo, first looking at the analogues of the 'no infinite antichains' condition and then the analogues of 'no infinite descending sequences' condition. This results in a classification of the different potential versions of $\mathcal{R}$-wqo into seven possible types. We give some examples of these different types. We then consider the corresponding notion of $\mathcal{R}$-bqo and prove that $\mathcal{R}$-bqos are closed under iterated power sets, similarly to bqos. We then see that under a certain extra condition on the shift maps, if two Ramsey spaces have the same corresponding notion of wqo, then they have the same corresponding notion of bqo.

- In Chapter 7 we consider Shelah's notion of better-quasi-orders for uncountable cardinals [49], and prove that the corresponding modification of his definition using fronts instead of barriers is equivalent. While this is easy in the countable case by just applying the Ramsey-theoretic Galvin and Prikry Theorem 2.1.6, it is non-trivial at an uncountable cardinal. Using this we can define a natural version of Simpson's definition of better-quasi-order for uncountable cardinals, even in the absence of any Ramsey-theoretic results.

- Finally in Chapter 8 we give a classification of fronts on $[\kappa]^\omega$ in an attempt to describe how far they are from being a κ-barrier. We then give a correspondence between existence of fronts on $[\kappa]^\omega$ of a given type in this classifcation and partition relations that involving κ. Thus giving a partial description of the cardinals κ at which fronts of this type can exist.

Chapter 2

Preliminaries

We will assume that the reader is familiar with basic set theory and its notation. In particular: the basic definitions and properties of ordinals, cardinals, relations and functions; as well as basic notions of topology such as the product topology on a space of infinite sequences; and later on the notion of a Ramsey cardinal. We cite [21] as a reference on these concepts.

2.1 Basic bqo theory

Definition 2.1.1. If A is an infinite subset of ω, let $[A]^\omega = \{X \subseteq A : |X| = \aleph_0\}$ and $[A]^{<\omega} = \{X \subseteq A : |X| < \aleph_0\}$. We equate $X \in [A]^\omega$ with the increasing enumeration of elements of X.

Definition 2.1.2.

- A class Q with a binary relation $\leqslant_Q$ on Q is called a *quasi-order* whenever $\leqslant_Q$ is transitive and reflexive.
- If Q is a quasi-order and $\leqslant_Q$ is antisymmetric then we call Q a *partial order*.
- For $a, b \in Q$ we write $a <_Q b$ iff $a \leqslant_Q b$ and $b \nleqslant_Q a$. We write $a \perp_Q b$ and call a and b *incomparable* iff $a \nleqslant_Q b$ and $b \nleqslant_Q a$.
- We write $a \geqslant_Q b$ iff $b \leqslant_Q a$ and $a >_Q b$ iff $b <_Q a$.

- $C \subseteq Q$ is a *chain* iff $\forall a, b \in C$, either $a <_Q b$, $a >_Q b$ or $a = b$.
- $A \subseteq Q$ is an *antichain* iff $\forall a, b \in A$, either $a \perp_Q b$ or $a = b$.
- A quasi-order Q is called *well-founded* iff there is no sequence $(q_n)_{n\in\omega}$ of elements of Q, such that $(\forall n \in \omega)$, $q_{n+1} <_Q q_n$. Such a sequence is called *descending*.
- A quasi-order Q is called *narrow* iff there is no infinite antichain $A \subseteq Q$.
- A quasi-order Q is called a *well-quasi-order* (wqo) iff Q is well-founded and narrow.

We write $\leqslant$, $<$ and $\perp$ in place of $\leqslant_Q$, $<_Q$ and $\perp_Q$ when the context is clear.

We note that narrow orders are also sometimes known as FAC (finite antichain condition) orders, and that there are generalisations of Haudorff's theorem on scattered linear orders [19] to these orders, see [1, 4].

Definition 2.1.3. If x and y are quasi-orders, then we call x and y *isomorphic* and write $x \cong y$ iff there is some bijection $\varphi : x \to y$ such that $(\forall a, b \in x), a \leqslant_x b \longleftrightarrow \varphi(a) \leqslant_y \varphi(b)$.

Definition 2.1.4.
- A function $f : [\omega]^\omega \to Q$ is called a *Q-array* if f is continuous, giving $[\omega]^\omega$ the product topology and Q the discrete topology.
- A Q-array $f : [\omega]^\omega \to Q$ is called *bad* if $\forall X \in [\omega]^\omega$ we have
$$f(X) \nleqslant f(X \setminus \{\min X\}).$$
- A Q-array $f : [\omega]^\omega \to Q$ is called *perfect* if $\forall X \in [\omega]^\omega$ we have
$$f(X) \leqslant f(X \setminus \{\min X\}).$$
- A quasi-order Q is called a *better-quasi-order* (bqo) iff there is no bad Q-array.

Remark 2.1.5. We note that we could replace 'continuous' in the definition of a Q-array with 'Borel measurable' and this would make no difference to the definition of bqo (see [53]). We can also consider arrays with domain $[A]^\omega$ for some $A \in [\omega]^\omega$.

We note that the motivation for bad arrays (originally used by Nash-Williams [39]), comes from contemplating why some transfinite constructions, such as the class of transfinite sequences $\tilde{Q}$ can fail to be bqo even when Q is bqo. For a detailed motivation as to why bad arrays arise naturally, see [42, 38].

The following is a well-known Ramsey-theoretic result due to Galvin and Prikry.

Theorem 2.1.6 (Galvin, Prikry [18])**.** *Given a Borel set B in $[\omega]^\omega$, there exists $A \in [\omega]^\omega$ such that either $[A]^\omega \subseteq B$ or $[A]^\omega \cap B = \emptyset$.*

Proof. See [18] or [53]. $\square$

Theorem 2.1.7 (Nash-Williams [39])**.** *If f is a Q-array, then there is $A \in [\omega]^\omega$ such that $f \upharpoonright [A]^\omega$ is either bad or perfect.*

Proof. Let $B = \{X \in [\omega]^\omega : f(X) \leqslant f(X \setminus \{\min X\})\}$. If B is Borel, then by Theorem 2.1.6 we will be done. Let $S : [\omega]^\omega \to [\omega]^\omega$ be the function $S(X) = X \setminus \{\min X\}$. Then and let $g : [\omega]^\omega \to Q \times Q$ be such that $g(X) = \langle f(X), f \circ S(X)\rangle$. Then g is continuous, since f and S are continuous. We also have that $B = g^{-1}(\leqslant)$, considering the relation $\leqslant$ as a subset of the discrete space $Q \times Q$. Therefore B is open and we are done. $\square$

Definition 2.1.8. Given a quasi-order Q, let $Q \cup \{-\infty\}$ be a new quasi-order defined by letting $p \leqslant q$ iff $p, q \in Q$ and $p \leqslant_Q q$, or $p = -\infty$.

Definition 2.1.9. Let Q_0 and Q_1 be quasi-orders, we define $Q_0 \times Q_1 = \{\langle q_0, q_1\rangle : q_0 \in Q_0, q_1 \in Q_1\}$ where for $\langle p_0, p_1\rangle, \langle q_0, q_1\rangle \in Q_0 \times Q_1$ we have

$$\langle p_0, p_1\rangle \leqslant \langle q_0, q_1\rangle \text{ iff } (p_0 \leqslant_{Q_0} q_0) \wedge (p_1 \leqslant_{Q_1} q_1).$$

Theorem 2.1.10 (Nash-Williams [39])**.** *If f is a bad $Q_0 \times Q_1$-array, then there is some $A \in [\omega]^\omega$ and g with dom$(g) = [A]^\omega$ such that either:*

- *g is a bad Q_0-array, and $g(X)$ is the first component of $f(X)$ for all $X \in [A]^\omega$.*
- *or g is a bad Q_1-array, and $g(X)$ is the second component of $f(X)$ for all $X \in [A]^\omega$.*

Proof. Define the Q_0-array f_0 and the Q_1-array f_1 so that for every $X \in [\omega]^\omega$ we have

$$f(X) = \langle f_0(X), f_1(X) \rangle.$$

Now apply Theorem 2.1.7 twice to restrict firstly so that f_0 is either bad or perfect and secondly so that f_1 is either bad or perfect. Then either we are done or the resulting restrictions of f_0 and f_1 are both perfect, which contradicts that f was bad. $\square$

Definition 2.1.11. Let Q be a quasi-order. We quasi-order the power set $\mathcal{P}(Q) = \{A : A \subseteq Q\}$ by letting $A \leqslant B$ iff $\exists f : A \to B$ such that $\forall a \in A$, $a \leqslant f(a)$.

We now iterate the power set operation on a quasi-order Q, defining the sets $\mathcal{P}_\alpha(Q)$ for $\alpha \in \mathrm{On}$ by recursion on α as follows:

$$\mathcal{P}_0(Q) = Q,$$

$$\mathcal{P}_{\alpha+1}(Q) = \mathcal{P}(\mathcal{P}_\alpha(Q)) \cup \mathcal{P}_\alpha(Q),$$

$$\lim(\lambda) \to \mathcal{P}_\lambda(Q) = \bigcup_{\gamma<\lambda} \mathcal{P}_\gamma(Q).$$

To aid notation we also define

$$\mathcal{P}_\infty(Q) = \bigcup_\gamma \mathcal{P}_\gamma.$$

We will now define the order on $\mathcal{P}_\alpha(Q)$ by first defining the notion of the transitive closure of a member of $\mathcal{P}_\infty(Q)$. If $A \in \mathcal{P}_\alpha(Q)$ then let $\mathrm{TC}_0(A) = A$ and for $n \in \omega$ define

$$\mathrm{TC}_{n+1}(A) = (\mathrm{TC}_n(A) \cap Q) \cup \bigcup (\mathrm{TC}_n(A) \setminus Q).$$

Then we define the *transitive closure* $\mathrm{TC}(A) = \bigcup_{n\in\omega} \mathrm{TC}(A)$.

If $\alpha = 0$ then the order on $\mathcal{P}_\alpha(Q)$ is already defined. If α is a limit ordinal, set $A \leqslant B$ iff $\exists \gamma < \alpha$ such that $A \leqslant_{\mathcal{P}_\gamma(Q)} B$. Otherwise, $\alpha = \beta + 1$ and we let $A \leqslant B$ iff either $A, B \in \mathcal{P}_\beta(Q)$ and $A \leqslant_{\mathcal{P}_\beta(Q)} B$, or at least one of A, B are in $\mathcal{P}_\alpha(Q) \setminus \mathcal{P}_\beta(Q)$ and one of the following occurs:

1. $A \notin Q$ and $\exists f : A \to B$ such that $\forall q \in A$, $q \leqslant_{\mathcal{P}_\beta(Q)} f(q)$; [1]

[1] This is well-defined since $A, B \subseteq \mathcal{P}_\beta(Q)$

2. $B = q \in Q$ and $(\forall t \in \mathrm{TC}(A) \cap Q), t \leqslant q$;

3. $A \in \mathcal{P}_\beta(Q)$ and $A \leqslant_{\mathcal{P}_\beta(Q)} A'$ for some $A' \in B$.

2.2 Concrete categories

Usually we will be interested in quasi-ordering classes of partial orders under embeddability, however we can keep the results more general with no extra difficulty by considering the notion of a *concrete category*. The idea is to add a little more meat to the notion of a quasi-order, considering classes of structures quasi-ordered by existence of some kind of embedding. This allows us to generate more complicated orders by colouring the elements of these structures with a quasi-order; enforcing that embeddings must increase values of this colouring. Then we can construct complicated objects from simple objects in a ranked way by iterating this colouring process, and the notion of *well-behaved* in a sense allows us to reduce back down through the ranks. The reader who is unfamiliar with categories may wish to refer to [34]. We shall now formalise these notions, similar to the definitions within [25] and [54].

Definition 2.2.1. A *concrete category* is a pair $\mathcal{O} = \langle \mathrm{obj}(\mathcal{O}), \mathrm{hom}(\mathcal{O}) \rangle$ such that:

1. each $\gamma \in \mathrm{obj}(\mathcal{O})$ has an associated *underlying set* U_γ;

2. for each $\gamma, \delta \in \mathrm{obj}(\mathcal{O})$ there are sets of *embeddings* $\mathrm{hom}_{\mathcal{O}}(\gamma, \delta)$ consisting of some functions from U_γ to U_δ;

3. $\mathrm{hom}_{\mathcal{O}}(\gamma, \gamma)$ contains the identity on γ for any $\gamma \in \mathcal{O}$;

4. for any $\gamma, \delta, \beta \in \mathcal{O}$, if $f \in \mathrm{hom}_{\mathcal{O}}(\gamma, \delta)$ and $g \in \mathrm{hom}_{\mathcal{O}}(\delta, \beta)$ then $f \circ g \in \mathrm{hom}_{\mathcal{O}}(\gamma, \beta)$;

5. $\mathrm{hom}(\mathcal{O}) = \bigcup \{\mathrm{hom}_{\mathcal{O}}(\gamma, \delta) : \gamma, \delta \in \mathrm{obj}(\mathcal{O})\}$.

Elements of $\mathrm{obj}(\mathcal{O})$ are called *objects* and elements of $\mathrm{hom}(\mathcal{O})$ are called *$\mathcal{O}$-morphisms* or *embeddings*. To simplify notation we write $\gamma \in \mathcal{O}$ for $\gamma \in \mathrm{obj}(\mathcal{O})$ and equate γ with U_γ .

Concrete categories are thus categories in the classical sense. They are precisely those categories with a faithful functor to the category of sets (i.e. send an object to its underlying set). This additional property allows us to think of concrete categories as classes of sets equipped with some additional structure, and embeddings as structure preserving functions.

Remark 2.2.2. Similar definitions to 2.2.1 appear within papers on bqo theory in [25], [54] and [33]. The first two enjoy a more category theoretic description and the last is in terms of structures and embeddings.

With the following definition, concrete categories will turn into quasi-ordered sets under *embeddability*; (3) and (4) of Definition 2.2.1 guaranteeing the reflexivity and transitivity properties respectively. This allows us to consider the bqo properties of concrete categories.

Definition 2.2.3. For $\gamma, \delta \in \mathcal{O}$, we say that

$$\gamma \leqslant_{\mathcal{O}} \delta \text{ iff } \hom_{\mathcal{O}}(\gamma, \delta) \neq \emptyset$$

i.e. $\gamma \leqslant_{\mathcal{O}} \delta$ iff there is an embedding from γ to δ. If $f \in \hom_{\mathcal{O}}(\gamma, \delta)$ then we say that f *witnesses* $\gamma \leqslant_{\mathcal{O}} \delta$.

Example 2.2.4. Let $\mathrm{obj}(\mathcal{P})$ be the class of partial orders. For any two partial orders x, y, let:

1. $U_x = x$,
2. $\hom_{\mathcal{P}}(x, y) = \{\varphi : x \to y : (\forall a, b \in x), a \leqslant_x b \longleftrightarrow \varphi(a) \leqslant_y \varphi(b)\}$,
3. $\hom(\mathcal{P}) = \bigcup\{\hom_{\mathcal{P}}(p, q) : p, q \in \mathrm{obj}(\mathcal{P})\}$,
4. $\mathcal{P} = \langle \mathrm{obj}(\mathcal{P}), \hom(\mathcal{P})\rangle$.

The category $\mathcal{P}$ of partial orders with embeddings is then a quintessential example of a concrete category and the order $\leqslant_{\mathcal{P}}$ is the usual embeddability ordering on the class of partial orders. We keep this example in mind since the majority of concrete categories used in this thesis are either subclasses of $\mathcal{P}$ or are derived from $\mathcal{P}$. We note that all $\mathcal{P}$-morphisms are injective.

Definition 2.2.5. Given a quasi-order Q and a concrete category $\mathcal{O}$, we define the concrete category $\mathcal{O}(Q)$ as follows.

- $\mathrm{obj}(\mathcal{O}(Q)) = \{f : f : \gamma \to Q, \gamma \in \mathrm{obj}(\mathcal{O})\}$.
- For $f \in \mathrm{obj}(\mathcal{O}(Q))$, we let $U_f = U_{\mathrm{dom}(f)}$.
- We define morphisms of $\mathcal{O}(Q)$ from $f : \gamma \to Q$ to $g : \delta \to Q$ to be embeddings $\varphi : \gamma \to \delta$ such that for every $x \in \gamma$ we have

$$f(x) \leqslant_Q g \circ \varphi(x).$$

Remark 2.2.6. We will use the convention of writing $\hat{\gamma} \in \mathcal{O}(Q)$ when we have $\hat{\gamma} : \gamma \to Q$. Indeed, if we have $\hat{\gamma} \in \mathcal{O}(Q)$ then we will use without specific declaration that $\gamma \in \mathcal{O}$ and $\gamma = \mathrm{dom}(\hat{\gamma})$.

The category $\mathcal{O}(Q)$ is thus a category of Q-*colourings* of $\mathcal{O}$, where we imagine labelling the elements of members of $\mathcal{O}$ with elements of Q. Embeddings must then not only be embeddings of $\mathcal{O}$, but also increase the value of every label pointwise.

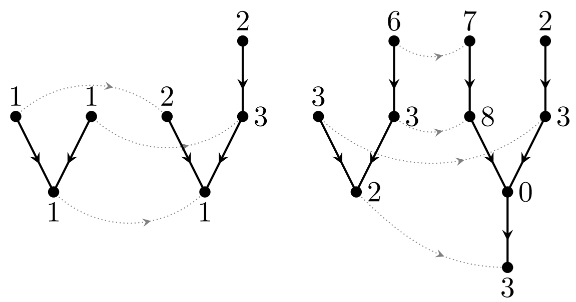

Figure 2.1: Embeddings between ω-coloured partial orders.

We are now ready to define the bqo preservation properties mentioned in Section 1.2, allowing us to pass from bad $\mathcal{O}(Q)$-arrays to bad Q-arrays.

Definition 2.2.7. Let $\mathcal{O}$ be a concrete category, then $\mathcal{O}$ *preserves bqo* iff for every quasi-order Q,

$$Q \text{ is a bqo } \longrightarrow \mathcal{O}(Q) \text{ is a bqo.}$$

Unfortunately, this simple definition fails to be particularly useful. Given a bad $\mathcal{O}(Q)$-array, preservation of bqo ensures the existence of a bad Q-array, but no link between these two arrays is guaranteed. The following definition remedies this situation and is extremely important within chapters 3, 4 and 5.

Definition 2.2.8. Let $\mathcal{O}$ be a concrete category, then $\mathcal{O}$ is *well-behaved* iff for any quasi-order Q and any bad array $f : [\omega]^\omega \to \mathcal{O}(Q)$, there is an $M \in [\omega]^\omega$ and a bad array

$$g : [M]^\omega \to Q$$

such that for all $X \in [M]^\omega$ there is some $v \in \mathrm{dom}(f(X))$ with

$$g(X) = f(X)(v).$$

We call g a *witnessing Q-array* for f.

Warning: this notion of well-behaved is the same as from [25]; it is different from the definition of well-behaved that appears in [54] which is in fact equivalent to Louveau and Saint-Raymond's notion of reflecting bad arrays [33].

Proposition 2.2.9. *$\mathcal{O}$ is well-behaved $\longrightarrow$ $\mathcal{O}$ preserves bqo $\longrightarrow$ $\mathcal{O}$ is bqo.*

Proof. If $\mathcal{O}$ is well-behaved then given a bad $\mathcal{O}(Q)$-array f, we have a bad Q-array. If Q were bqo this would give a contradiction and hence there is no such bad array f.

Now let $1 = \{0\}$ be the singleton quasi-order, clearly then 1 is bqo. Thus if $\mathcal{O}$ preserves bqo then $\mathcal{O}(1)$ is bqo. Clearly $\mathcal{O}(1)$ is order isomorphic to $\mathcal{O}$, therefore $\mathcal{O}$ is also bqo. $\square$

Remark 2.2.10. Note that the converse $\mathcal{O}$ is bqo $\longrightarrow$ $\mathcal{O}$ preserves bqo does not hold. For a counterexample let Z be the partial order consisting of points x_n and y_n for $n \in \omega$; ordered so that for $a, b \in Z$, we have $a \leqslant b$ iff $a = b$ or there is some $n \in \omega$ such that $a \in \{x_n, x_{n+1}\}$ and $b = y_n$. Then $\{Z\}$ is clearly bqo since it contains only one element, but it does not preserve bqo (see Figure 2.2).

It is not known whether or not the other converse holds, i.e. is it the case that

$$\mathcal{O} \text{ preserves bqo } \longrightarrow \mathcal{O} \text{ is well-behaved?}$$

This is an interesting technical question, which was asked by Thomas in [54].

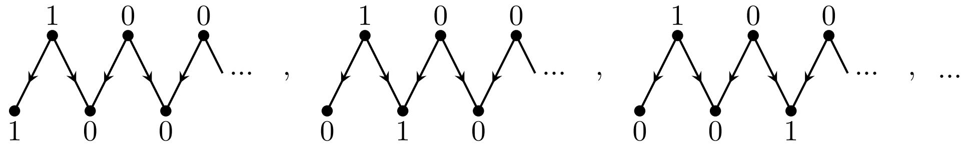

Figure 2.2: An antichain of $\{Z\}(A)$, where $A = \{0, 1\}$ with $0 \perp_A 1$.

Preserving bqos is an important definition historically, however it will not be used from now on, we opt instead for well-behaved.

Definition 2.2.11. Given a concrete category $\mathcal{O}$, a quasi-order Q and $\gamma \in \mathrm{obj}(\mathcal{O})$ we define

$$Q^{\gamma} = \{f : \gamma \to Q\} \subseteq \mathcal{O}(Q).$$

Lemma 2.2.12. *Let $\mathbb{P}$ be a finite set of finite partial orders, then $\mathbb{P}$ is well-behaved.*

Proof. Let Q be an arbitrary quasi-order and let f be a bad $\mathbb{P}(Q)$-array. Then since $\mathbb{P}$ is finite, we write $\mathbb{P} = \{P_0, ..., P_{n-1}\}$ for $n = |\mathbb{P}|$. For $i < n$ let $B_i = Q^{P_i} \subseteq \mathbb{P}(Q)$, then we can repeatedly apply the Galvin and Prikry Theorem 2.1.6 to each B_i $(i < n)$ in turn to find $A \in [\omega]^{\omega}$ such that for each $X, Y \in [A]^{\omega}$, we have that $f(X)$ and $f(Y)$ have the same underlying finite partial order $P = \{p_0, ..., p_{m-1}\}$.

Let $f_i : [A]^{\omega} \to Q$ be given by $f_i(X) = f(X)(p_i)$ for all $X \in [A]^{\omega}$. Let $A_0 = A$ and having defined A_j $(j < m)$ apply Theorem 2.1.7 to f_j, to find some $A_{j+1} \in [A_j]^{\omega}$ so that $f_j \restriction [A_{j+1}]^{\omega}$ is either a bad array or a perfect array. They cannot all be perfect otherwise $f \restriction [A_m]^{\omega}$ is perfect, which contradicts that f is bad. Therefore at least one of these arrays f_i $(i < m)$ is bad, and this is clearly a witnessing array for f. $\square$

2.3 Partial orders

Definition 2.3.1. We define Card as the class of cardinals, On as the class of ordinals and $\mathrm{On}^* = \{\alpha^* : \alpha \in \mathrm{On}\}$, where α^* is a reversed copy of α for every $\alpha \in \mathrm{On}$. These are considered as concrete categories whose morphisms are increasing injective maps.

Theorem 2.3.2 (Nash-Williams [40])**.** On *is well-behaved.*

Proof. See [53, 40]. $\square$

Definition 2.3.3. We let $1 = \{0\}$ be the partial order consisting of a single point. For $\kappa \in \mathrm{Card}$ we let $\mathrm{A}_\kappa = \{\alpha : \alpha < \kappa\}$ be the antichain of size κ. For $n \in \omega$ we let C_n be the chain of length n.

Definition 2.3.4. Let P be a partial order and $x \in P$, we define:

$$\downarrow x = \{y \in P : y \leqslant x\}, \quad \uparrow x = \{y \in P : y \geqslant x\},$$

$$\mapsdown x = \{y \in P : y < x\}, \quad \mapsup x = \{y \in P : y > x\}.$$

For $x, y \in P$, if it exists, we define the meet $x \wedge y$ to be the supremum of $\downarrow x \cap \downarrow y$.

Definition 2.3.5. We define the partial order $N = \{0, 1, 2, 3\}$ as follows. For $a, b \in N$ we let $a < b$ iff $a = 1$ and $b \in \{0, 2\}$ or $a = 3$ and $b = 2$ (see Figure 2.3).

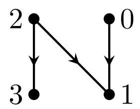

Figure 2.3: The partial order N.

Definition 2.3.6.

- A *linear order* is a partial order L with no incomparable elements.
- A linear order L is *scattered* if $\mathbb{Q} \not\leqslant L$.
- A linear order L is *σ-scattered* iff L can be partitioned into countably many scattered linear orders.
- We denote the class of scattered linear orders as $\mathscr{S}$.
- We denote the class of σ-scattered linear orders as $\mathscr{M}$.
- We denote the class of countable linear orders as $\mathscr{C}$.

Theorem 2.3.7 (Kříž, [25])**.** *$\mathscr{M}$ is well-behaved.*

Proof. See [25]. □

Definition 2.3.8. Given linear orders r and r' and sequences $k = \langle k_i : i \in r\rangle$ and $k' = \langle k'_j : j \in r'\rangle$, we denote by $\sqsubseteq$ the initial segment relation, and $\sqsubset$ the strict initial segment relation. That is

$$k \sqsubseteq k' \text{ iff } r \subseteq r' \text{ and if } j' \in r', j \in r, j' \leqslant j \text{ then } j' \in r, \text{ furthermore for all } j \in r, k_j = k'_j$$

and $k \sqsubset k'$ iff $k \sqsubseteq k'$ and $k \neq k'$. We denote by $k^\frown k'$ the concatenation of k and k'. We also define $\mathrm{ot}(k) = r$.

Definition 2.3.9. Let P be a partial order, and for each $p \in P$, let P_p be a partial order. We define the lexicographical P-sum of the P_p denoted by $\sum_{p\in P} P_p$ as the set $\bigsqcup_{p\in P} P_p$ ordered by letting $a \leqslant b$ iff

- there is some $p \in P$ such that $a, b \in P_p$ and $a \leqslant_{P_p} b$, or
- there are $p, q \in P$ such that $a \in P_p$, $b \in P_q$ and $p <_P q$.

We consider $\sum_P$ as a function $\sum_P : \mathscr{P}^P \to \mathscr{P}$, where $\sum_P(\hat{P}) = \sum_{p\in P} \hat{P}(p)$.

We note that each P_q $(q \in P)$ embeds into $\sum_{p\in P} P_p$ in the obvious way. We equate P_q with its image under this embedding, and write $P_q \subseteq \sum_{p\in P} P_p$ for inclusion as partial orders.

Definition 2.3.10. If $\mathbb{L}$ is a class of linear orders, we define $\overline{\mathbb{L}}$ as the least class containing $\mathbb{L}$ and closed under L-sums for all $L \in \overline{\mathbb{L}}$.

Theorem 2.3.11 (Hausdorff [19])**.** *If $\mathbb{L} = \mathrm{On} \cup \mathrm{On}^*$ then $\overline{\mathbb{L}} = \mathscr{S}$.*

Proof. See [19, 53]. □

2.4 Structured Trees

Definition 2.4.1. A partial order T is called a *tree* iff $(\forall t \in T)$, ${\downarrow}t$ is a well-order.

Definition 2.4.2. Let T be a tree, then we define as follows:

- $t \in T$ is called a *leaf* of T if there is no $t' \in T$ such that $t' > t$.
- T is *rooted* iff T has a minimal element, denoted $\mathrm{root}(T)$.
- T is *well-founded* iff every chain of T is finite.[2]
- The *height of* T is $\sup_{x \in T}\{\mathrm{ot}(\downarrow x)\}$.

We let $\mathscr{W}$ be the class of rooted well-founded trees; and let $\mathscr{R}$ be the class of rooted trees of height at most ω. We also let $\mathscr{T}$ be the concrete category of all trees, whose morphisms are partial order embeddings $\varphi : T \to S$ such that for all $x, y \in T$, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$. We consider subclasses of $\mathscr{T}$ to be concrete categories with the same embeddings.

Definition 2.4.3. Given a rooted well-founded tree T, and some $t \in T$ we define inductively[3]

$$\mathrm{rank}(t) = \sup\{\mathrm{rank}(s) + 1 : t <_T s\}.$$

We then define the *tree rank* of T as $\mathrm{rank}(T) = \mathrm{rank}(\mathrm{root}(T))$.

Definition 2.4.4. If T is a tree and $t \in T$ then let $\mathrm{succ}(t)$ be the set of *successors* of t, i.e. the set of minimal elements of $\uparrow t$. If $u \in \mathrm{succ}(t)$ then we call t the *predecessor* of u.

Definition 2.4.5. Let $\mathbb{T}$ be a class of trees, and let $\mathcal{O}$ be a concrete category. We define the new concrete category of *$\mathcal{O}$-structured trees of* $\mathbb{T}$, denoted $\mathbb{T}_{\mathcal{O}}$ as follows. The objects of $\mathbb{T}_{\mathcal{O}}$ consist of pairs $\langle T, l^T \rangle$ such that:

- $T \in \mathbb{T}$.
- $U_{\langle T, l^T \rangle} = T$.
- $l^T = \{l^T_v : v \in T\}$, where for each $v \in T$ there is some $\gamma_v \in \mathrm{obj}(\mathcal{O})$ such that $l^T_v : \mathrm{succ}(v) \longrightarrow \gamma_v$ is a bijection.

[2]Note that every tree is well-founded in the sense of Definition 2.1.2, considered as a quasi-order. Thus when we use the term 'well-founded tree' we always mean well-founded in the sense of Definition 2.4.2.

[3]For the base case we have that $\sup(\emptyset) = 0$.

If $t \in \updownarrow v \setminus \mathrm{succ}(v)$ and if t' is the (unique) element of $\mathrm{succ}(v)$ such that $t > t'$, then we will occasionally abuse notation and write $l_v^T(t)$ in place of $l_v^T(t')$.

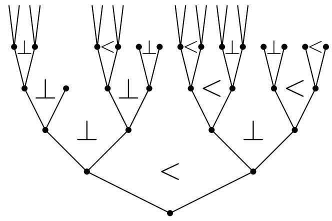

Figure 2.4: A tree structured by $\{\mathrm{C}_2, \mathrm{A}_2\}$.

For $\mathcal{O}$-structured trees $\langle T, l^T\rangle$ and $\langle T', l^{T'}\rangle$, we let $\varphi : T \to T'$ be an embedding whenever:

1. $x \leqslant y$ iff $\varphi(x) \leqslant \varphi(y)$,

2. $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$,

3. for any $v \in T$, if $\theta : \mathrm{range}(l_v^T) \to \mathrm{range}(l_{\varphi(v)}^{T'})$ is such that for all $x \in \mathrm{succ}(v)$
$$\theta(l_v^T(x)) = l_{\varphi(v)}^{T'}(\varphi(x));$$
then θ is an embedding of $\mathcal{O}$.

To simplify notation, we write T in place of $\langle T, l^T\rangle$ and always use $l^T = \{l_v^T : v \in T\}$.

Intuitively, when $\mathcal{O}$ is a class of partial orders, $\mathbb{T}_{\mathcal{O}}$ is obtained by taking $T \in \mathbb{T}$ and for each vertex $v \in T$, ordering the successors of v by some order in $\mathcal{O}$ as in Figure 2.4. An embedding for $\mathbb{T}_{\mathcal{O}}$ is then a tree embedding that preserves this ordering on the successors of v for every $v \in \mathbb{T}$. For example, in Figure 2.5 the map $\varphi : T \to U$ is a structured tree embedding because the induced map θ given by $l_a^T(x) \mapsto l_{\varphi(a)}^U(\varphi(x))$ for $x \in \{b, c\}$ is a partial order embedding.

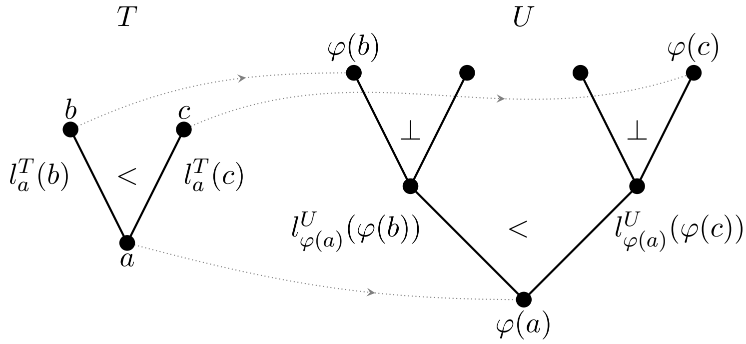

Figure 2.5: A structured tree embedding $\varphi : T \to U$.

Definition 2.4.6. Let T be an $\mathcal{O}$-structured tree, with $v \in T$ and $p \in \mathrm{range}(l_v^T)$ then we define

$$^p{\uparrow}v = \{t \in T : (\exists t' \in \mathrm{succ}(v)), t \geqslant t', l_v^T(t') = p\}.$$

It is clear that when $\mathbb{T}$ is a class of trees and $\mathcal{O}$ is a concrete category, then $\mathbb{T}_\mathcal{O}$ is a concrete category and hence we also have defined the Q-coloured, $\mathcal{O}$-structured trees of $\mathbb{T}$, denoted $\mathbb{T}_\mathcal{O}(Q)$. Finally we mention a theorem of Kříž that is fundamental to the results of this thesis.

Theorem 2.4.7 (Kříž, [25])**.** *If $\mathcal{O}$ is a well-behaved concrete category with injective morphisms, then $\mathscr{R}_\mathcal{O}$ is well-behaved.*

Proof. See [25]. □

Remark 2.4.8. Louveau and Saint-Raymond proved, using a modification of Nash-Williams' original method, that if $\mathcal{O}$ satisfies a slight weakening of well-behaved that they call *reflecting bad arrays* (which is stronger than preserving bqo) then $\mathscr{R}_\mathcal{O}$ also reflects bad arrays [33]. They were unable to attain full well-behavedness and Nash-Williams' method seems to be insufficient.

Chapter 3

Better-quasi-ordering partial orders

We note that the largest known classes of partial orders that preserve bqo are Laver's classes of σ-scattered trees and σ-scattered linear orders [30, 32] and Thomassé's class of countable N-free partial orders [55]. Each of these classes contains objects of a very different flavour than all of the objects in each of the other classes. It is therefore desirable to find a *natural* well-behaved class that incorporates all of these classes, unifying and expanding upon these very nice results. This is the aim of this section, which culminates with Theorem 3.5.12; where we prove for each $n \in \omega$ that a class of generalised σ-scattered orders is well-behaved. For $n = 1$ this class is $\mathscr{M}$, for $n = 2$ this class consists of transfinite N-free partial orders and for larger n the class contains partial orders that, for example, embed the partial order N (see Figure 3.1).

3.1 Operator construction

In this section we will define an operator algebra construction for partial orders similar to the one used by Pouzet in [44]. We make some modifications in order to use it to prove that the resulting class of partial orders is well-behaved, rather than just bqo.

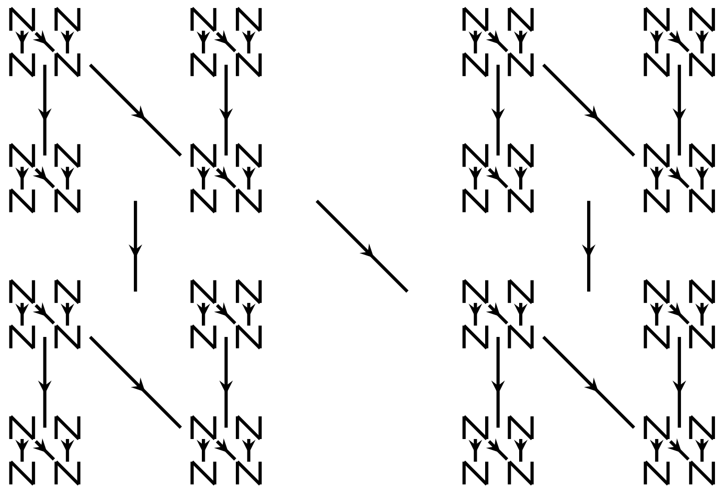

Figure 3.1: A generalised σ-scattered partial order.

Definition 3.1.1. An *admissible operator algebra* is a triple of the form $\mathcal{C} = \langle\{1\}, \mathcal{F}, \mathcal{A}\rangle$, where $\mathcal{A}$ is a class of partial orders[1] and $\mathcal{F}$ is the class of all functions of the form $\sum_P$ for $P \in \mathcal{A}$ (see Definition 2.3.9).

If $f \in \mathcal{F}$ and $f = \sum_P$ then we call P the *arity* $\mathbf{a}(f)$ of f.

Given an admissible operator algebra $\mathcal{C}$, we let $\tilde{\mathcal{C}}$ be the least class that contains the singleton partial order $1 = \{0\}$ and is closed under application of functions in $\mathcal{F}$. Thus, $\tilde{\mathcal{C}}$ enjoys the following inductive definition:

- $\mathcal{C}_0 = \{1\}$,
- $\mathcal{C}_{\alpha+1} = \{f(\hat{a}) : \hat{a} \in \mathcal{C}_\alpha^{\mathbf{a}(f)}, f \in \mathcal{F}\}$,[2]
- $\mathcal{C}_\lambda = \bigcup_{\gamma<\lambda} \mathcal{C}_\gamma$ for limit λ,
- $\tilde{\mathcal{C}} = \bigcup_{\gamma \in \mathrm{On}} \mathcal{C}_\gamma$.

Note also that $\tilde{\mathcal{C}} \subseteq \mathcal{P}$ and so $\tilde{\mathcal{C}}$ forms a concrete category whose morphisms are morphisms

[1]Considered as a concrete category whose morphisms are partial order embeddings

[2]Here $\hat{a}$ is as from Remark 2.2.6.

of $\mathscr{P}$.[3] For $\alpha \in \mathrm{On}$ we let

$$\mathcal{C}_{<\alpha} = \bigcup_{\gamma<\alpha} \mathcal{C}_\gamma.$$

Given $x \in \tilde{\mathcal{C}}$ we also define

$$\operatorname{rank}(x) = \min\{\alpha : x \in \mathcal{C}_\alpha\}.$$

Definition 3.1.2. Given a quasi-order Q and an admissible operator algebra $\mathcal{C} = \langle \{1\}, \mathcal{F}, \mathcal{A}\rangle$ and $\hat{x} \in \tilde{\mathcal{C}}(Q)$, we define $\hat{T} \in \mathscr{W}_{\mathcal{A}}(Q \cup \{-\infty\})$ a *decomposition tree for* $\hat{x}$ inductively as follows.

If $\hat{x} \in \mathcal{C}_0(Q)$, $x = 1$ and $T = \{t\}$, $\hat{T}(t) = \hat{x}(0)$ then we call $\hat{T}$ a decomposition tree for $\hat{x}$.

If $\alpha > 0$ and $\hat{x} \in \mathcal{C}_\alpha(Q)$, then $x = f(\hat{a})$ for some $f \in \mathcal{F}$ and some $\hat{a} \in \mathcal{C}_{<\alpha}^{\mathbf{a}(f)}$. Write $x_i = \hat{a}(i) \subseteq x$ for each $i \in \mathbf{a}(f)$ and pick decomposition trees $\hat{T}_i$ for each of the $\hat{x}_i = \hat{x} \restriction x_i$, $(i \in \mathbf{a}(f))$. Now let

$$T = \{t_0\} \cup \bigcup_{i \in \mathbf{a}(f)} T_i.$$

Order T so that for $s, t \in T$, we have $s < t$ iff either $s = t_0 \neq t$ or there is some $i \in \mathbf{a}(f)$ such that $s, t \in T_i$ and $s <_{T_i} t$. For $v \in T$ and $k \in \operatorname{succ}(v)$, define

$$l_v^T(k) = \begin{cases} l_v^{T_j}(k) & : v \in T_j \\ i & : v = t_0, k \in T_i \end{cases}.$$

Finally define $\hat{T} : T \to Q \cup \{-\infty\}$ and for $u \in T$,

$$\hat{T}(u) = \begin{cases} \hat{T}_j(u) & : u \in T_j \\ -\infty & : u = t_0 \end{cases}.$$

In this case we call $\hat{T}$ a decomposition tree for $\hat{x}$.

Lemma 3.1.3. *If $\hat{T}$ is a decomposition tree for $\hat{x} \in \tilde{\mathcal{C}}(Q)$ then for all $t \in T$, we have $\hat{T} \restriction \uparrow t$ is a decomposition tree for some $\hat{z} \in \tilde{\mathcal{C}}(Q)$.*

[3] Recall that $\mathscr{P}$ is the concrete category of partial orders, see Example 2.2.4.

Proof. By induction on $\operatorname{rank}(t)$. If $\operatorname{rank}(t) = 0$ then t is a leaf of T, so that $\uparrow t$ is a singleton. Thus $\hat{T} \restriction \uparrow t$ is a decomposition tree for $\hat{z} : 1 \to \{\hat{T}(t)\}$.

If $\operatorname{rank}(t) > 0$ then t is not a leaf of T. For each $i \in \operatorname{succ}(t)$ we have $\operatorname{rank}(i) < \operatorname{rank}(t)$ so by the induction hypothesis, the tree $\hat{T} \restriction \uparrow i$ is a decomposition tree for some $\hat{x}_i \in \tilde{\mathcal{C}}(Q)$. Furthermore, letting $a = \{l_t(i) : i \in \operatorname{succ}(t)\}$ we have $a \in \mathcal{A}$. Now let $f = \sum_a$, $\hat{a}(l_t(i)) = x_i$, $z = f(\hat{a})$ and for all $i \in \operatorname{succ}(t)$ and $j \in x_i$, let $\hat{z}(j) = \hat{x}_i(j)$. By construction then, $\hat{T} \restriction \uparrow t$ is a decomposition tree for $\hat{z} \in \tilde{\mathcal{C}}(Q)$. $\square$

Proposition 3.1.4. *Let P be a partial order, then for all $\hat{P} \in \mathcal{P}^P$ and all $i \in P$, we have*

$$\hat{P}(i) \leqslant \sum_P (\hat{P}).$$

Proof. The identity map is an embedding $\hat{P}(i) \to \sum_P (\hat{P})$. $\square$

In fact, Proposition 3.1.4 is implied by the following lemma.

Lemma 3.1.5. *Suppose that $\mathcal{C} = \langle \{1\}, \mathcal{F}, \mathcal{A} \rangle$ is an admissible operator algebra. Let $\hat{x}, \hat{y} \in \mathcal{P}(Q)$ be such that $x = f(\hat{a})$ and $y = g(\hat{b})$ for some $f, g \in \mathcal{F}$ and $\hat{a} \in \mathcal{P}^{\mathbf{a}(f)}$, $\hat{b} \in \mathcal{P}^{\mathbf{a}(g)}$. In this case if $\varphi : \mathbf{a}(f) \to \mathbf{a}(g)$ is an embedding such that for all $i \in \mathbf{a}(f)$,*

$$\hat{x} \restriction \hat{a}(i) \leqslant_{\mathcal{P}(Q)} \hat{y} \restriction (\hat{b} \circ \varphi(i)),$$

then $\hat{x} \leqslant \hat{y}$.

Proof. Let $\hat{x}$, $\hat{y}$, f, g, $\hat{a}$ and $\hat{b}$ be as described. Thus $x = \sum_{\mathbf{a}(f)} \hat{a}$ and $y = \sum_{\mathbf{a}(g)} \hat{b}$. Let $\varphi : \mathbf{a}(f) \to \mathbf{a}(g)$ be an embedding and for $i \in \mathbf{a}(f)$, suppose that φ_i witnesses $\hat{x} \restriction \hat{a}(i) \leqslant_{\mathcal{P}(Q)} \hat{y} \restriction (\hat{b} \circ \varphi(i))$. Now define $\psi : x \to y$ so that for all $j \in \hat{a}(i)$ we have

$$\psi(j) = \varphi_i(j) \in \hat{b} \circ \varphi(i) \subseteq y.$$

Then for $u, v \in x$ we have $u \leqslant v$ iff $u \in \hat{a}(i)$, $v \in \hat{a}(j)$, $i \leqslant_{\mathbf{a}(f)} j$ and $(i = j \longrightarrow u \leqslant_{\hat{a}(i)} v)$ iff

$$\psi(u) \in \hat{b} \circ \varphi(i), \quad \psi(v) \in \hat{b} \circ \varphi(j), \quad \varphi(i) \leqslant_{\mathbf{a}(g)} \varphi(j)$$

and

$$\varphi(i) = \varphi(j) \longrightarrow \varphi_i(u) \leqslant_{\hat{b} \circ \varphi(i)} \varphi_j(v)$$

iff $\psi(u) \leqslant \psi(v)$. Thus ψ is a partial order embedding. Furthermore, if $u \in x$ then $u \in \hat{a}(i)$ for some $i \in \mathbf{a}(f)$ so that $\hat{x}(u) \leqslant \hat{y} \circ \psi(u)$ as witnessed by φ_i. Thus ψ witnesses $\hat{x} \leqslant \hat{y}$. $\square$

Lemma 3.1.6. *Suppose that $\mathcal{C} = \langle \{1\}, \mathcal{F}, \mathcal{A} \rangle$ is an admissible operator algebra. If $\hat{T}_x$ and $\hat{T}_y$ are decomposition trees for $\hat{x}, \hat{y} \in \tilde{\mathcal{C}}(Q)$ respectively and $\exists t \in T_y$ such that $\hat{T}_x = \hat{T}_y \restriction \uparrow t$, then $\hat{x} \leqslant \hat{y}$.*

Proof. Let $L(t) \in \omega$ denote the level of T_y at which t appears, i.e. $t \in T_y$ has precisely $L(t)$ predecessors in T_y. We will prove the lemma by induction on $L(t)$. Suppose first that $L(t) = 0$, then $\hat{T}_x = \hat{T}_y$ and therefore since x and y can be constructed by precisely the same set of functions, and the colours of their leaves will be equal, a simple induction will show that $\hat{x} = \hat{y}$.

Suppose now that $L(t) = n + 1$ for some $n \in \omega$, and that the lemma holds for all members t' of a decomposition tree where t' has $\leqslant n$ predecessors. Then let t_0 be the root of T_y, so since $\hat{T}_y$ is a decomposition tree for $\hat{y}$, for some $f \in \mathcal{F}$ with $\mathbf{a}(f) = \mathrm{range}(l_{t_0}^{T_y})$ and some $\hat{a} \in \mathscr{P}^{\mathbf{a}(f)}$ we have $y = f(\hat{a})$. Let $i \in \mathrm{range}(l_{t_0}^{T_y})$ be such that $l_{t_0}^{T_y}(t) = i$. Since $\hat{T}_y$ is a decomposition tree, we see that ${}^{i}{\uparrow} t_0$ is a decomposition tree for $\hat{y} \restriction \hat{a}(i)$. But t has precisely n predecessors in ${}^{i}{\uparrow} t_0$, so by the induction hypothesis, $\hat{x} \leqslant \hat{y} \restriction \hat{a}(i)$. But since $\hat{a}(i) \subseteq f(\hat{a}) = y$ and by Proposition 3.1.4, we have:

$$\hat{x} \leqslant \hat{y} \restriction \hat{a}(i) \leqslant \hat{y} \restriction f(\hat{a}) = \hat{y}.$$

$\square$

Theorem 3.1.7. *Suppose that $\mathcal{C} = \langle \{1\}, \mathcal{F}, \mathcal{A} \rangle$ is an admissible operator algebra. If $\hat{T}_x$ and $\hat{T}_y$ are decomposition trees for $\hat{x}, \hat{y} \in \tilde{\mathcal{C}}(Q)$ respectively and $\hat{T}_x \leqslant \hat{T}_y$, then $\hat{x} \leqslant \hat{y}$.*

Proof. Suppose $\mathcal{C} = \langle \{1\}, \mathcal{F}, \mathcal{A} \rangle$ is as described and that $\hat{T}_x$ and $\hat{T}_y$ are decomposition trees for $\hat{x}, \hat{y} \in \tilde{\mathcal{C}}(Q)$ respectively and $\hat{T}_x \leqslant \hat{T}_y$. Let $\varphi : T_x \to T_y$ be an embedding witnessing $\hat{T}_x \leqslant \hat{T}_y$. We will prove $\hat{x} \leqslant \hat{y}$ by induction on $\mathrm{rank}(x)$.

If $\mathrm{rank}(x) = 0$ then $x = 1 = \{0\}$ and $T_x = \{t\}$ is a singleton. Then since φ witnesses $\hat{T}_x \leqslant \hat{T}_y$, we have

$$\hat{x}(0) = \hat{T}_x(t) \leqslant \hat{T}_y \circ \varphi(t).$$

So since $\hat{x}(0) \neq -\infty$, we have $\hat{T}_y \circ \varphi(t) \neq -\infty$ and therefore $\varphi(t)$ is a leaf of T_y. Now since $\hat{T}_y$ is a decomposition tree, by Lemma 3.1.3, it must be that $\hat{U} = \hat{T}_y \restriction \uparrow\varphi(t)$ is a decomposition tree for some $\hat{z} \in \tilde{\mathcal{C}}(Q)$, but then $U = \{\varphi(t)\}$ so that $\hat{z} \in \mathcal{C}_0(Q)$. Thus $z = 1 = x$ and $\hat{z}(0) = \hat{T}_y \circ \varphi(t) \geqslant \hat{x}(0)$, so the trivial embedding gives $\hat{x} \leqslant \hat{z}$. Now we can apply Lemma 3.1.6 to $\hat{U}$ and $\hat{T}_y$, to see that

$$\hat{x} \leqslant \hat{z} \leqslant \hat{y}.$$

Now suppose that $\mathrm{rank}(x) = \alpha > 0$. So $x = f(\hat{a})$ for some $\hat{a} \in \mathcal{C}_{<\alpha}^{\mathbf{a}(f)}$ with $\mathbf{a}(f) = \mathrm{range}(l_r^{T_x})$. Let $r = \mathrm{root}(T_x)$ and $s = \mathrm{root}(T_y)$ so by Lemma 3.1.6 we can assume without loss of generality that $\varphi(r) = s$. So because $\hat{T}_y$ is a decomposition tree for $\hat{y}$, we have $y = g(\hat{b})$ for some $g \in \mathcal{F}$, $\hat{b} \in \mathscr{P}^{\mathbf{a}(g)}$ with $\mathrm{range}(l_{\varphi(r)}^{T_y}) = \mathbf{a}(g)$.

Since φ is an embedding of structured trees, it induces an embedding

$$\varphi_r : \mathrm{range}(l_r^{T_x}) \longrightarrow \mathrm{range}(l_{\varphi(r)}^{T_y}).$$

In other words $\varphi_r : \mathbf{a}(f) \longrightarrow \mathbf{a}(g)$. Then for each $i \in \mathbf{a}(f)$, we have that $\hat{T}_x \restriction {}^{i}{\uparrow} r$ is a decomposition tree for $\hat{x} \restriction \hat{a}(i) \in \mathcal{C}_{<\alpha}(Q)$, and $\hat{T}_y \restriction {}^{\varphi_r(i)}{\uparrow} s$ is a decomposition tree for $\hat{y} \restriction (\hat{b} \circ \varphi_r(i)) \in \tilde{\mathcal{C}}(Q)$. Now since φ is a structured tree embedding we see that

$$\varphi({}^{i}{\uparrow} r) \subseteq {}^{\varphi_r(i)}{\uparrow} s.$$

Therefore by the induction hypothesis, for each $i \in \mathbf{a}(f)$, we have that

$$\hat{x} \restriction \hat{a}(i) \leqslant \hat{y} \restriction (\hat{b} \circ \varphi_r(i)).$$

But then by Lemma 3.1.5 we have $\hat{x} \leqslant \hat{y}$ as required. $\square$

Theorem 3.1.8. *Suppose that $\mathcal{C} = \langle \{1\}, \mathcal{F}, \mathcal{A} \rangle$ is an admissible operator algebra. If $\mathcal{A}$ is well-behaved then $\tilde{\mathcal{C}}$ is well-behaved.*

Proof. Suppose we have a bad $\tilde{\mathcal{C}}(Q)$-array f. Define the function g, with the same domain as f, such that $g(X)$ is a decomposition tree for $f(X)$. By Theorem 3.1.7 we have that g is a bad $\mathscr{W}_{\mathcal{A}}(Q \cup \{-\infty\})$-array. Theorem 2.4.7 tells us that if $\mathcal{A}$ is well-behaved, then

$\mathscr{W}_{\mathcal{A}} \subseteq \mathscr{R}_{\mathcal{A}}$ is well-behaved; whence there is a bad $Q \cup \{-\infty\}$-array h that is witnessing for g. Using Theorem 2.1.6 with the Borel set $h^{-1}(-\infty)$ we can now take a restriction h' of h that is a bad Q-array, and h' is still witnessing for g.

But each leaf of the tree $g(X)$ has the same colour as some element of $f(X)$, and hence h' is also witnessing for f. So every bad $\tilde{\mathcal{C}}(Q)$-array admits a witnessing bad Q-array, i.e. $\tilde{\mathcal{C}}$ is well-behaved. □

Corollary 3.1.9. *If $\mathbb{L}$ is a well-behaved class of linear orders, then $\overline{\mathbb{L}}$ is well-behaved.*

Proof. If $\mathcal{A} = \mathbb{L}$, $\mathcal{F} = \{\sum_L : L \in \mathbb{L}\}$ and $\mathcal{C} = \langle \{1\}, \mathcal{F}, \mathcal{A} \rangle$, then $\mathcal{C}$ is an admissible operator algebra and $\overline{\mathbb{L}} = \tilde{\mathcal{C}}$. So if $\mathbb{L}$ is well-behaved, then $\overline{\mathbb{L}}$ is well-behaved by Theorem 3.1.8. □

3.2 Scattered Partial Orders

Our aim is now to show that if we have a class of linear orders $\mathbb{L}$ and a class of partial orders $\mathbb{P}$, then whenever these are well-behaved[4], the corresponding scattered partial orders (with $\mathbb{L}$ and $\mathbb{P}$ as parameters) will be well-behaved too. First we will define an operator algebra which constructs these scattered orders internally, before giving a precise external description and proving equivalence.

3.2.1 The operator Algebra $\mathcal{S}_{\mathbb{P}}^{\mathbb{L}}$.

We begin by giving some machinery for building nested chains of intervals, for use with our operator algebra construction.

Definition 3.2.1. Suppose that $\mathbb{L}$ is a class of linear orders and $\mathbb{P}$ is a class of partial orders. Recall that $\mathrm{A}_2 = \{0, 1\}$ is the antichain of cardinality 2, as in Definition 2.3.3. Let

$$\mathbb{E}_{\mathbb{P}}^{\mathbb{L}} = \{\hat{r} \in \overline{\mathbb{L}}(\mathbb{P}(\mathrm{A}_2)) : (\forall i \in r), [\hat{r}(i) = \hat{a}_i \rightarrow (\exists! j \in a_i), \hat{a}_i(j) = 1]\}.$$

We consider $\mathbb{E}_{\mathbb{P}}^{\mathbb{L}}$ as a subcategory of $\overline{\mathbb{L}}(\mathbb{P}(\mathrm{A}_2))$.

[4]Considered as concrete categories with usual partial order embeddings, and with some other modest assumptions.

Definition 3.2.2. For $\hat{r} = \langle \hat{a}_i : i \in r \rangle \in \mathbb{E}_{\mathbb{P}}^{\mathbb{L}}$ and $i \in r$, let s_i be the unique element of a_i such that $\hat{a}_i(s_i) = 1$. Let $S_i = \emptyset$ if $i = \max r$ and $S_i = \{s_i\}$ otherwise. Now define the partial order

$$H_{\hat{r}} = \bigsqcup_{i \in r} (a_i \setminus S_i),$$

ordered so that for $u, v \in H_\eta$ we let $u < v$ iff $u \in a_i$, $v \in a_j$ and one of the following occurs:

- $i = j$ and $u <_{a_i} v$;
- $i < j$ and $u <_{a_i} s_i$;
- $i > j$ and $v >_{a_j} s_j$.

(See Figure 3.2.) Let $\mathbb{H}_{\mathbb{P}}^{\mathbb{L}} = \{H_{\hat{r}} : \hat{r} \in \mathbb{E}\}$, we consider $\mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ as a concrete category whose morphisms are partial order embeddings.

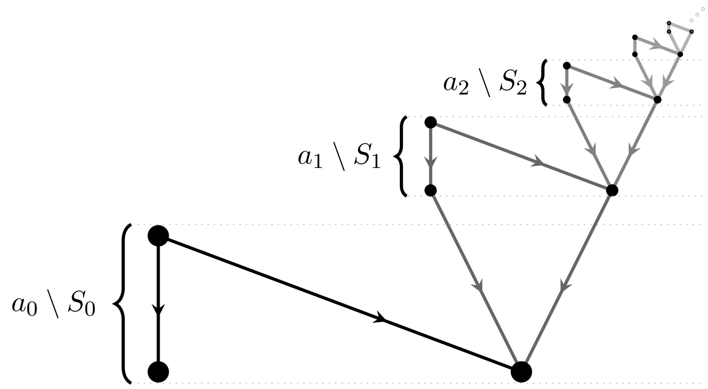

Figure 3.2: The partial order $H_{\hat{r}}$, for $\hat{r} = \langle \hat{N} : i \in \omega \rangle$, where $\hat{N}(x) = 1$ iff $x = 0$.

Definition 3.2.3. Let Q be a quasi-order and $\hat{H}_{\hat{r}} \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}(Q)$. Define

$$\Theta(\hat{H}_{\hat{r}}) \in \overline{\mathbb{L}}(\mathbb{P}(\mathrm{A}_2 \times (Q \cup \{-\infty\}))),$$

so that if $\hat{r} = \langle \hat{p}_i : i \in r \rangle$ then

$$\Theta(\hat{H}_{\hat{r}}) = \langle \hat{a}_i : i \in r \rangle,$$

where $a_i = p_i$ and for each $x \in a_i$, we have $\hat{a}_i(x) = \langle \hat{p}_i(x), q_{i,x} \rangle$, with

$$q_{i,x} = \begin{cases} \hat{H}_{\hat{r}}(x) & : \hat{a}_i(x) = 0 \text{ or } i = \max r \\ -\infty & : \hat{a}_i(x) = 1 \text{ and } i \neq \max r \end{cases}.$$

Lemma 3.2.4. *For all $\hat{H}_{\hat{r}}, \hat{H}_{\hat{u}} \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}(Q)$, if $\Theta(\hat{H}_{\hat{r}}) \leqslant \Theta(\hat{H}_{\hat{u}})$ then $\hat{H}_{\hat{r}} \leqslant \hat{H}_{\hat{u}}$.*

Proof. Suppose $\Theta(\hat{H}_{\hat{r}}) \leqslant \Theta(\hat{H}_{\hat{u}})$ and let $\hat{r} = \langle \hat{p}_i : i \in r \rangle$, $\hat{u} = \langle \hat{t}_j : j \in u \rangle$ and $\Theta(\hat{H}_{\hat{r}}) = \langle \hat{a}_i : i \in r \rangle$, $\Theta(\hat{H}_{\hat{u}}) = \langle \hat{b}_j : j \in u \rangle$. So for every $i \in r$ and $j \in u$ we have $p_i = a_i$ and $t_j = b_j$.

Since $\Theta(\hat{H}_{\hat{r}}), \Theta(\hat{H}_{\hat{u}}) \in \overline{\mathbb{L}}(\mathbb{P}(\mathrm{A}_2 \times (Q \cup \{-\infty\})))$, we have an embedding $\varphi : r \to u$ that witnesses $\Theta(\hat{H}_{\hat{r}}) \leqslant \Theta(\hat{H}_{\hat{u}})$. So for every $i \in r$ we have

$$\hat{a}_i \leqslant \hat{b}_{\varphi(i)}.$$

Let $\varphi_i : a_i \to b_{\varphi(i)}$ be an embedding witnessing this.

Given $x \in a_i$ and $y \in b_j$, let $q_{i,x}, q_{j,y} \in Q$ be such that $\hat{a}_i(x) = \langle \hat{p}_i(x), q_{i,x} \rangle$ and $\hat{b}_j(y) = \langle \hat{t}_j(y), q_{j,y} \rangle$. Therefore $\hat{p}_i(x) \leqslant \hat{t}_{\varphi(i)} \circ \varphi_i(x)$ as witnessed by φ_i, and thus

$$\hat{p}_i(x) = \hat{t}_{\varphi(i)} \circ \varphi_i(x), \tag{3.1}$$

because these are comparable elements of the antichain A_2. So by (3.1), if $\hat{p}_i(x) = 1$ and $i \neq \max r$ then $q_{i,x} = q_{\varphi(i),\varphi_i(x)} = -\infty$; and if $\hat{p}_i(x) = 0$ or $i = \max r$ then

$$\hat{H}_{\hat{r}}(x) \leqslant \hat{H}_{\hat{u}} \circ \varphi_i(x). \tag{3.2}$$

Now let $\psi : H_{\hat{r}} \longrightarrow H_{\hat{u}}$, be such that for all $x \in H_{\hat{r}} \subseteq \bigsqcup_{i \in r} a_i$ we have $\psi(x) = \varphi_i(x)$ whenever $x \in a_i$. So if ψ is an embedding, by (3.2) it will witness $\hat{H}_{\hat{r}} \leqslant \hat{H}_{\hat{u}}$. Thus it remains only to check that for $w, z \in H_{\hat{r}}$, we have $w \leqslant z$ iff $\psi(w) \leqslant \psi(z)$.

Let s_i be the unique element of a_i such that $\hat{p}_i(s_i) = 1$, and let s'_j be the unique element of b_j such that $\hat{t}_j(s'_j) = 1$. Then by (3.1), we have

$$\varphi_i(s_i) = s'_{\varphi(i)}.$$

For $i, j \in r$ let $w \in a_i$, $z \in a_j$. Then by definition of the order on $H_{\hat{r}}$, we have:

- if $i = j$ then $w \leqslant z$ iff $w \leqslant_{a_i} z$ iff $\varphi_i(w) = \varphi_i(z)$ iff $\psi(w) \leqslant \psi(z)$, since φ_i was an embedding.
- if $i < j$ then $w \leqslant z$ iff $w \leqslant_{a_i} s_i$ iff $\varphi_i(w) \leqslant_{b_{\varphi(i)}} \varphi_i(s_i) = s'_{\varphi(i)}$ iff $\psi(w) \leqslant \psi(z)$.
- if $i > j$ then $w \leqslant z$ iff $s_i \leqslant_{a_i} z$ iff $s'_{\varphi(i)} = \varphi_i(s_i) \leqslant_{b_{\varphi(i)}} \varphi_i(z)$ iff $\psi(w) \leqslant \psi(z)$.

Therefore ψ witnesses $\hat{H}_{\hat{r}} \leqslant \hat{H}_{\hat{u}}$. □

Lemma 3.2.5. *If $\mathbb{L}$ and $\mathbb{P}$ are well-behaved then $\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ is well-behaved.*

Proof. Suppose there is a bad $\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}(Q)$-array f. Now define the function g with the same domain as f, such that $g(X) = \Theta(f(X))$. Note that if $g(X) = \langle \hat{a}_i : i \in r \rangle$, and for $x \in a_i$ we have $\hat{a}_i(x) = \langle v, q \rangle$ for some $v \in \mathrm{A}_2$ and $q \in Q$, (i.e. $q \neq -\infty$) then $f(X)(x) = q$. Whence any witnessing Q-array for g will also be witnessing for f.

So by Lemma 3.2.4 we see that g is a bad $\overline{\mathbb{L}}(\mathbb{P}(\mathrm{A}_2 \times (Q \cup \{-\infty\})))$-array. Since both $\mathbb{L}$ and $\mathbb{P}$ are well-behaved and by Corollary 3.1.9, we obtain from g a witnessing bad $\mathrm{A}_2 \times (Q \cup \{-\infty\})$-array. By Theorem 2.1.10, and since A_2 is finite and therefore bqo, we obtain a witnessing bad $Q \cup \{-\infty\}$-array, and therefore by Theorem 2.1.6, we can restrict to find a bad Q-array, that is witnessing for g and thus for f. □

Definition 3.2.6. Suppose that $\mathbb{L}$ is a class of linear orders and $\mathbb{P}$ is a class of partial orders. Let $\mathcal{F} = \{\sum_H : H \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}\}$, and $\mathcal{A} = \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$. Then let $\mathcal{S}^{\mathbb{L}}_{\mathbb{P}} = \langle \{1\}, \mathcal{F}, \mathcal{A} \rangle$. It is clear then that $\mathcal{S}^{\mathbb{L}}_{\mathbb{P}}$ is an admissible operator algebra.

Theorem 3.2.7. *If $\mathbb{L}$ and $\mathbb{P}$ are well-behaved then $\tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ is well-behaved.*

Proof. By Lemma 3.2.5 $\mathcal{A} = \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ is well-behaved, hence the result follows by applying Theorem 3.1.8 to the admissible operator algebra $\mathcal{S}^{\mathbb{L}}_{\mathbb{P}}$. □

3.2.2 Intervals and indecomposable partial orders

We want to define the indecomposable partial orders which will serve as building blocks for larger partial orders, in order to do so we first require the notion of an *interval* (Fraïssé [17]).

Definition 3.2.8. Suppose that $a, b, c \in x \in \mathcal{C}$. We say that *a shares the same relationship* to b and c, and write $\mathrm{SSR}(a; b, c)$ iff for all $R \in \{<, >, \perp\}$ we have

$$aRb \text{ iff } aRc.$$

Definition 3.2.9. Let P be a partial order and $I \subseteq P$, then we call $I \neq \emptyset$ an *interval* of P if $\forall x, y \in I$ and $\forall p \in P \setminus I$ we have $\mathrm{SSR}(p; x, y)$.

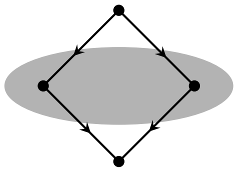

Figure 3.3: An interval of a partial order.

Definition 3.2.10. Let P be a partial order. Then P is called *indecomposable* if every interval of P is either P itself, or a singleton.

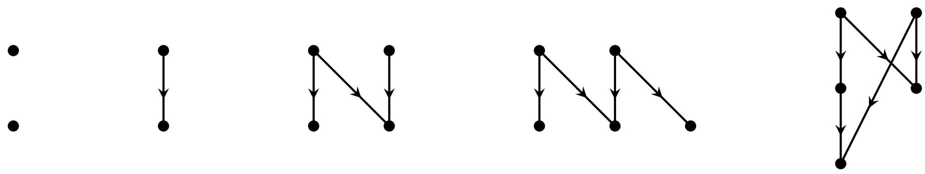

Figure 3.4: Examples of indecomposable partial orders.

Lemma 3.2.11. *Let $\langle I_j : j \in r\rangle$ be a chain of intervals of a partial order P under $\supseteq$. Then $\bigcup_{j\in r} I_j$ and $\bigcap_{j\in r} I_j$ are intervals.*

Proof. Let $a \in P \setminus \bigcup_{j\in r} I_j$ and $b, c \in \bigcup_{j\in r} I_j$. Then $b, c \in I_i$ for some $i \in r$, we know that I_i is an interval hence $\mathrm{SSR}(a; b, c)$ as required. The case of intersection is similar. $\square$

Proposition 3.2.12. *Let $\hat{r} = \langle \hat{a}_i : i \in r\rangle \in \mathbb{E}^{\mathbb{L}}_{\mathbb{P}}$ and $b_0, b_1, b_2 \in H_{\hat{r}}$, such that for each $i \in \{0, 1, 2\}$ we have $b_i \in a_{j_i}$ for $j_i \in r$ then*

$$j_0 < j_1, j_2 \longrightarrow \mathrm{SSR}(b_0; b_1, b_2).$$

Proof. Suppose that $j_0 < j_1, j_2$ and $b_i \in a_{j_i}$ for each $i \in \{0, 1, 2\}$. Let s be the unique element of a_{j_0} such that $\hat{a}_{j_0}(s) = 1$. Suppose that $b_0 < b_1$, then by definition of $H_{\hat{r}}$ and because $j_0 < j_1$, we have that $b_0 < s$. But then since we also have $j_0 < j_2$, we have that $b_0 < b_2$, so in this case $\mathrm{SSR}(b_0; b_1, b_2)$. The cases for when $b_0 > b_1$ and $b_0 \perp b_1$ are similar. $\square$

Definition 3.2.13. We let $2^{<\omega}$ be the binary tree consisting of all finite sequences of elements of $\{0, 1\}$ ordered by $\sqsubseteq$. We let $-2^{<\omega}$ be the partial order obtained by reversing the order on $2^{<\omega}$. We also define the partial order $2_\perp^{<\omega}$ with the same underlying set as $2^{<\omega}$. For $s, t \in 2_\perp^{<\omega}$, we define $s < t$ iff there is some sequence u such that $u^\frown\langle 0\rangle \sqsubseteq s$ and $u^\frown\langle 1\rangle \sqsubseteq t$. (See Figure 3.5.)

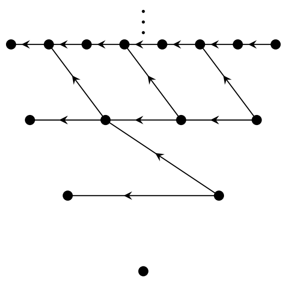

Figure 3.5: The partial order $2_\perp^{<\omega}$.

We are now able to define our class of partial orders that will be *scattered* in some sense.

Definition 3.2.14 (Scattered partial orders)**.** We define $\mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ to be the class of non-empty partial orders X with the following properties.

(i) Every indecomposable subset of X is isomorphic to a member of $\mathbb{P}$.

(ii) For every $x \in X$, there is a maximal chain of non-empty intervals of X with respect to $\supseteq$, with order type in $\overline{\mathbb{L}}$ that contains $\{x\}$.

(iii) $2^{<\omega}$, $-2^{<\omega}$ and $2_\perp^{<\omega}$ do not embed into X.

We let $\mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ be the class of those non-empty X satisfying (i) and (ii).

Remark 3.2.15. Using a result from [23], we have that any indecomposable partial order with at least three vertices embeds N (see also [55]). Thus for $\mathbb{P} = \{1, \mathrm{C}_2, \mathrm{A}_2\}$, condition 3.2.14 (i) is precisely that the order X is N-free. Furthermore, it is clear that the only indecomposable linear orders are 1 and C_2. So similarly for $\mathbb{P} = \{1, \mathrm{C}_2\}$, condition 3.2.14 (i) is precisely that the order X is linear.

3.3 Extending Hausdorff's theorem

The aim of this section is to prove the following theorem:

Theorem 3.3.1. *Let $\mathbb{P}$ be a class of indecomposable partial orders that do not embed any element of $\{2^{<\omega}, -2^{<\omega}, 2^{<\omega}_{\perp}\}$, that is closed under taking indecomposable subsets. Let $\mathbb{L}$ be a class of linear orders closed under taking subsets and reversing orders, such that* $\mathrm{On} \subseteq \mathbb{L}$. *Then $\tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}} = \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$.*

This can be seen to be an extension of Hausdorff's theorem 2.3.11, in the sense that it shows that members of a certain class of externally defined partial orders have a well-founded internal tree representation. Condition 3.2.14 (iii) seems especially reminiscent. In particular, when $\mathbb{P} = \{1, \mathrm{C}_2\}$ and $\mathbb{L} = \mathrm{On} \cup \mathrm{On}^*$, orders satisfying 3.2.14 (i) are linear and thus satisfy both (ii) and (iii) automatically of Definition 3.2.14. Therefore by 3.2.14 (ii) the class $\mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ is precisely $\overline{\mathbb{L}}$ which is in turn equal to the class of scattered linear orders $\mathscr{S}$ by Hausdorff's theorem 2.3.11.

For the rest of this chapter we will assume that $\mathbb{P}$ and $\mathbb{L}$ satisfy the assumptions of Theorem 3.3.1. The following two subsections contain the proof of Theorem 3.3.1, which will follow once we prove Theorem 3.3.13 and Theorem 3.3.31, that show each containment. Thus using Theorem 3.2.7 we will then immediately obtain the following scattered version of our main result as a corollary.

Theorem 3.3.2. *Let $\mathbb{P}$ be a class of indecomposable partial orders that do not embed any element of $\{2^{<\omega}, -2^{<\omega}, 2^{<\omega}_{\perp}\}$, that is closed under taking indecomposable subsets. Let $\mathbb{L}$*

be a class of linear orders closed under taking subsets, such that On $\subseteq \mathbb{L}$*. If* $\mathbb{L}$ *and* $\mathbb{P}$ *are well-behaved then* $\mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ *is well-behaved.*

Remark 3.3.3. The closure of $\mathbb{L}$ under reversing orders is not used to show that $\mathscr{S}_{\mathbb{P}}^{\mathbb{L}} \subseteq \tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$, so this assumption is omitted in the statement of Theorem 3.3.2. However, this assumption can essentially be taken for free because if $\mathbb{L}$ is well-behaved then its closure under reversing orders (i.e. $\mathbb{L} \cup \mathbb{L}^*$) is easily seen to be well-behaved by Theorem 2.1.6.

3.3.1 $\tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}} \subseteq \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$

We will show this inclusion by induction on the rank of a member of $\tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$, by first proving the induction step for each of the conditions (i), (ii) and (iii) of Definition 3.2.14. In particular, we wish to show that if we take some orders satisfying one of the conditions 3.2.14 (i), (ii) or (iii); any lexicographic H-sum (for $H \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$) of these orders will also satisfy that condition.

Lemma 3.3.4. *If* $Y = \sum_{x \in H} Y_x$ *where* $H \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ *and each* Y_x *satisfies 3.2.14 (i), then* Y *satisfies 3.2.14 (i).*

Proof. Suppose $H = H_{\hat{r}}$, $\hat{r} = \langle \hat{a}_i : i \in r \rangle \in \mathbb{E}_{\mathbb{P}}^{\mathbb{L}}$ and $i \in r$. If $A \subseteq H$ is such that elements of $A \cap a_i \neq \emptyset$ and $|A \cap \bigcup_{j>i} a_j^{\eta}| \geqslant 2$, then $A \cap \bigcup_{j>i} a_j^{\eta}$ is an interval by Proposition 3.2.12, and since it has at least two members, we see that A is not indecomposable. Thus any indecomposable subset I of H is a subset of $a_i \sqcup a_j$ for some $i, j \in r$ with $i < j$ and $|I \cap a_j^{\eta}| \leqslant 1$. So I has the same order type as a subset of $a_i \in \mathbb{P}$, which shows that I has order type in $\mathbb{P}$.

Thus if we take a subset $A \subseteq Y$ with at least two points inside a single Y_x and at least one point not in Y_x then since $A \cap Y_x$ is an interval of A we see that A is not indecomposable. So if J is an indecomposable subset Y then either J is entirely contained within some Y_x and hence J has order type in $\mathbb{P}$; or J contains at most one point of each of the Y_x that it intersects, and hence has the same order type as an indecomposable subset of H_η. Hence by the previous paragraph J has order type in $\mathbb{P}$, which completes the proof. $\square$

Lemma 3.3.5. *Suppose that:*

- *U is an indecomposable partial order with $|U| > 2$;*

- *$P = \sum_{u \in U} P_u$ for some non-empty partial orders P_u with $u \in U$;*

- *$I \subseteq P$ is an interval of P with $I \cap P_{v_0} \neq \emptyset$ and $I \cap P_{v_1} \neq \emptyset$ for some $v_0 \neq v_1 \in U$.*

Then $I = P$.

Proof. First we claim that

$$J = \{u : I \cap P_u \neq \emptyset\}$$

is an interval of U. To see this, we let $v \in U \setminus J$, $u_0, u_1 \in J$, $a \in P_v$ and $b_0 \in I \cap P_{u_0}$, $b_1 \in I \cap P_{u_1}$. Then $a \notin I$ (otherwise $v \in J$), so we have $\mathrm{SSR}(a; b_0, b_1)$ because I was an interval. But this implies $\mathrm{SSR}(v; u_0, u_1)$ since $P = \sum_{u \in U} P_u$, hence we have the claim that J is an interval of U.

Since U was indecomposable, we either have $|J| = 1$ or $J = U$. If $|J| = 1$ then this contradicts our assumption that $I \cap P_{v_0} \neq \emptyset$ and $I \cap P_{v_1} \neq \emptyset$ for some $v_0 \neq v_1 \in U$. Hence $J = U$.

Suppose for contradiction that there is some $a \in P \setminus I$, then $a \in P_v$ for some $v \in U$. For arbitrary $u_0, u_1 \in U$ with $v \notin \{u_0, u_1\}$ we have $b_0 \in I \cap P_{u_0}$ and $b_1 \in I \cap P_{u_1}$. Since $a \in P \setminus I$, $b_0, b_1 \in I$ and I is an interval, we then know that $\mathrm{SSR}(a; b_0, b_1)$. So since $P = \sum_{u \in U} P_u$ and $v \notin \{u_0, u_1\}$ we see that $\mathrm{SSR}(v; u_0, u_1)$. But then since u_0 and u_1 were arbitrary, it must be that $U \setminus \{v\}$ is an interval. Hence since U was indecomposable we have $|U \setminus \{v\}| = 1$ which means $|U| = 2$ which is a contradiction. □

Lemma 3.3.6. *Suppose that Y is a partial order that satisfies 3.2.14 (ii), $Y = \sum_{i \in L} K_i$ for some linear order L and $\forall i \in L$ there are no $W_0, W_1 \subseteq K_i$ such that $K_i = \sum_{j \in \mathrm{C}_2} W_j$. Then $L \in \overline{\mathbb{L}}$.*

Proof. Pick $x \in K_j$ for some $j \in L$, and find (using our assumption) a maximal chain $C = \langle C_j : j \in \mathrm{ot}(C) \rangle$ of intervals of Y containing $\{x\}$, with order type in $\overline{\mathbb{L}}$.

So C has a final segment consisting of some maximal chain of intervals of K_j and for all C_j not in this final segment, $C_j \setminus \bigcup_{j' > j} C_{j'} = K_l$ for some $l \in L$. In this case let $\tau(C_j) = l$.

Now let

$$C' = \{J \in C : K_j \subseteq J, \text{ and if } J' \in C, J' \subset J, \text{ then } \tau(J) < \tau(J')\}.$$

So by construction C' has order type equal to the initial segment $L' = \{j' \in L : j' \leqslant j\}$ of L. Furthermore since C' is a subset of C which has order type in $\overline{\mathbb{L}}$, and since $\mathbb{L}$ is closed under taking subsets, we see that C' has order type in $\overline{\mathbb{L}}$, and thus $L' \in \overline{\mathbb{L}}$.

Now choose a cofinal subset $S \subseteq L$ in some ordinal order type. So that for each $u \in S$ we have $\{v \in L : v \leqslant u\}$ has order type in $\overline{\mathbb{L}}$. Moreover

$$L_u = \{v \in L : (\forall u' \in S), u' < u \to u' < v \leqslant u\}$$

has order type in $\overline{\mathbb{L}}$. But then $L = \sum_{u \in S} L_u$ and therefore L has order type in $\overline{\mathbb{L}}$. $\square$

Lemma 3.3.7. *Suppose $Y = \sum_{u \in H} Y_u = \sum_{j \in L} K_j$ for some $H \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ and some linear order L. Suppose also that each Y_u $(u \in H)$ satisfies 3.2.14 (ii) and that $\forall i \in L$ there are no $W_0, W_1 \subseteq K_i$ such that $K_i = \sum_{j \in \mathrm{C}_2} W_j$. Then $L \in \overline{\mathbb{L}}$.*

Proof. Let $\hat{r} = \langle \hat{a}_i : i \in r\rangle \in \mathbb{E}^{\mathbb{L}}_{\mathbb{P}}$ be such that $H = H_{\hat{r}}$. For each $j \in r$ let s_j be the unique element of a_j such that $\hat{a}_j(s_j) = 1$. If $j \in r$ is such that $a_j \neq \mathrm{C}_2$ then pick $x \in a_j$ that is incomparable to s_j, which is possible since otherwise either $\downarrow s_j$, $\uparrow s_j$, $\downarrow\!\!\!\!\downarrow s_j$ or $\uparrow\!\!\!\!\uparrow s_j$ is a proper interval of a_j, which contradicts that $a_j \in \mathbb{P}$ is indecomposable. Thus for any $j' > j$ and any $u \in H \cap a_{j'}$ we have that any $y_0 \in Y_x$ is incomparable to any $y_1 \in Y_u$. Indeed since a_j cannot be written as a lexicographic C_2-sum of any partial orders (because it is indecomposable and not equal to C_2), there must be some $i \in L$ such that $Y_u \subseteq K_i$ for each $u \in a_i \cap H$ and furthermore,

$$\bigcup_{j' \geqslant j} \bigcup_{u \in a_{j'} \cap H} Y_u \subseteq K_i.$$

Let $r' \sqsubseteq r$ be longest such that $\forall j \in r'$, $a_j = \mathrm{C}_2$ then let

$$r_0 = \{i \in r' : a_i = \{x, s_i\}, x < s_i\}$$

and

$$r_1 = \{i \in r' : a_i = \{x, s_i\}, s_i < x\}.$$

Then $r_0, r_1 \subseteq r' \subseteq r$ and thus r_0 and r_1 have order type in $\overline{\mathbb{L}}$. We let r_1^* be the reversed order of r_1. For $i \in r' = r_0 \cup r_1$ let $Y_i = Y_{x_i}$ where x_i is the unique member of $a_i \setminus \{s_i\}$. If $r' = r$ and r has a maximal element, then let $Y_{i_0} = Y_{s_{\max r}}$ and $w = r_0{}^\frown\langle i_0\rangle{}^\frown r_1^*$. If $r = r'$ and r has no maximal element then let $w = r_0{}^\frown r_1^*$. If $r' \sqsubset r$ then let $Y_{i_0} = \bigcup_{j' \geqslant j} \bigcup_{u \in a_{j'} \cap H} Y_u$ and $w = r_0{}^\frown\langle i_0\rangle{}^\frown r_1^*$. Then since $H_{\hat{r}'} = H_{\hat{r}_0}{}^\frown H_{\hat{r}_1} = r_0{}^\frown r_1^*$ and by $Y = \sum_{i \in H_{\hat{r}}} Y_u$, we have

$$Y = \sum_{i \in w} Y_i = \sum_{j \in L} K_j.$$

Thus since the K_j cannot be partitioned into further lexicographic C_2-sums, we have for some partition L_i $(i \in w)$ that

$$Y = \sum_{i \in w} \sum_{j \in L_i} K_j.$$

Furthermore whenever $i \in w \setminus \{i_0\}$ we have $\sum_{j \in L_i} K_j \subseteq Y_{x_i}$ and therefore by Lemma 3.3.6 we have $L_i \in \overline{\mathbb{L}}$. But $\sum_{j \in L_{i_0}} K_j \subseteq \bigcup_{j' \geqslant j} \bigcup_{u \in a_{j'} \cap H} Y_u \subseteq K_{j_0}$ for some j_0 so that $K_{i_0} = 1 \in \overline{\mathbb{L}}$. Therefore $L = \sum_{i \in w} L_i$, with $w \in \overline{\mathbb{L}}$ and $L_i \in \overline{\mathbb{L}}$ for each $i \in w$; hence $L \in \overline{\mathbb{L}}$. $\square$

Lemma 3.3.8. *If $Y = \sum_{x \in H} Y_x$ where $H \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ and each Y_x satisfies 3.2.14 (ii), then Y satisfies 3.2.14 (ii).*

Proof. For some $\hat{r} = \langle \hat{a}_i : i \in r\rangle \in \mathbb{E}_{\mathbb{P}}^{\mathbb{L}}$ we have $H = H_{\hat{r}}$. For $j \in r$, let $\hat{r}_j = \langle \hat{a}_i : i \in r, j \leqslant i\rangle$ and

$$I_j = \sum_{x \in H_{\hat{r}_i}} Y_x \subseteq Y.$$

Then for all $j \in r$, $k \in Y \setminus I_j$ and $l_0, l_1 \in I_j$, we have by Proposition 3.2.12 that $\mathrm{SSR}(k; l_0, l_1)$, and hence I_j is an interval of Y.

For $i \in r$ let s_i be the unique element of a_i such that $\hat{a}_i(s_i) = 1$, and when $i \neq \max r$, let

$$Y_{s_i} = \bigcup_{j > i} I_j.$$

Note that Y_x is already defined for all $x \in H$, so with this definition we have defined Y_x for all $x \in \bigsqcup_{i \in r} a_i$. We will now prove the following claim.

Claim: If for some $i \in r$ and $u \in a_i$, we either have $X = Y_u$ or $X = \bigcup_{j>i} I_j$, then there is a maximal chain under $\supseteq$ of intervals of I_i that contain X, with order type in $\overline{\mathbb{L}}$.

Proof of claim: Suppose that $J \subseteq I_i$ is an interval of Y such that $X \subset J$. Then J is an interval of

$$I_i = \sum_{x \in a_i} Y_x,$$

and so by Lemma 3.3.5, since $a_i \in \mathbb{P}$ is indecomposable, if $|a_i| > 2$ then $J = I_i$; therefore $\{X, I_i\}$ is a maximal chain of such intervals and satisfies the statement of the claim.

Suppose that $|a_i| = 2$, so $a_i = \{y_0, y_1\}$ for some y_0 such that $X = Y_{y_0}$, and some y_1, so that $Y_{x_1} \cap J$ is an interval of Y_{x_1}. So for some $R \in \{<, >, \perp\}$ and for all $u \in Y_{y_1} = I_i \setminus X$ and $v \in X$, we have uRv. So for all $u' \in I_i \setminus J$ and for all $v' \in J$ we have also $u'Rv'$.

Suppose that $R = \perp$. For some cardinal κ, we can let U_α for $\alpha \in \kappa$ be some non-empty subsets of Y_{y_1} such that for any $\alpha \neq \beta$ and $u \in U_\alpha$, $v \in U_\beta$ we have $u \perp v$. Moreover, suppose that these U_α are maximal in the sense that for each $\alpha \in \kappa$ there are no non-empty $W_0, W_1 \subseteq U_\alpha$ with $U_\alpha = \sum_{j \in \mathrm{A}_2} W_j$. Now let

$$J_\alpha = X \cup \bigcup_{\gamma \geqslant \alpha} U_\alpha.$$

Then each J_α is an interval of Y since any element of $Y \setminus J_\alpha$ is incomparable to any point inside J_α. We also have that

$$\langle J_\alpha : \alpha \in \kappa \rangle^\frown \langle X \rangle$$

is a maximal chain of intervals that contain X, since if $J \cap U_\alpha \neq \emptyset$ then $U_\alpha \subseteq J$ (otherwise $W_0 = U_\alpha \cap J$ and $W_1 = U_\alpha \setminus J$ contradict our maximality of U_α). So we have the claim when $R = \perp$, since this chain has ordinal order type and $\mathrm{On} \subseteq \mathbb{L}$.

Now suppose that $R = <$. For some linear order L, and some $K_i \subseteq Y_{y_1}$ $(i \in L)$ we have

$$Y_{y_1} = \sum_{j \in L} K_j.$$

Furthermore we can suppose that for all $j \in L$ there are no non-empty $W_0, W_1 \subseteq K_j$ such that $K_j = \sum_{u \in C_2} W_u$.

If $X = \bigcup_{j>i} I_j$ then Y_{y_1} satisfies 3.2.14 (ii) so by Lemma 3.3.6 we see that $L \in \overline{\mathbb{L}}$. Otherwise $Y_{y_1} = \bigcup_{j>i} I_j = \sum_{u \in H} Y_u$ for some $H \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$, therefore $L \in \overline{\mathbb{L}}$ follows by Lemma 3.3.7.

Let J be an interval of I_i such that $X \subseteq J$. There can be no $y \in J$ and $z \in I_i \setminus J$ with $y < z$, since $\forall y' \in X$ we have $\mathrm{SSR}(z; y, y')$ and $z < y'$. Furthermore, if for some $j \in L$ we have $J \cap K_j \neq \emptyset$, then $K_j \subseteq J$ (otherwise $W_0 = K_j \setminus J$ and $W_1 = K_j \cap J$ contradict our assumption on K_j). Hence any such interval J is such that there is some final segment of L' of L with $J = X \cup \sum_{j \in L'} K_j$.

For $j \in L$, let

$$J_j = X \cup \sum_{j' \in L, j' \geqslant j} K_{j'} \subseteq I_i.$$

Then $\langle J_j : j \in L \rangle^\frown \langle X \rangle$ is a maximal chain of intervals of I_i, all of which contain X which has order type L and thus has order type in $\overline{\mathbb{L}}$.

If $R = >$ then the claim holds symmetrically. ■

Now let $u \in Y$, then $u \in Y_{x_0}$ for some $x_0 \in H$. Thus, there is a maximal chain C_0 of intervals of Y_{x_0} with respect to $\supseteq$ with $\mathrm{ot}(C_0) \in \overline{\mathbb{L}}$ and $\{u\} \in C_0$. Since C_0 is maximal, it has largest element Y_{x_0}. Applying the claim now to $X = Y_{x_0}$ and i such that $x_0 \in a_i$; there is a maximal chain C_1 of intervals of I_i that contain Y_{x_0}, with order type in $\overline{\mathbb{L}}$. Now for $j \in r$ with $j \leqslant i$, let C'_j be the maximal chain of intervals of I_j that contain $\bigcup_{j_0 > j} I_j$ given by the claim. Hence

$$C_0 \cup C_1 \cup \bigcup_{j<i} C'_j$$

is a chain of intervals that contain $\{u\} \in C_0$. It is maximal since each of its components were maximal. Moreover this chain has order type

$$\left(\sum_{j \in r, j<i} \mathrm{ot}(C'_j) \right) {}^\frown \mathrm{ot}(C_1)^\frown \mathrm{ot}(C_0) \in \overline{\mathbb{L}}.$$

Thus we can find a chain of intervals of Y that satisfies the lemma. □

Lemma 3.3.9. *If $Y = \sum_{x \in H} Y_x$ where $2^{<\omega}, -2^{<\omega}, 2_\perp^{<\omega} \nleqslant H$ and for each $x \in H$ we have $2^{<\omega}, -2^{<\omega}, 2_\perp^{<\omega} \nleqslant Y_x$, then $2^{<\omega}, -2^{<\omega}, 2_\perp^{<\omega} \nleqslant Y$.*

Proof. Fix $X \in \{2^{<\omega}, -2^{<\omega}, 2^{<\omega}_{\perp}\}$ and suppose that $X \leqslant Y$, with φ a witnessing embedding. For $s \in X$, let $\tau(s)$ be such that $\varphi(t) \in Y_{\tau(t)}$.

Let $i \in H$ and $s \in X$. We claim that there is some $t \in X$ with $s \sqsubseteq t$ such that for all $u \in X$ with $t \sqsubseteq u$ we have $\tau(u) \neq i$.

Suppose for contradiction that for all $t \in X$ with $s \sqsubseteq t$ there is some $\pi(t) \in X$ with $t \sqsubseteq \pi(t)$ and $\tau \circ \pi(t) = i$. We now define $\psi : X \to Y_i$ inductively by letting $\psi(\langle\rangle) = \varphi(\pi(s))$; and if we have defined $\psi(t) = \varphi(t')$ then for $m \in \{0, 1\}$ we let

$$\psi(t^\frown\langle m\rangle) = \varphi \circ \pi(t'^\frown\langle m\rangle).$$

Then ψ is an embedding since φ is an embedding, contradicting that $X \nleqslant Y_i$.

Let F be a finite subset of H, so applying the claim repeatedly for each $i \in F$, we have that for all $s \in X$ there is some $s_F \in y$ with $s \sqsubseteq s_F$ and for all $i \in F$ and all $z \in X$ with $s_F \sqsubseteq z$ we have $\tau(z) \neq i$.

Now define $\mu(\langle\rangle) = \varphi(\langle\rangle)$ and suppose inductively that we have defined μ on some sequences $t \in X$ so that $\mu(t) = \varphi(t')$ for some $t' \in X$. Let G be the set of t such that $\mu(t)$ is already defined. Let w be the lexicographically least element of

$$\{y \in X \setminus G : (\forall z \in X \setminus G), |y| \leqslant |z|\},$$

now let $v \in X$ and $m \in \{0, 1\}$ be such that $w = v^\frown\langle m\rangle$; so $\mu(v)$ is already defined. Let $w' = v'^\frown\langle m\rangle$ and $\mu(w) = \varphi(s_G)$. Thus μ is an embedding, and $\tau \circ \mu(t)$ is distinct for distinct t. Therefore $\tau \circ \mu : X \to H$ is an embedding, which is a contradiction. ☐

Proposition 3.3.10. *Let $Y \in \{2^{<\omega}, -2^{<\omega}, 2^{<\omega}_{\perp}\}$, and $s, s', t \in Y$ be such that $s \sqsubseteq s'$ and s and t are incomparable under $\sqsubseteq$. Then $\neg\mathrm{SSR}(s; s', t)$ and $\neg\mathrm{SSR}(s'; s, t)$.*

Proof. Let $Y \in \{2^{<\omega}, -2^{<\omega}, 2^{<\omega}_{\perp}\}$ and s, s', t be as described. Suppose that $Y = 2^{<\omega}$, then since $2^{<\omega}$ is just ordered by $\sqsubseteq$ we have $s \leqslant s'$ and $s \perp t$ hence $\neg\mathrm{SSR}(s; s', t)$. We also have that $s' \perp t$, and therefore $\neg\mathrm{SSR}(s'; s, t)$. If $Y = -2^{<\omega}$ then we have $s \geqslant s'$, $s \perp t$ and $s' \perp t$, and again we can conclude that $\neg\mathrm{SSR}(s; s', t)$ and $\neg\mathrm{SSR}(s'; s, t)$. If $Y = 2^{<\omega}_{\perp}$ then we have $s \perp s'$, and either $t > s$ and $t > s'$ or $t < s$ and $t < s'$. Hence again we can conclude $\neg\mathrm{SSR}(s; s', t)$ and $\neg\mathrm{SSR}(s'; s, t)$. ☐

Lemma 3.3.11. *Let $H \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ then $2^{<\omega}$, $-2^{<\omega}$, $2^{<\omega}_{\perp} \not\leqslant H$.*

Proof. Fix $Y \in \{2^{<\omega}, -2^{<\omega}, 2^{<\omega}_{\perp}\}$ with $Y \leqslant H = H_{\hat{r}} \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$, let φ be a witnessing embedding and $\hat{r} = \langle \hat{a}_i : i \in r \rangle$. Let s_i be the unique element of a_i such that $\hat{a}_i(s_i) = 1$, and $S_i = \{s_i\}$ if $i \neq \max r$, and $S_i = \emptyset$ if $i = \max r$. For every $u \in Y$ there is some unique $\tau(u) \in r$ such that $\varphi(u) \in a_{\tau(u)} \setminus S_{\tau(u)}$.

Claim: For any finite $F \subseteq r$ and $t \in Y$, there are $t \sqsubseteq t_0, t_1 \in Y$ with $t_0 \not\sqsubseteq t_1$ and $t_1 \not\sqsubseteq t_0$, and $\tau(t_0), \tau(t_1) \notin F$.

Proof of claim: Suppose not, then there is some $t \in Y$ and some finite $F \subseteq r$ such that $\forall t_0, t_1 \in Y$ with $t \sqsubseteq t_0, t_1$ and $t_0 \not\sqsubseteq t_1$, $t_1 \not\sqsubseteq t_0$, then either $\tau(t_0)$ or $\tau(t_1)$ is an element of F.

So for some $s \in Y$ such that $t \sqsubseteq s$ we have for every $s' \in Y$ with $s \sqsubseteq s'$ that $\tau(s') \in F$. Let $F = F_0$ and $s_0 = s$. Suppose for $k \in \omega$ that we have defined $F_k \subseteq F$ and $s_k \in Y$ such that $\forall u \in s_k$, $\tau(u) \in F_k$. Pick some $j_k \in F_k$. Suppose there is $s^{k+1} \in Y$ with $s^k \sqsubseteq s^{k+1}$ such that for every $u \in Y$ with $s^{k+1} \sqsubseteq u$, there exists some $\mu(u) \in Y$ with $\mu(u) \sqsubseteq u$ and $\tau(\mu(u)) = j_k$. Then let $\psi(\langle\rangle) = \varphi(\mu(s^{k+1}))$ and for $y \in Y$, if for some $y' \in Y$ we have defined $\psi(y) = \varphi(y')$, then for each $n \in \{0, 1\}$, let

$$\psi(y^\frown\langle n\rangle) = \varphi(\mu(y'^\frown\langle n\rangle)).$$

Thus $\psi : Y \to a_{j_k}$ is an embedding, which is a contradiction since $a_{j_k} \in \mathbb{P}$.

So there is some $s^{k+1} \in Y$ with $s^k \sqsubseteq s^{k+1}$ such that for every $u \in Y$ with $s^{k+1} \sqsubseteq u$, we have $\tau(u) \in F_k \setminus \{j_k\}$. So we can let $F_{k+1} = F_k \setminus \{j_k\}$, and we continue the induction. But then we have that $F_{|F|} = \emptyset$, so we have a contradiction which gives the claim. ■

By applying the claim to $t = \langle\rangle \in Y$ and $\{\tau(t)\} \subseteq r$, we obtain $t_0, t_1 \in Y$ with $t_0 \not\sqsubseteq t_1$ and $t_1 \not\sqsubseteq t_0$. We can assume without loss of generality also that $\tau(t_0) \neq \tau(t_1)$ by applying the claim to $\{\tau(t), \tau(t_0)\} \subseteq r$ and $t_0 \in Y$. Now apply the claim to $\{\tau(t), \tau(t_0), \tau(t_1)\}$ and t_0 to obtain t_{00} and t_{01} and similarly we can assume that $\tau(t_{00}) \neq \tau(t_{01})$. Applying the claim one more time to $\{\tau(\langle\rangle), \tau(t_0), \tau(t_1), \tau(t_{00}), \tau(t_{01})\}$

and t_1, we obtain t_{10} and t_{11} and similarly we can assume $\tau(t_{10}) \neq \tau(t_{11})$. Thus every element of $\{\tau(\langle\rangle), \tau(t_0), \tau(t_1), \tau(t_{00}), \tau(t_{01}), \tau(t_{10}), \tau(t_{11})\}$ is distinct.

We now use Proposition 3.2.12 in the following cases:

- $\tau(t_0) < \tau(t_1), \tau(t_{00})$ which implies $\mathrm{SSR}(\varphi(t_0); \varphi(t_1), \varphi(t_{00}))$,
- $\tau(t_{00}) < \tau(t_0), \tau(t_1)$ which implies $\mathrm{SSR}(\varphi(t_{00}); \varphi(t_0), \varphi(t_1))$,
- $\tau(t_1) < \tau(t_0), \tau(t_{11})$ which implies $\mathrm{SSR}(\varphi(t_1); \varphi(t_0), \varphi(t_{11}))$,
- $\tau(t_{11}) < \tau(t_0), \tau(t_1)$ which implies $\mathrm{SSR}(\varphi(t_{11}); \varphi(t_0), \varphi(t_1))$,
- $\tau(t_1) < \tau(t_0), \tau(t_{00})$ and $\tau(t_0) < \tau(t_1), \tau(t_{11})$ which implies $\tau(t_0) < \tau(t_1) < \tau(t_0)$ which is a contradiction.

Now note that any of the first four cases contradict Proposition 3.3.10. So we have a contradiction in every case, and our assumption that $Y \leqslant H$ must have been false. This gives the lemma. $\square$

Lemma 3.3.12. *If $Y = \sum_{x \in H} Y_x$ where $H \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ and each Y_x satisfies 3.2.14 (iii), then Y satisfies 3.2.14 (iii).*

Proof. By Lemma 3.3.9 and Lemma 3.3.11. $\square$

Theorem 3.3.13. *$\tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}} \subseteq \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$.*

Proof. If $Y \in \mathcal{S}_{\mathbb{P}0}^{\mathbb{L}}$ then Y is the singleton partial order, and hence trivially satisfies 3.2.14 (i), (ii) and (iii). Now suppose that $\mathcal{S}_{\mathbb{P}<\alpha}^{\mathbb{L}} \subseteq \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ and $Y \in \mathcal{S}_{\mathbb{P}\alpha}^{\mathbb{L}}$. Thus $Y = \sum_{x \in H} Y_x$ for some $H \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ and $Y_x \in \mathcal{S}_{\mathbb{P}<\alpha}^{\mathbb{L}}$. So by the induction hypothesis, for every $x \in H$ we have Y_x satisfies 3.2.14 (i), (ii) and (iii). So using lemmas 3.3.4, 3.3.8 and 3.3.9, we have that Y satisfies 3.2.14 (i), (ii) and (iii), i.e. $Y \in \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$. $\square$

3.3.2 $\tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}} \supseteq \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$

Our method for this direction will be to show that any partial order that satisfies conditions 3.2.14 (i) and (ii) but is not a member of $\tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$ must fail condition 3.2.14 (iii). To this end we will construct an internal structured tree representation for such an order, show that this structured tree must embed certain pathological structured trees (or else the order is a member of $\tilde{\mathscr{S}}^{\mathbb{L}}_{\mathbb{P}}$), before showing that embeddings of these pathological trees give rise to embeddings of either $2^{<\omega}$, $-2^{<\omega}$ or $2^{<\omega}_{\perp}$ into the order in question.

Definition 3.3.14. Let $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$, we call $\hat{T} \in \mathscr{T}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\mathscr{P}^{\mathbb{L}}_{\mathbb{P}})$ a *partial interval tree for X* iff

1. T has a root t_0 and $\hat{T}(t_0) = X$.
2. For all $t \in T$ we have $\hat{T}(t)$ is an interval of X.
3. If $t, s \in T$ with $t \leqslant s$, then $\hat{T}(t) \supseteq \hat{T}(s)$.
4. For all $t \in T$, if $H = \mathrm{range}(l^T_t) \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ and for each $u \in H$ if $t_u \in \mathrm{succ}(t)$ is such that $l^T_t(t_u) = u$, then
$$\hat{T}(t) = \sum_{u \in H} \hat{T}(t_u).$$
5. Moreover, if $H = H_{\hat{r}}$ for $\hat{r} = \langle \hat{a}_i : i \in r\rangle \in \mathbb{E}^{\mathbb{L}}_{\mathbb{P}}$, and for $j \in r$ we have $\hat{r}_j = \langle \hat{a}_i : i \in r, i \geqslant j\rangle$, then
$$\left\langle \sum_{u \in H_{\hat{r}_j}} \hat{T}(t_u) : j \in r \right\rangle$$
is a maximal chain of intervals of $\hat{T}(t)$.
6. For every leaf t of T we have $|\hat{T}(t)| = 1$.

Lemma 3.3.15. *If I is an interval of a partial order X that satisfies 3.2.14 (ii), then I satisfies 3.2.14 (ii).*

Proof. Let I be an interval of X and suppose that X satisfies 3.2.14 (ii). Let $x_0 \in I$ and using that X satisfies 3.2.14 (ii), pick a maximal chain C of non-empty intervals of X with order type under $\supseteq$ in $\overline{\mathbb{L}}$ that contains $\{x_0\}$. Thus $C' = \{J \cap I : J \in C\}$ is a chain of intervals of I, and furthermore it contains $\{x_0\}$ and its order type is isomorphic to a subset of $\overline{\mathbb{L}}$, i.e. $\mathrm{ot}(C) \in \overline{\mathbb{L}}$. If it were not maximal then there is some non-empty interval K of I such that for all $J \in C'$ we have either $J \subset K$ or $K \subset J$.

Consider

$$W = K \cup \bigcup \{J \in C : J \cap I \subset K\}.$$

We claim that W is an interval of X. Let $x \in X \setminus W$ and $y \in W$. If $y \in \bigcup\{J \in C : J \cap I \subset K\}$ then $\mathrm{SSR}(x; y, x_0)$ since $\bigcup\{J \in C : J \cap I \subset K\}$ is an interval of X by Lemma 3.2.11, and this interval contains x_0. If $x \in X \setminus I$ and $y, y' \in I$ then $\mathrm{SSR}(x; y, x_0)$ since I is an interval of X and $x_0 \in I$. If $x \in I$ and $y \in K$ then again $\mathrm{SSR}(x; y, x_0)$ since K is an interval of I and $x_0 \in K$. Therefore, for all $x \in X \setminus W$ and all $y \in W$ we have $\mathrm{SSR}(x; y, x_0)$, and thus for all $y' \in W$ we have $\mathrm{SSR}(x; y, y')$, i.e. W is an interval of X. But we also have that for all $J \in C$ either $J \subset W$ or $W \subset J$, which contradicts that C was maximal. $\square$

Lemma 3.3.16. *For every $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$, there is some $H \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ and some $Y_u \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ for $u \in H$ such that $X = \sum_{u \in H} Y_u$. Moreover if:*

- *C is a maximal chain of intervals of X with order type under $\supseteq$ in $\overline{\mathbb{L}}$ that contains a singleton,*
- *$H = H_{\hat{r}}$ for $\hat{r} = \langle \hat{a}_i : i \in r \rangle \in \mathbb{E}^{\mathbb{L}}_{\mathbb{P}}$,*
- *for $j \in r$ we have $\hat{r}_j = \langle \hat{a}_i : i \in r, i \geqslant j \rangle$,*

then $\left\langle \sum_{u \in H_{\hat{r}_j}} Y_u : j \in r \right\rangle = C$.

Proof. Let $C = \langle I_i : i \in r \rangle$ be as described, with $\{x\} \in C$. For $i \in r$, let

$$P_i = I_i \setminus \left(\bigcup_{j > i} I_j \setminus \{x\} \right).$$

Let $\mathcal{J}$ be the set of maximal chains of intervals of P_i that do not contain x, and

$$\mathcal{Z}_i = \left\{ \bigcup K : K \in \mathcal{J} \right\} = \{Z^i_\beta : \beta \in \gamma_i\},$$

where $\gamma_i = |\mathcal{Z}_i|$. Then for each $\beta \in \gamma_i$, pick some $z^i_\beta \in Z^i_\beta$, and let

$$Q_i = \{x\} \cup \{z^i_\beta : \beta \in \gamma_i\} \subseteq P_i.$$

Now $\bigcup_{i>j} I_j$ is an interval and each Z^i_β is an interval, hence if $Y^i(z^i_\beta) = Z^i_\beta$ for all $\beta \in \gamma_i$, and $Y^i(x) = \bigcup_{i>j} I_j$ then

$$I_i = \sum_{q \in Q_i} Y^i(q).$$

Claim: Q_i is indecomposable.

Proof of claim: We first claim that any interval of Q_i of size at least 2 contains x. If not, then there is an interval I of Q_i containing z^i_β and z^i_δ for some distinct $\beta, \delta \in \gamma_i$, and such that $x \notin I$. But then $Z^i_\beta \cup Z^i_\delta$ is an interval of P_i that doesnt contain x, which contradicts either that Z^i_β or Z^i_δ was the union of a maximal chain of such intervals.

Thus if I is a proper interval of Q_i with $|I| \geqslant 2$, then $x \in I$ and for some $\beta \in \gamma_i$ we have $z^i_\beta \notin I$. Now let

$$J = \bigcup_{z^i_\delta \in I} Z^i_\delta \cup \bigcup_{j>i} I_j \subset I_i.$$

Then using that $I_i = \sum_{q \in Q_i} Y^i(q)$, and I is an interval we can see that $J \subset I_i$ is an interval of I_i. Furthermore $\bigcup_{j>i} I_j \subset J$, and therefore C must not have been maximal since $J \notin C$. This contradiction gives the claim. ■

By the claim then, since $Q_i \subseteq X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ we have $Q_i \in \mathbb{P}$. Now define $H = \bigcup_{i \in r} Q_i \subseteq X$. Then for $z^i_\beta, z^j_\delta \in H$ we have $z^i_\beta \leqslant_H z^j_\delta$ iff:

- $i = j$ and $z^i_\beta \leqslant_{Q_i} z^i_\delta$ or;
- $i < j$ and $z^i_\beta \leqslant_{Q_i} x$ or;
- $i > j$ and $x \leqslant_{Q_j} z^j_\delta$.

Additionally, $z^i_\beta \leqslant_H x$ iff $z^i_\beta \leqslant_{Q_i} x$ and $x \leqslant_H z^i_\delta$ iff $x \leqslant_{Q_i} z^i_\delta$. Now let

$$\hat{r}_t = \langle \hat{Q} : i \in r \rangle \in \mathbb{E}^{\mathbb{L}}_{\mathbb{P}}$$

where for all $i \in r$ we have $\hat{Q}_i : Q_i \to \mathrm{A}_2$ is such that for $q \in Q_i$ we have $\hat{Q}_i(q) = 1$ iff $q = x$. Then we see that $H = H_{\hat{r}_t} \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$.

Now for $u \in H$ define

$$y_u = \begin{cases} \{x\} & : u = x \\ Z^i_\beta & : u = z^i_\beta \end{cases}.$$

Then we have

$$X = \sum_{u \in H} Y_u.$$

Clearly then each Y_u $(u \in H)$ is an interval of X. So by Lemma 3.3.15 we have Y_u satisfies 3.2.14 (ii). Furthermore since $Y_u \subseteq X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$, we have that Y_u must satisfy 3.2.14 (i); and hence $Y_u \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$. We also have by construction that $\left\langle \sum_{u \in H_{\hat{r}_j}} Y_u : j \in r \right\rangle = C$. $\square$

Lemma 3.3.17. *For every $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$, there is a partial interval tree for X.*

Proof. Let $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$. First we define T_0 as the singleton tree, and let $\hat{T}_0 : T_0 \to \{X\}$.

Suppose for $n \in \omega$ that we have defined $\hat{T}_n \in \mathscr{T}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\mathscr{P}^{\mathbb{L}}_{\mathbb{P}})$ that satisfies properties (1) to (5). For each leaf t of T_n, since $\hat{T}_n(t)$ satisfies 3.2.14 (ii) pick a maximal chain C of intervals of $\hat{T}_n(t)$ with order type in $\overline{\mathbb{L}}$ that contains a singleton. Then apply Lemma 3.3.16 to $\hat{T}_n(t) \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ so that $\hat{T}_n(t) = \sum_{u \in H^t} Y^t_u$ for some $H^t \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ and $Y^t_u \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ for $u \in H^t$.

We can now define

$$T_{n+1} = T_n \cup \{H^t : t \text{ is a leaf of } T_n, |\hat{T}_n(t)| > 1\},$$

where $\forall u, v \in T_{n+1} \setminus T_n$ with $u \neq v$ we let $u \perp_{T_{n+1}} v$, and if $s \in T_n$, $u \in H^t \subseteq T_{n+1}$ then let $s \not\geqslant_{T_{n+1}} t$ and $s <_{T_{n+1}} u$ iff $s \leqslant_{T_n} t$.

For $t \in T$ we let $l^{T_{n+1}}_t = l^{T_n}_t$ whenever t is not a leaf of T_n and $l^{T_{n+1}}_t : \mathrm{succ}(t) \to H^t$ be such that $l^{T_{n+1}}_t(u) = u$ for all $u \in \mathrm{succ}(t) = H^t$ whenever t is a leaf of T_n. We also let $\hat{T}_{n+1} \restriction T_n = \hat{T}_n$ and for $u \in H^t$ with t a leaf of T_n, we let $\hat{T}_{n+1}(u) = Y^t_u$.

So by construction, properties (1) to (5) hold for T_{n+1}, and we can inductively define T_n for every $n \in \omega$. Now let

$$T = \bigcup_{n \in \omega} T_n,$$

and $\hat{T}(t) = \hat{T}_n(t)$ whenever $t \in T_n$. It is clear then that properties (1) to (5) hold for T, as they are all witnessed by some $T_n \subseteq T$.

It remains to check property (5), so let $t \in T$ and thus there is some $n \in \omega$ with $t \in T_n$. If $|\hat{T}(t)| = |\hat{T}_n(t)| \neq 1$, then $H^t \neq \emptyset$ and hence we defined some successor of $t \in T_{n+1}$, i.e. t is not a leaf of T. So $\hat{T}$ is a partial interval tree for X. $\square$

Lemma 3.3.18. *For every $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}} \setminus \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$, any partial interval tree $\hat{T}$ for X is such that $T \notin \mathscr{W}$.*

Proof. Suppose that X has a partial interval tree $\hat{T}$ with $T \in \mathscr{W}$. We will prove by induction on $\operatorname{rank}(T)$ that $X \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. Firstly, if $\operatorname{rank}(T) = 0$ then T is the singleton tree, and thus its only point t_0 is both a leaf and the root of T. So by properties (1) and (5), we have that $X = \hat{T}(t_0) = \{x\}$, and hence $X \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$.

Suppose for some $\alpha \in \mathrm{On}$ that $\operatorname{rank}(T) = \alpha$ and whenever $Y \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ has a partial interval tree T' with $\operatorname{rank}(T') < \alpha$ we have $Y \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$.

Let t_0 be the root of T, $H = \operatorname{range}(l^T_{t_0}) \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ and for each $u \in H$ let $t_u \in \operatorname{succ}(t)$ be such that $l^T_t(t_u) = u$ then by properties (1) and (4) we have

$$X = \hat{T}(t_0) = \sum_{u \in H} \hat{T}(t_u). \tag{$\star$}$$

For each $u \in H$ let

$$\hat{T}_u = \hat{T} \restriction {\uparrow} t_u.$$

Then it is clear from the definition, and since T is a partial interval tree for X, that $\hat{T}_u$ is a partial interval tree for $\hat{T}(t_u)$. Moreover, $\operatorname{rank}(T_u) < \alpha$, and therefore $\hat{T}_u(t_u) \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. So using ($\star$) we see that $X \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. $\square$

Definition 3.3.19. Let $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$. We call $\hat{M} \in \mathscr{T}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\mathscr{P}^{\mathbb{L}}_{\mathbb{P}})$ a *full interval tree for X* iff

- There is a tree of transfinite sequences of ordinals $\mathcal{K}_M$ under $\sqsubseteq$, closed under initial segments.
- For each $s \in \mathcal{K}_M$ there is some interval X_s of X, and $X_{\langle\rangle} = X$.
- There is a partial interval tree $\hat{T}_s$ for X_s.

- For all $s, s' \in \mathcal{K}_M$ with $s' = s^\frown\langle\alpha\rangle$ there is some chain ζ of T_s with order type ω, such that
$$X_{s'} = \bigcap_{t\in\zeta} \hat{T}_s(t) \neq \emptyset.$$
- $M = \bigcup_{s\in\mathcal{K}_M} T_s$, ordered by $t < u$ iff $t \in T_s$, $u \in T_{s'}$ and either $s = s'$ and $t <_{T_s} u$ or for some $\alpha \in \mathrm{On}$, $s^\frown\langle\alpha\rangle \sqsubseteq s'$, and there is a maximal chain ζ of T_s such that $t \in \zeta$ and $X_{s^\frown\langle\alpha\rangle} = \bigcap_{t\in\zeta} \hat{T}_s(t)$.
- Whenever $t \in T_s$ we have $\hat{M}(t) = \hat{T}_s(t)$ and $l_t^M = l_t^{T_s}$.
- For any maximal chain ξ of M whose order type is a limit ordinal, we have
$$\bigcap_{t\in\xi} \hat{M}(t) = \emptyset.$$

Lemma 3.3.20. *For every $X \in \mathscr{P}_{\mathbb{P}}^{\mathbb{L}}$, there is a full interval tree M for X.*

Proof. Let $X \in \mathscr{P}_{\mathbb{P}}^{\mathbb{L}}$, let $\hat{T}_{\langle\rangle}$ be a partial interval tree for $X = X_{\langle\rangle}$. Suppose we have defined $\hat{T}_s$ for some sequences of ordinals s. Enumerate the maximal chains of $\hat{T}_s$ as ζ_s^α for $\alpha < \kappa_s \in \mathrm{Card}$. Now define
$$X_{s^\frown\langle\alpha\rangle} = \bigcap_{i\in\zeta_s^\alpha} \hat{T}_s(i),$$
and whenever $X_{s^\frown\langle\alpha\rangle} \neq \emptyset$ we let $\hat{T}_{s^\frown\langle\alpha\rangle}$ be a partial interval tree for $X_{s^\frown\langle\alpha\rangle}$. We let $\mathcal{K}_M$ be the set of sequences of ordinals s such that $X_s \neq \emptyset$.

We note that for $s, s' \in \mathcal{K}_M$ with $s \neq s'$, we have $X_s \neq X_{s'}$ and $X_s, X_{s'} \subseteq X$. Hence if $\mathcal{K}_M$ were a proper class, then X would have a proper class of distinct subsets and so X would be a proper class itself. So $\mathcal{K}_M$ is a set. Moreover $\mathcal{K}_M$ is a tree under $\sqsubseteq$.

Now define $M = \bigcup_{s\in\mathcal{K}_M} T_s$, ordered so that if $x \in T_s$, $y \in T_{s'}$, then $x <_M y$ iff $s = s'$ and $x <_{T_s} y$, or $s^\frown\langle\alpha\rangle \sqsubseteq s'$ and $x \in \zeta_s^\alpha$. If $t \in T_s$ then we let $\hat{M} : M \to \mathscr{P}_{\mathbb{P}}^{\mathbb{L}}$ be such that $\hat{M}(t) = \hat{T}_s(t)$ and set $l_t^M = l_t^{T_s}$. Clearly then $\hat{M}$ is a full interval tree for X. $\square$

Definition 3.3.21. Given a chain ζ and trees T_i^γ for each $i \in \zeta$, $\gamma < \kappa_i \in \text{Card}$,[5] we define the ζ-*tree-sum* of the T_i^γ (see Figure 3.6) as the set

$$\coprod_{i\in\zeta,\gamma<\kappa_i} T_i^\gamma = \zeta \sqcup \bigsqcup_{i\in\zeta,\gamma<\kappa_i} T_i^\gamma$$

ordered by letting $a \leqslant b$ iff

- $a, b \in \zeta$ with $a \leqslant_\zeta b$;
- or for some $i \in \zeta$, $\gamma < \kappa_i$ we have $a, b \in T_i^\gamma$ with $a \leqslant_{T_i^\gamma} b$;
- or $a \in \zeta$ and $b \in T_i^\gamma$ for some $i \in \zeta$ with $a <_\zeta i$ and $\gamma < \kappa_i$.

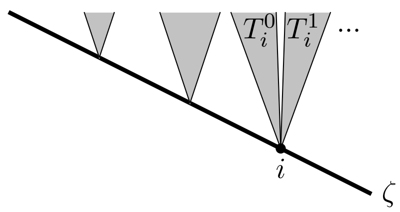

Figure 3.6: A ζ-tree-sum of the T_i^γ.

Lemma 3.3.22. *Suppose that $X \in \mathscr{P}_{\mathbb{P}}^{\mathbb{L}}$ has a full interval tree $\hat{M}$ where $M = \coprod_{t\in\zeta,\gamma<\kappa_t} T_t^\gamma$ for some chain $\zeta \subseteq M$. For all $t \in \zeta$ and $\gamma < \kappa_t$, suppose that $\hat{M}(root(T_t^\gamma)) \in \tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$. Then $X \in \tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$.*

Proof. For each $t \in \zeta$, let $s_t = l_t^M(t')$ where t' is the unique element of $\text{succ}(t) \cap \zeta$. We have $\text{range}(l_t^M) = H_{\hat{r}_t}$ for some $\hat{r}_t = \langle \hat{a}_i^t : i \in r_t \rangle \in \mathbb{E}_{\mathbb{P}}^{\mathbb{L}}$. So since

$$s_t \in H_{\hat{r}_t} \subseteq \bigsqcup_{i\in r_t} a_i^t,$$

there is some $\mu(t) \in r_t$ such that $s_t \in a_{\mu(t)}^t$. Now let

$$r_t^- = \langle i \in r_t : i \leqslant \mu(t) \rangle,$$

[5]The role of the κ_i is to allow multiple trees (i.e. κ_i many) to be added above each point $i \in \zeta$.

and

$$r = \sum_{t \in \zeta} r_t^- .$$

For $i \in r$, whenever $i \in r_t$, let $b_i^t = a_i^t$. If $i \neq \mu(t)$ then let $\hat{b}_i^t = \hat{a}_i^t$ and if $i = \mu(t)$ then let $\hat{b}_i^t : b_i^t \to \mathrm{A}_2$ be such that for $u \in b_i^t$, we have $\hat{b}_i^t(u) = 1$ iff $u = s_t$. Now define $\hat{r} : r \to \mathbb{P}(\mathrm{A}_2)$ so that for $t \in \zeta$ and $i \in r_t$ we have,

$$\hat{r}(i) = \hat{b}_i^t.$$

Then r has order type

$$\sum_{t \in \zeta} \mathrm{ot}(r_t^-) \in \overline{\mathbb{L}},$$

and hence $\hat{r} \in \mathbb{E}_{\mathbb{P}}^{\mathbb{L}}$. We aim to write X as a lexicographic $H_{\hat{r}}$-sum of partial orders $X_u \in \tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$.

Let s_t' be the unique element of $a_{\mu(t)}^t$ such that $\hat{a}_i^t(s_t') = 1$. Then for $i \in r_t^-$, $i < \mu(t)$ we have $\hat{r}_t^-(i) = \hat{b}_i^t = \hat{a}_i^t = \hat{r}_t(i)$.

Let $u \in H_{\hat{r}}$ and then let $t \in \zeta$, $i \in r_t$ be such that $u \in b_i^t$. If $u \neq s_t'$, then either $u = s_t$ and $t = \max \zeta$ or $u \neq s_t$ therefore in either case $T_u = {}^u{\upharpoonleft} t \subset M$ was one of the trees we used in the ζ-tree-sum and we let $X_u = \hat{M}(\mathrm{root}(T_u)) \in \tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$.

Suppose $u = s_t'$ so that $i = \mu(t)$. For $t \in \zeta$, let

$$r_t^+ = \langle i \in r_t : i > \mu(t) \rangle,$$

and

$$\hat{r}_t^+ = \langle \hat{a}_i^t : i \in r_t^+ \rangle \in \mathbb{E}_{\mathbb{P}}^{\mathbb{L}}.$$

Let $H(t) = H_{\hat{r}_t^+} \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ so that $H(t) \subseteq H_{\hat{r}_t} = \mathrm{range}(l_t^M)$, and for each $v \in H(t)$, let $\tau(v) \in \mathrm{succ}(t)$ be the element such that

$$l_t^M(\tau(v)) = v \in H(t).$$

Now define

$$X_u = \sum_{v \in H(t)} \hat{M}(\tau(v)).$$

For each $v \in H(t)$ we have $v \in H_{\hat{r}_t}$ and $v \neq s_t$, therefore $\uparrow\tau(v) \subseteq M$ was one of the trees we used in the ζ-tree-sum, so $\hat{M}(\tau(v)) \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. Hence $X_u \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ since $\tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ is closed under sums over elements of $\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$.

It remains to prove that

$$X = \sum_{u \in H_{\hat{r}'}} X_u.$$

First notice that the X_u ($u \in H_{\hat{r}'}$) partition X, since if $x \in X$ then there is a largest $t_x \in \zeta$ such that $x \in \hat{M}(t_x)$. For $i \in \zeta$, if $i = \max \zeta$ then let $S_i = \emptyset$, otherwise let $S_i = \{s_i\}$. So for some (unique) $v \in \mathrm{succ}(t_x) \setminus S_{t_x}$ we have $x \in \hat{M}(v)$. Let $u = l^M_{t_x}(v)$ so that either $u \in H_{\hat{r}^-_{t_x}}$ or $u \in H_{\hat{r}^+_{t_x}}$ and thus either $x \in X_u$ or $x \in X_{s'_{t_x}}$.

Now suppose that $x, y \in X$ with $x < y$. Let $u, v \in H_{\hat{r}'}$ be such that $x \in X_u$, $y \in X_v$. If $u = v$ then $x, y \in X_u$ which occurs iff $x < y$ in the sum too. Let $i, j \in r'$ be such that $u \in b_i$, $v \in b_j$, if $i = j$ then $u <_{a_i} v$ so $u <_{b_i} v$ and again this happens iff $x < y$ in the sum. If $i < j$ then $u <_{a_i} s'_i$, and if $i > j$ then $v <_{a_j} s'_j$. So in either case this occurs iff $x < y$ in the sum, and we conclude that $X = \sum_{u \in H_{\hat{r}'}} X_u \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. □

Definition 3.3.23. Let $\mathscr{U}_0 = \mathscr{W} \cup \{\emptyset\}$, and for $\alpha \in \mathrm{On}$ let $\mathscr{U}_{\alpha+1}$ be the class of ζ-tree-sums of trees of $\mathscr{U}_\alpha$ for some ordinal $\zeta \geqslant \omega$. For limit $\lambda \in \mathrm{On}$ we let $\mathscr{U}_\lambda = \bigcup_{\gamma<\lambda} \mathscr{U}_\gamma$, and finally set $\mathscr{U} = \bigcup_{\gamma \in \mathrm{On}} \mathscr{U}_\gamma$. For $T \in \mathscr{U}$ define the *scattered rank* of T, denoted $\mathrm{rank}_{\mathscr{U}}(T)$ as the least ordinal α such that $T \in \mathscr{U}_\alpha$. (See Figure 3.7.)

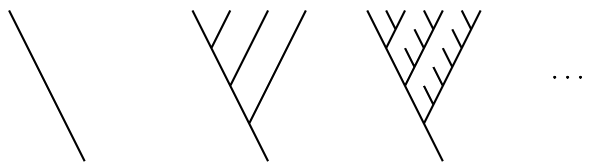

Figure 3.7: Trees of $\mathscr{U}$ of increasing scattered rank.

Lemma 3.3.24. *If $T \notin \mathscr{U}$ then there is some $u \in T$ and some $t_0, t_1 > u$ such that $\uparrow t_0 \cap \uparrow t_1 = \emptyset$ and $\uparrow t_0, \uparrow t_1 \notin \mathscr{U}$.*

Proof. Suppose $T \notin \mathscr{U}$ but there is no such $u \in T$. Thus for any maximal chain $\zeta \subseteq T$ there is some a minimal element t of $T \setminus \zeta$, such that $\uparrow t \notin \mathscr{U}$ (otherwise $T \in \mathscr{U}$).

So let ζ_0 be a maximal chain of T and t_0 be the minimal element of $T \setminus \zeta_0$ such that $\uparrow t \notin \mathscr{U}$. Suppose we have defined t_α as the minimal element of $T \setminus \zeta_\alpha$, for every $\alpha < \beta$, and that if $\alpha < \gamma < \beta$ then $t_\alpha < t_\gamma$. Then let ζ_β be a maximal chain of T containing $\downarrow t_\alpha$ for every $\alpha < \beta$, and let t_β be a minimal element of $T \setminus \zeta_\beta$ such that $\uparrow t_\beta \notin \mathscr{U}$.

Since $t_\beta \notin \zeta_\beta$ we cannot have $t_\alpha > t_\beta$ for any $\alpha < \beta$. Suppose that $t_\beta \perp t_\alpha$ for some $\alpha < \beta$. But then if we let $u = t_\beta \wedge t_\alpha$, $t_0 = t_\beta$ and $t_1 = t_\alpha$, then these satisfy the statement of the lemma. Otherwise $t_\alpha < t_\beta$ and we can continue the induction. So the induction continues for every ordinal. But then we have found proper class many distinct elements of T, namely t_α for $\alpha \in \mathrm{On}$. Thus T is a proper class, which is a contradiction. □

Theorem 3.3.25. *$T \in \mathscr{U}$ iff $T \in \mathscr{T}$ and $2^{<\omega} \not\leqslant T$.*

Proof. Let $T \in \mathscr{U}$, we will define $\varphi : 2^{<\omega} \to T$ by induction on the length of $s \in 2^{<\omega}$. Firstly, let $\varphi(\langle\rangle)$ be the element $u \in T$ given by Lemma 3.3.24. Suppose for $s \in 2^{<\omega}$, that we have defined $\varphi(s)$ such that there are $t_0, t_1 > \varphi(s)$ such that $\uparrow t_0, \uparrow t_1 \notin \mathscr{U}$. Then for $i \in \{0, 1\}$, let $\varphi(s^\frown\langle i\rangle)$ be the element $u \in T$ given by applying Lemma 3.3.24 to $\uparrow t_i$. This inductively defines φ, which is clearly an embedding.

For the other direction, firstly it is clear that $2^{<\omega} \not\leqslant U$ for any $U \in \mathscr{W}$, since $2^{<\omega}$ contains an infinite branch. Now suppose that $2^{<\omega} \not\leqslant U$ whenever $\mathrm{rank}_{\mathscr{U}}(U) < \alpha$. Then if $\mathrm{rank}_{\mathscr{U}}(T) = \alpha$, we have that T is a ζ-tree-sum of some lower ranked trees. If $2^{<\omega}$ embeds into T, then if any point in the range of this embedding is in one of the lower ranked trees, then $2^{<\omega}$ embeds into that tree, which cannot happen. Therefore $2^{<\omega}$ embeds into the chain ζ, which is again impossible, and therefore $2^{<\omega} \not\leqslant T$. □

Lemma 3.3.26. *For every $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}} \setminus \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ and every full interval tree $\hat{M}$ for X, we have $2^{<\omega} \leqslant M$.*

Proof. Let $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}} \setminus \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ and suppose there is a full interval tree $\hat{M}$ for X such that $2^{<\omega} \not\leqslant M$. Hence $\mathrm{rank}_{\mathscr{U}}(M)$ is a well-defined ordinal by Lemma 3.3.25. We will prove the lemma by induction $\mathrm{rank}_{\mathscr{U}}(M)$. If $\mathrm{rank}_{\mathscr{U}}(M) = 0$ then $M = T^X_{\langle\rangle} \in \mathscr{W}$, and hence $X \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ by Lemma 3.3.18.

Now suppose that $\mathrm{rank}_{\mathscr{U}}(M) = \alpha$ and whenever $Y \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ has a full interval tree $\hat{M}_Y$ with $\mathrm{rank}_{\mathscr{U}}(M_Y) < \alpha$ then $Y \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. So there is some chain ζ of M, such that for minimal $i \in (\bigcup_{t \in \zeta} \mathrm{succ}(\zeta)) \setminus \zeta$ if we let $M_i = \uparrow i \subseteq M$ then M is a ζ-tree-sum of the M_i, and $\mathrm{rank}_{\mathscr{U}} M_i < \alpha$. Thus for each such i we also have that M_i is a full interval tree for $\hat{M}(i) \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ and by the induction hypothesis $\hat{M}(i) \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. So we can apply Lemma 3.3.22 to see that $X \in \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. $\square$

Lemma 3.3.27. *Suppose that $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ has a full interval tree $\hat{M}$. Let $\varphi : 2^{<\omega} \to M$ be an embedding and for every $s \in 2^{<\omega}$ let $P_s \in \{\mathrm{A}_2, \mathrm{C}_2\}$ be the partial order isomorphic to*

$$\mathit{range}\left(l^M_{\varphi(s)} \restriction \{u \in \mathit{succ}(\varphi(s)) : u \leqslant \varphi(s^\frown\langle 0\rangle) \text{ or } u \leqslant \varphi(s^\frown\langle 1\rangle)\}\right).$$

Then for every $s \in 2^{<\omega}$, there is some $\tau(s) \in M$ and distinct $u_s, v_s \in \mathit{succ}(\tau(s))$ such that: $\varphi(s) \leqslant \tau(s)$; for some $t \in \mathit{range}(\varphi)$, $u_s \leqslant t$; and $l^M_{\tau(s)} \restriction \{u_s, v_s\} \ncong P_s$.

Proof. Suppose that the lemma does not hold. So let $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ have a full interval tree $\hat{M}$, let $\varphi : 2^{<\omega} \to M$ be an embedding, such that for some $s \in 2^{<\omega}$ and every $t \in M$ with $\varphi(s) \leqslant t$, if $u, v \in \mathrm{succ}(t)$ and $u \neq v$ then either $(\uparrow u \cup \uparrow v) \cap \mathrm{range}(\varphi) = \emptyset$ or $P_s \cong l^M_t \restriction \{u, v\}$. Without loss of generality, we can assume that $s = \langle\rangle$. Then in particular $P_{s_0} \cong P_{s_1}$ for all $s_0, s_1 \in 2^{<\omega}$.

Let $H = H_{\hat{r}} = \mathrm{range}(l^M_{t_0})$ with $\hat{r} = \langle \hat{a}_i : i \in r\rangle$ and for $j \in r$ let $\hat{r}_j = \langle \hat{a}_i : i \in r, i \geqslant j\rangle$. Let $P \cong P_{\langle\rangle}$ and $t_0 = \varphi(\langle\rangle)$. For $u \in H$ let $t_u \in \mathrm{succ}(t_0)$ be such that $l_{t_0}(t_u) = u$ and for $i \in r$ let $H_i = \{u_j : j \in r, j \geqslant i\}$. By property (5) in the definition of a partial interval tree, we have that:

$$C = \left\langle \sum_{u \in H_i} \hat{M}(t_u) : i \in r \right\rangle$$

is a maximal chain of intervals of $\hat{M}(t_0)$.

Let v_0, v_1 be the elements of $\mathrm{succ}(t_0)$ such that $v_0 \leqslant \varphi(\langle 0\rangle)$ and $v_1 \leqslant \varphi(\langle 1\rangle)$. For $i \in r$, let

$$Y_i = \bigcup_{j > i} \left(\sum_{u \in H_j} \hat{M}(t_u) \right)$$

and

$$X_i = \left(\sum_{u \in H_i} \hat{M}(t_u) \right) \setminus Y_i.$$

Let $u_0 = l^M_{t_0}(v_0) \in H$ and $u_1 = l^M_{t_0}(v_1) \in H$. Since $\uparrow v_0 \cap \mathrm{range}(\varphi) \neq \emptyset$ and $\uparrow v_1 \cap \mathrm{range}(\varphi) \neq \emptyset$ we have for every $u \in H$ that $\{u_0, u\} \cong P \cong \{u_1, u\}$. Let i_0 and i_1 be least such that $H_{i_0} \ni u_0$ and $H_{i_1} \ni u_1$. Now, u_0, u_1 are either both comparable or both incomparable to every other element of H. Hence $J_0 = \{u_0\} \cup \bigcup_{j > i_0} H_j$ and $J_1 = \{u_1\} \cup \bigcup_{j > i_1} H_j$ are intervals of H. Thus $J_0 = H_{i_0}$ and $J_1 = H_{i_1}$, otherwise for some $w \in \{0, 1\}$ we would have that the interval $\sum_{u \in J_w} \hat{M}(t_u)$ of $\hat{M}(t_0)$ contradicts that C is maximal. Therefore $H_{i_0} \setminus \bigcup_{j > i_0} H_j = \{u_0\}$ and $H_j \setminus \bigcup_{j > i_1} H_j = \{u_1\}$.

Now let $i = i_0$ and suppose without loss of generality that $i < i_1$ so that $v_1 \in Y_i$. Then since Y_i is an interval of X, for every $x \in X \setminus Y_i$ and every $y \in Y_i$, we have $\mathrm{SSR}(x; v_j, y)$. Let $t_x, t_y \in \mathrm{succ}(t_0)$ be such that there are $t'_x \geqslant t_x$ and $t'_y \geqslant t_y$ such that $\hat{M}(t'_x) = \{x\}$ and $\hat{M}(t'_y) = \{y\}$. These exist since for any maximal chain ζ of M we have $\bigcap_{t \in \zeta} \hat{M}(t) = \emptyset$ and therefore $\{\hat{M}(t) : t \text{ is a leaf of } M\}$ is a partition of X; and if t is a leaf of M then $|\hat{M}(t)| = 1$. Therefore we have $\mathrm{SSR}(l^M_{t_0}(t_x); u_j, l^M_{t_0}(t_y))$.

Since $\varphi(\langle 1 \rangle) \in \uparrow v_1 \cap \mathrm{range}(\varphi) \neq \emptyset$, by our assumption we have $\{l^M_{t_0}(t_x), u_1\} \cong P$, so that $\{l^M_{t_0}(t_x), l^M_{t_0}(t_y)\} \cong P$.

Since $H_i \setminus \bigcup_{j > i} H_j = \{u_0\}$, we have $\hat{M}(v_0) = X_i$. Let $w \in \mathrm{succ}(v_0)$ be such that $\uparrow w \cap \mathrm{range}(\varphi) \neq \emptyset$, thus for all $z \in \mathrm{succ}(v_0) \setminus \{w\} \neq \emptyset$ we have $\{l^M_{v_0}(w), l^M_{v_0}(z)\} \cong P$. Now:

- If $P = \mathrm{A}_2$ then let $K_0 = \hat{M}(w)$ and $K_1 = X_i \setminus K_0$.

- If $P = \mathrm{C}_2$, then $\mathrm{range}(l^M_{v_0}) = W_0 \cup W_1 \cup \{w\}$, where for all $x \in W_0$ and $y \in W_1$ we have $x < w < y$. For each $u \in \mathrm{range}(l^M_{v_0})$, let $J(u) = \hat{M}(t_u)$ where $t_u \in \mathrm{succ}(v_0)$ is such that $l^M_{v_0}(t_u) = u$. If $W_1 = \emptyset$ then let

  $$K_0 = \sum_{u \in W_0 \cup \{w\}} J(u) \text{ and } K_1 = \sum_{u \in W_1} J(u),$$

  otherwise let

  $$K_0 = \sum_{u \in W_0} J(u) \text{ and } K_1 = \sum_{u \in W_1 \cup \{w\}} J(u).$$

In either case we have some non-empty $K_0, K_1 \subset X$ such that

$$X_i = \sum_{j \in P} K_j.$$

Either $Y_i \cup K_0$ or $Y_i \cup K_1$ is an interval, since:

- if $P = \mathrm{A}_2$ and $x \in X_i \setminus (Y_i \cup K_0)$, $y \in Y_i$ then $x \perp y$ and also x is incomparable to any element of K_0;
- if $P = \mathrm{C}_2$ then for any $k_0 \in K_0$, $k_1 \in K_1$ we have $k_0 < k_1$, we also either have $x < y$ or $x > y$ for every $x \in X_i$, $y \in Y_i$ so that in the first case $K_1 \cup Y_i$ is an interval and in the second case $K_0 \cup Y_i$ is an interval.

But both $Y_i \cup K_0$ and $Y_i \cup K_1$ are then either proper subsets or proper supersets of every element of the chain C. One of them is an interval, so this contradicts that C is maximal. This contradiction gives the lemma. □

Definition 3.3.28. Let $\mathsf{S} \in \mathscr{T}$ be the set of of all finite sequences $s = \langle s_i : i < |s| \rangle$ of elements of $\{0, 1, 2, 3\}$ such that for all $i < |s|$:

- if i is even, then $s_i \in \{2, 3\}$,
- if i is odd, then $s_i \in \{0, 1\}$,
- if $s_i = 3$ then $i = |s| - 1$.

Thus S is a tree under $\sqsubseteq$.

We define $\mathsf{B}^+ \in \mathscr{T}_{\{\mathrm{A}_2, \mathrm{C}_2\}}$ to be the same tree as S, whose labels are defined as follows. If $|s|$ is even, then we let

$$l_s^{\mathsf{B}^+}(s^\frown\langle 3 \rangle) = \min(\mathrm{C}_2) \text{ and } l_s^{\mathsf{B}^+}(s^\frown\langle 2 \rangle) = \max(\mathrm{C}_2).$$

If $|s|$ is odd, the we let $\mathrm{range}(l_s^{\mathsf{B}^+}) = \mathrm{A}_2$. We define the tree $\mathsf{B}^- \in \mathscr{T}_{\{\mathrm{A}_2, \mathrm{C}_2\}}$ in the same way as B^+, with the only difference that

$$l_s^{\mathsf{B}^-}(s^\frown\langle 3 \rangle) = \max(\mathrm{C}_2) \text{ and } l_s^{\mathsf{B}^-}(s^\frown\langle 2 \rangle) = \min(\mathrm{C}_2).$$

We also define the tree $\mathsf{C} \in \mathscr{T}_{\{\mathrm{A}_2,\mathrm{C}_2\}}$ to have underlying set S. If $|s|$ is even, then we let $\mathrm{range}(l_s^{\mathsf{C}}) = \mathrm{A}_2$. If $|s|$ is odd then we let

$$l_s^{\mathsf{C}}(s^\frown\langle 0\rangle) = \min(\mathrm{C}_2) \text{ and } l_s^{\mathsf{C}}(s^\frown\langle 1\rangle) = \max(\mathrm{C}_2).$$

We now define $\mathsf{Q} \in \mathscr{T}_{\{\mathrm{C}_2\}}$ as a copy of $2^{<\omega}$ and for each $t \in \mathsf{Q}$ we have $\mathrm{range}(l_t^{\mathsf{Q}}) = \mathrm{C}_2$. Finally we define $\mathsf{A} \in \mathscr{T}_{\{\mathrm{A}_2\}}$ as a copy of $2^{<\omega}$, and for each $t \in \mathsf{A}$ we have $\mathrm{range}(l_t^{\mathsf{A}}) = \mathrm{A}_2$.

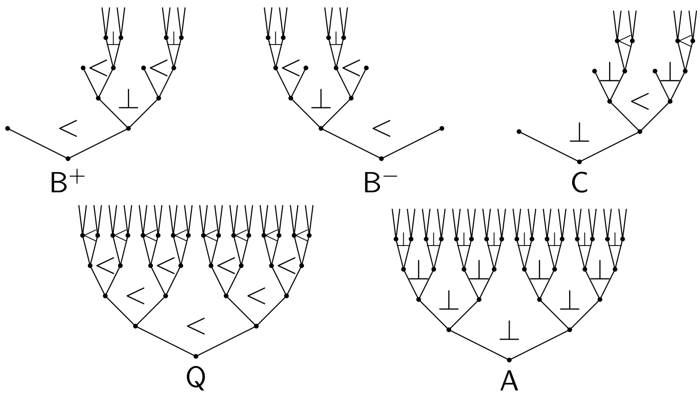

Figure 3.8: The structured trees B^+, B^-, C, Q and A.

Intuitively, the trees B^+, B^- and C will be decomposition trees for the partial orders $2^{<\omega}$, $-2^{<\omega}$ and $2^{<\omega}_\perp$ respectively. Then if $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$ has a full interval tree $\hat{M}$ with $2^{<\omega} \leqslant M$, then Q or A embed into M. Then, using Lemma 3.3.27 we will be able to find an embedding of B^+, B^- or C, so that X fails 3.2.14 (iii).

Lemma 3.3.29. *If $T \in \mathscr{T}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}$ is such that $2^{<\omega} \leqslant T$, then either $\mathsf{Q} \leqslant T$ or $\mathsf{A} \leqslant T$.*

Proof. Let $U \subseteq T$ be the range of the embedding given by $2^{<\omega} \leqslant T$. So for each $t \in U$, we have $|\mathrm{range}(l_t^U)| = 2$, i.e. $\mathrm{range}(l_t^U) \in \{\mathrm{A}_2, \mathrm{C}_2\}$. Let $\hat{U} : U \to 2$ be such that $\hat{U}(t) = 1$ iff $\mathrm{range}(l_t^U) = \mathrm{A}_2$. Then either there is a $t \in U$ such that $\hat{U}``{\uparrow}t = \{0\}$ (and thus $\mathsf{Q} \leqslant {\uparrow}t \leqslant U \leqslant T$), or for every $t \in U$ there is some $u \in U$ with $u > t$ such that $\hat{U}(u) = 1$.

In this case, pick $t_{\langle\rangle} \in U$ such that $\hat{U}(u) = 1$. Having defined t_s for some sequences $s \in 2^{<\omega}$, let $t'_{s^\frown\langle 0\rangle}$ and $t'_{s^\frown\langle 1\rangle}$ be the successors of t_s in U. So there are $t_{s^\frown\langle 0\rangle} > t'_{s^\frown\langle 0\rangle}$ and $t_{s^\frown\langle 1\rangle} > t'_{s^\frown\langle 1\rangle}$ such that $\hat{U}(t_{s^\frown\langle 0\rangle}) = \hat{U}(t_{s^\frown\langle 1\rangle}) = 1$. Therefore $\varphi : \mathsf{A} \to U$ given by $\varphi(s) = t_s$ is an embedding and $\mathsf{A} \leqslant U \leqslant T$. $\square$

Lemma 3.3.30. *Suppose that $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}} \setminus \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ has a full interval tree $\hat{M}$. Then either* B^{+}, B^{-} *or* C *embed into M.*

Proof. Let $\hat{M}$ be a full interval tree for $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}} \setminus \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$. By Lemma 3.3.26 we have $2^{<\omega} \leqslant M$, so by Lemma 3.3.29, there is $U \in \{\mathsf{A}, \mathsf{Q}\}$ such that $U \leqslant M$. Let φ be the embedding witnessing this. We also let $P \in \{\mathrm{A}_2, \mathrm{C}_2\}$ be such that for $t \in U$, $\mathrm{range}(l^U_t) = P$, and $P' \in \{\mathrm{A}_2, \mathrm{C}_2\} \setminus \{P\}$.

We will define the following embedding $\psi : \mathsf{S} \to M$ by repeatedly using Lemma 3.3.27. First we let $\psi(\langle\rangle) = \tau(\varphi(\langle\rangle))$. Now suppose that for some $s \in \{\langle\rangle\} \cup \{s'^\frown\langle i\rangle \in \mathsf{S} : i \in \{0,1\}\}$, we have defined $\psi(s) = \tau(\varphi(t))$. We then define $\psi(s^\frown\langle 3\rangle) = v_t$ and $\psi(s^\frown\langle 2\rangle) = \varphi(a)$ for $a \in 2^{<\omega}$ such that $\varphi(a) \geqslant u_t$ (which exists using Lemma 3.3.27). Finally suppose we have defined $\psi(s) = \varphi(t)$ when $s = s'^\frown\langle 2\rangle$ for some $s' \in \mathsf{S}$. Then for $i \in \{0,1\}$, let $\psi(s^\frown\langle i\rangle) = \tau(\varphi(t^\frown\langle i\rangle))$.

Thus ψ is an embedding. We also have for each $s \in \mathsf{S}$ with last element s_0,

$$\{l^M_{\psi(s)}(u) : u \in \mathrm{succ}(\psi(s)) \cap {\downarrow}\mathrm{range}(\psi)\} \cong \begin{cases} P & : s_0 = 2 \\ P' & : s = \langle\rangle \text{ or } s_0 \in \{0,1\} \\ \emptyset & : s_0 = 3 \end{cases}.$$

So if $P = \mathrm{C}_2$ then ψ witnesses that $\mathsf{C} \leqslant M$.

Suppose that $P = \mathrm{A}_2$. Let $s \in \mathsf{S}$ have last element s_0, and suppose either that $s = \langle\rangle$ or $s_0 \in \{0,1\}$. For $i \in \{2,3\}$ let t^s_i be the element of $\mathrm{succ}(\psi(s))$ such that $t^s_i \leqslant \psi(s^\frown\langle i\rangle)$. Then $\{l^M_{\psi(s)}(t^s_2), l^M_{\psi(s)}(t^s_3)\}$ is isomorphic to $P' = \mathrm{C}_2$.

Let $\hat{\mathsf{S}} : \mathsf{S} \to 3$ be defined as follows, when s_0 is the last element of $s \in \mathsf{S}$:

$$\hat{\mathsf{S}}(s) = \begin{cases} 0 & : s_0 \in \{2,3\} \\ 1 & : l^M_{\psi(s)}(t^s_2) < l^M_{\psi(s)}(t^s_3) \\ 2 & : l^M_{\psi(s)}(t^s_2) > l^M_{\phi(s)}(t^s_3) \end{cases}.$$

Either there is some $u \in \mathsf{S}$ such that $\hat{\mathsf{S}}``{\uparrow}u = \{0,1\}$ (in which case $\psi \restriction {\uparrow}u$ witnesses $\mathsf{B}^{+} \leqslant M$); or for every $u \in \mathsf{S}$ whose last element is not 3, there is some $\pi(u) \geqslant u$, such that $\hat{\mathsf{S}}(\pi(u)) = 2$.

In this case, let $\mu : \mathsf{S} \to \mathsf{S}$ be such that $\mu(\langle\rangle) = \pi(\langle\rangle)$, and if we have defined $\mu(s)$ for some s, define for $s^\frown\langle i\rangle \in \mathsf{S}$

$$\mu(s^\frown\langle i\rangle) = \begin{cases} \mu(s)^\frown\langle i\rangle & : i \in \{2,3\} \\ \pi(\mu(s)^\frown\langle i\rangle) & : i \in \{0,1\} \end{cases}.$$

So that $\mathrm{range}(\hat{S} \circ \mu) = \{0,2\}$, and $\psi \circ \mu$ is an embedding. Therefore $\psi \circ \mu$ witnesses $\mathsf{B}^- \leqslant M$. $\square$

Theorem 3.3.31. $\tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}} \supseteq \mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$

Proof. Let $X \notin \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ be a partial order. We will show that X fails to satisfy either 3.2.14 (i), (ii) or (iii). So suppose that X satisfies 3.2.14 (i) and (ii), i.e. $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}} \setminus \tilde{\mathcal{S}}^{\mathbb{L}}_{\mathbb{P}}$ and we will show that (iii) fails.

Let $\hat{M}$ be a full interval tree for X, so by Lemma 3.3.30, for some $T \in \{\mathsf{B}^+, \mathsf{B}^-, \mathsf{C}\}$ we have $T \leqslant M$. Let φ witness this embedding and let S be the set of elements of S with last element 3. For each $s \in S$, pick some element $x_s \in \hat{M}(\varphi(s))$ and consider $P = \{x_s : s \in S\}$. Let $\mu : S \to P$ be such that $\mu(s) = x_s$.

We claim that P embeds either $2^{<\omega}$, $-2^{<\omega}$ or $2^{<\omega}_{\perp}$. Let $\psi : 2^{<\omega} \to S$ be such that

$$\psi(\langle n_0, n_1, ..., n_m\rangle) = \langle 2, n_0, 2, n_1, ..., 2, n_m, 3\rangle.$$

Let $s \in \mathsf{S}$ be longest such that $s \sqsubseteq \psi(x), \psi(y)$. For $i \in \{0,1,2,3\}$, if $s^\frown\langle i\rangle \in T$ then let t_i be the element of $\mathrm{succ}(\varphi(s))$ such that $t_i \leqslant \varphi(s^\frown\langle i\rangle)$. We now have three cases:

- If $T = \mathsf{B}^+$ then let $x, y \in 2^{<\omega}$. If $x < y$ then $x \sqsubset y$ so $\psi(x) = s^\frown\langle 3\rangle$ and $s^\frown\langle 2\rangle \sqsubseteq \psi(y)$. So since φ is an embedding we have

  $$l^M_{\varphi(s)}(t_3) < l^M_{\varphi(s)}(t_2).$$

  Hence any element of the interval $\hat{M}(t_3)$ is below any element of the interval $\hat{M}(t_2)$. In particular, $\mu \circ \varphi \circ \psi(x) < \mu \circ \varphi \circ \psi(y)$.

  If $x \perp y$, then the last element of s is 2. Therefore

  $$l^M_{\varphi(s)}(t_0) \perp l^M_{\varphi(s)}(t_1),$$

  and so in particular $\mu \circ \varphi \circ \psi(x) \perp \mu \circ \varphi \circ \psi(y)$. Hence $\mu \circ \varphi \circ \psi$ witnesses $2^{<\omega} \leqslant P$.

- If $T = \mathsf{B}^-$ then let $x, y \in -2^{<\omega}$. If $x > y$ then $x \sqsubset y$ then $\psi(x) = s^\frown\langle 3\rangle$ and $s^\frown\langle 2\rangle \sqsubseteq \psi(y)$. So since φ is an embedding we have

  $$l^M_{\varphi(s)}(t_3) > l^M_{\varphi(s)}(t_2).$$

  Therefore $\mu \circ \varphi \circ \psi(x) > \mu \circ \varphi \circ \psi(y)$.

  If $x \perp y$, then we proceed exactly the same as in the previous case to see that $\mu\circ\varphi\circ\psi$ witnesses $-2^{<\omega} \leqslant P$.

- If $T = \mathsf{C}$ then let $x, y \in 2^{<\omega}_{\perp}$. If $x < y$ then there are some sequences s_0, x' and y' such that $x = s_0{}^\frown\langle 0\rangle^\frown x'$ and $y = s_0{}^\frown\langle 0\rangle^\frown y'$. Thus $s = s_0{}^\frown\langle 2\rangle$ and therefore

  $$l^M_{\varphi(s)}(t_0) < l^M_{\varphi(s)}(t_1).$$

  So in particular, $\mu \circ \varphi \circ \psi(x) < \mu \circ \varphi \circ \psi(y)$.

  If $x \perp y$, then either $x \sqsubset y$ or $y \sqsubset x$. In either case

  $$l^M_{\varphi(s)}(t_3) \perp l^M_{\varphi(s)}(t_2),$$

  and in particular $\mu \circ \varphi \circ \psi(x) \perp \mu \circ \varphi \circ \psi(y)$. Hence $\mu \circ \varphi \circ \psi$ witnesses $2^{<\omega}_{\perp} \leqslant P$.

Therefore $P \subseteq X$ fails 3.2.14 (iii) and hence $X \notin \mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$ which completes the proof. $\square$

This completes the proof of Theorem 3.3.1 and Theorem 3.3.2.

3.4 σ-scattered partial orders

Definition 3.4.1. Let $(X_n)_{n\in\omega}$ be a sequence of elements of $\mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$. We call $(X_n)_{n\in\omega}$ a *limiting sequence* iff for each $n \in \omega$ and each $x \in X_n$, there are partial orders $X^x_n \in \mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$ such that

$$X_{n+1} = \sum_{x\in X_n} X^x_n.$$

For every limiting sequence $(X_n)_{n\in\omega}$ and every $n \in \omega$, we consider $X_n \subseteq X_{n+1}$ by identifying every $x \in X_n$ with some point $x' \in X^x_n \subseteq X_{n+1}$.[6]

[6]So $\{x' : x \in X_n\} \subseteq X_{n+1}$ is a partial order isomorphic to X_n.

Definition 3.4.2 (σ-scattered partial orders)**.** We define the class $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ so that $X \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ iff X is a partial order and there is some limiting sequence $(X_n)_{n\in\omega}$ such that $X = \bigcup_{n\in\omega} X_n$. In this case we call X the *limit of* $(X_n)_{n\in\omega}$.

Proposition 3.4.3. *Every limiting sequence has a limit.*

Proof. If $(X_n)_{n\in\omega}$ is a limiting sequence, then we have made some identification between elements of X_n and X_{n+1}, so that $X_n \subseteq X_{n+1}$ and we have that $\bigcup_{n\in\omega} X_n$ is a limit of this sequence. $\square$

Lemma 3.4.4. *If $X \in \tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$ then there is a $\hat{T} \in \mathscr{W}_{\mathbb{H}_{\mathbb{P}}^{\mathbb{L}}}(\tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}})$ satisfying properties (1) to (4) and (5) of Definition 3.3.14 such that whenever $s, t \in T$ with $s < t$ then rank$(\hat{T}(s)) >$ rank$(\hat{T}(t))$.*

Proof. Let T_0 be a single point and $\hat{T}_0 : T_0 \to \{X\}$. Suppose that we have defined $\hat{T}_n \in \mathscr{W}_{\mathbb{H}_{\mathbb{P}}^{\mathbb{L}}}(\tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}})$. For each leaf $t \in T_n$ we have that $\hat{T}_n \in \tilde{\mathcal{S}}_{\mathbb{P}}^{\mathbb{L}}$, so there is a $H_t \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ and some intervals I_t^u $(u \in H_t)$ such that

$$\hat{T}_n(t) = \sum_{u\in H_t} I_t^u.$$

Now let

$$T_{n+1} = T_n \cup \{u \in H_t : t \text{ is a leaf of } T_n, |\hat{T}_n(t)| > 1\}.$$

For $s, t \in T_{n+1}$ we let $s < t$ iff either $s, t \in T_n$ and $s < t$ or $s \in T_n$ and $t \in H_s$. For $t \in T_{n+1}$ and $s \in \mathrm{succ}(t)$, let

$$l_t^{T_{n+1}}(s) = \begin{cases} l_t^{T_n}(s) & : s \in T_n \\ s & : s \in H_t \end{cases}.$$

We also let $\hat{T}_{n+1} \restriction T_n = \hat{T}_n$ and if $u \in H_t$ then let $\hat{T}_{n+1}(u) = I_t^u$. Let $T = \bigcup_{n\in\omega} T_n$ and for all $n \in \omega$ and $t \in T_n$, let $\hat{T}(t) = \hat{T}_n(t)$.

By construction then, $\hat{T}$ satisfies properties (1) to (4). We have that $T \in \mathscr{W}$, since if $s < t$ then $\mathrm{rank}(\hat{T}(s)) > \mathrm{rank}(\hat{T}(t))$. We also have property (5), since a successor of $t \in T$ was defined whenever $|\hat{T}(t)| > 1$. $\square$

Definition 3.4.5. For $X \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$, we call $\hat{T} \in \mathscr{R}_{\mathbb{H}_{\mathbb{P}}^{\mathbb{L}}}(\mathscr{M}_{\mathbb{P}}^{\mathbb{L}})$ a *regular interval tree for* X iff $\hat{T}$ satisfies (1) to (4) and (5) of Definition 3.3.14 and additionally it satisfies the following properties:

7. For every $x \in X$ there is some unique leaf $t_x \in T$ such that $\hat{T}(t_x) = \{x\}$.

8. For every chain $\zeta \subseteq T$ with $\mathrm{ot}(\zeta) = \omega$, we have

$$\bigcap_{t \in \zeta} \hat{T}(t) = \emptyset.$$

Definition 3.4.6. Let X be the limit of $(X_n)_{n\in\omega}$. So for each $n \in \omega$ we have $X_{n+1} = \sum_{x \in X_n} X_n^x$ for some partial orders X_n^x. Given $n \in \omega$ and $x \in X_n$, let $X_n^{x,0} = X_n^x$. Having defined $X_n^{x,m} \subseteq X_{n+m+1}$ for some $m \in \omega$, let

$$X_n^{x,m+1} = \sum_{y \in X_n^{x,m}} X_{n+m+1}^y.$$

Therefore $(X_n^{x,m})_{m\in\omega}$ is a limiting sequence and we let $X_n^{x,*}$ be its limit. We note that therefore, $X = \sum_{x \in X_n} X_n^{x,*}$. See Figure 3.9.

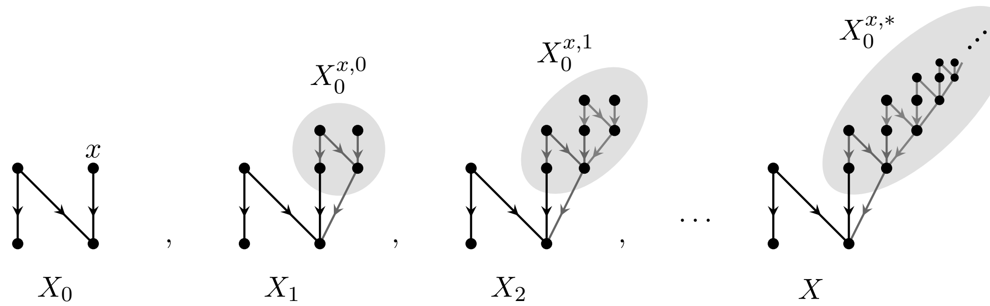

Figure 3.9: The subsets $X_n^{x,m}$ and $X_n^{x,*}$ of a limiting sequence $(X_n)_{n\in\omega}$.

Lemma 3.4.7. *If $X \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ and $x_0 \in X$, then there is some $H \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ and some partial orders $Y_u \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ for $u \in H$ such that $X = \sum_{u \in H} Y_u$ and for some $u_0 \in H$, we have $Y_{u_0} = \{x_0\}$.*

Proof. Let X be the limit of $(X_n)_{n\in\omega}$. Now let $n \in \omega$ be least such that $x_0 \in X_n$. Since $X_n \in \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ we can pick a maximal chain of intervals of X_n with order type in $\overline{\mathbb{L}}$ that contains $\{x_0\}$. Then using Lemma 3.3.16 there is some $H_n \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ and some partial orders Y_u^n for $u \in H_n$, such that $X_n = \sum_{u \in H_n} Y_u^n$. Therefore,

$$X = \sum_{u \in H_n} \sum_{x \in Y_u^n} X_n^{x,*}.$$

Since we used Lemma 3.3.16, we also have that if $H_n = H_{\hat{r}_n}$ with $\hat{r}_n = \langle \hat{a}_i : i \in r_n \rangle \in \mathbb{E}_{\mathbb{P}}^{\mathbb{L}}$, then $a_{\max r_n} = \{u_0^n\}$ and $Y_{u_0^n}^n = \{x_0\}$.

Now let $m > n$, and pick a maximal chain of intervals of $X_{m-1}^{x_0}$ that contains $\{x_0\}$. So we have again using Lemma 3.3.16 that there is some $H_m \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ and some partial orders Y_u^m for $u \in H_m$, such that $X_{m-1}^{x_0} = \sum_{u \in H_m} Y_u^m$ and therefore

$$X_{m-1}^{x_0,*} = \sum_{u \in H_m} \sum_{x \in y_u^m} X_m^{x,*}.$$

Furthermore if $H_m = H_{\hat{r}_m}$ and $\hat{r}_m = \langle \hat{a}_i : i \in r_m \rangle$, then $a_{\max r_m} = \{u_0^m\}$ and $Y_{u_0^m}^m = \{x_0\}$.

Now define as follows:

- $r'_m = r_m \setminus \{\max r_m\}$ for each $m \geqslant n$;
- $r = r'_n {}^\frown r'_{n+1} {}^\frown \dots {}^\frown \langle i_0 \rangle$;
- $\hat{r}(i) = \hat{r}_n(i)$ whenever $i \in r_n$;
- $\hat{r}(i_0) = \hat{a}_{i_0}$ where $a_{i_0} = \{u_0\}$, and $\hat{a}_{i_0}(u_0) = 1$.

Thus $H_{\hat{r}} \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ since $\mathrm{ot}(r)$ is a lexicographic $\omega + 1$-sum of order types in $\overline{\mathbb{L}}$ and we have

$$H_{\hat{r}} = \{u_0\} \sqcup \bigsqcup_{m \geqslant n} H_m \setminus a_{\max r_m}.$$

So for $u \in H_{\hat{r}}$ let $Y'_u = Y_u^m$ if $u \in H_m$ and $Y'_{u_0} = \{x_0\}$. We also let $X^{x,*} = X_m^{x,*}$ if $x \in Y_u^m$ for some $u \in H_m$ and we let $X^{x_0,*} = \{x_0\}$. Then letting $Y_u = \sum_{x \in Y'_u} X^{x,*}$ we have

$$X = \sum_{u \in H_{\hat{r}}} Y_u,$$

and we note that $Y_{u_0} = \{x_0\}$. $\square$

Lemma 3.4.8. *For every $X \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ there is a regular interval tree for X.*

Proof. Let T_0 be the singleton tree and $\hat{T}_0 : T_0 \to \{X\}$. Suppose that for $n \in \omega$ we have defined $\hat{T}_n \in \mathscr{W}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\mathscr{M}^{\mathbb{L}}_{\mathbb{P}})$, satisfying properties (1) to (4) of Definition 3.3.14.

For every $t \in T_n$ let $m(t) \in \omega$ be least such that $\hat{T}_n(t) \cap X_{m(t)} \neq \emptyset$. If $|\hat{T}_n(t) \cap X_{m(t)}| > 1$ then using Lemma 3.4.4 let $\hat{U}_t \in \mathscr{W}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\tilde{\mathscr{S}}^{\mathbb{L}}_{\mathbb{P}})$ satisfy properties (1) to (4) and (5) of Definition 3.3.14 for $X_{m(t)} \cap \hat{T}_n(t) \in \tilde{\mathscr{S}}^{\mathbb{L}}_{\mathbb{P}}$. Define $U'_t = U_t$ and for $v \in U'_t$ let

$$\hat{U}'_t(v) = \sum_{y \in \hat{U}_t(v)} X^{y,*}_{m(t)}.$$

Therefore $\hat{U}'_t(\mathrm{root}(U'_t)) = \hat{T}_n(t)$ and $\hat{U}'_t$ is a well-founded tree satisfying properties (1) to (4). We also have that for any leaf u of U'_t, that $\hat{U}'_t(u)$ contains precisely one point of $X_{m(t)}$.

If $|\hat{T}_n(t) \cap X_{m(t)}| = 1$ then let x_0 be the unique element of this set and use Lemma 3.4.7 to find some $H^t \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$ and partial orders Y^t_u $(u \in H^t)$ such that $\hat{T}_n(t) = \sum_{u \in H^t} Y^t_u$ and for some $u_0 \in H^t$, $Y^t_{u_0} = \{x_0\}$.

Now define

$$W_n = \bigcup \{U_t \setminus (\mathrm{root}(U_t)) : t \text{ is a leaf of } T_n, |\hat{T}_n(t) \cap X_{m(t)}| > 1\},$$

$$D_n = \bigcup \{H^t : t \text{ is a leaf of } T_n, |\hat{T}_n(t) \cap X_{m(t)}| = 1, |\hat{T}_n(t)| > 1\}$$

and $T_{n+1} = T_n \cup W_n \cup D_n$. For $u, v \in T_{n+1}$ let $u < v$ iff either

- $u, v \in T_n$ and $u <_{T_n} v$,
- $u, v \in U_t$ for some leaf t of T_n and $u <_{U_t} v$,
- $u \in T_n$, $u \leqslant_{T_n} t$, for some leaf t of T_n such that either $v \in U_t$ or $v \in H^t$.

For $t \in T_{n+1}$ and $u \in \mathrm{succ}(t)$ we define,

$$l^{T_{n+1}}_t(u) = \begin{cases} l^{T_n}_t(u) & : t \text{ is not a leaf of } T_n \\ l^{U_t}_{\mathrm{root}(U_t)}(u) & : t \text{ is a leaf of } T_n \text{ and } |\hat{T}_n(t) \cap X_{m(t)}| > 1 \\ u \in H^t & : t \text{ is a leaf of } T_n \text{ and } |\hat{T}_n(t) \cap X_{m(t)}| = 1 \\ l^{U_{t'}}_t(u) & : t \in U_{t'} \text{ for some leaf } t' \in T_n \end{cases}.$$

Finally for $t \in T_n$ let $\hat{T}_{n+1}(t) = \hat{T}_n(t)$, for $v \in U_t$, let $\hat{T}_{n+1}(v) = \hat{U}'_t(v)$ and for $u \in H^t$, let $\hat{T}_{n+1}(u) = Y^t_u$. This defines $\hat{T}_{n+1} \in \mathscr{W}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\mathscr{M}^{\mathbb{L}}_{\mathbb{P}})$ which satisfies properties (1) to (4).

Now let $T = \bigcup_{n\in\omega} T_n \in \mathscr{R}$ and $\hat{T}(t) = \hat{T}_n(t)$ whenever $t \in T_n$. Then $\hat{T}$ satisfies properties (1) to (4) with each instance witnessed by some $\hat{T}_n$ $(n \in \omega)$. We also have that $\hat{T}$ satisfies property (5), since if $|\hat{T}(t)| > 1$ then we have defined some successor $u \in T$ of t.

Now in order to show that $\hat{T}$ satisfies (6), pick some $x \in X$ and $m \in \omega$ be least such that $x \in X_m$. Consider the tree $U = \{t \in T : m(t) \leqslant m\}$. Then U is a well-founded tree. So by repeatedly using property (4) let $u \in U$ be largest such that $x \in \hat{U}(u)$. Then either $\hat{T}(u) = \{x\}$ or the predecessor $v \in U$ of u was such that $X_m \cap \hat{T}(v) = \{x\}$ but then there was a successor $t \in U$ of v such that $\hat{T}(t) = \{x\}$. Clearly then $t = u$ is a leaf of T because then we defined no successors of t in T.

If s and t are both leaves of T such that $\hat{T}(s) = \hat{T}(t) = \{x\}$ then $s \perp t$. But this contradicts property (4) at $s \wedge t$, since the point $x \in X$ is contained in two parts of a partition of X. Hence t is unique and $\hat{T}$ satisfies property (6).

Finally suppose there were a chain ζ if T such that $J = \bigcap_{t\in\zeta} \hat{T}(t) \neq \emptyset$. Then let $x \in J \subseteq X$, and s be the leaf of T such that $\hat{T}(s) = \{x\}$. Since s is a leaf, there is some $t \in \zeta$ with $s \perp t$. But this contradicts property (4) at $s \wedge t$, since the point $x \in X$ is contained in two parts of partition of X. This gives property (8) and therefore $\hat{T} \in \mathscr{R}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\mathscr{M}^{\mathbb{L}}_{\mathbb{P}})$ is a regular interval tree for X. $\square$

Definition 3.4.9. For $\hat{X} \in \mathscr{M}^{\mathbb{L}}_{\mathbb{P}}(Q)$, we call $\hat{T} \in \mathscr{T}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(Q \cup \{-\infty\})$ a *regular decomposition tree for* X iff there is some regular interval tree $\hat{T}' \in \mathscr{T}_{\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}}(\mathscr{M}^{\mathbb{L}}_{\mathbb{P}})$ such that $T = T'$; and for $x \in X$, if t_x is the unique element of T such that $\hat{T}(t_x) = \{x\}$, then

$$\hat{T}'(t) = \begin{cases} \hat{X}(x) & : t = t_x \text{ for some } x \in X \\ -\infty & : \text{otherwise} \end{cases}.$$

Proposition 3.4.10. *For any $\hat{X} \in \mathscr{M}^{\mathbb{L}}_{\mathbb{P}}(Q)$, there is a regular decomposition tree for $\hat{X}$.*

Proof. Use Lemma 3.4.8, and then define as in Definition 3.4.9. $\square$

Theorem 3.4.11. *If $\hat{T}_X$ and $\hat{T}_Y$ are regular decomposition trees for $\hat{X}, \hat{Y} \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}(Q)$ respectively and $\hat{T}_X \leqslant \hat{T}_Y$, then $\hat{X} \leqslant \hat{Y}$.*

Proof. Let $\hat{T}_X$ and $\hat{T}_Y$ be regular decomposition trees for $\hat{X}, \hat{Y} \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}(Q)$ respectively and let $\varphi : T_X \to T_Y$ witness $\hat{T}_X \leqslant \hat{T}_Y$. Since $\hat{T}_X$ and $\hat{T}_Y$ are regular decomposition trees, there are some regular interval trees $\hat{T}'_X$ and $\hat{T}'_Y$ used to define T_X and T_Y as in Definition 3.4.9.

So for every leaf $t \in T_X = T'_X$, we have that $\varphi(t)$ is a leaf of $T_Y = T'_Y$, because

$$-\infty \neq \hat{T}_X(t) \leqslant \hat{T}_Y \circ \varphi(t)$$

and therefore $\hat{T}_Y \circ \varphi(t) \neq -\infty$.

So since $\hat{T}'_X$ is a regular interval tree, for any $x \in X$, there is some $t_x \in T'_X$ such that $\hat{T}'_Y(t_x) = \{x\}$ and thus since $\hat{T}'_Y$ is a regular interval tree, we have $\hat{T}'_Y(\varphi(t_x)) = \{y_x\}$ for some $y_x \in Y$. So define $\psi : X \to Y$ so that $\psi(x) = y_x$ for all $x \in X$.

We claim that $\psi(x)$ is an embedding. For $a, b \in X$, let $t'_a, t'_b \in \mathrm{succ}(t_a \wedge t_b)$ such that $t'_a \leqslant t_a$ and $t'_b \leqslant t_b$. Then using property (4) we have $a < b$ iff

$$l^{T_X}_{t_a \wedge t_b}(t'_a) < l^{T_X}_{t_a \wedge t_b}(t'_b)$$

iff

$$l^{T_Y}_{\varphi(t_a \wedge t_b)}(\varphi(t'_a)) < l^{T_Y}_{\varphi(t_a \wedge t_b)}(\varphi(t'_b))$$

iff

$$l^{T_Y}_{\varphi(t_a) \wedge \varphi(t_b)}(\varphi(t'_a)) < l^{T_Y}_{\varphi(t_a) \wedge \varphi(t_b)}(\varphi(t'_b))$$

iff $\psi(a) < \psi(b)$.

We also have

$$\hat{X}(a) = \hat{T}_X(t_a) \leqslant \hat{T}_Y(\varphi(t_a)) = \hat{Y} \circ \psi(a).$$

Therefore ψ witnesses $\hat{X} \leqslant \hat{Y}$. $\square$

Theorem 3.4.12. *Let $\mathbb{P}$ be a class of indecomposable partial orders that do not embed any element of $\{2^{<\omega}, -2^{<\omega}, 2^{<\omega}_{\perp}\}$, that is closed under taking indecomposable subsets. Let $\mathbb{L}$ be a class of linear orders closed under taking subsets, such that* $\mathrm{On} \subseteq \mathbb{L}$. *Then if $\mathbb{L}$ and $\mathbb{P}$ are well-behaved then $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ is well-behaved.*

Proof. Suppose that f is a bad $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}(Q)$-array. Then let g have the same domain as f and let $g(X)$ be a regular decomposition tree for $f(X) \in \mathscr{M}_{\mathbb{P}}^{\mathbb{L}}(Q)$. Then g is a bad $\mathscr{T}_{\mathbb{H}_{\mathbb{P}}^{\mathbb{L}}}(Q \cup \{-\infty\})$-array by Theorem 3.4.11. We note that any witnessing bad Q-array for g will be a witnessing bad Q-array for f, since if $t \in \mathrm{dom}(g(X))$ and $g(X)(t) \neq -\infty$ then there is some $x \in f(X)$ such that $f(X)(x) = g(X)(t)$.

Now by Theorem 2.4.7 and Lemma 3.2.5, we have that $\mathscr{T}_{\mathbb{H}_{\mathbb{P}}^{\mathbb{L}}}$ is well-behaved and thus there is a witnessing bad $Q \cup \{\infty\}$-array for g, so by Theorem 2.1.6 there is a witnessing bad Q-array for g, which by the previous paragraph is also witnessing for f. $\square$

3.5 Corollaries and applications

We mention that the statement of Theorem 3.4.12 can be simplified using the following theorem.

Theorem 3.5.1. *If $\mathbb{P}$ is a well-behaved class of partial orders, then*

$$\breve{\mathbb{P}} = \{Y \in \mathscr{P} : \exists P \in \mathbb{P}, Y \subseteq P\}$$

is well-behaved.

Proof. Let Q be a quasi-order and $f : [\omega]^\omega \to \breve{\mathbb{P}}(Q)$ be a bad array. Then define $g : [\omega]^\omega \to \mathbb{P}(Q \cup \{-\infty\})$ so that $f(X) = \hat{Y}$ implies $g(X) = \hat{P}$ for some $P \in \mathbb{P}$ such that $Y \subseteq P$, and for $x \in P$ we have $\hat{P}(x) = \hat{Y}(x)$ whenever $x \in Y$ and $\hat{P}(x) = -\infty$ whenever $x \notin Y$. So since $\mathbb{P}$ is well-behaved there is some $A \in [\omega]^\omega$ and a witnessing bad array $h : [A]^\omega \to Q \cup \{-\infty\}$. Then by restricting further using Theorem 2.1.6 there is some $B \in [A]^\omega$ such that $h"[B]^\omega \subseteq Q$. Therefore $h \restriction [B]^\omega$ is a witnessing bad array for f. $\square$

Corollary 3.5.2. *If $\mathbb{P}$ is a class of partial orders and $2_{\perp}^{<\omega} \leqslant P \in \mathbb{P}$, then $\mathbb{P}$ is not well-behaved.*

Proof. Suppose that $2_{\perp}^{<\omega} \leqslant P \in \mathbb{P}$ and let $\varphi : 2_{\perp}^{<\omega} \to P$ be an embedding. Consider

$$Z = \{\varphi(\langle 0 \rangle), \varphi(\langle 0, 0 \rangle), ...\} \cup \{\varphi(\langle 1 \rangle), \varphi(\langle 1, 1 \rangle), ...\} \subseteq P.$$

We note that then Z is isomorphic to the partial order defined in Remark 2.2.10, thus similarly we can find an infinite antichain of $\breve{\mathbb{P}}(\mathrm{A}_2)$, so if $\mathbb{P}$ were well-behaved this would contradict Theorem 3.5.1. □

We can then obtain the following simplification of Theorem 3.4.12.

Theorem 3.5.3. *Let $\mathbb{P}$ be a class of indecomposable partial orders that do not embed $2^{<\omega}$ or $-2^{<\omega}$ and let $\mathbb{L}$ be a class of linear orders. Then if $\mathbb{L}$ and $\mathbb{P}$ are well-behaved then $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ is well-behaved.*

Proof. Since $\mathbb{P}$ is well-behaved, $2_{\perp}^{<\omega}$ does not embed into any element of $\mathbb{P}$ by Corollary 3.5.2. Now apply Theorem 3.4.12 using $\breve{\mathbb{P}} \cap \{P \in \mathscr{P} : P \text{ is indecomposable}\}$ and $\breve{\mathbb{L}} \cup \mathrm{On}$ (which are well-behaved by Theorem 3.5.1, Theorem 2.3.2 and Theorem 2.1.6). Then $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ is clearly a sub-class of the obtained well-behaved class, and is thus well-behaved. □

Furthermore, since we know now that under these assumptions $\mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ will be well-behaved, we can drop the condition that these orders must not embed $2_{\perp}^{<\omega}$ in (iii) of Definition 3.2.14 by Corollary 3.5.2. We also note that the definition of $\mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ (Definition 3.2.14) can be further simplified by considering the following alternate version of 3.2.14 (ii).[7]

(ii*) Every linear subset of X is isomorphic to a member of $\overline{\mathbb{L}}$.

Theorem 3.5.4. *If X is a partial order, $\mathrm{On} \subseteq \overline{\mathbb{L}}$ and $\mathbb{L}$ is closed under reversing orders, then X satisfies (ii*) implies X satisfies 3.2.14 (ii).*

Proof. Suppose that X satisfies (ii*) but fails (ii). So there is some $x \in X$ such that for any maximal chain $\langle I_i : i \in r\rangle$ of non-empty intervals of X under $\supseteq$ that contains $\{x\}$, we have $r \notin \overline{\mathbb{L}}$.

Pick such a chain $\langle I_i : i \in r\rangle$, and for each $i \in r$ let $P_i = I_i \setminus \bigcup_{j>i} I_j$. Consider $C_{\leqslant} = \{I_i : i \in r, (\exists y \in P_i), y \leqslant x\}$ then for each $I_i \in C_{\leqslant}$, pick $y_i \in P_i$ such that $y \leqslant x$. Thus since each I_i is an interval, we have $i < j$ iff $y_i < y_j$, therefore $\{y_i : I_i \in C_{\leqslant}\}$

[7]Thus we obtain the simpler characterisation of scattered orders described in the introduction.

is a linear subset of X and thus is isomorphic to a member of $\overline{\mathbb{L}}$ since X satisfies (ii*). Therefore, considering $C_{\leqslant}$ as a chain of intervals under $\supseteq$, we have $\mathrm{ot}(C_{\leqslant}) \in \overline{\mathbb{L}}$.

Now whenever $I_i \in C_{\leqslant}$, let

$$C^i = \{I_j : j \in r, (\forall I_k \in C_{\leqslant}), k > i \to I_i \supseteq I_j \supset I_k\}.$$

We consider the order $r \cup \{-\infty\}$ and let $C^{-\infty} = \{I_j : j \in r, (\forall I_k \in C_{\leqslant}), I_j \supset I_k\}$ and then for $i \in \{-\infty\} \cup \{i : I_i \in C_{\leqslant}\}$ we let,

$$C^i_{\geqslant} = \{I_j \in C^i : (\exists y \in P_j), y \geqslant x\}$$

For each $I_j \in C^i_{\geqslant}$, pick $y_j \in P_j$ such that $y \geqslant x$. Thus since each I_i is an interval, we have $i < j$ iff $y_i > y_j$, therefore $\{y_j : I_j \in C^i_{\geqslant}\}$ is a linear subset of X and thus is isomorphic to a member of $\overline{\mathbb{L}}$ since X satisfies (ii*). Therefore $\mathrm{ot}(C^i_{\geqslant}) \in \overline{\mathbb{L}}$ since $\overline{\mathbb{L}}$ is closed under reversing orders.

Now for $I_j \in C^i_{\geqslant}$, let

$$C^j_{\perp} = \{I_k : k \in r, (\forall I_l \in C^i_{\geqslant}), l > j \to I_j \supseteq I_k \supset I_l\}$$

We also define $C^{-\infty}_{\perp} = \{I_k : k \in r, (\forall I_l \in C^i_{\geqslant}), I_k \supset I_l\}$. So we have that if

$$I_i, I_j \notin C_{\leqslant} \cup C^{-\infty}_{\geqslant} \cup \bigcup_{I_i \in C_{\leqslant}} C^i_{\geqslant}$$

then $\forall y \in P_i, P_j$, $y \perp x$. So since I_i, I_j are intervals, if $i \neq j$ then $\forall y \in P_i$, $\forall z \in P_j$, $y \perp z$. So for each $I_j \in C^i_{\geqslant}$ or for $j = -\infty$, consider $\{P_k : I_k \in C^j_{\perp}\}$ and enumerate these in some ordinal order type so that $\{P_k : I_k \in C^j_{\perp}\} = \{P^{\gamma}_j : \gamma < |C^j_{\perp}|\}$. We also let $K_j = \bigcup_{k>j} I_k$ whenever $I_j \in C^i_{\geqslant}$ and $K_{-\infty} = \bigcup C^{-\infty}_{\geqslant} \cup \bigcup C_{\leqslant}$.

Thus for each $I_j \in C^i_{\geqslant}$ or for $j = -\infty$

$$D^j_{\perp} = \left\langle \bigcup_{\gamma \leqslant \alpha < |C^j_{\perp}|} P^{\alpha}_j \cup K_j : \gamma < |C^j_{\perp}| \right\rangle$$

is a chain of intervals of X under $\supseteq$, maximal in the sense that there is no other interval of X that is strictly contained in one of these intervals and not in another (otherwise our original chain of intervals would fail this property).

But now, whenever $I_j \in C^i_{\geqslant}$ or $j = -\infty$, we have $\mathrm{ot}(D^j_{\perp}) \in \mathrm{On} \subseteq \overline{\mathbb{L}}$. Furthermore if $\mathcal{I} = \{-\infty\} \cup \{i : I_i \in C_{\leqslant}\}$ and $\mathcal{J}_i = \{-\infty\} \cup \{j : I_j \in C^i_{\geqslant}\}$ then $D = \bigcup_{i \in \mathcal{I}} \bigcup_{j \in \mathcal{J}_i} D^j_{\perp}$ is a maximal chain of intervals of X under $\supseteq$, which contains $\{x\}$, and its order type is equal to $L = \sum_{i \in \mathrm{ot}(\mathcal{I})} \sum_{j \in \mathrm{ot}(\mathcal{J}_i)} \mathrm{ot}(D^j_{\perp})$. But $\mathrm{ot}(\mathcal{I})$ is equal to a C_2-sum of a single point (i.e. $-\infty$) and $\mathrm{ot}(C_{\leqslant}) \in \overline{\mathbb{L}}$, similarly each $\mathrm{ot}(\mathcal{J}_i)$ is equal to a C_2-sum of a single point and $\mathrm{ot}(C^i_{\geqslant}) \in \overline{\mathbb{L}}$. Therefore $L \in \overline{\mathbb{L}}$. But then the chain D contradicts that X fails (ii). $\square$

Remark 3.5.5. The assumptions that $\mathrm{On} \subseteq \overline{\mathbb{L}}$ and that $\mathbb{L}$ is closed under reversing orders can be taken for free. To see this, we have that On is well-behaved by Theorem 2.3.2, so that if $\mathbb{L}$ is well-behaved, then the class obtained by adjoining the ordinals and closing under reversing orders $(\mathbb{L} \cup \mathrm{On}) \cup (\mathbb{L}^* \cup \mathrm{On}^*)$ can be seen to be well-behaved, by applying Theorem 2.1.6 twice.

In light of Theorem 3.5.4 we see that under these modest assumptions, (ii) could be replaced by (ii*) in the definition of $\mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$ (Definition 3.2.14) with $\mathscr{M}^{\mathbb{L}}_{\mathbb{P}}$ defined analogously. Then we obtain that this $\mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$ is well-behaved under the same assumptions on $\mathbb{L}$ and $\mathbb{P}$ as in Theorem 3.4.12.

We mention the following natural question.

Question 3.5.6. *Are there assumptions on $\mathbb{L}$ and $\mathbb{P}$ under which $\mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$ consists of precisely the same class whether or not (ii) is replaced in Definition 3.2.14 by (ii*)?*

Definition 3.5.7. We define $\mathscr{C}_{\mathbb{P}}$ to be the class of countable partial orders such that every indecomposable subset is isomorphic to a member of $\mathbb{P}$.

Theorem 3.5.8. *If $\mathscr{C} \subseteq \overline{\mathbb{L}}$, then $\mathscr{C}_{\mathbb{P}} \subseteq \mathscr{M}^{\mathbb{L}}_{\mathbb{P}}$.*

Proof. Let $X \in \mathscr{C}_{\mathbb{P}}$, we will write X as the limit of some limiting sequence $(X_n)_{n \in \omega}$ that we will define. Pick an enumeration of $X = \{x_n : n \in \omega\}$. Since any chain of intervals of X must be countable, we have $X \in \mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$. So we can define a partial interval tree $\hat{T}$ for X by following the same method as Lemma 3.3.17, except that at each stage if we have defined $\hat{T}_n$ for an $n \in \omega$, then let C be a maximal chain that contains $\{x_n\}$.[8]

[8] In Lemma 3.3.17, C was an arbitrary maximal chain of intervals that contains a singleton, so this assumption makes no difference to the construction.

Therefore, for every $n \in \omega$ there is some leaf $t_n \in T$ with $|{\downarrow} t_n| \leqslant n+1$, such that $\hat{T}(t_n) = \{x_n\}$. Let $X_0 = \{x_0\}$ and $X^{x_0} = X$, also let t_{x_0} be the root of T.

Suppose that for $n \in \omega$ we have defined $X_n \subseteq X$, and some partial orders Y_n^x $(x \in X_n)$ so that:

- $X_n \in \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$,
- $X = \sum_{x \in X_n} Y_n^x$,
- $x_m \in Y_n^x \cap X_n = \{x_m\}$ where m is least such that $x_m \in Y_n^x$, and
- for each $x \in X_n$, there is some $t_x^n \in T$ with $\hat{T}(t_x^n) = Y_n^x$.

If $Y_n^x = \{z\}$ let $H_x = \{u\} \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$, $t_x^u = t_x^n$ and $y_x^u = z$. Thus $\hat{T}(t_x^u) = \{z\}$ in this case. If Y_n^x is not a singleton, we have for some $H_x = \mathrm{range}(l_{t_x^n}^T) \in \mathbb{H}_{\mathbb{P}}^{\mathbb{L}}$ that if $t_x^u \in \mathrm{succ}(t_x^n)$ is such that $l_{t_x^n}^T(t_x^u) = u$ then

$$Y_n^x = \hat{T}(t_x^n) = \sum_{u \in H_x} \hat{T}(t_x^u).$$

For each $x \in X_n$ and $u \in H_x$ let $y_x^u = x_m$ where m is least such that $x_m \in \hat{T}(t_x^u)$. Then let

$$X_{n+1} = \{y_x^u : x \in X_n, u \in H_x\}.$$

Then $X_{n+1} \in \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$, since $X_n \in \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ and X_{n+1} can be constructed internally using the same well-founded tree of functions as X_n, applied to $\sum_{u \in H_x} \{y_x^u\} \in \mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ for each $x \in X_n$.

Now $x \in X_{n+1}$ iff $x = y_{x'}^u$ for some $x' \in X_n$ and $u \in H_{x'}$. So for $x \in X_{n+1}$ with $x = y_{x'}^u$, let $Y_{n+1}^x = \hat{T}(t_{x'}^u)$ and $t_x^{n+1} = t_{x'}^u$. Therefore we have

$$X = \sum_{x \in X_n} Y_n^x = \sum_{x \in X_n} \left(\sum_{u \in H_x} \hat{T}(t_x^u) \right) = \sum_{x \in X_{n+1}} Y_{n+1}^x.$$

Thus we can inductively define X_n for every $n \in \omega$.

Now for $n \in \omega$ we have $X_n \subseteq X_{n+1}$, since every element x_m of X_n is contained in some Y_{n+1}^x for $x \in X_n$ and thus m is least such that $x_m \in Y_{n+1}^x$, i.e. $x_m \in X_{n+1}$.

Furthermore for each $n \in \omega$ and $x \in X_n$ let $X_n^x = X_{n+1} \cap Y_n^x$, then

$$X_{n+1} = X_{n+1} \cap \sum_{x \in X_n} Y_n^x = \sum_{x \in X_n} (X_{n+1} \cap Y_n^x) = \sum_{x \in X_n} X_n^x.$$

Therefore $(X_n)_{n\in\omega}$ is a limiting sequence. For each $x \in X$ we have $x = x_n$ for some $n \in \omega$ and therefore $x \in X_n$. Thus X must be the limit of $(X_n)_{n\in\omega}$ which means $X \in \mathscr{M}^{\mathbb{L}}_{\mathbb{P}}$. □

Corollary 3.5.9. *If $\mathbb{P}$ is a well-behaved set of countable indecomposable partial orders, closed under taking indecomposable subsets, then $\mathscr{C}_{\mathbb{P}}$ is well-behaved.*[9]

Proof. By Theorems 2.3.7, 3.5.8 and 3.4.12. □

Remark 3.5.10. If $\mathbb{P}$ is a set of finite partial orders then let $\tilde{\mathbb{P}}$ be the class of countable partial orders whose every finite restriction is in $\mathbb{P}$. In [44], Pouzet asked: if $\mathbb{P}$ preserves bqo, then is $\tilde{\mathbb{P}}$ bqo? As we have seen, well-behaved is a more useful concept than preserving bqo, so we modify the question to if $\mathbb{P}$ well-behaved. Corollary 3.5.9 brings us closer to a result of this kind, however fails to account for possible infinite indecomposable subsets of orders in $\tilde{\mathbb{P}}$. If we could prove that for any infinite indecomposable order X the set of finite indecomposable subsets of X is not well-behaved, then we would answer this version of Pouzet's question positively.

Definition 3.5.11. For $n \in \omega$ let $\mathscr{I}_n$ denote the set of indecomposable partial orders whose cardinality is at most n.

Theorem 3.5.12. *For any $n \in \omega$, the class $\mathscr{M}^{\mathscr{M}}_{\mathscr{I}_n}$ is well-behaved.*

Proof. $\mathscr{I}_n$ is a finite set of finite partial orders so by Lemma 2.2.12, is well-behaved. Furthermore, $\mathscr{M}$ is well-behaved by Theorem 2.3.7, so using Theorem 3.4.12 completes the proof. □

Corollary 3.5.13. *For any $n \in \omega$, the class $\mathscr{C}_{\mathscr{I}_n}$ is well-behaved.*

Proof. By Theorems 3.5.8 and 3.5.12. □

Note that $\bigcup_{n\in\omega} \mathscr{I}_n(\mathrm{A}_2)$ contains an antichain as in Figure 3.10, and as described by Pouzet in [44]. Hence $\bigcup_{n\in\omega} \mathscr{I}_n$ does not preserve bqo and is certainly not well-behaved. Thus Theorem 3.5.12 is optimal in some sense.

[9]This result was obtained independently by Christian Delhommé in as yet unpublished work [9]. The author thanks him for his private communication.

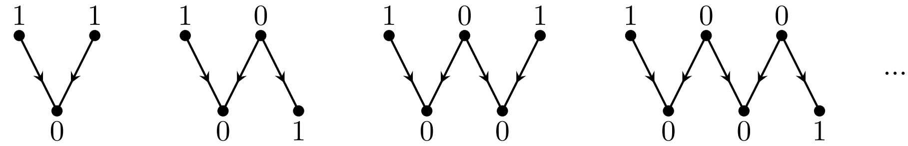

Figure 3.10: An antichain of $\bigcup_{n\in\omega} \mathscr{I}_n(\mathrm{A}_2)$.

In order to improve this result we would like to know the answers to questions such as:

- Is there consistently a well-behaved class of linear orders larger than $\mathscr{M}$? E.g. is the class of Aronszajn lines from [37] well-behaved under PFA?
- Is there an infinite well-behaved class of indecomposable partial orders?
- Is there an infinite indecomposable partial order P such that $\{P\}$ is well-behaved?

A positive answer to any of these questions would immediately improve Theorem 3.5.12.

Chapter 4

Better-quasi-ordering structured pseudo-trees

We will now aim to expand Kříž's structured tree theorem to a large class of pseudo-trees. This will give structured tree versions of both a transfinite analogue of Corominas' theorem on countable pseudo-trees [6] and a pseudo-tree analogue of Laver's theorem on σ-scattered trees [32]. The main result of this chapter will prove that a class $\mathbb{T}_{\mathcal{O}}$ of structured pseudo-trees is well-behaved (Corollary 4.3.5). Indeed this particular $\mathbb{T}$ will contain all *well-branched* members T of Laver's and Corominas' classes, i.e. those satisfying $(\forall x, y \in T)$, $x \wedge y$ exists. This property is required in order for the definition of a *structured* pseudo-trees to make sense. If we wanted to drop this condition then Theorem 3.5.12 already fully generalises Laver's and Corominas' results to such trees. The proof will be reminiscent of the proof of Theorem 3.4.12.

Pseudo-trees have been studied, along with their relationships with interval algebras in [5]. Applications to bqo theory and related areas can be found in [2, 3].

Throughout this chapter, $\mathbb{L}$ will be a class of linear orders closed under taking subsets with $\mathrm{On} \subseteq \mathbb{L}$. We will also let $\mathcal{O}$ be an arbitrary concrete category with injective morphisms.

4.1 σ-scattered pseudo-trees

Definition 4.1.1. A partial order T is called a *pseudo-tree* iff $(\forall t \in T)$, $\downarrow t$ is a linear order and for every $x, y \in T$, we have $x \wedge y$ exists.[1] If $\mathbb{L}$ is a class of linear orders, we call a pseudo-tree T a $\mathbb{L}$-tree iff every chain of T has order type in $\overline{\mathbb{L}}$.

We let $\mathscr{E}$ be the class of all pseudo-trees and $\mathscr{E}^{\mathbb{L}}$ be the class of all $\mathbb{L}$-trees. We consider $\mathscr{E}$ and $\mathscr{E}^{\mathbb{L}}$ as concrete categories whose morphisms are partial order embeddings φ such that $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$.

Definition 4.1.2. Let $\mathscr{U}_0^{\mathbb{L}} = \{\emptyset\}$ and for $\alpha \in \mathrm{On}$ let $\mathscr{U}_{\alpha+1}^{\mathbb{L}}$ be the class of ζ-tree-sums of pseudo-trees of $\mathscr{U}_\alpha^{\mathbb{L}}$ for some linear order $\zeta \in \overline{\mathbb{L}}$. For limit $\lambda \in \mathrm{On}$ we let $\mathscr{U}_\lambda^{\mathbb{L}} = \bigcup_{\gamma<\lambda} \mathscr{U}_\gamma^{\mathbb{L}}$, and finally set $\mathscr{U}^{\mathbb{L}} = \bigcup_{\gamma \in \mathrm{On}} \mathscr{U}_\gamma^{\mathbb{L}}$. For $T \in \mathscr{U}^{\mathbb{L}}$ define the *scattered rank* of T, denoted $\mathrm{rank}_{\mathscr{U}}(T)$ as the least ordinal α such that $T \in \mathscr{U}_\alpha^{\mathbb{L}}$. (See Figure 3.7.)

Lemma 4.1.3. *If $T \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$, then there is some $t \in T$ such that $\uparrow t \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$.*

Proof. Pick a maximal chain ζ of T, so that T is a ζ-tree-sum of some other pseudo-trees T_i^α for $i \in \zeta$ and $\alpha \in \kappa_i \in \mathrm{Card}$. The chain ζ has order type in $\overline{\mathbb{L}}$ thus at least one of these pseudo-trees T_i^α is not an element of $\mathscr{U}^{\mathbb{L}}$, since otherwise $T \in \mathscr{U}^{\mathbb{L}}$.

In accordance with Definition 3.3.21, i is incomparable to any $x \in T_i^\alpha$. If i were the least element of ζ then this would contradict either that ζ is maximal or that $x \wedge i \in T$. (Which is the case since $T \in \mathscr{E}^{\mathbb{L}}$.) Therefore let $t < i$ so that $t \in \zeta \subseteq T$ is such that $T_i^\alpha \subseteq \uparrow t$, therefore $\uparrow t \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$. ☐

Lemma 4.1.4. *If $T \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$ and ζ is a maximal chain of T, then there is some $t \in T \setminus \zeta$ such that $\uparrow t \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$.*

Proof. We have that T is a ζ-tree-sum of some other pseudo-trees T_i^α for $i \in \zeta$ and $\alpha \in \kappa_i \in \mathrm{Card}$ and ζ has order type in $\overline{\mathbb{L}}$. Thus at least one of these trees T_i^α is not an element of $\mathscr{U}^{\mathbb{L}}$, since otherwise $T \in \mathscr{U}^{\mathbb{L}}$. Using Lemma 4.1.3, there is some $t \in T_i^\alpha$ such that $\uparrow t \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$. Clearly then $t \in T \setminus \zeta$. ☐

[1] We consider only *well-branched* pseudo-trees and so define in this way in order to avoid repetition.

Lemma 4.1.5. *If $T \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$ then there is some $u \in T$ and some $t_0, t_1 > u$ such that $\uparrow t_0 \cap \uparrow t_1 = \emptyset$ and $\uparrow t_0, \uparrow t_1 \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$.*

Proof. Let ζ_0 be a maximal chain of T and t_0 be an element of $T \setminus \zeta_0$ such that $\uparrow t \notin \mathscr{U}$. Suppose we have defined t_α as an element of $T \setminus \zeta_\alpha$, for every $\alpha < \beta$, and that if $\alpha < \gamma < \beta$ then $t_\alpha < t_\gamma$. Then let ζ_β be a maximal chain of T containing $\downarrow t_\alpha$ for every $\alpha < \beta$, and let t_β be an element of $T \setminus \zeta_\beta$ such that $\uparrow t_\beta \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$, which exists by Lemma 4.1.4.

Since $t_\beta \notin \zeta_\beta$ we cannot have $t_\alpha \geqslant t_\beta$ for any $\alpha < \beta$. Suppose that $t_\beta \perp t_\alpha$ for some $\alpha < \beta$. But then if we let $u = t_\beta \wedge t_\alpha$, $t_0 = t_\beta$ and $t_1 = t_\alpha$, then these satisfy the statement of the lemma. Otherwise $t_\alpha < t_\beta$, and we can continue the induction. So the induction continues for every ordinal. But then we have found proper class many distinct elements of T, namely t_α for $\alpha \in \mathrm{On}$, thus T is a proper class, which is a contradiction. $\square$

Theorem 4.1.6. *$T \in \mathscr{U}^{\mathbb{L}}$ iff $T \in \mathscr{E}^{\mathbb{L}}$ and $2^{<\omega} \not\leqslant T$.*

Proof. Let $T \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$, we will define $\varphi : 2^{<\omega} \to T$ by induction on the length of $s \in 2^{<\omega}$. Firstly, let $\varphi(\langle\rangle)$ be the element $u \in T$ given by Lemma 4.1.5. Suppose for $s \in 2^{<\omega}$, that we have defined $\varphi(s)$ such that there are $t_0, t_1 > \varphi(s)$ such that $\uparrow t_0, \uparrow t_1 \in \mathscr{E}^{\mathbb{L}} \setminus \mathscr{U}^{\mathbb{L}}$. Then for $i \in \{0, 1\}$, let $\varphi(s^\frown\langle i\rangle)$ be the element $u \in T$ given by applying Lemma 4.1.5 to $\uparrow t_i$. This inductively defines φ, which is clearly an embedding.

For the other direction, firstly it is clear that $2^{<\omega} \not\leqslant \emptyset$. Now suppose that $2^{<\omega} \not\leqslant U$ whenever $\mathrm{rank}_{\mathscr{U}}(U) < \alpha$. Then if $\mathrm{rank}_{\mathscr{U}}(T) = \alpha$, we have that T is a ζ-tree-sum of some lower ranked trees. If $2^{<\omega}$ embeds into T, then if any point in the range of this embedding is in one of the lower ranked trees, then $2^{<\omega}$ embeds into that tree, which cannot happen. Therefore $2^{<\omega}$ embeds into the chain ζ, which is again impossible, and therefore $2^{<\omega} \not\leqslant T$.

We have that $\mathrm{ot}(\zeta) \in \overline{\mathbb{L}}$, so any chain γ of T is such that $\gamma = \zeta' \cup \xi$ for some $\zeta' \subseteq \zeta$ and some chain ξ of one of the lower ranked $\mathbb{L}$-trees. Thus γ is order equivalent to $\mathrm{ot}(\zeta') + \mathrm{ot}(\xi)$ which is a member of $\overline{\mathbb{L}}$ by the induction hypothesis and since we assumed that $\mathbb{L}$ was closed under taking subsets. We then have that $\mathrm{ot}(\gamma) \in \overline{\mathbb{L}}$, so that $T \in \mathscr{E}^{\mathbb{L}}$. $\square$

Definition 4.1.7. We call a sequence $(T_n)_{n\in\omega}$ *limiting* iff for each $n \in \omega$ we have $T_n \in \mathscr{U}^{\mathbb{L}}$, $T_n \subseteq T_{n+1}$ and $(\forall t \in T_{n+1} \setminus T_n)(\forall u \in T_n)$, $t \not\leqslant u$. We call $T = \bigcup_{n\in\omega} T_n$ the *limit* of $(T_n)_{n\in\omega}$, and let $\mathscr{T}^{\mathbb{L}}$ be the class of limits of limiting sequences of elements of $\mathscr{U}^{\mathbb{L}}$.

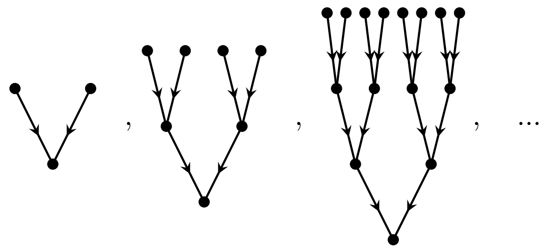

Figure 4.1: A limiting sequence of trees whose limit is $2^{<\omega}$.

4.2 Decomposition of structured pseudo-trees

We now extend the definition of structured trees to pseudo-trees.

Definition 4.2.1. Let $\mathbb{T}$ be a class of pseudo-trees. We define the new concrete category of *$\mathcal{O}$-structured pseudo-trees of* $\mathbb{T}$, denoted $\mathbb{T}_{\mathcal{O}}$ as follows. The objects of $\mathbb{T}_{\mathcal{O}}$ consist of pairs $\langle T, l^T \rangle$ such that:

- $T \in \mathbb{T}$.
- $U_{\langle T, l^T \rangle} = T$.
- $l^T = \{l_v^T : v \in T\}$, where for each $v \in T$ there is some $\gamma_v \in \mathrm{obj}(\mathcal{O})$ such that

$$l_v^T : \updownarrow v \longrightarrow \gamma_v$$

  and if $x, y \in T$ with $x > y > v$ then $l_v(x) = l_v(y)$.

For $\mathcal{O}$-structured trees $\langle T, l^T \rangle$ and $\langle T', l^{T'} \rangle$, we let $\varphi : T \to T'$ be an embedding whenever:

1. $x \leqslant y$ iff $\varphi(x) \leqslant \varphi(y)$,
2. $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$,

3. for any $v \in T$, if $\theta : \mathrm{range}(l_v^T) \to \mathrm{range}(l_{\varphi(v)}^{T'})$ is such that for all $x \in \upharpoonright v$

$$\theta(l_v^T(x)) = l_{\varphi(v)}^{T'}(\varphi(x));$$

then θ is an embedding of $\mathcal{O}$.

To simplify notation, we write T in place of $\langle T, l\rangle$ and always use l_v^T, $(v \in T)$ to denote the labelling functions.

Definition 4.2.2. We extend Definition 2.4.6 so that if T is an $\mathcal{O}$-structured pseudo-tree then for $t \in T$ and $u \in \mathrm{range}(l_v^T)$ we have

$$^{u}\!\upharpoonright t = \{v \in T : v > t, l_t^T(v) = u\}.$$

Definition 4.2.3. If $\mathcal{O}$ is a concrete category and $\mathbb{L}$ is a class of linear orders then define

$$\mathbb{F}_{\mathcal{O}}^{\mathbb{L}} = \{\hat{r} \in \overline{\mathbb{L}}(\mathcal{O}(\mathrm{A}_2)) : (\forall i \in r), (\exists! x \in \mathrm{dom}(\hat{r}(i))), \hat{r}(i)(x) = 1\}.$$

Definition 4.2.4. Given $\hat{r} \in \mathbb{F}_{\mathcal{O}}^{\mathbb{L}}$ with $\hat{r} = \langle \hat{a}_i : i \in r\rangle$, for $i \in r$ let s_i be the unique element of a_i such that $\hat{a}_i(s_i) = 1$. If $i \neq \max r$ then let $S_i = \{s_i\}$ and if $i = \max r$ then let $S_i = \emptyset$. We then define $B_{\hat{r}} \in \mathscr{U}_{\mathcal{O}}^{\mathbb{L}}$ as follows. First let

$$B_{\hat{r}} = r \cup \bigcup_{i \in r} a_i \setminus S_i,$$

we order $s < t$ iff either $s, t \in r$ and $s <_r t$ or $s \in r$, $t \in a_i$ and $i \geqslant_r s$. We let $l_i^{B_{\hat{r}}} : \upharpoonright i \to a_i$ so that $l_i^{B_{\hat{r}}}(t) = s_i$ if $\exists j \in r$ with $j > i$ and $t = j$ or $t \in a_j$. If no such $j \in r$ exists and $t > i$ then $t \in a_i$ and we let $l_i^{B_{\hat{r}}}(t) = t$.

We let $\mathbb{B}_{\mathcal{O}}^{\mathbb{L}} = \{B_{\hat{r}} : \hat{r} \in \mathbb{F}_{\mathcal{O}}^{\mathbb{L}}\}$ and consider $\mathbb{B}_{\mathcal{O}}^{\mathbb{L}}$ as a concrete category where $\varphi : B_{\hat{r}} \to B_{\hat{r}'}$ is an embedding if it is a $\mathscr{U}_{\mathcal{O}}^{\mathbb{L}}$-morphism and $\varphi"r \subseteq r'$.

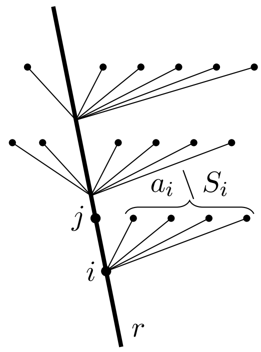

Figure 4.2: A structured pseudo-tree $B \in \mathbb{B}_{\mathcal{O}}^{\mathbb{L}}$.

We note that we would have $l_i^B(j) = s_i$ in Figure 4.2.

Definition 4.2.5. Suppose Q is a quasi-order, $B_{\hat{r}} \in \mathbb{B}_{\mathcal{O}}^{\mathbb{L}}$ and for each $u \in B_{\hat{r}}$ we have $\hat{T}_u \in \mathscr{E}_{\mathcal{O}}(Q)$, such that T_u is a singleton tree whenever $u \in r$.[2]

Then we define the *structured tree sum*

$$\sum_{u \in B_{\hat{r}}} \hat{T}_u = \hat{T} \in \mathscr{E}_{\mathcal{O}}$$

by letting

$$T = \sum_{u \in B_{\hat{r}}} T_u$$

and assigning labels so that for $v \in T_u$ if $u \notin r$ then $l_v^T = l_v^{T_u}$ and if $u \in r$ for $w \in \updownarrow t$ with $w \in T_u$ we have $l_v^T(w) = l_v^{B_{\hat{r}}}(u)$. For each $t \in T$, if $t \in T_u$ then let $\hat{T}(t) = \hat{T}_u(t)$.

Definition 4.2.6. Given $\hat{T} \in \mathscr{T}_{\mathcal{O}}^{\mathbb{L}}(Q)$ we call $\hat{C} \in \mathscr{R}_{\mathbb{B}_{\mathcal{O}}^{\mathbb{L}}}(\mathscr{T}_{\mathcal{O}}^{\mathbb{L}}(Q))$ a *construction tree for* $\hat{T}$ iff:

1. $\hat{C}(\text{root}(C)) = \hat{T}$.

2. If $\text{range}(l_t^C) = B_{\hat{r}}$ and for $u \in B_{\hat{r}}$ if $t_u \in \text{succ}(t)$ is such that $l_t^C(t_u) = u$, then

$$\hat{C}(t) = \sum_{u \in B_{\hat{r}}} \hat{C}(t_u).$$

[2]This condition ensures that the resultant $\hat{T}$ is a pseudo-tree.

3. Furthermore, if $u \in B_{\hat{r}}$ is such that $u \in r$ then $\mathrm{dom}(\hat{C}(t_u))$ is a singleton tree.

4. If t is a leaf of C then $\mathrm{dom}(\hat{C}(t))$ is either a singleton or is empty.

5. For each $x \in T$, there is some unique leaf $t_x \in C$ such that $\mathrm{dom}(\hat{C}(t_x)) = \{x\}$.

Lemma 4.2.7. *Given $\hat{T} \in \mathscr{E}^{\mathbb{L}}_{\mathcal{O}}(Q)$, and some maximal chain $\zeta \subseteq T$, there is some $\hat{r} \in \mathbb{F}^{\mathbb{L}}_{\mathcal{O}}$ and some $\hat{T}_u \in \mathscr{E}^{\mathbb{L}}_{\mathcal{O}}(Q)$ for each $u \in B_{\hat{r}}$, with $T_u \subseteq T$ such that for all $i \in r$ then T_i is a singleton tree, $\bigcup_{u \in r} T_u = \zeta$ and*

$$\hat{T} = \sum_{u \in B_{\hat{r}}} \hat{T}_u.$$

Proof. Suppose we have $\hat{T} \in \mathscr{E}^{\mathbb{L}}_{\mathcal{O}}(Q)$ and some maximal chain $\zeta \subseteq T$. Let $r = \zeta$ so that $\mathrm{ot}(r) \in \overline{\mathbb{L}}$ and for $i \in r$, let $a_i = \mathrm{range}(l^T_i) \in \mathcal{O}$. Now if $i \neq \max r$ then pick $j \in r$ with $j > i$ and let $s_i = l^T_i(j)$. If $i = \max r$ then pick some $s_i \in a_i$. Now let $\hat{a}_i \in \mathcal{O}(\mathrm{A}_2)$ be such that $\hat{a}_i(x) = 1$ iff $x = s_i$. Now let $\hat{r} = \langle \hat{a}_i : i \in r \rangle \in \mathbb{F}^{\mathbb{L}}_{\mathcal{O}}$.

For $u \in B_{\hat{r}} \setminus r$ we have $u \in a_i$ for some $i \in r$, so let $T_u = {}^{u}{\updownarrow} i \subseteq T$ and $\hat{T}_u = \hat{T} \restriction T_u$. When $u \in r$ then let $T_u = \{u\} \subseteq \zeta$, and again let $\hat{T}_u = \hat{T} \restriction T_u$.

By construction then, $\bigcup_{u \in r} T_u = \zeta$ and if $u \in r$ then T_u is a singleton tree. We also have that $\hat{T} = \sum_{u \in B_{\hat{r}}} \hat{T}_u$ as required. □

Lemma 4.2.8. *For every $\hat{T} \in \mathscr{T}^{\mathbb{L}}_{\mathcal{O}}(Q)$ there is a construction tree for $\hat{T}$.*

Proof. Suppose $\hat{T} \in \mathscr{T}^{\mathbb{L}}_{\mathcal{O}}(Q)$ is such that T is the limit of the limiting sequence $(T_n)_{n \in \omega}$. Let C_0 be the singleton tree and $\hat{C}_0 : C_0 \to \{\hat{T}\}$.

Suppose that for $n \in \omega$ we have defined some $\hat{C}_n \in \mathscr{R}_{\mathbb{B}^{\mathbb{L}}_{\mathcal{O}}}(\mathscr{T}^{\mathbb{L}}_{\mathcal{O}}(Q))$ satisfying properties (1) to (3) of Definition 4.2.6.

Let t be a leaf of C_n, such that $\mathrm{dom}(\hat{C}_n(t)) \subseteq T$ is neither empty nor a singleton, and let $m(t) \in \omega$ be least such that $\mathrm{dom}(\hat{C}_n(t)) \cap T_{m(t)} \neq \emptyset$. If $\mathrm{dom}(\hat{C}_n(t)) \cap T_{m(t)}$ is a singleton then let $\zeta'_t = \mathrm{dom}(\hat{C}_n(t)) \cap T_{m(t)}$, otherwise $\mathrm{rank}_{\mathscr{U}}(\mathrm{dom}(\hat{C}_n(t)) \cap T_{m(t)}) > 0$ so there is some chain ζ'_t for which $\mathrm{dom}(\hat{C}_n(t)) \cap T_{m(t)}$ is a ζ'_t-tree-sum of pseudo-trees of lower scattered rank. In either case let ζ_t be a maximal chain of $\mathrm{dom}(\hat{C}_n(t))$ that has ζ'_t as an initial segment.

Now using Lemma 4.2.7 with $\mathrm{dom}(\hat{C}_n(t))$ and ζ_t there is some $B^t = B_{\hat{r}} \in \mathbb{B}^{\mathbb{L}}_{\mathscr{O}}$ and for $u \in B_{\hat{r}}$ there are $\hat{T}_u$ such that

$$\hat{C}_n(t) = \sum_{u \in B_{\hat{r}}} \hat{T}_u.$$

We also have that if $\hat{r} = \langle \hat{a}_i : i \in r \rangle$ and $u \in B_{\hat{r}} \cap a_j$ with $j \in r$ and $T_j = \{x\} \subseteq \zeta'_t$ then $T_u = {}^u\!\!\upharpoonleft\!\! x$ and therefore $\mathrm{rank}_{\mathscr{U}}(T_u \cap T_{m(t)}) < \mathrm{rank}_{\mathscr{U}}(\mathrm{dom}(\hat{C}_n(t)) \cap T_{m(t)})$.

Now let

$$C_{n+1} = C_n \cup \bigcup \{B^t : t \text{ is a leaf of } C_n, |\mathrm{dom}(\hat{C}_n(t))| > 1\}.$$

For $t \in C_n$ not a leaf of C_n we let $l_t^{C_{n+1}} \restriction (\uparrow\! t \cap C_n) = l_t^{C_n}$ and for $x \in \uparrow\! t \cap C_{n+1} \setminus C_n$ we have $x \in B^u$ for some $u < x$, so let $l_t^{C_{n+1}}(x) = l_t^{C_n}(u)$. If t is a leaf of C_n and $|\mathrm{dom}(\hat{C}_n(t))| > 1$ then $\uparrow\! t = B^t$ so let $l_t^{C_{n+1}}(u) = u$. Now let $\hat{C}_{n+1} \restriction C_n = \hat{C}_n$ and for $u \in B^t$, let $\hat{C}_{n+1}(u) = \hat{T}_u$. Thus $\hat{C}_{n+1}$ satisfies properties (1) to (3) of Definition 4.2.6.

Finally let $C = \bigcup_{n \in \omega} C_n$ and for $t \in C$ and $u \in \uparrow\! t$ we let $l_t^C(u) = l_t^{C_n}(u)$ for n large enough so that $l_t^{C_n}(u)$ is defined. We also let $\hat{C} \restriction C_n = \hat{C}_n$ for every $n \in \omega$ and this is well-defined.

Therefore $\hat{C}$ satisfies properties (1) to (3) of Definition 4.2.6, with each case witnessed by C_n for some $n \in \omega$. We also have that $\hat{C}$ satisfies (4), since if $|\mathrm{dom}(\hat{C}(t))| > 1$ then we defined a successor of t in C.

Now let $x \in T$ and let $m \in \omega$ be such that $x \in T_m$. Consider $U = \{t \in C : \hat{C}(t) \cap T_m \neq \emptyset\}$, then this is a well-founded tree since if $w, z \in U$ with $w < z$ then either $m(w) < m(z) \leqslant m$ or $m(w) = m(z)$ and

$$\mathrm{rank}_{\mathscr{U}}(\mathrm{dom}(\hat{C}(w)) \cap T_{m(w)}) > \mathrm{rank}_{\mathscr{U}}(\mathrm{dom}(\hat{C}(z)) \cap T_{m(z)}).$$

So for some $t \in U$ we have $x \in \zeta'_t$. Therefore either $\hat{C}(t) = \{x\}$ or we defined a successor $u \in C$ of t such that $\hat{C}(t) = \{x\}$. If t were not unique then there is some other leaf t' such that $\hat{C}(t') = \{x\}$, but then this contradicts (2) at $t \wedge t'$. So we have that $\hat{C}$ also satisfies (5), and therefore $\hat{C}$ is a construction tree for $\hat{T}$. $\square$

Definition 4.2.9. Given $\hat{T} \in \mathscr{T}^{\mathbb{L}}_{\mathscr{O}}(Q)$ we call $\hat{D} \in \mathscr{R}_{\mathbb{B}^{\mathbb{L}}_{\mathscr{O}}}(Q \cup \{-\infty\})$ a *decomposition tree for* $\hat{T}$ iff there is some construction tree $\hat{C} \in \mathscr{R}_{\mathbb{B}^{\mathbb{L}}_{\mathscr{O}}}(\mathscr{T}^{\mathbb{L}}_{\mathscr{O}}(Q))$ for $\hat{T}$, such that $C = D$ and for $t \in D$ we have $\hat{D}(t) = -\infty$ whenever $\mathrm{dom}(\hat{C}(t))$ is not a singleton and if $\hat{C}(t) = \{x\}$ then

$$\hat{D}(t) = \hat{C}(t)(x) \in Q.$$

Proposition 4.2.10. *For every $\hat{T} \in \mathscr{T}_{\mathcal{O}}^{\mathbb{L}}(Q)$, there is a decomposition tree for $\hat{T}$.*

Proof. Use Lemma 4.2.8 to find a construction tree for $\hat{T}$, then define as in Definition 4.2.9. □

Theorem 4.2.11. *If $\hat{D}_T$ and $\hat{D}_U$ are decomposition trees for $\hat{T}, \hat{U} \in \mathscr{T}_{\mathcal{O}}^{\mathbb{L}}(Q)$ respectively, and $\hat{D}_T \leqslant \hat{D}_U$, then $\hat{T} \leqslant \hat{U}$.*

Proof. Suppose $\hat{T}$, $\hat{U}$, $\hat{D}_T$ and $\hat{D}_U$ are as described. Let $\hat{C}_T$ and $\hat{C}_U$ be corresponding construction trees used to define $\hat{D}_T$ and $\hat{D}_U$ respectively. Let φ be an embedding witnessing $\hat{D}_T \leqslant \hat{D}_U$.

If $x \in T$ then there is a unique leaf $t_x \in C_T$ such that $\mathrm{dom}(\hat{C}_T(t_x)) = \{x\}$. In particular this means that $\hat{D}_T(t_x) \neq -\infty$ and therefore $\hat{D}_U \circ \varphi(t_x) \neq -\infty$ since φ is an embedding. Therefore, $\mathrm{dom}(\hat{C}_U \circ \varphi(t_x)) = \{s_x\}$ for some $s_x \in U$. We now define $\psi : T \to U$ by letting $\psi(x) = s_x$, and we claim that ψ witnesses $\hat{T} \leqslant \hat{U}$.

Suppose that $x, y \in T$. Then $x <_T y$ iff

$$l^{D_T}_{t_x \wedge t_y}(t_x) < l^{D_T}_{t_x \wedge t_y}(t_y)$$

iff

$$l^{D_U}_{\varphi(t_x \wedge t_y)} \circ \varphi(t_x) < l^{D_U}_{\varphi(t_x \wedge t_y)} \circ \varphi(t_y)$$

iff

$$l^{D_U}_{\varphi(t_x) \wedge \varphi(t_y)} \circ \varphi(t_x) < l^{D_U}_{\varphi(t_x) \wedge \varphi(t_y)} \circ \varphi(t_y)$$

iff $\psi(x) <_U \psi(y)$. So ψ is a partial order embedding.

Consider $t_x \wedge t_y \in C_T$ and let $B_{\hat{r}} = \mathrm{range}(l^{C_T}_{t_x \wedge t_y})$ with $\hat{r} = \langle \hat{a}_i : i \in r \rangle$. We have

$$l^{C_T}_{t_x \wedge t_y}(t_x) \in \{i\} \cup a_i$$

and

$$l^{C_T}_{t_x \wedge t_y}(t_y) \in \{j\} \cup a_j$$

for some $i, j \in r$. Swapping the names of x and y if necessary, suppose without loss of generality that $i \leqslant j$. Since $\hat{C}_T$ is a construction tree, it satisfies property (2) of Definition 4.2.6. We also have that if $w, z \in B_{\hat{r}}$ and $w \in \{i\} \cup a_i$, $z \in \{j\} \cup b_j$ with $i \leqslant j$ then $w \wedge z = i$. Therefore

$$l^{C_T}_{t_x \wedge t_y}(t_{x \wedge y}) = i = l^{C_T}_{t_x \wedge t_y}(t_x) \wedge l^{C_T}_{t_x \wedge t_y}(t_y).$$

Let $B_{\hat{r}'} = \mathrm{range}(l^{C_U}_{\varphi(t_x) \wedge \varphi(t_y)})$ with $\hat{r}' = \langle \hat{a}'_i : i \in r' \rangle$. Suppose that

$$l^{C_U}_{\varphi(t_x) \wedge \varphi(t_y)} \circ \varphi(t_x) \in \{i'\} \cup a'_{i'}$$

and

$$l^{C_U}_{\varphi(t_x) \wedge \varphi(t_y)} \circ \varphi(t_y) \in \{j'\} \cup a'_{j'}$$

for $i', j' \in r'$. Then since φ induces an embedding $\mu : B_{\hat{r}} \to B_{\hat{r}'}$ we have that $i' \leqslant j'$, and thus if $v \in C_U$ is such that $\mathrm{dom}(\hat{C}_U(v)) = \{s_{\psi(x) \wedge \psi(y)}\}$ then similarly we have

$$l^{C_U}_{\varphi(t_x) \wedge \varphi(t_y)}(v) = i'.$$

Now μ is a structured pseudo-tree embedding and therefore for $w, z \in B_{\hat{r}}$ we have $\mu(w \wedge z) = \mu(w) \wedge \mu(z)$. Thus

$$\mu(i) = \mu(l^{C_T}_{t_x \wedge t_y}(t_x) \wedge l^{C_T}_{t_x \wedge t_y}(t_y)) = \mu(l^{C_T}_{t_x \wedge t_y}(t_x)) \wedge \mu(l^{C_T}_{t_x \wedge t_y}(t_y))$$

$$= \left(l^{C_U}_{\varphi(t_x) \wedge \varphi(t_y)} \circ \varphi(t_x)\right) \wedge \left(l^{C_U}_{\varphi(t_x) \wedge \varphi(t_y)} \circ \varphi(t_y)\right) = i'.$$

Since $i' \in r'$ it must be that v is a successor of $\varphi(t_x) \wedge \varphi(t_y)$ and furthermore

$$\{v\} = {}^{i'}\!\!\upharpoonleft \varphi(t_x) \wedge \varphi(t_y).$$

So since μ is induced by φ and $t_{x \wedge y} \in {}^{i}\!\!\upharpoonleft t_x \wedge t_y$, we have that $\varphi(t_{x \wedge y}) \in {}^{\mu(i)}\!\!\upharpoonleft \varphi(t_x) \wedge \varphi(t_y)$, i.e. $\varphi(t_{x \wedge y}) = v$. But this means that $\psi(x \wedge y) = \psi(x) \wedge \psi(y)$, as required, and thus ψ is a pseudo-tree embedding.

We will now show that ψ induces embeddings of the labels, so that ψ is a structured tree embedding. Let $x \in T$ we want to show that the embedding $\theta : \mathrm{range}(l^T_x) \to \mathrm{range}(l^U_{\psi(x)})$

induced by ψ is an embedding. So let $\theta : \mathrm{range}(l^T_x) \to \mathrm{range}(l^U_{\psi(x)})$ be such that for all $t \in \upharpoonleft x$

$$\theta(l^T_x(t)) = l^U_{\psi(x)}(\psi(t)).$$

If T is a singleton, then the result holds vacuously since $\upharpoonleft x = \emptyset$. So suppose that T is not a singleton, then for $x \in T$, $t_x \in C_T$ is not the root of C_T and we can let $t'_x \in C_T$ be the predecessor of t_x. Let $B_{\hat{r}} = \mathrm{range}(l^{C_T}_{t'_x})$ with $\hat{r} = \langle \hat{a}_i : i \in r\rangle$. So since t_x is a leaf, we have either that x is a leaf of T and so the result holds trivially as $\upharpoonleft x = \emptyset$, or that $l^{C_T}_{t'_x}(t_x) = i \in r \subseteq B_{\hat{r}}$. Therefore $\mathrm{range}(l^T_x) = a_i \in \mathcal{O}$, using property (2) of $\hat{C}_T$.

Now φ induces a structured pseudo-tree embedding μ from $B_{\hat{r}}$ to $\mathrm{range}(l^{C_T}_{\varphi(t'_x)}) = B_{\hat{r}'}$. Suppose that $\hat{r}' = \langle \hat{a}'_i : i \in r'\rangle$. So since $l^{C_T}_{t'_x}(t_x) \in r$, we have $l^{C_U}_{\varphi(t'_x)}(\varphi(t_x)) = \mu(i) \in r'$, since μ maps elements of $r \subseteq B_{\hat{r}}$ to $r' \subseteq B_{\hat{r}'}$. This means that $\varphi(t_x)$ is a successor of $\varphi(t'_x)$. Thus $\mathrm{range}(l^U_{\psi(x)}) = a'_{\mu(i)} \in \mathcal{O}$.

Now we have that μ maps $v \in a_i$ to the $w \in a'_{\mu(i)}$ such that if $z \in {}^{v}\!\upharpoonleft t'_x$ then $\varphi(w) \in {}^{w}\!\upharpoonleft \varphi(t'_x)$. Therefore $\theta(v) = w$ so θ is an embedding induced by the structured tree embedding μ.

Finally if $x \in T$ then $\hat{D}_T(t_x) \leqslant \hat{D}_U \circ \varphi(t_x)$ and therefore $\hat{T}(t) \leqslant \hat{U} \circ \psi(t)$. So ψ witnesses $\hat{T} \leqslant \hat{U}$, which gives the theorem. $\square$

4.3 The class $\mathscr{T}^{\mathbb{L}}_{\mathcal{O}}$ of structured pseudo-trees is well-behaved

Definition 4.3.1. If $\hat{B}_{\hat{r}} \in \mathbb{B}^{\mathbb{L}}_{\mathcal{O}}(Q)$, then let $\Gamma(\hat{B}_{\hat{r}}) \in \overline{\mathbb{L}}(\mathcal{O}(\mathrm{A}_2 \times Q))$ be defined as follows. If $\hat{r} = \langle \hat{a}_i : i \in r\rangle \in \mathbb{F}^{\mathbb{L}}_{\mathcal{O}}$, let s_i be the unique element of a_i such that $\hat{a}_i(s_i) = 1$. Then define $\Gamma(\hat{B}_{\hat{r}}) = \langle \hat{b}_i : i \in r\rangle$ where $a_i = b_i$ for each $i \in r$, and

$$\hat{b}_i(u) = \begin{cases} \langle 0, \hat{B}_{\hat{r}}(u)\rangle & : u \in a_i \setminus \{s_i\} \text{ for some } i \in r \\ \langle 1, \hat{B}_{\hat{r}}(i)\rangle & : u = s_i \text{ for some } i \in r \end{cases}.$$

Lemma 4.3.2. *If* $\Gamma(\hat{B}_{\hat{r}}) \leqslant \Gamma(\hat{B}_{\hat{r}'})$ *then* $\hat{B}_{\hat{r}} \leqslant \hat{B}_{\hat{r}'}$.

Proof. Let $\varphi : r \to r'$ witness $\Gamma(\hat{B}_{\hat{r}}) \leqslant \Gamma(\hat{B}_{\hat{r}'})$ and $\hat{r} = \langle \hat{a}_i : i \in r\rangle$, $\hat{r}' = \langle \hat{a}'_i : i \in r'\rangle$. For $i \in r$ let s_i be the unique element of a_i such that $\hat{a}_i(s_i) = 1$ and for $j \in r'$ let s'_j

be the unique element of a'_j such that $\hat{a}'_j(s'_j) = 1$. We also let $\Gamma(\hat{B}_{\hat{r}}) = \langle \hat{b}_i : i \in r \rangle$ and $\Gamma(\hat{B}_{\hat{r}'}) = \langle \hat{b}'_i : i \in r \rangle$.

Since φ witnesses $\Gamma(\hat{B}_{\hat{r}}) \leqslant \Gamma(\hat{B}_{\hat{r}'})$, for each $i \in r$ there is an embedding $\varphi_i : b_i \to b'_{\varphi(i)}$ witnessing $\hat{b}_i \leqslant \hat{b}'_{\varphi(i)}$. If $x \in B_{\hat{r}}$, then either $x \in r$ or $x \in a_i$ for some $i \in r$. If $x \in r$ then let $\psi(x) = \varphi(x) \in r' \subseteq B_{\hat{r}'}$. If $x \in a_i$ then let $\psi(x) = \varphi_i(x)$. We claim that ψ witnesses $\hat{B}_{\hat{r}} \leqslant \hat{B}_{\hat{r}'}$.

If $x \leqslant y \in B_{\hat{r}}$ then $x \in r$ and $y \in \{j\} \cup a_j$ for some $j \in r$ with $j \geqslant x$. Thus $\psi(y) \in \{\varphi(j)\} \cup a'_{\varphi(j)}$ so that $\psi(x) \leqslant \psi(y)$ as required. If $x \perp y \in B_{\hat{r}}$ then without loss of generality $x \in a_i$ for some $i \in r$ and $y \in \{j\} \cup a_j$ for some $j \in r$. Thus $\psi(x) \in a'_{\varphi(i)}$ and $\psi(y) \in \{\varphi(j)\} \cup a'_{\varphi(j)}$ so either $x \perp y$ or $i = j$. But if $i = j$ then $x, y \in a_i$ which means $\psi(x), \psi(y) \in a'_{\varphi(i)}$ and $x \neq y$. So since every $\mathcal{O}$-morphism is injective, we have $\psi(x) = \varphi_i(x) \neq \varphi_i(y) = \psi(y)$ and thus $\psi(x) \perp \psi(y)$.

Now for each $i \in r$ we have $\mathrm{range}(l_i^{B_{\hat{r}}}) = a_i$ and $\mathrm{range}(l_{\psi(i)}^{B_{\hat{r}}}) = a'_{\varphi(i)}$. Now φ_i witnesses $\hat{b}_i \leqslant \hat{b}'_{\varphi(i)}$ and thus since the first components of $\hat{b}_i(u)$ and $\hat{b}'_{\varphi(i)}(\varphi_i(u))$ must be equal, we have $\varphi_i(s_i) = s'_{\varphi(i)}$. Therefore φ_i is precisely the embedding induced by ψ, which is an $\mathcal{O}$-morphism as required.

We also have for $u \in B_{\hat{r}} \cap a_i$ that φ_i witnesses $\varphi_i : b_i \to b'_{\varphi(i)}$ so by comparing the second components we find $\hat{B}_{\hat{r}}(u) \leqslant \hat{B}_{\hat{r}'}(\varphi_i(u)) = \hat{B}_{\hat{r}'}(\psi(u))$.

Finally let $x, y \in B_{\hat{r}}$ be such that $x \in \{i\} \cup a_i$ and $y \in \{j\} \cup a_j$. Suppose without loss of generality that $i \leqslant j$. Then $x \wedge y = i$ and we have $\psi(x) \in \{\varphi(i)\} \cup a_{\varphi(i)}$ and $\psi(y) \in \{\varphi(j)\} \cup a_{\varphi(j)}$ with $\varphi(i) \leqslant \varphi(j)$, so that $\psi(x) \wedge \psi(y) = \psi(i) = \varphi(i) = \varphi(x \wedge y)$. Therefore ψ witnesses $\hat{B}_{\hat{r}} \leqslant \hat{B}_{\hat{r}'}$. $\square$

Theorem 4.3.3. *If $\mathbb{L}$ and $\mathcal{O}$ are well-behaved, then $\mathbb{B}_{\mathcal{O}}^{\mathbb{L}}$ is well-behaved.*

Proof. Suppose there is a bad $\mathbb{B}_{\mathcal{O}}^{\mathbb{L}}(Q)$-array f. For $X \in \mathrm{dom}(f)$ let $g(X) = \Gamma(f(X))$. So by Lemma 4.3.2 g is a bad $\overline{\mathbb{L}}(\mathcal{O}(\mathrm{A}_2 \times Q))$-array. So by Corollary 3.1.9 there is a witnessing bad $\mathcal{O}(\mathrm{A}_2 \times Q)$-array. Since $\mathcal{O}$ is well-behaved there is a witnessing bad $\mathrm{A}_2 \times Q$-array, thus since A_2 is bqo and by Theorem 2.1.10 there is a witnessing bad Q-array. Now by definition of Γ, every second component of $g(X)(i)(j)$ is equal to some $f(X)(y)$, therefore the bad Q-array that we found is witnessing for f. $\square$

Theorem 4.3.4. *Let $\mathbb{L}$ be a class of linear orders closed under taking subsets with* $\mathrm{On} \subseteq \mathbb{L}$ *and let $\mathcal{O}$ be a concrete category with injective morphisms. Then if $\mathbb{L}$ and $\mathcal{O}$ are well-behaved, then $\mathscr{T}_{\mathcal{O}}^{\mathbb{L}}$ is well-behaved.*[3]

Proof. Suppose there is a bad $\mathscr{T}_{\mathcal{O}}^{\mathbb{L}}(Q)$-array f. For $X \in \mathrm{dom}(f)$ let $g(X)$ be a decomposition tree for $f(X)$, which exists by Proposition 4.2.10. Thus by Theorem 4.2.11 g is a bad $\mathscr{R}_{\mathbb{B}_{\mathcal{O}}^{\mathbb{L}}}(Q \cup \{-\infty\})$-array. By Theorem 4.3.3, we have that $\mathbb{B}_{\mathcal{O}}^{\mathbb{L}}$ is well-behaved, and therefore by Theorem 2.4.7, we have $\mathscr{R}_{\mathbb{B}_{\mathcal{O}}^{\mathbb{L}}}$ is well-behaved. So from g we have a witnessing bad $Q \cup \{-\infty\}$-array, and using Theorem 2.1.6 we can restrict to find a bad Q-array h. Now for every $x \in \mathrm{dom}(g(X))$ with $g(X)(x) \in Q$, there is some $y \in f(X)$ such that $f(X)(y) = g(X)(x)$. Therefore h is witnessing for f, and thus any bad $\mathscr{T}_{\mathcal{O}}^{\mathbb{L}}(Q)$-array has a witnessing bad Q-array, i.e. $\mathscr{T}_{\mathcal{O}}^{\mathbb{L}}$ is well-behaved. $\square$

Corollary 4.3.5. *If $\mathcal{O}$ is well-behaved and has injective morphisms, then $\mathscr{T}_{\mathcal{O}}^{\mathscr{M}}$ is well-behaved.*

Proof. By theorems 4.3.4 and 2.3.7. $\square$

[3] This result was obtained independently by Christian Delhommé in as yet unpublished work [9]. The author thanks him for his private communication.

Chapter 5

Better-quasi-ordering graphs

Another area in which wqo and bqo theory has been extensively studied is graphs. The most famous theorem in this area is Robertson and Seymour's well-known Graph Minor Theorem, which states that the set of finite graphs is well-quasi-ordered under the minor relation [48]. Thomas considered versions of this theorem for infinite graphs [54].

Another quasi-order on graphs that has been considered is the induced subgraph relation. Damaschke [8] proved that the set of finite P_4-free[1] graphs ordered by induced subgraph is wqo,[2] which was later extended by Thomassé who proved in [55] that the set of countable P_4-free graphs preserves bqo under this ordering. Furthermore a result of Nešetřil and Ossona de Mendez is that any class of finite graphs with bounded tree depth, coloured by a wqo Q is wqo under the induced subgraph quasi-order (see Lemma 6.13 in [41]). Indeed many other classes of (finite) graphs with this ordering have been studied and shown either to be wqo or not (see [24, 11, 43]). For example Nicholas Korpelainen and Vadim V. Lozin proved that the P_7-free bipartite graphs are not wqo and the P_6-free bipartite graphs are wqo under the induced subgraph ordering [24].

In this chapter we will prove an analogue of Theorem 3.4.12 for graphs (Theorem 5.2.6). In particular, a class $\mathscr{G}$ of countable graphs (quasi-ordered by the induced sub-

[1] Also known as N-free or series-parallel.

[2] For $k \in \omega$ the graph P_k is a path of length k. I.e. $V(P_k) = \{0, ..., k-1\}$ and for $x, y \in V(P_k)$, $x \sim y$ iff $x + 1 = y$ or $y + 1 = x$.

graph relation) is well-behaved whenever the set of *indecomposable* induced subgraphs of members of $\mathscr{G}$ is well-behaved. This generalises Thomassé's theorem on countable P_4-free graphs from [55] and shows that many new classes of (transfinite) graphs are wqo under the induced subgraph relation.

5.1 Scattered Graphs

Definition 5.1.1. A *graph* G is a pair $G = \langle V(G), E(G)\rangle$ where $V(G)$ is a set of *vertices* and $E(G) \subseteq V(G) \times V(G)$ is such that $\langle x, y\rangle \in E(G) \to \langle y, x\rangle \in E(G)$ i.e. $E(G)$ is a symmetric binary relation on $V(G)$.

For $x, y \in V(G)$ we write $x \sim_G y$ iff $\langle x, y\rangle \in E(G)$ and $x \sim y$ if it is clear that x and y belong to $V(G)$. We call G a *singleton graph* if $|V(G)| = 1$.

We consider the class of graphs as a concrete category. A graph G has underlying set $U_G = V(G)$ and if G and H are graphs then $\varphi : V(G) \to V(H)$ is an embedding iff φ is injective and for all $x, y \in V(G)$ we have $x \sim_G y$ iff $\varphi(x) \sim_H \varphi(y)$.

Definition 5.1.2. We call H a *subgraph* of G iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We call H an *induced subgraph* of G and write $H \preceq G$ if $V(H) \subseteq V(G)$ and $\langle x, y\rangle \in E(H)$ iff $x, y \in V(H)$ and $\langle x, y\rangle \in E(G)$. We write $H \prec G$ if $V(H) \subset V(G)$ and $H \preceq G$.

If $S \subseteq V(G)$ then we define $G[S] = \langle S, E(G) \cap (S \times S)\rangle$. Thus $G[S]$ is the induced subgraph of G whose set of vertices is precisely S.

Remark 5.1.3. The order $\leqslant$ on the concrete category of graphs is essentially just the induced subgraph relation $\preceq$. Indeed, if $\varphi : G \to H$ is an embedding, then $H[\mathrm{range}(\varphi)]$ is isomorphic to G and is an induced subgraph of H.

Definition 5.1.4. If G is a graph then we call $I \preceq G$ an *interval* of G iff for all $x \in V(G) \setminus V(I)$ and all $y, y' \in V(I)$ we have $x \sim_G y$ iff $x \sim_G y'$.

Definition 5.1.5. We call a graph G *indecomposable* iff the only intervals of G are G itself, singleton graphs and $\emptyset$.

Definition 5.1.6. We define $\mathsf{G}_0 = \langle 2^{<\omega}, E(\mathsf{G}_0)\rangle$ where for $s, t \in 2^{<\omega}$ we have $\langle s, t\rangle \in E(\mathsf{G}_0)$ iff $s \sqsubseteq t$ or $t \sqsubseteq s$.

We define $\mathsf{G}_1 = \langle 2^{<\omega}, E(\mathsf{G}_1)\rangle$ where for $s, t \in 2^{<\omega}$ we have $\langle s, t\rangle \in E(\mathsf{G}_1)$ iff $s \not\sqsubseteq t$ and $t \not\sqsubseteq s$.

Definition 5.1.7. Suppose that $\mathbb{L}$ is a class of linear orders and $\mathbb{G}$ is a class of graphs. Let $\mathscr{H}_{\mathbb{G}}^{\mathbb{L}}$ be the class of graphs G such that:

1. Every indecomposable induced subgraph of G is isomorphic to a member of $\mathbb{G}$.

2. For every $x \in V(G)$ there is a maximal chain of intervals of G under $\succeq$ that contains $G[\{x\}]$ and has order type in $\overline{\mathbb{L}}$.

3. $\mathsf{G}_0, \mathsf{G}_1 \not\leq G$.

We also let $\mathscr{D}_{\mathbb{G}}^{\mathbb{L}}$ be the class of graphs satisfying (1) and (2).

For the rest of this chapter we will always let $\mathbb{L}$ be a class of linear orders that is closed under subsets with $\mathrm{On} \subseteq \mathbb{L}$ and $\mathbb{G}$ be a class of graphs which do not have G_0 or G_1 as induced subgraphs, which is closed under indecomposable induced subgraphs.

Definition 5.1.8. Given a collection of graphs G_i for $i \in X$ whose vertices are pairwise disjoint,[3] we define

$$\bigcup_{i\in X} G_i = \langle \bigcup_{i\in X} V(G_i), \bigcup_{i\in X} E(G_i)\rangle.$$

Definition 5.1.9. Given a graph H and graphs J_x for each $x \in V(H)$ whose vertices are pairwise disjoint, we define the *graph sum*

$$\sum_{x\in H} J_x = \langle \bigcup_{x\in V(H)} V(J_x), E\rangle$$

where for $w \in J_x$ and $z \in J_y$ we have $\langle w, z\rangle \in E$ iff $x = y$ and $w \sim_{J_x} z$ or $x \neq y$ and $x \sim_H y$.

[3]That is, $V(G_i) \cap V(G_j) = \emptyset$ for $i, j \in X$, $i \neq j$.

Lemma 5.1.10. *Let $C = \langle I_i : i \in r\rangle$ be a maximal chain under $\succeq$ of intervals of a graph G such that C contains a singleton graph. Then for each $i \in r$ there is an indecomposable graph $H_i \preceq G$ and for each $x \in V(H_i)$ there is some graph $J_i^x \preceq I_i$ so that $V(I_i) = \sum_{x\in H_i} J_i^x$. Furthermore, for each $i \in r$ there is some $x \in H_i$ such that $J_i^x = \bigcup_{j>i} I_j$.*

Proof. We proceed similarly to Lemma 3.3.16. $C = \langle I_i : i \in r\rangle$ be a maximal chain of intervals of a graph G under $\succeq$ with $G[\{x_0\}] \in C$ for some $x_0 \in V(G)$. For $i \in r$ let

$$P_i = I_i\left[\{x_0\} \cup \left(V(I_i) \setminus \bigcup_{j>i} V(I_j)\right)\right].$$

Let $\mathcal{J}$ be the set of maximal chains of intervals of P_i that do not contain x_0, and let

$$\mathcal{Z}_i = \{\bigcup_{J\in K} J : K \in \mathcal{J}\} = \{Z_\beta^i : \beta \in \gamma_i\},$$

where $\gamma_i = |\mathcal{Z}_i|$. For each $\beta \in \gamma_i$ pick some $z_\beta^i \in V(Z_\beta^i)$ and let

$$H_i = I_i[\{x_0\} \cup \{z_\beta^i : \beta \in \gamma_i\}].$$

Now for $x \in H_i$, if $x = x_0$ let $J_i^x = \bigcup_{j>i} I_j$ and if $x = z_\beta^i$ then let $J_i^x = Z_\beta^i$. Therefore we have for all $i \in r$ and $x \in H_i$ that $J_i^x \preceq I_i$ and since J_i^x is a union of chains of intervals of I_i, we have that each J_i^x is an interval. Thus since $\bigcup_{x\in H} V(J_i^x) = V(I_i)$ we conclude that for every $i \in r$,

$$I_i = \sum_{x\in H_i} J_i^x.$$

It remains only to show that H_i is indecomposable for every $i \in r$. So fix $i \in r$, and we claim that any interval of H_i of size at least 2 contains x_0. If not then there is an interval I of H_i that contains z_β^i and z_δ^i for some distinct $\beta, \delta \in \gamma_i$, with $x_0 \notin I$. But then $Z_\beta^i \cup Z_\delta^i$ is an interval of P_i that does not contain x_0, which contradicts that Z_β^i and Z_δ^i were unions of maximal chains of such intervals.

So if I is a proper interval of H_i with $|I| \geqslant 2$ then $x \in I$ and for some $\beta \in \gamma_i$ we have $z_\beta^i \notin I$. But then

$$\bigcup_{j>i} I_j \prec \bigcup_{z_\delta^i\in I} Z_\delta^i \cup \bigcup_{j>i} I_j \prec I_i,$$

which contradicts that C was maximal. This is a contradiction, so for every $i \in r$ we have that H_i is indecomposable as required. $\square$

Lemma 5.1.11. *If H is an interval of G and G satisfies (2) of Definition 5.1.7, then H also does.*

Proof. Suppose $H \preceq G$ and G satisfies 5.1.7 (2). Given $x_0 \in V(H)$ we want to find a maximal chain of intervals of H that contains $G[\{x_0\}]$ and has order type in $\overline{\mathbb{L}}$. We have $x_0 \in V(H) \subseteq V(G)$ so pick a maximal chain C of intervals of G that contains $G[\{x_0\}]$ and has order type in $\overline{\mathbb{L}}$. Now define

$$C' = \{G[V(I) \cap V(H)] : I \in C\}.$$

We claim that each element of C' is an interval of H. Let $I \in C$, $x \in V(H) \setminus V(I)$ and $y, y' \in V(I) \cap V(H)$. Then $x \in V(G) \setminus V(I)$ and $y, y' \in V(I)$, therefore since I is an interval of G we have $x \sim_G y$ iff $x \sim_G y'$. But H is an induced subgraph of G so we have $x \sim_H y$ iff $x \sim_H y'$, and indeed $G[V(I) \cap V(H)]$ is an interval of H.

If C' were not maximal then there is some non-empty interval J of H and $C_0, C_1 \subseteq C'$ with C_0 and C_1 initial and final segments of C' respectively and $C' = C_0 \cup C_1$, such that $\forall I_0 \in C_0$ and $\forall I_1 \in C_1$ we have $I_1 \prec J \prec I_0$. Consider the final segment $K = \{I \in C : V(I) \cap V(H) \in C_1\}$ of C and let

$$J' = J \cup \bigcup_{I \in K} I.$$

We claim that J' is an interval of G. Let $x \in V(G) \setminus V(J')$ and $y, y' \in V(J')$. Then we have the following.

- If $y, y' \in \bigcup_{I \in K} I$ then clearly $x \sim_G y$ iff $x \sim_G y'$, since $\bigcup_{I \in K} I$ is a union of a chain of intervals of G and hence is an interval of G itself.

- If $x \in V(G) \setminus V(H)$ and $y, y' \in J'$ then since H is an interval of G we have $x \sim_G y$ iff $x \sim_G y'$.

- If $x \in V(H) \setminus V(J') = V(H) \setminus V(J)$ and $y, y' \in J$ then since J is an interval of H we have $x \sim_G y$ iff $x \sim_G y'$.

Thus since $x_0 \in J \cap \bigcup_{I\in K} I$, in every case we have $x \sim_G y$ iff $x \sim_G x_0$ iff $x \sim_G y'$, and therefore J' is an interval of G. But then for all $I_1 \in K$ and all $I_0 \in C \setminus K$ we have $I_1 \prec J' \prec I_0$, which contradicts that C was maximal. $\square$

Definition 5.1.12. Let $G \in \mathscr{D}_{\mathbb{G}}^{\mathbb{L}}$, we call $\hat{T} \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}(\mathscr{D}_{\mathbb{G}}^{\mathbb{L}})$ a *decomposition tree for G* iff

1. T has a root t_0 and $\hat{T}(t_0) = G$.
2. For all $t \in T$ we have $\hat{T}(t)$ is an interval of G.
3. If $t, s \in T$ with $t \leqslant s$, then $\hat{T}(t) \succeq \hat{T}(s)$.
4. For all $t \in T$, if $H = \mathrm{range}(l_t^T) \in \mathbb{G}$ and for each $u \in H$ if $t_u \in \mathrm{succ}(t)$ is such that $l_t^T(t_u) = u$, then
$$\hat{T}(t) = \sum_{u\in H} \hat{T}(t_u).$$
5. For every leaf t of T we have $|V(\hat{T}(t))| = 1$.
6. For every $x \in V(G)$ there is some unique leaf $t_x \in T$ such that $\hat{T}(t_x) = G[\{x\}]$.

We note that decompositions of graphs and decomposition trees have been studied in the countable case by Courcelle and Delhommé in [7].

Theorem 5.1.13. *For every $G \in \mathscr{D}_{\mathbb{G}}^{\mathbb{L}}$ there is a decomposition tree $\hat{T}_G \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}$ for G.*

Proof. Given $G \in \mathscr{D}_{\mathbb{G}}^{\mathbb{L}}$ we define $\hat{T}_G \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}(\mathscr{D}_{\mathbb{G}}^{\mathbb{L}})$ as follows. Enumerate the elements of $V(G) = \{x_\alpha : \alpha \in |V(G)|\}$. Let T_0 be the singleton tree and $\hat{T}_0 : T_0 \to \{G\}$. Suppose that for some $\alpha \in \mathrm{On}$ we have defined $\hat{T}_\alpha \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}(\mathscr{D}_{\mathbb{G}}^{\mathbb{L}})$ such that for all $t \in T_\alpha$ we have $\hat{T}(t) \preceq G$.

For each leaf t of T_α such that $|V(\hat{T}_\alpha(t))| > 1$, let γ be least such that $x_\gamma \in V(\hat{T}(t))$. Using (2) pick a maximal chain $C = \langle I_i : i \in r\rangle$ of non-empty intervals of $\hat{T}(t)$ containing $G[\{x_\gamma\}]$ with order type under $\succeq$ in $\overline{\mathbb{L}}$. Then apply Lemma 5.1.10 to find, for each $i \in r$ an indecomposable graph $H_i \preceq G$, and for each $x \in V(H_i)$ some $J_i^x \preceq I_i$ such that
$$V(I_i) = \sum_{x\in H_i} J_i^x.$$

For each $i \in r$ let s_i be the element of $V(H_i)$ such that $J_i^{s_i} = \bigcup_{j>i} I_j$. Now define

$$U_t = \{\varepsilon_t\} \cup \bigsqcup_{i \in r} H_i,$$

and for $x, y \in U_t$ let $x \leqslant y$ iff for some $j \in r$, $x = s_j$ and $y = \varepsilon$ or $y \in H_{j'}$ for some $j' \geqslant j$. For $v = s_i \in U_t$ and $x \in \upharpoonleft v$ we define

$$l_v^{U_t}(x) = \begin{cases} x & : x \in H_i \\ s_i & : \text{otherwise} \end{cases}.$$

Therefore since G satisfies (1) and each $H_i \preceq G$ $(i \in r)$, we have $U_t \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}$. Now define $\hat{U}_t \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}(\mathscr{D}_{\mathbb{G}}^{\mathbb{L}})$ by letting $\hat{U}_t(x) = I_i$ if $x = s_i$ and $\hat{U}_t(x) = J_i^x$ if $x \in H_i \setminus \{s_i\}$, we also let $\hat{U}_t(\varepsilon) = G[\{x_\gamma\}]$.

Now let

$$T_{\alpha+1} = T_\alpha \cup \bigcup \{U_t : t \text{ is a leaf of } T_\alpha, |\hat{T}_\alpha(t)| > 1\}.$$

For $x, y \in T_{\alpha+1}$ we let $x < y$ iff either $x, y \in T_\alpha$ and $x <_{T_\alpha} y$; or there is a leaf t of T_α with $x, y \in U_t$ and $x <_{U_t} y$; or $x \in T_\alpha$, $y \in U_t$ and $x <_{T_\alpha} t$. If $v \in T_\alpha$ we let $l_v^{T_{\alpha+1}} \upharpoonright T_\alpha = l_v^{T_\alpha}$ and for $x \in U_t$ we let $l_v^{T_{\alpha+1}}(x) = l_v^{T_\alpha}(t)$. If $v \in U_t$ then let $l_v^{T_{\alpha+1}} = l_v^{U_t}$. So $T_{\alpha+1} \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}$.

Finally let $\hat{T}_{\alpha+1} \in \mathscr{E}_{\mathbb{G}}^{\mathbb{L}}(\mathscr{D}_{\mathbb{G}}^{\mathbb{L}})$ be such that $\hat{T}_{\alpha+1} \upharpoonright T_\alpha = \hat{T}_\alpha$ and for every leaf t of T_α with $|\hat{T}_\alpha(t)| > 1$, let $\hat{T}_{\alpha+1} \upharpoonright U_t = \hat{U}_t$.

Now suppose we have defined $\hat{T}_\alpha$ for every $\alpha < \lambda$ with λ a limit ordinal such that whenever $\alpha < \beta < \lambda$ we have: $T_\alpha \subseteq T_\beta$, $\hat{T}_\beta \upharpoonright T_\alpha = \hat{T}_\alpha$ and $l_v^{T_\beta} \upharpoonright T_\alpha = l_v^{T_\alpha}$ for all $v \in T_\alpha$. Then let $T_\lambda = \bigcup T_\alpha$; define for $v, x \in T_\alpha$ with $v < x$, $l_v^{T_\lambda}(x) = l_v^{T_\alpha}(x)$; and for all $\alpha < \lambda$ let $\hat{T}_\lambda \upharpoonright T_\alpha = \hat{T}_\alpha$.

Since $V(G)$ is not a proper class, there is some least $\alpha \in \mathrm{On}$ such that $\hat{T}_\alpha = \hat{T}_{\alpha+1}$. In this case let $\hat{T}_G = \hat{T}_\alpha$, which is a decomposition tree for G by construction. $\square$

Lemma 5.1.14. *If $\hat{T}$ is a decomposition tree for G and $2^{<\omega} \leqslant T$ then there is an embedding $\varphi : 2^{<\omega} \to T$ such that either for all $s \in 2^{<\omega}$, we have*

$$l_{\varphi(s)}^T \circ \varphi(s^\frown\langle 0 \rangle) \sim l_{\varphi(s)}^T \circ \varphi(s^\frown\langle 1 \rangle)$$

or for all $s \in 2^{<\omega}$ we have

$$l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \not\sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle).$$

Proof. Let $U \subseteq T$ be the range of the embedding given by $2^{<\omega} \leqslant T$. Let $\hat{U} : U \to 2$ be such that $\hat{U}(\varphi(s)) = 0$ iff $l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \not\sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle)$. Now proceed similarly to Lemma 3.3.29. □

Lemma 5.1.15. *If $\hat{T}$ is a decomposition tree for G and $\varphi : 2^{<\omega} \to T$ is an embedding, then $\forall s \in 2^{<\omega}$, $\exists \tau(s), u_s, v_s \in T$ such that $\varphi(s) \leqslant \tau(s) < u_s, v_s$; $\exists t \in range(\varphi)$ such that $u_s \leqslant t$; and*

$$l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle) \longleftrightarrow l^T_{\tau(s)}(u_t) \not\sim l^T_{\tau(s)}(v_t).$$

Proof. Let $\hat{T}$ and $\varphi : 2^{<\omega} \to T$ be as described. Suppose that the lemma fails, so there is some $s \in 2^{<\omega}$ such that $\forall x, u, v \in T$ with $\varphi(s) \leqslant x < u, v$; either $\uparrow u \cap \mathrm{range}(\varphi) = \emptyset$ and $\uparrow v \cap \mathrm{range}(\varphi) = \emptyset$; or

$$l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle) \longleftrightarrow l^T_x(u) \sim l^T_x(v).$$

Consider the graph

$$H = G\left[\left(\bigcup_{\varphi(s) < k \leqslant \varphi(s^\frown\langle 0\rangle)} V(\hat{T}(k))\right) \cup V(\hat{T} \circ \varphi(s^\frown\langle 1,0\rangle))\right].$$

Then if $y \in V(G) \setminus V(H)$ and $z \in V(H)$, let t_y and t_z be the elements of T such that $\hat{T}(t_y) = G[\{y\}]$ and $\hat{T}(t_z) = G[\{z\}]$. We have that $t_y \wedge t_z < u \leqslant t_z$ for some $u \in \{\varphi(s^\frown\langle 0\rangle), \varphi(s^\frown\langle 1,0\rangle)\}$ (since otherwise $y \in V(H)$). Therefore by our assumption we have

$$l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle) \longleftrightarrow l^T_{t_y \wedge t_z}(t_y) \sim l^T_{t_y \wedge t_z}(t_z).$$

So by definition of $\hat{T}_G$, if $W = \mathrm{range}(l^T_{t_y \wedge t_z})$ then $\hat{T}(t_y \wedge t_z) = \sum_{w \in W} J_w$ for some intervals J_w $(w \in W)$ of G with some distinct $d, e \in W$ such that $y \in V(J_d)$ and $z \in V(J_e)$. But this means that we have

$$l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle) \longleftrightarrow y \sim z.$$

But this does not depend on z and therefore H is an interval of G.

However the chain $\{\hat{T}(t) : t \in T, t \leqslant \varphi(s^\frown\langle 0\rangle)\}$ is a maximal chain of intervals I of G that satisfy $I \preceq \hat{T}(\varphi(s^\frown\langle 0\rangle))$. So there is some $a \in T$ with

$$\hat{T}(a) = G\left[\bigcup_{\varphi(s)<k\leqslant\varphi(s^\frown\langle 0\rangle)} V(\hat{T}(k))\right]$$

and therefore $\{k \in T : a < k < \varphi(s)\} = \emptyset$. However $\hat{T}(a) \prec H \prec \hat{T} \circ \varphi(s)$, which contradicts maximality. $\square$

Lemma 5.1.16. *If $\hat{T}$ is a decomposition tree for G and $2^{<\omega} \leqslant T$, then G fails 5.1.7 (3).*

Proof. Let $\hat{T}$ and suppose $2^{<\omega} \leqslant T$, so use Lemma 5.1.14 to find an embedding $\varphi : 2^{<\omega} \to T$ such that either for all $s \in 2^{<\omega}$, we have

$$l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle)$$

or for all $s \in 2^{<\omega}$ we have

$$l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 0\rangle) \not\sim l^T_{\varphi(s)} \circ \varphi(s^\frown\langle 1\rangle).$$

Suppose that we are in the first case and we aim to prove that $\mathsf{G}_1 \leqslant G$, the second case will follow in precisely the same way with G_0 in place of G_1.

We define $V \subseteq V(G)$ as follows. Let $V_0 = \emptyset$ and $T_n = \{\varphi(\langle\rangle)\}$. For $n \in \omega$, suppose that we have defined $V_n \subseteq V(G)$ and $T_n \subseteq \mathrm{range}(\varphi)$. Apply Lemma 5.1.15 to every leaf $\varphi(s)$ of T_n, to find $\tau(s), u_s, v_s \in T$ and $t_s \in \mathrm{range}(\varphi)$ with $\varphi(s) \leqslant \tau(s) < u_s, v_s$ and $u_s \leqslant t$ such that

$$l^T_{\tau(s)}(u_t) \not\sim l^T_{\tau(s)}(v_t).$$

Then $t_s = \varphi(s')$ for some $s' \in 2^{<\omega}$ with $s' > s$. Let

$$T_{n+1} = T_n \cup \{\varphi(s'^\frown\langle i\rangle) : \varphi(s) \text{ is a leaf of } T_n, i \in \{0,1\}\}.$$

Then for each leaf $\varphi(s)$ of T_n, choose $x_s \in V(\hat{T}(v_s))$ and let

$$V_{n+1} = \{x_s : \varphi(s) \text{ is a leaf of } T_n\}.$$

Let $V = \bigcup_{n\in\omega} V_n$ and let $H = G[V]$. We note also that the tree $T' = \bigcup_{n\in\omega} T_n \subseteq T$ is isomorphic to $2^{<\omega}$, and we let $\mu : 2^{<\omega} \to T'$ be an isomorphism.

We claim that $\psi : 2^{<\omega} \to H$ given by $\psi(s) = x_{\mu(s)}$ witnesses $\mathsf{G}_1 \leqslant H$. For $u, s \in 2^{<\omega}$, we have $\langle s, u\rangle \in E(\mathsf{G}_1)$ iff s and u are $\sqsubseteq$-incomparable. Now u and s are $\sqsubseteq$-incomparable iff $\mu(s)$ and $\mu(u)$ are $\sqsubseteq$-incomparable elements of T' which implies

$$l^T_{\mu(u)\wedge\mu(s)} \circ \mu(u) \sim l^T_{\mu(u)\wedge\mu(s)} \circ \mu(s),$$

and therefore $x_{\mu(u)} \sim_H x_{\mu(s)}$.

Now for each $s \in 2^{<\omega}$ we have $\mu(s) = \varphi(k)$ for some unique $k \in 2^{<\omega}$, we then define $v(s) = v_k$. Suppose $\langle s, u\rangle \notin E(\mathsf{G}_1)$, so without loss of generality $s \sqsubseteq u$, and thus $\mu(s) \leqslant \mu(u)$. Therefore

$$l^T_{\mu(s)} \circ \mu(u) \not\sim l^T_{\mu(s)}(v(s)),$$

which means that

$$x_{\mu(u)} \not\sim x_{\mu(s)}.$$

Therefore ψ is an embedding and $\mathsf{G}_1 \leqslant G$. □

5.2 σ-scattered graphs

Definition 5.2.1. Let $(G_n)_{n\in\omega}$, be a sequence of elements of $\mathscr{H}^{\mathbb{L}}_{\mathbb{G}}$. We call $(G_n)_{n\in\omega}$ a *limiting sequence* iff for every $n \in \omega$ and every $x \in G_n$ there are graphs $G^x_n \in \mathscr{H}^{\mathbb{L}}_{\mathbb{G}}$ such that

$$G_{n+1} = \sum_{x\in V(G_n)} G^x_n.$$

For every limiting sequence $(G_n)_{n\in\omega}$ and every $n \in \omega$ we consider $G_n \preceq G_{n+1}$ by identifying every $x \in V(G_n)$ with some point $x' \in V(G^x_n) \subseteq V(G_{n+1})$.[4]

Definition 5.2.2. Given a limiting sequence $(G_n)_{n\in\omega}$, we define its *limit* to be the graph $G = \bigcup_{n\in\omega} G_n$. We let $\mathscr{G}^{\mathbb{L}}_{\mathbb{G}}$ be the class of limits of limiting sequences in $\mathscr{H}^{\mathbb{L}}_{\mathbb{G}}$.

[4]Thus $V(G_n) \subseteq V(G_{n+1})$ and $G_n = G_{n+1}[V(G_n)]$.

Lemma 5.2.3. *If $G \in \mathscr{G}^{\mathbb{L}}_{\mathbb{G}}$, then the decomposition tree $\hat{T}_G$ given by Theorem 5.1.13 is such that $\hat{T}_G \in \mathscr{T}^{\mathbb{L}}_{\mathbb{G}}(\mathscr{D}^{\mathbb{L}}_{\mathbb{G}})$.*

Proof. Suppose that G is the limit of the limiting sequence $(G_n)_{n\in\omega}$. We aim to find a limiting sequence of trees $(T_n)_{n\in\omega}$ such that $T_G = \bigcup_{n\in\omega} T_n$. For $x \in V(G)$ let t_x be the element of T_G such that $\hat{T}(t_x) = G[\{x\}]$. Now for each $n \in \omega$, let $T_n = \bigcup_{x\in G_n} {\downarrow} t_x \subseteq T_G$. Then we claim that each T_n $(n \in \omega)$ is a member of $\mathscr{U}^{\mathbb{L}}_{\mathbb{G}}$. Firstly $T_n \in \mathscr{E}^{\mathbb{L}}$ since T_n is a subset of the underlying set of $T_G \in \mathscr{E}^{\mathbb{L}}$, and $\mathbb{L}$ closed under taking subsets.

So it remains to show that $2^{<\omega} \not\leqslant T_n$. We have $T_{G_n} \in \mathscr{U}^{\mathbb{L}}_{\mathbb{G}}$ and T_n consists of a copy of the underlying set of T_{G_n} with some added chains below each leaf. Thus if $2^{<\omega}$ were to embed into T_n, there would be an embedding of $2^{<\omega}$ either entirely into the underlying set of T_{G_n} or entirely into some chain; neither of which is possible.

Therefore $(T_n)_{n\in\omega}$ is a limiting sequence of trees and since $G = \bigcup_{n\in\omega} G_n$ we have that the underlying set of $T_G \in \mathscr{E}^{\mathbb{L}}_{\mathbb{G}}$ is precisely $\bigcup_{n\in\omega} T_n \in \mathscr{T}^{\mathbb{L}}$, which gives the lemma. $\square$

Definition 5.2.4. Given $\hat{G} \in \mathscr{D}^{\mathbb{L}}_{\mathbb{G}}(Q)$ we define $\Upsilon(\hat{G}) \in \mathscr{E}^{\mathbb{L}}_{\mathbb{G}}((\mathrm{A}_2 \times Q) \cup \{-\infty\})$ so that

$$\mathrm{dom}(\Upsilon(\hat{G})) = T_G$$

and for $x \in V(G)$ if t_x is the leaf of T_G such that $T_G(t_x) = G[\{x\}]$, then we define $\Upsilon(\hat{G})(t_x) = \langle i, \hat{G}(x)\rangle$ where $i = 0$ iff $\langle x, x\rangle \in E(G)$. If $t \in T_G$ is such that $t \neq t_x$ for any $x \in V(G)$, then let $\Upsilon(\hat{G})(t) = -\infty$.

Theorem 5.2.5. *If $\Upsilon(\hat{G}) \leqslant \Upsilon(\hat{H})$ then $\hat{G} \leqslant \hat{H}$.*

Proof. Suppose $\Upsilon(\hat{G}) \leqslant \Upsilon(\hat{H})$ and let φ be an embedding witnessing this. To simplify notation, let $\hat{T} = \hat{T}_G$. We have for all $x \in V(G)$ that there is some leaf $t_x \in T$ such that $\hat{T}(t_x) = G[\{x\}]$. In this case since φ is an embedding we have $\Upsilon(\hat{H})(\varphi(t_x)) \neq -\infty$ and thus $\Upsilon(\hat{H})(\varphi(t_x))$ is a leaf of $\Upsilon(\hat{H})$ and $\Upsilon(\hat{H})(\varphi(t_x)) = G[\{y_x\}]$ for some $y_x \in V(H)$.

So let $\psi : V(G) \to V(H)$ be such that $\psi(x) = y_x$ for all $x \in V(G)$. We claim that ψ is an embedding. We have that if $x, y \in V(G)$ with $x \neq y$ then $\langle x, y\rangle \in E(G)$ iff

$$l^T_{t_x \wedge t_y}(t_x) \sim l^T_{t_x \wedge t_y}(t_y),$$

iff

$$l^T_{\varphi(t_x)\wedge\varphi(t_y)} \circ \varphi(t_x) \sim l^T_{\varphi(t_x)\wedge\varphi(t_y)} \circ \varphi(t_y),$$

iff $\psi(x) \sim \psi(y)$. Using precisely the same argument we see that, $x \neq y$ iff $\psi(x) \neq \psi(y)$ which shows that ψ is injective.

We have $x \sim x$ iff

$$\Upsilon(\hat{G})(t_x) = \langle 0, \hat{G}(x)\rangle$$

iff

$$\Upsilon(\hat{H}) \circ \varphi(t_x) = \langle 0, \hat{H}(y_x)\rangle$$

iff $\psi(x) \sim \psi(x)$.

Finally let $q_0 \in Q$ be the second component of $\Upsilon(\hat{G})(t_x)$ and $q_1 \in Q$ be the second component of $\Upsilon(\hat{H}) \circ \varphi(t_x)$. Then $q_0 \leqslant q_1$ as witnessed by φ, therefore $\hat{G}(x) = q_0 \leqslant q_1 = \hat{H} \circ \psi(x)$. So we have shown that ψ witnesses $\hat{G} \leqslant \hat{H}$. □

Theorem 5.2.6. *Let $\mathbb{L}$ be a class of linear orders that is closed under subsets with* $\mathrm{On} \subseteq \mathbb{L}$ *and let $\mathbb{G}$ be a class of graphs which do not have G_0 or G_1 as induced subgraphs, which is closed under taking indecomposable induced subgraphs. Then if $\mathbb{L}$ and $\mathbb{G}$ are well-behaved then $\mathscr{G}^{\mathbb{L}}_{\mathbb{G}}$ is well-behaved.*

Proof. Suppose that f is a bad $\mathscr{G}^{\mathbb{L}}_{\mathbb{G}}(Q)$-array. Define $g(X) = \Upsilon(f(X))$ for all $X \in \mathrm{dom}(f)$. Then g is a bad $\mathscr{T}^{\mathbb{L}}_{\mathbb{G}}((\mathrm{A}_2 \times Q) \cup \{-\infty\})$-array by Lemma 5.2.3 and Theorem 5.2.5. Moreover, for any $t \in \mathrm{dom}(g(X))$ such that $g(X)(t) \neq -\infty$, we have that there is some $y \in \mathrm{dom}(f(X))$ such that $f(X)(y)$ is equal to the second component of $g(X)(t)$.

Now since $\mathbb{L}$ and $\mathbb{G}$ are well-behaved, we have that $\mathscr{T}^{\mathbb{L}}_{\mathbb{G}}$ is well-behaved by Theorem 4.3.4. So using theorems 2.1.6 and 2.1.10, we find a bad Q-array that is witnessing for f. □

5.3 Applications

We now apply Theorem 5.2.6 with some specific classes $\mathbb{L}$ and $\mathbb{G}$.

Definition 5.3.1. If $\mathbb{G}$ is a class of graphs, let $\mathscr{K}_{\mathbb{G}}$ be the class of countable graphs G such that every indecomposable induced subgraph of G is isomorphic to a member of $\mathbb{G}$.

Lemma 5.3.2. *If $\mathscr{C} \subseteq \overline{\mathbb{L}}$ then $\mathscr{K}_{\mathbb{G}} \subseteq \mathscr{G}_{\mathbb{G}}^{\mathscr{M}}$.*

Proof. Suppose that $\mathscr{C} \subseteq \overline{\mathbb{L}}$, let $G \in \mathscr{K}_{\mathbb{G}}$ and enumerate the elements of $V(G) = \{x_n : n \in \omega\}$. We will define a limiting sequence of graphs $(G_n)_{n\in\omega}$ whose limit is G.

First let $G_0 = G[\{x_0\}]$ and define $H_0^{x_0} = G$. Now suppose that for some $n \in \omega$ we have defined some induced subgraph G_n of G and for each $x \in V(G_n)$ we have defined H_n^x an induced subgraph of G such that $G = \sum_{x\in G_n} H_n^x$.

Let $m \in \omega$ be least such that $x_m \notin V(G_n)$. Thus there is some $y_n \in V(G_n)$ such that $x_m \in H_n^{y_n}$. Pick a maximal chain $\langle I_i : i \in r\rangle$ of intervals of $H_n^{y_n}$ (under $\succeq$) that contains $G[\{x_m\}]$. This chain must be countable, since otherwise $H_n^{y_n} \preceq G$ is uncountable. Thus r is a countable linear order.

For each $i \in r$ apply Lemma 5.1.10 to find an indecomposable graph $J_i \preceq G$ and for each $x \in V(J_i)$ some $J_i^x \preceq I_i$ such that

$$V(I_i) = \sum_{x\in J_i} J_i^x.$$

Moreover we have that for each $i \in r$ there is some $x \in H_i$ such that $J_i^x = \bigcup_{j>i} I_j$. Now let

$$G_{n+1} = G\left[(V(G_n) \setminus \{y_n\}) \cup \bigcup_{i\in r} V(J_i)\right].$$

If $x \in V(G_n) \setminus \{y_n\}$ then let $H_{n+1}^x = H_n^x$ and if $x \in V(J_i) \setminus \bigcup_{j>i} V(I_j)$ for some $i \in r$, then let $H_n^x = J_i^x$. Thus we have defined H_{n+1}^x for every $x \in V(G_{n+1})$. Now we have that

$$H_n^{y_n} = \sum_{x\in G[\bigcup_{i\in r} J_i]} H_{n+1}^x,$$

therefore

$$G = \sum_{x\in G_{n+1}} H_{n+1}^x$$

which completes the induction.

Now for all $n \in \omega$ and $x \in G_n$, define $G_n^x = G_{n+1}[V(G_{n+1}) \cap V(H_n^x)]$. Thus

$$G_{n+1} = G[V(G_{n+1}) \cap V(G)] = G\left[V(G_{n+1}) \cap V(\sum_{x \in G_n} H_n^x)\right] = \sum_{x \in G_n} G_n^x.$$

Now each G_n is a countable induced subgraph of G and therefore is a member of $\mathscr{D}_{\mathbb{G}}^{\mathbb{L}}$. Thus to see that $(G_n)_{n\in\omega}$ is a limiting sequence it remains to see that G_0 and G_1 are not isomorphic to induced subgraphs of any G_n $(n \in \omega)$.

We show this by induction on $n \in \omega$. Firstly it is clear that $\mathsf{G}_0, \mathsf{G}_1 \not\leqslant G_0$. Suppose that for $n \in \omega$, $\mathsf{G}_0, \mathsf{G}_1 \not\leqslant G_n$ and $B \in \{\mathsf{G}_0, \mathsf{G}_1\}$ is such that $B \leqslant G_{n+1}$. Thus $B \leqslant G[\bigcup_{i\in r} V(J_i)]$ since otherwise because $G[\bigcup_{i\in r} V(J_i)] \preceq H_n^{y_n}$ and $H_n^{y_n}$ is an interval we have $B \leqslant G[V(G_n) \setminus \{y_n\}]$ which contradicts the induction hypothesis.

So let φ be an embedding witnessing $B \leqslant G[\bigcup_{i\in r} V(J_i)]$. Suppose that $B = \mathsf{G}_0$ and the case for G_1 will hold analogously (with $\not\sim$ in place of $\sim$ and $\sim$ in place of $\not\sim$). For $s \in V(B) = 2^{<\omega}$, let $i_s \in r$ be such that $\varphi(s) \in V(J_{i_s})$.

Suppose that $s, t \in V(B)$ and $s \not\sim t$ and also suppose that $i_s < i_t$. Therefore for all $x \in \bigcup_{i>i_s} V(J_i)$ we have $\varphi(s) \not\sim x$. So in particular this means for all $u \in 2^{<\omega}$, $i_{s^\frown u} \leqslant i_s$. But if $i_{s^\frown u} < i_s$ then we have $\varphi(s^\frown u) \sim \varphi(t)$, which cannot be the case since $s \not\sim u$. Therefore $i_{s^\frown u} = i_s$, which means that B embeds into J_{i_s} which is an indecomposable induced subgraph of G and therefore is isomorphic to a member of $\mathbb{G}$, this is a contradiction since we had that G_0 and G_1 do not embed into any member of $\mathbb{G}$. Therefore whenever $s \not\sim t$, it must be that $i_s = i_t$, but then again we have that $B \leqslant J_{i_s}$, which again gives a contradiction.

Therefore $(G_n)_{n\in\omega}$ is a limiting sequence. Its limit is $\bigcup_{n\in\omega} G_n$ and since each G_n $(n \in \omega)$ is an induced subgraph of G we have $\bigcup_{n\in\omega} G_n \subseteq G$. Furthermore for each $n \in \omega$ we have that $x_n \in V(G_n)$. Therefore $G \subseteq \bigcup_{n\in\omega} G_n$, i.e. G is the limit of the limiting sequence $(G_n)_{n\in\omega}$ and therefore $G \in \mathscr{G}_{\mathbb{G}}^{\mathscr{M}}$. $\square$

Theorem 5.3.3. *If $\mathbb{G}$ is well-behaved then $\mathscr{K}_{\mathbb{G}}$ is well-behaved.*[5]

[5]This result was obtained independently by Christian Delhommé in as yet unpublished work [9]. The author thanks him for his private communication.

Proof. By Lemma 5.3.2, Theorem 2.3.7 and Theorem 5.2.6. □

Lemma 5.3.4. *Any finite set of finite graphs is well-behaved.*

Proof. Let $\mathscr{F}$ be a finite set of finite graphs and Q be a quasi-order. Suppose that $f : [\omega]^\omega \to \mathscr{F}(Q)$ is bad. By applying the Galvin and Prikry Theorem 2.1.6 finitely many times, there is some $A \in [\omega]^\omega$ such that for all $X, Y \in [A]^\omega$, we have $f(X)$ and $f(Y)$ have the same underlying graph G. Then again applying Theorem 2.1.6 at most $|G|$ times, we find that for some $B \in [A]^\omega$ and every $x \in V(G)$ we have $f_x : [B]^\omega \to Q$ given by $f_x(X) = f(X)(x)$ is bad (and therefore is a witnessing bad array). This is because otherwise there is some $B \in [A]^\omega$ such that $f \restriction [B]^\omega$ is perfect, which contradicts that f is bad. □

Definition 5.3.5. For $n \in \omega$ let $\mathscr{A}_n$ denote the set of indecomposable graphs with at most n vertices.

We are now ready to give two immediate applications of Theorem 5.2.6. Of course, other classes could be used in place of $\mathbb{G}$ and $\mathbb{L}$ to obtain other well-behaved classes of graphs.

Theorem 5.3.6. *For each $n \in \omega$, $\mathscr{G}^{\mathscr{M}}_{\mathscr{A}_n}$ is well-behaved.*

Proof. $\mathscr{A}_n$ is a finite set of finite graphs so by Lemma 5.3.4, is well-behaved. Furthermore, $\mathscr{M}$ is well-behaved by Theorem 2.3.7, so using Theorem 5.2.6 completes the proof. □

Corollary 5.3.7. *For each $n \in \omega$, $\mathscr{K}_{\mathscr{A}_n}$ is well-behaved.*

Proof. By Lemma 5.3.2 and Theorem 5.3.6. □

Chapter 6

Abstract well and better quasi-orders

In this chapter we try to find out what happens when the Ramsey space of infinite sequences of natural numbers (the *Ellentuck space*) is substituted for another Ramsey space in the definitions of wqo and bqo. Roughly speaking, Ramsey spaces are systems on which infinite dimensional Ramsey theory can be performed. The way in which we make this substitution is as follows.

We first consider the Nash-Williams style definition of bqo: Q is bqo iff there is no *bad* function from a *front* $\mathcal{F}$ to Q. A front here is a subset of $[A]^{<\omega}$ for some $A \in [\omega]^\omega$, such that any two elements of $\mathcal{F}$ are $\sqsubseteq$-incomparable and such that for all $X \in [A]^\omega$, there is some $a \in \mathcal{F}$ with $a \sqsubset X$. For $a, b \in \mathcal{F}$ we write $a \lhd b$ iff there is some $X \in [\omega]^\omega$ such that $a \sqsubset X$ and $b \sqsubset X \setminus \{\min X\}$. A function $f : \mathcal{F} \to Q$ is *bad* iff whenever $a \lhd b$ we have $f(a) \nleq f(b)$.

Now we note that given a Ramsey space $\mathcal{R}$, an analogous notion of a front $\mathcal{F}$ on some $A \in \mathcal{R}$ can be defined (see Definition 6.1.10). Fronts on an abstract Ramsey spaces have already been considered [56]. It is then easy to see how we will define our new version of bqo with respect to the Ramsey space $\mathcal{R}$. Again, Q is $\mathcal{R}$-bqo iff there is no *bad* function from a front $\mathcal{F}$ to Q. However this time $\mathcal{F}$ is a front on some $A \in \mathcal{R}$, and $a \lhd b$ iff there is some $X \in \mathcal{R}$ such that $a \sqsubset X$ and $b \sqsubset X^+$; where $\cdot^+ : \mathcal{R} \to \mathcal{R}$ is some *shift map*, which gives extra structure to the Ramsey space $\mathcal{R}$.

By doing this, firstly we will be able to study Ramsey spaces and their structure with respect to a given shift. We also generate many alternate, abstract versions of wqo each with a corresponding notion of bqo which may share some of the transfinite closure properties of the regular notion of bqo (for example, if Q is $\mathcal{R}$-bqo then $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-bqo for every ordinal α). The aim of this chapter will be to try to classify the possible types of $\mathcal{R}$-wqo, before proving that the notion of $\mathcal{R}$-bqo is closed under taking iterated power sets, a property shared by the standard notion of bqo.

6.1 Ramsey spaces

We recall the notion of a topological Ramsey space $(\mathcal{R}, \leqslant, r)$ as in [56]. We always let $\mathcal{R}$ be a nonempty set, $\leqslant$ be a quasi-order on $\mathcal{R}$ and

$$r : \mathcal{R} \times \omega \to \mathcal{AR}$$

be a sequence of "approximations". We will often refer to a Ramsey space $(\mathcal{R}, \leqslant, r)$ simply as $\mathcal{R}$.

We have in mind the example of the *Ellentuck space* $(\mathbb{N}^{[\infty]}, \subseteq, r)$, where $\mathbb{N}^{[\infty]} = [\omega]^\omega$. As before we consider $X \in \mathbb{N}^{[\infty]}$ as an infinite increasing sequence of elements of ω. Here $\mathcal{AR}$ is the set of finite subsets (or finite increasing sequences) of ω, and $r(X, n) = X \restriction n$, i.e. $r(X, n)$ is the set consisting of the least n elements of X. Our aim is to define generalisations $\mathcal{R}$-wqo and $\mathcal{R}$-bqo, this is done essentially by replacing the Ramsey space $\mathbb{N}^{[\infty]}$ in the definitions of wqo and bqo respectively, with a general Ramsey space $\mathcal{R}$. As such we will have that $\mathbb{N}^{[\infty]}$-wqo and $\mathbb{N}^{[\infty]}$-bqo coincide precisely with the usual notions of wqo and bqo respectively.

Definition 6.1.1. We make the following standard definitions as in [56]:

1. We let $r_n(\cdot) = r(\cdot, n)$ and $\mathcal{AR}_n$ be the range of r_n.

2. For $a, b \in \mathcal{AR}$, we let $a \sqsubseteq b$ iff $\exists X \in \mathcal{R}$ such that for some $n, m \in \omega$, $r_n(X) = a, r_m(X) = b$ and $n \leqslant m$. If $n < m$ here we say $a \sqsubset b$.

3. For $a \in \mathcal{AR}$ and $X \in \mathcal{R}$, we say $a \sqsubseteq X$ iff $a \sqsubset X$ iff $\exists n \in \omega$ such that $a = r_n(X)$.

4. For $a \in \mathcal{AR}, A \in \mathcal{R}$, let $[a, A] = \{X \in \mathcal{R} : a \sqsubset X, X \leqslant A\}$.

5. For $n \in \omega, A \in \mathcal{R}$, let $[n, A] = [r_n(A), A]$.

6. We also define a finitisation of $\leqslant$, which we denote $\leqslant_{\text{fin}}$, a relation on $\mathcal{AR}$, this will be defined by axiom A2.

7. From this, for $a \in \mathcal{AR}, X \in \mathcal{R}$, we define

$$\text{depth}_X(a) = \min\{n : a \leqslant_{\text{fin}} r_n(X)\}$$

8. We call $(\mathcal{R}, \leqslant, r)$ closed if, when we equate all $X \in \mathcal{R}$ to the corresponding sequence

$$\langle r_0(X), r_1(X), r_2(X), ...\rangle,$$

then $\mathcal{R}$ is closed when considered as a subset of the Tychonov cube $\mathcal{AR}^\omega$, where $\mathcal{AR}$ has the discrete topology.

9. If $a \in \mathcal{AR}$, we let $|a| = n$ whenever $\exists X \in \mathcal{R}, r_n(X) = a$. Given axiom A1 (3) it is easy to see that this is well-defined.

10. For $\mathcal{X} \subseteq \mathcal{AR}_n$ we define $\mathcal{X}^{\complement} = \mathcal{AR}_n \setminus \mathcal{X}$.

Definition 6.1.2. The topology for $\mathcal{R}$ given by basic open sets $[a, A]$ is called the *natural* (or *Ellentuck*) topology.

Indeed, the sets $[a, A]$ form a basis of $\mathcal{R}$. To see this, first we have that for any $X \in \mathcal{R}$, we have $X \in [\langle\rangle, X]$. Second, if $X \in [a, A] \cap [b, B]$ either $a \sqsubseteq b$ or $b \sqsubseteq a$. Let c be the longer element of $\{a, b\}$. Then since $c \sqsubseteq X$ we have $X \in [c, X]$, furthermore $[c, X] \subseteq [a, A] \cap [b, B]$.

Definition 6.1.3. For two sets A and B we define the *symmetric difference*

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

We call $\mathcal{M} \subseteq \mathcal{R}$ *meagre* iff for all $[a, X] \neq \emptyset$ there exists $a \sqsubseteq b \in \mathcal{AR}$ and some $Y \leqslant X$ such that $[b, Y] \neq \emptyset$ and $[b, Y] \cap \mathcal{M} = \emptyset$. A subset $\mathcal{X}$ of $\mathcal{R}$ has the *property of Baire* if, giving $\mathcal{R}$ the natural topology, $\mathcal{X} = \mathcal{O} \Delta \mathcal{M}$ for some open $\mathcal{O} \subseteq \mathcal{R}$ and meagre $\mathcal{M} \subseteq \mathcal{R}$.

Definition 6.1.4. A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey* if for every $[a, A] \neq \emptyset$ there is a $B \in [a, A]$ such that $[a, B] \subset \mathcal{X}$ or $[a, B] \subset \mathcal{X}^{\complement}$. A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey null* if for every $[a, A] \neq \emptyset$ there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

Definition 6.1.5. A triple $(\mathcal{R}, \leqslant, r)$ is a *topological Ramsey space* if every property of Baire subset of $\mathcal{R}$ is Ramsey and if every meagre subset of $\mathcal{R}$ is Ramsey null.

Since we will only be considering topological Ramsey spaces, we refer to them simply as Ramsey spaces.

We give the following axioms as from [56]:

A1. (1) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

A2. (1) $\{a \in \mathcal{AR} : a \leqslant_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$.

(2) $A \leqslant B$ iff $(\forall n \in \omega)(\exists m \in \omega), r_n(A) \leqslant_{\text{fin}} r_m(B)$.

(3) $(\forall a, b \in \mathcal{AR}), a \sqsubseteq b \wedge b \leqslant_{\text{fin}} c \to \exists d \sqsubseteq c, a \leqslant_{\text{fin}} d$.

A3. (1) If $\text{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\text{depth}_B(a), B]$.

(2) $A \leqslant B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

A4. If $\text{depth}_B(a) < \infty$, and if $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, then there is $A \in [\text{depth}_B(a), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^{\complement}$.

We then have the following important theorem.

Theorem 6.1.6 (Abstract Ellentuck Theorem [56])**.** *If $(\mathcal{R}, \leqslant, r)$ is closed, and satisfies the axioms A.1., A.2., A.3. and A.4., then every property of Baire subset of $\mathcal{R}$ is Ramsey and every meager subset is Ramsey null. In other words, $(\mathcal{R}, \leqslant, r)$ is a Ramsey space.*

Proof. See [56] chapter 5. $\square$

Theorem 6.1.7 (Abstract Galvin Prikry Theorem [56])**.** *Borel sets are Ramsey.*

Proof. See [56] chapter 5. $\square$

We will now give some standard examples of Ramsey spaces that are described by Todorčević in [56].

Example 6.1.8 ([56])**.** Let k be a positive integer, then define

$$\mathrm{FIN}_k = \{p : \mathbb{N} \to \{0, ..., k\} : k \in \mathrm{range}(p), |\{n : p(n) \neq 0\}| < \omega\}.$$

For $p \in \mathrm{FIN}_k$ let $\mathrm{supp}(p) = \{n : p(n) \neq 0\}$. A *block sequence* of members of FIN_k is a sequence $P = (p_n)_{n<|P|}$ with $|P| \in \omega+1$, such that for all $m < n < |P|$ and $a \in \mathrm{supp}(p_m)$, $b \in \mathrm{supp}(p_n)$ we have $a < b$. We let $\mathrm{FIN}_k^{[\infty]}$ be the collection of infinite block sequences of members of FIN_k.

Let for $i \leqslant k$ let $T^i : \mathrm{FIN}_k \to \mathrm{FIN}_{k-i}$ be defined by letting

$$T^i(p)(n) = \max\{p(n) - i, 0\}.$$

Given a block sequence $P = (p_n)_{n<|P|}$ of elements of FIN_k and some $j \in \omega$ with $1 \leqslant j \leqslant k$, let

$$[P]_j = \{T^{i_0}(p_{n_0}) \cup ... \cup T^{i_l}(p_{n_l}) \in \mathrm{FIN}_j : 0 \leqslant n_0 < ... < n_l < |P|, i_0, ..., i_l \in \{0, ..., k\}, (l \in \omega)\}.$$

For $P, Q \in \mathrm{FIN}_k$ with $P = (p_n)_{n<|P|}$ and $Q = (q_n)_{n<|Q|}$, we let $P \leqslant Q$ if $p_n \in [Q]_k$ for all $n < |P|$. If $P \leqslant Q$ then we call P a *block subsequence* of Q. Finally if $P = (p_n)_{n<|P|} \in \mathrm{FIN}_k^{[\infty]}$, and $m \in \omega$, define $r_m(P) = (p_n)_{n<m}$.

Then $(\mathrm{FIN}_k^{[\infty]}, \leqslant, r)$ is a Ramsey space. For a proof see Theorem 5.22 in [56].

Example 6.1.9 ([56])**.** Let $L = \bigcup_{n\in\omega} L_n$ be a fixed alphabet, where each L_n is a finite set. Let $v \notin L$ be a set that we call a *variable*. We let W_{L_v} be the set of *variable words* over L, i.e. finite non-empty sequences of elements of $L \cup \{v\}$ that contain at least one instance of v. If $X = (x_n)_{n<|X|}$ for $|X| \in \omega$ is a sequence of elements of W_{L_v}, then we call

X *rapidly increasing* iff $|x_n| > \sum_{i<n} |x_i|$ for all $n < |L|$. We let $W_{L_v}^{[\infty]}$ be the set of infinite rapidly increasing sequences of elements of W_{L_v}.

For $x \in W_{l_v}$ and $\lambda \in L \cup \{v\}$ let $x[\lambda]$ be the sequence obtained by replacing every occurrence of v in x with λ. Now for $X = (x_n)_{n<|X|} \in W_{L_v}^{[\infty]}$ define

$$[X]_{L_v} = \{x_{n_0}[\lambda_0]^\frown ...^\frown x_{n_k}[\lambda_k] \in W_{L_v} : k \in \omega, n_0 < ... < n_k, \lambda_i \in L_{n_i} \cup \{v\}(i \leqslant k)\}.$$

Then since X is rapidly increasing we have that for every $x \in [X]_{L_v}$, the set $\{n_0 < ... < n_k\}$ such that

$$x = x_{n_0}[\lambda_0]^\frown ...^\frown x_{n_k}[\lambda_k]$$

for some choice of $\lambda_i \in L_{n_i} \cup \{v\}$ $(i \leqslant k)$ is unique, so we let $\mathrm{supp}_X(x) = \{n_0, ..., n_k\}$. We then define the order $\leqslant$ on $W_{L_v}^{[\infty]}$ by letting $X = (x_n)_{n<\omega} \leqslant Y = (y_n)_{n<\omega}$ iff $x_n \in [Y]_{L_v}$ for all $n \in \omega$, and for $n < m$ and all $a \in \mathrm{supp}(x_n)$, $b \in \mathrm{supp}(y_n)$ we have $a < b$. Finally if $X = (x_n)_{n<|X|} \in \mathrm{FIN}_k^{[\infty]}$ and $m \in \omega$, define $r_m(X) = (x_n)_{n<m}$.

Then a theorem of Carlson is that $(W_{L_v}^{[\infty]}, \leqslant, r)$ is a Ramsey space. For a proof see Theorem 5.41 in [56].

6.1.1 Fronts

Definition 6.1.10. For a Ramsey space $\mathcal{R}$, we call $\mathcal{F} \subseteq \mathcal{AR}$ *Nash-Williams* iff it is an antichain under $\sqsubseteq$. For $A \in \mathcal{R}$ we call $\mathcal{F}$ a *front on* A iff $\mathcal{F}$ is Nash-Williams and for every $B \leqslant A$ there is some $n \in \omega$ such that $r_n(B) \in \mathcal{F}$.

Definition 6.1.11. Given a front $\mathcal{F}$, let $\overline{\mathcal{F}} = \{x \in \mathcal{AR} : (\exists y \in \mathcal{F}), x \sqsubseteq y\}$. Then $\overline{\mathcal{F}}$ is a well-founded tree under $\sqsubseteq$.

Definition 6.1.12. For a front $\mathcal{F}$ we define $\mathrm{rank}(\mathcal{F})$ as the tree rank $\overline{\mathcal{F}}$ (see Definition 2.4.3). We say call $\mathcal{F}$ *simple* iff $\mathrm{rank}(\mathcal{F}) = 1$.

Definition 6.1.13. If $\mathcal{F}$ is a front on $A \in \mathcal{R}$ and $B \leqslant A$, then define

$$\mathcal{F}|B = \{a \in \mathcal{F} : (\exists Y \leqslant B), a \sqsubseteq Y\}.$$

Theorem 6.1.14 (Abstract Nash-Williams Theorem [56]). *If $(\mathcal{R}, \leqslant, r)$ is a Ramsey space, then for any Nash-Williams $\mathcal{F} \subseteq \mathcal{AR}$, any partition $\mathcal{F}_0 \cup \mathcal{F}_1 = \mathcal{F}$ and any $X \in \mathcal{R}$ there is $Y \leqslant X$ and $i \in \{0,1\}$ such that $\mathcal{F}_i | Y = \emptyset$.*

Proof. See Theorem 5.17 in [56]. $\square$

6.1.2 Shift

We now can begin to generalise the theory of better-quasi-orders to a general Ramsey space $\mathcal{R}$. In order to define a notion of well and better quasi-orders for a general Ramsey space $\mathcal{R}$, essential is the notion of a *shift*. This will be a map that takes an $X \in \mathcal{R}$ to some X^+ with $X^+ \leqslant X$. This gives us an extra relational structure on fronts, that we will embed into the complement of a quasi-order relation via bad functions. The usual example for the Ellentuck space $\mathbb{N}^{[\infty]}$ is $X^+ = X \setminus \{\min X\}$; properties and surrounding Ramsey theory of this shift on this space have been studied by Di Prisco and Todorčević in [10], and of course this is the standard shift used in bqo theory (see e.g. [53]). We will also require that our shift has an appropriate finitisation; a version that can be applied to approximations.

Some precedent has been set with experimenting with unusual shifts on the Ellentuck space. Pequignot has shown in [42] that given any non-identity injective and increasing function $f : \omega \to \omega$, the map

$$\langle n_0, n_1, ...\rangle \mapsto \langle n_{f(0)}, n_{f(1)}, ...\rangle$$

is a shift on the Ellentuck space, whose corresponding notion of bqo that turns out to be equivalent to the usual one. Clearly for any shift defined in this way we have $X \leqslant X^+$.

Definition 6.1.15. Let $\mathcal{R}$ be a Ramsey space, we call $\cdot^+ : \mathcal{R} \cup \mathcal{AR} \to \mathcal{R} \cup \mathcal{AR}$ a *shift map* iff for all $X \in \mathcal{R}$:

1. $\cdot^+{}''\mathcal{R} \subseteq \mathcal{R}$ and $\cdot^+{}''\mathcal{AR} \subseteq \mathcal{AR}$,

2. $X^+ \leqslant X$,

3. $(\forall n \in \omega)(\exists n_X \in \omega)$, $(r_n(X))^+ = r_{n_X}(X^+)$

4. $\{n_X : n \in \omega\}$ is an unbounded subset of ω and if $n < m$ then $n_X \leqslant m_X$.

We also let $X^{(+)^0} = X$ and for $i \in \omega$ we let $X^{(+)^{i+1}} = (X^{(+)^i})^+$.

Definition 6.1.16. We define the *usual shift* on the Ellentuck space $\mathbb{N}^{[\infty]}$ so that if $X = \langle x_0, x_1, ...\rangle$ then $X^+ = \langle x_1, x_2, ...\rangle$. I.e. $X^+ = X \setminus \{\min X\}$.

We note that in the case of the Ellentuck space, for any possible shift we can give an appropriate finitisation: if $X^+ \subset X$ are infinite subsets of ω, then given an initial segment $a \sqsubseteq X$ we can define $a^+ = a \cap X^+$, which clearly satisfies conditions 3 and 4. Unless specifically mentioned otherwise, the shift that we take on $\mathbb{N}^{[\infty]}$ will be the usual shift.

Definition 6.1.17. For $(p_n)_{n\in\omega} \in \mathrm{FIN}_k^{[\infty]}$ we let $(p_n)^+_{n\in\omega} = (p_{n+1})_{n\in\omega}$. If $(p_n)_{n<m}$ is a finite block sequence of members of FIN_k then let $(p_n)^+_{n<m} = (p_{n+1})_{n<m-1}$. Therefore $\cdot^+$ is a shift map for the space $\mathrm{FIN}_k^{[\infty]}$.

For $(x_n)_{n\in\omega} \in W_{L_v}^{[\infty]}$ we let $(x_n)^+_{n\in\omega} = (x_{n+1})_{n\in\omega}$. If $(x_n)_{n<m}$ is a finite rapidly increasing sequence of members of W_{L_v} then let $(x_n)^+_{n<m} = (x_{n+1})_{n<m-1}$. Therefore $\cdot^+$ is a shift map for the space $W_{L_v}^{[\infty]}$.

We will use these shifts for the spaces $\mathrm{FIN}_k^{[\infty]}$ and $W_{L_v}^{[\infty]}$ respectively.

From now on we fix a Ramsey space $\mathcal{R}$ with a shift map $\cdot^+$.

Definition 6.1.18. Let $a, b \in \mathcal{AR}$ and $X \in \mathcal{R}$, we write $a \lhd_X b$ iff $a \sqsubseteq X$ and $b \sqsubseteq X^+$. We write $a \lhd b$ iff $(\exists Y \in \mathcal{R})$, $a \lhd_Y b$.

Lemma 6.1.19. *If $a \lhd b$ then $a^+ \sqsubseteq b$ or $b \sqsubset a^+$.*

Proof. Suppose that $a \lhd_X b$, i.e. $a = r_n(X) \sqsubset X$ and $b \sqsubset X^+$. Then $a^+ = (r_n(X))^+ = r_{n_X}(X^+)$ so a^+ and b are $\sqsubseteq$-comparable as required. □

Lemma 6.1.20. *If $a \sqsubseteq b$ then $a^+ \sqsubseteq b^+$.*

Proof. Let $X \in \mathcal{R}$ be such that $a = r_{|a|}(X)$ and $b = r_{|b|}(X)$. So $a^+ = r_{|a|_X}(X^+)$ and $b^+ = r_{|b|_X}(X^+)$. The result now follows since $|a| \leqslant |b|$ implies $|a|_X \leqslant |b|_X$. □

Definition 6.1.21. For all $a, b \in \mathcal{AR}$ such that $a \lhd_Y b$ let $n \in \omega$ be least such that $n_Y \geqslant |b|$ and $n \geqslant |a|$. Then define $a \cup^*_Y b = r_n(Y)$.

Lemma 6.1.22. *For all $a, b \in \mathcal{AR}$ such that $a \lhd_Y b$ we have that $a \cup^*_Y b \in \mathcal{AR}$ is such that $a \cup^*_Y b \sqsubset Y$, $a \sqsubseteq a \cup^*_Y b$ and $b \sqsubseteq (a \cup^*_Y b)^+$.*

Proof. Let $n \in \omega$ be least such that $n_Y \geqslant |b|$ and $n \geqslant |a|$. So $a \cup^*_Y b = r_n(Y)$, thus $a \sqsubseteq a \cup^*_Y b$ and $b = r_{|b|}(Y^+) \sqsubseteq r_{n_Y}(Y^+) = (r_n(Y))^+ = (a \cup^*_Y b)^+$. $\square$

Lemma 6.1.23. *For all $a \in \mathcal{AR}$ and any $X \in \mathcal{R}$ such that $a \sqsubset X$ we have $a = a \cup^*_X a^+$.*

Proof. Suppose that $a = r_n(X)$. We then have that $a^+ = r_{n_X}(X^+)$ therefore $|a^+| = n_X$. Now, $a \cup^*_X a^+ = r_m(X)$ where m is least such that $m \geqslant n$ and $m_X \geqslant n_X$. In particular $n_X \geqslant n_X$ so $m = n$, which implies $a = a \cup^*_X a^+$. $\square$

Definition 6.1.24. Let $\mathcal{F}$ be a front on $A \in \mathcal{R}$, then we define

$$\mathcal{F}^2 = \{a \cup^*_Y b : a, b \in \mathcal{F}, Y \leqslant A, a \lhd_Y b\}.$$

Definition 6.1.25. Define $\pi_0, \pi_1 : \mathcal{F}^2 \to \mathcal{F}$ so that for $e \in \mathcal{F}^2$ we have $\pi_0(e)$ is the initial segment of e in $\mathcal{F}$ and $\pi_1(e)$ is the initial segment of e^+ in $\mathcal{F}$. This is well-defined by Lemma 6.1.22.

Lemma 6.1.26. *Let $\mathcal{F}$ be a front on $A \in \mathcal{R}$, then*

1. *$\mathcal{F}^2$ is a front on A.*

2. *$\forall a \cup^*_Y b \in \mathcal{F}^2$, $\pi_0(a \cup^*_Y b) = a$ and $\pi_1(a \cup^*_Y b) = b$.*

Proof. 1. Suppose that $\mathcal{F}^2$ is not Nash-Williams, i.e. there are $a, b, c, d \in \mathcal{F}$ and $X, Y \in \mathcal{R}$ such that $a \cup^*_X b \sqsubset c \cup^*_Y d$. Hence $a, c \sqsubseteq c \cup^*_Y d$ and thus a and c are $\sqsubseteq$-comparable. By Lemma 6.1.20 we have that $(a \cup^*_X b)^+ \sqsubseteq (c \cup^*_Y d)^+$. Therefore $b, d \sqsubseteq (c \cup^*_Y d)^+$ and hence b and d are $\sqsubseteq$-comparable. So since $a, b, c, d \in \mathcal{F}$ and $\mathcal{F}$ is Nash-Williams, we have $a = c$ and $b = d$, which contradicts our assumption.

   Now in order to see that $\mathcal{F}^2$ is a front, let $X \leqslant A$ and we will find an initial segment in $\mathcal{F}^2$. Let a be the initial segment of X in $\mathcal{F}$ and b be the initial segment of X^+ in $\mathcal{F}$. Then $a \lhd_X b$ so that for some $n \in \omega$, $r_n(X) = a \cup^*_X b \in \mathcal{F}^2$.

2. By Lemma 6.1.22 we have $a \sqsubseteq a \cup_Y^* b$ and $b \sqsubseteq (a \cup_Y^* b)^+$, furthermore $a, b \in \mathcal{F}$, so $\pi_0(a \cup_Y^* b) = a$ and $\pi_0(a \cup_Y^* b) = b$.

□

6.2 Abstract well-quasi-orders

From now on we will always let $\mathcal{R}$ denote a Ramsey space and $\cdot^+$ be a shift map on $\mathcal{R}$.

Definition 6.2.1. If $\mathcal{F}$ is a front on $\mathcal{R}$ then:

- $f : \mathcal{F} \to Q$ is *bad* iff $\forall a, b \in \mathcal{F}$ with $a \lhd b$, $f(a) \not\leq f(b)$.
- $f : \mathcal{F} \to Q$ is *bad*$_\perp$ iff $\forall a, b \in \mathcal{F}$ with $a \lhd b$, $f(a) \perp f(b)$.
- $f : \mathcal{F} \to Q$ is *bad*$_>$ iff $\forall a, b \in \mathcal{F}$ with $a \lhd b$, $f(a) > f(b)$.
- If $\mathcal{F}$ is a simple front then we call $f : \mathcal{F} \to Q$ *simple*.

Definition 6.2.2.
- Q is $\mathcal{R}$*-well-quasi-ordered*, or $\mathcal{R}$*-wqo* iff there is no simple bad function to Q.
- Q is $\mathcal{R}$-wqo$_\perp$ iff there is no simple bad$_\perp$ function to Q.
- Q is $\mathcal{R}$-wqo$_>$ iff there is no simple bad$_>$ function to Q.

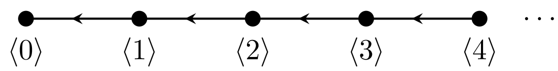

Figure 6.1: A simple front on $\mathbb{N}^{[\infty]}$ ordered by $\lhd$.

Example 6.2.3. For any quasi-order Q and $k \in \omega$, we have Q is $\mathbb{N}^{[\infty]}$-wqo iff Q is $\mathrm{FIN}_k^{[\infty]}$-wqo.

To see this, let $\mathcal{F}$ be a simple front on $\mathrm{FIN}_k^{[\infty]}$. Let $\varphi : \mathcal{F} \to \omega$ be such that $\varphi(a) = \max(\mathrm{supp}(a))$. Then $\mathcal{G} = \{\langle n \rangle : n \in \mathrm{range}(\varphi)\}$ is a simple front on $\mathrm{range}(\varphi) \in \mathbb{N}^{[\infty]}$. For $a, b \in \mathcal{F}$, $a \lhd b$ iff $\forall n \in \mathrm{supp}(a)$, $\forall m \in \mathrm{supp}(b)$ we have $n < m$. Therefore

$$a \lhd b \longrightarrow \varphi(a) < \varphi(b) \longrightarrow \langle \varphi(a) \rangle \lhd \langle \varphi(b) \rangle.$$

If we have a simple bad function to Q, then since any two simple fronts on $\mathbb{N}^{[\infty]}$ are order-isomorphic when ordered by $\lhd$, we have a bad $f : \mathcal{G} \to Q$. Therefore $f \circ \varphi : \mathcal{F} \to Q$ is simple and bad, hence $\mathrm{FIN}_k^{[\infty]}$-wqo implies $\mathbb{N}^{[\infty]}$-wqo.

Now suppose that $\mathcal{G}$ is a simple front on some $A \in \mathrm{FIN}_k^{[\infty]}$, and that $g : \mathcal{G} \to Q$ is bad. Suppose that $A = (a_m)_{m\in\omega}$ and let $\psi : [\omega]^1 \to \mathcal{G}$ be such that for each $n \in \omega$, $\psi(\langle n \rangle) = a_n$. Therefore for $\langle n \rangle, \langle m \rangle \in [\omega]^1$ we have

$$\langle n \rangle \lhd \langle m \rangle \longrightarrow n < m \longrightarrow a_n \lhd a_m.$$

Therefore $g \circ \psi : [\omega]^1 \to Q$ is simple and bad, so that $\mathbb{N}^{[\infty]}$-wqo implies $\mathrm{FIN}_k^{[\infty]}$-wqo.

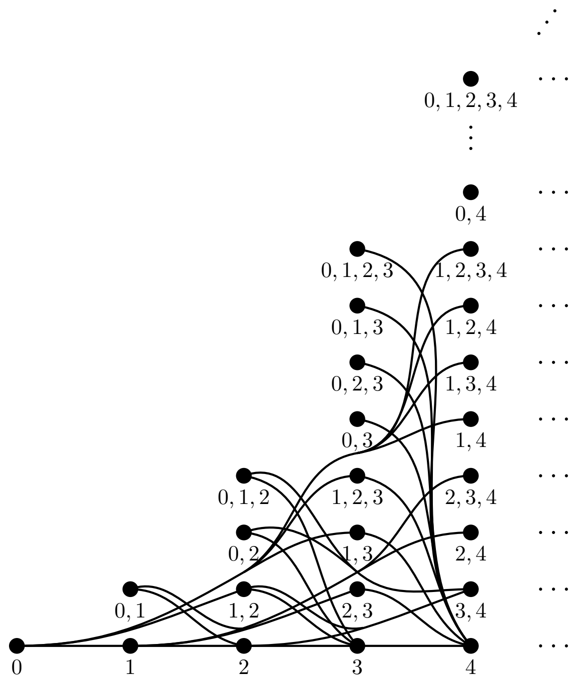

Figure 6.2: A simple front on $\mathrm{FIN}_1^{[\infty]}$ ordered by $\lhd$, each point is labelled by its support.

Note that the fronts on $\mathrm{FIN}_1^{[\infty]}$ are isomorphic to the set of finite subsets of ω ordered by domination i.e. $X < Y$ iff $(\forall x \in X)(\forall y \in Y), x < y$, see Figure 6.2.

Example 6.2.4. For any quasi-order Q, we have Q is $\mathbb{N}^{[\infty]}$-wqo iff Q is $W_{L_v}^{[\infty]}$-wqo.

Let $\mathcal{F}$ be a simple front on $W_{L_v}^{[\infty]}$, and let $\varphi : \mathcal{F} \to \omega$ be such that $\varphi(a) = |a|$. Since sequences of $W_{L_v}^{[\infty]}$ are rapidly increasing, we have for $a, b \in \mathcal{F}$ that $a \lhd b$ implies $\varphi(a) = |a| < |b| = \varphi(b)$. So similarly to the previous example $W_{L_v}^{[\infty]}$-wqo implies $\mathbb{N}^{[\infty]}$-wqo.

Now suppose that $\mathcal{G}$ is a simple front on some $A \in W_{L_v}^{[\infty]}$, and that $g : \mathcal{G} \to Q$ is bad. Suppose that $A = (a_m)_{m\in\omega}$ and let $\psi : [\omega]^1 \to \mathcal{G}$ be such that for each $n \in \omega$, $\psi(\langle n \rangle) = a_n$. Again similarly to the previous example, we can conclude $\mathbb{N}^{[\infty]}$-wqo implies $W_{L_v}^{[\infty]}$-wqo.

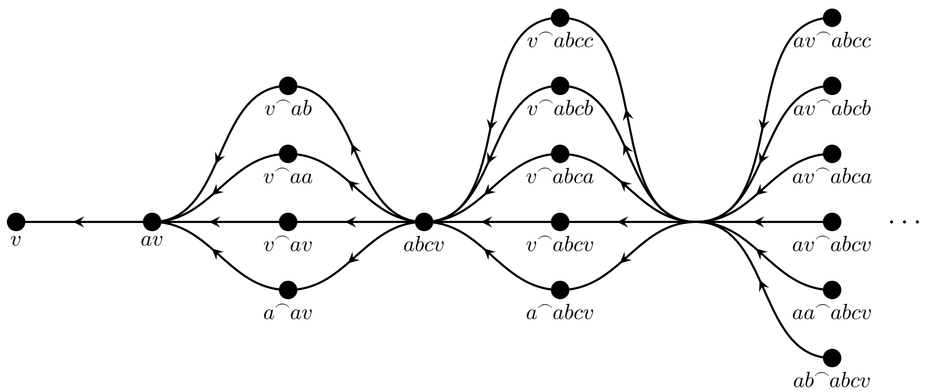

Figure 6.3: A simple front on $\langle v, av, abcv, ...\rangle \in W_{L_v}^{[\infty]}$ ordered by $\lhd$.

6.2.1 Basic results

Proposition 6.2.5. *If $f : \mathcal{F} \to Q$ is bad, then there is $X \in \mathcal{R}$ such that $f \restriction (\mathcal{F}|X)$ is either bad$_\perp$ or bad$_>$.*

Proof. Consider the subset $B = \{a \in \mathcal{F}^2 : f(\pi_0(a)) \perp f(\pi_1(a))\}$ of the front $\mathcal{F}^2$. Applying the Abstract Nash-Williams Theorem 6.1.14 to the set B gives an $X \in \mathcal{R}$ so that either $\mathcal{F}^2|X \subseteq B$ or $(\mathcal{F}^2|X) \cap B = \emptyset$. Therefore $f \restriction (\mathcal{F}|X)$ is as required. $\square$

The previous proposition gives us an analogue of the characterisation of the notion of wqo as 'well-founded and narrow'. That is to say that if $f : \mathcal{F} \to Q$ is bad, then we obtain either an embedding of the digraph $(\mathcal{F}, \lhd)$ into $(Q, >)$ or into $(Q, \perp)$. We will investigate what can happen in these cases separately in an attempt to classify the possible notions of $\mathcal{R}$-wqo, for different Ramsey spaces $\mathcal{R}$.

Theorem 6.2.6. *Q is $\mathbb{N}^{[\infty]}$-wqo iff Q is well-founded and narrow.*

Proof. By Proposition 6.2.5, if we have a bad function to Q then we have either a $\text{bad}_\perp$ or $\text{bad}_>$ function to Q. Clearly then this function enumerates either an infinite antichain or descending sequence.

If Q has a descending sequence or infinite antichain, then the function enumerating this sequence composed with the function $\langle n \rangle \mapsto n$ is a bad since $\langle n \rangle \lhd \langle m \rangle$ iff $n < m$. Furthermore this function has domain equal to the simple front $[\omega]^1$ on $\omega \in \mathbb{N}^{[\infty]}$. $\square$

Lemma 6.2.7. *If $\mathcal{R}$ is finite, then any quasi-order Q is $\mathcal{R}$-wqo.*

Proof. If $\mathcal{R}$ is finite then any front $\mathcal{G}$ is also finite. By repeated restriction using Theorem 6.1.14, there is a front $\mathcal{F} \subseteq \mathcal{G}$ on some $X \in \mathcal{R}$ with $|\mathcal{F}| = 1$. Therefore $r_1(X)$ is the unique element of this front. Then $X^+ \leqslant X$ and therefore X^+ has an initial segment in $\mathcal{F}$, hence $r_1(X) = r_1(X^+)$. But since the order on any quasi-order Q is reflexive, we have $f(r_1(X)) \leqslant f(r_1(X^+))$ for *any* function $f : \mathcal{G} \to Q$, therefore no such function can be bad, and hence Q is $\mathcal{R}$-wqo. $\square$

Lemma 6.2.8. *For any Ramsey space $\mathcal{R}$, any well-order Q is $\mathcal{R}$-wqo.*

Proof. If Q is not $\mathcal{R}$-wqo, then by Proposition 6.2.5 for some simple front $\mathcal{F}$ on $A \in \mathcal{R}$, there is $f : \mathcal{F} \to Q$ that is either $\text{bad}_\perp$ or $\text{bad}_>$. In the former case considering $a \in \mathcal{F}$ with $a \sqsubseteq X$, let b be the initial segment of X^+ in $\mathcal{F}$, then $a \lhd b$ and thus $f(a) \perp f(b)$, this is a contradiction because then $f(a) \neq f(b)$ and thus Q is not linear.

So suppose f is $\text{bad}_>$. Pick the least element q_0 of $\text{im}(f)$ with respect to the order on Q. Let $a \in f^{-1}(q_0)$, and $a \sqsubseteq X \leqslant A$, and let b be the initial segment of X^+ in $\mathcal{F}$. Therefore $a \lhd b$ and so $f(a) > f(b)$, but this contradicts that q_0 was the least element of $\text{im}(f)$. $\square$

Lemma 6.2.9. *Every finite quasi-order is $\mathcal{R}$-wqo.*

Proof. Suppose not, then for some $n \in \omega$ there is a simple bad function f to Q, a set of size n. Considering each element of Q in turn, using the Abstract Nash-Williams Theorem

6.1.14 at most n times, we can find some $A \in \mathcal{R}$ so that $|f"(\mathcal{F}|A)| = 1$. Then $f \restriction (\mathcal{F}|A)$ is still bad, thus $f"(\mathcal{F}|A)$ is not $\mathcal{R}$-wqo, but $f"(\mathcal{F}|A)$ is a singleton, thus a well-founded linear order, and is therefore $\mathcal{R}$-wqo by Lemma 6.2.8. □

6.2.2 Abstract antichains

Definition 6.2.10. Let $\mathcal{R}$ be a Ramsey space and $\mathcal{F}$ be a front on $A \in \mathcal{R}$. We say that $\mathcal{F}$ *has loops* iff there is an $n \in \omega$ and $a, a_1, ..., a_n \in \mathcal{F}$ such that

$$a \lhd a_1 \lhd ... \lhd a_n \lhd a.$$

In this case we call $\{a, a_1, ..., a_n\}$ a *loop*. We say that $\mathcal{F}$ *has no loops* if no subset of $\mathcal{F}$ is a loop.

Lemma 6.2.11. *If every simple front on $\mathcal{R}$ has loops, then every quasi-order Q is $\mathcal{R}$-wqo$_{>}$.*

Proof. Suppose there were a simple bad$_{>}$ function $f : \mathcal{F} \to Q$. We know that $\mathcal{F}$ has a loop $\{a, a_1, ..., a_n\}$, so that $f(a) > f(a_1) > ... > f(a_n) > f(a)$. Clearly this is impossible. □

Definition 6.2.12. For a Ramsey space $\mathcal{R}$ we define the class of graphs

$$\mathfrak{F}(\mathcal{R}) = \{\langle \mathcal{F}, \lhd \cup \rhd \rangle : \mathcal{F} \text{ is a simple front on } \mathcal{R}\}.$$

Let $\mathfrak{G}(\mathcal{R})$ be the set of graphs $G \in \mathfrak{F}(\mathcal{R})$ which have arbitrarily large finite complete subgraphs but do not have an infinite complete subgraph.[1]

Lemma 6.2.13. *Q is not $\mathcal{R}$-wqo$_{\perp}$ iff some graph of $\mathfrak{F}(\mathcal{R})$ is isomorphic to a subgraph of $\langle Q, \perp \rangle$.*

Proof. Suppose that Q is not $\mathcal{R}$-wqo$_{\perp}$. Thus there is a simple bad$_{\perp}$ function $f : \mathcal{F} \to Q$. Consider the graphs $\langle \mathcal{F}, \lhd \cup \rhd \rangle$, and

$$G = \langle \text{range}(f), \{\langle f(a), f(b) : a, b \in \mathcal{F}, a \lhd b \text{ or } b \lhd a\} \rangle.$$

Clearly f is an isomorphism between these graphs. Moreover since $a \lhd b$ or $b \lhd a$ imply that $f(a) \perp f(b)$, we have that G is a subgraph of $\langle Q, \perp \rangle$.

[1] Recall that a graph G is complete iff $(\forall x, y \in V(G)), x \sim y$.

Now suppose that some graph $\langle \mathcal{F}, \lhd \cup \rhd \rangle \in \mathfrak{F}(\mathcal{R})$ is isomorphic to a subgraph of $\langle Q, \perp \rangle$, and let $f : \mathcal{F} \to Q$ be an isomorphism. Thus if $a \lhd b$ we have $f(a) \perp f(b)$, and thus f is simple bad$_{>}$. $\square$

Lemma 6.2.14. *Suppose that there is some member of $\mathfrak{F}(\mathcal{R})$ whose complete subgraphs have size bounded by some $n \in \omega$, then every quasi-order is $\mathcal{R}$-wqo.*

Proof. Suppose that the complete subgraphs of $\langle \mathcal{F}, \lhd \cup \rhd \rangle \in \mathfrak{F}(\mathcal{R})$ are bounded by $n \in \omega$. Now apply abstract Nash-Williams Theorem 6.1.14 at most n times to find some $A \in \mathcal{R}$ such that $\langle \mathcal{F}|A, \lhd \cup \rhd \rangle$ has complete subgraphs of size at most 1. Therefore for no distinct $a, b \in \mathcal{F}|A$ do we have $a \lhd b$. However we know that $a \lhd a^+$ and therefore $a = a^+$ for all $a \in \mathcal{F}|A$. Thus if $f : \mathcal{F} \to Q$ is bad, we have $f(a) \not\leqslant f(a^+) = f(a)$, which contradicts that the order on Q is reflexive. $\square$

Lemma 6.2.15. *If $|\mathcal{AR}_1| \leqslant \aleph_0$ and $\mathfrak{G}(\mathcal{R}) \neq \emptyset$ then either every quasi-order Q is $\mathcal{R}$-wqo, or we have that a quasi-order Q is not $\mathcal{R}$-wqo$_\perp$ iff $\mathfrak{G}(\mathcal{R})$ contains an element that is isomorphic to a subgraph of $\langle Q, \perp \rangle$.*

Proof. If Q is a quasi-order then by Lemma 6.2.13 we have that Q is not $\mathcal{R}$-wqo$_\perp$ iff some graph $G \in \mathfrak{F}(\mathcal{R})$ is isomorphic to a subgraph of $\langle Q, \perp \rangle$.

Suppose that a quasi-order Q is not $\mathcal{R}$-wqo$_\perp$. Therefore every element of $\mathfrak{F}(\mathcal{R})$ has arbitrarily large finite complete subgraphs by Lemma 6.2.14. So if G does not contain an infinite complete subgraph, then $G \in \mathfrak{G}(\mathcal{R})$. If G contains an infinite complete subgraph, then so does $\langle Q, \perp \rangle$. But then since $|\mathcal{AR}_1| \leqslant \aleph_0$ we have that every front is countable, so every $H \in \mathfrak{G}(\mathcal{R}) \neq \emptyset$ is countable. This means there is a subgraph of the infinite complete subgraph of $\langle Q, \perp \rangle$ that is isomorphic to H.

Now if $\mathfrak{G}(\mathcal{R})$ contains a graph that is isomorphic to a subgraph of $\langle Q, \perp \rangle$ then $\mathfrak{F}(\mathcal{R})$ contains this graph too. Hence Q is not $\mathcal{R}$-wqo$_\perp$ by Lemma 6.2.13. $\square$

Theorem 6.2.16. *For any Ramsey space $\mathcal{R}$ such that $|\mathcal{AR}_1| \leqslant \aleph_0$, either:*

- *Q is $\mathcal{R}$-wqo whenever Q is a quasi-order.*

- *$\mathfrak{G}(\mathcal{R}) \neq \emptyset$ and Q is $\mathcal{R}$-wqo$_\perp$ iff no element of $\mathfrak{G}(\mathcal{R})$ is isomorphic to a subgraph of $\langle Q, \perp \rangle$.*
- *$\mathfrak{G}(\mathcal{R}) = \emptyset$ and Q is $\mathcal{R}$-wqo$_\perp$ iff Q is narrow*

Proof. Firstly by Lemma 6.2.15, either: $\mathfrak{G}(\mathcal{R}) = \emptyset$; every quasi-order Q is $\mathcal{R}$-wqo; or $\mathfrak{G}(\mathcal{R})$ contains an element that is isomorphic to a subgraph of $\langle Q, \perp \rangle$.

If $\mathfrak{G}(\mathcal{R}) = \emptyset$ then by Lemma 6.2.14, every quasi-order Q is $\mathcal{R}$-wqo or every element of $\mathfrak{F}(\mathcal{R})$ contains an infinite complete subgraph. In this case, by Lemma 6.2.13 if Q is $\mathcal{R}$-wqo$_\perp$, then there is some $G \in \mathfrak{F}(\mathcal{R})$ that is isomorphic to a subset of $\langle Q, \perp \rangle$. Therefore Q contains an infinite antichain, since G contains an infinite complete subgraph.

Finally if Q is not narrow then since $|\mathcal{AR}_1| \leqslant \aleph_0$ we have that every graph $G \in \mathfrak{F}(\mathcal{R})$ is countable, therefore isomorphic to a subgraph of $\langle Q, \perp \rangle$, since this contains a countable complete subgraph. Thus Q is not $\mathcal{R}$-wqo$_\perp$ by Lemma 6.2.13. □

6.2.3 Abstract descending sequences

Lemma 6.2.17. *If Q is well-founded then Q is $\mathcal{R}$-wqo$_>$.*

Proof. If Q is not $\mathcal{R}$-wqo$_>$ then let $f : \mathcal{F} \to Q$ be simple and bad$_>$. We notice that $\mathcal{F}$ is a front on some $A \in \mathcal{R}$ which has no loops, otherwise for some $b \in \mathcal{F}$ we would have $f(b) > f(b)$. For $i \in \omega$ let a_i be the initial segment of $A^{(+)^i}$ in $\mathcal{F}$. Therefore

$$a_0 \lhd a_1 \lhd a_2 \lhd ...$$

and so

$$f(a_0) > f(a_1) > f(a_2) > ...$$

is an infinite descending chain in Q since $\mathcal{F}$ had no loops. □

Definition 6.2.18. For $a, b \in \mathcal{F}$, let $a \lhd' b$ iff $a = b$ or $\exists x_0, x_1, ..., x_n \in \mathcal{F}$ such that

$$a \lhd x_0 \lhd x_1 \lhd ... \lhd x_n \lhd b.$$

If $\mathcal{F}$ has no loops then $\lhd'$ is a partial order. Reflexivity and transitivity are immediate from the definition. For antisymmetry, suppose there were $a \lhd' b$ and $b \lhd' a$ with $a \neq b$

then there are $a \lhd x_0 \lhd ... \lhd x_n \lhd b \lhd y_0 \lhd ... \lhd y_m \lhd a$ so that $\mathcal{F}$ has loops which is a contradiction.

We let $\preceq_{\mathcal{F}}$ be a linear extension of the reverse order of $\lhd'$, which exists by Zorn's lemma. We also let $a \prec_{\mathcal{F}} b$ iff $a \preceq_{\mathcal{F}} b$ and $a \neq b$

Proposition 6.2.19. *For any $\mathcal{F}$ that does not have loops, $(\mathcal{F}, \preceq_{\mathcal{F}})$ is not $\mathcal{R}$-wqo$_>$.*

Proof. For any $a, b \in \mathcal{F}$ we have $a \lhd b$ implies $b \prec_{\mathcal{F}} a$. Thus the identity map on $\mathcal{F}$ is bad$_>$ for this ordering. $\square$

Lemma 6.2.20. *If $f : \mathcal{F} \to \mathbb{Q}$ is bad$_>$ then there is some $B \in \mathcal{R}$ such that $\mathcal{F}|B$ well-founded with respect to $\lhd$.*

Proof. The set $\{\frac{p}{q} : p, q \in \mathbb{N} \setminus \{0\}, \gcd(p, q) = 1\}$ ordered by $<$ is clearly a dense linear order without end points, thus it is isomorphic to $\mathbb{Q}$ so we will use $\mathbb{Q}$ to denote this set. Suppose $f : \mathcal{F} \to \mathbb{Q}$ is bad$_>$. Define:

$$\mathcal{O} = \left\{ c \in \mathcal{F}^2 : f(\pi_0(c)) = \frac{p_0}{q_0}, f(\pi_1(c)) = \frac{p_1}{q_1} \text{ and } q_0 \geqslant q_1 \right\}.$$

By the abstract Nash-Williams theorem 6.1.14, there is a $B \in \mathcal{R}$ such that $\mathcal{F}^2|B \subseteq \mathcal{O}$ or $(\mathcal{F}^2|B) \cap \mathcal{O} = \emptyset$. Suppose $\mathcal{F}^2|B \subseteq \mathcal{O}$. If $a, b \in \mathcal{F}|B$ with $a \lhd b$, then when written in their lowest form,[2] the denominator of $f(b)$ is at most equal to the denomintor of $f(a)$. We also know that $\mathcal{F}$ does not have loops because otherwise f could not be bad$_>$. So for $X \leqslant B$ and $i \in \omega$, let $a_i = r_1(X^{(+)^i}) \in \mathcal{F}|B$. Thus $(f(a_i))_{i \in \omega}$ is a descending sequence of rationals $(\frac{p_i}{q_q})_{i \in \omega}$ such that $(q_i)_{i \in \omega}$ is also descending. Notice that whenever $q_i = q_{i+1}$ we have $p_i > p_{i+1}$, thus either $(q_i)_{i \in \omega}$ or $(p_i)_{i \in \omega}$ has a strictly descending subsequence. However these are sequences in the well-founded order $\mathbb{N} \setminus \{0\}$ so this is impossible.

Therefore $(\mathcal{F}^2|B) \cap \mathcal{O} = \emptyset$, and thus when $a, b \in \mathcal{F}$ with $a \lhd b$, when written in their lowest form the denominator of $f(a)$ is strictly less than the denominator of $f(b)$. Suppose there were an infinite $\lhd$-descending sequence in $\mathcal{F}|B$ i.e. $a_0 \rhd a_1 \rhd ...$ with each $a_i \in \mathcal{F}|B$ $(i \in \omega)$. But this means that the sequence of denominators of $f(a_i)$ $(i \in \omega)$ is a descending sequence in $\mathbb{N} \setminus \{0\}$. $\square$

[2] i.e. when coprime.

Theorem 6.2.21. *If $(\mathbb{Q},<)$ is not $\mathcal{R}$-wqo$_>$ then there is an ordinal α such that α^* is not $\mathcal{R}$-wqo$_>$.*

Proof. Suppose $f : \mathcal{F} \to \mathbb{Q}$ is bad$_>$ so by Lemma 6.2.20 we can assume without loss of generality that $\mathcal{F}$ is well-founded with respect to $\lhd$. Let F_0 be the set of $\lhd$-minimal elements of $\mathcal{F}$, i.e. $d \in F_0$ iff $d \in \mathcal{F}$ and there is no $e \in \mathcal{F}$ such that $e \lhd d$. We know then that $F_0 \neq \emptyset$, otherwise either $\mathcal{F} = \emptyset$, $\mathcal{F}$ has loops or $\mathcal{F}$ has an infinite $\lhd$-descending sequence, none of which are possible. Also notice that no two elements of F_0 are comparable under $\lhd$. Let $\tau_0 = |F_0|$ and $f_0 : F_0 \to \tau_0$ be a bijection.

Suppose for induction that we have defined $F_\alpha \subseteq \mathcal{F}$, τ_α an ordinal and $f_\alpha : \bigcup_{\beta \leqslant \alpha} F_\beta \to \tau_\alpha$ for every $\alpha < \gamma$. Suppose also that $f_\alpha \upharpoonright \bigcup_{\beta<\delta} = f_\delta$ for all $\delta < \alpha$, and for all $a, b \in \bigcup_{\beta<\alpha} F_\beta$ with $a \lhd b$ we have $f_\alpha(a) < f_\alpha(b)$. Now define:

$$F_\gamma = \left\{ a \in \mathcal{F} \setminus \bigcup_{\alpha<\gamma} F_\alpha : (\forall x \in \mathcal{F}), x \lhd a \to x \in \bigcup_{\alpha<\gamma} F_\alpha \right\}.$$

Notice that F_γ is pairwise $\lhd$-incomparable, since if $a \in \mathcal{F}_\gamma$ and $x \lhd a$ then $x \notin \mathcal{F}_\gamma$. Let τ_γ be the ordinal $\bigcup_{\alpha<\gamma} \tau_\alpha + |F_\gamma|$, and define $f_\gamma : \bigcup_{\alpha \leqslant \gamma} F_\alpha \to \tau_\alpha$ by letting $f_\gamma \upharpoonright \bigcup_{\beta \leqslant \delta} = f_\delta$ for every $\delta < \gamma$; and $f_\gamma \upharpoonright F_\gamma$ be a bijection to $\tau_\gamma \setminus \bigcup_{\alpha<\gamma} \tau_\alpha$.

But if for no $\gamma < |\mathcal{F}|^+$ is $F_\gamma = \emptyset$, then $\bigcup_{\alpha<\omega_1} F_\alpha$ is a subset of $\mathcal{F}$ of cardinality at least $|\mathcal{F}|^+$ which is clearly a contradiction. Thus for some $\gamma < |\mathcal{F}|^+$ we have $F_\gamma = \emptyset$.

Now for this γ we claim that $\mathcal{F} = \bigcup_{\alpha<\gamma} F_\alpha$. So suppose $F_\gamma = \emptyset$ and let $a \in \mathcal{F} \setminus \bigcup_{\alpha<\gamma} F_\alpha$. If all $c \in \mathcal{F}$ with $c \lhd a$ were in $\bigcup_{\alpha<\gamma} F_\alpha$ then $a \in \mathcal{F}_\gamma \neq \emptyset$. So there must be some $b_0 \in \mathcal{F} \setminus \bigcup_{\alpha<\gamma} F_\alpha$ with $b_0 \lhd a$. But a was an arbitrary element of $\mathcal{F} \setminus \bigcup_{\alpha<\gamma} F_\alpha$ so we can conclude there is a $b_1 \in \mathcal{F} \setminus \bigcup_{\alpha<\gamma} F_\alpha$ such that $b_1 \lhd b_0$. Similarly we find $b_0, b_1, b_2, ... \in \mathcal{F}$ such that

$$a \rhd b_0 \rhd b_1 \rhd b_2 \rhd ...$$

and therefore there is either an infinite $\lhd$-descending sequence or a loop in $\mathcal{F}$, neither of which are possible, proving the claim.

Therefore if $a, b \in \mathcal{F} = \bigcup_{\alpha<\gamma} F_\alpha$, with $a \lhd b$, then for some ordinals $\alpha < \beta$, we have $a \in F_\alpha$ and $b \in F_\beta$ and therefore $f(a) < f(b)$. Therefore by reversing the order on τ_γ, the same map f witnesses that τ_γ^* is not wqo$_>$. $\square$

Theorem 6.2.22. *If $f : \mathcal{F} \to \sum_{q\in Q} Q_q$ is bad, then there is either a bad function to Q or there is some $P \subseteq Q$ and $A \in \mathcal{R}$ as well as disjoint $\mathcal{F}_p \subseteq \mathcal{F}$ $(p \in P)$ with $\mathcal{F}|A = \bigcup_{p\in P} \mathcal{F}_p$, $f``\mathcal{F}_p \subseteq Q_p$ and for every $a \in \mathcal{F}_p$ and $b \in \mathcal{F}_{p'}$, if $a \lhd b$ then $p = p'$.*

Proof. Suppose that $f : \mathcal{F} \to \sum_{q\in Q} Q_q$ is bad and define $g : \mathcal{F} \to Q$ so that $g(a) = q$ whenever $f(a) \in Q_q$. Let

$$\mathcal{F}_0^2 = \{a \cup_Y^* b \in \mathcal{F}^2 : g(a) \leqslant g(b)\},$$

and $\mathcal{F}_1^2 = \mathcal{F}^2 \setminus \mathcal{F}_0^2$. So by Lemma 6.1.26 and Theorem 6.1.14 there is some $A \in [\kappa]^\kappa$ such that $\mathcal{F}^2|A \subseteq \mathcal{F}_0^2$ or $\mathcal{F}^2|A \subseteq \mathcal{F}_1^2$. In the second case, we have that $g \restriction \mathcal{F}|A$ is a bad function to Q. In the first case, if $a, b \in \mathcal{F}|A$ and $a \lhd b$ then $g(a) \leqslant g(b)$, suppose that $g(a) \in Q_q$ and $g(b) \in Q_p$. So $g(a) = q \leqslant p = g(b)$ and thus if $q \neq p$ we have $f(a) \leqslant f(b)$ which contradicts that f is bad. So whenever $a, b \in \mathcal{F}|A$ and $a \lhd b$, we have $g(a) = g(b)$. Thus $P = \mathrm{range}(f)$ and $\mathcal{F}_p = f^{-1}(Q_p)$ $(p \in P)$ satisfies the statement of the lemma. □

Definition 6.2.23. An ordinal γ is *decomposable* iff there are $\alpha, \beta < \gamma$ such that $\gamma = \beta \cdot \alpha$. An ordinal γ is *indecomposable* iff it is not decomposable.[3]

Lemma 6.2.24. *Let γ be a decomposable ordinal. If $f : \mathcal{F} \to \gamma^*$ bad, then there is an ordinal $\alpha < \gamma$ and some $A \in \mathcal{R}$ such that $f : \mathcal{F}|A \to \alpha^*$ bad.*

Proof. If γ is a decomposable ordinal, let $\alpha, \beta < \gamma$ be such that $\gamma = \beta \cdot \alpha$. Therefore $\gamma^* \cong \sum_{i\in\alpha^*} \beta_i^*$ where each $\beta_i^* \cong \beta_i$. If γ^* is not $\mathcal{R}$-wqo, let $f : \mathcal{F} \to \gamma^*$ be bad. Now apply Theorem 6.2.22. If α^* is not $\mathcal{R}$-wqo then we are done. Otherwise there is some $\delta^* \subseteq \alpha^*$ and $A \in \mathcal{R}$ such that for $i \in \delta^*$ there are some $\mathcal{F}_i \subseteq F$, that partition $\mathcal{F}|A$, and such that $f``\mathcal{F}_i \subseteq \beta_i^*$ and if $a \in \mathcal{F}_i$, $b \in \mathcal{F}_j$ and $a \lhd b$, then $i = j$. Now every β_i^* is isomorphic to β^*, so let $g_i : \beta_i^* \to \beta^*$ be an isomorphism. Now let $h : \mathcal{F}|A \to \beta^*$ be such that $h(a) = g_i \circ f$ whenever $f(i) \in \mathcal{F}_i$. This is well-defined since the $\mathcal{F}_i$ $(i \in \delta^*)$ form a partition of $\mathcal{F}|A$, and since $f``\mathcal{F}_i \subseteq \beta_i^*$ for each $i \in \delta^*$. Now if $a, b \in \mathcal{F}|A$ are such that $a \lhd b$ then $a, b \in \mathcal{F}_i$ for

[3] These properties are sometimes known as multiplicatively decomposable and multiplicatively indecomposable respectively.

some $i \in \delta^*$, therefore $f(a), f(b) \in \beta_i^*$ and $f(a) \not\leq f(b)$, and thus $h(a) \not\leq h(b)$. Therefore h is bad and β^* is not $\mathcal{R}$-wqo. $\square$

Corollary 6.2.25. *If α is the least ordinal such that α^* is not $\mathcal{R}$-wqo, then α is indecomposable.*

Proof. Follows immediately by Lemma 6.2.24. $\square$

Lemma 6.2.26. *Suppose ω^* is $\mathcal{R}$-wqo, $\alpha > \omega$ is a countable limit ordinal, $f : \mathcal{F} \to \alpha^*$ is bad$_>$, and Q is a quasi-order that embeds β^* for all β in some unbounded subset of α. Then there is some $A \in \mathcal{R}$ and a bad$_>$ function $g : \mathcal{F}|A \to Q$.*

Proof. Suppose that $\alpha > \omega$ is a countable limit ordinal and and $f : \mathcal{F} \to \alpha^*$ is bad$_>$. Since α is a countable limit ordinal, we have that $\alpha = \sum_{i\in\omega} \beta_i$, and so $\alpha^* = \sum_{i\in\omega^*} \beta_i^*$. By Theorem 6.2.22, since ω^* is $\mathcal{R}$-wqo, there is some $A \in \mathcal{R}$ such that for $i \in \omega^*$ there are some $\mathcal{F}_i \subseteq F$, that partition $\mathcal{F}|A$, and such that $f"\mathcal{F}_i \subseteq \beta_i^*$ and if $a \in \mathcal{F}_i$, $b \in \mathcal{F}_j$ and $a \lhd b$, then $i = j$.

Let Q be a quasi-order and suppose that for all $\beta < \alpha$ we have β^* embeds into Q, letting $h_\beta : \beta^* \to Q$ be an embedding. Define $g : \mathcal{F}|A \to Q$ by letting $g(a) = h_{\beta_i} \circ f(a)$ whenever $a \in \mathcal{F}_i$. Now if $a \lhd b$, then $a, b \in \mathcal{F}_i$ for some $i \in \omega^*$. Therefore

$$g(a) = h_{\beta_i} \circ f(a) \not\leq h_{\beta_i} \circ f(b) = g(b),$$

so g is bad, and Q is not $\mathcal{R}$-wqo$_>$. $\square$

Lemma 6.2.27 (Cantor)**.** *Any countable linear order embeds into $\mathbb{Q}$.*

Proof. Let $\mathbb{Q} \cup \{-\infty, \infty\}$ be ordered by extending the order on $\mathbb{Q}$, letting $-\infty < q < \infty$ for all $q \in \mathbb{Q}$. Let L be a countable linear order, and fix an enumeration $L = \{x_n : n \in \omega\}$. Let $\varphi : L \to \mathbb{Q}$ be defined by induction on $n \in \omega$ as follows. Having defined $\varphi(x_m)$ for all $m < n$, let $\mu_0(n) = \max(\{\varphi(x_m) : m < n, x_m < x_n\} \cup \{-\infty\})$ and $\mu_1(n) = \min(\{\varphi(x_m) : m < n, x_m > x_n\} \cup \{\infty\})$. Then pick $\varphi(x_n)$ inside the interval $(\mu_0(n), \mu_1(n)) \subseteq \mathbb{Q}$, which is possible since $\mathbb{Q}$ is dense. Clearly then φ is an embedding. $\square$

Definition 6.2.28. For a Ramsey space $\mathcal{R}$ such that $\alpha > \omega$ is the least ordinal such that α^* is not $\mathcal{R}$-wqo, we define the class of partial orders

$$\mathfrak{H}(\mathcal{R}) = \{(\mathcal{F}, \lhd') : \mathcal{F} \text{ is a simple front on } \mathcal{R}\}.$$

We say that a quasi-order P can be *weakly embedded* into a quasi-order P' iff there is a function $\varphi : P \to P'$ such that for all $a, b \in P$ with $a < b$ we have $\varphi(a) < \varphi(b)$. In this case we call φ *order preserving.*

We let $\mathfrak{P}(\mathcal{R})$ be the set of well-founded partial orders $P \in \mathfrak{H}(\mathcal{R})$ such that:

- $P = \bigcup_{i\in\omega} P_i$ for some disjoint partial orders P_i.
- For $a, b \in P$, $a < b$ implies $a, b \in P_i$ for some $i \in \omega$ and $a <_{P_i} b$.
- If β_i is the least ordinal such that P_i can be weakly embedded into β_i, then $\gamma = \bigcup_{i\in\omega} \beta_i$ is indecomposable, $\alpha \leqslant \gamma$ and $(\forall i \in \omega)$, $\beta_i < \gamma$.

Theorem 6.2.29. *If $|\mathcal{AR}_1| \leqslant \aleph_0$ and $\alpha > \omega$ is the least ordinal such that α^* is not $\mathcal{R}$-wqo, then Q is $\mathcal{R}$-wqo$_>$ iff no element of $\mathfrak{P}(\mathcal{R})$ weakly embeds into $(Q, \geqslant)$.*

Proof. Suppose that $(\mathcal{F}, \lhd') \in \mathfrak{P}(Q)$ weakly embeds into $(Q, \geqslant)$, so there is some order preserving $f : \mathcal{F} \to Q$. If $a, b \in \mathcal{F}$ with $a \lhd b$ then $a \lhd' b$ and $a \neq b$ so $f(a) > f(b)$, and thus f is bad$_>$.

Now suppose that $f : \mathcal{F} \to Q$ is bad. Since $|\mathcal{AR}_1| \leqslant \aleph_0$, we have that range$(f)$ is countable, so pick a countable linear extension of range(f), and embed this into $\mathbb{Q}$ using Lemma 6.2.27. Composing embedding then gives a bad $g : \mathcal{F} \to \mathbb{Q}$, and hence by Lemma 6.2.20 there is some $A \in \mathcal{R}$ such that $\mathcal{F}|A$ is well-founded with respect to $\lhd$.

Thus it is possible to weakly embed $(\mathcal{F}|A, \lhd')$ into some least ordinal γ. Without loss of generality (renaming γ and A if necessary), let γ be least such that there is some $B \leqslant A$ such that $(\mathcal{F}|B, \lhd')$ weakly embeds into γ. If $\gamma < \alpha$ then the given order preserving map witnesses that γ^* is not $\mathcal{R}$-wqo, which contradicts that α was minimal. Now by applying Lemma 6.2.24 we see that γ is indecomposable, since otherwise γ would not be least.

Let $\beta_i < \gamma$ $(i \in \omega)$ be such that $\gamma = \bigcup_{i\in\omega} \beta_i$. Since $\sum_{i\in \mathrm{A}_{\aleph_0}} \beta_i^*$ embeds every β_i^* $(i \in \omega)$ and $f : \mathcal{F}|A \to \gamma^*$ is bad, we have by Lemma 6.2.26 that there is some $B \leqslant A$ and a bad$_>$ function

$$g : \mathcal{F}|B \to \sum_{i\in \mathrm{A}_{\aleph_0}} \beta_i^*.$$

Partition $\mathcal{F}|B$ into maximal disconnected components, i.e. $\mathcal{F}|B = \bigcup_{i\in\omega} \mathcal{F}_i$ for some disjoint $\mathcal{F}_i \subseteq \mathcal{F}$ $(i \in \omega)$ such that $a, b \in \mathcal{F}|B$ and $a \lhd b$ implies $a, b \in \mathcal{F}_i$ for some $i \in \omega$, and there is no finer partition satisfying this condition. Let δ_i $(i \in \omega)$ be the least ordinal such that there is an order preserving map from $(\mathcal{F}_i, \lhd')$ into δ_i.

Since γ was least, there is an order preserving map from $\mathcal{F}|B$ to γ. Thus, either for some $i \in \omega$ we have $\delta_i = \gamma$ or $\bigcup_{i\in\omega} \delta_i = \gamma$ with each $\delta_i < \gamma$ $(i \in \omega)$. However if some $\delta_i = \gamma$ then this contradicts that g was bad, since γ^* does not weakly embed into $\sum_{i\in \mathrm{A}_{\aleph_0}} \beta_i^*$.

Therefore, $(\mathcal{F}|B, \lhd') \in \mathfrak{P}(\mathcal{R})$ and finally $f : \mathcal{F}|B \to Q$ is bad, hence f is an order preserving map from $(\mathcal{F}|B, \lhd')$ into $(Q, \geqslant)$. □

Theorem 6.2.30. *For any Ramsey space $\mathcal{R}$ such that $|\mathcal{AR}_1| \leqslant \aleph_0$, either:*

- *Every simple front on $\mathcal{R}$ has loops and every quasi-order is $\mathcal{R}$-wqo$_>$.*
- *There is a simple front on $\mathcal{R}$ with no loops and Q is $\mathcal{R}$-wqo$_>$ iff Q is well-founded.*
- *There is a simple front on $\mathcal{R}$ with no loops and some indecomposable ordinal $\alpha > \omega$ least such that α^* is not $\mathcal{R}$-wqo. Furthermore, Q is $\mathcal{R}$-wqo$_>$ iff no element of $\mathfrak{P}(\mathcal{R})$ weakly embeds into $(Q, \geqslant)$.*

Proof. Suppose every simple front $\mathcal{F}$ has loops, in which case by Lemma 6.2.11 there are no simple bad$_>$ functions to any quasi-order. Hence every quasi-order is $\mathcal{R}$-wqo$_>$.

Now suppose there is a simple front $\mathcal{F}$ on $\mathcal{R}$ with no loops. By Corollary 6.2.19, the linear order $(\mathcal{F}, \preceq_{\mathcal{F}})$ is not $\mathcal{R}$-wqo$_>$. Since $|\mathcal{AR}_1| \leqslant \aleph_0$ we also know that $\mathcal{F}$ is countable. Hence by Lemma 6.2.27 there is a bad$_>$ function to $\mathbb{Q}$. Therefore by Theorem 6.2.21 there is a bad function to some α^*, for some countable ordinal α. Suppose without loss of generality that α is least such that α^* is not $\mathcal{R}$-wqo.

If Q is a quasi-order that is not $\mathcal{R}$-wqo$_>$ then by Lemma 6.2.17 Q must have an infinite descending chain. Thus if $\alpha = \omega$ then Q is $\mathcal{R}$-wqo$_>$ iff Q is well-founded.

If $\alpha > \omega$ then apply Theorem 6.2.29 to see that Q is $\mathcal{R}$-wqo$_>$ iff no element of $\mathfrak{P}(\mathcal{R})$ weakly embeds into $(Q, \geqslant)$. $\square$

6.2.4 Possible notions of $\mathcal{R}$-wqo

We have thus narrowed down the possible versions of $\mathcal{R}$-wqo for various Ramsey spaces $\mathcal{R}$ that satisfy $|\mathcal{AR}_1| = \aleph_0$. Theorems 6.2.16 and 6.2.30 allow us to classify the possibilities into seven types:

1. Q is $\mathcal{R}$-wqo whenever Q is any quasi-order.

2. $\mathfrak{G}(\mathcal{R}) \neq \emptyset$ and Q is $\mathcal{R}$-wqo iff $\mathfrak{G}(\mathcal{R})$ contains an element that is isomorphic to a subgraph of $\langle Q, \perp \rangle$.

3. Q is $\mathcal{R}$-wqo iff Q is narrow.

4. $\mathfrak{G}(\mathcal{R}) \neq \emptyset$ and Q is $\mathcal{R}$-wqo iff Q is well-founded and $\mathfrak{G}(\mathcal{R})$ contains an element that is isomorphic to a subgraph of $\langle Q, \perp \rangle$.

5. Q is $\mathcal{R}$-wqo iff Q is well-founded and narrow.

6. $\mathfrak{G}(\mathcal{R}) \neq \emptyset$ and there is some indecomposable ordinal $\alpha > \omega$ least such that α^* is not $\mathcal{R}$-wqo. Furthermore Q is $\mathcal{R}$-wqo iff no element of $\mathfrak{P}(\mathcal{R})$ weakly embeds into $(Q, \geqslant)$ and $\mathfrak{G}(\mathcal{R})$ contains an element that is isomorphic to a subgraph of $\langle Q, \perp \rangle$.

7. There is some indecomposable ordinal $\alpha > \omega$ least such that α^* is not $\mathcal{R}$-wqo, and Q is $\mathcal{R}$-wqo iff no element of $\mathfrak{P}(\mathcal{R})$ weakly embeds into $(Q, \geqslant)$ and Q is narrow.

The reasoning that results in this classification is as follows. Fix some arbitrary Ramsey space $\mathcal{R}$ and let Q be a quasi-order. Suppose that every front on $\mathcal{R}$ has loops, so by Lemma 6.2.11 and Proposition 6.2.5 we have that Q is $\mathcal{R}$-wqo iff Q is $\mathcal{R}$-wqo$_\perp$. Thus Theorem 6.2.16 gives us cases 1, 2 and 3. Now suppose that there is a simple front on $\mathcal{R}$ with no loops and we note that by Proposition 6.2.5, Q is $\mathcal{R}$-wqo iff Q is $\mathcal{R}$-wqo$_>$ and

$\mathcal{R}$-wqo$_\perp$. Suppose that Q is $\mathcal{R}$-wqo$_>$ iff Q is well-founded, thus Theorem 6.2.16 gives us cases 4 and 5. Finally by Theorem 6.2.30, the only remaining case is that there is some indecomposable ordinal $\alpha > \omega$ that is least such that α^* is not $\mathcal{R}$-wqo, and Theorem 6.2.16 gives us cases 6 and 7.

Examples of Ramsey spaces $\mathcal{R}$ with exotic[4] $\mathcal{R}$-wqo seem to be hard to find. Indeed, for all of the examples of topological Ramsey spaces $\mathcal{R}$ given in [56] (with a 'natural' shift), we have that $\mathcal{R}$-wqo is equivalent to wqo (i.e. is of type 5). This can be seen similarly to examples 6.2.3 and 6.2.4.

Thus, it seems likely that this classification can be refined further. In the next subsection we will see an example of a Ramsey space $\mathcal{H}$, which falls into type 4. However $\mathcal{H}$-wqo turns out to be equivalent to $\mathbb{N}^{[\infty]}$-bqo$_2$ (see Definition 6.3.1). Similarly for every $k \in \omega$ we could define Ramsey spaces whose version of wqo is equivalent to $\mathbb{N}^{[\infty]}$-bqo$_k$, however this gives no new functionality and added complexity. Thus we ask the following question.

Question 6.2.31. *Is there a Ramsey space $\mathcal{R}$ such that $\mathcal{R}$-wqo is neither of type 1 nor equivalent to $\mathbb{N}^{[\infty]}$-bqo$_k$ for any $k \in \omega$?*

Furthermore, despite numerous fruitless attempts, we have not found any Ramsey space of type other than 1, 4 and 5. So we also ask the following question.

Question 6.2.32. *Is there a Ramsey space of type other than 1, 4 and 5?*

More generally, we ask which further conditions can be added to the definitions of the sets $\mathfrak{G}(\mathcal{R})$ and $\mathfrak{P}(\mathcal{R})$ for different Ramsey spaces $\mathcal{R}$ that could give more precise characterisations of $\mathcal{R}$-wqo.

6.2.5 Examples

We now give some examples to differentiate some of the different cases.

Example 6.2.33. Firstly we consider the most trivial Ramsey space possible. Let $X = \langle 1, 1, 1, 1, ... \rangle$ and r_n be the usual restriction, then $(\{X\}, =, r)$ is a Ramsey space, all of the

[4] I.e. of a type other than 1 and 5.

axioms are satisfied trivially. The only choice of shift possible is $X^+ = X$ and therefore Q is $\mathcal{R}$-wqo for every quasi-order Q. Hence this extremely trivial Ramsey space falls into type 1.

Example 6.2.34. The Ellentuck space $(\mathbb{N}^{[\infty]}, \subseteq, r)$ with $X^+ = X \backslash \min(X)$ and $(r_n(X))^+ = r_{n-1}(X^+)$ gives the usual notion of wqo by Theorem 6.2.6. Hence this Ramsey space falls into type 3.

Definition 6.2.35. Let $\mathcal{H}$ be the set of sequences of pairs of natural numbers $(\langle n_i, n_{i+1}\rangle)_{i\in\omega}$ such that for all $i \in \omega$, $n_i < n_{i+1}$.

For all $(\langle n_i, n_{i+1}\rangle)_{i\in\omega}, (\langle m_i, m_{i+1}\rangle)_{i\in\omega} \in \mathcal{H}$, we let $(\langle n_i, n_{i+1}\rangle)_{i\in\omega} \leqslant (\langle m_i, m_{i+1}\rangle)_{i\in\omega}$ iff

$$\{n_i : i \in \omega\} \subseteq \{m_i : i \in \omega\}.$$

We also define

$$r_m((\langle n_i, n_{i+1}\rangle)_{i\in\omega}) = (\langle n_i, n_{i+1}\rangle)_{i<m}.$$

We define the shift on $\mathcal{H}$ as by letting $(\langle n_i, n_{i+1}\rangle)^+_{i\in\omega} = (\langle n_{i+1}, n_{i+2}\rangle)_{i\in\omega}$ and for all $m \in \omega$, $(\langle n_i, n_{i+1}\rangle)^+_{i<m} = (\langle n_{i+1}, n_{i+2}\rangle)_{i<m-1}$.

Theorem 6.2.36. *$\langle \mathcal{H}, \leqslant, r\rangle$ is a Ramsey space.*

Proof. A1 and A2 are easily verified. For A3(1), suppose that $A \in [\text{depth}_B(a), B]$. Let $b = r_{\text{depth}_B(a)}(B)$ now clearly there is an element $A' \in \mathcal{R}$ containing only numbers that are in A but not in $b \setminus a$, so that $A' \in [a, A]$.

For A3(2), suppose that $A \leqslant B$ and $[a, A] \neq \emptyset$. Let $b = r_{\text{depth}_B(a)}(B)$ and now consider the unique sequence A' in $\mathcal{R}$ containing every number in A and b. Then $b \sqsubset A' \leqslant B$ and $\emptyset \neq [a, A'] \subseteq [a, A]$ as required.

For A4, suppose that $\text{depth}_B(a) < \infty$ and $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$. Consider

$$O = \{\langle n, m\rangle : a^\frown\langle\langle n, m\rangle\rangle \in r_{|a|+1}[a, B] \cap \mathcal{O}\}$$

and

$$O^\complement = \{\langle n, m\rangle : a^\frown\langle\langle n, m\rangle\rangle \in r_{|a|+1}[a, B] \cap \mathcal{O}^\complement\}$$

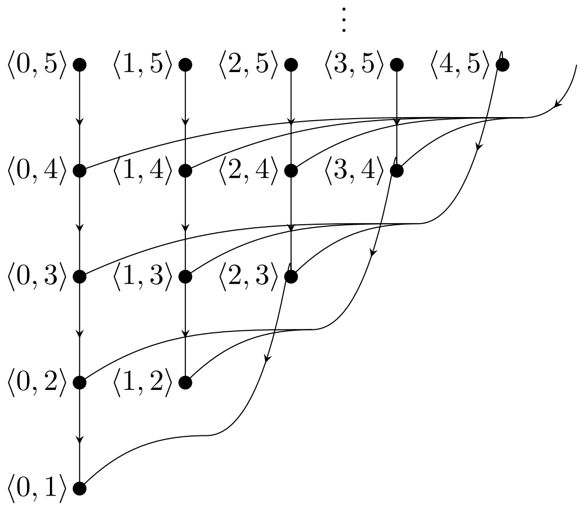

Figure 6.4: Rado's poset $\mathfrak{R}$.

If $\langle n, m\rangle \in O \cup O^{\complement}$ then n must be equal to the last element of the last pair of a, we call this number n_0. Now at least one of the sets $M = \{m : \langle n_0, m\rangle \in O\}$ or $M^{\complement} = \{m : \langle n_0, m\rangle \in O^{\complement}\}$ is infinite. Suppose without loss of generality that M is infinite. Then consider the unique $A \in \mathcal{R}$ whose members contain numbers that are either in M or in $r_{\mathrm{depth}_B(a)}(B)$. Thus $r_{|a|+1}[a, A] = \{a^\frown\langle\langle n_0, m\rangle\rangle : m \in M\} \subseteq \mathcal{O}$ as required. □

Definition 6.2.37 (Rado's poset [47]). Let $\mathfrak{R}$ be the the partial order consisting of pairs $\langle n, m\rangle \in \omega \times \omega$ such that $n < m$. We order $\langle n_0, m_0\rangle \leqslant \langle n_1, m_1\rangle$ iff either $n_0 = n_1$ and $m_0 \leqslant m_1$ or $m_0 < n_1$.

Theorem 6.2.38. *$\mathfrak{R}$ is not $\mathcal{H}$-wqo$_\perp$, yet it has no infinite antichain.*

Proof. First observe that $\langle n_0, m_0\rangle \perp \langle n_1, m_1\rangle$ iff

- $n_0 \neq n_1$,
- $m_0 \geqslant n_1$,
- $m_1 \geqslant n_0$.

Thus $\mathfrak{R}$ has no infinite antichain since any antichain has only boundedly many possible first elements by the second two conditions, and therefore only boundedly many elements by the first condition.

Now let $\mathcal{F}$ be a simple front on $A \in \mathcal{H}$. Then

$$\mathcal{F} = \{\langle\langle n, m\rangle\rangle : (\exists n', m' \in \omega), \langle n, n'\rangle \in A, \langle m, m'\rangle \in A, n < m\}.$$

Let $f : \mathcal{F} \to \mathfrak{R}$ be defined by $f(\langle\langle n, m\rangle\rangle) = \langle n, m\rangle$. We claim that f is bad$_\perp$. Suppose that $\langle\langle n_0, m_0\rangle\rangle \lhd \langle\langle n_1, m_1\rangle\rangle$, then $m_0 = n_1$, which implies that $\langle m_0, n_0\rangle \perp \langle m_1, n_1\rangle$. $\square$

Theorem 6.2.39. *Q is $\mathcal{H}$-wqo iff there is no bad $f : \mathcal{F} \to Q$ where $\mathcal{F}$ is a front on $\mathbb{N}^{[\infty]}$ of rank 2.*

Proof. Suppose that Q is not $\mathcal{H}$-wqo, so let $f : \mathcal{G} \to Q$ be bad with $\mathcal{G}$ a simple front on some $A = \langle\langle a_0, a_1\rangle, \langle a_1, a_2\rangle, ...\rangle \in \mathcal{H}$. Let $A' = \langle a_i : i \in \omega\rangle \in \mathbb{N}^{[\infty]}$, so that

$$\mathcal{G} = \{\langle\langle n, m\rangle\rangle : n, m \in A', n < m\}.$$

Let $\mathcal{F} = \{\langle n, m\rangle : n, m \in A', n < m\}$, thus $\mathcal{F}$ is a front on A' of rank 2. Let $g : \mathcal{F} \to \mathcal{G}$ be given by $g(\langle n, m\rangle) = \langle\langle n, m\rangle\rangle$ for all $n, m \in A'$, $n < m$. Therefore $\langle n, m\rangle \lhd \langle k, l\rangle$ iff $g(\langle n, m\rangle) \lhd g(\langle k, l\rangle)$ and therefore $f \circ g$ is bad.

Now let $f : \mathcal{F} \to Q$ where $\mathcal{F}$ is a front on $A \in \mathbb{N}^{[\infty]}$ of rank 2. Let $\mathcal{F}'$ be the set of all length 2 increasing sequences of elements of A, then $\mathcal{F}'$ is a rank 2 front on A and $f' : \mathcal{F}' \to Q$ so that for all $a \in \mathcal{F}'$, $f'(a) = f(c)$ where $c \in \mathcal{F}$ is such that $c \sqsubseteq a$. Now let $\mathcal{G} = \{\langle a\rangle : a \in \mathcal{F}'\}$ so $\mathcal{G}$ is a rank 1 front of $\mathcal{H}$. Set $g : \mathcal{G} \to \mathcal{F}'$ such that $g(\langle a\rangle) = a$, so since $\langle a\rangle \lhd \langle b\rangle$ whenever $a \lhd b$, we have that $g \circ f'$ is bad, and Q is not $\mathcal{H}$-wqo. $\square$

We mention that $\mathcal{H}$ falls into type 4. To see this, notice that no front on $\mathcal{H}$ has loops and ω^* is not $\mathcal{H}$-wqo, because $f : \{\langle n, m\rangle : n < m < \omega\} \to \omega^*$ given by $f(\langle n, m\rangle) = n$ is bad. Thus by Theorem 6.2.30, Q is $\mathcal{H}$-wqo$_>$ iff Q is well-founded. Furthermore, since $\mathfrak{R}$ is not $\mathcal{H}$-wqo and has no infinite antichain, thus $\mathcal{H}$ is not of type 5, and therefore must be of type 4.

6.3 Abstract better-quasi-orders

Definition 6.3.1. Q is *$\mathcal{R}$-better-quasi-ordered* or *$\mathcal{R}$-bqo* iff there is no bad function $f : \mathcal{F} \to Q$, for any front $\mathcal{F}$ on $\mathcal{R}$. Let α be an ordinal, if $\mathrm{rank}(\mathcal{F}) = \alpha$ then we call $f : \mathcal{F} \to Q$ *rank* α. We say that Q is $\mathcal{R}$-bqo$_\alpha$ iff there is no rank α bad function $f : \mathcal{F} \to Q$, for any front $\mathcal{F}$ on $\mathcal{R}$.

We now have two aims, firstly we will show that (just as with the usual notion of bqo), for a general Ramsey space $\mathcal{R}$, the notion of $\mathcal{R}$-bqo is closed under taking iterated power sets as in Definition 2.1.11. I.e. if Q is $\mathcal{R}$-wqo, then $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-wqo for every ordinal α. Secondly, we aim to show that given a Ramsey space $\mathcal{R}$ its corresponding notion of $\mathcal{R}$-bqo is determined by its notion of $\mathcal{R}$-wqo, however for this we require that the shift on $\mathcal{R}$ has an extra property.

Proposition 6.3.2. *Q is $\mathcal{R}$-bqo iff for every ordinal α, Q is $\mathcal{R}$-bqo$_\alpha$.*

Proof. This follows easily since $\mathrm{rank}(\mathcal{F}) \in \mathrm{On}$ is well-defined for every front $\mathcal{F}$ of $\mathcal{R}$. ☐

Proposition 6.3.3. *Q is $\mathcal{R}$-wqo iff Q is $\mathcal{R}$-bqo$_1$.*

Proof. This follows trivially since a front $\mathcal{F}$ is simple iff $\mathrm{rank}(\mathcal{F}) = 1$. ☐

Proposition 6.3.4. *If $\alpha < \beta$ then Q is $\mathcal{R}$-bqo$_\beta$ implies Q is $\mathcal{R}$-bqo$_\alpha$.*

Proof. Let $\mathcal{F}$ be a front on some $A \in \mathcal{R}$ such that $\mathrm{rank}(\mathcal{F}) = \alpha$. Then by extending of elements of $\mathcal{F}$ we can find a front $\mathcal{G}$ such that $(\forall a \in \mathcal{G})(\exists b \in \mathcal{F})$, $b \sqsubseteq a$. This b must be unique since $\mathcal{F}$ is Nash-Williams, so define $g(a) = b$ in this case. Suppose there were a bad $f : \mathcal{F} \to Q$, and then let $h : \mathcal{G} \to Q$ be such that $h = f \circ g$, then h is bad and rank β. ☐

Theorem 6.3.5. *If Q is $\mathcal{R}$-bqo$_\alpha$ for any ordinal α then it is $\mathcal{R}$-wqo.*

Proof. By the two previous propositions. ☐

6.3.1 Closure under power sets

Theorem 6.3.6. *If Q is $\mathcal{R}$-bqo then so is $\mathcal{P}(Q)$.*

Proof. Suppose $\mathcal{P}(Q)$ is not $\mathcal{R}$-bqo. So let $f : \mathcal{F} \to \mathcal{P}(Q)$ be bad, with $\mathcal{F}$ a front on some $A \in \mathcal{R}$. If $X, Y \in \mathcal{P}(Q)$ are such that $X \not\leqslant Y$ then there is no function $X \to Y$ that increases every element with respect to the order on Q. In other words, there must be an element $x \in X$ such that for all $y \in Y$, $x \not\leqslant y$.

Let $g : \mathcal{P}(Q) \times \mathcal{P}(Q) \to Q$ be any function so that $g(X, Y) = x$ for such an x. So $g(X, Y) \in X$ and $\forall y \in Y$, $g(X, Y) \not\leqslant y$. Let $h : \mathcal{F}^2 \to Q$ be given by

$$h(a) = g(f(\pi_0(a)), f(\pi_1(a))).$$

We claim that h is bad. Let $s, t \in \mathcal{F}^2$ with $s \lhd t$, so that $s = s_0 \cup^*_S s_1$, $t = t_0 \cup^*_T t_1$ for some $s_0, s_1, t_0, t_1 \in \mathcal{F}$. We have that $s \lhd t$ implies either $s^+ \sqsubseteq t$ or $t \sqsubset s^+$ by Lemma 6.1.19. We also know that $s_2 \sqsubseteq s^+$ and $t_1 \sqsubseteq t$, thus s_2 and t_1 are $\sqsubseteq$-comparable, and because they are both members of $\mathcal{F}$, we have that $s_2 = t_1$.

We have now that $h(t) \in h(t_1) = h(s_2)$ and for all $q \in h(s_2)$ we have $h(s) \not\leqslant q$. Therefore $h(s) \not\leqslant h(t)$ and thus h is bad since s and t were arbitrary. Therefore Q is not $\mathcal{R}$-bqo. $\square$

Lemma 6.3.7. *Let $\mathcal{F}$ be a front on $A \in \mathcal{R}$ and let $C \subseteq \mathcal{F}$. Then define*

$$\mathcal{F}^C = C \cup \{x \cup^*_Z y : x \in \mathcal{F} \setminus C, y \in \mathcal{F}, Z \leqslant A, x \lhd_Z y\}.$$

Then $\mathcal{F}^C$ is a front on A.

Proof. Let $a, b \in \mathcal{F}^C$ and suppose that $a \sqsubset b$.

- If $a, b \in C$ then $a, b \in \mathcal{F}$ which contradicts that $\mathcal{F}$ is Nash-Williams.
- If $a \in C$, $b \in \mathcal{F}^C \setminus C$ then $a \sqsubset b_0 \cup^*_Z b_1 = b$, so a and b_0 are $\sqsubseteq$-comparable and therefore $a = b_0$ since both are members of the front $\mathcal{F}$. But $a \in C$ and $b_0 \in \mathcal{F} \setminus C$, which is a contradiction.

- If $a \in \mathcal{F}^C \setminus C$, $b \in C$ then $a_0 \sqsubset a = a_0 \cup^*_Z a_1 \sqsubset b$ for some $a_0, a_1 \in \mathcal{F}$. But then $a_0 \sqsubset b$ and $a_0, b \in \mathcal{F}$, contradicts that $\mathcal{F}$ is Nash-Williams.

- If $a, b \in \mathcal{F}^C \setminus C$ then a contradiction follows by the same argument as for $\mathcal{F}^2$, Lemma 6.1.26.

So $\mathcal{F}^C$ is Nash-Williams. Now let $S \leqslant A$. Choose $a, b \in \mathcal{F}$ such that $a \sqsubset S$ and $b \sqsubset S^+$, which is possible since $\mathcal{F}$ is a front. If $a \in C$ then $a \in \mathcal{F}^C$. If not, then we know that $a \lhd_S b$ and so $a \cup^*_S b \in \mathcal{F}^C$. Therefore in either case there is an initial segment of S contained in $\mathcal{F}^C$. $\square$

The following proof is essentially due to Shelah [49] but has been modified for use in this abstract context.

Theorem 6.3.8. *If Q is $\mathcal{R}$-bqo, then so is $\mathcal{P}_\alpha(Q)$ for any ordinal α.*

Proof. Suppose that Q is $\mathcal{R}$-bqo and $\mathcal{P}_\alpha(Q)$ is not. So let $f : \mathcal{F} \to \mathcal{P}_\alpha(Q)$ be bad, with $\mathcal{F}$ a front on $A \in \mathcal{R}$. We will find a contradiction by constructing a bad function to Q.

We will define by induction on $n \in \omega$, fronts $\mathcal{F}_n$ on A and bad $f_n : \mathcal{F}_n \to \mathcal{P}_\alpha(Q)$ as follows. Let $\mathcal{F}_0 = \mathcal{F}$, $f_0 = f$, and for $n \in \omega$ define $C_n = \{a \in \mathcal{F}_n : f_n(x) \in Q\}$ and set $\mathcal{F}_{n+1} = \mathcal{F}_n^{C_n}$. So for $a, b \in \mathcal{F}_n$ with $a \lhd b$ and $f(a) \notin Q$ there is some $Z \leqslant A$ such that $a \cup^*_Z b \in \mathcal{F}_{n+1}$. It remains now to define f_{n+1}.

Let $g : \mathcal{P}_\alpha(Q) \times \mathcal{P}_\alpha(Q) \to \mathcal{P}_\alpha(Q)$ be such that for $U, V \in \mathcal{P}_\alpha(Q)$, if $U \not\leqslant V$ then, similarly to the proof of Theorem 6.3.6, let $g(U, V)$ be such that $g(U, V) \in U$ and either $V \in Q$ and $g(U, V) \not\leqslant V$ or $V \notin Q$ and $\forall W \in V$ we have $g(U, W) \not\leqslant V$. Now if $f_n(a) \in Q$ then let $f_{n+1}(a) = f_n(a)$, otherwise let

$$f_{n+1}(a \cup^*_X b) = g(f_n(a), f_n(b)).$$

We assumed that $\mathcal{F}_0$ was a front, moreover if $\mathcal{F}_n$ is a front then $\mathcal{F}_{n+1} = \mathcal{F}_n^{C_n}$ is a front by Lemma 6.3.7. So for every $n \in \omega$ we have that $\mathcal{F}_n$ is a front. We now claim that for each $n \in \omega$, f_n is bad. Let $a, b \in \mathcal{F}_{n+1}$ be such that $a \lhd b$, thus for some $Z \leqslant A$ we have $a \lhd_Z b$. We now have the following cases:

- If $a, b \notin C_n$ then $a = a_0 \cup_S^* a_1$ and $b = b_0 \cup_T^* b_1$ for some $a_0, a_1, b_0, b_1 \in \mathcal{F}_n$ and $S, T \leqslant A$. Then we know that $a_1 \sqsubseteq a^+$ and $b_0 \sqsubseteq b$ and that a^+ and b are $\sqsubseteq$-comparable (by Lemma 6.1.19). Therefore a_1 and b_0 are $\sqsubseteq$-comparable, and hence $a_1 = b_0$ because both are members of the front $\mathcal{F}$. So $f_{n+1}(a) = g(f_n(a_0), f_n(a_1))$ and $f_{n+1}(b) = g(f_n(a_1), f_n(b_1))$ so $f_{n+1}(b) \in f_n(a_1)$ and therefore $f_{n+1}(a) \nleqslant f_{n+1}(b)$.

- If $a \notin C_n$ and $b \in C_n$ then $a = a_0 \cup_S^* a_1$ for some $a_0, a_1 \in \mathcal{F}_n$ and $S \leqslant A$. We know that $a \sqsubseteq Z$ so $a_1 \sqsubseteq a^+ \sqsubseteq Z^+$. Hence a_2 and b are $\sqsubseteq$-comparable and so $a_2 = b$. So $f_{n+1}(a) = g(f_n(a_0), f_n(a_1))$ and $f_{n+1}(a) \nleqslant f_n(b) = f_{n+1}(b)$.

- If $a \in C_n$ and $b \notin C_n$ then $b = b_0 \cup_T^* b_1$ for some $b_0, b_1 \in \mathcal{F}_n$ and $T \leqslant A$. Now $f_{n+1}(a) = f_n(a) \nleqslant f_n(b_0)$ no element of $f_n(b_0)$ can be larger than $f_{n+1}(a)$. Therefore $f_{n+1}(a) \nleqslant f_{n+1}(b)$.

- If $a, b \in C_n$ then $f_{n+1}(a) = f_n(a) \nleqslant f_n(b) = f_{n+1}(b)$.

So we see that f_n is bad for every $n \in \omega$.

We see from the definitions of $\mathcal{F}_{n+1}$ and f_{n+1} that for any $n \leqslant m \leqslant \omega$, if $x \in \mathcal{F}_n$ is such that $f_n(x) \in Q$ then $x \in \mathcal{F}_m$ and $f_m(x) = f_n(x)$. So we can define

$$\mathcal{F}^* = \{x : (\exists n \in \omega), x \in \mathcal{F}_n, f_n(x) \in Q\}.$$

We also define $f^*(x) = f_n(x)$ whenever $f_n(x) \in Q$. We claim that $\mathcal{F}^*$ is a front. Suppose $a, b \in \mathcal{F}^*$ are such that $a \sqsubset b$. Then $a \in \mathcal{F}_n$ and $b \in \mathcal{F}_m$ and $f_n(a) \in Q$, $f_m(b) \in Q$ so that $a, b \in \mathcal{F}_{\max\{n,m\}}$ and therefore $\mathcal{F}_{\max\{n,m\}}$ is not Nash-Williams. This contradiction shows that $\mathcal{F}^*$ is Nash-Williams.

Now let $S \leqslant A$ and $\mathcal{F}_n(S)$ be the unique initial segment of S contained in $\mathcal{F}_n$. If there were no initial segment of S in $\mathcal{F}^*$, then for every $k \in \omega$ we have $f_k(\mathcal{F}_k(S)) \notin Q$. So we have that $f_{n+1}(\mathcal{F}_{n+1}(S)) = g(f_n(a), f_n(b))$ for $a = \pi_0(\mathcal{F}_{n+1}(S)) \in \mathcal{F}_n$, $b \in \mathcal{F}_n$. We know that $a \sqsubseteq \mathcal{F}_{n+1}(S)$ so that $a = \mathcal{F}_n(S)$. Now by definition of g, we have that

$$f_{n+1}(\mathcal{F}_{n+1}(S)) \in f_n(a) = f_n(\mathcal{F}_n(S)).$$

Hence $(f_n(\mathcal{F}_n(S)))_{n\in\omega}$ is a sequence which contradicts the well-foundedness of $\in$. So we conclude that $\mathcal{F}^*$ is a front.

Finally we show that $f^* : \mathcal{F}^* \to Q$ is bad. Let $a, b \in \mathcal{F}^*$ be such that $a \lhd b$, so there are $n, m \in \omega$ such that $a \in \mathcal{F}_n$, $b \in \mathcal{F}_m$. Therefore $a, b \in \mathcal{F}_{\max\{n,m\}}$ and

$$f^*(a) = f_{\max\{n,m\}}(a) \not\leqslant f_{\max\{n,m\}}(b) = f^*(b).$$

So indeed f^* is bad and clearly the image of f^* is contained in Q. So Q is not $\mathcal{R}$-bqo, which is a contradiction. $\square$

6.3.2 Strong shifts

Definition 6.3.9. Given a Ramsey space $\mathcal{R}$ we say that a shift map $\cdot^+$ is *strong* iff for every $a, b, c \in \mathcal{AR}$ and $A \in \mathcal{R}$ with $|a| \leqslant |b|$, $a \lhd_X b$ and $b \sqsubseteq c \leqslant A$ we have that $a \sqsubset a \cup_X^* b \lhd_Y c$ for some $Y \leqslant A$.

Example 6.3.10. The usual shift on $\mathbb{N}^{[\infty]}$ is strong. If $|a| \leqslant |b|$ and $a \lhd_X b$ then if $X = \langle x_0, x_1, ...\rangle$ we have $a = \langle x_0, ..., x_{|a|-1}\rangle$ and $b = \langle x_1, ..., x_{|b|-1}\rangle$. We then have $c = \langle x_1, ..., x_{|b|-1}, y_0, ..., y_n\rangle$ for some $n \in \omega$. So that $a \cup_X^* b = \langle x_0, ..., x_{|b|-1}\rangle \lhd_Y c$ with $Y = \langle x_0, ..., x_{|b|-1}, y_0, ..., y_n, y_{n+1}, ...\rangle$ for some choice of $y_{n+1} < y_{n+2} < ...$ from A, which can be easily found.

Example 6.3.11. Consider the shift on $\mathbb{N}^{[\infty]}$ which removes the second element of any sequence and leaves the first alone. Then this is not a strong shift since for any $a, b \in \mathcal{AR}$ both of length one, we have $a \lhd_X b$ iff $a = b = \langle x_0\rangle$ and therefore $a = a \cup_X^* b$ so $a \not\sqsubset a \cup_X^* b$.

Theorem 6.3.12. *If the shift on $\mathcal{R}$ is strong and if Q is such that $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-wqo for every ordinal α, then Q is $\mathcal{R}$-bqo.*

Proof. Suppose for every ordinal α, that $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-wqo and the shift on $\mathcal{R}$ is strong. Suppose for contradiction that Q is not $\mathcal{R}$-bqo, so there is a bad $f : \mathcal{F} \to Q$ for some front $\mathcal{F}$ on $A \in \mathcal{R}$. Using the abstract Nash-Williams Theorem 6.1.14, without loss of generality we can assume that for any $a, b \in \mathcal{F}$ with $a \lhd b$ we have $|a| \leqslant |b|$ (otherwise we could restrict so that $a \lhd b$ implies $|a| > |b|$ and thus the lengths of the initial segments of $A, A^+, A^{++}, ...$ in $\mathcal{F}$ would be an infinite descending sequence of natural numbers).

Now define a function $h : \overline{\mathcal{F}} \to \mathcal{P}_\infty(Q)$ recursively by rank as follows. Let $h \restriction \mathcal{F} = f$, then for all $a \in \overline{\mathcal{F}} \setminus \mathcal{F}$ let

$$h(a) = \{h(b) : b \in \overline{\mathcal{F}}, |b| = |a| + 1, a \sqsubset b\}.$$

For $n \in \omega$ let $\mathcal{F}_n = \{a : (\exists b \in \mathcal{F}), a \sqsubset b, |a| = n\} \cup \{c \in \mathcal{F} : |c| \leqslant n\}$. Now we have that $h \restriction \mathcal{F}_1$ is a map from a front of rank 1 to $\mathcal{P}_\gamma(Q)$ for some ordinal γ.[5] Thus by our assumption, $h \restriction \mathcal{F}_1$ is good. So let $x_0, y_0 \in \mathcal{F}_1$ be such that $x_0 \lhd_{B_0} y_0$ for some $B_0 \leqslant A$ and $h(x_0) \leqslant h(y_0)$.

Now we will define by induction $x_n, y_n \in \mathcal{F}_n$ and $B_n \leqslant A$ such that:

1. $x_n \lhd_{B_n} y_n$,
2. $|x_n| \leqslant |y_n|$,
3. $h(x_n) \leqslant h(y_n)$.

Suppose that $x_n \in \overline{\mathcal{F}} \setminus \mathcal{F}$. Since the shift on $\mathcal{R}$ is strong, for any $a, b \in \mathcal{AR}$ with $a \lhd_X b$ we have $a \sqsubset a \cup^*_X b$ so we can let $x_{n+1} = r_{n+1}(x_n \cup^*_{B_n} y_n)$ which is a member of $\overline{\mathcal{F}}$ since it is an initial segment of $B_n \leqslant A$ and also $x_n \in \overline{\mathcal{F}} \setminus \mathcal{F}$, and $|x_{n+1}| = |x| + 1$. Then since $h(x_n) \leqslant h(y_n)$ either:

1. $\exists g : h(x_n) \to h(y_n)$ such that $\forall q \in h(x_n)$, $q \leqslant g(q)$. So let $y_{n+1} = g \circ h(x_{n+1})$, hence $h(x_{n+1}) \leqslant h(y_{n+1})$ as required.

2. $h(y_n) \in Q$ and $(\forall q \in \mathrm{TC}(h(x_n)))$, $q \leqslant h(y_n)$. So let $y_{n+1} = y_n$. Thus since $x_{n+1} \in x_n$ we have $h(x_{n+1}) \leqslant h(y_{n+1})$ as required.

3. For some $q \in h(y_n)$, $h(x_n) \leqslant q$. So by definition of h we have $q = h(a)$ for some $a \in \mathcal{AR}$ with $y_n \sqsubset a$. Let $y_{n+1} = a$, then we have $h(x_{n+1}) \in h(x_n)$ and therefore $h(x_{n+1}) \leqslant h(x_n) \leqslant h(y_{n+1})$ as required.

[5]Since it is a map to $\mathcal{P}_\infty(Q)$ whose range is a set and is therefore is contained within some $\mathcal{P}_\gamma(Q)$.

Now we have $|x_n| \leqslant |y_n|$ and $x_n \lhd y_n$ and in each case we have $y_n \sqsubseteq y_{n+1} \leqslant A$ therefore since the shift on $\mathcal{R}$ is strong, we have that $x_n \cup^*_{B_n} y_n \lhd_{B_{n+1}} y_{n+1}$ for some $B_{n+1} \leqslant A$. Now since $x_{n+1} \sqsubseteq x_n \cup^*_{B_n} y_n$ we also have $x_{n+1} \lhd_{B_{n+1}} y_{n+1}$ as required.

Now since $x_n \sqsubset x_{n+1}$ we have that $h(x_{n+1}) \in h(x_n)$ therefore if α is least such that $h(x_n) \in \mathcal{P}_\alpha(Q)$ and β is least such that $h(x_{n+1}) \in \mathcal{P}_\beta(Q)$ then $\beta < \alpha$. Therefore we can let m be least such that $h(x_m) \in \mathcal{P}_0(Q) = Q$. Thus $x_m \in \mathcal{F}$ by definition of h, and the induction stops after m stages.

So we have $x_m \lhd y_m$, with $h(x_m) \leqslant h(y_m)$, $|x_m| \leqslant |y_m|$ and $h(x_m) \in Q$. If $h(y_m) \in Q$ then $y_m \in \mathcal{F}$ and hence $f(x_m) \leqslant f(y_m)$ which contradicts that f was bad. Otherwise since $h(x_m) \in Q$ and $h(x_m) \leqslant h(y_m)$ we can find $y_{m+1} \in \overline{\mathcal{F}}$ with $h(x_m) \leqslant h(y_{m+1})$. Repeating this process we can find let $y \in \mathcal{F}$ with $y_m \sqsubseteq y$ and $h(x_m) \leqslant h(y)$. Then $x_m \cup^*_{B_m} y_m \lhd y$ so that $x_m \lhd y$, but then $f(x_m) = h(x_m) \leqslant h(y) = f(y)$, again contradicting that f was bad. $\square$

Theorem 6.3.13. *Suppose that the shift on $\mathcal{R}$ is strong. Then the following are equivalent:*

- *Q is $\mathcal{R}$-bqo,*
- *For every ordinal α, $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-bqo,*
- *For every ordinal α, $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-wqo.*

Proof. By theorems 6.3.8, 6.3.5 and 6.3.12. $\square$

Theorem 6.3.14. *Let $\mathcal{R}$ and $\mathcal{S}$ be Ramsey spaces, suppose that for every quasi-order Q, Q is $\mathcal{R}$-wqo implies Q is $\mathcal{S}$-wqo, and that the shift on $\mathcal{S}$ is strong. Then Q is $\mathcal{R}$-bqo implies Q is $\mathcal{S}$-bqo.*

Proof. If Q is $\mathcal{R}$-bqo, then by Theorem 6.3.8, for every ordinal α, $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-bqo. So $\mathcal{P}_\alpha(Q)$ is $\mathcal{R}$-wqo by Theorem 6.3.5 and therefore $\mathcal{P}_\alpha(Q)$ is $\mathcal{S}$-wqo, by our assumption. So since the shift on $\mathcal{S}$ is strong, by Lemma 6.3.13, Q is $\mathcal{S}$-bqo. $\square$

Corollary 6.3.15. *Let $\mathcal{R}$ and $\mathcal{S}$ be Ramsey spaces, suppose that for every quasi-order Q, Q is $\mathcal{R}$-wqo iff Q is $\mathcal{S}$-wqo, and that the shifts on both $\mathcal{R}$ and $\mathcal{S}$ are strong. Then for every quasi-order Q, Q is $\mathcal{R}$-bqo iff Q is $\mathcal{S}$-bqo.*

Proof. Apply Theorem 6.3.14 twice, in either direction. $\square$

Example 6.3.16. We claim that Q is $\mathbb{N}^{[\infty]}$-bqo iff Q is $\mathrm{FIN}_k^{[\infty]}$-bqo. By examples 6.2.3 and 6.3.10 and Corollary 6.3.15, it remains only to show that the shift on $\mathrm{FIN}_k^{[\infty]}$ is strong.

Let $\mathcal{R} = \mathrm{FIN}_k^{[\infty]}$ and $a, b, c \in \mathcal{AR}$ be such that $|a| \leqslant |b|, a \lhd_X b$ and $b \sqsubseteq c \leqslant A \in \mathcal{R}$. So there are $a_0, ..., a_{|c|} \in \mathrm{FIN}_k$ such that $a = (a_i)_{i<|a|}$, $b = (a_{i+1})_{i<|b|}$ and $c = (a_{i+1})_{i<|c|}$. Pick an infinite block sequence $Y \leqslant A$ of elements of FIN_k such that $\langle a_0 \rangle ^\frown c \sqsubseteq Y$ (clearly this is possible). Then $a \sqsubset \langle a_0, ..., a_{|b|} \rangle = a \cup_X^* b \lhd_Y c$ as required. So indeed the shift on $\mathrm{FIN}_k^{[\infty]}$ is strong, and Q is $\mathbb{N}^{[\infty]}$-bqo iff Q is $\mathrm{FIN}_k^{[\infty]}$-bqo.

Example 6.3.17. We claim that Q is $\mathbb{N}^{[\infty]}$-bqo iff Q is $W_{L_v}^{[\infty]}$-bqo. By examples 6.2.4 and 6.3.10 and Corollary 6.3.15, it remains only to show that the shift on $W_{L_v}^{[\infty]}$ is strong, which can be seen similarly to Example 6.3.16.

Chapter 7

Better-quasi-orders for uncountable cardinals

In this chapter we consider Shelah's notion of κ-bqo [49] for an uncountable cardinal κ. We aim to simplify his definition and give a version of Simpson's definition of bqo (as in [53]) for κ-bqo. This is not so straightforward, the main obstacle is that Shelah uses an unusual property to define his *κ-barriers* which we call the *barrier property*. We must somehow remove this property from the definition, even in the absence of any Ramsey theory at κ.

7.1 Fronts on κ and κ-barriers

Definition 7.1.1. For $A \subseteq \kappa \in \text{Card}$ and $\lambda \leqslant \kappa$ we define $[A]^\lambda = \{X \subseteq A : \text{ot}(A) = \lambda\}$ and $[A]^{<\lambda} = \{X \subseteq A : \text{ot}(X) < \lambda\}$. We equate $X \subseteq A$ with the increasing enumeration of elements of X. For $X \in [A]^\omega$ and $n \in \omega$ we let $X \restriction n$ be the set containing the least n elements of X and write

$$X = \langle X_0, X_1, X_2 ... \rangle.$$

Definition 7.1.2. Fix a cardinal κ and let $A \in [\kappa]^\kappa$ and $\mathcal{F} \subseteq [A]^{<\omega}$. We call $\mathcal{F}$ *Nash-Williams* if $\mathcal{F}$ is an antichain under $\sqsubseteq$. We call $\mathcal{F}$ a *front* on A iff $\mathcal{F}$ is Nash-Williams and for all $X \in [A]^\omega$, there is some $a \in \mathcal{F}$ such that $a \sqsubset X$.

If $\mathcal{F}$ is a front on $A \in [\kappa]^\kappa$, then we define

$$\overline{\mathcal{F}} = \{a \in [A]^{<\omega} : (\exists b \in \mathcal{F}), a \sqsubseteq b\}.$$

Thus $\overline{\mathcal{F}}$ is a well-founded tree under $\sqsubseteq$. We define $\operatorname{rank}(\mathcal{F})$ as the tree rank of $\overline{\mathcal{F}}$. If $a \in \mathcal{F}$ then we also let $\operatorname{rank}_{\mathcal{F}}(a)$ be the rank of a considered as an element of the tree $\overline{\mathcal{F}}$ (see Definition 2.4.3).

If $X \in [A]^\omega$ and $\mathcal{F}$ is a front on A then we define $\mathcal{F}(A)$ as the unique initial segment of A that is contained in $\mathcal{F}$. If $B \in [A]^\kappa$ then define $\mathcal{F}|B = \mathcal{F} \cap [B]^{<\omega}$ and we call $\mathcal{F}|B$ a *restriction* of $\mathcal{F}$.

Definition 7.1.3. If $X \in [\kappa]^{<\kappa}$ then define $X^+ = X \setminus \{\min X\}$. For $a, b \in [\kappa]^{<\omega}$ we define $a \lhd b$ iff either $b \sqsubset a^+$ or $a^+ \sqsubseteq b$ and $\min a < \min b$.

Remark 7.1.4. The relation $\lhd$ may seem unusual. Note that in particular it is not transitive: $\langle 1,2,3\rangle \lhd \langle 2,3\rangle \lhd \langle 3,4,5\rangle$, but it is not the case that $\langle 1,2,3\rangle \lhd \langle 3,4,5\rangle$. The notion of a bad function (and hence of bqo) relies upon traversing infinite sequences by removing their first member $X \longrightarrow X \setminus \{\min X\}$. Intuitively, $\lhd$ is the corresponding traversal that is used when defining bqo in terms of finite sequences and fronts.

Definition 7.1.5 (Shelah [49])**.** We define a *κ-barrier* $\mathcal{B}$ to be a front on $[A]^\omega$, for some $A \subseteq \kappa$ whose order type is κ, with the extra property:

$$(\forall a, b \in \mathcal{B}), b \not\sqsubset a^+.$$

We will refer to this property as the *barrier property.*

It is worth mentioning that the barrier property is implied by the *Sperner property*, i.e. $(\forall a, b \in \mathcal{B}), a \not\subset b$. This is more usually seen in the definition of barriers [56, 39]. With this definition, we can now define what it means to be κ-bqo.

Definition 7.1.6 (Shelah [49])**.** A quasi-order Q is called *κ-bqo* iff there is no function $f : \mathcal{B} \to Q$, for $\mathcal{B}$ a κ-barrier, such that $\forall a, b \in \mathcal{B}$,

$$a \lhd b \longrightarrow f(a) \not\leq f(b).$$

When $\kappa = \omega$, κ-bqo is equivalent to bqo. One direction follows since any front on ω satisfying the Sperner property also satisfies the barrier property. The other direction holds since any κ-barrier can be restricted so that it satisfies the barrier property, using the Galvin and Prikry Theorem 2.1.6, and thus the relevant bad functions can be found by taking their corresponding restriction.

7.2 Simplifying the definition of κ-bqo

We now ask the question as to whether or not the barrier property is necessary in this definition. That is, for an arbitrary quasi-order Q, if we have $A \in [\kappa]^\kappa$, some $\mathcal{F}$ a front on $[A]^\omega$ and a bad function $f : \mathcal{F} \to Q$; then does this imply that Q is not κ-bqo?

When we are able to use the usual Ramsey techniques (for example when κ is a Ramsey cardinal or when $\kappa = \omega$), by taking a restriction of the given front $\mathcal{F}$ we could find some restriction of f whose domain is a barrier, and hence Q is not κ-bqo. This is because by using Ramsey techniques either we can restrict a front $\mathcal{F}$ that satisfies $(\forall a, b \in \mathcal{F}), a \triangleleft b \to b \not\sqsubseteq a^+$ or restrict to a front $\mathcal{F}$ that satisfies $(\forall a, b \in \mathcal{F}), a \triangleleft b \to b \sqsubseteq a^+$. In the first case the barrier property holds, since if not then for some $a, b \in \mathcal{F}$ we have $b \sqsubseteq a^+$; hence $a \triangleleft b$ and thus $b \not\sqsubseteq a^+$, a contradiction. In the second case, picking $a_0 \in \mathcal{F}$ and defining a_{n+1} as the initial segment of a_n^+ in $\mathcal{F}$ makes $(|a_n|)_{n \in \omega}$ an infinite descending sequence of natural numbers. So this case cannot happen and we can always find some restriction satisfying the barrier property. However for general κ, we have no Ramsey theory to work with, and so this method does not work. Thus we will need a more nuanced argument.

We first define the notion of κ-bqo without the barrier property.

Definition 7.2.1. A quasi-order Q is called κ-bqo$'$ iff there is no function $f : \mathcal{F} \to Q$, for $\mathcal{F}$ a front on some $A \in [\kappa]^\kappa$, such that $\forall a, b \in \mathcal{B}$,

$$a \triangleleft b \longrightarrow f(a) \not\leq f(b).$$

To proceed we need to stratify the levels of κ-bqo and κ-bqo$'$ by ranks as follows:

Definition 7.2.2. A quasi-order Q is called κ-bqo$_\alpha$ iff there is no bad function $f : \mathcal{B} \to Q$, for $\mathcal{B}$ a κ-barrier of rank $\leqslant \alpha$.

A quasi-order Q is called κ-bqo$'_\alpha$ iff there is no bad function $f : \mathcal{F} \to Q$, for a front $\mathcal{F}$ on some $A \in [\kappa]^\kappa$ of rank $\leqslant \alpha$.

In order to give a Simpson style definition of κ-bqo we first need to remove the barrier property from our definition of κ-bqo. Thus we aim to prove the following theorem.

Theorem 7.2.3. *For any cardinal κ and any quasi-order Q, we have that Q is κ-bqo iff Q is κ-bqo$'$.*

In order to prove this theorem we first we define for an ordinal $\alpha > 0$,

$$-1+\alpha = \begin{cases} \alpha - 1 & : \alpha < \omega \\ \alpha & : \alpha \geqslant \omega \end{cases}.$$

We then in fact aim to prove the following stronger theorem.

Theorem 7.2.4. *For any cardinal κ, any ordinal $\alpha > 0$ and any quasi-order Q, the following are equivalent:*

1. *Q is κ-bqo$_\alpha$,*

2. *$\mathcal{P}_{-1+\alpha}(Q)$ is κ-bqo$_1$,*

3. *$\mathcal{P}_{-1+\alpha}(Q)$ is κ-bqo$'_1$,*

4. *Q is κ-bqo$'_\alpha$.*

We will prove this theorem in a series of lemmas. Firstly we note that 2 implies 3, since any front of rank 1 trivially satisfies the barrier property, because every element has length 1. We also see that 4 implies 1 because a bad function witnessing the failure of 1 will also witness the failure of 4. It remains to show that 1 implies 2 and that 3 implies 4.

The proof of Theorem 7.2.4 will rely on constructing bad functions across $\overline{\mathcal{F}}$ for some front $\mathcal{F}$ of rank α. In Figure 7.1 elements of $\overline{\mathcal{F}}$ will correspond accordingly to sets in the given part of $\mathcal{P}_\infty(Q)$.

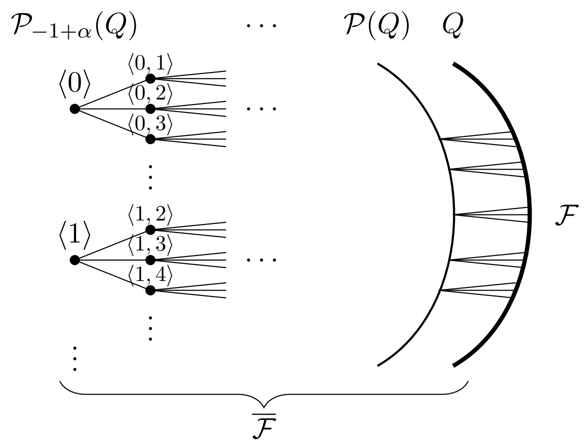

Figure 7.1: The correspondence between elements of $\overline{\mathcal{F}}$ and sets in $\mathcal{P}_\infty(Q)$.

In the next three lemmas we will show that 1 implies 2.

Definition 7.2.5. Let $\mathcal{F}$ be a front on $A \in [\kappa]^\kappa$, and let $C \subseteq \mathcal{F}$ then define

$$\mathcal{F}^C = C \cup \{x \cup y : x \in \mathcal{F} \setminus C, y \in \mathcal{F}, x \lhd y\}.$$

The following lemma is due to Shelah. I present a detailed version of the proof which is sketched in [49].

Lemma 7.2.6 (Shelah [49])**.** *Let $\mathcal{F}$ be a front on $A \in [\kappa]^\kappa$, and let $C \subseteq \mathcal{F}$. Then $\mathcal{F}^C$ is a front on A with rank$(\mathcal{F}^C) \leqslant$ rank$(\mathcal{F}) + 1$. Moreover, if $\mathcal{F}$ is a κ-barrier, then so is $\mathcal{F}^C$.*

Proof. If $\mathcal{F} = \{\langle\rangle\}$ then the lemma holds trivially, so suppose that this is not the case.

Let $x, y \in \mathcal{F}^C$ be such that $x \sqsubset y$. Then either:

- $x, y \in C \subseteq \mathcal{F}$, which contradicts that $\mathcal{F}$ was a front.
- $x \in C, y \in \mathcal{F}^C \setminus C$, so that $y = a \cup b$ for some $a, b \in \mathcal{F}$. Therefore $x \sqsubset a \cup b$ and so x and a are $\sqsubseteq$-comparable, and hence equal since they're both members of the front $\mathcal{F}$. But then since $y \notin C$ it must have been that $a \notin C$, even though $x \in C$, this is clearly a contradiction.
- $x \in \mathcal{F}^C \setminus C$, $y \in C$. Then $x = a \cup b$ for some $a, b \in \mathcal{F}$. So $a \sqsubseteq x \sqsubset y$, which contradicts that $\mathcal{F}$ was a front since $a, y \in \mathcal{F}$.

- $x, y \in \mathcal{F}^C \setminus C$. So there are $a, b, c, d \in \mathcal{F}$ such that $x = a \cup b$, $y = c \cup d$, $a \lhd b$ and $c \lhd d$. So $a \sqsubseteq a \cup b \sqsubset c \cup d$ and $c \sqsubseteq c \cup d$, thus $a = c$ since $\mathcal{F}$ is Nash-Williams. We also have $x^+ \sqsubseteq y^+$ so similarly $b, d \sqsubseteq (c \cup d)^+$ and thus $b = d$. Therefore $x = y$, which contradicts $x \sqsubset y$.

So $\mathcal{F}^C$ is Nash-Williams. We claim that it is a front on A. So let $S \subseteq A$ and $x, y \in \mathcal{F}$ be such that $x \sqsubset S$ and $y \sqsubset S^+$.[1] If $x \in C$ then $x \in \mathcal{F}^C$ so that $\mathcal{F}^C$ contains an initial segment of S. Otherwise $x \lhd y$ so $x \cup y \in \mathcal{F}^C$ and $x \cup y$ is also an initial segment of S. Hence $\mathcal{F}^C$ is a front on A.

To prove $\operatorname{rank}(\mathcal{F}^C) \leqslant \operatorname{rank}(\mathcal{F}) + 1$, we will first prove by induction on $\alpha \in \mathrm{On}$ the following claim.

Claim: For $a, b \in \overline{\mathcal{F}}$ with $a \lhd b$; if $\operatorname{rank}_{\mathcal{F}}(a), \operatorname{rank}_{\mathcal{F}}(b) \leqslant \alpha$, then $\operatorname{rank}_{\mathcal{F}^\emptyset}(a \cup b) \leqslant \alpha$.

Proof of claim: If $\alpha = 0$ then $a, b \in \mathcal{F}$ so $a \cup b \in \mathcal{F}^\emptyset$ and thus $\operatorname{rank}_{\mathcal{F}^\emptyset}(a \cup b) = 0$. Now suppose that $\alpha > 0$ and the claim holds for all $\beta < \alpha$. For all $i \in A \setminus \max(a \cup b)$, let $a_i \sqsubseteq a \cup b^\frown\langle i \rangle$ and $b_i \sqsubseteq (a \cup b^\frown\langle i \rangle)^+$ be longest possible so that also $a_i, b_i \in \overline{\mathcal{F}}$. Now either $\operatorname{rank}_{\mathcal{F}}(a) > \operatorname{rank}_{\mathcal{F}}(a_i)$ or $a = a_i \in \mathcal{F}$ and thus $\operatorname{rank}_{\mathcal{F}}(a_i) = 0$. In either case we have $\operatorname{rank}_{\mathcal{F}}(a_i) < \alpha$. Similarly we have $\operatorname{rank}_{\mathcal{F}}(b_i) < \alpha$ and furthermore $a_i, b_i \in \overline{\mathcal{F}}$ and $a_i \lhd b_i$.

So by the induction hypothesis we have that for all $i \in A \setminus \max(a \cup b)$,

$$\operatorname{rank}_{\mathcal{F}^\emptyset}(a_i \cup b_i) \leqslant \max\{\operatorname{rank}_{\mathcal{F}}(a_i), \operatorname{rank}_{\mathcal{F}}(b_i)\}.$$

If $\operatorname{rank}_{\mathcal{F}^\emptyset}(a \cup b) = 0 < \alpha$ then we are done. If $\operatorname{rank}_{\mathcal{F}^\emptyset}(a \cup b) > 0$ then we have that $a \cup b^\frown\langle i \rangle \in \overline{\mathcal{F}^\emptyset}$ for all $i \in A \setminus \max(a \cup b)$. So there are some $a', b' \in \overline{\mathcal{F}}$ with $a' \lhd b'$ and $a' \cup b' = a \cup b^\frown\langle i \rangle$. Thus $a' \sqsubseteq a \cup b^\frown\langle i \rangle$ and $b' \sqsubseteq (a \cup b^\frown\langle i \rangle)^+$ so since a_i and b_i were the

[1] Note that is possible to find such x and y since $\mathcal{F}$ is a front.

longest satisfying this, it must be that $a_i \cup b_i = a \cup b^\frown\langle i\rangle$. Therefore,

$$\begin{aligned}
\operatorname{rank}_{\mathcal{F}^\emptyset}(a \cup b) &= \bigcup\{\operatorname{rank}_{\mathcal{F}^\emptyset}(a \cup b^\frown\langle i\rangle) + 1 : i \in A \setminus \max(a \cup b)\} \\
&= \bigcup\{\operatorname{rank}_{\mathcal{F}^\emptyset}(a_i \cup b_i) + 1 : i \in A \setminus \max(a \cup b)\} \\
&\leqslant \bigcup\{\max\{\operatorname{rank}_{\mathcal{F}}(a_i), \operatorname{rank}_{\mathcal{F}}(b_i)\} + 1 : i \in A \setminus \max(a \cup b)\} \\
&= \max\{\operatorname{rank}_{\mathcal{F}}(a), \operatorname{rank}_{\mathcal{F}}(b)\} \leqslant \alpha
\end{aligned}$$

This gives the claim. ■

So we have:

$$\begin{aligned}
\operatorname{rank}(\mathcal{F}^C) &\leqslant \operatorname{rank}(\mathcal{F}^\emptyset) \\
&= \operatorname{rank}_{\mathcal{F}^\emptyset}(\langle\rangle) \\
&= \bigcup\{\operatorname{rank}_{\mathcal{F}^\emptyset}(\langle i\rangle) + 1 : i \in A\} \\
&\leqslant \bigcup\{\max\{\operatorname{rank}_{\mathcal{F}}(\langle\rangle), \operatorname{rank}_{\mathcal{F}}(\langle i\rangle)\} + 1 : i \in A\} \\
&= \operatorname{rank}(\mathcal{F}) + 1.
\end{aligned}$$

Finally, suppose that $\mathcal{F}$ has the barrier property and that there are $x, y \in \mathcal{F}^C$ with $x \sqsubset y^+$. We now have the following cases:

- $x, y \in C$, so $x, y \in \mathcal{F}$ which contradicts the barrier property for $\mathcal{F}$.
- $x \in C$ and $y \in \mathcal{F}^C \setminus C$, then $y = a \cup b$ for some $a, b \in \mathcal{F}$. But $a \cup b = a$ or $a \cup b = \langle j\rangle^\frown b$ for some $j \in A$. So either $x \sqsubset a^+$, which contradicts the barrier property for $\mathcal{F}$, or $x \sqsubset b$ which contradicts that $\mathcal{F}$ is Nash-Williams.
- If $x \in \mathcal{F}^C \setminus C$ and $y \in C$ then $x = a \cup b$ so that $a \sqsubset y^+$, which contradicts the barrier property for $\mathcal{F}$.
- $x, y \in \mathcal{F}^C \setminus C$. So we have for some $a, b, c, d \in \mathcal{F}$ that $x = a \cup b$ and $y = c \cup d$. Now $a \sqsubseteq a \cup b \sqsubset (c \cup d)^+$ and $d \sqsubseteq (c \cup d)^+$, so a and d are $\sqsubseteq$-comparable and therefore $a = d$ since $a, d \in \mathcal{F}$. Now $a \cup b \sqsubset (c \cup a)^+$ so it must be that $|c| > |a|$. But then $c \cup a = c$, and we have $a \sqsubset a \cup b \sqsubset c^+$, which contradicts the barrier property for $\mathcal{F}$.

So there are no such $x, y \in \mathcal{F}^C$, hence $\mathcal{F}^C$ satisfies the barrier property. $\square$

The following lemma is also due to Shelah [49]. I present a version of the proof which I will extend when proving Theorem 7.2.8.

Lemma 7.2.7 (Shelah [49]). *If Q is κ-bqo then so is $\mathcal{P}_\gamma(Q)$ for any ordinal γ.*

Proof. Suppose that $\mathcal{P}_\gamma(Q)$ is not κ-bqo. We will show that Q is not κ-bqo, by constructing a bad function. So let $f : \mathcal{B} \to \mathcal{P}_\gamma(Q)$ be bad, with $\mathcal{B}$ a κ-barrier on some $A \in [\kappa]^\kappa$.

We will define by induction, κ-barriers $\mathcal{B}_n$ and bad $f_n : \mathcal{B}_n \to \mathcal{P}_\gamma(Q)$ as follows. Firstly $\mathcal{B}_0 = \mathcal{B}$ and $f_0 = f$. Now define

$$C_n = \{x \in \mathcal{B}_n : f_n(x) \in Q\}$$

and let $\mathcal{B}_{n+1} = \mathcal{B}_n^{C_n}$. So for $x, y \in \mathcal{B}_n$ with $x \lhd y$ and $f_n(x) \notin Q$ we have $x \cup y \in \mathcal{B}_{n+1}$.

Let $g : \mathcal{P}_\infty(Q) \times \mathcal{P}_\infty(Q) \to \mathcal{P}_\infty(Q)$ be such that whenever $X \not\leqslant Y$ and $X \notin Q$ we have:

- $g(X, Y) \in X$,
- $Y \in Q$ implies $g(X, Y) \not\leqslant Y$,
- $Y \notin Q$ implies $(\forall Z \in Y)$, $g(X, Y) \not\leqslant Z$.

To see that such a $g(X, Y)$ always exists, assume that $X \not\leqslant Y$ and $X \notin Q$. If $Y \in Q$ then $X \not\leqslant Y$ iff $(\forall x \in X), x \not\leqslant Y$, so we can let $g(X, Y)$ be any element of X. If $Y \notin Q$ then either we can find a valid value for $g(X, Y)$, or for every $x \in X$ there is some $y_x \in Y$ such that $x \leqslant y_x$. But then let $f(x) = y_x$, so that $f : X \to Y$ witnesses that $X \leqslant Y$, a contradiction.

So for $s \in \mathcal{B}_{n+1}$ with $s \notin C_n$ we have $s = x \cup y$ for some $x, y \in \mathcal{B}_n$. In this case we define $f_{n+1}(s) = g(f_n(x), f_n(y))$. If $s \in C_n$ then $s \in \mathcal{B}_n$ so let $f_{n+1}(s) = f_n(s)$.

By Lemma 7.2.6 we have for every $n \in \omega$ that $\mathcal{B}_n$ is a κ-barrier. We now wish to show by induction that each f_n $(n \in \omega)$ is bad.

So suppose that f_n is bad and let $a \lhd b$ with $a, b \in \mathcal{B}_{n+1}$. Suppose that $a, b \notin C_n$ so $a = a_0 \cup a_1$ and $b = b_0 \cup b_1$ for some $a_0, a_1, b_0, b_1 \in \mathcal{F}_n$ with $a_0 \lhd a_1$, $b_0 \lhd b_1$ and $a_0, b_0 \notin C_n$.

So $a_1 \sqsubseteq a^+$, $b_0 \sqsubseteq b$ and a^+ and b are $\sqsubseteq$-comparable. Therefore a_1 and b_0 are $\sqsubseteq$-comparable and thus $a_1 = b_0$ since they are both members of $\mathcal{B}_n$.

Now $f_{n+1}(a) = g(f_n(a_0), f_n(a_1))$ and $f_{n+1}(b) = g(f_n(a_1), f_n(b_1))$ so $f_{n+1}(b) \in f_n(a_1)$ and $\forall q \in f_n(a_1), f_{n+1}(a) \not\leqslant q$. So indeed $f_{n+1}(b) \not\leqslant f_{n+1}(a)$ whenever $a \lhd b$, i.e. f_{n+1} is bad.

We also note that if $x \in \mathcal{B}_n$ and $f_n(x) \in Q$ then $x \in \mathcal{B}_m$ and $f_m(x) = f_n(x)$ for every $m \geqslant n$; this is clear by the definitions of $\mathcal{B}_n$ and f_n. Using this we can define

$$\mathcal{B}^* = \{x : (\exists n \in \omega), x \in \mathcal{F}_n \wedge f_n(x) \in Q\},$$

and $f^* : \mathcal{B}^* \to Q$ by setting $f^*(x) = f_n(x)$ whenever $f_n(x) \in Q$.

We claim that $\mathcal{B}^*$ is a κ-barrier. If $a, b \in \mathcal{B}^*$ then we can choose $n \in \omega$ large enough so that $a, b \in \mathcal{B}_n$. So we see that for any such a, b we cannot have $a \sqsubset b$ or $a \sqsubset b^+$. Now let $S \subseteq A$, and denote by $\mathcal{B}_n(S)$ the unique initial segment of S in $\mathcal{B}_n$. Suppose that there is no initial segment of S in $\mathcal{B}^*$, then $f_n(\mathcal{B}_n(S)) \notin Q$ for every $n \in \omega$. Thus $f_{n+1}(\mathcal{B}_{n+1}(S)) = g(f_n(a), f_n(b))$ for $a, b \in \mathcal{B}_n$ with $a \cup b = \mathcal{B}_{n+1}(S)$. So $a \sqsubseteq \mathcal{B}_{n+1}(S)$ which means that $a = \mathcal{B}_n(S)$. But by the definition of g we have that

$$f_{n+1}(\mathcal{B}_{n+1}(S)) \in \mathcal{B}_n(a) = f_n(\mathcal{B}_n(S)).$$

But then $(f_n(\mathcal{B}_n(S)))_{n\in\omega}$ is a descending $\in$-sequence, which contradicts well-foundedness of $\in$. So we conclude that $\mathcal{B}^*$ is a κ-barrier on A.

It remains to show that f^* is bad. If $a, b \in \mathcal{B}^*$ then let $n \in \omega$ be large enough so that $a, b \in \mathcal{B}_n$. Thus $f^*(a) = f_n(a) \not\leqslant f_n(b) = f^*(b)$, and so f^* is bad, and Q is not κ-bqo. $\square$

Shelah mentions in [49] that it should be possible to compute some $\chi : \mathrm{On} \to \mathrm{On}$ so that if Q is κ-bqo$_{\chi(\alpha)}$ then $\mathcal{P}_\alpha(Q)$ is κ-bqo$_1$. However he never actually made the computation. We do this in the next theorem.

Theorem 7.2.8. *If Q is κ-bqo$_\alpha$ then $\mathcal{P}_{-1+\alpha}(Q)$ is κ-bqo$_1$.*

Proof. We follow the same proof as Theorem 7.2.7, keeping the same notation, but with the extra assumptions; $\mathrm{rank}(\mathcal{B}) = 1$ and $\gamma = -1+\alpha$. So assuming $\mathcal{P}_{-1+\alpha}(Q)$ is not κ-bqo$_1$, we obtain a bad function $f^* : \mathcal{B}^* \to Q$ and we have $\{\langle i\rangle : \langle i\rangle \sqsubseteq x \in \mathcal{B}^*\} = [A]^1 = \mathcal{B}$. It now suffices to show that $\mathrm{rank}(\mathcal{B}^*) = \alpha$. To simplify notation, we will write $\mathrm{rank}(a)$ in place of $\mathrm{rank}_{\mathcal{B}^*}(a)$, and $\mathcal{P}(\beta)$ in place of $\mathcal{P}_\beta(Q)$.

We define $c : \overline{\mathcal{B}^*} \to \mathcal{P}(\gamma)$ by induction on $\mathrm{rank}(a)$. If $\mathrm{rank}(a) = 0$ then $a \in \mathcal{B}^*$, so set $c \restriction \mathcal{B}^* = f^*$. Otherwise set

$$c(a) = \{c(b) : b \in \overline{\mathcal{B}^*}, |b| = |a| + 1, a \sqsubset b\}.$$

This is well-defined since if $a \sqsubset b$ then necessarily $\mathrm{rank}(a) > \mathrm{rank}(b)$.

Now let $\Lambda : \mathcal{P}(\gamma) \to \gamma + 1$ be such that $\Lambda(A)$ is the least δ such that $A \in \mathcal{P}(\delta)$.

Claim:

$$\Lambda \circ c(a) = -1 + \mathrm{rank}(a) + 1.$$

Proof of claim: We will prove the claim by induction on $\mathrm{rank}(a)$. For the base case, if $\mathrm{rank}(a) = 0$ then $a \in \mathcal{B}^*$ so $c(a) \in Q = \mathcal{P}(-1 + 0 + 1)$. Clearly $\delta = 0$ is the least possible, so $\Lambda \circ c(a) = 0$ here.

Suppose that $\mathrm{rank}(a) > 0$. If $b \in \overline{\mathcal{B}^*}$, $|b| = |a| + 1$ and $a \sqsubset b$, then by the induction hypothesis we have

$$c(b) \in \mathcal{P}(-1 + \mathrm{rank}(b) + 1) \subseteq \mathcal{P}(-1 + \mathrm{rank}(a)).$$

So $c(a) \subseteq \mathcal{P}(-1 + \mathrm{rank}(a))$, i.e. $c(a) \in \mathcal{P}(-1 + \mathrm{rank}(a) + 1)$ and thus $\Lambda \circ c(a) \leqslant -1 + \mathrm{rank}(a) + 1$.

We now show by induction that $\Lambda \circ c(a) \geqslant -1 + \mathrm{rank}(a) + 1$, i.e. $c(a) \notin \mathcal{P}(-1 + \mathrm{rank}(a))$. Suppose that $\mathrm{rank}(a') \leqslant \beta$ implies $c(a') \notin \mathcal{P}(-1 + \mathrm{rank}(a'))$. If $\mathrm{rank}(a) = \beta + 1$ and $A \in c(a)$, then by the induction hypothesis we have $A \notin \mathcal{P}(-1 + \beta)$. But if we had $c(a) \in \mathcal{P}(-1 + \beta + 1)$ then some $A' \in c(a)$ is in $\mathcal{P}(-1 + \beta)$, which is a contradiction.

Now suppose that $-1 + \mathrm{rank}(a) = \mathrm{rank}(a) = \lambda$ for some limit ordinal λ. Then for some $\mu \in \mathrm{On}$ and $i \in \mu$ there are $c(b_i) \in c(a)$ such that $\mathrm{rank}(b_i) = \delta_i < \lambda$ and $\bigcup_{i\in\mu} \delta_i = \lambda$. So by the induction hypothesis $c(b_i) \in \mathcal{P}(-1 + \delta_i + 1)$ and $c(b_i) \notin \mathcal{P}(-1 + \delta_i)$. Suppose that $c(a) \in \mathcal{P}(\lambda)$. Then it must be that $c(a) \subseteq \mathcal{P}(\delta)$ for some $\delta < \lambda$. But $c(b_i) \notin \mathcal{P}(\delta)$ whenever $\delta_i > \delta$, and hence $c(a) \nsubseteq \mathcal{P}(\delta)$ since $c(b_i) \in c(a)$. This is a contradiction, so $c(a) \notin \mathcal{P}(\lambda) = \mathcal{P}(-1 + \operatorname{rank}(a))$. ■

We notice that

$$\bigcup_{n \in \omega} \mathcal{B}_n = \overline{\mathcal{B}^*},$$

since $\mathcal{B}_0 = \mathcal{B}$ consists only of length 1 elements, and in general if $x \in \mathcal{B}_n$ and $f_n(x) \notin Q$ then $\mathcal{B}_{n+1}$ contains every length $|x| + 1$ extension of x that is in $\overline{\mathcal{B}^*}$.

We now define $h : \overline{\mathcal{B}^*} \to \mathcal{P}_\infty(Q)$ so that $h(a) = f_n(a)$ whenever $a \in \mathcal{B}_n$. In order to see that this is well-defined; notice that if $a \in \mathcal{B}_n$ and $f_n(a) \notin Q$ then $a \notin \mathcal{B}_m$ for any $m > n$, moreover if $f_n(a) \notin Q$ then $f_{m'}(a') \notin Q$ for any $a' \sqsubseteq a$ with $a' \in \mathcal{B}_{m'}$. Hence any $a \in \mathcal{B}_n \cap \mathcal{B}_m$ is such that $f_n(a) \in Q$ and therefore $f_m(a) \in Q$.[2] From here it follows that the functions f_n and f_m agree on $\mathcal{B}_n \cap \mathcal{B}_m$, and therefore h is well-defined.

Claim: For any $a \in \overline{\mathcal{B}^*}$,

$$\Lambda \circ h(a) \geqslant \Lambda \circ c(a).$$

Proof of claim: This is clearly the case when $\operatorname{rank}(a) = 0$ since then $h(a) = f^*(a) = c(a)$. Assume for induction that $\Lambda \circ h(a) \geqslant \Lambda \circ c(a)$ for every a with $\operatorname{rank}(a) < \alpha$. If $a \sqsubset b$ and $|a| + 1 = |b|$ then $h(a) \in h(b)$ by the definition of f_i. So using the induction hypothesis we have

$$\Lambda \circ h(a) > \Lambda \circ h(b) \geqslant \Lambda \circ c(b).$$

The definition of c gives that

$$\Lambda \circ c(a) = \bigcup_{b \in \{x : a \sqsubset x, |x| = |a| + 1\}} \Lambda \circ c(b) + 1$$

So if $\Lambda \circ c(a) = \xi + 1$ then there is some $b \in \mathcal{B}^*$ with $a \sqsubset b$ and $|b| = |a| + 1$ such that $\Lambda \circ c(b) = \xi$. Therefore $\Lambda \circ h(b) \geqslant \xi$ by the induction hypothesis, which means that $\Lambda \circ h(a) > \xi$, i.e. $\Lambda \circ h(a) \geqslant \xi + 1 = \Lambda \circ c(a)$ as required.

[2] This was proved in Lemma 7.2.7.

If $\Lambda \circ c(a) = \lambda$ for a limit ordinal λ then there are b_i $(i < \lambda)$ with $a \sqsubset b_i$ and $|b_i| = |a| + 1$ such that $\Lambda \circ c(b_i) = \delta_i$ for some $\delta_i < \lambda$ with $\bigcup_{i<\lambda} \delta_i = \lambda$. So $\Lambda \circ h(b_i) \geqslant \delta_i$ by the induction hypothesis. But also $h(b_i) \in h(a)$ for every $i < \lambda$, so $\Lambda \circ h(a) > \delta_i$ for every $i < \lambda$. Therefore

$$\Lambda \circ h(a) \geqslant \bigcup_{i<\lambda} \delta_i = \lambda = \Lambda \circ c(a).$$

This completes the induction and gives the claim. ■

Now if $a \in \mathcal{B}_0$ then $h(a) = f(a) \in \mathcal{P}(\gamma)$, hence $\Lambda \circ h(a) \leqslant \gamma$. We also note that by definition,

$$\operatorname{rank}(\mathcal{B}^*) = \bigcup_{|a|=1} (\operatorname{rank}(a) + 1).$$

So if $\operatorname{rank}(\mathcal{B}^*) = \zeta + 1$ then there is some $b \in \mathcal{B}_0$ such that $\operatorname{rank}(b) = \zeta$. Using the two claims we see that

$$-1 + \zeta + 1 = \Lambda \circ c(b) \leqslant \Lambda \circ h(b) \leqslant \gamma = -1 + \alpha.$$

Therefore indeed, $\operatorname{rank}(\mathcal{B}^*) \leqslant \alpha$ whenever $\operatorname{rank}(\mathcal{B}^*)$ is a successor ordinal. Finally if $\operatorname{rank}(\mathcal{B}^*)$ is a limit ordinal, then since for any $a \in \mathcal{B}_0$ we have $\operatorname{rank}(a) < \operatorname{rank}(\mathcal{B}^*)$. Thus by the two claims, it follows that

$$\begin{aligned}
\operatorname{rank}(\mathcal{B}^*) &= \bigcup_{a\in[A]^1} (\operatorname{rank}(a) + 1) \\
&= \bigcup_{a\in[A]^1} (-1 + \operatorname{rank}(a) + 1) \\
&= \bigcup_{a\in[A]^1} \Lambda \circ c(a) \\
&\leqslant \bigcup_{a\in[A]^1} \Lambda \circ h(a) \\
&\leqslant \gamma = -1 + \alpha = \alpha.
\end{aligned}$$

Therefore $\operatorname{rank}(\mathcal{B}^*) \leqslant \alpha$, as required. □

We now aim to prove that 3 implies 4 in Theorem 7.2.4. The following is a modified version of the method used by Shelah [49], which works in the absence of the barrier property.

Lemma 7.2.9. *If $\mathcal{P}_{-1+\alpha}(Q)$ is κ-bqo$'_1$, then Q is κ-bqo$'_\alpha$.*

Proof. Assume that Q is not κ-bqo, we aim to show that $\mathcal{P}_{-1+\alpha}(Q)$ is not κ-bqo$'_1$. So suppose $\mathcal{F}$ is a front on $A \in [\kappa]^\kappa$ with $\mathrm{rank}(\mathcal{F}) \leqslant \alpha$, and $f : \mathcal{F} \to Q$ is bad.

To simplify notation, we will again write $\mathrm{rank}(a)$ in place of $\mathrm{rank}_{\mathcal{B}^*}(a)$, and $\mathcal{P}(\beta)$ in place of $\mathcal{P}_\beta(Q)$.

Claim: For any $a \in \overline{\mathcal{F}} \setminus \mathcal{F}$ and $b \in \overline{\mathcal{F}}$, if $a \lhd b$ and $|a| = |b|$ then $a \cup b \in \overline{\mathcal{F}}$.

Proof of claim: For such a and b we have that $a^+ \sqsubset b$. Now because $|a| = |b|$ we have $a \cup b = a^\frown\langle i\rangle$ for some $i \in A$. So since $a \in \overline{\mathcal{F}} \setminus \mathcal{F}$, we have $a \cup b \in \overline{\mathcal{F}}$. ■

Now for any $a \in [\kappa]^{<\omega}$ let $g(a) = -1 + \mathrm{rank}(a) + 1$. Then $g(a) = 0$ iff $\mathrm{rank}(a) = 0$. Define $c : \overline{\mathcal{F}} \to \mathcal{P}_\infty(Q)$ by induction on $\mathrm{rank}(a)$. If $\mathrm{rank}(a) = 0$ then $a \in \mathcal{F}$, so set $c \upharpoonright \mathcal{F} = f$. Otherwise set

$$c(a) = \{c(b) : b \in \overline{\mathcal{F}}, |b| = |a| + 1, a \sqsubset b\}.$$

This is well-defined since if $a \sqsubset b$ then necessarily $\mathrm{rank}(a) > \mathrm{rank}(b)$.

Claim: For any $a \in \overline{\mathcal{F}}$ we have $c(a) \in \mathcal{P}(g(a))$.

Proof of claim: We prove the claim by induction, when $\mathrm{rank}(a) = 0$ we have $c(a) \in \mathcal{P}(0) = \mathcal{P}(g(a))$. If $\mathrm{rank}(a) > 0$, and assuming the claim holds for any b with $a \sqsubset b$, then it follows that

$$c(b) \in \mathcal{P}(g(b)) \subseteq \mathcal{P}(-1 + \mathrm{rank}(a)).$$

Hence $c(a) \subseteq \mathcal{P}(-1 + \mathrm{rank}(a))$ follows from the definition of c. Therefore

$$c(a) \in \mathcal{P}(-1 + \mathrm{rank}(a) + 1) = \mathcal{P}(g(a)).$$

■

For $a \in [\kappa]^1$ we have

$$g(a) = -1 + \mathrm{rank}(a) + 1 \leqslant -1 + \mathrm{rank}(\mathcal{F}) \leqslant -1 + \alpha$$

hence by the second claim, $c(a) \in \mathcal{P}(-1+\alpha)$. So in order to show that $\mathcal{P}(-1+\alpha)$ is not κ-bqo$'_1$ it suffices to show that $r : [A]^1 \to \mathcal{P}(-1+\alpha)$ given by $r(a) = c(a)$ is bad.

Suppose for contradiction that r is good, i.e. there are $a_0, b_0 \in [A]^1$ such that $a_0 \lhd b_0$ and $r(a_0) \leqslant r(b_0)$. We will define inductively on $i \in \omega$, elements $a_i, b_i \in \overline{\mathcal{F}}$ such that for every $i \in \omega$:

- $a_i \lhd b_i$,
- $c(a_i) \leqslant c(b_i)$,
- $|a_i| = |b_i| = i + 1$,
- $a_i \sqsubset a_{i+1}$ and $b_i \sqsubset b_{i+1}$.

Thus a_0 and b_0 suffice for the base case. Suppose $0 < i \in \omega$ and we have defined a_i and b_i. By the induction hypothesis we have $a_i \lhd b_i$, if $a_i \in \overline{\mathcal{F}} \setminus \mathcal{F}$, we can define $a_{i+1} = a_i \cup b_i$ so that $a_{i+1} \in \overline{\mathcal{F}}$ by the first claim. Then $a_i \sqsubset a_{i+1}$ and $|a_{i+1}| = |a_i| + 1 = i + 2$ as required. Now since $c(a_i) \leqslant c(b_i)$ either:

- $\exists h : c(a_i) \to c(b_i)$ such that for all $q \in c(a_i)$, $q \leqslant h(q)$. In this case let b_{i+1} be such that $c(b_{i+1}) = h \circ c(a_{i+1})$, hence $c(a_{i+1}) \leqslant c(b_{i+1})$ as required.
- For some $q \in c(b_i)$ we have $c(a_i) \leqslant q$. Then by definition of c we have $q = c(x)$ for some $x \in \overline{\mathcal{F}}$ with $|x| = |b_i| + 1$ and $b_i \sqsubset x$. Now set $b_{i+1} = x$ so since $c(a_{i+1}) \in c(a_i)$, we have $c(a_{i+1}) \leqslant c(a_i) \leqslant c(b_{i+1})$ as required.
- $c(b_i) \in Q$, which implies that $b_i \in \mathcal{F}$.

Now in the first two cases, we also have that $|b_{i+1}| = |b_i| + 1$ and $b_i \sqsubset b_{i+1}$, whence it follows that $a_{i+1} \lhd b_{i+1}$, and we can continue the induction.

So now we have that the induction stops at $n \in \omega$, when either $a_n \in \mathcal{F}$ or $b_n \in \mathcal{F}$. Suppose that $a_n \in \mathcal{F}$, then since $c(a_n) \leqslant c(b_n)$ there is some element $q \in Q \cap \mathrm{TC}(c(b_n))$ such that $c(a_n) \leqslant q$. So by definition of c, there is some $b \in \mathcal{F}$ with $b_n \sqsubseteq b$ such that $c(b) = q$. But then since $|a_n| = |b_n|$ and $b_n \sqsubseteq b$ we have $a_n \lhd b$. Furthermore,

$$f(a_n) = c(a_n) \leqslant c(b) = f(b)$$

which contradicts that f was bad.

So it must have been the case that $a_n \in \overline{\mathcal{F}} \setminus \mathcal{F}$ and the induction stops because $b_n \in \mathcal{F}$. But then since $c(a_n) \leqslant c(b_n)$ we have that every $q \in Q \cap \mathrm{TC}(c(a_n))$ is such that $q \leqslant c(b_n)$. Pick $a \in \mathcal{F}$ such that $a_n \cup b_n \sqsubseteq a$. Then $a \lhd b_n$ and furthermore $c(a) \in Q \cap \mathrm{TC}(c(a_n))$ which implies that

$$f(a) = c(a) \leqslant c(b_n) = f(b_n)$$

again contradicting that f was bad.

Hence we have obtained a contradiction of our assumption that r was good, which completes the proof. □

This completes the proof of Theorem 7.2.4, from which Theorem 7.2.3 follows. Now Theorem 7.2.3 gives rise to a version of Simpson's definition of bqo for κ-bqos (similar to Definition 2.1.4, and that originally found in [53]) as follows.

Definition 7.2.10. For a cardinal κ, a quasi-order Q is called κ-bqo iff for any $A \in [\kappa]^\kappa$ there is no continuous bad function $f : [A]^\omega \to Q$, giving $[A]^\omega$ the product topology and Q the discrete topology.

Theorem 7.2.11. *The two definitions of κ-bqo are equivalent.*

Proof. Suppose that there is a bad function $f : \mathcal{F} \to Q$ for some $\mathcal{F}$ a front on $A \in [\kappa]^\kappa$. Then define $g : [A]^\omega \to Q$ so that $g(X) = f(\mathcal{F}(X))$. Then if $x \sqsubset X$ and $y \sqsubset X^+$ we have that $x \lhd y$, therefore if also $x, y \in \mathcal{F}$, we have $g(X) = f(x) \leqslant f(y) = g(X^+)$ and therefore g is bad. Moreover g is continuous since if $X, Y \in [A]^\omega$ are such that $\mathcal{F}(X) = \mathcal{F}(Y)$ then $g(X) = g(Y)$.

Now suppose that there is a continuous bad function $f : [A]^\omega \to Q$. Since f is continuous, for each $X \in [A]^\omega$, there is some $x \sqsubset X$ such that $|f([x, A])| = 1$. Let $h : [A]^\omega \to [A]^{<\omega}$ be given by $h(X) = x$ whenever x is shortest such that $x \sqsubset X$ and $|f([x, A])| = 1$. Let $\mathcal{F} = \mathrm{range}(h)$ and we claim that $\mathcal{F}$ is a front. Firstly, $\mathcal{F}$ is Nash-Williams since if $x \sqsubset y$ with $x, y \in \mathcal{F}$, then $y = h(Y)$ for some $Y \in [A]^\omega$. However, we can see that $x \sqsubset y \sqsubset Y$ and $|f([x, A])| = 1$ so by definition of h we have in fact that $h(Y) = x$, which is clearly a contradiction. Now clearly for any $X \in [A]^\omega$ we have $h(X) \in \mathcal{F}$ and $h(X) \sqsubset X$. So indeed $\mathcal{F}$ is a front.

Now let $g : \mathcal{F} \to Q$ be defined by $g(x) = f(X)$ for any $X \in [x, A]$. This is well-defined since $x \in \mathrm{range}(h)$ so by definition of h we had that $|f([x, A])| = 1$. We have that g is bad because if $x, y \in \mathcal{F}$ with $x \lhd y$ then we can let X be such that $x \sqsubset X$ and $y \sqsubset X^+$, and thus

$$g(x) = f(X) \nleq f(X^+) = g(y).$$

The conclusion of the theorem now follows from Theorem 7.2.3. $\square$

Chapter 8

Classifying fronts on κ.

We will now somewhat depart from Shelah's bqo theory at uncountable κ, and consider the combinatorics of the barrier property. We have seen that if Q is a quasi-order and there is some bad $f : \mathcal{F} \to Q$ with $\mathcal{F}$ a front on some $A \in [\kappa]^\kappa$, then there is in fact a bad $g : \mathcal{B} \to Q$ with $\mathcal{B}$ a κ-barrier. Indeed, Theorem 7.2.4 implies that we can even assume that $\mathcal{F}$ and $\mathcal{B}$ have the same rank. However the relationship between the two bad functions f and g is rather indirect. More specifically, we would like to ask the question - how 'close' is $\mathcal{F}$ to being a κ-barrier?

8.1 Extending fronts to barriers

Suppose that we have two fronts $\mathcal{F}, \mathcal{G}$ on $A \in [\kappa]^\kappa$, such that for every $a \in \mathcal{G}$ there is some $b \in \mathcal{F}$ such that $b \sqsubseteq a$. Then whenever we have a bad function $f : \mathcal{F} \to Q$, we can define a new function $g : \mathcal{G} \to Q$ by letting $g(a) = f(b)$ whenever $b \sqsubseteq a$. Then the function g will also be bad. In fact, the corresponding functions $f' : [A]^\omega \to Q$ and $g' : [A]^\omega \to Q$ given by the alternative Simpson style definition (defined as in the proof of Theorem 7.2.11) will be equal, and so in a sense f and g are equivalent.

In this way we can consider any operation that extends elements of a fronts as somehow invariant for bad functions. That is to say that if we take a front $\mathcal{F}$, and extend some of its elements to generate another front $\mathcal{G}$ then any bad function with domain $\mathcal{F}$ will be equivalent in this sense to a bad function with domain $\mathcal{G}$. In a similar way, when we restrict a front $\mathcal{F}$ on A to some $\mathcal{F}|B$ for $B \in [A]^\kappa$ we have that $f \restriction (\mathcal{F}|B)$ is equivalent to part of f.

8.1.1 Stars and $\mho$

So we have two methods of obtaining a κ-barrier from a general front $\mathcal{F}$. Restriction to a κ-barrier, which is not always possible for uncountable κ, and extension of $\mathcal{F}$ into a κ-barrier which we will investigate in this chapter. For example, consider the front

$$\mathcal{F} = \{\langle 1\rangle\} \cup ([\kappa]^3 \setminus \{\langle 1\rangle^\frown a : a \in [\kappa \setminus \{0,1\}]^2\}).$$

Then $\mathcal{F}$ is not a κ-barrier since $\langle 1\rangle \sqsubset \langle 0,1,2\rangle^+$ witnesses the failure of the barrier property. However, every element of $\mathcal{F}$ is an initial segment of an element of $[\kappa]^3$, which is a κ-barrier.

Example 8.1.1. Indeed, for some specific cardinals κ there are fronts $\mathcal{F}$ on κ which cannot be restricted to a κ-barrier, but can be extended to one.

Let $<$ be the lexicographic ordering of ${}^\kappa 2$, i.e. for $x, y : \kappa \to 2$, $x < y$ iff $x(\alpha) < y(\alpha)$ where α is least such that $x(\alpha) \neq y(\alpha)$. Enumerate ${}^\kappa 2$ as $\{x_\alpha : \alpha < 2^\kappa\}$, and let $f : [\kappa^+]^2 \to 2$ be such that $f(\langle\alpha,\beta\rangle) = 0$ iff $x_\alpha < x_\beta$. We call f *the Sierpiński function* (see Proposition 7.5 in [22], originally [51, 29]). Sierpiński showed that this f satisfies the following property:

$$(\forall A \in [\kappa^+]^{\kappa^+}), f\text{"}[A]^2 = 2. \tag{8.1}$$

Now let

$$\mathcal{F} = \{a^\frown(X \restriction 2f(a)) : a \in [\kappa^+]^2, X \in [\kappa^+ \setminus (\max a)]^\omega\}.$$

Then it is simple to check that $\mathcal{F}$ is a front. Let $A \in [\kappa^+]^{\kappa^+}$ and suppose that $\mathcal{F}|A$ satisfies the barrier property. So there are no $a, b \in \mathcal{F}|A$ with $a \sqsubset b^+$ thus for every $X \in [A]^\omega$ we have $|\mathcal{F}(X)| \leqslant |\mathcal{F}(X^+)|$, and therefore for any $x, y, z \in A$ such that $x < y < z$, we have

$$f(\langle x, y\rangle) \leqslant f(\langle y, z\rangle).$$

Since $f"[A]^2 = 2$, there is some $\langle x, y\rangle \in [A]^2$ such that $f(\langle x, y\rangle) = 1$. Now for any $z_0, z_1 \in A$ with $y < z_0 < z_1$ we have

$$1 = f(\langle x, y\rangle) \leqslant f(\langle y, z_0\rangle) \leqslant f(\langle z_0, z_1\rangle) \leqslant 1.$$

It follows that for any $B \in [A \setminus y]^2$ we have $f(B) = 1$. But this contradicts property 8.1.

Therefore $\mathcal{F}|A$ does not satisfy the barrier property for any $A \in [\kappa^+]^{\kappa^+}$. However since the length of elements of $\mathcal{F}$ is bounded above by 4, we have $[\kappa]^4$ is an extension of $\mathcal{F}$, which is clearly a κ-barrier.

The most direct way of extending elements of a front in order to attempt to find a κ-barrier is the following.

Definition 8.1.2. Given a front $\mathcal{F}$ on $A \in [\kappa]^\kappa$, define

$$\mho\mathcal{F} = \{(a^\frown X) \restriction n \in [A]^{<\omega} : a \in \mathcal{F},$$
$$n = \sup(\{|a|\} \cup \{|b| - 1 : b \in \mathcal{F}, (\exists * \in [\kappa]^1), b \sqsubset *^\frown a^\frown X\})\}.$$

We also define $\mho^0\mathcal{F} = \mathcal{F}$ and $\mho^{n+1}\mathcal{F} = \mho(\mho^n\mathcal{F})$ for all $n \in \omega$.

Now we notice that for the front $\mathcal{F}$ in Example 8.1.1, $\mho\mathcal{F}$ is a κ-barrier. However $\mho\mathcal{F}$ will not always be a front. The problem is that there could be some $a \in \mathcal{F}$ such that

$$\sup(\{|a|\} \cup \{|b| - 1 : b \in \mathcal{F}, (\exists * \in [\kappa]^1), b \sqsubset *^\frown a^\frown X\}) = \omega.$$

When this happens we call $\mathcal{F}$ *inextensible*, and if this never happens we call $\mathcal{F}$ *extensible*.

The operator $\mho$ tries to improve an extensible front $\mathcal{F}$ by extending its elements to form a new front but meanwhile removing cases where the barrier property fails. So as before, any bad function from $\mathcal{F}$ will induce the same bad function in our version of Simpson's definition as its extended counterpart from $\mho\mathcal{F}$.

Definition 8.1.3. For $a, b \in [\kappa]^{<\omega}$, if $a \sqsubset b^+$ then we call b a $*$-extension of a.

So b is a $*$-extension of a when b consists of an extension of a with an element adjoined to the beginning (see Figure 8.1).

$$\begin{aligned} b &= \langle * \rangle^\frown \langle a_0, a_1, a_2, ..., a_n, b_0, b_1, ...b_n \rangle \\ a &= \qquad \langle a_0, a_1, a_2, ..., a_n \rangle \end{aligned}$$

Figure 8.1: A $*$-extension b of a.

Lemma 8.1.4. *If $\mathcal{F}$ is an extensible front on $A \in [\kappa]^\kappa$, then $\mho\mathcal{F}$ is a front.*

Proof. Suppose that there are $(a^\frown X) \restriction n, (b^\frown Y) \restriction m \in \mho\mathcal{F}$ with

$$(a^\frown X) \restriction n \sqsubset (b^\frown Y) \restriction m.$$

Therefore $a = b$ and $n < m$, moreover since $\mathcal{F}$ is extensible, for some $*, *' \in [\kappa]^1$ we have

$$*^\frown (a^\frown X) \restriction n,\ \ *'^\frown (a^\frown Y) \restriction m \ \ \in \mathcal{F}.$$

But then

$$*'^\frown (a^\frown X) \restriction n \sqsubset *'^\frown (a^\frown Y) \restriction m.$$

But since $\mathcal{F}$ is a front this means that for some $n' > n$ we have

$$*'^\frown (a^\frown (X \setminus a)) \restriction n' \in \mathcal{F}.$$

But n was the supremum of a set which contains n', so that $n' \geqslant n$, clearly this is a contradiction. Hence we conclude that $\mho\mathcal{F}$ is Nash-Williams.

Now let $X \in [A]^\omega$. Let a be the initial segment of X in $\mathcal{F}$. Now either $a \in \mho X$ in which case we are done, or $(a^\frown X) \restriction n \in \mho\mathcal{F}$ for some $n < \omega$ since $\mathcal{F}$ is extensible. So we conclude that $\mho\mathcal{F}$ is a front. $\square$

We note however that $\mho\mathcal{F}$ may not be a κ-barrier, since some of the extended elements may cause new failures of the barrier property. If this is the case then perhaps iterating $\mho$ can generate a κ-barrier. This intuition gives rise to the following definition.

Definition 8.1.5. A front $\mathcal{F}$ on $A \in [\kappa]^\kappa$ is said to have $\mho$-rank $k \in \omega$, iff k is least such that $\mho^k\mathcal{F} = \mho^{k+1}\mathcal{F}$. We say that $\mathcal{F}$ has $\mho$-rank ∞ if there is no such k.

Notice that $\mho^k \mathcal{F} = \mho^{k+1}\mathcal{F}$ iff $\mho^k\mathcal{F}$ is a κ-barrier. This follows from Lemma 8.1.4, and since $\mho^k = \mho^{k+1}$ iff for every $a \in \mathcal{F}$ and $X \in [A]^\omega$,

$$\sup\{|b| - 1 : b \in \mathcal{F}, (\exists * \in [\kappa]^1), b \sqsubset *^\frown a^\frown X\} \leqslant |a|$$

which is equivalent to saying that a has no $*$-extensions. So the $\mho$-rank of $\mathcal{F}$ is the least number of iterations of $\mho$ required to apply to $\mathcal{F}$ so that the result will be a κ-barrier.

Now consider some front $\mathcal{F}$ of $\mho$-rank $k \in \omega$. We want work out what makes it have $\mho$-rank k, or more precisely, which are the elements of $\mathcal{F}$ that mean that k is least such that $\mho^k\mathcal{F}$ is a κ-barrier. For example, in our original example, the elements $\langle 1\rangle$ and $\langle 0, 1, 2\rangle$ cause $\mathcal{F}$ to have $\mho$-rank 1.

If $\mathcal{F}$ has $\mho$-rank $k > 0$, then there must be an element $a \in \mho^k\mathcal{F}$ that is not in $\mho^{k-1}\mathcal{F}$. In order for this to happen, $\mho^{k-1}\mathcal{F}$ must contain both a proper initial segment a' of a and a $*$-extension b of this initial segment, with $|b| = |a| + 1$. Now since $a \notin \mho^{k-1}\mathcal{F}$, it must be that $b \notin \mho^{k-2}\mathcal{F}$, and so there are similar implications for b as there were for a, i.e. $\mho^{k-2}\mathcal{F}$ contains an initial segment of b and a $*$-extension of this initial segment. Ultimately, using an inductive argument, the fact that $a \in \mho^k\mathcal{F}$ but $a \notin \mho^{k-1}\mathcal{F}$ implies existence of a certain system of $*$-extensions and initial segments that are contained in $\mathcal{F} = \mho^0\mathcal{F}$. Initial segments are easily found since applying $\mho$ cannot reduce the length of elements, so we need only worry about iterated $*$-extensions. This gives rise to the following definition.

Definition 8.1.6. Given $X \in [\kappa]^\kappa$ and $k, n \in \omega$, we call a $c \in \mathcal{F}$ (k, n, X)-*critical* iff either there is some finite sequence $* \in [\kappa]^k$ such that

$$c = *^\frown(X \restriction n).$$

Given a (k, n, X)-critical $c \in \mathcal{F}$, we say that c is (k, n, X)-*maximal in* $\mathcal{F}$ if there is no (l, m, X)-critical element of $\mathcal{F}$ for $l \leqslant n$ and $m \geqslant n$ with at least one of these inequalities strict.

Lemma 8.1.7. *If a front* $\mathcal{F}$ *has a* (k, n, X)*-critical element, then* $X \restriction n' \in \mho^k\mathcal{F}$ *for some* $n' \geqslant n$.

Proof. Let $c \in \mathcal{F}$ be (k, n, X)-critical. We will prove the lemma by induction on $k \in \omega$. If $k = 0$ then $c = X \restriction n$ and hence clearly the lemma holds.

Suppose that $k > 0$ and the lemma holds for $k - 1$. Then $c = *^\frown(X \restriction n)$ for some $* \in [\kappa]^k$. Let $*_0 \in [\kappa]^{k-1}$ and $*_1 \in [\kappa]^1$ be such that $* = *_0{}^\frown *_1$. Then c is $(k-1, n+1, *_1{}^\frown X)$-critical, so by the induction hypothesis $(*_1{}^\frown X) \restriction n_0 \in \mho^{k-1}\mathcal{F}$ for some $n_0 \geqslant n$. Now $\mho^{k-1}\mathcal{F}$ also contains $X \restriction m$ for some $m \in \omega + 1$. Therefore, by definition of $\mho$, we have that $X \restriction n' \in \mho^k\mathcal{F}$ where $n' \geqslant \max\{m, n_0\} \geqslant n$. Which completes the induction. $\square$

Theorem 8.1.8. *For $X \in [\kappa]^\kappa$ and $k, n \in \omega$, with $k > 0$ suppose that $\mho^k\mathcal{F}$ is a front. Then we have that $X \restriction n \in \mho^k\mathcal{F}$ and $X \restriction n \notin \mho^{k-1}\mathcal{F}$ iff $\mathcal{F}$ has a (k, n, X)-maximal element.*

Proof. $(\longrightarrow)$

We will first prove that there is some (k, n, X)-critical element by induction on $k \in \omega$. Clearly it holds when $k = 0$ since then $X \restriction n \in \mathcal{F}$ is the only possible (k, n, X)-critical element.

Suppose that the lemma holds for all $k < r$. Now assume that $X \restriction n \in \mho^r\mathcal{F}$ and $X \restriction n \notin \mho^{r-1}\mathcal{F}$. So since elements of a front can only be extended after applying $\mho$ we have that $X \restriction n_0 \in \mho^{r-1}\mathcal{F}$ for some $n_0 < n + 1$. Furthermore, by definition of $\mho$, we see that $n = \sup\{|b| - 1 : b \in \mathcal{F}, (\exists * \in [\kappa]^1), b \sqsubset *^\frown X\}$, and thus since $\mho^k\mathcal{F}$ is a front we have $n \in \omega$ so there is some $* \in [\kappa]^1$ such that

$$*^\frown(X \restriction n) \in \mho^{r-1}\mathcal{F}.$$

If $k = 1$ then this element is (k, n, X)-critical. Now if $k > 1$ then suppose that

$$*^\frown(X \restriction n) \in \mho^{r-2}\mathcal{F}.$$

But then by definition of $\mho$, we have that $n_0 \geqslant n$, which is a contradiction. So

$$*^\frown(X \restriction n) \notin \mho^{r-2}\mathcal{F},$$

which means we can apply the induction hypothesis here, which gives a $(r-1, 1+n, *_0{}^\frown X)$-critical element $c = *_1{}^\frown((*_0{}^\frown X) \restriction 1 + n)$ (for some $*_1 \in [\kappa]^{r-1}$). Then c is also (r, n, X)-critical, as required.

It remains to show that c is maximal. Suppose not, then there is some (r', n', X)-critical subset for $r' \leqslant r$ and $n' \geqslant n$ with at least one of these strict. So by Lemma 8.1.7, this means that $X \restriction n'' \in \mho^{r'}\mathcal{F}$, where $n'' \geqslant n'$. If $n'' > n$ then this contradicts that $X \restriction n \in \mho^k\mathcal{F}$. But if $k' < k$ and $n'' = n$ then either $X \restriction n \in \mho^{r-1}\mathcal{F}$ which is a contradiction of our original assumption, or $X \restriction l \in \mho^{r-1}\mathcal{F}$ for some $l > n$, which contradicts that $X \restriction n \in \mho^r\mathcal{F}$.

$(\longleftarrow)$

Now assume there is a (k, n, X)-maximal element $c \in \mathcal{F}$. Then by Lemma 8.1.7 we have $X \restriction n' \in \mho^k\mathcal{F}$ for some $n' \geqslant n$. If $n' > n$ then let $k' \leqslant k$ be least such that there is some $m \geqslant n$ with $X \restriction m \in \mho^{k'}\mathcal{F}$. Therefore $(\longrightarrow)$ implies that there is a (k', m, X)-maximal element, but we know that $k' \leqslant k$ and $m \geqslant n$ so since c was (k, n, X)-maximal, we have that $k' = k$ and $m = n$. But this means that $s \restriction n \in \mho^k\mathcal{F}$ and k is least such that this occurs, i.e. $s \restriction n \notin \mho^k\mathcal{F}$. □

Corollary 8.1.9. *If the lengths of elements of $\mathcal{F}$ are bounded, then $\mathcal{F}$ has finite $\mho$-rank.*

Proof. Suppose that $\mathcal{F}$ has infinite $\mho$-rank, so by Theorem 8.1.8, $\mathcal{F}$ contains (k, n, X)-maximal elements for arbitrarily large k, and some $n \in \omega$, $X \in [\kappa]^\kappa$. But if c is (k, n, X)-maximal then $|c| = k + n$, and therefore $\mathcal{F}$ contains arbitrarily long elements. □

8.1.2 Classifying fronts by extensibility

Now that we have seen how $\mho$ can behave on fronts on $A \in [\kappa]^\kappa$, we will classify such fronts by their behaviour when $\mho$ is applied repeatedly.

Definition 8.1.10. Let $\mathcal{F}$ be a front on $A \in [\kappa]^\kappa$. Then define:

- $\mathcal{F}$ is $\mho^r$ if $r \in \omega$ is least such that $\mho^r\mathcal{F} = \mho^{r+1}\mathcal{F}$ and $\mho^r\mathcal{F}$ is a front.
- $\mathcal{F}$ is $\mho^\infty$ iff there is some $X \in [A]^\omega$ and for each $i \in \omega$ there are $k_i, n_i \in \omega$ such that $\mathcal{F}$ contains a (k_i, n_i, X)-critical element, $k_i < k_{i+1}$, $n_i < n_{i+1}$ and $\mho^i\mathcal{F}$ is a front.
- $\mathcal{F}$ is $\mho^{<\infty}$ iff $\mho^r\mathcal{F}$ is a front for every $r \in \omega$, but $\mathcal{F}$ is not $\mho^\infty$ or $\mho^k$ for any $k \in \omega$.

- $\mathcal{F}$ is $\mho_\times$ iff $\mho\mathcal{F}$ is not a front.

- $\mathcal{F}$ is $\mho_\times^r$ iff r is least such that $\mho^r\mathcal{F}$ is not a front.

For $\Gamma \in \{\mho^r, \mho^\infty, \mho^{<\infty}, \mho_\times, \mho_\times^r\}$, we call Γ the *type* of $\mathcal{F}$ and say that $\mathcal{F}$ is of type Γ iff $\mathcal{F}$ is Γ.

Proposition 8.1.11. *Every front on $A \in [\kappa]^\kappa$ is precisely one of the following: $\mho^r$, $\mho^\infty$, $\mho^{<\infty}$ or $\mho_\times^r$ for some $r \in \omega$.*

Proof. Suppose that $\mho^r\mathcal{F}$ is a front for every $r \in \omega$. Then $\mathcal{F}$ is either $\mho^\infty$, $\mho^{<\infty}$ or $\mho^r$, for some $r \in \omega$. If it exists, let $r \in \omega$ be least such that $\mho^r\mathcal{F} = \mho^{r+1}\mathcal{F}$, so that $\mathcal{F}$ is $\mho^r$. Otherwise $\mathcal{F}$ is either $\mho^\infty$ or $\mho^{<\infty}$. By definition if it is $\mho^\infty$ then it cannot be $\mho^{<\infty}$. Now the only other case is if $\mho^r\mathcal{F}$ is not a front for some $r \in \omega$. Thus we can let r be least such that this happens, so that $\mathcal{F}$ is $\mho_\times^r$. $\square$

We will now give some examples of each type.

Example 8.1.12. Any κ-barrier is $\mho^0$. For example $[\kappa]^n$ for $n \in \omega$.

Example 8.1.13. Let

$$\begin{aligned}\mathcal{F} =& \{\langle 0,1\rangle^\frown a : a \in [\kappa \setminus \{0,1\}]^2\}\\ &\cup \{\langle 1\rangle^\frown a : a \in [\kappa \setminus \{0,1\}]^1\}\\ &\cup \{\langle 0\rangle^\frown a : a \in [\kappa \setminus \{0,1\}]^1\} \cup [\kappa \setminus \{0,1\}]^1.\end{aligned}$$

Then $\mathcal{F}$ is a front on κ. Furthermore let $X = \langle 2,3,4,5,...\rangle \in [\kappa]^\omega$, then the element $\langle 0,1,2,3\rangle$ is $(2,2,X)$-critical. Any other $(2,n,X)$-critical element has length 4 and therefore $n = 2$. Any $(1,n,X)$-critical element has length 1 or 2 and hence $n < 2$. Furthermore $\langle 2\rangle$ is the only $(0,n,X)$-critical element for any $n \in \omega$. Hence $\langle 0,1,2,3\rangle$ is $(2,2,X)$-maximal.

Now if for some $Y \in [\kappa]^\omega$ there were a $(3,n,Y)$-maximal element a then $n > 1$ since there is an initial segment of Y in $\mathcal{F}$ of at least length 1 (and otherwise this element would contradict that a is $(3,n,Y)$-maximal. Therefore $|a| = 3 + n > 4$, but every element of $\mathcal{F}$ has length at most 4. Therefore no such a exists.

Therefore $\mho\mathcal{F} \neq \mho^2\mathcal{F} = \mho^3\mathcal{F}$. Clearly the length of every element of $\mho^2\mathcal{F}$ is bounded above by 4, hence $\mho^2\mathcal{F}$ is a front, and therefore $\mathcal{F}$ is $\mho^2$.

By introducing (k, n, X)-critical elements for $k > 2$ we could also give examples of $\mho^k$ fronts. If we introduce such elements for different X and arbitrarily large k, then we could make a $\mho^{<\infty}$ front.

Example 8.1.14. Let

$$\mathcal{F} = \{\langle n\rangle^\frown a : n \in \omega, a \in [\kappa \setminus \omega]^n\} \cup \{\langle n, m\rangle : n < m < \omega\} \cup [\kappa \setminus \omega]^1.$$

Then $\mathcal{F}$ is $\mho_\times$ since if $a = \langle\omega\rangle$ and $X = \langle\omega + 1, \omega + 2, ...\rangle$ then

$$\sup\{|b| - 1 : b \in \mathcal{F}, (\exists * \in [\kappa]^1), b \sqsubset *^\frown a^\frown X\} = \omega.$$

Example 8.1.15. For some $r \in \omega$, let

$$\mathcal{F} = \{\langle n_0, ..., n_{r-1}\rangle^\frown a : n_0 < ... < n_{r-1} < \omega, a \in [\kappa \setminus \omega]^{n_{r-1}}\} \cup$$

$$[\omega]^{r+1} \cup \{a^\frown b : a \in [\omega]^{<r}, b \in [\kappa \setminus \omega]^1\}.$$

Then $\mathcal{F}$ is a front. Notice that if $r > 1$ then $\mathcal{F}$ does not contain arbitrarily long $*$-extensions of any element, and that then $\mho\mathcal{F}$ contains some $a^\frown b$ such that $a \in [\omega]^{r-1}$ and the b are arbitrarily long. Whence we see that $\mho^{r-1}\mathcal{F}$ contains arbitrarily long $*$-extensions of an element, so that $\mho^r\mathcal{F}$ is not a front. This means that $\mathcal{F}$ is $\mho^r_\times$.

Example 8.1.16. Let $\mathcal{F}_0$ consist of the elements:

$$\langle\omega\rangle,$$

$$\langle 1, \omega, \omega + 1\rangle,$$

$$\langle 2, 3, \omega, \omega + 1, \omega + 2\rangle, \langle 3, \omega, \omega + 1\rangle,$$

$$\langle 4, 5, 6, \omega, \omega + 1, \omega + 2, \omega + 3\rangle, \langle 5, 6, \omega, \omega + 1, \omega + 2\rangle, \langle 6, \omega, \omega + 1\rangle,$$

$$\vdots$$

For $X \in [\kappa]^\kappa$, if there is no $a \in \mathcal{F}_0$ such that $a \sqsubset X$, then let b_X be the shortest initial segment of X such that no $a' \in \mathcal{F}_0$ is $\sqsubseteq$-comparable with b_X, otherwise let b_X be the initial segment of X in $\mathcal{F}_0$. For example, if the first element of X is $\alpha > \omega$, then $b_X = \langle \alpha \rangle$. If $\langle 4, 5, 6, \omega, \alpha \rangle \sqsubseteq X$ for $\alpha \neq \omega+1$, then let $b_X = \langle 4, 5, 6, \omega, \alpha \rangle$. Then let $\mathcal{F} = \{b_X : X \in [\kappa]^\kappa\}$. It is simple to check that $\mathcal{F}$ is a front on κ and $\mathcal{F}_0 \subseteq \mathcal{F}$.

Set $Y = \langle \omega, \omega+1, ... \rangle$. Then $\langle \omega \rangle, \langle 1, \omega, \omega+1 \rangle, \langle 2, 3, \omega, \omega+1, \omega+2 \rangle, \langle 4, 5, 6, \omega, \omega+1, \omega+2, \omega+3 \rangle, ...$ are $(k, k+1, Y)$-maximal, for $k = 0, 1, 2, 3, ...$, because they each contain the longest part of Y for each given k.

We also see that no element has arbitrarily large $*$-extensions and that this property will be preserved when applying $\mho$. So $\mathcal{F}$ is $\mho^\infty$.

Now we want to be able to describe the relationships between these types of front, with respect to extendibility.

Definition 8.1.17. Given two fronts $\mathcal{F}$ and $\mathcal{G}$, we say that $\mathcal{G}$ is an extension of $\mathcal{F}$ iff for any $a \in \mathcal{G}$ there is some $b \in \mathcal{F}$ such that $b \sqsubseteq a$.

Definition 8.1.18. For two types $\Upsilon, \Gamma \in \{\mho^r, \mho^\infty, \mho^{<\infty}, \mho_\times, \mho^r_\times : r \in \omega\}$ we say that $\Upsilon \sqsubseteq \Gamma$ iff for any front $\mathcal{F}$ of type Υ there is a front $\mathcal{G}$ of type Γ such that $\mathcal{G}$ is an extension of $\mathcal{F}$.

Proposition 8.1.19. *For every $r \in \omega$, $\mho^{r+1} \sqsubseteq \mho^r$ and $\mho^{r+1}_\times \sqsubseteq \mho^r_\times$.*

Proof. If $\mathcal{F}$ is $\mho^{r+1}$ then $\mho\mathcal{F}$ is an extension of $\mathcal{F}$, and it is $\mho^r$. Similarly if $\mathcal{F}$ is $\mho^{r+1}_\times$ then $\mho\mathcal{F}$ is an extension of $\mathcal{F}$, and it is $\mho^r_\times$. □

Theorem 8.1.20. $\mho^0 \sqsubseteq \mho^r$ *for any* $r \in \omega$*.*

Proof. Let $\mathcal{B}$ be a κ-barrier which is (without loss of generality) a front on κ, and $0 < r \in \omega$. The theorem holds if we can find an extension of $\mathcal{B}$ that is $\mho^r$.

Let $X = \langle \omega, \omega+1, \omega+2, ... \rangle$, let x be the initial segment of X in $\mathcal{B}$, and let y be the initial segment of $\langle 0, 1, 2, ..., r-1 \rangle^\frown X$ in $\mathcal{B}$. We also let $z \in [\omega]^{<\omega}$ be such that if $y \sqsubset \langle 0, 1, 2, ..., r-1 \rangle$ then $y^\frown z = \langle 0, 1, 2, ..., r-1 \rangle$, otherwise let $z = \langle \rangle$. Then set

$$\mathcal{F} = (\mathcal{B} \setminus \{y\}) \cup \{y^\frown u^\frown a : u \sqsubseteq z^\frown x, |a| = 1, u^\frown a \not\sqsubseteq z^\frown x\}.$$

Then $\mathcal{F}$ is a front, and we will show that it has an extension that is $\mho^r$. Let b be the initial segment of $\langle 0,1,2,...,r-1\rangle^\frown X$ in $\mathcal{F}$. Then b is $(r,|x|+1,X)$-critical, and we claim that it is $(r,|x|+1,X)$-maximal.

Let $c = *^\frown(X\restriction n) \in \mathcal{F}$ be (k,n,X)-critical, so $* \in [\kappa]^k$. If $*$ is not $\sqsubseteq$-comparable with y then $c \in \mathcal{B}$, so by Lemma 8.1.7 we have that $X \restriction n' \in \mho^k\mathcal{B}$ for some $n' \geqslant n$. But we have that $\mathcal{B}$ is a κ-barrier and therefore $\mho^k\mathcal{B} = \mathcal{B}$, which gives that $n' = |x|$. So in this case we have $n < |x|+1$ so c cannot contradict that b is $(r,|x|+1,X)$-maximal. Now if $*$ is $\sqsubseteq$-comparable with y, then by definition of $\mathcal{F}$, we either have $* \sqsubset \langle 0,1,...,r-1\rangle$ and $n = 1 < |x|+1$ or we have $c = b$, neither of which can contradict that b is $(r,|x|+1,X)$-maximal. Therefore b is indeed $(r,|x|+1,X)$-maximal and therefore $\mho^{r-1}\mathcal{F} \neq \mho^r\mathcal{F}$ by Theorem 8.1.8. So it remains only to check that $\mathcal{F}$ is not $\mho^k_\times$, $\mho^\infty$, $\mho^{<\infty}$ or $\mho^{r'}$ for any $k \in \omega$ and $r' > r$.

Now since the elements of $\mathcal{F} \setminus \mathcal{B}$ have bounded length, we must not have added any (k,n,Y)-critical elements for arbitrarily large n and k and $Y \in [\kappa]^\omega$ (since such elements have length $n+k$). So since there were not any such critical elements in $\mathcal{B}$ we have that $\mathcal{F}$ is not $\mho^k_\times$, $\mho^\infty$ or $\mho^{<\infty}$. Therefore it must be that $\mathcal{F}$ is $\mho^{r'}$ for some $r' \geqslant r$, so that $\mho^{r'-r}\mathcal{F}$ is $\mho^r$. $\square$

Theorem 8.1.21. $\mho^{<\infty} \sqsubseteq \mho^0$.

Proof. Let $\mathcal{F}$ be a $\mho^{<\infty}$ front on κ. It suffices to find an extension of $\mathcal{F}$ which is a κ-barrier. Since $\mathcal{F}$ is $\mho^{<\infty}$ we have that $\mho^i\mathcal{F}$ is defined for every $i \in \omega$. Define

$$\mho^\infty\mathcal{F} = \{a \in [\kappa]^{<\omega} : a \in \mho^i\mathcal{F} \text{ for infinitely many } i \in \omega\}.$$

Clearly then $\mho^\infty\mathcal{F}$ is an extension of $\mathcal{F}$. We claim that it is a κ-barrier, and we begin by showing it is a front.

Let $X \in [\kappa]^\omega$ be such that there is no initial segment of X in $\mho^\infty\mathcal{F}$. This means that there are infinitely many $i,n \in \omega$ such that $X \restriction n \in \mho^i\mathcal{F}$ and $X \restriction n \notin \mho^{i-1}\mathcal{F}$. But then by Theorem 8.1.8, for each such i,n there is a (i,n,X)-maximal element of $\mathcal{F}$. But this implies that $\mathcal{F}$ is $\mho^\infty$, which is a contradiction.

Now if $a \in \mho^{\infty}\mathcal{F}$, then there is some $k \in \omega$ such that $a \in \mho^{j}\mathcal{F}$ for every $j \geqslant k$. This is because if $a \in \mho^{k}\mathcal{F}$ and $a \notin \mho^{k+1}\mathcal{F}$ then $a \notin \mho^{j}\mathcal{F}$ for any $j > k$, which means that $\{i : a \in \mho^{i}\mathcal{F}\}$ is finite so $a \notin \mho^{\infty}\mathcal{F}$. So if $a, b \in \mho^{\infty}\mathcal{F}$ with $a \sqsubset b$, then let $k, l \in \omega$ be such that $a \in \mho^{j}\mathcal{F}$ for every $j \geqslant k$, and $b \in \mho^{r}\mathcal{F}$ for every $r \geqslant l$. Then $a, b \in \mho^{\max\{k,l\}}\mathcal{F}$, which contradicts that $\mho^{\max\{k,l\}}\mathcal{F}$ is a front.

Therefore $\mho^{\infty}\mathcal{F}$ is a front. It remains to check that it is a κ-barrier. So suppose that $a, b \in \mho^{\infty}\mathcal{F}$ are such that $a \sqsubset b^{+}$. Then similarly there is some $k \in \omega$ be such that $a, b \in \mho^{j}\mathcal{F}$ for every $j > k$. But then b is a $*$-extension of a and therefore $a \notin \mho^{k+1}\mathcal{F}$, which is a contradiction. Hence $\mho^{\infty}\mathcal{F}$ is a κ-barrier. $\square$

Theorem 8.1.22. *For any $r \in \omega$ and $\Upsilon \in \{\mho^{n}, \mho^{<\infty}, \mho^{\infty}, \mho_{\times}, \mho^{m}_{\times} : n \in \omega, m > r\}$, we have $\Upsilon \sqsubseteq \mho^{r}_{\times}$.*

Proof. Let $r \in \omega$ and $\mathcal{F}$ be a front on κ of type

$$\Upsilon \in \{\mho^{n}, \mho^{<\infty}, \mho^{\infty}, \mho_{\times}, \mho^{m}_{\times} : n \in \omega, m > r\}.$$

In particular, note that $\mho^{r}\mathcal{F}$ is a front. It suffices to find an extension $\mathcal{G}$ of $\mathcal{F}$ such that $\mho^{n}\mathcal{G}$ is not a front and $\mho^{n-1}\mathcal{G}$ is a front. First, by extending if necessary, suppose without loss of generality that all elements of $\mathcal{F}$ are at least length $r+1$.

Let $\mathcal{F}(Y)$ be the initial segment of Y in $\mathcal{F}$, and for $Y \in [\kappa]^{\omega}$, let $Y = \langle Y_0, Y_1, ..., Y_{r-1}, ...\rangle$. Now define:

$$W = \{b^\frown Z : b \in [\omega]^{r}, Z \in [\kappa \setminus \omega]^{\omega}\},$$

$$\mathcal{G}_0 = \{a \in \mathcal{F} : (\forall Y \in W), a \not\sqsubseteq Y\},$$

$$G_1 = \{Y \restriction m : Y \in W, m = \max\{|\mathcal{F}(Y)|, r + Y_{r-1}\}\},$$

and

$$\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1.$$

We claim that $\mathcal{G}$ is a front. If $X \in [\kappa]^{\omega}$ then either $X \notin W$ in which case there is an initial segment of X in $\mathcal{G}_0$, otherwise $X \in W$ so there is an initial segment of X in $\mathcal{G}_1$. Now if $a, b \in \mathcal{G}$ with $a \sqsubset b$ then we have the following cases:

- $a, b \in \mathcal{G}_0$ which contradicts that $\mathcal{F}$ is a front.

- $a \in \mathcal{G}_0$, $b \in \mathcal{G}_1$, but then there is some $Y \in W$ with $a \sqsubset b \sqsubset Y$ which contradicts $a \in \mathcal{G}_0$.

- $a \in \mathcal{G}_1$, $b \in \mathcal{G}_0$, so that the first $r+1$ elements of b are elements of ω, yet the $r+1$th element of a not an element of ω, which is a contradiction.

- $a, b \in \mathcal{G}_1$, so we have $a = A \restriction m$ and $b = B \restriction m'$ for some $A, B \in W$. First, we must have that $A_{r-1} = B_{r-1}$. Now since $a \sqsubset b$, and $\mathcal{F}$ is a front, it must be that $\mathcal{F}(A) = \mathcal{F}(B)$. Hence $m = m'$, contradicting that $a \sqsubset b$.

So $\mathcal{G}$ is indeed a front.

It remains to show that $\mathcal{G}$ is $\mho^r_\times$. First we will show that $\mho^{r-1}\mathcal{G}$ is a front. If not, then for some $l < r-1$ there is some $t \in \mho^l\mathcal{G}$ with arbitrarily long $*$-extensions in $\mho^l\mathcal{G}$. Without loss of generality let l be the least such that this occurs. Then since l is least, these $*$-extensions are not contained in $\mho^{l-1}\mathcal{G}$, and hence by Theorem 8.1.8 there are (l, n, X)-maximal elements of $\mathcal{G}$ for each of these $*$-extensions, for some $X \in [\kappa]^\omega$ and some unbounded set of possible n, with the length such an element equal to $l+n$. Let $C \subseteq \mathcal{G}$ be the set of these (l, n, X)-maximal elements. If some subset of C with arbitrarily long members was a subset of $\mathcal{F}$, then since $\mathcal{G}$ is an extension of $\mathcal{F}$ the elements of C are still maximal, so using Theorem 8.1.8 we can see that $\mho^l\mathcal{F}$ is not a front, which contradicts that $\mho^r\mathcal{F}$ is a front.

So $C \cap (\mathcal{G} \setminus \mathcal{F}) = C \cap \mathcal{G}_1$ contains (l, n, X)-maximal elements for $\mathcal{G}$ for some $X \in [\kappa]^\omega$ and unbounded n. But since the length of an element $Y \restriction m \in \mathcal{G}_1$ is

$$\max\{|\mathcal{F}(Y)|, r + Y_{r-1}\},$$

and $\mho^{r-1}\mathcal{F}$ is a front, we have for any $c \in C \cap \mathcal{G}_1$ which is (l, n, X)-critical, that

$$|c| = r + c_{r-1} = l + n$$

where $c = \langle c_0, ..., c_{|c|-1}\rangle$. So therefore since there were such (l, n, X)-maximal elements of $C \cap \mathcal{G}_1$ for arbitrarily large $n \in \omega$ we have that $\{c_{r-1} : c \in C \cap \mathcal{G}_1\}$ is unbounded in ω.

But since $l < r-1$ we have c_{r-1} is fixed, so this cannot be the case, and we conclude that $\mho^{r-1}\mathcal{G}$ is a front.

Now consider $A = \langle \omega, \omega+1, \omega+2, ... \rangle \in [\kappa]^\omega$. Let for $n \geqslant r$, let

$$c_n = \mathcal{G}(\langle 0, 1, ..., r-2, n\rangle^\frown A).$$

Then since $|c_n| \geqslant n$ we have $\{|c_n| : n \in \omega\}$ is unbounded. Now c_n is a $(r-1, |c_n| - r + 1, \langle n\rangle^\frown A)$-critical subset of $\mathcal{G}$. Hence Lemma 8.1.7 implies that $\mho^{r-1}\mathcal{G}$ contains $\langle n\rangle^\frown(A \upharpoonright n')$ for some $n' \geqslant |c_n| - r + 1$. But then since $\{|c_n| : n \in \omega\}$ is unbounded, this means that $\{n' : n \in \omega\}$ is unbounded, so if a is the initial segment of A in $\mho^{r-1}\mathcal{F}$ and $a^\frown X = A$, we have

$$\sup\{|b| - 1 : b \in \mho^{r-1}\mathcal{F}, (\exists * \in [\kappa]^1), b \sqsubset *^\frown a^\frown X\} = \sup\{n' : n \in \omega\} = \omega.$$

Hence $\mho^r\mathcal{G}$ is not a front and r is least such that this is the case, i.e. $\mathcal{G}$ is $\mho^r_\times$. $\square$

Theorem 8.1.23. $\mho^0 \sqsubseteq \mho^{<\infty}$

Proof. Let $\mathcal{B}$ be a κ-barrier, i.e. $\mathcal{B}$ is a front on κ of type $\mho^0$. It suffices to find an extension of $\mathcal{B}$ that is $\mho^{<\infty}$. For $n \in \omega$, define as follows:

$$X_n = \langle n\omega, n\omega + 1, n\omega + 2, ...\rangle,$$

$$a_n = \langle n^2, n^2 + 1, ..., n^2 + n - 1\rangle,$$

let x_n be the initial segment of X_n in $\mathcal{B}$ and let y_n be the initial segment of $a_n{}^\frown X_n$ in $\mathcal{B}$. Now let

$$\mathcal{F} = \{y \in \mathcal{B} : (\forall n \in \omega), y \neq y_n\} \cup$$

$$\{y_n{}^\frown u^\frown w \in [\kappa]^{<\omega} : y_n{}^\frown u \sqsubseteq a_n{}^\frown x_n, y_n{}^\frown u^\frown w \not\sqsubseteq a_n{}^\frown x_n, |w| = 1\}.$$

Then $\mathcal{F}$ is a front on κ and it remains to show that $\mathcal{F}$ is $\mho^{<\infty}$. First we show that $\mathcal{F}$ is not $\mho^r$ for any $r \in \omega$. For $n \in \omega$ consider

$$c_n = a_n{}^\frown(X_n \upharpoonright |x| + 1) \in \mathcal{F},$$

and notice that c_n is $(n, |x_n| + 1, X_n)$-critical.

We claim that c_n is $(n, |x_n|+1, X_n)$-maximal. Suppose there is a (n', m, X_n)-critical $c'_n \in \mathcal{F}$ for $n' \leqslant n$ and $m \geqslant |x_n|+1$. If $c'_n \in \mathcal{B}$, then by Lemma 8.1.7 we have that m is at most equal to the length of the initial segment of X_n in $\mho^{n'}\mathcal{B}$. But since $\mathcal{B}$ is $\mho^0$ we have that $\mho^{n'}\mathcal{B} = \mathcal{B}$ and thus $m \leqslant |x_n| < |x_n|+1$ which is a contradiction. So the longer element c'_n must be in $\mathcal{F} \setminus \mathcal{B}$, and therefore for some $i \in \omega$ we have $y_i \sqsubseteq c'_n$. However since each X_n $(n \in \omega)$ has a different first element and c'_n was (n', m, X_n)-critical, it must be that $i = n$. Now considering the length of the longest possible extensions of y_n in $\mathcal{F}$ we have

$$|a_n| + |x_n| + 1 \geqslant |c'_n| = m + n'.$$

But then since c'_n is (n', m, X_n)-critical, n' is equal to the number of elements in the sequence c'_n that are less than the first element of X_n, i.e. $n' = n$. This implies that $m \leqslant |x_n|+1$ and so $m = |x_n|+1$. Thus c_n is indeed $(n, |x_n|+1, X_n)$-maximal. Therefore by Theorem 8.1.8, $\mho^r\mathcal{F} \neq \mho^{r+1}\mathcal{F}$ for all $r \in \omega$, which implies that $\mathcal{F}$ is not $\mho^r$ for any $r \in \omega$.

If we were to have the correct critical elements s_n that make $\mathcal{F}$ is $\mho^\infty$ or $\mho^r_\times$ for some $r \in \omega$, then these are of form $s_n = *_n{}^\frown\langle\alpha\rangle^\frown b_n$ for some $*_n \in [\kappa]^{<\omega}$, some $\alpha \in \kappa$ and some $b_n \in [\kappa]^{<\omega}$ with $|b_n| < |b_{n+1}|$. We note that α is fixed since if s_n is (k, l, Y)-critical then α is the first element of Y. Since $\mathcal{B}$ is a κ-barrier, infinitely many must be in $\mathcal{F} \setminus \mathcal{B}$ (otherwise $\mathcal{B}$ would be $\mho^\infty$ or $\mho^r_\times$) so assume without loss of generality that $s_n \in \mathcal{F} \setminus \mathcal{B}$ for every $n \in \omega$. Now α is an element of each s_n $(n \in \omega)$ and therefore by definition of $\mathcal{F}$, this is either the last element of s_n (and hence $|b_n| = 0$ so $n = 0$ because the b_n have increasing length); or there is some fixed $j \in \omega$ such that y_j is an initial segment of this s_n. So y_j is an initial segment of s_n for every $n > 0$. But the length of possible extensions of y_j is bounded by $|a_j|+|x_j|+1$, which contradicts that the lengths of b_n were unbounded. Therefore such critical elements s_n for $n \in \omega$ cannot exist, which implies that $\mathcal{F}$ is $\mho^{<\infty}$. □

Theorem 8.1.24. $\mho^0 \sqsubseteq \mho^\infty$

Proof. Let $\mathcal{B}$ be a κ-barrier. It suffices to find an extension of $\mathcal{B}$ that is $\mho^\infty$. Let

$$X = \langle \omega, \omega+1, ... \rangle,$$

$$a_n = \langle n^2, n^2+1, ..., n^2+n-1 \rangle,$$

let x be the initial segment of X in $\mathcal{B}$, let y_n be the initial segment of $a_n{}^\frown X$ in $\mathcal{B}$, and let $x_n = X \upharpoonright (|x|+n)$. Now define

$$\mathcal{F} = \{y \in \mathcal{B} : (\forall n \in \omega), y \neq y_n\} \cup$$

$$\{y_n{}^\frown u^\frown w \in [\kappa]^{<\omega} : y_n{}^\frown u \sqsubseteq a_n{}^\frown x_n, y_n{}^\frown u^\frown w \not\sqsubseteq a_n{}^\frown x_n, |w| = 1\}.$$

Then $\mathcal{F}$ is a front and it remains to show that $\mathcal{F}$ is $\mho^\infty$. Let $c_n = a_n{}^\frown(X \upharpoonright |x_n|+1)$ then c_n is a $(n, |x|+n+1, X)$-critical element of $\mathcal{F}$. Then since $n < n+1$ and $|x|+n+1 < |x|+n+2$ it remains only to check that $\mho^i\mathcal{F}$ is defined for every $i \in \omega$.

If $\mho^i\mathcal{F}$ is not a front for some $i \in \omega$, then by Theorem 8.1.8 there are $(i-1, s_j, Y)$-maximal elements c'_j of $\mathcal{F}$ for some $i \in \omega$ and $s_j \in \omega$ unbounded for $j \in \omega$. Then for any $j \in \omega$, the $j+1$th element of c'_j is equal to the first element of Y. Now since at most finitely many c'_j were subsets of $\mathcal{B}$ (else $\mathcal{B}$ is not a κ-barrier), we can assume without loss of generality that $c'_j \in \mathcal{F} \setminus \mathcal{B}$ for every $j \in \omega$. Suppose that the first element of Y is less than ω, then for some $n \in \omega$ and every $j \in \omega$, we have $y_n \sqsubseteq c'_j$, but the length of possible elements of $\mathcal{F}$ with initial segment y_n is bounded by $|a_n| + |x_n| + 1$, but then

$$|a_n| + |x_n| + 1 \geqslant |c'_j| = i - 1 + s_j.$$

Hence $\{s_j : j \in \omega\}$ is bounded which is a contradiction.

So the first element of Y is at least ω. Since $c'_j \in \mathcal{F} \setminus \mathcal{B}$, for each $j \in \omega$ either $n_j = 1$ or there is some $n \in \omega$ such that $a_n \sqsubseteq c'_j$. But the number of elements of c'_j that are less than the first element of Y is precisely $i-1$. So since $|a_n| = n$, and $a_n \in [\omega]^{<\omega}$, we have

$$|\{a_n : (\exists j \in \omega), a_n \sqsubseteq c'_j\}| < \omega.$$

But then the lengths of these c'_j is bounded by $|a_m| + |x_m| + 1$ where

$$m = \max\{|a_n| : (\exists j \in \omega), a_n \sqsubseteq c'_j\}.$$

This means that $\{s_j : j \in \omega\}$ is again bounded which is a contradiction and therefore $\mho^i\mathcal{F}$ is defined, hence $\mathcal{F}$ is $\mho^\infty$. $\square$

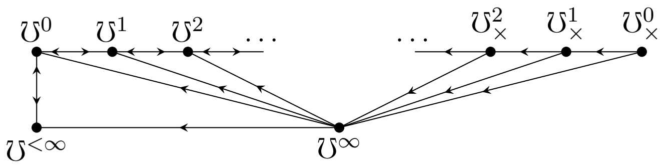

Figure 8.2: The digraph of the relation $\sqsubseteq$ on the set $\{\mho^n, \mho^{<\infty}, \mho^\infty, \mho^n_\times : n \in \omega\}$.

8.1.3 Other extensions

We know that for $r \in \omega$ we can extend $\mho^r$ fronts to κ-barriers by repeatedly applying $\mho$. We also know that we can extend $\mho^{<\infty}$ fronts to κ-barriers by applying $\mho^\infty$ as in Theorem 8.1.21. We know that $\mho$ cannot however extend $\mho^\infty$ or $\mho^r_\times$ to κ-barriers. However $\mho$ is just one possible type of extension, so we ask if there is any method of extending $\mho^\infty$ and $\mho^r_\times$ fronts to κ barriers. I.e. is $\mho^r_\times \sqsubseteq \mho^0$ and is $\mho^\infty \sqsubseteq \mho^0$? We answer both of these questions negatively.

Lemma 8.1.25. *If $\mathcal{B}$ is a κ-barrier, then for all $X \in [\kappa]^\omega$ and all $* \in [\kappa]^{<\omega}$, we have that*

$$|\mathcal{B}(X)| \geqslant |\mathcal{B}(*^\frown X)| - |*|.$$

Proof. Let $X \in [\kappa]^\omega$ and $l = |\mathcal{G}(X)|$. We will prove by induction on $k \in \omega$ that $l \geqslant |\mathcal{G}(*^\frown X)| - k$ for all $* \in [\kappa]^k$. Firstly, $l = |\mathcal{G}(X)|$ so we have the base case.

Let $* \in [\kappa]^{k+1}$ be such that $*^\frown X \in [\kappa]^\omega$. Then since $\mathcal{G}$ satisfies the barrier property, we have that

$$\mathcal{G}(*^\frown X)^+ \sqsubseteq \mathcal{G}(*^{+\frown} X).$$

But $*^+ \in [\kappa]^k$, so comparing lengths and by the induction hypothesis we have

$$|\mathcal{G}(*^\frown X)| - 1 - k \leqslant \mathcal{G}(*^{+\frown} X) - k \leqslant l.$$

So the induction holds. $\square$

Theorem 8.1.26. *For any $\mho^{\infty}$ front $\mathcal{F}$ there is no κ-barrier $\mathcal{G}$ such that $\mathcal{G}$ is an extension of $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a $\mho^{\infty}$ front on κ and suppose that $\mathcal{F}$ has an extension $\mathcal{G}$ which is a κ-barrier. Since $\mathcal{F}$ is $\mho^{\infty}$ there is some $X \in [\kappa]^{\omega}$ and for each $i \in \omega$ there are $k_i, n_i \in \omega$ such that $\mathcal{F}$ has a (k_i, n_i, X)-critical element c_i and furthermore $k_i < k_{i+1}$ and $n_i < n_{i+1}$.

But then $c_i = *_i{}^\frown(X \restriction n_i)$ for some $* \in [\kappa]^{k_i}$. Thus by Lemma 8.1.25 we have

$$l \geqslant |\mathcal{G}(*_i{}^\frown X)| - k_i \geqslant |c_i| - k_i = n_i.$$

But the n_i were unbounded and therefore $l \geqslant \omega$, which is a contradiction. □

Theorem 8.1.27. *For any $r \in \omega$ and any $\mho^{r}_{\times}$ front $\mathcal{F}$ there is no κ-barrier $\mathcal{G}$ such that $\mathcal{G}$ is an extension of $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a $\mho^{r}_{\times}$ front on κ and suppose that $\mathcal{F}$ has an extension $\mathcal{G}$ which is a κ-barrier. Since $\mathcal{F}$ is $\mho^{r}_{\times}$ there is some $X \in [\kappa]^{\omega}$ and for each $i \in \omega$ there are $n_i \in \omega$ such that $\mathcal{F}$ has a $(r-1, n_i, X)$-maximal element c_i and $n_i < n_{i+1}$.

But then $c_i = *_i{}^\frown(X \restriction n_i)$ for some $* \in [\kappa]^{r-1}$. Thus by Lemma 8.1.25 we have

$$l \geqslant |\mathcal{G}(*_i{}^\frown X)| - r + 1 = |c_i| - r + 1 = n_i.$$

But the n_i were unbounded and therefore $l \geqslant \omega$, which is a contradiction. □

8.1.4 Combining restriction and extension

Of course, if we want to answer the question 'how close is a front $\mathcal{F}$ on $A \in [\kappa]^{\kappa}$ to being a barrier?' we should not neglect the standard way of moving from fronts to barriers. That is to say, that when $\kappa = \omega$ and we have Ramsey methods available to us, it is always possible to find some $B \in [A]^{\kappa}$, so that $\mathcal{F} \cap [B]^{<\omega}$ is a κ-barrier. For an uncountable cardinal κ this is not so clear, so now we will classify fronts by whether or not some repeated (but finite) iteration of extensions and restrictions will allow us to move from said front to a κ-barrier.

So we will be concerned with fronts $\mathcal{F}$ which are one of the types $\mho^{r}$, $\mho^{<\infty}$, $\mho^{\infty}$ or $\mho^{r}_{\times}$ $(r \in \omega)$ *in every restriction.*

Definition 8.1.28. Let $\mathcal{F}$ be a front on $A \in [\kappa]^\kappa$, and $r \in \omega$ then:

- $\mathcal{F}$ is $\hat{\mho}^r$ iff $(\forall B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^r$.
- $\mathcal{F}$ is $\hat{\mho}^{<\infty}$ iff $(\forall B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^{<\infty}$.
- $\mathcal{F}$ is $\hat{\mho}^{\infty}$ iff $(\forall B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^{\infty}$.
- $\mathcal{F}$ is $\hat{\mho}^r_\times$ iff $(\forall B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^r_\times$.
- $\mathcal{F}$ is $\hat{\mho}^{<\infty}_\times$ iff $(\forall B \in [A]^\kappa), (\exists k \in \omega)$, $\mathcal{F}|B$ is $\mho^k_\times$.
- $\mathcal{F}$ is $\hat{\mho}^{\leqslant r}$ iff $(\forall B \in [A]^\kappa), (\exists l \leqslant r)$, $\mathcal{F}|B$ is $\mho^l$.
- $\mathcal{F}$ is $\hat{\mho}^{\geqslant r}$ iff $(\forall B \in [A]^\kappa), (\exists l \geqslant r)$, $\mathcal{F}|B$ is $\mho^l$.
- $\mathcal{F}$ is $\hat{\mho}^{\leqslant r}_\times$ iff $(\forall B \in [A]^\kappa), (\exists l \leqslant r)$, $\mathcal{F}|B$ is $\mho^l_\times$.
- $\mathcal{F}$ is $\hat{\mho}^{\geqslant r}_\times$ iff $(\forall B \in [A]^\kappa), (\exists l \geqslant r)$, $\mathcal{F}|B$ is $\mho^l_\times$.
- $\mathcal{F}$ is $\hat{\mho}^{\dot{\leqslant} r}$ iff $\mathcal{F}$ is $\hat{\mho}^{\leqslant r}$ and $(\exists B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^n$.
- $\mathcal{F}$ is $\hat{\mho}^{\dot{\geqslant} r}$ iff $\mathcal{F}$ is $\hat{\mho}^{\geqslant r}$ and $(\exists B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^n$.
- $\mathcal{F}$ is $\hat{\mho}^{\dot{\leqslant} r}_\times$ iff $\mathcal{F}$ is $\hat{\mho}^{\leqslant r}_\times$ and $(\exists B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^n_\times$.
- $\mathcal{F}$ is $\hat{\mho}^{\dot{\geqslant} r}_\times$ iff $\mathcal{F}$ is $\hat{\mho}^{\geqslant r}_\times$ and $(\exists B \in [A]^\kappa)$, $\mathcal{F}|B$ is $\mho^n_\times$.

Fortunately we will be able to eliminate most of these cases through restriction or extension!

Lemma 8.1.29. *Suppose that $\mathcal{F}$ contains the critical and maximal elements required to be one of the following types:*

$$\mho_\times \quad \mho^2_\times \quad \mho^3_\times \quad \cdots \qquad\qquad \mho^\infty \quad \mho^{<\infty} \qquad\qquad \cdots \quad \mho^2_\times \quad \mho^1_\times \quad \mho^0_\times$$

then $\mathcal{F}$ is not any of the types further right of that type.

Proof. For $r \in \omega$, if a front $\mathcal{F}$ contains the maximal elements for a $\mho^r_\times$ front, then $\mho^r \mathcal{F}$ is not a front, hence $\mathcal{F}$ cannot be $\mho^\infty, \mho^{<\infty}$ or $\mho^n$ for any $n \in \omega$.

If $\mathcal{F}$ has the critical elements for a $\mho^\infty$ front, then by definition $\mathcal{F}$ is not $\mho^{<\infty}$. Furthermore, either for some $r \in \omega$ we have $\mho^r \mathcal{F}$ is not a front, or we have that $\mho^n \mathcal{F} \neq \mho^{n+1}\mathcal{F}$ for every $n \in \omega$, and hence $\mathcal{F}$ is not $\mho^n$ for any $n \in \omega$.

If $\mathcal{F}$ has the maximal elements to be $\mho^{<\infty}$, then either for some $r \in \omega$ we have $\mho^r \mathcal{F}$ is not a front, or we have that $\mho^n \mathcal{F} \neq \mho^{n+1}\mathcal{F}$ for every $n \in \omega$, and hence $\mathcal{F}$ is not $\mho^n$ for any $n \in \omega$.

Finaly if $\mathcal{F}$ has the maximal elements to be a $\mho^r$ front, then either for some $i \in \omega$ $\mho^i \mathcal{F}$ is not a front, or we have that $\mho^{r-1}\mathcal{F} \neq \mho^r \mathcal{F}$, and hence $\mathcal{F}$ is not $\mho^n$ for any $n < r$. □

Lemma 8.1.30. *If $\mathcal{F}$ is a $\hat{\mho}^{\dot{\leqslant} r}_\times$ front on $A \in [\kappa]^\kappa$ for some $r \in \omega$, then there is some $B \in [A]^\kappa$ such that $\mathcal{F}|B$ is $\hat{\mho}^r_\times$.*

Proof. If $\mathcal{F}$ is as described then by definition there is some $B \in [A]^\kappa$ such that $\mathcal{F}|B$ is $\mho^r_\times$. But then for any $C \in [B]^\kappa$ we have $\mathcal{F}|C$ is $\hat{\mho}^{\leqslant l}_\times$ for some $l \leqslant r$. But if for any such C we had $l < r$, then we have that $\mathcal{F}|C$ contains the maximal elements required for $\mathcal{F}|C$ to be $\mho^i$ for $i \leqslant l$. But then these maximal elements are also in $\mathcal{F}|B$ so by Lemma 8.1.29, $\mathcal{F}|B$ cannot be $\mho^r_\times$, contradicting our assumption. □

Corollary 8.1.31. *If $\mathcal{F}$ is a $\hat{\mho}^{\leqslant r}_\times$ front on $A \in [\kappa]^\kappa$ for some $r \in \omega$, then there is some $B \in [A]^\kappa$ and $l \leqslant r$ such that $\mathcal{F}|B$ is $\hat{\mho}^l_\times$.*

Proof. If $\mathcal{F}$ is $\hat{\mho}^{\leqslant r}_\times$ then either there is a restriction that is $\mho^r$ so $\mathcal{F}$ is $\hat{\mho}^{\dot{\leqslant} r}_\times$ and we can apply the previous lemma; or not, in which case $\mathcal{F}$ is $\mho^{\leqslant r-1}_\times$. The corollary thus follows inductively. □

Lemma 8.1.32. *If $\mathcal{F}$ is a $\hat{\mho}^{\dot{\geqslant} r}$ front on $A \in [\kappa]^\kappa$ for some $r \in \omega$, then there is some $B \in [A]^\kappa$ such that $\mathcal{F}|B$ is $\hat{\mho}^r$.*

Proof. Similar to Lemma 8.1.30. □

Corollary 8.1.33. *If $\mathcal{F}$ is a $\hat{\mho}^{\geqslant r}$ front on $A \in [\kappa]^\kappa$ for some $r \in \omega$, then there is some $B \in [A]^\kappa$ and $l \geqslant r$ such that $\mathcal{F}|B$ is $\hat{\mho}^l$.*

Proof. Since $\mathcal{F}$ is a $\hat{\mho}^{\geqslant r}$, either there is a restriction of $\mathcal{F}$ to a $\mho^r$ front and we can apply the Lemma 8.1.32; or $\mathcal{F}$ is $\hat{\mho}^{\geqslant r+1}$. We continue this inductively, but the process must stop at some point since $\mathcal{F}$ itself is $\mho^l$ for some $l \in \omega$. □

Corollary 8.1.34. *If $\mathcal{F}$ is a $\hat{\mho}^{\leqslant r}$ front on $A \in [\kappa]^\kappa$ for some $r \in \omega$, then there is some $B \in [A]^\kappa$ and $l \leqslant r$ such that $\mathcal{F}|B$ is $\hat{\mho}^l$.*

Proof. Since every restriction of $\mathcal{F}$ is $\mho^l$ for some $l \in \omega$, pick a restriction that gives the least possible l. So this restriction is $\hat{\mho}^{\dot{\geqslant} l}$ and hence by Lemma 8.1.32 we can restrict to a $\hat{\mho}^l$ front. □

Lemma 8.1.35. *If $\mathcal{F}$ is a $\hat{\mho}_\times^{\geqslant r}$ front on $A \in [\kappa]^\kappa$ for some $r \in \omega$, then there is a $B \in [A]^\kappa$ such that $\mathcal{F}|B$ is either $\mho_\times^l$ for some $l \in \omega$, $l \geqslant r$ or $\mathcal{F}$ is $\hat{\mho}_\times^{<\infty}$.*

Proof. If $\mathcal{F}$ is a $\hat{\mho}_\times^{\geqslant r}$ front on $A \in [\kappa]^\kappa$, then suppose that $\mathcal{F}$ is not $\hat{\mho}_\times^{<\infty}$. Therefore there is some $B \in [A]^\kappa$ such that $\mathcal{F}|B$ is $\mho_\times^n$ for some $n \geqslant r$ and furthermore no further restriction of $\mathcal{F}|B$ is $\mho_\times^m$ for any $m > n$. Hence $\mathcal{F}|B$ is $\hat{\mho}_\times^{\dot{\leqslant} n}$. So by 8.1.30 we can restrict further to a $\hat{\mho}_\times^l$ front for some $l \leqslant n$. But then also $l \geqslant r$ otherwise $\mathcal{F}$ is not $\hat{\mho}_\times^{\geqslant r}$. □

Theorem 8.1.36. *For every front $\mathcal{F}$ on $A \in [\kappa]^\kappa$, there is some $B \in [A]^\kappa$ such that $\mathcal{F}|B$ is either $\mho^r$, $\hat{\mho}^{<\infty}$, $\hat{\mho}^\infty$, $\hat{\mho}_\times^r$ or $\hat{\mho}_\times^{<\infty}$ (for some $r \in \omega$).*

Proof. Let $\mathcal{F}$ be a front on $A \in [\kappa]^\kappa$. If there is some $r \in \omega$ such that for every $B \in [A]^\kappa$ we have that $\mho^r(\mathcal{F}|B)$ is not a front, then $\mathcal{F}$ is $\mho_\times^{\leqslant r}$ which by Lemma 8.1.31 means that $\mathcal{F}$ has a $\hat{\mho}_\times^l$ restriction for some $l \leqslant r$.

Otherwise, for every $r \in \omega$ there is $B \in [A]^\kappa$ such that $\mho^r(\mathcal{F}|B)$ is a front. Now suppose there is no $C \in [B]^\kappa$ such that $\mho^n(\mathcal{F}|C)$ is defined for every $n \in \omega$. Thus $\mathcal{F}|B$ is $\hat{\mho}_\times^{\geqslant n}$ and therefore by Lemma 8.1.35 $\mathcal{F}|B$ is either $\mho_\times^m$ for some $m \in \omega$ or $\mathcal{F}$ is $\hat{\mho}_\times^{<\infty}$.

Now suppose there is some $C \in [B]^\kappa$ such that $\mho^n(\mathcal{F}|C)$ is defined for every $n \in \omega$. Now if for every $D \in [C]^\kappa$ and every $i \in \omega$, we have

$$\mho^i(\mathcal{F}|D) \neq \mho^{i+1}(\mathcal{F}|D),$$

then every such $\mathcal{F}|D$ is either $\mho^\infty$ or $\mho^{<\infty}$. If for some $E \in [C]^\kappa$ we have $\mathcal{F}|E$ is $\mho^{<\infty}$ then no further restriction of $\mathcal{F}|E$ can be $\mho^\infty$, since then $\mathcal{F}|E$ contains the critical elements required for $\mathcal{F}|E$ to be $\mho^\infty$, which contradicts that $\mathcal{F}|E$ was $\mho^{<\infty}$. Hence $\mathcal{F}|E$ is $\hat{\mho}^{<\infty}$. If no such E exists, then $\mathcal{F}|D$ is $\hat{\mho}^\infty$.

So either we are done or there is some $H \in [C]^\kappa$ and some $i \in \omega$ such that

$$\mho^i(\mathcal{F}|H) \neq \mho^{i+1}(\mathcal{F}|H),$$

and therefore $\mathcal{F}|H$ is $\mho^j$ for some $j \leqslant i$. □

Theorem 8.1.37. *For any front $\mathcal{F}$ on $A \in [\kappa]^\kappa$, there is some $B \in [A]^\kappa$ such that $\mathcal{F}|B$ can either be extended to a barrier or is $\hat{\mho}^\infty$, $\hat{\mho}^{<\infty}_\times$ or $\hat{\mho}^r_\times$ for some $r \in \omega$.*

Proof. By Theorem 8.1.36, Theorem 8.1.21 and Proposition 8.1.19. □

8.1.5 Extending before restricting

Theorems 8.1.27 and 8.1.26 tell us that blocks of type $\mho^\infty$ and $\mho^r_\times$ for $r \in \omega$ do not have any κ-barrier extensions. It follows that any restriction of a $\hat{\mho}^r_\times$, $\hat{\mho}^{<\infty}_\times$ or $\hat{\mho}^\infty$ front cannot be extended to a κ-barrier. But what if we extend first? Is there a process of extending and restricting of such a front, that will yield a κ-barrier? For fronts of types $\hat{\mho}^r_\times$, $\hat{\mho}^{<\infty}_\times$ or $\hat{\mho}^\infty$, we answer this question negatively.

Lemma 8.1.38. *Suppose that $\mathcal{F}$ is a front on $A \in [\kappa]^\kappa$ of type $\hat{\mho}^\infty$. Then for any front $\mathcal{G}$ on A which is an extension of $\mathcal{F}$, and any $B \in [A]^\kappa$, we have that $\mathcal{G}|B$ is either $\mho^\infty$ or $\mho^r_\times$ for some $r \in \omega$.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be fronts on $A \in [\kappa]^\kappa$ such that $\mathcal{G}$ is an extension of $\mathcal{F}$ and $\mathcal{F}$ is $\hat{\mho}^\infty$. If $B \in [A]^\kappa$, then $\mathcal{F}|B$ is $\mho^\infty$, and hence for some $X \in [B]^\omega$ and for each $i \in \omega$, there are $n_i, k_i \in \omega$ and (k_i, n_i, X)-critical elements $c_i \in \mathcal{F}|B$ with $n_i < n_{i+1}$ and $k_i < k_{i+1}$. Let $c'_i = \mathcal{G}((c \restriction k_i)^\frown X)$. Then $c_i \sqsubseteq c'_i$ since $\mathcal{G}$ is an extension of $\mathcal{F}$ and we also have that c'_i is a (k_i, n'_i, X)-critical element of $\mathcal{G}|B$. The lemma now follows by Lemma 8.1.29. □

Lemma 8.1.39. *Suppose that $\mathcal{F}$ is a front on $A \in [\kappa]^\kappa$ of type $\hat{\mho}_\times^{<\infty}$ or $\hat{\mho}_\times^k$ for some $k \in \omega$. Then for any front $\mathcal{G}$ on A which is an extension of $\mathcal{F}$ and any $B \in [A]^\kappa$, we have that $\mathcal{G}|B$ is $\mho_\times^r$ for some $r \in \omega$.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be fronts on $A \in [\kappa]^\kappa$ such that $\mathcal{G}$ is an extension of $\mathcal{F}$ and $\mathcal{F}$ is either $\hat{\mho}_\times^{<\infty}$ or $\hat{\mho}_\times^k$ for some $k \in \omega$. If $B \in [A]^\kappa$ then $\mathcal{F}|B$ is $\mho_\times^k$ for some $k \in \omega$, and hence for some $X \in [B]^\omega$ and for each $i \in \omega$, there are $n_i, r \in \omega$ and (k, n_i, X)-critical elements $c_i \in \mathcal{F}|B$, with $n_i < n_{i+1}$. Let $c_i' = \mathcal{G}((c \restriction k)^\frown X)$. Then $c_i \sqsubseteq c_i'$ since $\mathcal{G}$ is an extension of $\mathcal{F}$ and we also have that c_i' is a (k, n_i', X)-critical element of $\mathcal{G}|B$. The lemma now follows by Lemma 8.1.29. $\square$

Theorem 8.1.40. *Let $\mathcal{F}_0$ be a front on $A \in [\kappa]^\kappa$ of type $\hat{\mho}^\infty$, $\hat{\mho}_\times^{<\infty}$ or $\hat{\mho}_\times^k$ for $k \in \omega$. Then for every $n \in \omega$ there are no $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ such that for each $i < n$, $\mathcal{F}_{i+1}$ is an extension of $\mathcal{F}_i$ or $\mathcal{F}_{i+1} = \mathcal{F}_i|B$ for some $B \in [A]^\kappa$ where $\mathcal{F}_n$ is a κ-barrier.*

Proof. If such $\mathcal{F}_i$ for $i \leqslant n \in \omega$ existed, then each is a restriction of an extension of a $\hat{\mho}^\infty$ or $\hat{\mho}_\times^k$ front. Thus by Lemma 8.1.38 or Lemma 8.1.39, we have that $\mathcal{F}_n$ is either $\hat{\mho}^\infty$ or $\hat{\mho}_\times^r$ for some $r \in \omega$. Therefore $\mathcal{F}_n$ is not a κ-barrier. $\square$

8.2 Existence of badly behaved fronts on κ

So by Theorem 8.1.37, a front on κ is either essentially equivalent to a κ-barrier, or it is one of the types $\hat{\mho}^\infty$, $\hat{\mho}_\times^{<\infty}$ or $\hat{\mho}_\times^k$ for some $k \in \omega$. By Theorem 8.1.40 we can never obtain a κ-barrier from such a front by a process of extending and restricting. Thus we are interested in knowing for which cardinals κ do these types of front exist.

In the introduction of [49], Shelah claims without proof that for 'every block contains a barrier'[1] to hold at κ, then κ has to be Ramsey. In fact, the following partition relation classifies precisely when this occurs, and is implied when κ is Ramsey. Thus for a Ramsey cardinal κ, no $\hat{\mho}^r$, $\hat{\mho}^{<\infty}$, $\hat{\mho}^\infty$, $\hat{\mho}_\times^r$, or $\hat{\mho}_\times^{<\infty}$ fronts on $A \in [\kappa]^\kappa$ can exist for any $r \in \omega$.

[1] By this we mean that for every front $A \in [\kappa]^\kappa$ there is $B \in [A]^\kappa$ such that $\mathcal{F}|B$ is a κ-barrier.

Definition 8.2.1. For cardinals λ, γ and κ, define

$$\kappa \xrightarrow{\text{open}} (\lambda)^{\omega}_{\gamma}$$

iff for all $f : [\kappa]^{\omega} \to \gamma$ such that f is continuous with respect to the product topology on $[\kappa]^{\omega}$ and the discrete topology on γ, there exists $A \in [\kappa]^{\lambda}$ such that $|f\text{"}[A]^{\omega}| = 1$.

We also define

$$\kappa \xrightarrow{\text{open}} (\lambda)^{\omega} \text{ iff } \kappa \xrightarrow{\text{open}} (\lambda)^{\omega}_{2}.$$

Lemma 8.2.2. *$\kappa \xrightarrow{open} (\lambda)^{\omega}_{\gamma}$ iff for every front $\mathcal{F}$ on κ and every $f : \mathcal{F} \to \gamma$, there exists $A \in [\kappa]^{\lambda}$ such that $|f\text{"}(\mathcal{F}|A)| = 1$.*

Proof. ($\longrightarrow$) If $f : \mathcal{F} \to \gamma$ then similarly to the proof of Theorem 7.2.11, there is a continuous function $g : [\kappa]^{\omega} \to \gamma$, such that $g(X) = f(\mathcal{F}(X))$ for every $X \in [\kappa]^{\omega}$. Hence for some $A \in [\kappa]^{\kappa}$ we have that $|g\text{"}[A]^{\omega}| = 1$, which means that $|f\text{"}(\mathcal{F}|A)| = 1$ as required.

($\Longleftarrow$) If $g : [\kappa]^{\omega} \to \gamma$ is continuous, then similarly to the proof of Theorem 7.2.11 there is a front $\mathcal{F}$ on κ and a function $f : \mathcal{F} \to \gamma$ such that $g(X) = f(\mathcal{F}(X))$ for every $X \in [\kappa]^{\omega}$. Hence for some $A \in [\kappa]^{\kappa}$ we have that $|f\text{"}(\mathcal{F}|A)| = 1$, which means that $|g\text{"}[A]^{\omega}| = 1$ as required. ☐

Theorem 8.2.3. *Let $\kappa > \omega$ be a cardinal, then the following are equivalent:*

1. *For every front $\mathcal{F}$ on $A \in [\kappa]^{\kappa}$, there is some $B \in [A]^{\kappa}$ such that the elements of $\mathcal{F}|B$ all have the same length.*
2. *For every front $\mathcal{F}$ on $A \in [\kappa]^{\kappa}$, there is some $B \in [A]^{\kappa}$ such that $\mathcal{F}|B$ is a κ-barrier.*
3. *$\kappa \xrightarrow{open} (\kappa)^{\omega}$*
4. *$\kappa \xrightarrow{open} (\kappa)^{\omega}_{\gamma}$ for any $\gamma < \kappa$.*

Proof. ($1 \longrightarrow 2$). This is trivial since if every element of a front $\mathcal{F}|B$ is the same length then $\mathcal{F}|B$ is a κ-barrier.

($2 \longrightarrow 3$). Suppose that $\kappa \xrightarrow{\text{open}} (\kappa)^{\omega}$ fails. So there is a front $\mathcal{F}$ on κ and some $f : \mathcal{F} \to 2$ such that $\forall A \in [\kappa]^{\kappa}$,

$$|f\text{"}(\mathcal{F}|A)| = 2.$$

Now assuming 3. there is some $B \in [\kappa]^\kappa$ such that $\mathcal{F}|B$ is a κ-barrier. Now define:

$$\mathcal{G} = \{a \in \mathcal{F}|B : f(a) = 0\} \cup \{X \restriction n : X \in [B]^\omega, f(\mathcal{F}(X)) = 1, n = |\mathcal{F}(X^+)| + 2\}.$$

We claim that $\mathcal{G}$ is a front. Firstly, for any $X \in [B]^\omega$, either $f(\mathcal{F}(X)) = 0$ hence $\mathcal{F}(X) \in \mathcal{G}$, or $f(\mathcal{F}(X)) = 1$ hence $X \restriction (|\mathcal{F}(X^+)| + 2) \in \mathcal{G}$.

Now if $a, b \in \mathcal{G}$ with $a \sqsubset b$. Since $\mathcal{G}$ is an extension of $\mathcal{F}$, if $b \in \mathcal{F}$, then there is some $a' \in \mathcal{F}$ with $a' \sqsubseteq a \sqsubset b$, a contradiction. Hence there are some $a', b' \in \mathcal{F}$ with $a' \sqsubset a$, $b' \sqsubset b$ and $f(b') = 1$. But since $a \sqsubset b$ we have that a' and b' are $\sqsubseteq$-comparable and therefore equal. So $f(a') = 1$, and thus $a \notin \mathcal{F}$.

Therefore $a = X \restriction (|\mathcal{F}(X^+)| + 2)$ and $b = Y \restriction (|\mathcal{F}(Y^+)| + 2)$ for some $X, Y \in [B]^\omega$. Thus since $a \sqsubset b$, for some $x, y \in [\kappa]^1$ we have,

$$\mathcal{F}(X^+)^\frown x = a^+ \sqsubset b^+ = \mathcal{F}(Y^+)^\frown y.$$

Therefore $\mathcal{F}(X^+) \sqsubset \mathcal{F}(Y^+)$ which contradicts that $\mathcal{F}$ is a front and from this we conclude that $\mathcal{G}$ is a front on B.

Now if $X \in [B]^\omega$ is such that $f(\mathcal{F}(X)) = 1$ but $f(\mathcal{F}(X^+)) = 0$, then $|\mathcal{G}(X)| = |\mathcal{F}(X^+)| + 2$ and $\mathcal{G}(X^+) = \mathcal{F}(X^+)$. But then we see that

$$|\mathcal{G}(X)| = |\mathcal{G}(X^+)| + 2,$$

and therefore

$$\mathcal{G}(X^+) \sqsubset \mathcal{G}(X)^+,$$

i.e. $\mathcal{G}$ fails the barrier property.

Using 3. we have that there is some $C \in [B]^\kappa$ such that $\mathcal{G}|B$ is a κ-barrier. Therefore for every $X \in [C]^\omega$, if $f(\mathcal{F}(X)) = 1$ then $f(\mathcal{F}(X^+)) = 1$.

Now we know that $|f``(\mathcal{F}|C)| = 2$, so there is some $Y \in [C]^\omega$ such that $f(\mathcal{F}(Y)) = 1$. Let $Z \in [C]^\omega$ be such that $\min Z > \max \mathcal{F}(Y)$ and let $W = \mathcal{F}(Y)^\frown Z \in [C]^\omega$. So

$$1 = f(\mathcal{F}(Y)) = f(\mathcal{F}(W)) = f(\mathcal{F}(W^+)) = f(\mathcal{F}(W^{++})) = ... = f(\mathcal{F}(Z)).$$

In other words, for $D = C \setminus (\max \mathcal{F}(Y))$ we have that if $Z \in [D]^\omega$ then $f(\mathcal{F}(Z)) = 1$, i.e.

$$f``(\mathcal{F}|D) = \{1\}.$$

But this is a contradiction since we had $|f``(\mathcal{F}|D)| = 2$.

($3 \longrightarrow 4$) The following is a modified version of Proposition 7.14 (c) in [22]. We use the characterisation of these partition relations given by Lemma 8.2.2. Suppose $\kappa \overset{\text{open}}{\longrightarrow} (\kappa)^\omega$, let $\gamma < \kappa$ and $f : \mathcal{F} \to \gamma$ for some front $\mathcal{F}$ on κ. Define:

$$\mathcal{G} = \{a^\frown b : a, b \in \mathcal{F}, \max a < \min b\}.$$

Then we claim that $\mathcal{G}$ is a front on κ. First, for any $X \in [\kappa]^\omega$ we have that $\mathcal{F}(X)^\frown \mathcal{F}(X \setminus (\max \mathcal{F}(X))) \sqsubset X$ is a member of $\mathcal{G}$. Now if $a^\frown c \sqsubseteq b^\frown d$ with $a, b, c, d \in \mathcal{F}$, then a and b are $\sqsubseteq$-comparable and hence equal, which means that b and d are $\sqsubseteq$-comparable and hence equal, so that $a^\frown c = b^\frown d$, and indeed $\mathcal{G}$ is a front.

Now let $g : \mathcal{G} \to 2$ be given by $g(a^\frown b) = 1$ iff $f(a) = f(b)$. Using $\kappa \overset{\text{open}}{\longrightarrow} (\kappa)^\omega$ there is some $H \in [\kappa]^\kappa$ such that $|g``(\mathcal{G}|H)| = 1$. Now since $\gamma < \kappa$ there are $a, b \in \mathcal{F}|H$ such that $\max a < \min b$ and $f(a) = f(b)$. Hence by homogeneity

$$|g``(\mathcal{G}|H)| = \{1\}.$$

Now let $c, d \in \mathcal{F}|H$ and pick $e \in \mathcal{F}|H$ with $\min e > \max c, \max d$. Then

$$g(c^\frown e) = 1 = g(d^\frown e),$$

and therefore

$$f(c) = f(e) = f(d).$$

We thus conclude that $|f``(\mathcal{F}|H)| = 1$.

($4 \longrightarrow 1$) Let $\mathcal{F}$ be a front on $A \in [\kappa]^\kappa$. Now let $f : \mathcal{F} \to \omega$ be given by $f(a) = |a|$. So using our partition relation, we can find $B \in [A]^\kappa$ such that $|f``(\mathcal{F}|B)| = 1$ and therefore every element of $\mathcal{F}|B$ has the same length. $\square$

Theorem 8.2.4. *If κ is Ramsey then $\kappa \overset{\text{open}}{\longrightarrow} (\kappa)^\omega$.*

Proof. If κ is Ramsey then $\kappa \longrightarrow (\kappa)^{<\omega}_\omega$. Let $\mathcal{F}$ be a front on $A \in [\kappa]^\kappa$. Now define $f : [A]^{<\omega} \to \omega$ by $f(a) = |b|$ if $b \in \mathcal{F}$ is such that $b \sqsubseteq a$, and $f(a) = 0$ if no such b exists. Thus since κ is Ramsey, there is some $B \in [A]^\kappa$ such that for each $n \in \omega$, we have $|f"[B]^n| = 1$. Therefore every element of $\mathcal{F}|B$ has the same length. We can then use Theorem 8.2.3. $\square$

Question 8.2.5. *If $\kappa \xrightarrow{open} (\kappa)^\omega$ holds then is κ Ramsey?*

We now attempt to find a necessary condition for there to be no $\hat{\mho}^i_\times$ front for some $i \in \omega$.

Definition 8.2.6. For λ, κ, γ cardinals and $r \in \omega$ we say that

$$\kappa \to [\lambda]^r_{\gamma,<\gamma}$$

iff for every $f : [\kappa]^r \to \gamma$ there is some $A \in [\kappa]^\lambda$ such that $|f"[A]^{<\gamma}| < \gamma$.

Theorem 8.2.7. *Suppose that for some $i \leqslant r \in \omega$ there are no $\hat{\mho}^i_\times$ fronts. Then $\kappa \to [\lambda]^r_{\omega,<\omega}$ for every $\lambda \leqslant \kappa$.*

Proof. Suppose for some $i \leqslant r \in \omega$ there are no $\hat{\mho}^i_\times$ fronts and that $\kappa \not\to [\lambda]^r_{\omega,<\omega}$, for some $\lambda \leqslant \kappa$. So there is some $f : [\kappa]^r \to \omega$ such that for every $A \in [\kappa]^\lambda$ we have $|f"[A]^{<\omega}| = \omega$. Let

$$\mathcal{F} = \{a^\frown(X \upharpoonright f(a)) \in [\kappa]^\omega : a \in [\kappa]^r, X \in [\kappa]^\omega\}.$$

Then clearly $\mathcal{F}$ contains an initial segment of every $Y \in [\kappa]^\omega$. If $a, b \in \mathcal{F}$ are such that $a \sqsubseteq b$ then the first r elements of a and b are equal, hence $|a| = |b|$ because their length is determined by the first r elements, so $a \not\sqsubset b$, which implies that $\mathcal{F}$ is a front.

Now, $a^\frown(X \upharpoonright f(a))$ is $(r, f(a), X)$-critical. Hence for all $A \in [\kappa]^\kappa$, since $|f"[A]^{<\omega}| = \omega$ we have that $\mathcal{F}|A$ contains (r, n, X)-critical elements for an unbounded set of $n \in \omega$. Thus $\mathcal{F}$ is $\hat{\mho}^{\leqslant r}_\times$ and therefore by Corollary 8.1.31 we see that $\mathcal{F}|A$ can be restricted to some $\hat{\mho}^i_\times$ front for $i \leqslant r$. $\square$

Chapter 9

Questions and conclusions

Many open questions and areas of possible future interest have been posed within this thesis. We will collect and summarise them here.

9.1 Constructing better-quasi-orders

9.1.1 Partial orders

We first mention again the question reguarding when (ii*) can be fully used in place of (ii) in Definition 3.2.14. A positive answer would likely simplify the statement of the full form of Theorem 3.4.12. For a minimum assumption we would require that every member of $\mathbb{P}$ satisfies (ii*).

Question 9.1.1. *Are there assumptions on $\mathbb{L}$ and $\mathbb{P}$ under which $\mathscr{S}_{\mathbb{P}}^{\mathbb{L}}$ consists of precisely the same class whether or not 3.2.14 (ii) is replaced in its definition by (ii*)?*

Theorem 3.5.12 tells us that some of the largest classes[1] of partial orders known already to preserve bqo are also well-behaved. Thus well-behaved classes of partial orders seem to be just as ubiquitous as classes that preserve bqo. This observation motivates again the question of whether or not these two notions are equivalent, a positive answer to

[1] Such as the class of countable N-free partial orders [55] and the class of σ-scattered linear orders [30].

this question would also imply they are both equivalent to Louveau and Saint-Raymond's notion of *reflecting bad arrays* [33].

Question 9.1.2 (Thomas [54])**.** *Suppose that $\mathcal{O}$ is a concrete category. If $\mathcal{O}$ preserves bqo, then is $\mathcal{O}$ well-behaved?*

We now reiterate some questions that were mentioned in section 3.5, elaborating first on Remark 3.5.10. For a finite set of partial orders $\mathbb{P}$, let $\tilde{\mathbb{P}}$ be the class of countable partial orders whose every finite subset (with the restricted ordering) is isomorphic to a member of $\mathbb{P}$. Pouzet then asked the following question.

Question 9.1.3 (Pouzet [44])**.** *If $\mathbb{P}$ preserves bqo, then is $\tilde{\mathbb{P}}$ bqo?*

Since well-behavedness is much more useful than preservation of bqos, in the absence of an answer to Question 9.1.2, we modify Pouzet's question to the following.

Question 9.1.4. *If $\mathbb{P}$ is well-behaved, then is $\tilde{\mathbb{P}}$ bqo?*

Corollary 3.5.9 bring us substantially closer to a solution of this challenging question. The problem here is that $\tilde{\mathbb{P}}$ could contain a partial order with an infinite indecomposable subset. Indeed if no element of $\tilde{\mathbb{P}}$ contains an infinite indecomposable subset then we have $\tilde{\mathbb{P}} \subseteq \mathscr{C}_{\mathbb{P}}$, since every indecomposable subset of $\tilde{\mathbb{P}}$ is finite and thus is a member of $\mathbb{P}$. Therefore in this case Corollary 3.5.9 answers this question positively.

So what is required is a study of the possible infinite indecomposable partial orders, in particular whether or not they are well-behaved. We note that by the argument of the previous paragraph, a negative solution to the following question (mentioned in section 3.5) would solve Pouzet's question positively.

Question 9.1.5. *Is there an infinite indecomposable partial order P such that $\{P\}$ is well-behaved?*

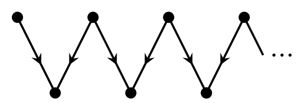

Figure 9.1: An infinite indecomposable partial order P such that $\{P\}$ is not well-behaved.

If the answer to Question 9.1.5 is positive, Pouzet's question is reduced by Corollary 3.5.9 to the following question.

Question 9.1.6. *Suppose that $\mathbb{X}$ is a set of countably infinite, indecomposable partial orders. Is $\mathbb{X}$ well-behaved whenever*

$$\mathbb{X}_0 = \{Q \subseteq P : P \in \mathbb{X}, |Q| < \aleph_0, Q \textit{ is indecomposable}\}$$

is well-behaved?

To see this, assume that Question 9.1.6 has been answered positively and let $\mathbb{P}$ be a well-behaved set of finite partial orders. Then let $\mathbb{X}$ be the class of infinite, indecomposable subsets of members of $\tilde{\mathbb{P}}$. Thus $\tilde{\mathbb{P}} = \mathscr{C}_{\mathbb{P}\cup\mathbb{X}}$ and $\mathbb{X}_0 \subseteq \mathbb{P}$. Therefore $\mathbb{X}_0$ is well-behaved, so by our assumption $\mathbb{X}$ is well-behaved. Then by a simple application of the Galvin and Prikry Theorem 2.1.6, we have that $\mathbb{P} \cup \mathbb{X}$ is well-behaved, so by Corollary 3.5.9 we have that $\mathscr{C}_{\mathbb{P}\cup\mathbb{X}} = \tilde{\mathbb{P}}$ is indeed well-behaved.

A study of the well-behavedness of infinite indecomposable partial orders could not only move us towards an answer of Pouzet's question however. Theorem 3.5.12 is the best we can do so far, however if we can answer either Question 9.1.5 or any of the following questions positively, then by then applying Theorem 3.4.12 with even larger classes of indecomposable partial orders or linear orders, we can improve Theorem 3.5.12 and hence extend Fraïssé's conjecture even further.

Question 9.1.7. *Is there an infinite well-behaved class of indecomposable partial orders?*

Question 9.1.8. *Is there a well-behaved class of linear orders larger than $\mathscr{M}$?*

In answering these questions it is likely that we will come up against the bounds of what is provable with the axioms of ZFC alone, so we are also curious about consistent answers of the previous two questions.

There are consistently larger *bqo* classes of linear orders than $\mathscr{M}$, for example $\mathscr{M} \cup \mathscr{B}$, where $\mathscr{B}$ is the class of Borel linear orders that embed into the lexicographic ordering of $2^{\omega\cdot n}$ for some $n \in \omega$. The class $\mathscr{B}$ is proved to be bqo under projective determinacy by

Louveau and Saint-Raymond in [33]. However as stated in this paper, assuming the axiom of choice, the class $\mathscr{B}$ does not preserve bqo and so it is certainly not well-behaved.

One hope could be to further investigate the result of Martinez-Ranero [37], that the class of Aronszajn lines is bqo under PFA. This motivates the following question, a positive answer to which would consistently extend Theorem 3.5.12 substantially, just by applying Theorem 3.4.12.

Question 9.1.9. *Is the class of Aronszajn lines well-behaved under PFA?*

9.1.2 2-structures

In chapters 3, 4 and 5 we constructed some very large transfinite classes of objects. In particular, the main results of chapters 3 and 5 had a lot in common. The definitions of the two classes $\mathscr{M}_{\mathbb{P}}^{\mathbb{L}}$ and $\mathscr{G}_{\mathbb{G}}^{\mathbb{L}}$ were indeed very similar and both classes share a similar theorem stating that they will be well-behaved when their parameters are too (i.e. Theorem 3.4.12 and Theorem 5.2.6). This seems to suggest that there is a possible generalisation to more abstract objects.

The correct notion here seems to be that of 2-*structures*. These are essentially binary relational structures, with a given labelling, that serves to give extra structure similarly to the labellings for structured trees. For a given class Λ of *labels*, a Λ-2-*structure* is a pair $\langle V, l\rangle$ where $l : V_*^2 \to \Lambda$ is a Λ-colouring of the set $V_*^2 = \{\langle x, y\rangle \in V \times V : x \neq y\}$. For different Λ, the class of Λ-2-structures can be isomorphic to for example, the class of partial orders or the class of graphs, among many others. For a comprehensive reference on 2-structures see [12].

Indeed, if Λ is quasi-order then the class of Λ-2-structures has a notion of embedding, where embeddings must increase the values of labels. This class then a forms a concrete category and thus can be quasi-ordered as usual under embeddability (similarly to partial orders and graphs). One can then generalise the notions of a sub-Λ-2-strucutre, an interval of a Λ-2-strucutre and an indecomposable Λ-2-strucutre. The bqo theory of classes of 2-structures has been studied independently by Christian Delhommé in as yet unpublished work. In private communication, he kindly shared with the author an early version of a paper in this area [9], in which he proved the following theorem.

Theorem 9.1.10 (Delhommé, [9])**.** *Let Λ be a quasi-ordered class of labels and $\mathscr{X}$ be a class of countable Λ-2-structures such that $\mathscr{X}$ contains all of its sub-Λ-2-structures. Then $\mathscr{X}$ is well-behaved whenever the class of its indecomposable members is well-behaved.*

This is a generalisation of Corollary 3.5.9 to 2-structures. Indeed it seems extremely likely that with the correct Hausdorff type theorem as in Section 3.3, we could obtain a similar generalisation of Theorem 3.4.12, that also generalises Delhommé's result into the transfinite. It is more than likely that a method similar to that of Chapter 5 would give a proof.

9.2 Abstract wqo and bqo

While our classification given in Section 6.2.4 narrows down the possible situations significantly, we still feel that this result can be improved upon in the future. Either by refining the classification or by finding examples of Ramsey spaces of each of the different types. We mention again the questions that were posed in this section.

Question 9.2.1. *Is there a Ramsey space $\mathcal{R}$ such that $\mathcal{R}$-wqo is neither of type 1 nor equivalent to $\mathbb{N}^{[\infty]}$-bqo$_k$ for any $k \in \omega$?*

Question 9.2.2. *Is there a Ramsey space of type other than 1, 4 and 5?*

The following is of particular interest.

Question 9.2.3. *Is there a Ramsey space $\mathcal{R}$ and some $\alpha > \omega$ least such that α^* is not $\mathcal{R}$-wqo?*

The least α for which this is possible is ω^ω (ordinal exponentiation), which can be seen by Corollary 6.2.25. We hope that at least that this motivation could inspire the discovery of new and interesting Ramsey spaces, with the results of Section 6.2 helping to narrow down the possibilities.

While most *natural* shifts seem to be strong shifts, (such as those of $\mathbb{N}^{[\infty]}$, $\mathrm{FIN}_k^{[\infty]}$ and $W_{L_v}^{[\infty]}$) it is not difficult to find shifts even on these spaces that are not strong. We note that Pequignot has shown in [42] that there are many shifts on the Ellentuck space, whose corresponding notion of bqo is equivalent to the usual one, many of which are not strong. One such shift is that from Example 6.3.11.

This hints towards one possible future direction. It may be possible to find a weakening of the notion of strong shift,[2] such that for a fixed Ramsey space $\mathcal{R}$, any such shift on $\mathcal{R}$ results an equivalent notion of $\mathcal{R}$-bqo. Afterwards, one could attempt to prove that there is always one strong shift for each Ramsey space $\mathcal{R}$. If this is the case, then by Theorem 6.3.13 every notion of $\mathcal{R}$-bqo with respect to a shift satisfying the weaker property depends only on the notion of $\mathcal{R}$-wqo with respect to the strong shift. Furthermore, by Corollary 6.3.15, any two Ramsey spaces with shifts satisfying the weaker property would have equivalent notions of bqo, as soon as their respective wqo notions with their respective strong shifts are equivalent.

9.3 Fronts and barriers on an uncountable cardinal

In Chapter 8 we found that for any front $\mathcal{F}$ on some $A \in [\kappa]^\kappa$, either there is a finite process of restriction and extension, the result of which is a κ-barrier, in which case $\mathcal{F}$ is of type $\hat{\mho}^\infty$, $\hat{\mho}_\times^{<\infty}$ or $\hat{\mho}_\times^r$ (Theorem 8.1.37) in which case there is no such process (Theorem 8.1.40).

This makes types $\hat{\mho}^\infty$, $\hat{\mho}_\times^{<\infty}$ and $\hat{\mho}_\times^r$ the most interesting. Motivating the following general question, an answer to which would characterise the fronts which contain an extension to a κ-barrier.

Question 9.3.1. *For which cardinals $\kappa > \omega$ are there fronts of type $\hat{\mho}^\infty$, $\hat{\mho}_\times^{<\infty}$ or $\hat{\mho}_\times^r$?*

Theorems 7.2.4 and 8.2.7 give initial results towards an answer to this question: none of these types of fronts on an $A \in [\kappa]^\kappa$ can exist when $\kappa \xrightarrow{\text{open}} (\lambda)^\omega$; and if there is no $\hat{\mho}_\times^r$ front on any $A \in [\kappa]^\kappa$, then $\kappa \to [\lambda]^r_{\omega,<\omega}$ for every $\lambda \leqslant \kappa$. We are curious about possible

[2] An ideal result would be just for any shift.

analogues of Theorem 8.2.7 to the types $\hat{\mho}^{\infty}$ and $\hat{\mho}^{<\infty}_{\times}$. It also seems likely that these results can be strengthened, giving a more precise description of the uncountable cardinals at which these types of front exist.

Our interest is also sparked by Shelah's comment in the introduction of [49] - that if 'every block contains a barrier' holds at κ, then κ *'has to be Ramsey'*. He could simply have meant that the method of the original proof requires κ to be Ramsey (indeed, it does). But we ask the following question which would settle whether or not it is possible for every block to contain a barrier at non-Ramsey cardinal (by Theorem 7.2.4).

Question 9.3.2. *If $\kappa \overset{open}{\longrightarrow} (\kappa)^{\omega}$ holds then is κ Ramsey?*

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List of Symbols

$-1+\alpha$	$\alpha - 1$ when $\alpha < \omega$, α when $\alpha \geqslant \omega$. 160
$-\infty$	Adjoined minimal element of a quasi-order. 30
$2^{<\omega}$, $-2^{<\omega}$	Finite sequences of $\{0,1\}$, ordered by $\sqsubseteq$ and $\sqsupseteq$. 53
$2^{<\omega}_{\perp}$	Finite sequences of $\{0,1\}$, $s < t$ iff $\exists u$, $u^\frown\langle 0\rangle \sqsubseteq s$, $u^\frown\langle 1\rangle \sqsubseteq t$. 53
$B_{\hat{r}}$, $\mathbb{B}^{\mathbb{L}}_{\mathcal{O}}$	Structured r-chain, $\{B_{\hat{r}} : \hat{r} \in \mathbb{F}^{\mathbb{L}}_{\mathcal{O}}\}$. 98
$G[S]$	$\langle S, E(G) \cap (S \times S)\rangle$. 108
$H_{\hat{r}}$, $\mathbb{H}^{\mathbb{L}}_{\mathbb{P}}$	Nested interval partial order, $\{H_{\hat{r}} : \hat{r} \in \mathbb{E}^{\mathbb{L}}_{\mathbb{P}}\}$. 49
N	The partial order that looks like a letter N. 37
Q^{γ}	$\{f : \gamma \to Q\} \subseteq \mathcal{O}(Q)$. 36
U_{γ}	Underlying set of an object γ from a concrete category. 32
$V(G), E(G)$	Vertex set and edge relation of a graph $G = \langle V(G), E(G)\rangle$. 108
$W^{[\infty]}_{L_v}$	Ramsey space of infinite sequences of variable words. 127
$X \restriction n$	Least n elements of X. 157
X_n	$X = \langle X_0, X_1, X_2, ...\rangle$. 157
$X^{x,m}_n$, $X^{x,*}_n$	Subsets of a σ-scattered partial order X. 82
$[A]^{\lambda}, [A]^{<\lambda}$	$\{X \subseteq A : \lvert A\rvert = \lambda\}$, $\{X \subseteq A : \lvert A\rvert < \lambda\}$. 28, 157
$[a, A], [n, A]$	$\{X \in \mathcal{R} : a \sqsubset X, X \leqslant A\}$, $[r_n(A), A]$. 124
X^{+}	For $X \in [\kappa]^{<\kappa}$, $X^{+} = X \setminus \min X$. 158
$.^{+}$, $.^{(+)^n}$	Shift map, iterated shift map. 128, 129

$\mathcal{AR}, \mathcal{AR}_n$	Set of approximations for a given Ramsey space $\mathcal{R}$, the range of r_n. 123
$\mathcal{F}^2$	For $\mathcal{F}$ a front on $A \in \mathcal{R}$, $\mathcal{F}^2 = \{a \cup^*_Y b : a, b \in \mathcal{F}, Y \leqslant A, a \lhd_Y b\}$. 130
$\mathcal{F}^C$	For $\mathcal{F}$ a front on $A \in \mathcal{R}$ and $C \subseteq \mathcal{F}$, $\mathcal{F}^C = C \cup \{x \cup^*_Z y : x \in \mathcal{F} \setminus C, y \in \mathcal{F}, Z \leqslant A, a \lhd_Z b\}$; for $\mathcal{F}$ a front on $A \in [\kappa]^\kappa$ and $C \subseteq \mathcal{F}$, $\mathcal{F}^C = C \cup \{x \cup y : x \in \mathcal{F} \setminus C, y \in \mathcal{F}, a \lhd b\}$. 150, 161
$\mathcal{F}\vert B$	For a front $\mathcal{F}$ on $A \in \mathcal{R}$, $\mathcal{F}\vert B = \{a \in \mathcal{F} : (\exists Y \leqslant B), a \sqsubseteq Y\}$. 127, 158
$\mathrm{FIN}_k^{[\infty]}$	Ramsey space of infinite block sequences of vectors. 126
$\mathcal{R}$	Ramsey space $(\mathcal{R}, \leqslant, r)$. 123
$\mathrm{SSR}(a; b, c)$	$(\forall R \in \{<, >, \perp\})$, $aRb \longleftrightarrow aRc$. 52
$\mathrm{rank}_{\mathscr{U}}(T)$	For $T \in \mathscr{U}$, $\min\{\alpha : T \in \mathscr{U}_\alpha\}$; for $T \in \mathscr{U}^{\mathbb{L}}$, $\min\{\alpha : T \in \mathscr{U}^{\mathbb{L}}_\alpha\}$. 72, 95
$\mathrm{root}(T)$	The root of T. 39
$a \cup^*_Y b$	$r_n(Y)$ where $n \in \omega$ is least such that $n_Y \geqslant \vert b\vert$ and $n \geqslant \vert a\vert$. 130
$\Upsilon \sqsubseteq \Gamma$	For $\Upsilon, \Gamma \in \{\mho^r, \mho^\infty, \mho^{<\infty}, \mho_\times, \mho^r_\times : r \in \omega\}$ $\Upsilon \sqsubseteq \Gamma$ iff any front of type Υ can be extended to a front of type Γ. 182
α^*, On^*	A reversed copy of α, $\{\alpha^* : \alpha \in \mathrm{On}\}$ 16. 36
$\mathbf{a}(f)$	The arity of f, i.e. $f = \sum_{\mathbf{a}(f)}$. 43
$\mathsf{G}_0, \mathsf{G}_1$	$V(\mathsf{G}_0) = V(\mathsf{G}_1) = 2^{<\omega}$; $\langle s, t\rangle \in E(\mathsf{G}_0) \leftrightarrow s \sqsubseteq t$ or $t \sqsubseteq s$; $\langle s, t\rangle \in E(\mathsf{G}_1) \leftrightarrow s \not\sqsubseteq t$ or $t \not\sqsubseteq s$. 109
$\bigcup_{i \in X} G_i$	$\langle \bigcup_{i \in X} V(G_i), \bigcup_{i \in X} E(G_i)\rangle$. 109
$\overline{\mathcal{F}}$	For a front $\mathcal{F}$, $\overline{\mathcal{F}} = \{x : (\exists y \in \mathcal{F}), x \sqsubseteq y\}$. 127, 158
$\overline{\mathbb{L}}$	Least class containing $\mathbb{L}$ and closed under L-sums for $L \in \overline{\mathbb{L}}$. 38
$\cong$	Isomorphic. 29

$\mathscr{C}_{\mathbb{P}}$	Class of countable partial orders whose indecomposable subsets are isomorphic to members of $\mathbb{P}$. 90
$\mathrm{depth}_X(a)$	$\min\{n : a \leqslant_{\text{fin}} r_n(X)\}$. 124
$\downarrow x$, $\uparrow x$, $\downarrow x$, $\uparrow x$	$\{y : y \leqslant x\}$, $\{y : y \geqslant x\}$, $\{y : y < x\}$, $\{y : y > x\}$. 37
$\hat{\gamma}$	$\hat{\gamma} \in \mathcal{O}(Q) \quad \longrightarrow \quad \gamma \in \mathcal{O}$, $\hat{\gamma} : \gamma \to Q$. 34
$\sqsubseteq, \sqsubset$	Initial segment, strict initial segment of sequences. 38
$\sqsubseteq, \sqsubset$	For $a, b \in \mathcal{AR}$, $a \sqsubseteq b$ iff $(\exists X \in \mathcal{R})(\exists n, m \in \omega)$, $a = r_n(X)$, $b = r_m(X)$, $n \leqslant m$. $a \sqsubset b$ iff $a \sqsubseteq b$ and $n < m$; for $X \in \mathcal{R}$, $a \sqsubset X$ iff $a \sqsubseteq X$ iff $(\exists n \in \omega), a = r_n(X)$. 124
$\kappa \xrightarrow{\text{open}} (\lambda)^\omega$	$\kappa \xrightarrow{\text{open}} (\lambda)^\omega_2$. 196
$\kappa \xrightarrow{\text{open}} (\lambda)^\omega_\gamma$	$\forall$ continuous $f : [\kappa]^\omega \to \gamma$, $\exists A \in [\kappa]^\lambda$ such that $\lvert f\text{"}[A]^\omega \rvert = 1$. 196
$\lhd'$	For $a, b \in \mathcal{F}$, $a \lhd' b$ iff $a = b$ or $\exists x_0, ..., x_n \in \mathcal{F}$ such that $a \lhd x_0 \lhd ... \lhd b$. 137
$\lhd_X, \lhd$	For $a, b \in \mathcal{AR}$, $a \lhd_X b$ iff $a \sqsubseteq X$ and $b \sqsubseteq X^+$, $a \lhd b$ iff $(\exists Y \in \mathcal{R})$, $a \lhd_Y b$; for $a, b \in [\kappa]^{<\omega}$, $a \lhd b$ iff $b \sqsubset a^+$ or, $a^+ \sqsubseteq b$ and $\min a < \min b$. 129
$\mathbb{E}^{\mathbb{L}}_{\mathbb{P}}$	$\{\hat{r} \in \overline{\mathbb{L}}(\mathbb{P}(\mathrm{A}_2)) : (\forall i \in r), [\hat{r}(i) = \hat{p}_i \to (\exists! j \in p_i), \hat{p}_i(j) = 1]\}$. 48
$\mathbb{F}^{\mathbb{L}}_{\mathcal{O}}$	$\{\hat{r} \in \overline{\mathbb{L}}(\mathcal{O}(\mathrm{A}_2)) : (\forall i \in r), (\exists! x \in \mathrm{dom}(\hat{r}(i))), \hat{r}(i)(x) = 1\}$. 98
$\mathbb{N}^{[\infty]}$	Ellentuck space $([\omega]^\omega, \subseteq, r)$. 123
$\mathbb{T}_{\mathcal{O}}$	If $\mathbb{T} \subseteq \mathscr{T}$, the concrete category of $\mathcal{O}$-structured trees of $\mathbb{T}$; if $\mathbb{T} \subseteq \mathscr{E}$, concrete category of $\mathcal{O}$-structured pseudo-trees of $\mathbb{T}$. 39, 97
$\mathcal{C}_\alpha, \mathcal{C}_{<\alpha}$	Stratification of levels of $\tilde{\mathcal{C}}$. 43
$\mathcal{O}(Q)$	Q-coloured members of $\mathcal{O}$. 34
$\mathcal{P}(Q)$, $\mathcal{P}_\alpha(Q)$, $\mathcal{P}_\infty(Q)$	Power set quasi-order, iterated power set, $\bigcup_\alpha \mathcal{P}_\alpha(Q)$. 31
$\mathcal{S}^{\mathbb{L}}_{\mathbb{P}}$	The admissible operator algebra $\langle\{1\}, \{\sum_H : H \in \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}\}, \mathbb{H}^{\mathbb{L}}_{\mathbb{P}}\rangle$. 51

$\mathcal{X}^{\complement}$	For $\mathcal{X} \subseteq \mathcal{AR}_n$, $\mathcal{X}^{\complement} = \mathcal{AR}_n \setminus \mathcal{X}$. 124
$\mathfrak{F}(\mathcal{R})$	$\{\langle \mathcal{F}, \lhd \cup \rhd \rangle : \mathcal{F}$ is a simple front on $\mathcal{R}\}$. 135
$\mathfrak{G}(\mathcal{R})$	The graphs of $\mathfrak{F}(\mathcal{R})$ with arbitrarily large finite complete subgraphs, but with no infinite complete subgraph. 135
$\mathfrak{H}(\mathcal{R})$	$\{(\mathcal{F}, \lhd') : \mathcal{F}$ is a simple front on $\mathcal{R}\}$. 142
$\mathfrak{P}(\mathcal{R})$	A subset of $\mathfrak{H}(\mathcal{R})$ with additional properties. 142
$\mathscr{P}$	The concrete category of partial orders with embeddings. 33
$\mathrm{TC}(A)$	Transitive closure of $A \in \mathcal{P}_\alpha(Q)$. 31
$\mathscr{D}^{\mathbb{L}}_{\mathbb{G}}$	Class of graphs with indecomposable subsets in $\mathbb{G}$ and chains of intervals in $\overline{\mathbb{L}}$. 109
$\mathscr{E}$, $\mathscr{E}^{\mathbb{L}}$	Concrete categories of all pseudo-trees, and $\mathbb{L}$-trees. 95
$\mathscr{G}^{\mathbb{L}}_{\mathbb{G}}$	Class of generalised σ-scattered graphs. 116
$\mathscr{H}^{\mathbb{L}}_{\mathbb{G}}$	Class of generalised scattered graphs. 109
$\mathscr{K}_{\mathbb{G}}$	Class of countable graphs whose indecomposable induced subgraphs are isomorphic to members of $\mathbb{G}$. 119
$\mathscr{P}^{\mathbb{L}}_{\mathbb{P}}$	Class of partial orders with indecomposable subsets in $\mathbb{P}$ and chains of intervals in $\overline{\mathbb{L}}$. 54
S, B^+, B^-, C, Q, A	Pathological decomposition trees. 76
$\mathrm{Card}, \mathrm{On}$	The classes of cardinals and ordinals. 36
$\hom_{\mathcal{O}}(\gamma, \delta)$	The class of $\mathcal{O}$-morphisms from γ to δ. 32
$\mathrm{obj}(\mathcal{O})$, $\hom(\mathcal{O})$	Objects and morphisms of a concrete category $\mathcal{O}$. 32
$\mathrm{rank}_{\mathcal{F}}(a)$	$\mathrm{rank}(a)$, considering a as a member of the tree $\overline{\mathcal{F}}$ ordered by $\sqsubseteq$. 158
$\hat{\mho}^{\infty}$, $\hat{\mho}^{<\infty}$, $\hat{\mho}^{r}_{\times}$, $\hat{\mho}^{<\infty}_{\times}$	Classification types of a front on $A \in [\kappa]^\kappa$. 191
$\mho\mathcal{F}$	$\{(a^\frown X) \restriction n \in [A]^{<\omega} : a \in \mathcal{F}, n = \sup(\{\lvert a\rvert\} \cup \{\lvert b\rvert - 1 : b \in \mathcal{F}, (\exists * \in [\kappa]^1), b \sqsubset {*}^\frown a^\frown X\})\}$. 175
$\mho^n\mathcal{F}$	$\mho^0\mathcal{F} = \mathcal{F}$, $\mho^{n+1}\mathcal{F} = \mho(\mho^n\mathcal{F})$. 175
$\mho^r$, $\mho^\infty$, $\mho^{<\infty}$, $\mho_\times$, $\mho^r_\times$	Classification types of a front on $A \in [\kappa]^\kappa$. 179

π_0, π_1	Projection functions for $\mathcal{F}^2$; $\pi_0(a \cup^*_Y b) = a$, $\pi_1(a \cup^*_Y b) = b$. 130
$\preceq, \prec$	Induced subgraph, $H \preceq G$ iff $V(H) \subseteq V(G)$ and $(\forall x, y \in V(H))$, $x \sim_H y \leftrightarrow x \sim_G y$; $H \prec G$ iff $H \preceq G$ and $V(H) \subset V(G)$. 108
$\preceq_{\mathcal{F}}$	A linear extension of the reverse of $\lhd'$. 138
$\prod_{i\in\zeta,\gamma<\kappa_i} T_i^\gamma$	ζ-tree-sum. 70
$\mathrm{rank}(\mathcal{F})$	For a front $\mathcal{F}$, $\mathrm{rank}(\mathcal{F})$ is the tree rank of $\overline{\mathcal{F}}$ ordered by $\sqsubseteq$. 127, 158
$\mathrm{rank}(t), \mathrm{rank}(T)$	$t \in T$, $\sup\{\mathrm{rank}(s)+1 : t <_T s\}$, $\mathrm{rank}(\mathrm{root}(T))$. 39
$\mathrm{rank}(x)$	For $x \in \tilde{\mathcal{C}}$, $\min\{\alpha : x \in \mathcal{C}_\alpha\}$. 44
$\mathscr{S}$, $\mathscr{M}$, $\mathscr{C}$	The classes of scattered, σ-scattered and countable linear orders. 37
$\mathscr{S}^{\mathbb{L}}_{\mathbb{P}}$	Class of generalised scattered partial orders. 53
$\mathscr{U}$, $\mathscr{U}_\alpha$	Scattered trees, stratification of scattered trees. 72
$\mathscr{U}^{\mathbb{L}}$, $\mathscr{U}^{\mathbb{L}}_\alpha$	Scattered $\mathbb{L}$-trees, stratification of scattered $\mathbb{L}$-trees. 95
$\sim_G, \sim$	For $x, y \in V(G)$, $x \sim y$ iff $x \sim_G y$ iff $\langle x, y\rangle \in E(G)$. 108
$\mathscr{M}^{\mathbb{L}}_{\mathbb{P}}$	Class of generalised σ-scattered partial orders. 81
$\mathrm{succ}(t)$	Set of immediate successors of t. 39
$\sum_{p\in P} P_p$, $\sum_P$	Lexicographic P-sum. 38
$\sum_{u\in B_{\hat{r}}} \hat{T}_u$	Structured pesudo-tree sum. 99
$\sum_{x\in H} G_x$	Graph sum. 109
$\tilde{\mathcal{C}}$	Closure of $\{1\}$ under functions of $\mathcal{F}$, for $\mathcal{C} = \langle\{1\}, \mathcal{F}, \mathcal{A}\rangle$. 43
$\mathscr{W}$, $\mathscr{R}$, $\mathscr{T}$	The concrete categories of rooted well-founded trees, rooted trees of height at most ω, and all trees. 39
$k^\frown k'$, $\mathrm{ot}(k)$	Concatenation, order type. 38
l^T_v	For $T \in \mathscr{T}_{\mathcal{O}}$, labelling function $\mathrm{succ}(v) \to \gamma_v$ at a point $v \in T$; for $T \in \mathscr{E}_{\mathcal{O}}$, labelling function $\upharpoonleft v \to \gamma_v$ at a point $v \in T$. 40, 98

n_X	$\lvert(r_n(X))^+\rvert$. 129
r_n	nth approximation $r_n(X) = r(X, n)$ for $X \in \mathcal{R}$, for $(\mathcal{R}, \leqslant, r)$. 123
$x \wedge y$	$\sup(\downarrow x \cap \downarrow y)$. 37
${}^{u}\!\downarrow v$	For $v \in T \in \mathscr{T}_{\mathcal{O}}$, $\{t \in T : (\exists t' \in \mathrm{succ}(v)), t \geqslant t', l_v^T(t') = u\}$; for $v \in T \in \mathscr{E}_{\mathcal{O}}$, $\{t \in T : t > v, l_v^T(t) = u\}$. 41, 98
$\lvert a\rvert$	For $a \in \mathcal{AR}$, $\lvert a\rvert = n$ whenever $\exists X \in \mathcal{R}$ such that $a = r_n(X)$. 124
A_κ,C_n	Antichain of size κ, chain of length n. 37

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