OPTIMAL REALISATIONS OF TWO-DIMENSIONAL, TOTALLY-DECOMPOSABLE METRICS

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Abstract. A realisation of a metric d on a finite set X is a weighted graph (G, w) whose vertex set contains X such that the shortest-path distance between elements of X considered as vertices in G is equal to d. Such a realisation (G, w) is called optimal if the sum of its edge weights is minimal over all such realisations. Optimal realisations always exist, although it is NP-hard to compute them in general, and they have applications in areas such as phylogenetics, electrical networks and internet tomography. In [*Adv. in Math.* **53**, 1984, 321-402] A. Dress showed that the optimal realisations of a metric d are closely related to a certain polytopal complex that can be canonically associated to d called its tight-span. Moreover, he conjectured that the (weighted) graph consisting of the zero- and one-dimensional faces of the tight-span of d must always contain an optimal realisation as a homeomorphic subgraph. In this paper, we prove that this conjecture does indeed hold for a certain class of metrics, namely the class of totally-decomposable metrics whose tight-span has dimension two. As a corollary, it follows that the minimum Manhattan network problem is a special case of finding optimal realisations of two-dimensional totally-decomposable metrics.

Keywords: optimal realisations, totally-decomposable metrics, tight-span, Manhattan network problem, Buneman complex

2000 MSC: 05C12, 30L05, 52B99, 54E99.

1. Introduction

Let (X, d) be a finite metric space, that is, a finite set X, $|X| \geq 2$, together with a metric d (i.e., a symmetric map $d : X \times X \to \mathbb{R}_{\geq 0}$ that vanishes precisely on the diagonal and that satisfies the triangle inequality). A *realisation* (G, w) of (X, d) consists of a graph $G = (V(G), E(G))$ with X a subset of the vertex set $V(G)$ of G, together with a weighting $w : E(G) \to \mathbb{R}_{>0}$ on the edge set $E(G)$ of G such that for all $x, y \in X$ the length of any shortest path in (G, w) between x and y equals $d(x, y)$. A realisation (G, w) of d is called *optimal* if $\sum_{e\in E(G)} w(e)$ is minimal amongst all realisations of (X, d).

Realising metrics by graphs has applications in fields such as phylogenetics, electrical networks and internet tomography. Optimal realisations were introduced by Hakimi and Yau [11] who also gave a polynomial algorithm for their computation in the special case where the metric space has a (necessarily unique) optimal realisation that is a tree. Every finite metric space has an optimal realisation [6, 17], although they are not necessarily unique [1, 6]. In general, it is NP-hard to compute optimal realisations [1, 22], although recently some progress has been made in deriving heuristics for their computation [13, 14].

In [6], Dress pointed out an intriguing connection between optimal realisations and tight-spans, which we now recall. The *tight-span* $T(d)$ of the metric space (X, d) [6, 18] is the set of all minimal elements (with respect to the product order) of the polyhedron

$$P(d) := \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X\}.$$

Note that, in particular, $T(d)$ consists of the union of the bounded faces of $P(d)$. Moreover, the map d_∞, given by $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$ for all $f, g \in P(d)$, is a metric on $T(d)$ and the *Kuratowski map*

$$\kappa : X \to T(d) : x \to h_x; \quad h_x(y) := d(x, y), \text{ for all } x \in X,$$

gives an isometric embedding of (X, d) into $(T(d), d_\infty)$; that is, κ is injective and preserves distances.

In [6, Theorem 5], Dress showed that the (necessarily finite and connected) weighted graph G_d consisting of the zero- and one-dimensional faces of $T(d)$ and weighting w_∞ defined by $w_\infty(\{f, g\}) := d_\infty(f, g)$, f, g zero-dimensional faces of $T(d)$, is homeomorphic to a realisation of d (see Section 2 for relevant definitions). Moreover, he showed that if (G, w) is any optimal realisation of (X, d), then there exists a certain map $\psi : V(G) \to T(d)$ of the vertices of G into $T(d)$ [6, Theorem 5] (see also Theorem 2.3 below). This led him to suspect that every optimal realisation of (X, d) is homeomorphic to a subgraph of (G_d, w_∞). Even though this conjecture was disproven by Althöfer [1], the following related conjecture is still open:

Conjecture 1.1 (cf. (3.20) in [6])**.** Let (X, d) be a finite metric space. Then there exists an optimal realisation of (X, d) that is homeomorphic to a subgraph of (G_d, w_∞).

Apart from having an intrinsic mathematical interest, if this conjecture were true, it could provide new strategies for computing optimal realisations, as it would provide a "search space" (albeit a rather large one in general) in which to systematically search for optimal realisation [12].

Conjecture 1.1 is known to hold for metrics d that can be realised by a tree since in this case (G_d, w_∞) is precisely the tree that realises d uniquely [17]. In this paper, we show that it also holds for a certain class of metrics that generalise tree metrics. More specifically, for a finite metric space (X, d) as above, define, for any four elements $x, y, u, v \in X$,

$$\beta(x, y; u, v) := \max\{d(x, u) + d(y, v), d(x, v) + d(y, u)\} - d(x, y) - d(u, v)$$

and put $\alpha(x, y; u, v) := \max(\beta(x, y; u, v), 0)$. The metric d is called *totally-decomposable* if for all $t, x, y, u, v \in X$ the inequality $\beta(x, y; u, v) \le \alpha(x, t; u, v) + \alpha(x, y; u, t)$ holds [2]. Such metrics are commonly used to understand genetic data in phylogenetic analysis. Defining the *dimension* of d to be the dimension of $T(d)$ (regarded as a subset of $\mathbb{R}^X$), we shall prove the following result.

Theorem 1.2. *Let (X, d) be a totally-decomposable finite metric space with dimension two. Then there exists an optimal realisation of (X, d) that is homeomorphic to a subgraph of (G_d, w_∞).*

In fact this immediately follows from a somewhat stronger theorem that we shall prove (Theorem 4.1), which shows that a certain special type of optimal realisation of a two-dimensional, totally-decomposable metric d can be found as a homeomorphic subgraph of (G_d, w_∞). Note also that Theorem 1.2 implies that the optimal realisation problem for l_1-planar metrics is equivalent to the Minimum Manhattan Network (MMN) problem; since the MMN problem is NP-hard [4], the optimal realisation problem for two-dimensional metrics is also NP-hard (see [12, Section 5] for more details and some algorithmic consequences).

Our proof of Theorem 1.2 heavily relies on the two-dimensionality of the tight-span, and we do not know how to extend our arguments to totally-decomposable metrics. Even so, it might be of interest to try and extend our result to two-dimensional metrics in general, especially as a great deal is known concerning the structure of their tight-spans (e.g. [15, 19]). Indeed, our proof of Theorem 1.2 relies on a close relationship between tight-spans and so-called Buneman or median complexes, and so results concerning median complexes and folder complexes [3] could potentially help yield a more general result for two-dimensional metrics.

The remainder of this paper is organised as follows. We recall some definitions and results in Section 2. We will then present a theorem about embeddings of realisations into the Buneman complex in Section 3 which uses the new notions of split-flow digraphs and split potentials. Finally, we establish our main result in Section 4, from which Theorem 1.2 follows.

2. Preliminaries and previous results

In this section, we will state the known definitions and results that are used in the rest of the paper.

2.1. **Graphs.** A *weighted graph* (G, w) is a graph G with vertex set $V(G)$ and edge set $E(G) \subseteq \binom{V(G)}{2}$ together with a weight function $w : E(G) \to \mathbb{R}_{>0}$ that assigns a positive weight or *length* to each edge. A weighted graph (G', w') is a *subgraph* of (G, w) if $V(G') \subseteq V(G)$, $E(G') \subseteq \{e \in E(G) \,|\, e \subseteq V(G')\}$ and $w' = w|_{E(G')}$. The *length* of (G, w) is $l(G, w) := \sum_{e \in E(G)} w(e)$. A *path* P from u to v in G is a sequence $u = v_0, v_1, \ldots, v_{k-1}, v_k = v$ of distinct vertices in G such that $\{v_{i-1}, v_i\} \in E(G)$ for all $1 \le i \le k$. Note that u and v will be referred to as the *ends* of P and the *length* of P is defined as $w(P) := \sum_{i=1}^{k} w(\{v_{i-1}, v_i\})$. It is easily observed that for any $W \subseteq V(G)$ the map setting $d_{(G,w)}(u, v)$ to be the length of a shortest path between u and v defines a metric space $(W, d_{(G,w)})$.

We *suppress* a vertex of degree two in a weighted graph when we remove it and replace its two incident edges by a single edge whose length is equal to the sum of their lengths. Given two weighted graphs (G_1, w_1) and (G_2, w_2), they are *isomorphic* if there exists an isomorphism between G_1 and G_2 that also preserves the length of each edge; they are *homeomorphic* if there exist two isomorphic weighted graphs (G'_1, w'_1) and (G'_1, w'_2) such that (G'_i, w'_i) $(i = 1, 2)$ can be obtained from (G_i, w_i) by suppressing a sequence of degree two vertices.

As mentioned in the introduction, a weighted graph (G, w) with $X \subseteq V(G)$ and $d = d_{(G,w)}$ is called a *realisation* of the metric space (X, d). The elements in $V(G) \setminus X$ are called *auxiliary vertices* of the realisation, and throughout this paper we will use the convention that all auxiliary vertices of degree two are suppressed.

2.2. **Optimal realisations and geodesics.** We now recall some well-known observations concerning optimal realisations.

Lemma 2.1 (Lemma 2.1 in [1]). *Let* (G, w) *be an optimal realisation of a finite metric space* (X, d)*. Then*

(1) *For any edge* $e \in E(G)$*, there exist two elements* $x, x' \in X$ *such that* e *belongs to all shortest paths between* x *and* x'*.*
(2) *For any two edges in* $E(G)$ *that share a common vertex, there exists a shortest path between two elements of* X *that contains these edges.*

As an immediate consequence of Lemma 2.1(2), we have:

Corollary 2.2. *Let* (G, w) *be an optimal realisation of a finite metric space* (X, d)*. Then* G *is triangle-free.*

As mentioned in the introduction, not all optimal realisations are homeomorphic to subgraphs of the tight-span. However, we will now present some properties of optimal realisations that will guarantee this property. If (G, w) is a weighted graph and $A \subseteq V(G)$, we denote by $\Gamma(G, w; A)$ the set of all pairwise distinct shortest paths in G connecting elements of A. By [21, Proposition 7.1], for each metric space (X, d) there exists a *path-saturated* optimal realisation (G, w) of (X, d) such that $|\Gamma(G, w; X)| \geq |\Gamma(G', w'; X)|$ holds for all optimal realisations (G', w') of (X, d). If in addition the number of vertices $V(G)$ is minimal among all path-saturated realisations of (X, d), then (G, w) is called a *minimal path-saturated realisation* of (X, d).

Now, if (X, d) is a (not necessarily finite) metric space, a function $\gamma : [0, 1] \to X$ is called a *geodesic* in (X, d) if for all $a < b < c \in [0, 1]$ one has $d(\gamma(a), \gamma(c)) = d(\gamma(a), \gamma(b)) + d(\gamma(b), \gamma(c))$. Note that this implies that γ is continuous. A map $\psi : X \to X'$ between two arbitrary metric spaces (X, d) and (X', d') is called *non-expansive*, if $d'\big(\psi(x_1), \psi(x_2)\big) \leq d(x_1, x_2)$ for all $x_1, x_2 \in X$. If ψ^{-1} exists and is non-expansive, too, ψ is an *isometry* and (X, d) and (X', d') are said to be *isometric*.

For a weighted graph (G, w) we denote by $||(G, w)||$ its *geometric realisation*, that is, the metric space obtained by regarding each edge $e \in E(G)$ as a real interval of length $w(e)$ and gluing them together at the vertices of G (see, e.g., Daverman and Sher [5, p. 547] for details of this construction). For $X \subset V(G)$ a function $\gamma : [0, 1] \to ||(G, w)||$ is called an X-*geodesic* if it is a geodesic between two points of X interpreted as points in $||(G, w)||$.

The following theorem gives us a way to relate the geometric realisation of an optimal realisation of a metric with its tight-span. The first part is due to Dress [6, Theorem 5], and the second part is given in [21, Proposition 7.1].

Theorem 2.3. *Let (G, w) be an optimal realisation of a finite metric space (X, d). Then there exists a non-expansive map ψ from $||(G, w)||$ to $(T(d), d_\infty)$ such that $\psi(x) = \kappa(x)$ for all $x \in X$. If, in addition, (G, w) is path-saturated, then ψ is injective.*

2.3. **Splits and total-decomposability.** A *split* $S = \{A, B\}$ of a finite set X is a bipartition of X, that is $A \cup B = X$ and $A \cap B = \emptyset$. A *weighted split system* $(\mathcal{S}, \alpha)$ on X is a pair consisting of a set $\mathcal{S}$ of splits of X, and a weight function $\alpha : \mathcal{S} \to \mathbb{R}_{>0}$. For all $x, y \in X$, we set $\mathcal{S}(x, y) = \{\{A, B\} \in \mathcal{S} \mid x \in A, y \in B \text{ or } x \in B, y \in A\}$ and define

$$d_{(\mathcal{S},\alpha)}(x, y) = \sum_{S \in \mathcal{S}(x,y)} \alpha(S)\,.$$

If $\mathcal{S}(x, y) \neq \emptyset$ for all distinct $x, y \in X$, the pair $(X, d_{(\mathcal{S},\alpha)})$ becomes a finite metric space.

Two splits $\{A, B\}$ and $\{A', B'\}$ of X are called *incompatible* if none of the four intersections $A \cap A'$, $A \cap B'$, $B \cap A'$ and $B \cap B'$ is empty. A weighted split system $(\mathcal{S}, \alpha)$ is called (1) *two-compatible* if $\mathcal{S}$ does not contain three pairwise incompatible splits, (2) *weakly compatible* if for any three splits S_1, S_2, S_3 in $\mathcal{S}$, there exist $A_i \in S_i$, for each $i \in \{1, 2, 3\}$, such that $A_1 \cap A_2 \cap A_3 = \emptyset$, and (3) *octahedral-free* if there exists no partition $X = X_1 \cup \cdots \cup X_6$ of X into six non-empty disjoint subsets X_i, $1 \leq i \leq 6$, such that each one of the following four splits:

$$S_1 = \{X_1 \cup X_2 \cup X_3, X_4 \cup X_5 \cup X_6\}, \quad S_2 = \{X_2 \cup X_3 \cup X_4, X_5 \cup X_6 \cup X_1\},$$
$$S_3 = \{X_3 \cup X_4 \cup X_5, X_6 \cup X_1 \cup X_2\}, \quad S_4 = \{X_1 \cup X_3 \cup X_5, X_2 \cup X_4 \cup X_6\}$$

belongs to $\mathcal{S}$. Note that it is easily seen that two-compatible split systems are octahedral-free.

It can be shown (cf. [9]) that a metric space (X, d) is totally-decomposable if and only if there exists a weakly compatible weighted split system $(\mathcal{S}, \alpha)$ on X such that $d = d_{(\mathcal{S},\alpha)}$. Furthermore, if (X, d) is totally-decomposable, then d has dimension at most two if and only if $(\mathcal{S},\alpha)$ is two-compatible (cf. [9]). From now on we will call two-dimensional, totally-decomposable metric spaces *two-decomposable.*

2.4. **Polytopal complexes.** We now recall some definitions about polytopes (see [23] for further details). A *polyhedron* P is the intersection of a finite collection of halfspaces in a real vector space $\mathbb{V}$, and a *polytope* is a bounded polyhedron. For any linear functional $L : \mathbb{V} \to \mathbb{R}$ the set $F = \{x \in P \,|\, L(x) = \max_{y\in P} L(y)\}$ is called a *face* of P, as is the empty set. The zero- and one-dimensional faces are called *vertices* and *edges* of P and they naturally gives rise to a *graph* of the polyhedron. A *cell complex* (or *polytopal complex*) P is a finite collection of polytopes (called *cells*) such that each face of a member of P is itself a member of P, and the intersection of two members of P is a face of each. We denote the set of vertices of P by $V(\mathrm{P})$. Two cell complexes P, P' are *isomorphic* if there exists an bijection π (called *cell-complex isomorphism*) between them such that for all $F, F' \in P$ the cell F is a face of F' if and only if $\pi(F)$ is a face of $\pi(F')$.

For a finite metric space (X, d), the set $P(d)$ defined in the introduction is obviously a polyhedron and it can be easily observed that $T(d)$ is the union of bounded faces of $P(d)$ (cf. [6]) and hence naturally carries the structure of a cell complex $\mathbb{T}(d)$.

2.5. **The Buneman complex.** The Buneman complex, also known as a median complex [3], is a cell complex that can be associated to any weighted split system, and that has proven useful in, for example, understanding the structure of the tight-span of a totally-decomposable metric [10]. Given a weighted split system $(\mathcal{S},\alpha)$ on X we define its *support* to be the set $\mathrm{supp}(\alpha) := \{A \subseteq X \,|\, \text{there exists } S \in \mathcal{S} \text{ with } A \in S\}$. Consider the polytope (which is a hypercube)

$$H(\mathcal{S},\alpha) := \left\{\mu \in \mathbb{R}_{\geq 0}^{\mathrm{supp}(\alpha)} \;\middle|\; \mu(A) + \mu(B) = \alpha(S) \text{ for all } S \in \mathcal{S}\right\}.$$

Its subset

$$B(\mathcal{S},\alpha) := \{\mu \in H(\alpha) \mid \text{“}\mu(A) \neq 0 \neq \mu(B) \,\text{and}\, A \cup B = X\text{”} \Rightarrow A \cap B = \emptyset\}$$

carries the structure of a cell complex and is the *Buneman complex* $\mathbb{B}(\mathcal{S},\alpha)$ of $(\mathcal{S},\alpha)$. Obviously, the set of vertices of $H(\mathcal{S},\alpha)$ consists of those $\mu \in H(\mathcal{S},\alpha)$ with $\mu(A) \in \{0, \alpha(\{A, X \setminus A\})\}$ for all $A \in \mathrm{supp}(\alpha)$. It is easily seen that the set $V(B(\mathcal{S},\alpha))$ of vertices of $B(\mathcal{S},\alpha)$ consists of the $\mu \in B(\mathcal{S},\alpha)$ with this property.

Setting

$$d_1(\mu,\nu) := \frac{1}{2} \sum_{A\in \mathrm{supp}(\alpha)} |\mu(A) \setminus \nu(A)|$$

for all $\mu, \nu \in B(\mathcal{S}, \alpha)$, we obtain a metric space $(B(\mathcal{S}, \alpha), d_1)$ and the map $\Phi : X \to B(\mathcal{S}, \alpha)$ defined via

$$\Phi(x)(A) := \begin{cases} \alpha(\{A, X \setminus A\}) & \text{if } x \in A; \\ 0 & \text{else,} \end{cases} \tag{1}$$

for any $x \in X$ and $A \in \text{supp}(\alpha)$ is an isometric embedding from $(X, d_{(\mathcal{S},\alpha)})$ to $(B(\mathcal{S}, \alpha), d_1)$.

There exists a natural map $\Lambda : \mathbb{R}^{\text{supp}(\alpha)} \to \mathbb{R}^X, \mu \mapsto f_\mu$ where

$$f_\mu(x) = \sum_{A \in \text{supp}(\alpha)} \mu(A) \quad \text{for all } x \in X\,.$$

It is easily seen that $\Lambda(\Phi(x)) = \kappa(x)$ for all $x \in X$. Depending on properties of the split system $(\mathcal{S}, \alpha)$, this map takes elements from $H(\mathcal{S}, \alpha)$ to elements of $P(d_{(\mathcal{S},\alpha)})$ or even from $B(\mathcal{S}, \alpha)$ to elements of $T(d_{(\mathcal{S},\alpha)})$; see [10, 7, 8] for details. In case $(\mathcal{S}, \alpha)$ is weakly compatible and octahedral-free the following holds:

Theorem 2.4 (Theorem 3.1 in [10])**.** *If $(\mathcal{S}, \alpha)$ is a weakly compatible, octahedral-free weighted split system on X, then the map $\Lambda|_{B(\mathcal{S},\alpha)}$ is a bijection onto $T(d_{(\mathcal{S},\alpha)})$ that induces a cell-complex isomorphism $\Lambda' : \mathbb{B}(\mathcal{S}, \alpha) \to \mathbb{T}(d_{(\mathcal{S},\alpha)})$.*

Recall that if $\mathcal{S}$ is two-compatible, then it is weakly compatible and octahedral-free. In fact, in the two-compatible case we know even more about the relation between $B(\mathcal{S}, \alpha)$ and $T(d_{(\mathcal{S},\alpha)})$.

Theorem 2.5. *Suppose that $(\mathcal{S}, \alpha)$ is a two-compatible weighted split system on X. Then the following hold:*

(1) *The cell complex $\mathbb{B}(\mathcal{S}, \alpha)$ is at most two-dimensional and all two dimensional cells are quadrangles.*
(2) *The map $\Lambda|_{B(\mathcal{S},\alpha)} : B(\mathcal{S}, \alpha) \to T(d_{(\mathcal{S},\alpha)})$ is an isometry of the metric spaces $(B(\mathcal{S}, \alpha), d_1)$ and $(T(d_{(\mathcal{S},\alpha)}), d_\infty)$.*

Proof. (1) By [9, Lemma 2.1], the dimension of $B(\mathcal{S}, \alpha)$ is bounded by two if $(\mathcal{S}, \alpha)$ is two-compatible. Corollary 7.3 in [16] states that for weakly compatible split systems $(\mathcal{S}, \alpha)$ all cells in $\mathbb{T}(d_{(\mathcal{S},\alpha)})$ are isomorphic to either hypercubes or rhombic dodecahedra. Hence (1) follows from Theorem 2.4.
(2) This follows from (1) in connection with the equivalence of (i) and (vi) in [9, Theorem 1.1 (c)]. □

2.6. **The tight-span.** We have seen in the introduction that any metric space (X, d) can be embedded into its tight-span and that $d(x, y) = d_\infty(x, y) = d_{(G_d, w_\infty)}(x, y)$ holds for all $x, y \in X$. A key in the proof of our main theorem will be the following observation by Hirai [15], which shows that for two-decomposable metrics the latter equality holds for general vertices of the tight-span. For the sake of completeness, a proof is included here. Note that the result is not true in general for metrics with dimension 3 or more (see, e.g. [20, Theorem 3]).

Proposition 2.6. [15, Proposition 4.2] *Let (X, d) be a two-decomposable metric space. Then for any two elements $f, g \in V(T(d))$, we have $d_\infty(f, g) = d_{(G_d, w_\infty)}(f, g)$.*

Proof. For any geodesic $\gamma : [0, 1] \to T(d)$, let $\Omega(\gamma)$ be the set of all cells C in $\mathbb{T}(d)$ with dimension two or more such that their intersection with γ is not contained in the union of the vertices and edges of $\mathbb{T}(d)$. By Theorem 2.5 (1), all elements of $\Omega(\gamma)$ are quadrangles. It now suffices to show that for any distinct $f, g \in V(T(d))$ there exists a geodesic γ between f and g with $\Omega(\gamma) = \emptyset$.

Suppose that is not the case and let γ be a geodesic between f and g such that the set $\Omega(\gamma)$ has minimal cardinality among all those geodesics. For any $C \in \Omega(\gamma)$, denote the minimal (resp. maximal) element $t \in [0, 1]$ with $\gamma(t) \in C$ by t_C^- (resp. t_C^+). Since γ is a geodesic, $\gamma(t) \in C$ holds if and only if $t \in [t_C^-, t_C^+]$.

Let $C \in \Omega(\gamma)$ be the first two-dimensional cell met by γ, that is, the one with minimal t_C^-. Clearly, $\gamma(t_C^-) \in V(T(d))$ and, since C is a quadrangle and $\gamma(t_C^+)$ belongs to one incident side of C, there exists a geodesic segment γ' between $\gamma(t_C^-)$ and $\gamma(t_C^+)$ using only the boundary edges of C. This allows us to construct a geodesic γ^* between f and g with $|\Omega(\gamma^*)| = |\Omega(\gamma)| - 1$, a contradiction. □

3. Embeddings in the Buneman complex

We now start to investigate embeddings of optimal realisations into the Buneman complex given by Theorem 2.3 and show that they have the following useful property (where Φ is the map defined in Section 2.5):

Theorem 3.1. *Let $(\mathcal{S}, \alpha)$ be a weighted split system on X and (G, w) a minimal path-saturated optimal realisation of $(X, d_{(\alpha,\mathcal{S})})$. Then for any non-expansive map $\psi : ||(G, w)|| \to B(\mathcal{S}, \alpha)$ with $\psi(x) = \Phi(x)$ for all $x \in X$ we have $\psi(V(G)) \subseteq V(B(\mathcal{S}, \alpha))$.*

Note that this theorem holds for all split system. We shall prove this theorem at the end of this section, after introducing and investigating the notions of split-flow digraphs and split potentials.

We begin with split-flow digraphs. To this end, recalling the following notation for digraphs. A *directed graph* (or *digraph* for short) $D = (V, \mathcal{A})$ is a pair consisting of a set V of vertices and a subset $\mathcal{A} \subseteq V \times V$, whose elements are called the *arcs* of D. A sequence $v_0, v_1, \cdots, v_k$ of vertices of D is called a *directed path* (from v_0 to v_k) in D if $(v_{i-1}, v_i) \in \mathcal{A}$ holds for all $i = 1, \ldots, k$. A *strongly connected component* in D is a maximal subset C of V such that for all distinct $u, v \in C$ there exists a directed path from u to v. Obviously, the set of strongly connected components forms a partition of V.

Given a realisation (G,w) of a finite metric space (X,d) and $A \subseteq X$, the *split-flow digraph* $D(G,w;A)$ is the digraph with vertex set $V(G)$ and arc set

$$\begin{aligned}\mathcal{A} := \big\{(u,v),(v,u) \in V \times V \,\big|\, &\text{there exists } x,y \in A \text{ or } x,y \in X \setminus A \\ &\text{such that } \{u,v\} \text{ belongs to a shortest path from } x \text{ to } y \text{ in } (G,w)\big\} \\ \bigcup \big\{(u,v) \in V \times V \,\big|\, &\text{there exists } x \in A \text{ and } y \in X \setminus A \\ &\text{such that } \{u,v\} \text{ belongs to a shortest path from } x \text{ to } y \text{ in } (G,w)\big\}.\end{aligned}$$

If (G,w) is an optimal realisation, Lemma 2.1 implies that for each $\{u,v\} \in E(G)$ we have $(u,v) \in \mathcal{A}$ or $(v,u) \in \mathcal{A}$ and both hold if and only if $\{u,v\}$ belongs to a shortest path between two elements of A or between two elements of $X \setminus A$. Furthermore, this implies that there exist strongly connected components C_A and $C_{X\setminus A}$ such that $A \subseteq C_A$ and $X \setminus A \subseteq C_{X\setminus A}$. (Note that C_A and $C_{X\setminus A}$ might be equal.) Note that for a split-flow digraph, there might strongly connected components other than C_A and $C_{X\setminus A}$ (see Fig. 1 for an example). But we shall show that for the case of minimal path-saturated optimal realisations, these are the only strongly connected components of $D(G,w;A)$.

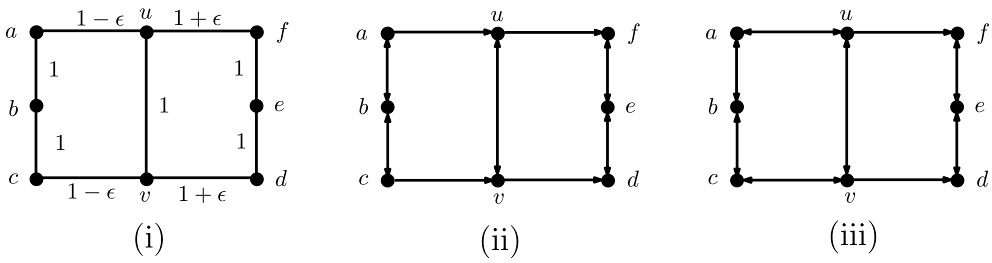

FIGURE 1. Examples of split-flow digraphs: (i) An optimal realisation (G,w_ϵ) $(-1/2 \le \epsilon \le 1/2)$ for the metric in [1] on $X = \{a,b,c,d,e,f\}$; (ii) the split-flow digraph $D(G,w_0;A)$ with $A = \{a,b,c\}$; (iii) the split-flow digraph $D(G,w_{1/2};A)$ with $A = \{a,b,c\}$. Note that the number of shortest paths in $(G,w_{1/2})$ with ends in X is larger than that in (G,w_0), and the split-flow digraph in (ii) has three strongly connected components while that in (iii) has two.

Theorem 3.2. *Let (G,w) be a minimal path-saturated optimal realisation of a metric space (X,d) and $A \subseteq X$. Then the strongly connected components of the split-flow digraph $D(G,w;A)$ are C_A and $C_{X\setminus A}$. In particular, the number of strongly connected components of $D(G,w;A)$ is either one or two.*

Proof. Let V and $\mathcal{A}$ be the vertex and arc set of $D(G,w;A)$, respectively. Assume that there exists a strongly connected component W of $D(G,w;A)$ that is distinct from both C_A and $C_{X\setminus A}$. It suffices to show that it leads to a contradiction.

The first stage of the proof is to show that

$$\Delta(G,w) = \min\{w(P) - d(x,y) : P \notin \Gamma(G,w;X) \text{ is a path in } (G,w) \text{ whose ends } x \text{ and } y \text{ are contained in } X\}$$

is strictly positive. To see this, consider a vertex u in W. Since the degree of u is at least three, applying Lemma 2.1 to the edges incident with u shows that u is contained in a path P_u connecting two elements $x_u, y_u \in X$ so that we have either $\{x_u, y_u\} \subseteq A$ or $\{x_u, y_u\} \subseteq X \setminus A$. As u is contained in neither C_A nor $C_{X\setminus A}$, we have $w(P_u) - d(x,y) > 0$. Since there are finite many paths in (G,w), it follows that $\Delta(G,w) > 0$, as required.

The second stage of the proof is to show that there exists a family of optimal realisations of (X,d) which can be parametrised by an interval and derived from (G,w) by perturbing the weights of a certain set of edges determined by the strongly connected component W. To this end, define the map $\delta : E(G) \to \mathbb{R}$ as

$$\delta(\{u,v\}) := \begin{cases} +1, & \text{if } (u,v) \in \mathcal{A} \text{ with } u \in V \setminus W, v \in W, \\ -1, & \text{if } (u,v) \in \mathcal{A} \text{ with } u \in W, v \in V \setminus W, \\ 0, & \text{otherwise.} \end{cases}$$

So for any $e \in E(G)$ we have $\delta(e) = +1$ (resp. $\delta(e) = -1$) if e induces an arc in $D(G,w;A)$ that is *entering* (resp. *leaving*) W. By exchanging A and $X \setminus A$ if necessary, we can assume that the difference N of the number of arcs entering and leaving W is non-negative. Since W is a strongly connected component, for all $u \in V \setminus W$ and $v \in W$ with $\{u,v\} \in E(G)$, we have either $(u,v) \in \mathcal{A}$ or $(v,u) \in \mathcal{A}$, but not both. Therefore the above map δ is well defined.

Let $t = \min\{w(e) \mid e \in E(G) \text{ and } \delta(e) = -1\}$. Then we have $t > 0$. For any $\epsilon \in (0,t)$, the map $w_\epsilon : E(G) \to \mathbb{R}_{>0}$ defined as

$$w_\epsilon(e) := w(e) + \epsilon\delta(e)$$

is a weight function on $E(G)$, and hence (G, w_ϵ) is a weighted graph. Since $X \cap W = \emptyset$, the numbers of edges entering and leaving W are equal for every path in $\Gamma(G,w;X)$. Therefore we have

$$w(P) = w_\epsilon(P) \text{ for all } P \in \Gamma(G,w;X) \text{ and } 0 < \epsilon < t\,, \tag{2}$$

and hence also

$$d_{(G,w_\epsilon)}(x,y) \leq d_{(G,w)}(x,y) = d(x,y) \text{ for all } x,y \in X \text{ and } 0 < \epsilon < t\,. \tag{3}$$

In addition, we have the following

Claim-A: If (G, w_ϵ) is a realisation of (X,d) for some $0 < \epsilon < t$, then (G, w_ϵ) is an optimal realisation with $\Gamma(G,w;X) \subseteq \Gamma(G,w_\epsilon;X)$.

Indeed, we have $l(G, w_\epsilon) = l(G, w) - \epsilon N$, which, since (G, w) is optimal, implies $N = 0$ and that (G, w_ϵ) is optimal. Moreover, by Eq. (2) we have$\Gamma(G, w; X) \subseteq \Gamma(G, w_\epsilon; X)$, and hence the claim follows.

The last step of the second stage is to show that for all

$$0 < \epsilon < \min\left\{t, \frac{\Delta(G, w)}{|E(G)|}\right\},$$

(G, w_ϵ) is an optimal realisation of (X, d) (the existence of such an ϵ follows from $\Delta(G, w) > 0$, as established in the first stage). To this end, consider two arbitrary elements x, y in X and a path P between them in G. If P is contained in $\Gamma(G, w; X)$, then we have $w_\epsilon(P) = w(P) \geq d(x, y)$ in view of Eq. (2). Otherwise, P is not contained in $\Gamma(G, w; X)$, and by the definition of $\Delta(G, w)$ we have

$$\begin{aligned} w_\epsilon(P) &\geq w(P) - |E(P)|\epsilon \\ &> w(P) - \frac{|E(P)|}{|E(G)|}\Delta(G, w) \\ &\geq w(P) - \Delta(G, w) \\ &\geq d(x, y). \end{aligned}$$

This implies that for all $x, y \in X$, we have $d_{(G,w_\epsilon)}(x, y) \geq d(x, y)$ and hence $d_{(G,w_\epsilon)}(x, y) = d(x, y)$ in view of Eq. (3). Together with **Claim-A**, it follows that (G, w_ϵ) is an optimal realisation of (X, d), completing the proof of the second stage.

In the final stage of the proof, we shall obtain a contradiction by considering optimal realisations corresponding to the right extremal point in the parameter interval that we found in the second stage of the proof. To this end, let

$$\epsilon' = \sup\left\{\epsilon \in (0, t) \mid (G, w_\epsilon) \text{ is a realisation of } (X, d)\right\}.$$

Then we have $\epsilon' > 0$ by the second stage. Now we have the following two cases to consider.

Case 1: $0 < \epsilon' < t$. We shall first show that $(G, w_{\epsilon'})$ is an optimal realisation of (X, d). By **Claim-A** it suffices to prove that $(G, w_{\epsilon'})$ is a realisation of (X, d). Indeed, if $(G, w_{\epsilon'})$ is not a realisation, then by Eq. (3) there exist two elements x_0, y_0 in X and a path P_0 between x_0 and y_0 in G such that

$$w_{\epsilon'}(P_0) = d_{(G,w_{\epsilon'})}(x_0, y_0) < d(x_0, y_0).$$

On the other hand, for all $0 < \epsilon < \epsilon'$ we have $w_\epsilon(P_0) \geq d(x_0, y_0)$ because (G, w_ϵ) is a realisation of (X, d). Since $w_\epsilon(P_0)$ is a continuous function for $0 < \epsilon \leq \epsilon'$, we have $w_{\epsilon'}(P_0) \geq d(x_0, y_0)$. This is a contradiction, and hence $(G, w_{\epsilon'})$ must be an optimal realisation of (X, d).

Next, fix some ϵ_1 in (ϵ', t). Since the weighted graph (G, w_{ϵ_1}) is not a realisation of (X, d), there exists a path P_1 with ends $x_1, y_1 \in X$ so that $w_{\epsilon_1}(P_1) < d(x_1, y_1)$.

Note that this implies $P_1 \notin \Gamma(G, w; X)$ by Eq. (2). On the other hand, since $w_\epsilon(P_1) \geq d(x_1, y_1)$ for all $0 < \epsilon < \epsilon'$ and $w_\epsilon(P_1)$ is a continuous function for $0 < \epsilon \leq \epsilon'$, we have $w_{\epsilon'}(P_1) = d(x_1, y_1)$, and hence $P_1 \in \Gamma(G, w_{\epsilon'}; X)$. Noting that $\Gamma(G, w; X) \subseteq \Gamma(G, w_{\epsilon'}; X)$ in view of **Claim-A**, we have $|\Gamma(G, w_{\epsilon'}; X)| > |\Gamma(G, w; X)|$, which is a contradiction to the fact that (G, w) is path-saturated. This completes the proof of Case 1.

Case 2: $\epsilon' = t$. Let

$$M := \{\{u, v\} \in E(G) \mid w(\{u, v\}) = t \text{ and } \delta(\{u, v\}) = -1\}.$$

By definition of t, M is not empty. Next, we will show that M is a matching, that is, no two edges in M share a common vertex. Suppose M contains $e_1 = \{u, v\}$ and $e_2 = \{v, u'\}$ for some $u, v, u' \in V(G)$. We may assume $v \in W$ and $u, u' \in V \setminus W$, as the other case (i.e., $u, u' \in W$ and $v \in V \setminus W$) is similar. By Lemma 2.1 (2), this implies that e_1 and e_2 are contained in a shortest path between two elements of X. By the construction of $D(G, w; A)$, this is a contradiction to the assumption that W is a strongly connected component.

Now let G' be the graph obtained from G by *contracting* all edges in M. That is, for every edge $e = \{u, v\}$ in M, where $u \in W$, we add an edge $\{u', v\}$ for all $u' \neq v$ that is adjacent to u, and delete u and all edges incident to it. Since M is a matching, the graph G' is well-defined. We define the weight function $w' : E(G') \to \mathbb{R}_{>0}$ by setting $w'(e) = w(e) + t\delta(e)$ if $e \in E(G)$ and $w'(e) = w(e^*) + t\delta(e^*)$ otherwise, where $e^* = \{u', u\}$ for the unique vertex u in G so that $\{u, v\} \in M$.

Since contracting length-zero edges does not change any path lengths, an argument similar to the one in Case 1 shows that (G', w') is an optimal realisation of (X, d).

Now consider two arbitrary paths $P_1, P_2 \in \Gamma(G, w; X)$ and let $P_1', P_2' \in \Gamma(G', w'; X)$ be the paths obtained by contracting all its edges contained in M. Then it remains to show that $P_1' \neq P_2'$, because this implies that $|\Gamma(G', w'; X)| \geq |\Gamma(G, w; X)|$, and hence (G', w') is a path-saturated optimal realisation of (X, d) with $|V(G')| < |V(G)|$, a contradiction as required. Indeed, if $P_1' = P_2'$, then there exists some edge $e = \{u, v\} \in M$ such that e is contained in exactly one path among P_1, P_2, say P_1. Switching the role of u and v if necessarily, we have $(u, v) \in \mathcal{A}$. Let s be the other vertex in P_1 that is adjacent to u. Since no edges in M share a common vertex, we have $\{s, u\} \notin M$ and hence $\{s, v\} \in P_1' = P_2'$. Again using the fact that no two edges in M share a common vertex, we get $\{s, v\} \in P_2$ and so $\{u, v\}, \{s, v\}, \{s, u\} \in E(G)$, contradicting Corollary 2.2. Therefore we have $P_1' \neq P_2'$, which completes the proof of Case 2, as well as the theorem. □

Let (G, w) be a realisation of (X, d). A map $\lambda : ||(G, w)|| \to [0, 1]$ is called a *split potential* on $||(G, w)||$ if

(1) $\lambda(x) \in \{0, 1\}$ for all $x \in X$, and
(2) $\lambda \circ \gamma : [0, 1] \to [0, 1]$ is monotonic for all X-geodesics $\gamma : [0, 1] \to ||(G, w)||$.

Lemma 3.3. *Let (G, w) be a minimal path-saturated optimal realisation of (X, d) and λ a split potential on $||(G, w)||$. Then $\lambda(v) \in \{0, 1\}$ holds for all $v \in V(G)$.*

Proof. Let A be the set of all $x \in X$ that are mapped to 0 by λ and let $D = (V, \mathcal{A})$ be the split-flow digraph $D(G, w; A)$.

We will now show that $(u, v) \in \mathcal{A}$ implies that $\lambda(u) \leq \lambda(v)$. This implies that λ restricted to any strongly connected component of D is a constant, and the lemma then follows from Theorem 3.2. Indeed, fix $a \in A$ and $b \in X - A$; then we have $\{f(a), f(b)\} \subseteq \{0, 1\}$ and Theorem 3.2 implies that we have $f(v) \in \{f(a), f(b)\}$ for each $v \in V(G)$.

So let $(u, v) \in \mathcal{A}$. If $u \in A$ or $v \in X \setminus A$, then $\lambda(u) \leq \lambda(v)$ obviously holds. Otherwise, by Lemma 2.1(i) there exists $x, y \in X$ and a shortest path P in (G, w) from x to y such that $\{u, v\}$ is an edge in P and either $x \in A$ or $y \in X \setminus A$. Here we shall consider the case $x \in A$ since the other one is similar. The path P induces an X-geodesic γ between x and y that passes first through u and then through v. Since $\lambda \circ \gamma$ is monotonic and $\lambda(\gamma(0)) = \lambda(x) = 0 < 1 = \lambda(y) = \lambda(\gamma(1))$, we get $\lambda(u) \leq \lambda(v)$, as required. □

We now present a specific way to define split potentials using the Buneman complex, which will allow us to prove Theorem 3.1.

Let $(\mathcal{S}, \alpha)$ be a weighted split system on X. For any $A \in \mathrm{supp}(\alpha)$, define the map $\lambda_A : B(\mathcal{S}, \alpha) \to [0, 1]$ by putting

$$\lambda_A(\mu) := \frac{\mu(A)}{\alpha(\{A, X \setminus A\})} \quad \text{for all } \mu \in B(\mathcal{S}, \alpha)\,.$$

Lemma 3.4. *Let $(\mathcal{S}, \alpha)$ be a weighted split system on X, $A \in \mathrm{supp}(\alpha)$ and (G, w) a realisation of $(X, d_{(\alpha,\mathcal{S})})$. Then for any non-expansive map $\psi : ||(G, w)|| \to B(\mathcal{S}, \alpha)$ with $\psi(x) = \Phi(x)$ for all $x \in X$, the function $\lambda_A \circ \psi$ is a split potential on $||(G, w)||$.*

Proof. Let $\gamma : [0, 1] \to ||(G, w)||$ be an X-geodesic in $||(G, w)||$ and set $\lambda'_A := \lambda_A \circ \psi \circ \gamma$. Since $\psi(x) = \Phi(x)$, we have $(\lambda_A \circ \psi)(x) \in \{0, 1\}$ for all $x \in X$, so it remains to show that λ'_A is monotonic.

Since ψ is non-expansive, the map $\psi \circ \gamma : [0, 1] \to B(\mathcal{S}, \alpha)$ is a geodesic in $(B(\mathcal{S}, \alpha), d_1)$. Therefore, for any $0 \leq t_1 < t_2 < t_3 \leq 1$, we have

$$d_1(\nu_1, \nu_3) = d_1(\nu_1, \nu_2) + d_1(\nu_2, \nu_3)\,,$$

where $\nu_i := \psi \circ \gamma(t_i)$ for $i = 1, 2, 3$. By the definition of the metric d_1, this gives us

$$\sum_{B \in \mathrm{supp}(\alpha)} |\nu_1(B) - \nu_3(B)| = \sum_{B \in \mathrm{supp}(\alpha)} \big(|\nu_1(B) - \nu_2(B)| + |\nu_2(B) - \nu_3(B)|\big).$$

Since, by definition, for all $B \in \mathrm{supp}(\alpha)$, we have

$$|\nu_1(B) - \nu_3(B)| \leq |\nu_1(B) - \nu_2(B)| + |\nu_2(B) - \nu_3(B)|\,,$$

this implies that

$$|\nu_1(A) - \nu_3(A)| = |\nu_1(A) - \nu_2(A)| + |\nu_2(A) - \nu_3(A)|$$

and hence

$$\big(\nu_1(A) - \nu_2(A)\big) \cdot \big(\nu_2(A) - \nu_3(A)\big) \geq 0.$$

Together with

$$\lambda'_A(t_i) = \lambda_A(\nu_i) = \frac{\nu_i(A)}{\alpha(\{A, X \setminus A\})}$$

for $i = 1, 2, 3$, this implies

$$\big(\lambda'_A(t_1) - \lambda'_A(t_2)\big) \cdot \big(\lambda'_A(t_2) - \lambda'_A(t_3)\big) \geq 0\,,$$

hence λ'_A is monotonic. □

With the above results, we are now in a position to present the proof of the theorem stated at the beginning of this section.

Proof of Theorem 3.1. Lemmas 3.3 and 3.4 show that for all $A \in \mathrm{supp}(\alpha)$ and $v \in V(G)$ we have $\lambda_A(\psi(v)) \in \{0, 1\}$. However, by the definition of the map λ_A, we have $\lambda_A(\mu) \in \{0, 1\}$ if and only if $\mu(A) \in \{0, \alpha(\{A, X \setminus A\})\}$ for all $\mu \in B(\mathcal{S}, \alpha)$. So $\psi(v)$ is a vertex of $B(\mathcal{S}, \alpha)$. □

4. The main theorem

We are now ready to prove the main result of this paper, from which Theorem 1.2 follows immediately.

Theorem 4.1. *Let (X, d) be a totally-decomposable finite metric space with dimension two and (G, w) a minimal path-saturated optimal realisation of (X, d). Then (G, w) is homeomorphic to a subgraph of (G_d, w_∞).*

Proof. Since (G, w) is path-saturated, by Theorem 2.3 there exists a non-expansive injection ψ from $||(G, w)||$ to $T(d)$ satisfying $\psi(x) = \kappa(x)$ for all $x \in X$, where κ is the embedding of (X, d) into its tight-span.

The first stage of the proof is to show that the injection ψ maps vertices of G to vertices of $T(d)$, that is,

$$\psi(V) \subseteq V(T(d)) = V(G_d). \tag{4}$$

To see this, consider the weighted 2-compatible split system $(\mathcal{S}, \alpha)$ on X such that $d = d_{(\mathcal{S},\alpha)}$. Then (X, d) is embedded into the Buneman complex $B(\mathcal{S}, \alpha)$ via the map Φ (see Section 2.5). By Theorem 2.5 (2), there exists an isometry Λ from $B(\mathcal{S}, \alpha)$ to $T(d)$ such that $\Lambda(\Phi(x)) = \kappa(x)$ holds for all $x \in X$. Noting that the map ϕ is non-expansive, it follows that the map $\psi' := \Lambda^{-1} \circ \psi$ is a non-expansive map from $||(G, w)||$ to $B(\mathcal{S}, \alpha)$ with $\psi'(x) = \Phi(x)$ for all $x \in X$. By Theorem 3.1 we have $\psi'(V) \subseteq V(B(\mathcal{S}, \alpha))$, and hence $\Lambda(\psi'(V)) = \psi(V) \subseteq V(T(d)) = V(G_d)$. Hence Eq. (4) holds.

The second stage of the proof is to construct an optimal realisation of (X, d) that is a subgraph of (G_d, w_∞).

To simplify the notation, we set $\widetilde{u} := \psi(u)$ for all $u \in V(G)$ (note that $\widetilde{x} = x$ for all $x \in X$). Now, for each edge $e = \{u, v\} \in E(G)$, since $\widetilde{u}$ and $\widetilde{v}$ are vertices in G_d by Eq. (4), we fix a shortest path P_e in G_d between $\widetilde{u}$ and $\widetilde{v}$ and let $\mathcal{P} := \{P_e \,|\, e \in E(G)\}$ be the collection of all such paths. We now consider the subgraph (G^*, w^*) of (G_d, w_∞) defined as

$$V(G^*) = \{u \in V(G_d) \mid u \text{ is contained in some path } P \in \mathcal{P}\}\ ,$$
$$E(G^*) = \{\{u, v\} \in E(G_d) \mid u \text{ and } v \text{ are adjacent in some path } P \in \mathcal{P}\}$$

and $w^* = w_\infty|_{E(G^*)}$. The aim of the second stage is to show that (G^*, w^*) is an optimal realisation of (X, d).

Firstly, note that there exists a map τ associating each path $P = v_0, v_1, \ldots, v_k$ ($k \geq 1$) in $\Gamma(G, w; X)$ with the walk[1] $\tau(P)$ in (G^*, w^*) between $\widetilde{v}_0$ to $\widetilde{v}_k$ that is obtained from P by replacing each edge $e_i = \{v_{i-1}, v_i\}$ ($1 \leq i \leq k$) with the path P_{e_i} in $\mathcal{P}$. In particular, if P contains only one edge e, then $\tau(P) = P_e$.

Secondly, we have

$$w(e) = d_{(G,w)}(u, v) \geq d_\infty(\widetilde{u}, \widetilde{v}) = d_{(G_d, w_\infty)}(\widetilde{u}, \widetilde{v}) = w_\infty(P_e) = w^*(P_e) \qquad (5)$$

for all edge $e = \{u, v\}$ in $E(G)$, where the inequality follows from ψ being non-expansive, and the second equality follows from Proposition 2.6.

Moreover, we claim that for each pair of distinct elements $x, y \in X$, we have

$$d_{(G^*, w^*)}(x, y) \leq d_{(G,w)}(x, y). \qquad (6)$$

To see this, fix a shortest path $P_{x,y}$ in G (and hence $P_{x,y} \in \Gamma(G, w, ; X)$) and consider the walk $\tau(P_{x,y})$ between x and y in (G^*, w^*). Then we have

$$d_{(G^*, w^*)}(x, y) \leq w^*(\tau(P_{x,y})) = \sum_{i=1}^{k} w^*(P_{e_i}) \leq \sum_{i=1}^{k} w(e_i) = w(P_{x,y}) = d_{(G,w)}(x, y),$$

where the second inequality follows from Eq. (5). Hence Eq. (6) holds as claimed.

Next, since (G_d, w_∞) and (G, w) are both realisations of (X, d), by Eq. (6) for all $x, y \in X$ we have

$$d(x, y) = d_{(G_d, w_\infty)}(x, y) \leq d_{(G^*, w^*)}(x, y) \leq d_{(G,w)}(x, y) = d(x, y)\,.$$

Hence

$$d(x, y) = d_{(G_d, w_\infty)}(x, y) = d_{(G^*, w^*)}(x, y) = d_{(G,w)}(x, y) = w^*(\tau(P_{x,y})) \qquad (7)$$

holds for every shortest path $P_{x,y}$ between x and y in G. Therefore (G^*, w^*) is a realisation of (X, d). In addition, this implies $\tau(P_{x,y})$ is a shortest path between x and y in (G^*, w^*), and hence τ is a map from $\Gamma(G, w; X)$ to $\Gamma(G^*, w^*; X)$.

[1]Note that a walk in G^* is a sequence of not necessarily distinct vertices in G^* with each consecutive pair forming an edge in G^*; we shall show later in the proof that $\tau(P)$ is actually a path in $\Gamma(G^*, w^*; X)$.

Finally, we have

$$l(G^*, w^*) \le \sum_{P \in \mathcal{P}} w_\infty(P) = \sum_{e \in E(G)} w_\infty(P_e) \le \sum_{e \in E(G)} w(e) = l(G, w),$$

where the the first inequality follows from construction and the second one from Eq. (5). Since (G, w) is optimal, this implies that (G^*, w^*) is also optimal, and that

$$l(G^*, w^*) = \sum_{P \in \mathcal{P}} w_\infty(P) = \sum_{e \in E(G)} w_\infty(P_e) = \sum_{e \in E(G)} w(e) = l(G, w). \tag{8}$$

This complete the second stage of the proof.

In the third and final stage of the proof, we shall show that (G^*, w^*) is homeomorphic to (G, w). Suppose first that for any two distinct paths P_{e_1} and P_{e_2} in $\mathcal{P}$ with $e_i = (u_i, v_i)$ for $i = 1, 2$, we have

$$V(P_{e_1}) \cap V(P_{e_2}) \subseteq \{\widetilde{u}_1, \widetilde{v}_1\} \cap \{\widetilde{u}_2, \widetilde{v}_2\}. \tag{9}$$

Then the weighted graph obtained from (G^*, w^*) by suppressing all vertices with degree two in $V(G^*) \setminus \psi(V)$ is isomorphic to (G, w). That is, (G, w) is homeomorphic to (G^*, w^*), a subgraph of (G_d, w_∞), as desired. So it remains to show that Inclusion (9) always holds.

To this end, note first that two distinct paths in $\mathcal{P}$ do not share a common edge in (G_d, w_∞), because otherwise we have $l(G^*, w^*) < \sum_{P \in \mathcal{P}} w_\infty(P)$, a contradiction to Eq. (8). Therefore, the map $\tau : \Gamma(G, w, ; X) \to \Gamma(G^*, w^*; X)$ is injective.

Secondly, we must have $|V(P_{e_1}) \cap V(P_{e_2})| < 2$. Indeed, if this were not the case, there would exist two vertices u and v in $V(P_{e_1}) \cap V(P_{e_2})$. Let P_1 and P_2 denote the subpath from u to v induced by P_{e_1} and P_{e_2}, respectively. Now consider the path $P^\circ_{e_2}$ obtained from P_{e_2} by replacing P_2 by P_1 and let $\mathcal{P}^\circ$ be the collection of paths obtained from $\mathcal{P}$ by replacing P_{e_2} with $P^\circ_{e_2}$. Since two distinct paths in $\mathcal{P}$ do not share a common edge, we have

$$\sum_{P \in \mathcal{P}^\circ} w_\infty(P) < \sum_{P \in \mathcal{P}} w_\infty(P) = l(G^*, w^*). \tag{10}$$

On the other hand, using an argument similar to showing that (G^*, w^*) is an optimal realisation of (X, d), we know that the graph (G°, w°) obtained as the union of the paths in $\mathcal{P}^\circ$ is also an optimal realisation of (X, d), a contradiction to Eq. (10) and that (G^*, w^*) is optimal. Hence $|V(P_{e_1}) \cap V(P_{e_2})| < 2$ as claimed.

So, to complete the proof, assume that $(V(P_{e_1}) \cap V(P_{e_2})) \setminus (\{\widetilde{u}_1, \widetilde{v}_1\} \cap \{\widetilde{u}_2, \widetilde{v}_2\}) = \{v\}$ for some $v \in V(G)$. For $i = 1, 2$, let f_i and f'_i be the two edges in P_{e_i} that are incident with v. Note that $\{f_1, f'_1\} \cap \{f_2, f'_2\} = \emptyset$. Since (G^*, w^*) is isomorphic to an optimal realisation, Lemma 2.1 implies that there exists a path P^* in $\Gamma(G^*, w^*; X)$ that contains the edges f_1 and f_2. Since for each path P in $\Gamma(G, w; X)$, the path $\tau(P)$ in (G^*, w^*) can contain at most two edges incident with v and the set of these two edges must be $\{f'_1, f_1\}$ or $\{f'_2, f_2\}$ (but not both), we know that P^* is not the image of any path P in $\Gamma(G, w; X)$ under the map τ. Therefore, the map $\tau : \Gamma(G, w, ; X) \to \Gamma(G^*, w^*; X)$ is not surjective. But τ is injective, and so $|\Gamma(G^*, w^*; X)| > |\Gamma(G, w; X)|$, a contradiction to the assumption that (G, w) is path-saturated. This completes the proof of Eq. (9), and hence also the theorem. □

Acknowledgements. Work of SH was supported by a fellowship within the Post-doc-Programme of the German Academic Exchange Service (DAAD) and the University of East Anglia. Part of the work of TW was supported by the Singapore MOE grant R-146-000-134-112. JHK was partially supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0008138), and JHK was also partially supported by the '100 talents' program of the Chinese Academy of Sciences. Finally, we thank the comments and suggestions of two anonymous reviewers that have led to a substantial improvement of this paper.

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