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Wave Forces On A Vertical Cylinder With Non-Circular Cross Section

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1 Introduction

We consider the linear problem of water waves scattering by a vertical cylinder with non-circular cross section extending from the sea bottom to the free surface in water of finite depth h. We assume a plane wavetrain incident from $x \sim -\infty$ and propagating at an angle α to the positive $x-$direction toward a vertical cylinder whose cross section is described by the equation $r = R + \varepsilon f(\theta)$ with $\varepsilon \ll 1$. The function $f(\theta)$ describes the deviation of the shape of the cylinder from the circular one, $f(\theta) = 0$ corresponds to the circular cylinder with radius R. The problem of wave scattering by a nearly circular cylinder was formulated in [1]. The top view of the problem is shown in Figure 1. The problem of wave diffraction by a vertical cylinder has been solved by a number of researchers for many different shapes. The challenge of the present study is to solve the complex body geometries with less effort. The abstract presents the results which have been obtained for one simple geometry: a cylinder with elliptic cross section.

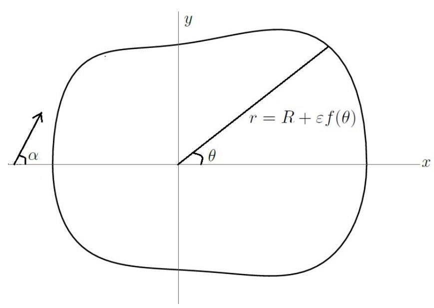

Figure 1: Top view of the problem configuration.

2 Mathematical Formulation of The Problem

The linear boundary problem is formulated with respect to the velocity potential $\Phi(r, \theta, z, t)$

$$\Phi(r, \theta, z, t) = \Re\left\{\frac{gA}{\omega}\frac{\cosh[k(z+h)]}{\cosh(kh)}\phi(r,\theta)e^{-iwt}\right\},$$

where ϕ satisfies the Helmholtz equation $(\nabla^2 + k^2)\phi = 0$ in the flow region, A is the incident wave amplitude, $k = \frac{2\pi}{\lambda}$ is the wave number, λ is the incident wave length, ω is the wave frequency related to the wave number k by the dispersion relation $\omega^2 = gk\tanh(kh)$, where g is the gravitational acceleration. The coordinate system (r, θ, z) is used with the origin at the free surface and the z-axis directed upwards. The axis of the corresponding circular cylinder with $\varepsilon = 0$ coincides with the z-axis.

The boundary condition on the cylinder $r = R + \varepsilon f(\theta)$ is

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad r = R + \varepsilon f(\theta), \quad -h < z < 0, \tag{1}$$

where $\vec{n}$ is the unit normal vector on the cylinder. This boundary condition can be written as

$$\frac{\partial \phi}{\partial r}(R+\varepsilon f(\theta),\theta) - \frac{\varepsilon f'(\theta)}{\big[R+\varepsilon f(\theta)\big]^2}\frac{\partial \phi}{\partial \theta}(R+\varepsilon f(\theta),\theta) = 0. \tag{2}$$

We approximate the boundary condition (2) up to $\mathcal{O}(\varepsilon^5)$ using the Taylor expansion at $r = R$ and substitute the fifth order asymptotic expansion of the potential ϕ

$$\phi(r,\theta) = \phi_0(r,\theta) + \varepsilon\phi_1(r,\theta) + \varepsilon^2\phi_2(r,\theta) + \varepsilon^3\phi_3(r,\theta) + \varepsilon^4\phi_4(r,\theta) + \mathcal{O}(\varepsilon^5), \tag{3}$$

into the boundary condition (2) with the result

$$\begin{aligned}
&\phi_{0,r} + \varepsilon\left[\phi_{1,r} + f(\theta)\phi_{0,rr} - \frac{f'(\theta)}{R^2}\phi_{0,\theta}\right] \\
&+\varepsilon^2\left[\phi_{2,r} + f(\theta)\phi_{1,rr} - \frac{f'(\theta)}{R^2}\phi_{1,\theta} + \frac{f^2(\theta)}{2}\phi_{0,rrr} + \frac{2f(\theta)f'(\theta)}{R^3}\phi_{0,\theta} - \frac{f(\theta)f'(\theta)}{R^2}\phi_{0,r\theta}\right] \\
&+\varepsilon^3\left[\phi_{3,r} + f(\theta)\phi_{2,rr} - \frac{f'(\theta)}{R^2}\phi_{2,\theta} + \frac{f^2(\theta)}{2}\phi_{1,rrr} + \frac{2f(\theta)f'(\theta)}{a^3}\phi_{1,\theta} - \frac{f(\theta)f'(\theta)}{R^2}\phi_{1,r\theta}\right. \\
&\qquad\left. + \frac{2f^2(\theta)f'(\theta)}{R^3}\phi_{0,r\theta} + \frac{f^3(\theta)}{6}\phi_{0,rrrr} - \frac{3f^2(\theta)f'(\theta)}{R^4}\phi_{0,\theta} - \frac{f^2(\theta)f'(\theta)}{2R^2}\phi_{0,rr\theta}\right] \\
&+\varepsilon^4\left[\phi_{4,r} + f(\theta)\phi_{3,rr} - \frac{f'(\theta)}{R^2}\phi_{3,\theta} + \frac{f^2(\theta)}{2}\phi_{2,rrr} - \frac{f(\theta)f'(\theta)}{a^2}\phi_{2,r\theta} + \frac{2f(\theta)f'(\theta)}{R^3}\phi_{2,\theta}\right. \\
&\quad - \frac{3f^2(\theta)f'(\theta)}{R^4}\phi_{1,\theta} - \frac{f^2(\theta)f'(\theta)}{2a^2}\phi_{1,rr\theta} + \frac{2f^2(\theta)f'(\theta)}{a^3}\phi_{1,r\theta} + \frac{f^3(\theta)}{6}\phi_{1,rrrr} \\
&\quad\left. - \frac{3f^3(\theta)f'(\theta)}{R^4}\phi_{0,r\theta} + \frac{f^3(\theta)f'(\theta)}{a^3}\phi_{0,rr\theta} - \frac{f^3(\theta)f'(\theta)}{6a^2}\phi_{0,rrr\theta} + \frac{f^4(\theta)}{24}\phi_{0,rrrrr} + \frac{4f^3(\theta)f'(\theta)}{R^5}\phi_{0,\theta}\right] = 0,
\end{aligned} \tag{4}$$

where the functions are computed at $r = R$. Since the right hand side of this equation is zero, the coeffcients of ε^i, $i = 0,1,2,3,4$, on the left hand side of (4) are equal to zero. Using (4) we derive five boundary conditions for five unknown potentials $\phi_i(r,\theta)$, $i = 0,1,2,3,4$, first two of them are:

$$\phi_{0,r}(R,\theta) = 0, \tag{5}$$

$$\phi_{1,r}(R,\theta) = \frac{1}{R^2}f'(\theta)\phi_{0,\theta}(R,\theta) - f(\theta)\phi_{0,rr}(R,\theta). \tag{6}$$

It is clear that $\phi_0(r,\theta)$ is the velocity potential of the diffraction problem for the circular cylinder $r = R$ with the solution (see [1])

$$\phi_0(r,\theta) = \sum_{m=0}^{\infty}\epsilon_m i^m\left[J_m(kr) - \frac{J_m'(kR)}{H_m^{(1)'}(kR)}H_m^{(1)}(kr)\right]\cos[m(\theta-\alpha)],$$

which satisfies equation (5), where ϵ_m is the Neumann symbol which is given by $\epsilon_0 = 1$, $\epsilon_m = 2$, $m \geq 1$. The series converges exponentially as $m \to \infty$.

The most general representations of $\phi_i(r,\theta)$, $i = 1,2,3,4$, which satisfy the radiation condition at infinity are

$$\phi_i(r,\theta) = \sum_{m=0}^{\infty}\Big[C_{i,m}\cos[m(\theta-\alpha)] + D_{i,m}\sin[m(\theta-\alpha)]\Big]H_m^{(1)}(kr),$$

where the evanescent modes are not included (see [1]) and $C_{i,m}$ and $D_{i,m}$, $i = 0,1,2,3,4$, are unknown coeffcients. The coefficients can be determined using the boundary conditions (5),(6) and the other 3 conditions and hence we can find the velocity potentials with the accuracy $\mathcal{O}(\varepsilon^5)$.

We also assume that $f(\theta)$ can be written as a Fourier series

$$f(\theta) = \frac{f_0^c}{2} + \sum_{m=1}^{\infty} [f_m^c \cos(m\theta) + f_m^s \sin(m\theta)],$$

where the coeffcients f_i^c and f_i^s, $i = 0, 1, 2, \ldots$ depend on a particular shape of the vertical cylinder in waves. If the function $f(\theta)$ is independent of ε then the right hand side of the conditions (5), (6) and the other 3 conditions depends only on θ. So we can write these conditions as $\phi_{i,r}(a, \theta) = G_i(\theta)$, $i = 0, 1, 2, 3, 4$, where $G_i(\theta)$ are represented by their Fourier series. After writing $G_i(\theta)$ as Fourier series, we can find the unknown coefficients $C_{i,m}$ and $D_{i,m}$, $i = 0, 1, 2, 3, 4$.

If the function $f(\theta, \varepsilon)$ depends on θ and ε, then we can use the asymptotic expansion of $f(\theta, \varepsilon)$ as $\varepsilon \to 0$:

$$f(\theta, \varepsilon) = f_0(\theta) + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \varepsilon^3 f_3(\theta) + \varepsilon^4 f_4(\theta) + \mathcal{O}(\varepsilon^5),$$

or higher order, and substituting this into (4) and then applying the same procedure as in the previous case we can find the unknown coefficients. As an example of this case, we have solved a problem for the cylinder with elliptic cross section of small eccentricity in the next section and calculated the hydrodynamic forces acting on this cylinder. The x and y components of the hydrodynamic force due to the fluid motion are given by

$$F_x = \Re\left\{ \frac{-i\rho g A \tanh(kh)}{k} \left[\int_0^{2\pi} \phi(a + \varepsilon f(\theta), \theta)[\varepsilon f'(\theta) \sin\theta + [a + \varepsilon f(\theta)] \cos\theta] \, \mathrm{d}\theta \right] e^{-iwt} \right\}, \quad (7)$$

$$F_y = \Re\left\{ \frac{-i\rho g A \tanh(kh)}{k} \left[\int_0^{2\pi} \phi(a + \varepsilon f(\theta), \theta)[-\varepsilon f'(\theta) \cos\theta + [a + \varepsilon f(\theta)] \sin\theta] \, \mathrm{d}\theta \right] e^{-iwt} \right\}. \quad (8)$$

Dividing F_x and F_y by $\rho g A \pi a^2 \tanh(kh)$, we arrive at the non-dimensionalized force components $\tilde{F}_x$ and $\tilde{F}_y$.

3 Example: Elliptic cylinder

The ellipse's equation in the polar coordinates with the origin at the focus reads

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}, \quad (9)$$

where $e = \sqrt{1 - \frac{b^2}{a^2}}$, $0 < e < 1$, is the eccentricity of the ellipse, a is semi-major axis, b is semi-minor axis. Assuming $e \ll 1$ and setting $e = \varepsilon$ we can write (9) in the form of Fourier series and then using Taylor expansion about $\varepsilon = 0$ we obtain

$$\begin{aligned} r = R + \varepsilon f(\theta) &= a\sqrt{1 - \varepsilon^2} + 2a\sqrt{1 - \varepsilon^2} \sum_{n=1}^{\infty} \left[\frac{\varepsilon}{1 + \sqrt{1 - \varepsilon^2}} \right]^n \cos(n\theta) \\ &= a + \varepsilon a \cos\theta - \varepsilon^2 a \sin^2\theta - \varepsilon^3 a \cos\theta \sin^2\theta + \varepsilon^4 a \cos^2\theta \sin^2\theta + \mathcal{O}(\varepsilon^5). \end{aligned}$$

Hence,

$$\begin{aligned} R &= a, \\ f(\theta) &= a\cos\theta - \varepsilon a \sin^2\theta - \varepsilon^2 a \cos\theta \sin^2\theta + \varepsilon^3 a \cos^2\theta \sin^2\theta + \mathcal{O}(\varepsilon^4) \end{aligned} \quad (10)$$

in (2). Rewriting conditions (5),(6) and the other 3 conditions for the function (10) we get:

$$\begin{aligned} \phi_{0,r}(a, \theta) =& 0, \\ \phi_{1,r}(a, \theta) =& -\frac{1}{a} \sin\theta \phi_{0,\theta}(a, \theta) - a\cos\theta \phi_{0,rr}(a, \theta), \\ \phi_{2,r}(a, \theta) =& -a\cos\theta \phi_{1,rr}(a, \theta) + a\sin^2\theta \phi_{0,rr}(a, \theta) - \frac{a^2\cos^2\theta}{2} \phi_{0,rrr}(a, \theta) - \frac{1}{a} \sin\theta \phi_{1,\theta}(a, \theta). \end{aligned}$$

In the same manner, we can write the conditions for $\phi_{3,r}(a, \theta)$ and $\phi_{4,r}(a, \theta)$.

Substituting (10) into the equations (7), (8) and dividing them by $\rho g A \pi a^2 \tanh(kh)$, we get non-dimensionalized force components $\tilde{F}_x$, $\tilde{F}_y$ for elliptic cylinder.

4 Results and Conclusions

We have applied the asymptotic analysis described above to scattering problems of one-dimensional waves by a vertical cylinder with non-circular cross section. For an elliptic cylinder, we compared our results with the results by Williams [2] who used the expansion of the exact expressions for the forces which are given by the Mathieu functions for small values of the elliptic eccentricity parameter. The present asymptotic approach provides a good approximation for the forces exerted on the elliptic cylinder with eccentricity $\varepsilon = 0.5$ to the incident wave for values of $\alpha = 0°$ and $\alpha = 90°$ (see Figure 2, Figure 3). We found that, if the incident wave makes zero angle with the positive x-axis and $ka \to 0$, then

$$\tilde{F}_x = \left[2 - \frac{3}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \mathcal{O}(\varepsilon^6)\right]\cos\omega t,$$

and if the incident wave makes the angle of $\frac{\pi}{2}$ with the positive x-axis and $ka \to 0$, then

$$\tilde{F}_y = \left[2 - \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \mathcal{O}(\varepsilon^6)\right]\cos\omega t,$$

which coincide with the asymptotic formulae from [2].

In conclusion, for different shapes of a vertical cylinder, $r = R + \varepsilon f(\theta)$, we can find the wave forces acting on the cylinder by using the formulae (7) and (8).

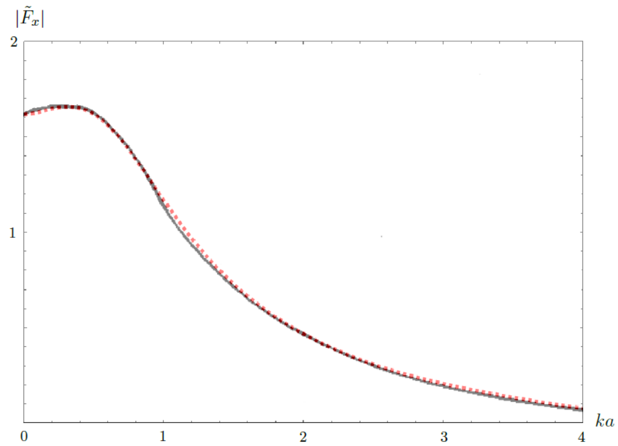

Figure 2: The x-component of the non-dimensionlized force on elliptic cylinder for $\varepsilon = 0.5$. (Solid curve is from [2], dotted curve is by the present method).

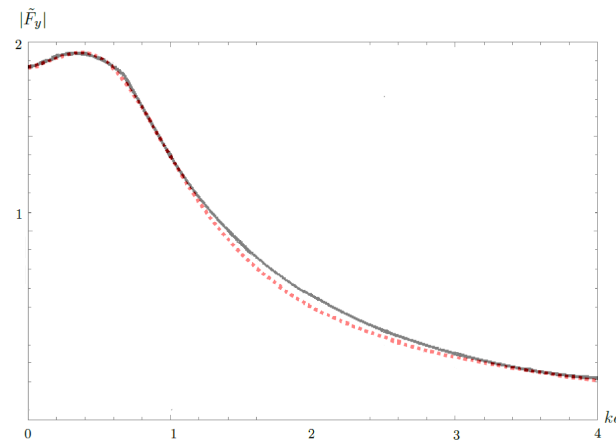

Figure 3: The y-component of the non-dimensionlized force on elliptic cylinder for $\varepsilon = 0.5$. (Solid curve is from [2], dotted curve is by the present method).

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