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ISOTYPIC FAITHFUL 2-REPRESENTATIONS
OF J-SIMPLE FIAT 2-CATEGORIES

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ABSTRACT. We introduce the class of isotypic 2-representations for finitary 2-categories and the notion of inflation of 2-representations. Under some natural assumptions we show that isotypic 2-representations are equivalent to inflations of cell 2-representations.

1. INTRODUCTION

Classification problems are one of the driving forces in representation theory. With the advent of 2-representation theory, a canonical major problem is to classify all 2-representations of suitable 2-categories. Another direction is the study of uniqueness questions, which has been pursued, for example, in [CR, LW, MM1]. One natural class of 2-categories to consider is that of *finitary* 2-categories, which provide 2-analogues of finite dimensional algebras.

A systematic study of abstract 2-representation theory of finitary 2-categories was initiated in [MM1]–[MM5], [Xa]. Inspired by [BFK, BG, CR, KL, Ro, So], among others, we especially focused on the class of *fiat* 2-categories, which can be considered as 2-analogues of algebras with involution. Fiat 2-categories also appear, sometimes in disguise, in [EW, EL, MT, SS] and many other papers. The first question in representation theory is that of classifying simple representations. Its 2-analogue was addressed in [MM5], where we proposed the notion of a *simple transitive* 2-representation as an appropriate 2-substitute for the classical notion of a simple representation. Under certain combinatorial assumptions, which are satisfied for most of the inspiring examples mentioned above, we obtained a complete classification of simple transitive 2-representations up to equivalence, showing them to be *cell* 2-representations, previously constructed in [MM1, MM2].

In the present article, we take the first step in the direction of classifying more complicated 2-representations. The starting point was an attempt to classify all transitive 2-representations. However, it turned out that our methods yield stronger results. We introduce the class of *isotypic* 2-representations. In full analogy with the corresponding classical notion, these are 2-representations with only one equivalence class of simple transitive weak Jordan-Hölder subquotients. Our main result, Theorem 4, asserts that, under the assumption that a weakly fiat 2-category is J-simple and the 2-representation is faithful, each such 2-representation is obtained, up to equivalence, from a cell 2-representation by an easy procedure which we call *inflation*, see Section 3.6. In the special case where the 2-category concerned is a finitary quotient of a 2-Kac-Moody algebra, this recovers results from [Ro2, Section 4.3.4].

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Let us now briefly describe the structure of the article. In Section 2, we collect all necessary preliminaries on finitary and weakly fiat 2-categories. Section 3 recalls the basics on 2-representations. It also contains a combinatorial result, Proposition 1, which renders superfluous the numerical assumption appearing in [MM1, Theorem 43], [MM3, Section 4], [MM3, Theorems 14,16] and [MM5, Theorem 18]. Furthermore, we introduce the notion of isotypicality of a 2-representation and describe the construction of inflation. The last subsection contains the statement of our main result, Theorem 4. Section 4, which constitutes the core of the paper, gives a proof of the main theorem in the special case of a particularly nice class of 2-categories, the 2-categories of projective functors for finite dimensional self-injective algebras. In Section 5 we deduce the general case. To round things off, Section 6 contains several new constructions of finitary and fiat 2-categories, providing many additional examples to which our theorems apply. In an appendix, we indicate how to generalize our previous results on fiat 2-categories to the weakly fiat case.

2. Preliminaries on fiat 2-categories and 2-representations

2.1. **Notation.** Throughout this paper, we work over a fixed algebraically closed field $\Bbbk$.

By a 2-*category* we mean a category which is enriched over the category of all small categories. Thus, a 2-category $\mathscr{C}$ consists of a collection of objects, denoted $\mathtt{i}, \mathtt{j}, \mathtt{k}, \ldots$; for each pair $(\mathtt{i}, \mathtt{j})$ of objects, a set $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of 1-morphisms, denoted $\mathrm{F}, \mathrm{G}, \mathrm{H}, \ldots$; and for each pair (F, G) of 1-morphisms in a fixed $\mathscr{C}(\mathtt{i}, \mathtt{j})$, a set $\mathrm{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(\mathrm{F}, \mathrm{G})$ of 2-morphisms, denoted $\alpha, \beta, \gamma, \ldots$. For $\mathtt{i} \in \mathscr{C}$, the identity 1-morphism in $\mathscr{C}(\mathtt{i}, \mathtt{i})$ is denoted $\mathbb{1}_{\mathtt{i}}$ and, for a 1-morphism F, the corresponding identity 2-morphism in $\mathrm{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(\mathrm{F}, \mathrm{F})$ is denoted id_{F}. We write $\circ$ for composition of 1-morphisms (which we often omit), $\circ_0$ for horizontal composition of 2-morphisms and $\circ_1$ for vertical composition of 2-morphisms. We set **Cat** to be the 2-category of small categories.

2.2. **Finitary** 2-**categories.** We call an additive $\Bbbk$-linear category *finitary* provided that it has split idempotents, finitely many isomorphism classes of indecomposable objects and sets of morphisms form finite dimensional $\Bbbk$-vector spaces. We denote by $\mathfrak{A}_{\Bbbk}^{f}$ the 2-category whose

- objects are finitary additive $\Bbbk$-linear categories,

- 1-morphisms are additive $\Bbbk$-linear functors,

- 2-morphisms are natural transformations of functors.

Now we define a *finitary* 2-category (over $\Bbbk$) to be a 2-category $\mathscr{C}$ such that

- it has finitely many objects;

- for each pair $\mathtt{i}, \mathtt{j}$ of objects, $\mathscr{C}(\mathtt{i}, \mathtt{j})$ belongs to $\mathfrak{A}_{\Bbbk}^{f}$ and horizontal composition is additive and $\Bbbk$-linear;

- for every $\mathtt{i} \in \mathscr{C}$, the corresponding 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable.

We refer the reader to [Le, McL] for generalities on abstract 2-categories and to [MM1]–[MM5], [Xa] for more information on finitary 2-categories.

2.3. 2-**representations.** For a finitary 2-category $\mathscr{C}$, a 2-*representation* of $\mathscr{C}$ is a strict 2-functor from $\mathscr{C}$ to the 2-category **Cat**. A *finitary* 2-*representation* of $\mathscr{C}$ is a strict 2-functor from $\mathscr{C}$ to the 2-category $\mathfrak{A}_{\Bbbk}^{f}$. We usually denote 2-representations by $\mathbf{M}, \mathbf{N}, \ldots$, in particular, $\mathbf{P}_{\mathtt{i}}$ stands for the $\mathtt{i}$-th *principal* 2-representation $\mathscr{C}(\mathtt{i}, \_)$, where $\mathtt{i} \in \mathscr{C}$. All finitary 2-representations of $\mathscr{C}$ form a 2-category, which we denote by $\mathscr{C}$-afmod. In this 2-category, 1-morphisms are 2-natural transformations and 2-morphisms are modifications (see [Le, MM3] for details).

Two 2-representations $\mathbf{M}$ and $\mathbf{N}$ of $\mathscr{C}$ are said to be *equivalent* provided that there is a 2-natural transformation $\Phi : \mathbf{M} \to \mathbf{N}$ such that $\Phi_{\mathtt{i}}$ is an equivalence of categories for each $\mathtt{i}$.

Abusing notation, we write $\mathrm{F}\, X$ instead of $\mathbf{M}(\mathrm{F})\, X$ for a 1-morphism F.

Consider a 2-representation $\mathbf{M}$ of $\mathscr{C}$ and assume that $\mathbf{M}(\mathtt{i})$ is additive and idempotent split for each $\mathtt{i} \in \mathscr{C}$. For every collection $X_i \in \mathbf{M}(\mathtt{i}_i)$ of objects, where $i \in I$, the additive closure $\mathrm{add}(\{\mathrm{F} X_i\})$, where $i \in I$ and F runs through the set of all 1-morphisms of $\mathscr{C}$, has the structure of a 2-representation of $\mathscr{C}$ by restriction. We write $\mathbf{G}_{\mathbf{M}}(\{X_i : i \in I\})$ for this 2-subrepresentation of $\mathbf{M}$.

2.4. **Combinatorics of finitary** 2-**categories.** For a finitary 2-category $\mathscr{C}$, we denote by $\mathcal{S}(\mathscr{C})$ the set of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$, which forms a multisemigroup by [MM2, Section 3]. This multisemigroup comes equipped with several natural preorders. For two 1-morphisms F and G, we say $\mathrm{G} \geq_L \mathrm{F}$ in the *left preorder* if there is a 1-morphism H such that G appears, up to isomorphism, as a direct summand of $\mathrm{H} \circ \mathrm{F}$. A *left cell* is an equivalence class for this preorder. Similarly one defines the *right* and *two-sided* preorders $\geq_R$ and $\geq_J$ and the corresponding *right* and *two-sided* cells, respectively.

Note that $\geq_L$ defines a genuine partial order on the set of left cells. A similar statement holds for $\geq_R$ and right cells, and for $\geq_J$ and two-sided cells.

2.5. **Weakly fiat and fiat** 2-**categories.** For a 2-category $\mathscr{C}$, we denote by $\mathscr{C}^{\mathrm{op}}$ the 2-category obtained from $\mathscr{C}$ by reversing both 1- and 2-morphisms.

We call a finitary 2-category $\mathscr{C}$ *weakly fiat* if

- $\mathscr{C}$ is equipped with a weak equivalence $* : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$;

- for any pair $\mathtt{i}, \mathtt{j}$ of objects and every 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, there are 2-morphisms $\alpha : \mathrm{F} \circ \mathrm{F}^* \to \mathbb{1}_{\mathtt{j}}$ and $\beta : \mathbb{1}_{\mathtt{i}} \to \mathrm{F}^* \circ \mathrm{F}$ with the property that $\alpha_{\mathrm{F}} \circ_1 \mathrm{F}(\beta) = \mathrm{id}_{\mathrm{F}}$ and $\mathrm{F}^*(\alpha) \circ_1 \beta_{\mathrm{F}^*} = \mathrm{id}_{\mathrm{F}^*}$.

If $*$ is a weak involution, we call $\mathscr{C}$ a *fiat* 2-category, see [MM1, MM2].
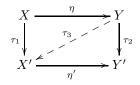
The equivalence $*$ sends a 1-morphism F to F* and the image of F under an inverse of $*$ will be denoted *F. Note that both F $\mapsto$ F* and F $\mapsto$ *F are isomorphisms between $\leq_L$ and $\leq_R$.

We note that the last condition in the paragraph above mirrors the notion of *rigidity* in [EGNO, ENO], see also references therein.

2.6. 2-ideals. Given a a 2-category $\mathscr{C}$, a *left* 2-*ideal* $\mathscr{I}$ of $\mathscr{C}$ is a 2-semicategory on the same objects as $\mathscr{C}$ and in which, for each pair $\mathtt{i}, \mathtt{j}$ of objects, the set $\mathscr{I}(\mathtt{i}, \mathtt{j})$ is an ideal in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ such that $\mathscr{I}$ is closed under left horizontal multiplication with both 1- and 2-morphisms in $\mathscr{C}$. *Right* 2-*ideals* and *two-sided* 2-*ideals* (also just called 2-*ideals*) are defined similarly. For instance, principal 2-representations are left 2-ideals in $\mathscr{C}$.

For a 2-category $\mathscr{C}$ and a 2-representation **M** of $\mathscr{C}$, an *ideal* **I** in **M** consists of an ideal **I**($\mathtt{i}$) in each **M**($\mathtt{i}$), for $\mathtt{i} \in \mathscr{C}$, which is stable under the action of $\mathscr{C}$.

2.7. Abelianization. For a finitary additive $\Bbbk$-linear category $\mathcal{A}$, its *abelianization* is the abelian category $\overline{\mathcal{A}}$, in which objects are diagrams $X \xrightarrow{\eta} Y$, for $X, Y \in \mathcal{A}$ and $\eta \in \mathcal{A}(X, Y)$, and in which morphisms are equivalence classes of solid commutative diagrams of the form

modulo the equivalence relation $\Bbbk$-linearly spanned by diagrams for which there is τ_3 as shown by the dashed arrow with $\eta' \tau_3 = \tau_2$, see [Fr]. The category $\overline{\mathcal{A}}$ is equivalent to the left module category for the finite dimensional $\Bbbk$-algebra

$$\mathrm{End}_{\mathcal{A}}(P)^{\mathrm{op}} \quad \text{where} \quad P := \bigoplus_{Q \in \mathrm{Ind}(\mathcal{A})/\cong} Q$$

where $\mathrm{Ind}(\mathcal{A})$ denotes the collection of indecomposable objects in $\mathcal{A}$.

Given a 2-category $\mathscr{C}$ and a finitary 2-representation **M** of $\mathscr{C}$, the *abelianization* of **M** is the 2-representation $\overline{\mathbf{M}}$ of $\mathscr{C}$ which sends each $\mathtt{i} \in \mathscr{C}$ to the category $\overline{\mathbf{M}(\mathtt{i})}$ and defines the action of $\mathscr{C}$ on diagrams component-wise. Thus the action of each 1-morphism in $\mathscr{C}$ on an abelianized finitary 2-representation is right exact.

2.8. The 2-category $\mathscr{C}_A$. Fix a basic, self-injective, non-semisimple, connected, finite dimensional $\Bbbk$-algebra A and a small category $\mathcal{A}$ which is equivalent to A-mod. Define, following [MM1, Section 7.3], the 2-category $\mathscr{C}_A$ to have

- one object $\clubsuit$ (identified with $\mathcal{A}$);

- direct sums of functors with summands isomorphic to the identity functor or to tensoring with projective A-A-bimodules as 1-morphisms;

- natural transformations of functors as 2-morphisms.

By a *projective functor*, we mean a functor which is isomorphic to tensoring with a projective A-A-bimodule.

We fix a set $\{e_1, e_2, \ldots, e_n\}$ of primitive, pairwise orthogonal idempotents in A which sum to the identity element 1 in A, and denote, for $i, j \in \{1, 2, \ldots, n\}$, by F_{ij} an indecomposable 1-morphism given by tensoring with the A-A-bimodule $Ae_i \otimes e_j A$. Then the 2-category $\mathscr{C}_A$ has two two-sided cells: the minimal one, consisting of the isomorphism class of the identity morphism, and the maximal one, which we call $\mathcal{J}$, consisting of the isomorphism classes of F_{ij} for $i, j \in \{1, 2, \ldots, n\}$. The two-sided cell $\mathcal{J}$ decomposes into left respectively right cells

$$\mathcal{L}_j := \{\mathrm{F}_{ij} : i \in \{1, 2, \ldots, n\}\} \quad \text{and} \quad \mathcal{R}_i := \{\mathrm{F}_{ij} : j \in \{1, 2, \ldots, n\}\},$$

where $i, j \in \{1, 2, \ldots, n\}$. We denote by $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ the *Nakayama* bijection which is defined by requiring $\operatorname{soc} Ae_i \cong \operatorname{top} Ae_{\sigma(i)}$ or, equivalently, $Ae_i \cong \operatorname{Hom}_{\Bbbk}(e_{\sigma(i)}A, \Bbbk)$. The isomorphism

$$\operatorname{Hom}_A(Ae_i \otimes_{\Bbbk} e_j A, \_) \cong \operatorname{Hom}_{\Bbbk}(e_j A, \Bbbk) \otimes_{\Bbbk} e_i A \otimes_A \_,$$

see for example [MM1, Section 7.3], implies that $(\mathrm{F}_{ij}, \mathrm{F}_{\sigma^{-1}(j)i})$ form an adjoint pair of functors. Hence $\mathscr{C}_A$ is weakly fiat, where $*$ is given on 1-morphisms by $\mathrm{F}_{ij}^* = \mathrm{F}_{\sigma^{-1}(j)i}$. Moreover, $\mathscr{C}_A$ is fiat if and only if A is weakly symmetric, that is, σ is the identity map.

Setting $\mathrm{F} := \bigoplus\limits_{i,j=1}^{n} \mathrm{F}_{ij}$ we have $\mathrm{F} \circ \mathrm{F} \cong \mathrm{F}^{\oplus \dim(A)}$ and $\mathrm{F}^* \cong \mathrm{F}$.

Every nonzero two-sided 2-ideal in $\mathscr{C}_A$ necessarily contains the identity 2-morphism on each indecomposable 1-morphism not isomorphic to the identity, see [Ag, Section 3.5], which means that $\mathscr{C}_A$ is $\mathcal{J}$-*simple* in the sense of [MM2, Section 6.2].

3. Transitive 2-representations

3.1. Cell 2-representations.
Let $\mathscr{C}$ be a finitary 2-category. Given a left cell $\mathcal{L}$ in $\mathscr{C}$, there is $\mathtt{i} = \mathtt{i}_{\mathcal{L}} \in \mathscr{C}$ with the property that every 1-morphism in $\mathcal{L}$ has domain $\mathtt{i}$. This allows us to define $\mathbf{N} := \mathbf{G}_{\mathbf{P}_{\mathtt{i}}}(\mathcal{L})$.

The 2-representation $\mathbf{N}$ has a unique maximal ideal $\mathbf{I}$ which does not contain id_{F} for any $\mathrm{F} \in \mathcal{L}$, see [MM5, Lemma 3]. The corresponding quotient 2-functor $\mathbf{C}_{\mathcal{L}} := \mathbf{N}/\mathbf{I}$ is said to be the *(additive) cell* 2-*representation* of $\mathscr{C}$ associated to $\mathcal{L}$.

3.2. Strongly regular cells.
Let $\mathscr{C}$ be a weakly fiat 2-category and $\mathcal{J}$ a two-sided cell in $\mathscr{C}$. We say that $\mathcal{J}$ is *strongly regular*, see [MM1, Section 4.8], provided that the following two conditions are satisfied:

- any two left (resp. right) cells in $\mathcal{J}$ are not comparable with respect to the left (resp. right) order;

- the intersection between any right and any left cell in $\mathcal{J}$ contains precisely one isomorphism class of indecomposable 1-morphisms.

If $\mathcal{L}$ is a left cell belonging to a strongly regular two-sided cell, then $\mathcal{L}$ contains a unique element $\mathrm{G} = \mathrm{G}_{\mathcal{L}}$ such that $\mathrm{G}^* \in \mathcal{L}$, see Proposition 28. This element is called the *Duflo involution* of $\mathcal{L}$.

If $\mathcal{J}$ is a strongly regular two-sided cell and $\mathrm{F} \in \mathcal{J}$, then the component of ${}^*\mathrm{F} \circ \mathrm{F}$ which belongs to $\mathcal{J}$ is in the same left cell as F and in the same right cell as F^*, and the intersection of these two cells contains a unique element H. Hence there is a positive integer $\mathbf{m}_{\mathrm{F}}$ such that ${}^*\mathrm{F} \circ \mathrm{F} \cong \mathrm{H}^{\oplus \mathbf{m}_{\mathrm{F}}} \oplus \mathrm{K}$, where no indecomposable direct summand of K belongs to $\mathcal{J}$.

3.3. The numerical condition.

Let $\mathscr{C}$ be a weakly fiat 2-category and $\mathcal{L}$ a left cell in $\mathscr{C}$ which belongs to a strongly regular two-sided cell. Set $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$ and consider the corresponding cell 2-representation $\mathbf{C}_{\mathcal{L}}$, as defined in Section 3.1, as well as its abelianization $\overline{\mathbf{C}}_{\mathcal{L}}$. For each $\mathtt{j} \in \mathscr{C}$, isomorphism classes of indecomposable projective respectively simple objects in $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{j})$ are indexed by isomorphism classes of 1-morphisms in $\mathcal{L} \cap \mathscr{C}(\mathtt{i}, \mathtt{j})$ and denoted by P_{F} and L_{F}, respectively.

By Proposition 27(ii) and Proposition 26(i), respectively, for all $\mathrm{F}, \mathrm{H} \in \mathcal{L}$ we have

$$\text{(1)} \qquad \mathrm{F}\, L_{\mathrm{H}} \cong \begin{cases} P_{\mathrm{F}} & \text{if } \mathrm{H} \cong \mathrm{G}_{\mathcal{L}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1. *Let $\mathscr{C}$ be a weakly fiat 2-category and $\mathcal{J}$ a strongly regular two-sided cell. Then the function*

$$\begin{array}{ccc} \mathcal{J} & \overset{\mathbf{m}}{\longrightarrow} & \mathbb{Z} \\ \mathrm{F} & \mapsto & \mathbf{m}_{\mathrm{F}} \end{array}$$

is constant on right cells of $\mathcal{J}$.

Proof. Let $\mathcal{L}$ be a left cell in $\mathcal{J}$. Consider the corresponding 2-representations $\mathbf{C}_{\mathcal{L}}$ and $\overline{\mathbf{C}}_{\mathcal{L}}$. From (1) and [MM5, Lemma 13], it follows that each $\mathrm{F} \in \mathcal{L}$ is represented in $\overline{\mathbf{C}}_{\mathcal{L}}$ by an indecomposable projective functor.

For each $\mathtt{j} \in \mathscr{C}$, denote by $A_{\mathtt{j}}$ the basic algebra whose module category is equivalent to $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{j})$, and fix some set $\{e_1^{\mathtt{j}}, e_2^{\mathtt{j}}, \ldots, e_{n_{\mathtt{j}}}^{\mathtt{j}}\}$ of primitive pairwise orthogonal idempotents in $A_{\mathtt{j}}$ which sum to the identity. Then, without loss of generality, we may assume that the projective functors corresponding to $\mathrm{F} \in \mathcal{L}$ are precisely $A_{\mathtt{j}} e_k^{\mathtt{j}} \otimes_{\Bbbk} e_1^{\mathtt{i}} A_{\mathtt{i}}$, where $k = 1, 2, \ldots, n_{\mathtt{j}}$.

By Proposition 30(i), the algebra $A_{\mathtt{j}}$ is self-injective. The computation in [MM1, Lemma 45] shows that F^* are represented by projective functors corresponding to $A_{\mathtt{i}} e_{\sigma^{-1}(1)}^{\mathtt{i}} \otimes_{\Bbbk} e_k^{\mathtt{j}} A_{\mathtt{j}}$. Hence $m_{\mathrm{F}^*} = \dim(e_1^{\mathtt{i}} A_{\mathtt{i}} e_{\sigma^{-1}(1)}^{\mathtt{i}})$ and this clearly does not depend on the choice of $\mathrm{F} \in \mathcal{L}$. Since $\mathrm{F} \mapsto \mathrm{F}^*$ defines a bijection from $\mathcal{L}$ to the right cell in $\mathcal{J}$ containing G^*, the function $\mathbf{m}$ is constant on this right cell. The fact that every right cell contains a Duflo involution, which follows from Proposition 28(c), completes the proof. $\square$

Remark 2. Proposition 1 implies that the additional numerical condition which appears in [MM1, Theorem 43], [MM3, Section 4], [MM3, Theorems 14, 16] and [MM5, Theorem 18] is redundant.

3.4. **Transitive and simple transitive 2-representations.** A finitary 2-representations $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is called *transitive* if, for each $\mathtt{i}$ and each non-zero object $X \in \mathbf{M}(\mathtt{i})$, we have $\mathbf{G}_{\mathbf{M}}(\{X\}) = \mathbf{M}$. If $\mathbf{M}$ is transitive, then, by [MM5, Lemma 4], it has a unique maximal ideal $\mathbf{I}$ which does not contain any identity morphisms with the exception of the one for the zero object. We call $\mathbf{M}$ *simple transitive* if $\mathbf{I} = 0$. In general, the quotient $\underline{\mathbf{M}}$ of $\mathbf{M}$ by $\mathbf{I}$ is called the *simple transitive quotient* of $\mathbf{M}$.

If $\mathscr{C}$ is weakly fiat, then [MM5, Proposition 6], [MM1, Theorem 43] and Section 7 show that all cell 2-representations associated to left cells of strongly regular two-sided cells are simple transitive. Furthermore, [MM5, Theorem 18] and Theorem 33 assert that, under the assumption that all two-sided cells of $\mathscr{C}$ are strongly regular, each simple transitive 2-representation of $\mathscr{C}$ is equivalent to a cell 2-representation.

3.5. **Simple transitive subquotients of finitary 2-representations.** Let $\mathscr{C}$ be a finitary 2-category. For a finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$, denote by $\mathrm{Ind}(\mathbf{M})$ the set of isomorphism classes of indecomposable objects in $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$. Assume that $\mathbf{X} \subset \mathrm{Ind}(\mathbf{M})$ is stable under the induced action of $\mathscr{C}$, that is any indecomposable direct summand of $\mathrm{F}\,X$ belongs to $\mathbf{X}$ for any $X \in \mathbf{X}$ and any 1-morphism $\mathrm{F} \in \mathscr{C}$. Then the additive closure in $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$ of all objects X whose isomorphism class is in $\mathbf{X}$ inherits the structure of a 2-representation of $\mathscr{C}$ by restriction. We will denote this 2-representation by $\mathbf{M}_{\mathbf{X}}$.

If $\mathbf{X} \subset \mathbf{Y} \subset \mathrm{Ind}(\mathbf{M})$ are such that both $\mathbf{M}_{\mathbf{X}}$ and $\mathbf{M}_{\mathbf{Y}}$ are defined and the quotient $\mathbf{M}_{\mathbf{Y}}/\mathbf{M}_{\mathbf{X}}$ is transitive, then the simple transitive quotient of $\mathbf{M}_{\mathbf{Y}}/\mathbf{M}_{\mathbf{X}}$ is called a *simple transitive subquotient* of $\mathbf{M}$. By [MM5, Theorem 8], the multiset of simple transitive subquotients of $\mathbf{M}$ (up to equivalence) is an invariant of $\mathbf{M}$. For example, for weakly fiat $\mathscr{C}$ with strongly regular two-sided cells, simple transitive subquotients of principal 2-representations are exactly cell 2-representations.

3.6. **Isotypic 2-representations.** A finitary 2-representation $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is called *isotypic* provided that all simple transitive subquotients of $\mathbf{M}$ are equivalent. For example, all transitive 2-representations of $\mathscr{C}$ are isotypic and so are direct sums of copies of the same transitive 2-representation.

We now give a more sophisticated example. We first recall the construction of a tensor product of $\Bbbk$-linear categories, see [GK, Section (1.1)] or [KV, § 5.18]. For two finitary $\Bbbk$-linear categories $\mathcal{A}$ and $\mathcal{B}$, their *tensor product* $\mathcal{A} \boxtimes \mathcal{B}$ is the category with objects

$$\bigoplus_{i=1}^{n} X_i \boxtimes Y_i, \quad \text{where} \quad X_i \in \mathcal{A}, Y_i \in \mathcal{B},$$

and the morphism space

$$\mathrm{Hom}_{\mathcal{A} \boxtimes \mathcal{B}}\left(\bigoplus_{i=1}^{n} X_i \boxtimes Y_i, \bigoplus_{j=1}^{m} U_j \boxtimes V_j\right)$$

is given by the matrix

$$\left[\mathrm{Hom}_{\mathcal{A}}(X_i, U_j) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{B}}(Y_i, V_j)\right]_{i=1,2,\ldots,n}^{j=1,2,\ldots,m}$$

with the obvious composition. In the above, we use the usual abbreviation

$$\bigoplus_{i=1}^{n} W_i := W_1 \oplus W_2 \oplus \cdots \oplus W_n,$$

which, in particular, means that $X_1 \boxtimes Y_1 \oplus X_2 \boxtimes Y_2$ and $X_2 \boxtimes Y_2 \oplus X_1 \boxtimes Y_1$ are different but isomorphic objects.

Let now $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$ and $\mathcal{A}$ any finitary $\Bbbk$-linear category. Define the $\mathcal{A}$-*inflation* $\mathbf{M}^{\boxtimes \mathcal{A}}$ *of* $\mathbf{M}$ as follows:

- for $\mathtt{i} \in \mathscr{C}$, set $\mathbf{M}^{\boxtimes \mathcal{A}}(\mathtt{i}) := \mathbf{M}(\mathtt{i}) \boxtimes \mathcal{A}$;

- for a 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, set $\mathbf{M}^{\boxtimes \mathcal{A}}(\mathrm{F}) := \mathbf{M}(\mathrm{F}) \boxtimes \mathrm{Id}_{\mathcal{A}}$;

- for a 2-morphism $\alpha : \mathrm{F} \to \mathrm{G}$, define $\mathbf{M}^{\boxtimes \mathcal{A}}(\alpha) := \mathbf{M}(\alpha) \boxtimes \mathrm{id}_{\mathrm{Id}_{\mathcal{A}}}$.

Proposition 3. *Any inflation of any transitive 2-representation $\mathbf{M}$ of $\mathscr{C}$ is isotypic.*

Proof. Let $\mathcal{A}$ be a finitary $\Bbbk$-linear category. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be full and irredundant lists of representatives of isomorphism classes of indecomposable objects in $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$ and $\mathcal{A}$, respectively.

For $k = 0, 1, 2, \ldots, m$, let $\mathbf{X}_k$ denote the subset of $\mathrm{Ind}(\mathbf{M}^{\boxtimes \mathcal{A}})$ with representatives $X_i \boxtimes Y_j$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. Then, for $k = 1, 2, \ldots, m$, we have the corresponding 2-subrepresentation $\mathbf{M}_{\mathbf{X}_k}^{\boxtimes \mathcal{A}}$ and the quotient

$$\mathbf{M}_{\mathbf{X}_k}^{\boxtimes \mathcal{A}} / \mathbf{M}_{\mathbf{X}_{k-1}}^{\boxtimes \mathcal{A}}.$$

By transitivity of $\mathbf{M}$, the latter quotient is transitive.

Now we claim that the simple transitive quotient of $\mathbf{M}_{\mathbf{X}_k}^{\boxtimes \mathcal{A}} / \mathbf{M}_{\mathbf{X}_{k-1}}^{\boxtimes \mathcal{A}}$ is equivalent to $\underline{\mathbf{M}}$. Without loss of generality, we may assume $k = m$. Denote by $\mathcal{I}$ the ideal of $\mathcal{A}$ generated by the identity morphisms on all Y_j with $j < m$, together with all radical morphisms of the local algebra $\mathrm{End}_{\mathcal{A}}(Y_m)$. Then

$$\mathbf{I} := \coprod_{\mathtt{i} \in \mathscr{C}} (\mathbf{M}(\mathtt{i}) \boxtimes \mathcal{I})$$

is an ideal in $\mathbf{M}^{\boxtimes \mathcal{A}}$ invariant under the action of $\mathscr{C}$. Sending $X_i \boxtimes Y_m$ to X_i, for $i = 1, 2, \ldots, n$, defines a 2-natural transformation from $\mathbf{M}^{\boxtimes \mathcal{A}} / \mathbf{I}$ to $\mathbf{M}$, which is an equivalence by construction. Consequently, the simple transitive quotients of $\mathbf{M}^{\boxtimes \mathcal{A}} / \mathbf{I}$ and $\mathbf{M}$ are equivalent.

The above, together with the Weak Jordan-Hölder Theorem [MM5, Theorem 8], shows that any simple transitive subquotient of $\mathbf{M}^{\boxtimes \mathcal{A}}$ is equivalent to $\underline{\mathbf{M}}$. Therefore, $\mathbf{M}^{\boxtimes \mathcal{A}}$ is isotypic. $\square$

3.7. **Classification of isotypic faithful 2-representations of $\mathcal{J}$-simple weakly fiat 2-categories.** The following is the main result of this paper.

Theorem 4. *Let $\mathscr{C}$ be a weakly fiat 2-category with a unique maximal two-sided cell $\mathcal{J}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. Assume that $\mathcal{J}$ is strongly regular and that $\mathscr{C}$ is $\mathcal{J}$-simple. Then any isotypic faithful 2-representation of $\mathscr{C}$ is equivalent to an inflation of $\mathbf{C}_{\mathcal{L}}$.*

There are many examples of 2-categories which satisfy all assumptions of Theorem 4, obtained from the 2-categories of Soergel bimodules in type A, 2-Kac-Moody algebras and projective functors on self-injective finite dimensional algebras. Namely, choosing any left cell in any of these 2-categories, and factoring out the annihilator of the corresponding cell 2-representation, we obtain a weakly fiat $\mathcal{J}$-simple 2-category with a unique maximal two-sided cell $\mathcal{J}$, and this cell is, moreover, strongly regular.

4. Proof of Theorem 4 for $\mathscr{C} = \mathscr{C}_A$

4.1. **Annihilation filtrations.** Let A be a connected self-injective finite dimensional $\Bbbk$-algebra and $\mathscr{C}_A$ the corresponding weakly fiat 2-category as defined in Section 2.8. Let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}_A$ and $\overline{\mathbf{M}}$ its abelianization. For $F, G \in \mathscr{C}(\clubsuit, \clubsuit)$, $\alpha : F \to G$ and $X \in \overline{\mathbf{M}}(\clubsuit)$, we say that α *annihilates* X provided that the linear map $\alpha_X : F X \to G X$ is zero.

Lemma 5.

(i) *If α annihilates Y, and X is a subobject of Y, then α annihilates Y.*

(ii) *If α annihilates Y, and Z is a quotient of Y, then α annihilates Z.*

(iii) *Any $X \in \overline{\mathbf{M}}(\clubsuit)$ has a unique maximal submodule and a unique maximal quotient which are annihilated by α.*

Proof. A short exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ gives rise to the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F X & \longrightarrow & F Y & \longrightarrow & F Z & \longrightarrow & 0 \\
& & \downarrow{\alpha_X} & & \downarrow{\alpha_Y} & & \downarrow{\alpha_Z} & & \\
0 & \longrightarrow & G X & \longrightarrow & G Y & \longrightarrow & G Z & \longrightarrow & 0
\end{array}
$$

with exact rows. If α_Y is zero, then so are α_X and α_Z, which proves claims (i) and (ii).

We prove the second part of claim (iii), the first being proved similarly. For $i = 1, 2$, consider short exact sequences $K_i \hookrightarrow X \twoheadrightarrow Y_i$ and assume that $\alpha_{Y_i} = 0$ for $i = 1, 2$. Note that $\alpha_{Y_1 \oplus Y_2} = 0$ by additivity. Then the isomorphism theorem implies that $X/(K_1 \cap K_2)$ is a subobject of $Y_1 \oplus Y_2$ and hence is annihilated by α thanks to claim (i). □

For a collection $\boldsymbol{\alpha} := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of 2-morphisms, we say that $\boldsymbol{\alpha}$ *annihilates* X if α_i annihilates X for each i. Similarly, for a finite dimensional vector space of 2-morphisms, we will say that it *annihilates* X if some finite generating set of this space annihilates X.

Let $\boldsymbol{\alpha}$ be a fixed collection or a finite dimensional vector space of 2-morphisms. A filtration

$$(2) \qquad\qquad 0 = X_0 \subset X_1 \subset \cdots \subset X_k = X$$

is called an *annihilation filtration* for $\boldsymbol{\alpha}$ provided that each subquotient X_i/X_{i-1} is annihilated by $\boldsymbol{\alpha}$. An annihilation filtration is called *minimal*, if there does not exist any annihilation filtration of strictly smaller length (not necessarily refining the

given one). The length of a minimal annihilation filtration is called the *annihilation length* of X and denoted al(X).

Set $\mathrm{sub}_{\boldsymbol{\alpha}}^0(X) := 0$. Denote by $\mathrm{sub}_{\boldsymbol{\alpha}}(X) = \mathrm{sub}_{\boldsymbol{\alpha}}^1(X)$ the largest submodule of X annihilated by $\boldsymbol{\alpha}$, which exists by Lemma 5(iii). For $i \in \{2, 3, \dots\}$, define $\mathrm{sub}_{\boldsymbol{\alpha}}^i(X)$ as the full preimage of $\mathrm{sub}_{\boldsymbol{\alpha}}(X/\mathrm{sub}_{\boldsymbol{\alpha}}^{i-1}(X))$ in X. The filtration

$$0 = \mathrm{sub}_{\boldsymbol{\alpha}}^0(X) \subset \mathrm{sub}_{\boldsymbol{\alpha}}^1(X) \subset \mathrm{sub}_{\boldsymbol{\alpha}}^2(X) \subset \dots$$

is called the *upper* annihilation filtration of X.

Set $\mathrm{und}_{\boldsymbol{\alpha}}^0(X) := X$. Denote by $\mathrm{und}_{\boldsymbol{\alpha}}(X) = \mathrm{und}_{\boldsymbol{\alpha}}^1(X)$ the minimal submodule of X such that $X/\mathrm{und}_{\boldsymbol{\alpha}}(X)$ is annihilated by $\boldsymbol{\alpha}$, which exists by Lemma 5(iii). For $i \in \{2, 3, \dots\}$, define $\mathrm{und}_{\boldsymbol{\alpha}}^i(X)$ as the minimal submodule of $\mathrm{und}_{\boldsymbol{\alpha}}^{i-1}(X)$ such that $\mathrm{und}_{\boldsymbol{\alpha}}^{i-1}(X)/\mathrm{und}_{\boldsymbol{\alpha}}^i(X)$ is annihilated by $\boldsymbol{\alpha}$, The filtration

$$\dots \subset \mathrm{und}_{\boldsymbol{\alpha}}^2(X) \subset \mathrm{und}_{\boldsymbol{\alpha}}^1(X) \subset \mathrm{und}_{\boldsymbol{\alpha}}^0(X) = X$$

is called the *lower* annihilation filtration of X.

Proposition 6.

(i) *If* (2) *is an annihilation filtration, then, for any* $i = 1, 2, \dots, k$, *we have*

$$\mathrm{und}_{\boldsymbol{\alpha}}^{k-i}(X) \subset X_i \subset \mathrm{sub}_{\boldsymbol{\alpha}}^i(X).$$

(ii) *If* (2) *is a minimal annihilation filtration, then* al$(X) = k$ *is also the length of both the upper and the lower annihilation filtration.*

Proof. Claim (i) follows directly from the definitions. Claim (ii) follows from claim (i). $\square$

Remark 7. Existence of annihilation filtration does depend on the choice of X and $\boldsymbol{\alpha}$. For example, choosing $\boldsymbol{\alpha}$ to contain some identity 2-morphism, it is easy to construct an example where no annihilation filtration exists (in these cases the upper and the lower annihilation filtrations stabilize before reaching X or 0, respectively).

Example 8. Consider the natural action of $\mathscr{C}_A$ on A-mod and let $\boldsymbol{\alpha} = 1 \otimes \mathrm{rad}(A^{op})$ be a subspace of $\mathrm{End}_{\mathscr{C}_A}(\mathrm{F})$, see Section 2.8. Then it is easy to check that the upper annihilation filtration of $_AA$ with respect to $\boldsymbol{\alpha}$ coincides with the socle series of $_AA$ and the lower annihilation filtration of $_AA$ with respect to $\boldsymbol{\alpha}$ coincides with the radical series of $_AA$.

4.2. **Auxiliary statements.** We first introduce some notation that will remain in place for the remainder of this section. Let A be a self-injective finite dimensional $\Bbbk$-algebra and $\mathscr{C}_A$ the corresponding weakly fiat 2-category as defined in Section 2.8. We recall that F stands for the multiplicity free direct sum of all indecomposable projective functors. Let $\mathbf{M}$ be an isotypic faithful 2-representation of $\mathscr{C}_A$ and $\overline{\mathbf{M}}$ its abelianization. Isotypicality and faithfulness of $\mathbf{M}$, together with [MM5, Theorem 15], imply that all simple transitive subquotients of $\mathbf{M}$ are equivalent to $\mathbf{C}_{\mathcal{L}}$.

Let $P_1, P_2, \dots, P_r$ be a complete and irredundant list of representatives of isomorphism classes of indecomposable projective objects in $\overline{\mathbf{M}}(\clubsuit)$ and $L_1, L_2, \dots, L_r$ their corresponding simple tops. For $i, j = 1, 2, \dots, r$, let a_{ij} denote the multiplicity of P_i as a direct summand of $\mathrm{F} P_j$, and let b_{ij} denote the composition multiplicity of L_i in $\mathrm{F} L_j$. Then, since $\mathrm{F}^* \cong \mathrm{F}$, we have $b_{ij} = a_{ji}$ by [MM5, Lemma 10] and we can

set up the matrix $[\mathrm{F}]_{\mathbf{M}} := (a_{ij})_{i,j=1,\ldots,r}$. We denote by $[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}}$ a similarly defined matrix for the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}_A$ corresponding to the left cell $\mathcal{L} = \mathcal{L}_1$ as in Section 2.8.

Lemma 9. *We can rearrange the ordering of $P_1, P_2, \ldots, P_r$ such that $[\mathrm{F}]_{\mathbf{M}}$ has the diagonal form*

$$[\mathrm{F}]_{\mathbf{M}} = \begin{pmatrix} [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} & 0 & \cdots & 0 \\ 0 & [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} \end{pmatrix}.$$

Proof. Since $\mathbf{M}$ is isotypic, without loss of generality we may assume that $[\mathrm{F}]_{\mathbf{M}}$ has the form

$$(3) \qquad [\mathrm{F}]_{\mathbf{M}} = \begin{pmatrix} [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} & * & \cdots & * \\ 0 & [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} \end{pmatrix}.$$

Let us first prove the claim for the smallest possible non-trivial case

$$[\mathrm{F}]_{\mathbf{M}} = \begin{pmatrix} [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} & Q \\ 0 & [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} \end{pmatrix}.$$

Since $\mathrm{F} \circ \mathrm{F} \cong \mathrm{F}^{\oplus \dim(A)}$, we see that

$$[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} Q + Q[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} = \dim(A)Q.$$

Multiplying with $[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}}$ on the left and using $[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}}^2 = \dim(A)[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}}$, we obtain

$$\dim(A)[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} Q + [\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} Q[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} = \dim(A)[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} Q,$$

which implies

$$[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} Q[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}} = 0.$$

As $[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}}$ is a matrix with positive coefficients and Q is a matrix with non-negative coefficients, we get $Q = 0$.

The general case follows from the above baby case by double induction on the columns from left to right, and on the entries within each column from bottom to top. $\square$

Set $\hat{P} := P_1 \oplus P_2 \oplus \cdots \oplus P_r$ and, for $X \in \overline{\mathbf{M}}(\clubsuit)$, define

$$\dim(X) := \dim \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(\hat{P}, X).$$

If L is a simple A-module, then $\mathrm{F}\, L \cong {}_A A$ and hence $\dim(\mathrm{F}\, L) = \dim(A) \dim(L)$. By exactness of F, this extends to all A-modules and hence to any object in $\mathbf{C}_{\mathcal{L}}(\clubsuit)$. Using Lemma 9, we thus obtain

$$(4) \qquad \dim(\mathrm{F}\, X) = \dim(A) \dim(X)$$

for any $X \in \overline{\mathbf{M}}(\clubsuit)$.

As explained in Section 2.8, the matrix $[\mathrm{F}]_{\mathbf{C}_{\mathcal{L}}}$ has size $n \times n$. We now choose a special ordering of $P_1, P_2, \ldots, P_r$. First of all, we need $[\mathrm{F}]_{\mathbf{M}}$ to have the form given by Lemma 9. We note that $r = nk$ for some $k \in \{1, 2, \ldots\}$. For every $i = 1, 2, \ldots, k$, we have a transitive 2-representation of $\mathscr{C}_A$ on the additive closure of $P_{n(i-1)+1}, P_{n(i-1)+2}, \ldots, P_{ni}$. The simple transitive quotients are equivalent for

all i. We choose arbitrarily the ordering of $P_1, P_2, \ldots, P_n$ and assume that each such equivalence induces simply a shift by n on indices of the indecomposable projectives in these transitive 2-representations.

Set $P := P_1 \oplus P_2 \oplus \cdots \oplus P_n$ and define $L := P/\mathrm{rad}(P)$. Define $\boldsymbol{\alpha}$ to be $1 \otimes \mathrm{rad}(A^{op})$ considered as a subspace of the space $\mathrm{End}_{\mathscr{C}_A}(\mathrm{F})$ of 2-endomorphisms of F. Note that $\boldsymbol{\alpha}$ is nilpotent, and hence any object in $\overline{\mathbf{M}}(\clubsuit)$ has an annihilation filtration with respect to $\boldsymbol{\alpha}$. Furthermore, since the nilpotency degree of $\boldsymbol{\alpha}$ is exactly the Loewy length $\mathbf{c} := \mathrm{ll}(_A A)$, we have $\mathrm{al}(X) \leq \mathbf{c}$ for any $X \in \overline{\mathbf{M}}(\clubsuit)$. We set

$$N_i := P_i/\mathrm{und}^1_{\boldsymbol{\alpha}}(P_i) \text{ for } i = 1, 2, \ldots, r, \qquad N := \bigoplus_{i=1}^{n} N_i,$$

and note that $N \twoheadrightarrow L$ by nilpotency of $\boldsymbol{\alpha}$.

Lemma 10. *Let Q be a basic projective generator in $\mathbf{C}_{\mathcal{L}}(\clubsuit)$. Then $\mathrm{al}(Q) = \mathbf{c}$.*

Proof. By [MM5, Proposition 9], we may assume that $\mathbf{C}_{\mathcal{L}}(\clubsuit)$ is the defining 2-representation of $\mathscr{C}_A$ and that $Q = {}_A A$. As $\mathscr{C}_A$ is $\mathcal{J}$-simple, $\mathbf{C}_{\mathcal{L}}$ is faithful and hence $\mathbf{C}_{\mathcal{L}}(\boldsymbol{\alpha})$ has nilpotency degree exactly $\mathbf{c}$. This means that $\mathrm{al}(_A A) \geq \mathbf{c}$ implying the claim. $\qquad \square$

Lemma 11. *For each $i = 1, 2, \ldots, n$, we have $\mathrm{al}(\mathrm{F}\, L_i) = \mathrm{al}(\mathrm{F}\, N_i) = \mathrm{al}(\mathrm{F}\, P_i) = \mathbf{c}$.*

Proof. As $P_i \twoheadrightarrow N_i \twoheadrightarrow L_i$, it is enough to prove that $\mathrm{al}(\mathrm{F}\, L_i) = \mathbf{c}$. Note that $\mathrm{F}\, L_i \neq 0$ since F is exact and $\mathbf{M}$ is faithful. This induces a finitary 2-representation, $\mathbf{N}$, of $\mathscr{C}_A$ on $\mathrm{add}(\mathrm{F}\, L_i)$. Since F does not annihilate $\mathrm{F}\, L_i$, at least one of the simple transitive subquotients of $\mathbf{N}$ is equivalent to $\mathbf{C}_{\mathcal{L}}$. Using Lemma 10, we thus deduce that $\mathrm{F}\, L_i$ has a subquotient X such that $\mathrm{al}(X) = \mathbf{c}$. Therefore, Lemma 5 yields that $\mathrm{al}(\mathrm{F}\, L_i) \geq \mathbf{c}$ and we are done. $\qquad \square$

Let $\mathbf{K}$ denote the restriction of $\mathbf{M}$ to $\mathrm{add}(P)$. Then $\mathbf{K}$ is transitive and its simple transitive quotient is equivalent to $\mathbf{C}_{\mathcal{L}}$. Denote by $\mathbf{I}$ the corresponding maximal $\mathscr{C}_A$-stable ideal of $\mathbf{K}$. Denote by I the subspace of $\mathrm{End}(P)$ belonging to $\mathbf{I}$. Then [MM5, Proposition 9] implies that $\mathrm{End}_{\overline{\mathbf{M}}(\clubsuit)}(P)/I \cong A$. For $i = 1, 2, \ldots, n$, we set $\tilde{P}_i := P_i/(P_i \cap IP)$.

Lemma 12. *For $i, j, s = 1, 2, \ldots, n$, we have*

$$\mathrm{F}_{ij}\, L_s \cong \begin{cases} \tilde{P}_i & \text{if } s = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows by comparing the action of F_{ij} on $\mathbf{K}/\mathbf{I}$ and on A-mod, the latter being equivalent to $\mathbf{C}_{\mathcal{L}}$ by [MM5, Proposition 9]. $\qquad \square$

Lemma 13.

(i) *For $i, j, k = 1, 2, \ldots, n$, we have $\mathrm{F}_{ij}\, N_k = 0$ unless $j = k$.*

(ii) *For $i, j = 1, 2, \ldots, n$, we have $P_i \twoheadrightarrow \mathrm{F}_{ij}\, N_j$.*

Proof. By adjunction,

$$\mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(\mathrm{F}_{ij}\, N_k, L_s) \cong \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(N_k, \mathrm{F}_{\sigma^{-1}(j)i}\, L_s),$$

and the object $\mathrm{F}_{\sigma^{-1}(j)i} L_s$ cannot have L_k as a composition subquotient unless $s \in \{1, 2, \ldots, n\}$, moreover, for such s we have $\mathrm{F}_{\sigma^{-1}(j)i} L_s = 0$ unless $s = i$. If $s = i$, then, by Lemma 12, we have

$$\mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(N_k, \mathrm{F}_{\sigma^{-1}(j)i} L_i) \cong \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(N_k, \tilde{P}_{\sigma^{-1}(j)}).$$

As $\boldsymbol{\alpha}$ annihilates N_k, the image of N_k under such a homomorphism is contained in $\mathrm{sub}_{\boldsymbol{\alpha}}(\tilde{P}_{\sigma^{-1}(j)})$, which is isomorphic to L_j, see Example 8. Hence $\mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(N_k, \tilde{P}_{\sigma^{-1}(j)})$ is zero unless $j = k$, and in the latter case this space is one-dimensional. Both claims of the lemma follow. $\qquad\square$

Lemma 14. *The object N is a progenerator for the full subcategory $\mathrm{sub}_{\boldsymbol{\alpha}}(\overline{\mathbf{M}}(\clubsuit))$ of $\overline{\mathbf{M}}(\clubsuit)$ consisting of all objects isomorphic to $\mathrm{sub}_{\boldsymbol{\alpha}}(X)$ for some $X \in \overline{\mathbf{M}}(\clubsuit)$.*

Proof. Let $X \hookrightarrow Y \twoheadrightarrow Z$ be a short exact sequence in $\mathrm{sub}_{\boldsymbol{\alpha}}(\overline{\mathbf{M}}(\clubsuit))$. By projectivity of P, we have a short exact sequence

$$(5) \qquad 0 \to \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(P, X) \to \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(P, Y) \to \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(P, Z) \to 0.$$

As $\boldsymbol{\alpha}$ annihilates X, Y and Z, any homomorphism from P to any of these objects factors over N. Therefore (5) induces an exact sequence

$$0 \to \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(N, X) \to \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(N, Y) \to \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(N, Z) \to 0,$$

which implies that N is relatively projective in $\mathrm{sub}_{\boldsymbol{\alpha}}(\overline{\mathbf{M}}(\clubsuit))$. As $N \twoheadrightarrow L$, it is even a progenerator. $\qquad\square$

4.3. Analysis of $\mathrm{F} N_s$ and $\hat{P}$.

Proposition 15. *For any $s = 1, 2, \ldots, n$, the object $\mathrm{F} N_s$ has a filtration of length $\dim(A)$ in which all subquotients are isomorphic to N_i for some $i \in \{1, 2, \ldots, n\}$. Moreover, each such N_i appears as a subquotient in this filtration l_i times, where l_i is the composition multiplicity of L_i in $_A A$.*

Proof. Any composition series

$$(6) \qquad\qquad 0 = C_0 \subset C_1 \subset \cdots \subset C_{\dim(A)} = {}_A A$$

gives rise to the filtration

$$(7) \qquad\qquad 0 = X_0 \subset X_1 \subset \cdots \subset X_{\dim(A)} = \mathrm{F} N_s,$$

where X_i is the image of $(C_i \otimes A^{\mathrm{op}})_N$ in $\mathrm{F} N_s$. As $\boldsymbol{\alpha}$ annihilates N_s, it also annihilates any subquotient X_i/X_{i-1}. Using Lemma 13(ii) and Lemma 14, we see that each subquotient X_i/X_{i-1} is a quotient of a sum of copies of N.

Now we would like to estimate the number of tops in each X_i/X_{i-1}. There is a unique $t \in \{1, 2, \ldots, n\}$ such that Ae_t surjects onto C_i/C_{i-1}. This induces a morphism of functors

$$\bigoplus_{j=1}^{n} \mathrm{F}_{tj} \to \mathrm{F}$$

which, evaluated at N_s, induces a surjection $\mathrm{F}_{ts} N_s \twoheadrightarrow X_i/X_{i-1}$ by Lemma 13(i). By Lemma 13(ii), $\mathrm{F}_{ts} N_s$ has simple top L_t and hence

$$(8) \qquad\qquad N_t \twoheadrightarrow X_i/X_{i-1}.$$

Therefore the number of times L_t appears as simple top in some subquotient in filtration (7) is at most l_t.

Now choose j such that $\dim(N_j)$ is maximal possible. From the above, by additivity, we have

$$(9) \qquad \dim(\mathrm{F}\, N_j) \leq \sum_{t=1}^{n} l_t \dim(N_t) \leq \sum_{t=1}^{n} l_t \dim(N_j) = \dim(A) \dim(N_j)$$

since $\sum_{t=1}^{n} l_t = \dim(A)$. If there were t such that $\dim(N_t) < \dim(N_j)$, then the second inequality in (9) would be strict. At the same time, we know that $\dim(\mathrm{F}\, N_j) = \dim(A) \dim(N_j)$ by (4), a contradiction. Therefore $\dim(N_t) = \dim(N_j)$ for all t.

Hence, the total number of simple tops in all subquotients of filtration (7) is exactly $\sum_{t=1}^{n} l_t = \dim(A)$. By (4) and the above, we have

$$\dim(\mathrm{F}\, N_s) = \dim(\mathrm{F}\, N_j) = \dim(A) \dim(N_j).$$

By comparing dimensions, we see that the surjection (8) is, in fact, an isomorphism. The claim follows. $\qquad\square$

For any two 1-morphisms G and H in $\mathscr{C}$, and any 2-morphism $\beta : \mathrm{G} \to \mathrm{H}$, we consider the right exact endofunctor $\mathrm{Coker}(\overline{\mathbf{M}}(\beta))$ of $\overline{\mathbf{M}}(\clubsuit)$. For $i, j \in \{1, 2, \ldots, n\}$, we denote by Q_{ij} the endofunctor $\mathrm{Coker}(\overline{\mathbf{M}}(\beta_{ij}))$ where $\beta_{ij} : \mathrm{G}_{ij} \to \mathrm{F}_{ij}$ corresponds to a presentation for the simple quotient of $Ae_i \otimes_{\Bbbk} e_j A$ in the category of A-A-bimodules.

Lemma 16.

(i) For $i, j \in \{1, 2, \ldots, n\}$, we have

$$\mathrm{Q}_{ij} \circ \mathrm{Q}_{kl} \cong \begin{cases} \mathrm{Q}_{il} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For $i, j, s \in \{1, 2, \ldots, n\}$, we have

$$\mathrm{Q}_{ij} \, N_s \cong \begin{cases} N_i & \text{if } j = s, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) For $i \in \{1, 2, \ldots, n\}$, the restriction of

$$Q := \bigoplus_{i=1}^{n} \mathrm{Q}_{ii}$$

to $\mathrm{sub}_{\alpha}(\overline{\mathbf{M}}(\clubsuit))$ is isomorphic to the identity functor.

Proof. The composition $\mathrm{Q}_{ij} \circ \mathrm{Q}_{kl}$ is isomorphic to the cokernel of the map

$$\mathrm{G}_{ij}\mathrm{F}_{kl} \oplus \mathrm{F}_{ij}\mathrm{G}_{kl} \xrightarrow{\;((\beta_{ij})_{\mathrm{F}_{kl}}, \mathrm{F}_{ij}(\beta_{kl}))\;} \mathrm{F}_{ij}\mathrm{F}_{kl},$$

compare [MM1, Section 3.5]. On the level of A-A-bimodules, it is easy to check that the cokernel of $((\beta_{ij})_{\mathrm{F}_{kl}}, \mathrm{F}_{ij}(\beta_{kl}))$ is isomorphic to the cokernel of β_{il} if $k = j$ and to zero otherwise. Claim (i) follows.

That $\mathrm{Q}_{ij} \, N_s = 0$ for $j \neq s$ follows from Lemma 13(i). If $j = s$, then $\mathrm{Q}_{is} \, N_s = N_s$ follows from the proof of Proposition 15. This proves claim (ii).

On the level of A-A-bimodules, the quotient of A modulo its radical is isomorphic to the direct sum of the cokernels of β_{ii} for $i = 1, 2, \ldots, n$. This gives an epimorphic

natural transformation $\gamma : \mathrm{Id}_{\overline{\mathbf{M}}(\clubsuit)} \twoheadrightarrow Q$. For every $s = 1, 2, \ldots, n$, the epimorphism γ_{N_s} is, in fact, an isomorphism due to claim (ii). As N is a progenerator in $\overline{\mathbf{M}}(\clubsuit)$ by Lemma 14, claim (iii) follows and the proof is complete. $\qquad\square$

Lemma 17.

(i) *For $i, s \in \{1, 2, \ldots, n\}$ such that $\mathrm{al}(\mathrm{F}_{is}\, N_s) = \mathbf{c}$, we have*
$$\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, N_s) = \mathrm{sub}_{\boldsymbol{\alpha}}^{\mathbf{c}-1}(\mathrm{F}_{is}\, N_s) \quad and \quad \mathrm{F}_{is}\, N_s/\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, N_s) \cong N_i.$$

(ii) *We have $(\mathrm{F}_{is}\, P) \cong P_i^{\oplus \dim(Ae_{\sigma^{-1}(s)})}$ and hence*
$$(\mathrm{F}_{is}\, P)/\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, P) \cong N_i^{\oplus \dim(Ae_{\sigma^{-1}(s)})}.$$

Proof. From Proposition 15, we obtain an exact sequence $\mathrm{Ker} \hookrightarrow \mathrm{F}_{is}\, N_s \twoheadrightarrow N_i$ and the proof of Proposition 15 shows that $\mathrm{Ker} \subset \mathrm{sub}_{\boldsymbol{\alpha}}^{\mathbf{c}-1}(\mathrm{F}_{is}\, N_s)$. On the other hand, $(\mathrm{F}_{is}\, N_s)/\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, N_s) \twoheadrightarrow N_i$ as N_i is annihilated by $\boldsymbol{\alpha}$. By relative projectivity of N_i, see Lemma 14, it is a direct summand of $(\mathrm{F}_{is}\, N_s)/\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, N_s)$ and hence coincides with the latter as both have simple tops. This proves claim (i).

To see claim (ii), note that $\mathrm{F}_{is}\, P$ is projective, and that, using adjunction,
$$(10) \qquad \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(\mathrm{F}_{is}\, P, L_k) \cong \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(P, \mathrm{F}_{\sigma^{-1}(s)i}\, L_k)$$
is zero unless $k = i$ by Lemma 12, implying that $\mathrm{F}_{is}\, P$ is a direct sum of copies of P_i. If $k = i$, the space in (10) is isomorphic to $\mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(P, \tilde{P}_{\sigma^{-1}(s)})$, again by Lemma 12, which has dimension $\dim(\tilde{P}_{\sigma^{-1}(s)}) = \dim(Ae_{\sigma^{-1}(s)})$. Both claims in (ii) follow. $\qquad\square$

Proposition 18. *We have $\mathrm{F}\, N_s \cong P$ for every $s \in \{1, 2, \ldots, n\}$.*

Proof. Consider the diagram

(11)

$$\begin{array}{ccccc}
P & \xrightarrow{\ \ f_1\ \ } & \mathrm{F}\, N_s & \xrightarrow{\ \ f_2\ \ } & \mathrm{F}\, L_s \\
& & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} \\
\mathrm{sub}_{\boldsymbol{\alpha}}(P) & \xrightarrow{\ \ h_1\ \ } & \mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, N_s) & \xrightarrow{\ \ h_2\ \ } & \mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, L_s)
\end{array}$$

in which f_1 is an epimorphism given by Lemma 13(ii), and f_2 is an epimorphism induced by the natural projection $N_s \twoheadrightarrow L_s$ using exactness of F. Comparing the action of F on $\mathbf{K}/\mathbf{I}$ and on A-mod, the latter being equivalent to $\mathbf{C}_{\mathcal{L}}$ by [MM5, Proposition 9], and using Example 8, we see that $\mathrm{F}\, L_s$ has isomorphic top and socle and that this socle coincides with $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, L_s)$. Let $\varphi : A \to A$ be an endomorphism of A which induces and isomorphism between the top and the socle of ${}_A A$. Define $g_2 := (\varphi \otimes 1)_{N_s}$ and $g_3 := (\varphi \otimes 1)_{L_s}$ and let h_2 be the restrictions of f_2. Then the right square of the diagram commutes. The fact that the map g_3 is an epimorphism again follows by comparing the action of F on $\mathbf{K}/\mathbf{I}$ and on A-mod. This implies that h_2 is an epimorphism. Finally, g_2 is an epimorphism since the top of $\mathrm{F}\, N_s$ maps isomorphically on the top of $\mathrm{F}\, L_s$ by f_2, the latter top maps isomorphically onto $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, L_s)$ by g_3, and the top of $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, N_s)$ maps isomorphically onto $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, L_s)$ by h_2.

Define h_1 to be the restriction of f_1. The isomorphism theorem yields
$$\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, L_s) \cong (\mathrm{sub}_{\boldsymbol{\alpha}}(P) + IP)/IP \cong \mathrm{sub}_{\boldsymbol{\alpha}}(P)/(\mathrm{sub}_{\boldsymbol{\alpha}}(P) \cap IP)$$

whence $h_2 h_1$ is an epimorphism. Since the map h_2 is just factoring out the radical, the map h_1 must be an epimorphism.

The lower row of (11) belongs to $\mathrm{sub}_{\boldsymbol{\alpha}}(\overline{\mathbf{M}}(\clubsuit))$. We have $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, N_s) \in \mathrm{add}(N)$ by Proposition 15, moreover, $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, N_s) \twoheadrightarrow L$ as $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, L_s) \twoheadrightarrow L$. As the map $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, N_s) \twoheadrightarrow L$ induces an isomorphism modulo the radical, we obtain $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, N_s) \cong N$. Since N is relatively projective in $\mathrm{sub}_{\boldsymbol{\alpha}}(\overline{\mathbf{M}}(\clubsuit))$ by Lemma 14, the object $\mathrm{sub}_{\boldsymbol{\alpha}}(\mathrm{F}\, N_s) \cong N$ splits off as a direct summand of $\mathrm{sub}_{\boldsymbol{\alpha}}(P)$.

For $i \in \{1, 2, \ldots, n\}$, applying F_{is} to the short exact sequence

$$0 \to N_s \to P \to \mathrm{Coker} \to 0,$$

produces the short exact sequence

$$0 \to \mathrm{F}_{is}\, N_s \to \mathrm{F}_{is}\, P \to \mathrm{F}_{is}\, \mathrm{Coker} \to 0.$$

Using Lemma 11 and additivity, there is i such that $\mathrm{al}(\mathrm{F}_{is}\, N_s) = \mathrm{al}(\mathrm{F}_{is}\, P) = \mathbf{c}$.
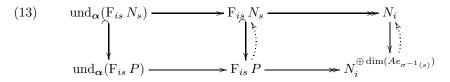
Now we claim that

$$(12) \quad \mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, N_s) \subset \mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, P) \cap (\mathrm{F}_{is}\, N_s) \subset$$
$$\subset \mathrm{sub}_{\boldsymbol{\alpha}}^{\mathbf{c}-1}(\mathrm{F}_{is}\, P) \cap (\mathrm{F}_{is}\, N_s) \subset \mathrm{sub}_{\boldsymbol{\alpha}}^{\mathbf{c}-1}(\mathrm{F}_{is}\, N_s).$$

Indeed, the first inclusion is due to the fact that

$$(\mathrm{F}_{is}\, N_s)/\big(\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, P) \cap (\mathrm{F}_{is}\, N_s)\big) \hookrightarrow (\mathrm{F}_{is}\, P)/\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, P)$$

by the isomorphism theorem and the right hand side is annihilated by $\boldsymbol{\alpha}$. Therefore the left hand side is annihilated by $\boldsymbol{\alpha}$ and is hence a quotient of $(\mathrm{F}_{is}\, N_s)/\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, N_s)$ by the universal property of the latter. The second inclusion follows from Proposition 6(i) and the last inclusion is obvious. By Lemma 17, all inclusions in (12) are, in fact, equalities.

Consider the solid commutative diagram

$$(13)$$

with exact rows given by Lemma 17. Here the middle vertical arrow is the natural monomorphism, which induces both other vertical arrows. Since all inclusions in (12) are equalities, in particular $\mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, N_s) = \mathrm{und}_{\boldsymbol{\alpha}}(\mathrm{F}_{is}\, P) \cap (\mathrm{F}_{is}\, N_s)$, the left square of (13) is a pullback, implying that the morphism between cokernels on the right is a monomorphism. The object N_i being indecomposable, a Loewy length argument shows that the cokernel morphism $N_i \hookrightarrow N_i^{\oplus \dim(Ae_{\sigma^{-1}(s)})}$ splits as shown by the right dotted arrow. By projectivity of $\mathrm{F}_{is}\, P$, this induces the middle dotted arrow in (13). Since $\mathrm{F}_{is}\, N_s$ has simple top, this dotted arrow is a splitting of the embedding $\mathrm{F}_{is}\, N_s \hookrightarrow \mathrm{F}_{is}\, P$. Therefore, $\mathrm{F}_{is}\, N_s$ is a direct summand of a projective object and is hence projective itself.

Now we claim that $\mathrm{F}_{js}\, N_s$ is projective for any j. This follows from the facts that all F_{ab} send projectives to projectives (as each of them is left adjoint to an exact functor), and all F_{js} appear as direct summand in FF_{is}. Taking this into account, an application of Lemma 13 completes the proof. $\square$

For $s = 1, 2, \ldots, n$, set $\hat{N}_s := N_s \oplus N_{s+n} \oplus \cdots \oplus N_{n(k-1)+s}$. Using Proposition 18, we have

$$(14) \qquad\qquad \mathrm{F}\,\hat{N}_s \cong \hat{P} \qquad \text{for each } s.$$

By construction, $\mathrm{End}_{\mathscr{C}}(\mathrm{F}) \cong A \otimes_{\Bbbk} A^{\mathrm{op}}$, so we may identify A with the subalgebra $A \otimes 1$ in $\mathrm{End}_{\mathscr{C}}(\mathrm{F})$. Set $B_s := \mathrm{End}_{\overline{\mathbf{M}}(\clubsuit)}(\hat{N}_s)$.

Lemma 19. *For any $s, t = 1, 2, \ldots, n$ there is an algebra isomorphism $B_s \cong B_t$.*

Proof. By Lemma 16, the functor Q_{st} induces an equivalence between $\mathrm{add}(\hat{N}_t)$ and $\mathrm{add}(\hat{N}_s)$, which implies the claim. $\qquad\square$

Thanks to Lemma 19, we can write B for B_s. Since $\boldsymbol{\alpha}$ annihilates $\hat{N}_s$, the natural map $\mathrm{End}_{\mathscr{C}}(\mathrm{F}) \otimes_{\Bbbk} B \to \mathrm{End}_{\overline{\mathbf{M}}(\clubsuit)}(\hat{P})$ factors through the morphism

$$(15) \qquad\qquad A \otimes_{\Bbbk} B \to \mathrm{End}_{\overline{\mathbf{M}}(\clubsuit)}(\hat{P}).$$

Proposition 20. *The morphism in (15) is an isomorphism.*

Proof. First we show that the dimensions of both algebras agree. The algebra on the right hand side has dimension $\dim(\hat{P})$ which is equal to $\dim(A)\dim(\hat{N}_s)$ by the calculation in (9). Since $\hat{N}_s$ is a basic progenerator in B-mod, we have $\dim(\hat{N}_s) = \dim(B)$.

It remains to show that the morphism in (15) is injective. Choose a basis $\{\varphi_1, \varphi_2, \ldots, \varphi_{\dim(A)}\}$ in A compatible with the filtration in (6). In analogy to (7), we obtain a filtration

$$0 = Y_1 \subset Y_1 \subset \cdots \subset Y_{\dim(A)} = \mathrm{F}\,\hat{N}_s = \hat{P},$$

where Y_i is the image of $(C_i \otimes A^{\mathrm{op}})_{\hat{N}_s}$. For a non-zero $\psi \in B$ consider the endomorphism $\varphi_i \otimes \psi$ of $\hat{P}$. The proof of Proposition 15 shows that the quotient of the image of $\varphi_i \otimes \psi$ modulo Y_{i-1} coincides with the image of $\mathrm{Q}_{ts}(\psi)$ considered as an endomorphism of $\hat{N}_t$. By Lemma 16, the latter is non-zero. This implies the claim. $\qquad\square$

4.4. The proof for $\mathscr{C} = \mathscr{C}_A$.

Proof of Theorem 4 for $\mathscr{C}_A$. For a fixed $s \in \{1, 2, \ldots, n\}$ set $\mathcal{B} := \mathrm{add}(\hat{N}_s)$. Consider the inflation $\mathbf{P}_{\clubsuit} \boxtimes \mathcal{B}$ and define the 2-natural transformation

$$\Phi : \mathbf{P}_{\clubsuit} \boxtimes \mathcal{B} \to \overline{\mathbf{M}}$$

in the following way:

- send $\mathrm{G} \boxtimes X$ to $\mathrm{G}\,X$ for $X \in \mathcal{B}$ and a 1-morphism $\mathrm{G} \in \mathscr{C}_A$;

- send $\alpha \otimes f : \mathrm{G}_1 \boxtimes X_1 \to \mathrm{G}_2 \boxtimes X_2$ to $\mathrm{G}_2(f) \circ \alpha_{X_1} = \alpha_{X_2} \circ \mathrm{G}_1(f)$.

It is easy to check that this is a well-defined functor and, moreover, defines a 2-natural transformation.

For our fixed left cell $\mathcal{L} = \mathcal{L}_1$, consider the 2-subrepresentation $\mathbf{N}$ of $\mathbf{P}_{\clubsuit}$ as defined in Section 3.1. Restricting Φ to $\mathbf{N} \boxtimes \mathcal{B}$ and using (14), we get a 2-natural transformation from $\mathbf{N} \boxtimes \mathcal{B}$ to $\mathbf{G}_{\overline{\mathbf{M}}}(\mathrm{add}(\hat{P}))$. Since the ideal $\mathbf{I}$ defined in Section 3.1

is generated by $\boldsymbol{\alpha}$ (see, for example, the proof of [MM5, Lemma 14]), the ideal $\mathbf{I}$ annihilates $\hat{N}_s$ by construction. Hence we get the induced 2-natural transformation

$$\overline{\Phi} : \mathbf{C}_{\mathcal{L}} \boxtimes \mathcal{B} \to \mathbf{G}_{\overline{\mathbf{M}}}(\mathrm{add}(\hat{P})).$$

Proposition 20 shows that $\overline{\Phi}$ is an equivalence of categories. This completes the proof. □

5. Proof of Theorem 4 in the general case

5.1. **Obvious generalization of Section 4.** Let A be as in Section 2.8. For certain central subalgebras X in A, in [MM3, Section 4.5] we constructed certain fiat 2-subcategories $\mathscr{C}_{A,X}$ in $\mathscr{C}_A$. The only difference between $\mathscr{C}_{A,X}$ and $\mathscr{C}_A$ is that the endomorphism algebra of the identity 1-morphism in $\mathscr{C}_{A,X}$ is X. Similarly one defines $\mathscr{C}_{A,X}$ in the weakly fiat case. Since we did not use the endomorphism algebra of the identity 1-morphism in our arguments in Section 4, the claim of Theorem 4 is true for 2-representations of $\mathscr{C}_{A,X}$.

Let A be as in Section 2.8, but not necessarily connected, say

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_m.$$

As above, choose some central subalgebras X_i in each A_i. With such A and $X = (X_1, X_2, \ldots, X_m)$ one associates a 2-category $\mathscr{C}_{A,X}$ with m objects defined similarly to the connected case, see [MM1, Section 7.3] and [MM3, Section 4.5] for details. The arguments in Section 4 work mutatis mutandis in this more general setup.

5.2. **The proof.**

Proof of Theorem 4. Let $\mathscr{C}$ be a weakly fiat 2-category with a unique maximal two-sided cell $\mathcal{J}$. Assume that $\mathcal{J}$ is strongly regular and that $\mathscr{C}$ is $\mathcal{J}$-simple. Let $\mathbf{M}$ be an isotypic faithful 2-representation of $\mathscr{C}$.

Denote by $\mathscr{C}_{\mathcal{J}}$ the 2-full 2-subcategory of $\mathscr{C}$ whose 1-morphisms are those in $\mathcal{J}$ as well as the identity 1-morphisms on their respective sources and targets. Restricting $\mathbf{M}$ to $\mathscr{C}_{\mathcal{J}}$ produces a faithful 2-representation, which we denote by $\mathbf{M}_{\mathcal{J}}$.

Let $\mathcal{L}$ be a left cell in $\mathcal{J}$. Our assumptions and Theorem 31 imply that all simple transitive subquotients of $\mathbf{M}$ are equivalent to $\mathbf{C}_{\mathcal{L}}$, in particular, they are not annihilated by any 1-morphism in $\mathcal{J}$. By Theorem 33, the 2-category $\mathscr{C}_{\mathcal{J}}$ has only one (up to equivalence) simple transitive 2-representation which is not annihilated by 1-morphisms in $\mathcal{J}$. This means that $\mathbf{M}_{\mathcal{J}}$ is an isotypic 2-representation of $\mathscr{C}_{\mathcal{J}}$.

By Theorem 32, the 2-category $\mathscr{C}_{\mathcal{J}}$ is a 2-category of the form $\mathscr{C}_{A,X}$ for (not necessarily connected) A and X as in Section 5.1. Therefore we can construct the object $\hat{N}_s$ as in Section 4. Using the principal 2-representation of $\mathscr{C}$ instead of the principal 2-representation of $\mathscr{C}_{\mathcal{J}}$, the proof is now completed by the same argument as in Section 4.4. □

6. New examples

6.1. **Tensor product of finitary 2-categories.** The tensor product of categories as described in Section 3.6 can be used to produce new examples of finitary and fiat 2-categories. This is motivated by the 2-category of Soergel bimodules associated to Coxeter systems with several irreducible components.

Let $\mathscr{A}$ and $\mathscr{C}$ be two finitary 2-categories. Define the 2-category $\mathscr{A} \boxtimes \mathscr{C}$ whose objects are pairs $(\mathtt{i}, \mathtt{j})$ where $\mathtt{i}$ is an object of $\mathscr{A}$ and $\mathtt{j}$ is an object of $\mathscr{C}$, and whose morphism categories are given by

$$(\mathscr{A} \boxtimes \mathscr{C})\big((\mathtt{i}, \mathtt{j}), (\mathtt{k}, \mathtt{l})\big) := \mathscr{A}(\mathtt{i}, \mathtt{k}) \boxtimes \mathscr{C}(\mathtt{j}, \mathtt{l}).$$

Composition is defined component-wise and $\mathbb{1}_{(\mathtt{i},\mathtt{j})} = \mathbb{1}_{\mathtt{i}} \boxtimes \mathbb{1}_{\mathtt{j}}$.

Proposition 21. *The 2-category $\mathscr{A} \boxtimes \mathscr{C}$ is finitary, moreover, it is (weakly) fiat if both $\mathscr{A}$ and $\mathscr{C}$ are.*

Proof. As the tensor product of two local algebras is local, the tensor product of an indecomposable 1-morphisms in $\mathscr{A}$ and an indecomposable 1-morphisms in $\mathscr{C}$ will have a local endomorphism algebra. This implies that $(\mathscr{A} \boxtimes \mathscr{C})\big((\mathtt{i}, \mathtt{j}), (\mathtt{k}, \mathtt{l})\big)$ is idempotent split. All other axioms of finitary 2-categories follow directly from the definition.

If both $\mathscr{A}$ and $\mathscr{C}$ are (weakly) fiat, we can define $*$ on $\mathscr{A} \boxtimes \mathscr{C}$ component-wise. Since all compositions are defined component-wise, adjunction morphisms for $\mathscr{A} \boxtimes \mathscr{C}$ are constructed by tensoring adjunction morphisms for $\mathscr{A}$ with adjunction morphisms for $\mathscr{C}$. $\qquad\square$

If $\mathcal{L}_1$ is a left cell in $\mathscr{A}$ and $\mathcal{L}_2$ is a left cell in $\mathscr{C}$, then

$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 := \{\mathrm{F} \boxtimes \mathrm{G} \,:\, \mathrm{F} \in \mathcal{L}_1, \mathrm{G} \in \mathcal{L}_2\}$$

is a left cell in $\mathscr{A} \boxtimes \mathscr{C}$. The partial order on the set of left cells in $\mathscr{A} \boxtimes \mathscr{C}$ is the direct product of the partial orders on the sets of left cells in $\mathscr{A}$ and $\mathscr{C}$. Similar statements hold for right and two-sided cells. It is worth pointing out that the direct product of strongly regular two-sided cells is strongly regular. Furthermore, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are two-sided cells in $\mathscr{A}$ respectively $\mathscr{C}$ such that $\mathscr{A}$ and $\mathscr{C}$ are $\mathcal{J}_1$-respectively $\mathcal{J}_2$-simple, then $\mathscr{A} \boxtimes \mathscr{C}$ is $\mathcal{J}_1 \boxtimes \mathcal{J}_2$-simple.

As a special case of the above construction, consider two self-injective algebras A and B. The finitary 2-category $\mathscr{C}_A \boxtimes \mathscr{C}_B$ can be realized as an "extension" of $\mathscr{C}_{A \otimes B}$ by adding the (additive closure of) 1-morphisms given by $A \otimes B$-$A \otimes B$-bimodules

$$A \otimes_{\Bbbk} (B \otimes_{\Bbbk} B) \quad \text{and} \quad (A \otimes_{\Bbbk} A) \otimes_{\Bbbk} B$$

and all possible natural transformations between all functors.

6.2. **Isotypic 2-representations for certain tensor products.** Let $\mathscr{C}$ be as in Theorem 4 and $\mathcal{J}$ the unique maximal two-sided cell in $\mathscr{C}$. For a local commutative finite dimensional $\Bbbk$-algebra B denote by $\mathscr{Y}_B$ the finitary 2-category having

- one object $\spadesuit$;

- direct sums of copies of $\mathbb{1}_{\spadesuit}$ as 1-morphisms;

- B as the endomorphism algebra of $\mathbb{1}_{\spadesuit}$.

Consider the 2-category $\mathscr{C} \boxtimes \mathscr{Y}_B$. This 2-category has $\hat{\mathcal{J}} := \mathcal{J} \boxtimes \mathbb{1}_{\spadesuit}$ as its unique maximal two-sided cell. Note that $\mathscr{C} \boxtimes \mathscr{Y}_B$ is not $\hat{\mathcal{J}}$-simple unless $B \cong \Bbbk$. Given a finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ and a finitary 2-representation $\mathbf{N}$ of $\mathscr{Y}_B$, the tensor product $\mathbf{M} \boxtimes \mathbf{N}$ carries the natural structure of a 2-representation of $\mathscr{C} \boxtimes \mathscr{Y}_B$ defined component-wise.

A finitary 2-representation $\mathbf{N}_{\mathcal{C},\varphi}$ of $\mathscr{Y}_B$ is given, up to equivalence, by the data of a finitary category $\mathcal{C}$ and a fixed homomorphism φ from B to the center of $\mathcal{C}$.

Proposition 22. *Every isotypic 2-representation of $\mathscr{C} \boxtimes \mathscr{Y}_B$ which does not annihilate any 1-morphism in $\hat{\mathcal{J}}$ is equivalent to $\mathbf{C}_{\mathcal{L}} \boxtimes \mathbf{N}_{\mathcal{C},\varphi}$ for some left cell $\mathcal{L}$ in $\mathcal{J}$ and appropriate $\mathcal{C}$ and φ as above.*

Proof. Let $\mathbf{M}$ be an isotypic 2-representation of $\mathscr{C} \boxtimes \mathscr{Y}_B$ which does not annihilate any 1-morphism in $\hat{\mathcal{J}}$. Restricting $\mathbf{M}$ to the canonical copy of $\mathscr{C}$ in $\mathscr{C} \boxtimes \mathscr{Y}_B$ gives a faithful isotypic 2-representation of $\mathscr{C}$. Consider the category $\mathrm{add}(\hat{N}_s)$. It carries the natural structure of a 2-representation of $\mathscr{Y}_B$ by restriction (since any indecomposable 1-morphism in $\mathscr{Y}_B$ is isomorphic to the identity). We denote the resulting 2-representation of $\mathscr{Y}_B$ by $\mathbf{N}$. By the above remark, $\mathbf{N}$ is equivalent to $\mathbf{N}_{\mathcal{C},\varphi}$ for some $\mathcal{C}$ and φ.

Let $\mathcal{L}$ be a left cell in $\mathcal{J}$ and $\mathtt{i} := \mathtt{i}_{\mathcal{L}}$. Consider now the 2-representation $\mathbf{P}_{\mathtt{i}} \boxtimes \mathbf{N}$ of $\mathscr{C} \boxtimes \mathscr{Y}_B$. Mapping $\mathrm{F} \boxtimes X$ to $\mathrm{F}\, X$ defines a 2-natural transformation from $\mathbf{P}_{\mathtt{i}} \boxtimes \mathbf{N}$ to $\mathbf{M}$. Similarly to Section 4.4, this induces an equivalence between $\mathbf{C}_{\mathcal{L}} \boxtimes \mathbf{N}$ and $\mathbf{M}$. $\qquad\square$

6.3. **Finitary 2-categories associated with trivial extensions.** Let B be a basic connected finite dimensional algebra and let M be a B-B-bimodule in $\mathrm{add}\{B, B \otimes_{\Bbbk} B\}$. Denote by A the *trivial extension* of B by M, that is, the set

$$\left\{ \begin{pmatrix} b & m \\ 0 & b \end{pmatrix} : b \in B, m \in M \right\}$$

with the obvious (matrix) multiplication. Denote by $\mathscr{C}(B, M)$ the 2-category obtained by adding to $\mathscr{C}_A$ all functors given by tensoring with bimodules in $\mathrm{add}(A \otimes_B A)$ and all natural transformations between all our functors.

Proposition 23. *The 2-category $\mathscr{C}(B, M)$ is finitary.*

Proof. The only thing we need to check is that $\mathscr{C}(B, M)$ is closed with respect to composition of 1-morphisms. Since ${}_B A_B \cong B \oplus M \in \mathrm{add}\{B, B \otimes_{\Bbbk} B\}$, we have

$$(A \otimes_B A) \otimes_A (A \otimes_B A) \cong (A \otimes_B A) \oplus (A \otimes_B M \otimes_B A)$$

and

$$A \otimes_B M \otimes_B A \in \mathrm{add}\{A \otimes_B A, A \otimes_{\Bbbk} A\}$$

by our choice of M.

Next we claim that both $(A \otimes_B A) \otimes_A (A \otimes_{\Bbbk} A)$ and $(A \otimes_{\Bbbk} A) \otimes_A (A \otimes_B A)$ are projective A-A-bimodules. We prove the first part and the second part is similar. We have to show that $A \otimes_B A$ is projective as a left A-module. As

$$A \otimes_B A \cong A \otimes_B (B \oplus M) \in \mathrm{add}\{A \otimes_B B, A \otimes_B B \otimes_{\Bbbk} B\},$$

the claim follows and the proof of the proposition is complete. $\qquad\square$

This construction generalizes in the obvious way to the case of a not necessarily connected algebra B.

6.4. Fiat 2-categories associated to iterated trivial extensions of symmetric algebras.

Proposition 24. *Let B be a connected finite dimensional symmetric algebra and $M = B = B^*$. Then $\mathscr{C}(B, M)$ is a fiat 2-category.*

We will need the following observation.

Lemma 25. *The A-A-bimodule $A \otimes_B A$ is indecomposable.*

Proof. Denote by Z the center of B. Note that Z is local as B is connected. Using adjunction on both sides, we see

$$\operatorname{Hom}_{A\text{-}A}(A \otimes_B A, A \otimes_B A) \cong \operatorname{Hom}_{B\text{-}B}(B, A \otimes_B A) \cong Z^{\oplus 4}$$

since $A \otimes_B A \cong B^{\oplus 4}$ as a B-B-bimodule and $Z \cong \operatorname{End}_{B\text{-}B}(B)$.

Next, a direct calculation shows that the subalgebra

$$\hat{Z} := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in Z \right\}$$

of A is central. We denote by $\hat{Z}_1$ the subalgebra of $\hat{Z}$ consisting of diagonal elements and by $\hat{Z}_2$ the subalgebra of $\hat{Z}$ consisting of upper triangular elements. Thus $\hat{Z} = \hat{Z}_1 \oplus \hat{Z}_2$ as a direct sum of two subspaces. We also note that $\hat{Z}_1$ is isomorphic to Z.

The commutative algebra $\tilde{Z} := \hat{Z} \otimes_{\hat{Z}_1} \hat{Z}$ acts faithfully by endomorphisms of the A-A-bimodule $A \otimes_B A$ where the left component acts by multiplication on the left and the right component acts by multiplication on the right. This induces an isomorphism $\hat{Z} \otimes_{\hat{Z}_1} \hat{Z} \cong \operatorname{End}_{A\text{-}A}(A \otimes_B A)$ since both are isomorphic to $Z^{\oplus 4}$ as Z-modules.

Finally, note that both $\hat{Z}_2 \otimes_{\hat{Z}_1} \hat{Z}$ and $\hat{Z} \otimes_{\hat{Z}_1} \hat{Z}_2$ are nilpotent in $\tilde{Z}$ of degree 2 and thus belong to the radical of $\tilde{Z}$. The quotient $\tilde{Z}/(\hat{Z}_2 \otimes_{\hat{Z}_1} \hat{Z} + \hat{Z} \otimes_{\hat{Z}_1} \hat{Z}_2)$ is isomorphic to Z and is thus local. This implies that $\tilde{Z}$ is local, completing the proof. $\qquad\square$

Proof of Proposition 24. First we note that the trivial extension A of B by B^* is symmetric, in particular, weakly symmetric, so the 2-subcategory $\mathscr{C}_A$ of $\mathscr{C}(B, M)$ is fiat. In view of Lemma 25, it only remains to check that $A \otimes_B A$ is self-adjoint. This follows from the chain of isomorphisms

$$
\begin{aligned}
\operatorname{Hom}_{A\text{-}}(A \otimes_B A, A) &\cong \operatorname{Hom}_{B\text{-}}(A, A) & \\
&\cong \operatorname{Hom}_{B\text{-}}(A, B) \otimes_B A & (\text{as } {}_B A \cong B \oplus B) \\
&\cong \operatorname{Hom}_{\Bbbk}(A, \Bbbk) \otimes_B A & (\text{as } B^* \cong B) \\
&\cong A \otimes_B A & (\text{as } A^* \cong A).
\end{aligned}
$$

$\square$

Proposition 24 implies that for a symmetric algebra B the construction can be iterated. Set $A_0 := B$ and for $k = 1, 2, \ldots$ defined A_k recursively as the trivial extension of A_{k-1} by $A_{k-1}^* \cong A_{k-1}$. This leads to a fiat 2-category $\mathscr{C}_{B,k}$ which is obtained by adding to $\mathscr{C}_{A_k}$ all functors given by tensoring with bimodules in

add($A_k \otimes_{A_i} A_k$) for all $i < k$ and all natural transformations between all our functors. The arguments in the proof of Proposition 24 generalize mutatis mutandis to show that $\mathscr{C}_{B,k}$ is a fiat 2-category. Clearly, all two-sided cells in $\mathscr{C}_{B,k}$ are strongly regular.

Combining this construction with the one proposed in Section 6.1, we obtain numerous new examples of fiat 2-categories in which all two-sided cells are strongly regular.

7. APPENDIX: WEAKLY FIAT POTPOURRI

In this section, we collect appropriate reformulations of statements from [MM1, MM2, MM3] (originally for fiat 2-categories) in the more general setup of weakly fiat 2-categories. Let $\mathscr{C}$ be a weakly fiat 2-category as defined in Section 2.5 with a weak equivalence $*$.

We write $\mathbf{P}$ for the direct sum of all principal representations of $\mathscr{C}$. Isomorphism classes of indecomposable projective and simple objects in $\overline{\mathbf{P}}$ are indexed by elements in $\mathcal{S}(\mathscr{C})$ and denoted by $\hat{P}_{\mathrm{F}}$ and $\hat{L}_{\mathrm{F}}$.

Proposition 26. *Let* $\mathrm{F}, \mathrm{G}, \mathrm{H} \in \mathcal{S}(\mathscr{C})$.

(i) *The inequality* $\mathrm{F}\,\hat{L}_{\mathrm{G}} \neq 0$ *is equivalent to* $^{*}\mathrm{F} \leq_{R} \mathrm{G}$ *and also to* $\mathrm{F} \leq_{L} \mathrm{G}^{*}$.

(ii) *The inequality* $[\mathrm{F}\,\hat{L}_{\mathrm{G}} : \hat{L}_{\mathrm{H}}] \neq 0$ *implies* $\mathrm{H} \leq_{L} \mathrm{G}$.

(iii) *If* $\mathrm{H} \leq_{L} \mathrm{G}$, *then there is* $\mathrm{K} \in \mathcal{S}(\mathscr{C})$ *such that* $[\mathrm{K}\,\hat{L}_{\mathrm{G}} : \hat{L}_{\mathrm{H}}] \neq 0$.

(iv) *If* $\hat{L}_{\mathrm{F}}$ *occurs in the top or socle of* $\mathrm{H}\,\hat{L}_{\mathrm{G}}$, *then* F *is in the same left cell as* G.

(v) *If* $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, *then there is a unique (up to scalar) non-zero homomorphism* $\hat{P}_{\mathbb{1}_{\mathtt{i}}} \to \mathrm{F}^{*}\,\hat{L}_{\mathrm{F}}$, *in particular,* $\mathrm{F}^{*}\,\hat{L}_{\mathrm{F}} \neq 0$.

Proof. Mutatis mutandis [MM1, Lemmata 12-15]. □

Proposition 27. *Let* $\mathcal{L}$ *be a left cell of* $\mathscr{C}$ *and* $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$.

(i) *There is a unique submodule* K *of* $\hat{P}_{\mathbb{1}_{\mathtt{i}}}$ *such that*

   (a) *every simple subquotient of* $\hat{P}_{\mathbb{1}_{\mathtt{i}}}/K$ *is annihilated by any* $\mathrm{F} \in \mathcal{L}$;

   (b) *the module* K *has simple top* $\hat{L}_{\mathrm{G}_{\mathcal{L}}}$ *for some* $\mathrm{G}_{\mathcal{L}} \in \mathcal{S}(\mathcal{C})$ *and* $\mathrm{F}\,\hat{L}_{\mathrm{G}_{\mathcal{L}}} \neq 0$ *for any* $\mathrm{F} \in \mathcal{L}$.

(ii) *For any* $\mathrm{F} \in \mathcal{L}$ *the module* $\mathrm{F}\,\hat{L}_{\mathrm{G}_{\mathcal{L}}}$ *has simple top* $\hat{L}_{\mathrm{F}}$.

(iii) *We have* $\mathrm{G}_{\mathcal{L}} \in \mathcal{L}$.

(iv) *For any* $\mathrm{F} \in \mathcal{L}$ *we have* $^{*}\mathrm{F} \leq_{R} \mathrm{G}_{\mathcal{L}}$ *and* $\mathrm{F} \leq_{L} \mathrm{G}_{\mathcal{L}}^{*}$.

(v) *We have* $\mathrm{G}_{\mathcal{L}}^{*} \in \mathcal{L}$.

Proof. Mutatis mutandis [MM1, Proposition 17]. □

The element $G_{\mathcal{L}} \in \mathcal{L}$ is called the *Duflo involution* in $\mathcal{L}$.

Proposition 28. *Let $\mathcal{J}$ be a strongly regular two-sided cell in $\mathscr{C}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. Then for $G \in \mathcal{L}$ the following assertions are equivalent.*

(a) $G = G_{\mathcal{L}}$.

(b) $G^* \in \mathcal{L}$.

(c) $\{G\} = \mathcal{L} \cap {}^*\mathcal{L}$.

(d) $G = {}^*H$ *where* $\{H\} = \mathcal{L} \cap \mathcal{L}^*$.

(e) $G \, \hat{L}_G \neq 0$.

Proof. We have

- (a)$\Rightarrow$(b) by Proposition 27(v);

- (b)$\Rightarrow$(d) as $|\mathcal{L} \cap \mathcal{L}^*| = 1$;

- (d)$\Rightarrow$(c) applying $F \mapsto {}^*F$;

- (c) $\Rightarrow$(b) applying $F \mapsto F^*$;

- (b)$\Rightarrow$(a) again as $|\mathcal{L} \cap \mathcal{L}^*| = 1$.

Finally, (b)$\Leftrightarrow$(e) follows from strong regularity of $\mathcal{J}$ and Proposition 26(i). $\square$

Let $\mathcal{J}$ be a strongly regular two-sided cell in $\mathscr{C}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. To simplify computational expressions, we assume that $\mathcal{J}$ is a maximal two-sided cell. In the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ as defined in Section 3.1, we have indecomposable projective objects P_F, indecomposable injective objects I_F and simple objects L_F, where $F \in \mathcal{L}$. Set $G = G_{\mathcal{L}}$.

Lemma 29. *For any $F \in \mathcal{L}$ we have $FG \cong \mathbf{m}_G F$.*

Proof. Strong regularity of $\mathcal{J}$ implies that $GG = mG$ for some non-negative integer m. Applying this to L_G, using exactness of G and a character argument, we see that
$$m = [P_G : L_G] = \dim \operatorname{End}(P_G).$$
At the same time, $\dim \operatorname{End}(P_G) = \dim \operatorname{Hom}(G \, L_G, G \, L_G)$ equals $\mathbf{m}_G$ by adjunction. Thus $m = \mathbf{m}_G$.

Strong regularity of $\mathcal{J}$ again yields $FG \cong kF$ for some non-negative integer k. Moreover, k is, in fact, positive, since $FG \, L_G \neq 0$. Now, computing FGG in two different ways using associativity, and then dividing by k, we obtain $k = \mathbf{m}_G$. $\square$

Proposition 30. *Let $F \in \mathcal{L}$ and $H \in \mathcal{J}$.*

(i) *The projective object P_F is injective.*

(ii) *The object $H \, L_F$, when non-zero, has a non-zero projective-injective summand.*

(iii) *We have $F^* L_F = I_G$.*

(iv) *The object* $\mathrm{H}\,L_\mathrm{F}$ *is both projective and injective.*

(v) *The object* $\mathrm{H}\,L_\mathrm{F}$ *is indecomposable or zero.*

Proof. By adjunction, L_G injects into F^*L_F. Consider an injective object I and let L_K be a simple quotient of I, where $\mathrm{K} \in \mathcal{L}$. Then L_G is a subquotient of the injective object $\mathrm{K}^*\,I$. As $\mathcal{J}$ is strongly regular, it follows from Proposition 26(i) that G annihilates all simples except for L_G. Hence L_G appears in the top of the injective object $\mathrm{GK}^*\,I$. Applying F, we obtain that P_F is a quotient (and hence a direct summand) of the injective object $\mathrm{FGK}^*\,I$ and is therefore injective. This proves claim (i).

Assume that $\mathrm{H}\,L_\mathrm{F} \neq 0$. The argument in the previous paragraph shows that $\mathrm{H}'\,L_\mathrm{F}$ has a non-zero projective-injective summand for some H' in the same left cell as H. Claim (ii) is now deduced by multiplying the latter on the left with elements in $\mathcal{J}$ and using strong regularity of $\mathcal{J}$.

Using adjunction and strong regularity of $\mathcal{J}$, we see that F^*L_F has simple socle L_G. Therefore claim (iii) follows from claim (ii). Claim (iv) is implied by claim (iii) as any inequality $\mathrm{H}\,L_\mathrm{F} \neq 0$ means, by strong regularity of $\mathcal{J}$, that H and F^* are in the same left cell and hence $\mathrm{H}\,L_\mathrm{F}$ is a direct summand of the projective-injective object $\mathrm{KF}^*L_\mathrm{F}$ for some K.

It remains to prove claim (v). By strong regularity of $\mathcal{J}$, there is a unique element $\tilde{\mathrm{G}}$ in the right cell of F such that $\tilde{\mathrm{G}}\,L_\mathrm{F} \neq 0$. This implies that $\tilde{\mathrm{G}}\,\hat{L}_\mathrm{F} \neq 0$ and thus $\tilde{\mathrm{G}}\,\hat{L}_{\tilde{\mathrm{G}}} \neq 0$ by Proposition 26(i), as $\tilde{\mathrm{G}}$ and F are in the same right cell. In particular, by Proposition 28, $\tilde{\mathrm{G}}$ is the Duflo involution in its left cell.

Using claim (iv) and strong regularity of $\mathcal{J}$, we deduce $\tilde{\mathrm{G}}\,L_\mathrm{F} \cong kP_\mathrm{F}$ for some non-negative integer k. We compute $\mathrm{F}^*\tilde{\mathrm{G}}\,L_\mathrm{F}$ in two different ways. On the one hand,

$$\mathrm{F}^*\tilde{\mathrm{G}}\,L_\mathrm{F} \cong k\mathrm{F}^*\,P_\mathrm{F} \cong k\mathrm{F}^*\mathrm{F}\,L_\mathrm{F} \cong k\mathbf{m}_\mathrm{F}P_{\mathrm{G}^*},$$

where the last isomorphism follows from the isomorphism $\mathrm{F}^*\mathrm{F} \cong (^*\mathrm{FF})^*$ and the fact that $\{\mathrm{G}^*\} = \mathcal{L} \cap \mathcal{L}^*$. On the other hand, F^* is in the same left cell as $\tilde{\mathrm{G}}^*$ and hence in the same left cell as $\tilde{\mathrm{G}}$ as the latter is the Duflo involution. Therefore, Lemma 29 and claim (iii) give

$$\mathrm{F}^*\tilde{\mathrm{G}}\,L_\mathrm{F} \cong \mathbf{m}_{\tilde{\mathrm{G}}}P_{\mathrm{G}^*}.$$

From Proposition 1, we obtain $\mathbf{m}_{\tilde{\mathrm{G}}} = \mathbf{m}_\mathrm{F}$ which yields $k = 1$ and proves claim (v) in the case $\mathrm{H} = \tilde{\mathrm{G}}$.

Now, in the general case, assume $\mathrm{H}\,L_\mathrm{F} \cong kP_\mathrm{K}$ for some $\mathrm{K} \in \mathcal{L}$ and a positive integer k (note that K and H are then in the same right cell). Then $^*\mathrm{HH}\,L_\mathrm{F} \neq 0$ by adjunction and hence $^*\mathrm{HH} \cong \mathbf{m}_\mathrm{H}\tilde{\mathrm{G}}$. Let us now compute the dimension of $\mathrm{End}(\mathrm{H}\,L_\mathrm{F})$. On the one hand, by adjunction and the fact that $\tilde{\mathrm{G}}\,L_\mathrm{F} \cong P_\mathrm{F}$, proved in the previous paragraph, this dimension equals $\mathbf{m}_\mathrm{H}$. On the other hand, it equals $k^2\,\mathrm{dim}\,\mathrm{End}(P_\mathrm{K})$ which, in turn, by adjunction, equals $k^2\mathbf{m}_\mathrm{K}$. Proposition 1 implies $\mathbf{m}_\mathrm{K} = \mathbf{m}_\mathrm{H}$ and hence $k = 1$, completing the proof. $\square$

Proposition 30 shows that indecomposable 1-morphisms in $\mathcal{J}$ act, under $\mathbf{C}_\mathcal{L}$, as indecomposable projective functors for some self-injective algebra. In analogy to [MM1, Theorem 43], this implies the following.

Theorem 31. *Let $\mathscr{C}$ be a weakly fiat 2-category and $\mathcal{J}$ a strongly regular two-sided cell in $\mathscr{C}$.*

    (i) For any left cell $\mathcal{L}$ in $\mathcal{J}$, the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ is strongly simple in the sense of [MM1, Section 6.2].

    (ii) If $\mathcal{L}$ and $\mathcal{L}'$ are two right cells in $\mathcal{J}$, then the cell 2-representations $\mathbf{C}_{\mathcal{L}}$ and $\mathbf{C}_{\mathcal{L}'}$ are equivalent.

Using this, and following the arguments in [MM3], we can generalize [MM3, Theorem 13] to all weakly fiat 2-categories (in the notation of Section 5).

Theorem 32. *Let $\mathscr{C} = \mathscr{C}_{\mathcal{J}}$ be a skeletal weakly fiat $\mathcal{J}$-simple 2-category for a strongly regular $\mathcal{J}$. Then $\mathscr{C}$ is biequivalent to $\mathscr{C}_{A,X}$ for appropriate self-injective A and $X \subset Z(A)$.*

Similarly, we obtain the following generalization of [MM5, Theorem 8].

Theorem 33. *Let $\mathscr{C}$ be a weakly fiat 2-category such that all two-sided cells in $\mathscr{C}$ are strongly regular. Then any simple transitive 2-representation of $\mathscr{C}$ is equivalent to a cell 2-representation.*

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