

Some Banach spaces added by a Cohen real

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Abstract

We study certain Banach spaces that are added in the extension by one Cohen real. Specifically, we show that adding just one Cohen real to any model adds a Banach space of density \aleph_1 which does not embed into any such space in the ground model (Theorem 1.1). Moreover, such a Banach space can be chosen to be UG (Theorem 1.6). This has consequences on the isomorphic universality number for Banach spaces of density \aleph_1 , which is hence equal to \aleph_2 in the standard Cohen model and the same is true for UG spaces. Analogous universality results for Banach spaces are true for other cardinals, by a different proof (Theorem 2.10(1)).¹

Keywords: one Cohen real, Banach spaces, isomorphic universality, UG spaces

0 Introduction

It is a mixture of sadness and pleasure to be writing an article remembering the great mathematician and friend, Mary Ellen Rudin, whom I have had the pleasure to know for many years. She is missed in many ways but the inspiration she left remains. Writing this paper I chose results that I thought I could share with her, as they fit her taste of using combinatorial set theory in a context that involves topology and analysis. I also hope that by exposing a completely combinatorial way of using the forcing extension by one Cohen real in the context of Banach spaces of weight \aleph_1 , this work will attract the attention of Banach space theorists looking for consistent examples of non-separable Banach spaces constructed without the necessity to use involved set-theoretic techniques.

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1 The extension by one Cohen real and an unembeddable UG space

In this section we show two constructions of interesting Banach spaces that are added in the extension by one Cohen real. The results are about Banach spaces but they build on the spirit of a number of earlier results in set theory, namely that adding just one Cohen real causes the universe to contain non-trivial objects of size \aleph_1 . The first such result is due to Roitman who showed in [8] that $MA(\aleph_1)$ is false in this extension. This result was improved by Shelah in [9] who proved that there is no Suslin tree in this model, the alternative proofs of which then were given by Todorćević [10] and Mark Bickford, as given in Velleman [13].

The general technique is to use a neat simplified $(\omega, 1)$ -morass (constructed in ZFC by Velleman ², as explained below) in order to inductively construct a Boolean algebra \mathfrak{A} of size \aleph_1 and with properties required by the construction. The point of the morass is that it lets us construct our algebra, an object of size \aleph_1 , by controlling the *finite* pieces of the algebra. Once \mathfrak{A} is constructed we obtain our space as $C(\text{St}(\mathfrak{A}))$. The reason to use the extension by one Cohen real is that that particular forcing interacts very nicely with the morass to let us have a lot freedom in choosing the relations between the generators of \mathfrak{A} , which at the end translate into the properties of the Banach space. The idea to extend Cohen-like conditions in incomparable ways are present in the proof by Brech and Koszmider ([2], Theorem 3.2) that l^∞/c_0 is not a universal Banach space for density \mathfrak{c} in the extension by \aleph_2 Cohen reals, and the earlier proof by Džamonja and Shelah ([4], Theorem 3.4) that in the model considered there there is no universal normed vector space over \mathbb{Q} of size \aleph_1 under isomorphic vector space embeddings. In our case we use further properties of the Cohen forcing to control objects of size \aleph_1 in the extension by controlling the ones that are already in the ground model, see Lemma 1.3.

Theorem 1.1 *Forcing with one Cohen real adds a Boolean algebra \mathfrak{A} of size \aleph_1 such that in the extension the Banach space $C(\text{St}(\mathfrak{A}))$ does not isomorphically embed into any space X_* which is in the ground model and has density \aleph_1 .*

Proof. Let us recall that a *neat simplified* $(\omega, 1)$ -morass is a system $\langle \theta_\alpha : \alpha \leq \omega \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \omega \rangle$ such that

1. for $\alpha < \omega$, θ_α is a finite number > 0 , and $\theta_\omega = \omega_1$,
2. for $\alpha < \beta \leq \omega$, $\mathcal{F}_{\alpha,\beta}$ is a set of order preserving functions from θ_α to θ_β ,
3. $\bigcup_{\alpha < \omega} \bigcup_{f \in \mathcal{F}_{\alpha,\omega}} f''\theta_\alpha = \omega_1$,
4. for all $\alpha < \beta < \gamma \leq \omega$ we have that $\mathcal{F}_{\alpha,\gamma} = \{f \circ g : g \in \mathcal{F}_{\alpha,\beta} \text{ and } f \in \mathcal{F}_{\beta,\gamma}\}$,

²who was a student of Mary Ellen Rudin and Ken Kunen

5. for all $\alpha < \omega$ we have that $\mathcal{F}_{\alpha, \alpha+1} = \{\text{id}_\alpha, h_\alpha\}$ for some h_α such that there is a *splitting point* k with $h_\alpha \upharpoonright k = \text{id}_\alpha \upharpoonright k$ and $h_\alpha(k) > \theta_\alpha$, where id_α is the identity function on θ_α ,
6. for every $\beta_0, \beta_1 < \omega$ and $f_l \in \mathcal{F}_{\beta_l, \omega}$ for $l < 2$ there is $\gamma < \omega$ with $\beta_0, \beta_1 < \gamma$, function $g \in \mathcal{F}_{\gamma, \omega}$ and $f'_l \in \mathcal{F}_{\beta_l, \gamma}$ such that $f_l = g \circ f'_l$ for $l < 2$.

(Expanded) simplified $(\kappa, 1)$ -morasses and their improvement to neat simplified $(\kappa, 1)$ -morasses, as defined here for $\kappa = \omega$, were introduced by Velleman in [12], Definition 2.10 and just after Theorem 3.8. respectively. He showed in [11] that a neat simplified $(\omega, 1)$ -morass exist in ZFC. We shall use the terminology *neat morass* to refer to neat simplified $(\omega, 1)$ -morasses.

For future use in the proof we need the following observation which easily follows from the properties of a neat morass:

Lemma 1.2 *Suppose that $\langle \theta_\alpha : \alpha \leq \omega \rangle, \langle \mathcal{F}_{\alpha, \beta} : \alpha < \beta \leq \omega \rangle$ is a neat morass. Then:*

- (1) *Suppose that $\alpha \leq \beta \leq \gamma < \omega$. Then $\bigcup_{f \in \mathcal{F}_{\alpha, \gamma}} \text{rng}(f) \subseteq \bigcup_{f \in \mathcal{F}_{\beta, \gamma}} \text{rng}(f)$.*
- (2) *The sequence $\langle \theta_\alpha \rangle_{\alpha < \omega}$ is strictly increasing and $\theta_\alpha \notin \bigcup_{f \in \mathcal{F}_{\alpha, \alpha+1}} \text{rng}(f)$.*
- (3) *For any $\alpha, \gamma < \omega$ and $1 \leq n < \omega$, if $\gamma - \alpha \geq n$ then $|\theta_\gamma \setminus \bigcup_{f \in \mathcal{F}_{\alpha, \gamma}} \text{rng}(f)| \geq n$.*

Proof of the Lemma. (1) This follows from item 4. in the definition of a neat morass, since $\bigcup_{f \in \mathcal{F}_{\alpha, \gamma}} \text{rng}(f) = \bigcup_{g \in \mathcal{F}_{\alpha, \beta}, f \in \mathcal{F}_{\beta, \gamma}} \text{rng}(f \circ g) \subseteq \bigcup_{f \in \mathcal{F}_{\beta, \gamma}} \text{rng}(f)$.

(2) These observations follow easily from item 5. in the definition of a neat morass.

(3) The proof is by induction on $n \geq 1$. The initial stage $n = 1$ is taken care of (2), as $\theta_\alpha \notin \bigcup_{f \in \mathcal{F}_{\alpha, \alpha+1}} \text{rng}(f)$, which suffices by (1). For the induction step $n + 1$, let $\beta = \gamma - 1$. As before, $\theta_\beta \notin \bigcup_{f \in \mathcal{F}_{\beta, \gamma}} \text{rng}(f)$, so by (2), $\theta_\beta \notin \bigcup_{f \in \mathcal{F}_{\alpha, \gamma}} \text{rng}(f)$.

Let $\{s_0, \dots, s_{n-1}\}$ be a 1-1 enumeration of a subset of θ_β disjoint from $\bigcup_{f \in \mathcal{F}_{\alpha, \beta}} \text{rng}(f)$, which exists by the induction hypothesis. We shall show that for every $s \in \{s_0, \dots, s_{n-1}\}$ we have that $s \notin \bigcup_{f \in \mathcal{F}_{\alpha, \gamma}} \text{rng}(f)$, which together with the point θ_β will give a set of $n + 1$ elements of $\theta_\gamma \setminus \bigcup_{f \in \mathcal{F}_{\alpha, \gamma}} \text{rng}(f)$. So fix such s .

Suppose that $s \in \text{rng}(h)$ for some $h \in \mathcal{F}_{\alpha, \gamma}$ and let $g \in \mathcal{F}_{\alpha, \beta}$ and $g \in \mathcal{F}_{\beta, \gamma}$ be such that $h = f \circ g$. Let $t \in \text{rng}(g)$ be such that $s = f(t)$. If $f = \text{id}_\beta$ then $s = t$, so $s \in \text{rng}(g)$, contrary to the choice of the set $\{s_0, \dots, s_{n-1}\}$. If $f = h_\beta$, let k be the splitting point of h . If $t < k$ then again $s = t$, a contradiction as before. If $t \geq k$ then $s = h_\beta(t) > \theta_\beta$, a contradiction. $\star_{1.2}$

For the rest of the proof, let \mathbf{V} denote the ground model, so any model of ZFC, and let $\mathbf{V}[G]$ denote the extension of \mathbf{V} by one Cohen real. Unless stated otherwise, our arguments take place in \mathbf{V} . Fixing a neat morass as in the above definition, we shall define a Boolean algebra \mathfrak{A} as follows. Let $\mathcal{L} = \{\leq^*, d\}$ be a language in which both \leq^* and d are binary relation symbols and let φ be the \mathcal{L} -sentence stating that \leq^* is a partial order, d is symmetric and antireflexive and that for all x, y , the statements $d(x, y)$ and $\leq^*(x, y)$ are contradictory. Our aim is to obtain a model I of φ on ω_1 , and then

interpreting \leq^* as the Boolean \leq and d as the disjointness relation in the language of Boolean algebras, generate \mathfrak{A} by ω_1 freely except for the relations in I . This is possible by the compactness theorem and the fact that we have defined φ so that it is consistent with the axioms of a Boolean algebra.

To achieve this we first by induction on $\alpha < \omega$ define a model I_α of φ on θ_α . The basic requirement of the induction will be:

(i) if $\alpha < \beta$ and $f \in \mathcal{F}_{\alpha,\beta}$ then f gives rise to an \mathcal{L} -embedding from I_α to I_β .

Then we shall define \leq^* on ω_1 by letting $i \leq^* j$ iff there are some $\alpha < \omega$, $f \in \mathcal{F}_{\alpha,\omega}$ and $i', j' \in \theta_\alpha$ such that $i' \leq^* j'$ holds in θ_α and $f(i') = i$, $f(j') = j$. We similarly define d . Requirement 6. in the definition of a neat morass and part (i) of the inductive hypotheses give that this constructions defines a well defined model of φ .

For the main part of the proof, suppose that in the ground model \mathbf{V} we have a Banach space X_* with density \aleph_1 and a fixed dense set $\{z_i : i < \omega_1\}$ of X_* . We shall guarantee that for all natural numbers $n_* \geq 3$ and for all $j : \omega_1 \rightarrow \omega_1$ there are $i_0, i_1 < \dots < i_{n_*}$ such that:

$$\begin{aligned} &\text{if } \|z_{j(i_0)} + z_{j(i_1)} + \dots + z_{j(i_{n_*})}\| < n_* - 1, \text{ then } i_0, i_1, \dots, i_{n_*} \text{ are disjoint in } \mathfrak{A} \text{ and} \\ &\text{if } \|z_{j(i_0)} + z_{j(i_1)} + \dots + z_{j(i_{n_*})}\| \geq n_* - 1, \text{ then } i_0 <_{\mathfrak{A}} i_1 \dots <_{\mathfrak{A}} i_{n_*}. \end{aligned} \quad (*)$$

To see that our algebra, once constructed, has the required properties, suppose that T is an isomorphic embedding of $C(\text{St}(\mathfrak{A}))$ into some X_* as above and that $(*)$ holds for a dense set $\{z_i : i < \omega_1\}$ of X_* . Let $x_i = T(\chi_{[i]})$ for $i < \omega_1$ and let $n_* \geq 3$ be large enough so that for each $x \in C(\text{St}(\mathfrak{A}))$ we have that

$$\frac{1}{n_*} \|x\| < \|T(x)\| < n_* \|x\|,$$

as is guaranteed to exist by the definition of an isomorphic embedding. For each $i < \omega_1$ let us choose $j(i)$ such that $\|x_i - z_{j(i)}\| < \frac{1}{n_*^2 + 1}$. Let $i_0, i_1 < \dots < i_{n_*}$ be as guaranteed by $(*)$. Then in the case $\|z_{j(i_0)} + z_{j(i_1)} + \dots + z_{j(i_{n_*})}\| < n_* - 1$ we have that

$$\|x_{i_0} + x_{i_1} + \dots + x_{i_{n_*}}\| \leq \|z_{j(i_0)} + z_{j(i_1)} + \dots + z_{j(i_{n_*})}\| + \sum_{k \leq n_*} \|x_{i_k} - z_{j(i_k)}\| < n_* - 1 + \frac{n_*^2 + 1}{n_*^2 + 1} = n_*,$$

yet $\frac{1}{n_*} \|\chi_{[i_0]} + \dots + \chi_{[i_{n_*}]}\| = \frac{n_*^2 + 1}{n_*} > n_*$, in contradiction with the choice of n_* . The other case is similar.

Now we claim that to guarantee the condition $(*)$ for any fixed X_* and $\{z_i : i < \omega_1\}$, it suffices to assure that for all

$n_* \geq 3, A \in [\omega_1]^{\omega_1} \cap \mathbf{V}, j : A \rightarrow \omega_1 \in \mathbf{V}$ there are $i_0, i_1 < \dots < i_{n_*} \in A$ exemplifying $(*)$. (**)

For this we shall use the following well known Lemma 1.3, which is the combinatorial heart of most arguments about the forcing with one Cohen real.

Lemma 1.3 *For every $j : \omega_1 \rightarrow \omega_1$ in $\mathbf{V}[G]$, there is $A \in [\omega_1]^{\omega_1}$ in \mathbf{V} and $j_0 : A \rightarrow \omega_1$ in \mathbf{V} such that $j \upharpoonright A = j_0$.*

Proof of the Lemma. Suppose that j is a name in the Cohen forcing for a function $j : \omega_1 \rightarrow \omega_1$. Then for every $\alpha < \omega_1$ there is $p \in G$ deciding the value of $j(\alpha)$. Since the forcing notion is countable, there is p^* in G such that the set A of all α for which p decides the value of α is uncountable, and then it suffices to define j_0 on A by $j_0(\alpha) = i$ iff p forces $j(\alpha) = i$. $\star_{1.3}$

In order to guarantee the requirement (**), we first define by induction on n an increasing sequence of natural numbers α_n such that $\alpha_0 = 0$ and $\alpha_{n+1} \geq \alpha_n + (n + 1)$. By Lemma 1.2(3), for each $n < \omega$ we can find a set $A_n \subseteq \theta_{\alpha_{n+1}}$ of size $n + 1$ such that for any $\beta \leq \alpha_n$ and $f \in \mathcal{F}_{\beta, \alpha_{n+1}}$, the intersection $A_n \cap \text{rng}(f)$ is empty.

Our second requirement of the induction will be as follows, where r is the generic Cohen real, viewed as a function from ω to 2 :

(ii) Let $n < \omega$, suppose that I_{α_n} has been defined and let $\{i_0, i_1, \dots, i_n\}$ be the increasing enumeration of A_n . Then if $r(n) = 0$ we have that i_0, i_1, \dots, i_n are pairwise in the relation d and if $r(n) = 1$ then $i_0 \leq^* i_1 \leq^* \dots \leq^* i_n$. For $\beta \in (\alpha_n, \alpha_{n+1})$ we let I_β be the restriction of $I_{\alpha_{n+1}}$ to θ_β .

By the choice of the set A_n it follows that that conditions (i) and (ii) can be met in a simple inductive construction, as there will be no possible contradiction between (i) and the requirements that (ii) puts on the elements of A_n . Let us now show that the resulting algebra \mathfrak{A} is as required. Hence let us fix $n_*, X_*, \{z_i : i < \omega_1\}, A$ and $j = j_0$ as in (**). Is it is then sufficient to observe that the following set is dense in the Cohen forcing (note that the set is in the ground model):

$$\left\{ p : (\exists n \in \text{dom}(p))(n \geq n_*^2 \text{ and } p(n) = 0 \iff \|z_{j(i_0)} + z_{j(i_1)} + \dots + z_{j(i_{n_*^2})}\| < n_* - 1 \right\},$$

where $\{i_0, i_1, \dots, i_{n_*^2}\}$ is the increasing enumeration of the first $n_*^2 + 1$ elements of A_n .

$\star_{1.1}$

The following Theorem 1.6 shows that the construction from Theorem 1.1 can be refined so that the resulting Banach space is UG. For completeness we recall the definition of such a space, but we do not define the Gâteaux differentiability as we do not need it in the rest of the paper.

Definition 1.4 *A Banach space X is called UG if it admits a uniformly Gâteaux differentiable renorming.*

A Banach space $C(K)$ of continuous real functions on a compact space K is UG iff K is a uniform Eberlein compact, as shown by Fabian, Godefroy and Zizler in [5]. Our approach to constructing the desired space $X = C(K)$ will follow the schema described in the introduction to this section and used in the proof of Theorem 1.1. We shall not recall the definition of a uniformly Eberlein compact space, but only the following notion of a c -algebra.

Definition 1.5 *A subset C of a Boolean algebra \mathfrak{A} has the nice property if for no finite $F \subseteq C$ do we have $\bigvee F = 1$. A Boolean algebra \mathfrak{A} is a c -algebra iff there is a family $\{B_n : n < \omega\}$ of pairwise disjoint antichains of \mathfrak{A} whose union has the nice property and generates \mathfrak{A} .*

Bell showed in [1] that the Stone space of a c -algebra \mathfrak{A} is a uniform Eberlein compact which therefore implies that for such an algebra the Banach space $C(\text{St}(\mathfrak{A}))$ is an UG space. We prove:

Theorem 1.6 *Forcing with one Cohen real adds a c -algebra \mathfrak{A} of size \aleph_1 such that in the extension the UG Banach space $C(\text{St}(\mathfrak{A}))$ does not isomorphically embed into any space X_* which is in the ground model and has density \aleph_1 .*

To avoid repetitions, we present the proof as a variant of the proof of Theorem 1.1, to which we refer throughout.

Proof. Fixing a neat morass as in the proof of Theorem 1.1, we shall define a Boolean algebra \mathfrak{A} similarly as in the proof of that theorem. Let us change the definition of \mathcal{L} from the proof of Theorem 1.1 to be a more complex language $\mathcal{L} = \{\leq^*, d, B_n : n < \omega\}$ in which both \leq^* and d are binary relation symbols and all B_n are unary relation symbols. Let T be the following set of \mathcal{L} -sentences:

- \leq^* is a partial order,
- d is symmetric and antireflexive,
- for all x, y , the statements $d(x, y)$ and $\leq^*(x, y)$ are contradictory,
- for each pair $n \neq m$, for every x , $B_n(x)$ and $B_m(x)$ are contradictory,
- for each n , and for every $x \neq y$ with $B_n(x), B_n(y)$ we have $d(x, y)$.

A part of our aim is to obtain a model I of φ on ω_1 , and then interpreting \leq^* as the Boolean \leq and d as the disjointness relation in the language of Boolean algebras, generate \mathfrak{A} by ω_1 freely except for the relations in I . As in the proof of Theorem 1.1, this is possible by the compactness theorem since we have defined T so that it is consistent with the axioms for being a set of generators for a Boolean algebra- this can be checked by taking any c -algebra as an example. The sentences of T guarantee that I is a disjoint union of B_n s and that each B_n is an antichain. However, we shall have to work a bit harder to obtain that I has the nice property. For this we shall modify the definition of the ordinals α_n so that we shall now inductively choose $\alpha_n < \omega$ so that $\alpha_{n+1} \geq \alpha_n + n + 2$. By Lemma 1.2(3) we can fix sets A_n as before, consisting of some $n + 1$ elements of $\theta_{\alpha_{n+1}} \setminus \bigcup_{\beta \leq \alpha_n, f \in \mathcal{F}_{\beta, \alpha_{n+1}}} \text{rng}(f)$ and let also a_n be an element of $\theta_{\alpha_{n+1}} \setminus (\bigcup_{\beta \leq \alpha_n, f \in \mathcal{F}_{\beta, \alpha_{n+1}}} \text{rng}(f) \cup A_n)$. The requirements of the induction will be the **(i)**, **(ii)** and **(iii)** from the proof of Theorem 1.1, with the additional requirement

(iv) a_n is in the d relation with every element of $\bigcup_{\beta \leq \alpha_n, f \in \mathcal{F}_{\beta, \alpha_{n+1}}} \text{rng}(f)$.

To see that it is possible to meet the inductive requirements, for (i)- (iii) we use basically the same argument as before, except that we make sure that every element is in some B_k and for (iv) we simply choose a large enough m so that the B_m relation is still empty and we declare a_n a member of B_m .

It remains to verify that the set I obtained in the limit of the construction has the nice property. So let F be a finite subset of I and let α be such that for some $h \in \mathcal{F}_{\alpha, \omega}$ we have that $F \subseteq \text{rng}(h)$. Finding such α is possible by noting that if some $i < \omega_1$ belongs to $\text{rng}(h)$ for some $h \in \mathcal{F}_{\alpha, \omega}$ and some $\alpha < \omega$ then for every $\beta \in [\alpha, \omega)$, $i \in \text{rng}(h')$ for some $h' \in \mathcal{F}_{\beta, \omega}$, by the property 4. of a neat morass. By a similar reasoning we can assume that $\alpha = \alpha_n$ for some n . Let $H = h^{-1}(F)$. Again applying property 4. of a neat morass, we can decompose $h = f \circ g$ for some $g \in \mathcal{F}_{\alpha_n, \alpha_{n+1}}$ and some $f \in \mathcal{F}_{\alpha_{n+1}, \omega}$. Notice that a_n is in the d -relation with every element of $g^{-1}(H)$. By the choice of f and the definition of I it follows that $f(a_n)$ is in the d relation with every element of F , hence this positive element $f(a_n)$ in \mathfrak{A} is disjoint from every element of F , so witnessing that $\bigvee F \neq 1$. $\star_{1.6}$

2 Negative universality results for UG spaces in the iterated Cohen extension and further results on larger densities

We shall now explain how the results proved in Section 1 have as a consequence that in the standard Cohen model for violating CH there are no universal Banach spaces, and moreover that this can be witnessed by a UG space. We also show that the same is true at larger densities and in the extensions obtained by Cohen-like forcing. It is a well known theorem of Shelah (see [7], Appendix, for a proof) that in the extension obtained by adding a regular $\kappa \geq \lambda^{++}$ number of Cohen subsets to a regular λ over a model of GCH, the universality number for models of size λ^+ for any unstable complete first order theory is $\kappa = 2^\lambda$, so the maximal possible. This in particular applies to Boolean algebras. This result is implied by our results since, as we now explain, if there is no universal Banach space of a given density θ , then there is no universal Boolean algebra of size θ . This is the case because standard arguments (see Fact 1.1 in [2] imply that every Banach space can be embedded into a Banach space of the same density and the form $C(K)$ for some 0-dimensional space K . Combining that with the Stone representation theorem and further standard observations about preservation of isomorphisms, we obtain that the universality number of Banach spaces under isomorphism is never larger than the universality number of Boolean algebras under the isomorphism of Boolean algebras.

2.1 Negative universality results for UG space in the standard Cohen model

By the standard Cohen model we mean a model obtained by forcing over a model \mathbf{V} of *GCH* to add \aleph_2 Cohen reals to obtain a model where $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ (other values of 2^{\aleph_0} are possible to obtain in the same way, of course). We claim that the universality number for Banach spaces of density \aleph_1 in this model is \aleph_2 , which is the maximal possible value by the above remarks and the fact that in the model there are only \aleph_2 pairwise non-isomorphic Boolean algebras of size \aleph_1 . Moreover, this is witnessed by UG spaces (hence their universality number is also \aleph_2), as we shall show in Theorem 2.1.

Theorem 2.1 *In the standard Cohen model for $2^{\aleph_0} = \aleph_2$ there is a family \mathcal{F} of \aleph_2 many UG spaces of density \aleph_1 such that no family of $< \aleph_2$ Banach spaces of density \aleph_1 suffices to isomorphically embed all members of \mathcal{F} .*

Proof. Let as stated \mathbf{V} be a model of *GCH* and let \mathbb{P} denote the single step Cohen forcing, which we shall for concreteness conceive as the partial order consisting of finite partial functions from ω to 2, ordered by extension. Let \mathbb{Q} be the iteration of ω_2 steps of \mathbb{P} . Then in the extension by \mathbb{Q} we have that $2^{\aleph_0} = \aleph_2$ and the new reals are added throughout the iteration. Note that by the same considerations mentioned above and the fact that any \aleph_1 Boolean algebras of size \aleph_1 can easily be embedded into a single Boolean algebra of size \aleph_1 , the universality number for Banach spaces of density \aleph_1 is either 1 or \aleph_2 , so it suffices to show that there is no single Banach space X_* of density \aleph_1 which embeds all UG spaces of density \aleph_1 . Suppose for a contradiction that there is such a space X_* .

We can by the same argument as above assume that $X_* = C(\text{St}(\mathfrak{A}_*))$ for some Boolean algebra \mathfrak{A}_* of size \aleph_1 . By standard arguments about the iteration of forcing, we can find an intermediate universe in the iteration, call it \mathbf{V}' for simplicity, which contains \mathfrak{A}_* . If we could say that $C(\text{St}(\mathfrak{A}_*))$ is also in \mathbf{V}' , then we could apply Theorem 1.6 to conclude that for the generic *c*-algebra \mathfrak{B} added to \mathbf{V}' by the next iterand in the Cohen iteration, we have that $C(\text{St}(\mathfrak{B}))$ does not embed into $C(\text{St}(\mathfrak{A}_*))$ and we would be done. Note that by the factoring properties of the Cohen iteration it suffices to work with the case $\mathbf{V}' = \mathbf{V}$. The problem is that since we keep adding reals, $C(\text{St}(\mathfrak{A}_*))$ keeps changing from \mathbf{V} to the final universe and hence there is no contradiction. We resolve this difficulty by a use of simple functions with rational coefficients.

Lemma 2.2 *Let G be a generic for \mathbb{Q} . Then in $\mathbf{V}[G]$ no Banach space $C(\text{St}(\mathfrak{A}_*))$ where \mathfrak{A}_* is in \mathbf{V} isomorphically embeds all UG spaces of density \aleph_1 .³*

Proof. Suppose that \mathfrak{A}_* invalidates the lemma and let \mathfrak{A} be the *c*-algebra of size \aleph_1 added by \mathbb{P} as in the proof of Theorem 1.6, with K its Stone space. Let $p^* \in G$ force \dot{T}

³A preliminary version of this lemma was stated for isometric embeddings and upon hearing about it, Saharon Shelah suggested that it should work for the isomorphic embeddings as well.

to be an isomorphic embedding from the $C(K)$ to $X_* = C(\text{St}(\mathfrak{A}_*))$ with $\|T\| \leq c < n_*$. Let $\varepsilon > 0$ be small enough, precisely $\varepsilon < \min \left\{ \frac{n_* - c}{n_*}, \frac{n_*^2 + 1 - cn_*}{cn_*^2 + 1} \right\}$. We shall think of the model I obtained in the construction of \mathfrak{A} in the proof of Theorem 1.6 as being enumerated as $\{i : i < \omega_1\}$, so we have $\omega_1 \subseteq \mathfrak{A}$ and the notation $[i]$ refers to the basic clopen set in K induced by i . For $i < \omega_1$ let $p_i \geq p^*$ force that h_i is a simple function with rational coefficients (so $h_i \in \mathbf{V}$) satisfying $\|T(\chi_{[i]}) - h_i\| < \varepsilon$. Since the Cohen forcing is countable, we can assume by passing to an uncountable set of conditions that the first coordinate $p_i(0)$ is a fixed condition r^* in the Cohen forcing, and by applying standard arguments using the Δ -system Lemma we can also assume that for any finite $F \subseteq \omega_1$ the set $\{p_i : i \in F\}$ has a least common upper bound $\bigcup_{i \in F} p_i$ in \mathbb{Q} . Let m be such that $\text{dom}(r^*) \subseteq m$. Now applying (a simplified version) of the reasoning behind the choice of $\langle \alpha_n : n < \omega \rangle$ in the proof of Theorem 1.6, we can find a large enough $n > n_*^2$ such that $m \leq \alpha_n$. Hence among the first elements of A_n we can find a set $F = \{i_0, i_1, \dots, i_{n_*^2}\}$ of size $n_*^2 + 1$ such that r^* has two extensions r' and r'' with r' forcing in \mathbb{P} that i_k s are disjoint in \mathfrak{A} and r'' forcing that i_k s are increasing in \mathfrak{A} . Now we can find two extensions q' and q'' of $\bigcup_{i \in F} p_i$ in \mathbb{Q} such that $q'(0) \geq r'$ and $q''(0) \geq r''$.

Without loss of generality, by extending conditions p_i further if necessary, we can assume that $\text{dom}(p_i(0))$ is such that for some $\alpha \in \text{dom}(p_i(0))$ and some $f_i \in \mathcal{F}_{\alpha, \omega}$ (using the notation of the proof of Theorem 1.6) we have $i \in \text{rng}(f_i)$. However the value of $\|\sum_{k=0}^{n_*^2} h_{i_k}\|$ is decided in \mathbf{V} , and if $\|\sum_{k=0}^{n_*^2} h_{i_k}\| \geq n^*$, it follows by the choice of ε that there cannot be an extension of $\bigcup_{k=0}^{n_*^2} p_{i_k}$ forcing i_k s and to be disjoint, a contradiction, and if $\|\sum_{k=0}^{n_*^2} h_{i_k}\| < n^*$ then it follows that there cannot be an extension of $\bigcup_{k=0}^{n_*^2} p_{i_k}$ forcing i_k to be increasing with k , again a contradiction. $\star_{2.2}$

$\star_{2.1}$

2.2 Negative universality results for densities larger than \aleph_1

Let us now consider the situation of adding Cohen subsets to cardinals λ possibly larger than ω . In this situation we cannot adapt the above techniques to prove that just adding one such subset adds interesting spaces of density λ^+ . The major difficulties are that the analogues of neat morass are considerably more complicated and do not just exist in ZFC (although they do exist in the constructible universe \mathbf{L} , for example) and that the Cohen forcing for adding a subset to λ no longer satisfies the convenient property given by Lemma 1.3. However, we can still get negative universality results by generically adding Boolean algebras in an iteration of Cohen-like forcing, as we now show.

Let $\lambda = \lambda^{<\lambda}$. Fix a set $A = \{a_i : i < \lambda^+\}$ of indices which we shall assume forms an unbounded co-unbounded subset of λ^+ .

Definition 2.3 *The forcing $\mathbb{P}(\lambda)$ consists of Boolean algebras p generated on a subset of λ^+ by some subset w_p of A of size $< \lambda$ satisfying $p \cap A = w_p$. The ordering on \mathbb{P} is*

given by $p \leq q$ if p is embeddable as a subalgebra of q and the embedding fixes w_p , where in our notation q is the stronger condition.

Let us check some basic properties of the forcing $\mathbb{P}(\lambda)$, reminding the reader of the following notions:

Definition 2.4 (1) A forcing notion \mathbb{P} is said to be λ^+ -stationary cc if for every set $\{p_i : i < \lambda^+\}$ of conditions in \mathbb{P} , there is a club C and a regressive function f on C satisfying that for every $i, j < \lambda^+$ of cofinality λ satisfying $f(i) = f(j)$, the conditions p_i and p_j are compatible.

(2) A subset Q of a partial order P is directed if every two elements of Q have an upper bound in P . A forcing notion \mathbb{P} is $(< \lambda)$ -directed closed if every directed $Q \subseteq \mathbb{P}$ of size $< \lambda$, has an upper bound in \mathbb{P} .

Lemma 2.5 (1) $\mathbb{P}(\lambda)$ satisfies the λ^+ -stationary cc and is $(< \lambda)$ -directed closed.

(2) $\mathbb{P}(\lambda)$ adds a Boolean algebra of size λ^+ generated by $\{a_i : i < \lambda^+\}$.

Proof. For part (1), suppose that $\{p_i : i < \lambda^+\}$ are conditions in $\mathbb{P}(\lambda)$. Let us denote by w_i the set w_{p_i} and let us consider it in its increasing enumeration. Let the *isomorphism type* of w_i be determined by the order type of w_i and the Boolean algebra equations satisfied between the elements of w_i , denote this by $t(w_i)$. Note that by $\lambda^{<\lambda} = \lambda$ the cardinality of the set \mathcal{I} of isomorphism types is exactly λ , so let us fix a bijection $g : \mathcal{I} \times [\lambda^+]^{<\lambda} \rightarrow \lambda^+$. Note that there is a club C of $\lambda^+ \setminus 1$ such that for every point γ in C of cofinality λ we have $i < \gamma$ iff $w_i \subseteq \gamma$ and $g(t(w_i) \times \gamma) \subseteq \gamma$. Define f on C by letting $f(j) = g(t(w_j), w_j \cap j)$ for j of cofinality λ and 0 otherwise. Hence f is regressive on C and if $i < j$ both in C have cofinality λ and satisfy $f(i) = f(j)$ then we have that $w_i \cap i = w_j \cap j = w_i \cap w_j$ and that p_i and p_j satisfy the same equations on this intersection. Hence p_i and p_j are compatible.

For the closure, note that a family of conditions in $\mathbb{P}(\lambda)$ being $(< \lambda)$ -directed in particular implies that the conditions in any finite subfamily agree on the equations involving any common elements. Therefore the union of the family generates a Boolean algebra which is a common upper bound for the entire family.

For part (2), note that if $p \in \mathbb{P}$ and a_i is not in w_p , then we can extend p to q which is freely generated by $w_p \cup \{a_i\}$, except for the equations present in p . Hence the set $\mathcal{D}_i = \{p : a_i \in w_p\}$ is dense. Note that if $p, q \in G$ satisfy $p \cap A = q \cap A$, then p and q are isomorphic over w_p . Let H be a subset of G which is obtained by taking one representative of each isomorphism class with the induced ordering, hence H is still a filter in \mathbb{P} and it intersects every \mathcal{D}_i . By the downward closure of G it follows that for every F a subset of A of size $< \lambda$, there is exactly one element p_F of H with $w_{p_F} = F$. (Note that no new bounded subsets of λ are added by \mathbb{P} , since (1) holds). Now note that $\{p_F : F \in [A]^{<\lambda}\}$ with the relation of embeddability over A form a directed system of Boolean algebras directed by $([A]^{<\lambda}, \subseteq)$. Let B be the limit of this system (see [6] for an explicit construction of this object), so B is generated by $\{a_\alpha : \alpha < \lambda\}$ and it embeds

every element of H . (We shall call this algebra “the” generic algebra. It is unique up to isomorphism). ★_{2.5}

Note that in the case of $\lambda = \aleph_0$, we in particular have that the forcing is ccc.

Theorem 2.6 *If \mathfrak{A} is the generic Boolean algebra for some $\mathbb{P}(\lambda)$, then there is no Banach space X_* in the ground model such that $C(\text{St}(\mathfrak{A}))$ isomorphically embeds into X_* in the extension by $\mathbb{P}(\lambda)$.*

Proof. Let us fix a λ . Suppose for a contradiction that there is an X_* and embedding T in the extension contradicting the statement of the theorem. Let p^* force that T is an isomorphic embedding and, without loss of generality p^* decides a positive constant c such that $1/c \cdot \|x\| \leq \|Tx\| \leq c \cdot \|x\|$ for every x . Let n_* be a positive integer such that $c < n_*$. For each $i < \lambda^+$ let $p_i \geq p^*$ be such that $a_i \in w_{p_i}$, and without loss of generality p_i decides the value $x_i = T(\chi_{[a_i]})$. Now let us perform a similar argument as in the proof of the chain condition, and in particular by passing to a subfamily of the same cardinality we can assume that the sets w_{p_i} form a Δ -system with a root w^* . Since $a_i \in w_{p_i}$, we can assume that $w_i \neq w^*$, and in particular that $a_i \in w_i \setminus w^*$. Let us take distinct i_0, i_1, \dots, i_{n_*} . Since there are no equations in p_i or p_j that connect a_i and a_j for $i \neq j$, we can find two conditions q' and q'' which both extend $p_{i_0}, \dots, p_{i_{n_*}}$ and such that in q' we have $a_{i_0} \leq \dots \leq a_{i_{n_*}}$ and in q'' we have that $a_{i_0}, \dots, a_{i_{n_*}}$ are pairwise disjoint. Hence q' forces that $\chi_{[a_{i_0}]} + \dots + \chi_{[a_{i_{n_*}}]}$ has norm $n_* + 1$ and q'' forces that it has norm 1. Therefore the vector $x_{i_0} + \dots + x_{i_{n_*}}$ must have norm both $> n_*$ and $< n_*$ in X_* , which is impossible. ★_{2.6}

We would now like to extend Theorem 2.6 by using “classical tricks” with iterations to obtain the non-existence of an isomorphically universal Banach space of density λ^+ in an iterated extension, as we did with in the case of a standard Cohen model. It turns out that the cases $\lambda = \aleph_0$ which corresponds to adding one Cohen real and larger λ are different. In the situation in which we add a generic Boolean algebra of cardinality $\lambda^+ \geq \aleph_2$ with conditions of size $< \lambda$, the forcing is countably closed and therefore it does not add any new reals, making the proof of the analogue of Lemma 2.2 trivial- modulo Lemma 2.6. However, to finish the proof we need an iteration theorem for the forcing, in order to make sure that the cardinals are preserved (which in the case of $\lambda = \aleph_0$ is simply a well known theorem about the iteration with finite supports of ccc forcing retaining the ccc property, and hence preserving cardinals). Paper [3] reviews various known forcing axioms and building up on them proves the following Theorem 2.7, which is perfectly suitable for our purposes.

Theorem 2.7 (Cummings, Džamonja, Magidor, Morgan and Shelah) *Let $\lambda = \lambda^{<\lambda}$. Then, the iterations with $(< \lambda)$ -supports of $(< \lambda)$ -closed stationary λ^+ -cc forcing which is countably parallel-closed, are $(< \lambda)$ -closed stationary λ^+ -cc.*

Here, the property of countable parallel-closure is defined as follows:

Definition 2.8 Two increasing sequences $\langle p_i : i < \omega \rangle$ and $\langle q_i : i < \omega \rangle$ of conditions in a forcing \mathbb{P} are said to be pointwise compatible if for each $i < \omega$ the conditions p_i, q_i are compatible. The forcing \mathbb{P} is said to be countably parallel-closed if for every two ω -sequences of pointwise compatible conditions as above, there is a common upper bound to $\{p_i, q_i : i < \omega\}$ in \mathbb{P} .

Lemma 2.9 The forcing $\mathbb{P}(\lambda)$ is countably parallel-closed.

Proof. Suppose that $\langle p_i : i < \omega \rangle$ and $\langle q_i : i < \omega \rangle$ are pairwise compatible increasing sequences. In particular this means that, on the one hand, each p_i, q_i agree on their intersections, and on the other hand that $\bigcup_{i < \omega} p_i$ and $\bigcup_{i < \omega} q_i$ each form a Boolean algebra. Furthermore $\bigcup_{i < \omega} p_i$ and $\bigcup_{i < \omega} q_i$ agree on their intersection, and hence their union can be used to generate a Boolean algebra, which will then be a common upper bound to the two sequences. $\star_{2.9}$

Theorem 2.10 Let $\lambda = \lambda^{<\lambda}$, let $\text{cf}(\kappa) \geq \lambda^{++}$ and let G be a generic for the iteration with $(< \lambda)$ - supports of length κ of the forcing to add the generic Boolean algebra of size λ^+ by conditions of size $< \lambda$ over a model of GCH. Then in $\mathbf{V}[G]$:

- (1) the universality number of the class of Banach spaces of density λ^+ under isomorphisms is $\kappa = 2^{\lambda^+}$.
- (2) there are 2^{λ^+} many pairwise non-isomorphic Banach spaces of density λ^+ .

Proof. (1) For the case of $\lambda = \aleph_0$ we use Theorem 2.2. In the case of uncountable λ , it follows from Lemma 2.5, Lemma 2.9 and Theorem 2.7 that the iteration of the forcing $\mathbb{P}(\lambda)$ described in the statement of the Theorem, preserves cardinals. By the countable closure of the forcing no reals are added and hence the analogue of Lemma 2.2 holds by Lemma 2.6, giving the desired conclusion.

(2) The claim is that the Banach spaces $C(\text{St}(\mathfrak{A}))$ added at the individual steps of the iteration are not pairwise isomorphic. To prove this, we only need to notice that an embedding of a Banach space into another is determined by the restriction of an embedding onto a dense set, hence in our case a set of size λ^+ , and then to argue as in (1). $\star_{2.10}$

Using the same reasoning as in Theorem 2.10 with $\lambda = \aleph_0$ and relating this to Theorem 1.6 we obtain

Theorem 2.11 In the Cohen model for $2^{\aleph_0} = \kappa$, where $\text{cf}(\kappa) \geq \aleph_2$, the universality number of the UG Banach spaces is at least $\text{cf}(\kappa)$ and there are $\text{cf}(\kappa)$ pairwise non-isomorphic UG Banach spaces.

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