Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 21/02

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April 2021

URN: urn:nbn:de:gbv:ilm1-2021200042

Impressum:
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$\mathcal{PT}$-SYMMETRIC HAMILTONIANS AS COUPLINGS OF DUAL PAIRS

Volodymyr Derkach, Philipp Schmitz, and Carsten Trunk

Dedicated to our friend and colleague Seppo Hassi on the occasion of his 60th birthday

1 Introduction

In the seminal paper (Bender & Boettcher, 1998) a new view of quantum mechanics was proposed. This new view differs from the old one in that the restriction on the Hamiltonian to be Hermitian is relaxed: now the Hamiltonian is $\mathcal{PT}$-symmetric. Here $\mathcal{P}$ is parity and $\mathcal{T}$ is time reversal. Since 1998, $\mathcal{PT}$-symmetric Hamiltonians have been analyzed intensively by many authors. In Mostafazadeh (2002) $\mathcal{PT}$-symmetry was embedded into the more general mathematical framework of pseudo-Hermiticity or, what is the same, self-adjoint operators in Kreĭn spaces, see (Langer & Tretter, 2004; Azizov & Trunk, 2012; Hassi & Kuzhel, 2013; Leben & Trunk, 2019). For a general introduction to $\mathcal{PT}$-symmetric quantum mechanics we refer to the overview paper of Mostafazadeh (2010) and to the books of Moiseyev (2011) and Bender (2019).

A prominent class consists of the $\mathcal{PT}$-symmetric Hamiltonians

$$H := \frac{1}{2}p^2 - (iz)^{N+2},$$

where N is a positive integer, see (Bender, Brody & Jones, 2002). The associated eigenvalue problem is defined on a contour Γ in the complex plane which is contained in a specific area in the complex plane, the so-called Stokes wedges, see (Bender & Boettcher, 1998),

$$-y''(z) - (iz)^{N+2}y(z) = \lambda y(z), \quad z \in \Gamma, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ is the eigenvalue parameter. Recall that a *Stokes wedge* S_k, $k = 0, \ldots, N+3$, is an open sector in the plane with vertex zero,

$$S_k := \left\{ z \in \mathbb{C} : -\frac{N+2}{2N+8}\pi + \frac{2k-2}{4+N}\pi < \arg(z) < -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \right\},$$

see (Bender et al., 2006). The boundary of S_k consists of two rays from the origin, the so-called *Stokes lines*. $\mathcal{PT}$-symmetry forces Γ to lie in two Stokes wedges, which are symmetric with respect to the imaginary axis.

In Mostafazadeh (2005) the contour Γ in equation (1.1) was parameterized by a real parameter. In Bender et al. (2006) and in Jones & Mateo (2006) this approach was extended to different parameterizations and contours. Here we choose, for simplicity, Γ to be a wedge-shaped contour,

$$\Gamma := \{xe^{i\phi \operatorname{sgn} x} : x \in \mathbb{R}\}, \tag{1.2}$$

for some angle $\phi \in (-\pi/2, \pi/2)$, see Figure 1.

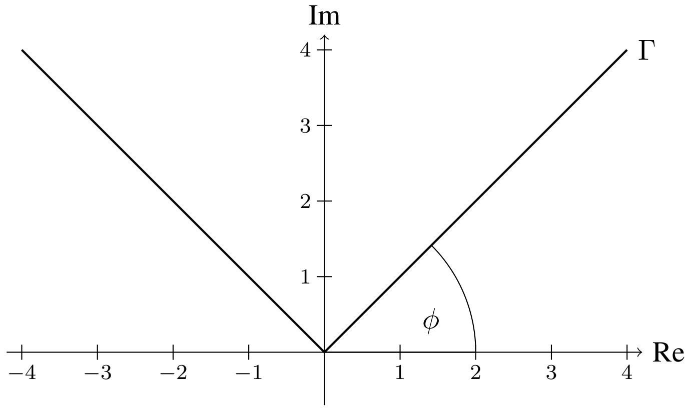

Figure 1. The complex contour Γ.

Let $z : \mathbb{R} \to \mathbb{C}$ parameterize Γ via $z(x) := xe^{i\phi \operatorname{sgn} x}$. Then y solves (1.1) for $z \neq 0$ if and only if the pair of functions u_+ and u_-, given by $u_\pm(x) := y(z(x))$, $x \in \mathbb{R}_\pm$, solves

$$\mathfrak{a}_-[u_-] = \lambda u_-, \quad x \in \mathbb{R}_-, \qquad \mathfrak{a}_+[u_+] = \lambda u_+, \quad x \in \mathbb{R}_+, \tag{1.3}$$

where the differential expressions $\mathfrak{a}_\pm$ are given by

$$\mathfrak{a}_\pm[u_+] = -e^{\mp 2i\phi} u_\pm'' - (ix)^{N+2} e^{\pm i(N+2)\phi} u_\pm. \tag{1.4}$$

In what follows we assume that Γ lies in Stokes wedges and then, by Leben & Trunk (2019), the differential expressions $\mathfrak{a}_\pm$ are in the limit-point case at $\pm\infty$ according to the classification in Brown et al. (1999), which is a refinement of the classification in Sims (1957). We mention, that the limit-circle case can be treated in a similar way as in Azizov & Trunk (2010; 2012).

The theory of $\mathcal{PT}$-symmetry claims that the main object, the Hamiltonian, commutes under the joint action of the parity $\mathcal{P}$ and the time reversal $\mathcal{T}$,

$$(\mathcal{P}f)(x) := f(-x), \qquad (\mathcal{T}f)(x) := \overline{f(x)}. \tag{1.5}$$

The time reversal $\mathcal{T}$ applied to the differential expressions $\mathfrak{a}_\pm$ gives rise to new differential expressions $\mathfrak{b}_\pm = \mathcal{T}\mathfrak{a}_\pm\mathcal{T}$ defined on $\mathbb{R}_\pm$

$$\mathfrak{b}_\pm[v_\pm] = -e^{\pm 2i\phi} v_\pm'' - (-ix)^{N+2} e^{\mp i(N+2)\phi} v_\pm. \tag{1.6}$$

In Section 3 we introduce the minimal operators $A_\pm$ and $B_\pm$ associated with $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ in $L^2(\mathbb{R}_\pm)$ and show that

$$\langle A_\pm f, g\rangle_\pm = \langle f, B_\pm g\rangle_\pm, \qquad \text{for all } f \in \operatorname{dom} A_\pm,\ g \in \operatorname{dom} B_\pm. \tag{1.7}$$

Here $\langle\cdot,\cdot\rangle_\pm$ stands for the usual inner products in the Hilbert spaces $L^2(\mathbb{R}_\pm)$. Condition (1.7) shows that the pairs (A_+, B_+) and (A_-, B_-) form dual pairs, see Section 2.1 for details. An extension theory for dual pairs based on the boundary triple technique was developed by Malamud & Mogilevskiĭ (2002). This is a generalization of the boundary triple approach to the extension theory of symmetric operators which was developed by Calkin (1939); Kočhubeĭ (1975); Gorbachuk & Gorbachuk (1991); Derkach & Malamud (1991), and others. For recent developments of the method of boundary triples and its application to the extension theory of differential operators, see the monographs by Derkach & Malamud (2017) and by Behrndt, Hassi, & de Snoo (2020).

Following this approach, we construct in Theorem 3.1 boundary triples for dual pairs (A_+, B_+) and (A_-, B_-). As our interest is focused on the Hamiltonian in $L^2(\mathbb{R})$ and not on the differential expressions $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$, which are defined on the semi-axes, we extend the coupling method for symmetric operators from Derkach et al. (2000) to the case of dual pairs and create a new dual pair (A, B) of operators defined on $\mathbb{R}$. This dual pair (A, B) is called the coupling of the dual pairs (A_+, B_+) and (A_-, B_-), see Theorem 2.5 and Definition 2.6 below.

We show that the operator $\mathcal{PT}$ intertwines the dual pairs (A_+, B_+) and (A_-, B_-), i.e.,

$$\mathcal{PT}A_+ = A_-\mathcal{PT} \quad \text{and} \quad \mathcal{PT}B_+ = B_-\mathcal{PT}.$$

Due to our construction of the coupling, these relations imply that the operator A is $\mathcal{PT}$-*symmetric*

$$\mathcal{PT}A = A\mathcal{PT}.$$

Moreover, the operator A turns out to be $\mathcal{P}$-symmetric in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ with the fundamental symmetry $\mathcal{P}$ in $\mathfrak{H} = L^2(\mathbb{R})$. In Leben & Trunk (2019) it was shown that the extension H_0 of A, defined as a restriction of the adjoint A^+ to the domain

$$\text{dom } H_0 = \left\{ u_+ \oplus u_- \in \text{dom } A^+ : \, u_+(0) - u_-(0) = e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = 0 \right\},$$

is a $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint operator in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$. Here A^+ stands for the adjoint with respect to the Kreĭn space inner product $[\cdot,\cdot]$. In Theorem 3.2 below, which is the main result of this note, we find a one-parameter family $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ of $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ with domain

$$\text{dom } H_\alpha = \left\{ u_+ \oplus u_- \in \text{dom } A^+ : \, u_+(0) - u_-(0) = 0, \, e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = \alpha u_+(0) \right\}.$$

Theorem 3.2 is based on the abstract construction of the coupling (A, B) of two dual pairs (A_+, B_+) and (A_-, B_-) in Theorem 2.5 and the description of all $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A given in Theorem 2.14.

Summing up, the results presented here promote the use of boundary triple techniques for dual pairs and techniques from Sturm–Liouville theory for complex potentials in the study of $\mathcal{PT}$-symmetric quantum mechanics. This is in line with Leben & Trunk (2019) and it is, to some extent, a surprise that in the physical literature the techniques presented here were never exploited. It is the aim of this paper to recall those techniques and, hence, provide a mathematically sound setting of the (nowadays) classical Bender–Boettcher-theory.

2 Coupling of dual pairs and parity

In this section we recall known facts about dual pairs of linear operators, their boundary triples and corresponding Weyl functions, and coupling from Malamud & Mogilevskiĭ (2002). However, our notations differ slightly from that paper; we mainly follow the notations of Baidiuk, Derkach & Hassi (2021).

Moreover, throughout this paper we use the following notations. By $\mathbb{R}_+$ and $\mathbb{R}_-$ we denote the set of all positive and negative reals, respectively. For $z \in \mathbb{C}$, $\overline{z}$ denotes the complex conjugate of z. All operators in this paper are densely defined linear operators in some Hilbert spaces. For such operators T, we use the common notation $\mathrm{dom}\ T$, $\mathrm{ran}\ T$, and $\ker T$ for the domain, the range, and the null-space, respectively, of T. Moreover, as usual, $\rho(T)$, $\sigma(T)$, and $\sigma_p(T)$ stand for the resolvent set, the spectrum, and the point spectrum, respectively, of T. The inner product in a Hilbert space is usually denoted by $\langle\cdot,\cdot\rangle$ and the adjoint of the operator T by T^*. The set of all bounded and everywhere defined operators in a Hilbert space $\mathfrak{H}$ is denoted by $\mathcal{L}(\mathfrak{H})$.

2.1 Dual pairs of linear operators and Weyl functions

Definition 2.1. A pair (A, B) of densely defined closed linear operators A and B in a Hilbert space $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ is called a *dual pair*, if

$$\langle Af, g\rangle - \langle f, Bg\rangle = 0 \qquad \text{for all } f \in \mathrm{dom}\ A,\ g \in \mathrm{dom}\ B. \tag{2.1}$$

The equality (2.1) means that

$$A \subset B^* \quad \text{and} \quad B \subset A^*.$$

Clearly, if (A, B) is a dual pair, then (B, A) is also a dual pair.

Definition 2.2. Let (A, B) be a dual pair in a Hilbert space $\mathfrak{H}$, let $\mathcal{H}_1$, $\mathcal{H}_2$ be auxiliary Hilbert spaces, and let

$$\Gamma^B = \begin{pmatrix} \Gamma_1^B \\ \Gamma_2^B \end{pmatrix} : \mathrm{dom}\ B^* \to \mathcal{H}_1 \times \mathcal{H}_2 \quad \text{and} \quad \Gamma^A = \begin{pmatrix} \Gamma_1^A \\ \Gamma_2^A \end{pmatrix} : \mathrm{dom}\ A^* \to \mathcal{H}_1 \times \mathcal{H}_2 \tag{2.2}$$

be linear operators. Then the triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ is called a *boundary triple for the dual pair* (A, B) if:

(1) the mappings Γ^B and Γ^A in (2.2) are surjective;

(2) the following identity holds for every $f \in \mathrm{dom}\ B^*$, $g \in \mathrm{dom}\ A^*$,

$$\langle B^* f, g\rangle - \langle f, A^* g\rangle = \langle \Gamma_1^B f, \Gamma_1^A g\rangle_{\mathcal{H}_1} - \langle \Gamma_2^B f, \Gamma_2^A g\rangle_{\mathcal{H}_2}.$$

It is easily seen that if a triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ is a boundary triple for a dual pair (A, B), then the following identity also holds

$$\langle A^* g, f\rangle - \langle g, B^* f\rangle = \langle \Gamma_2^A g, \Gamma_2^B f\rangle_{\mathcal{H}_2} - \langle \Gamma_1^A g, \Gamma_1^B f\rangle_{\mathcal{H}_1}, \quad f \in \mathrm{dom}\ B^*,\ g \in \mathrm{dom}\ A^* \tag{2.3}$$

and, hence, the triple

$$(\mathcal{H}_2 \times \mathcal{H}_1, (\Gamma^B)^T, (\Gamma^A)^T) := \left(\mathcal{H}_2 \times \mathcal{H}_1, \begin{pmatrix} \Gamma_2^B \\ \Gamma_1^B \end{pmatrix}, \begin{pmatrix} \Gamma_2^A \\ \Gamma_1^A \end{pmatrix}\right) \tag{2.4}$$

is a boundary triple for the dual pair (B, A). The boundary triple (2.4) is called *transposed* with respect to the boundary triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$.

A linear operator $\widetilde{A}$ is called a *proper extension* of a dual pair (A, B) if

$$A \subset \widetilde{A} \subset B^*.$$

The proper extension A_2 of A is defined as the restriction of B^* to the set

$$\text{dom } A_2 = \{f \in \text{dom } B^* : \Gamma_2^B f = 0\}. \tag{2.5}$$

Similarly, the proper extension B_1 of B is defined as the restriction of A^* to the set

$$\text{dom } B_1 = \{f \in \text{dom } A^* : \Gamma_1^A f = 0\}. \tag{2.6}$$

For every $z \in \rho(A_2)$ the following decomposition holds

$$\text{dom } B^* = \text{dom } A_2 \dotplus \mathfrak{N}_z(B^*), \qquad \text{where} \quad \mathfrak{N}_z(B^*) := \ker (B^* - zI),$$

and, consequently, the mapping $\Gamma_2^B|_{\mathfrak{N}_z(B^*)} : \mathfrak{N}_z(B^*) \to \mathcal{H}_2$ is boundedly invertible, see (Malamud & Mogilevskiĭ, 2002) for details. In a similar way, for every $z \in \rho(B_1)$ the following decomposition holds

$$\text{dom } A^* = \text{dom } B_1 \dotplus \mathfrak{N}_z(A^*), \qquad \text{where} \quad \mathfrak{N}_z(A^*) := \ker (A^* - zI),$$

and, hence, the mapping $\Gamma_1^A|_{\mathfrak{N}_z(A^*)} : \mathfrak{N}_z(A^*) \to \mathcal{H}_1$ is boundedly invertible for $z \in \rho(B_1)$.

Moreover, in light of (2.3), (2.5), and (2.6), one has that $B_1 = A_2^*$ and, hence, in particular the following identity holds

$$\rho(B_1) = \overline{\rho(A_2)}.$$

Definition 2.3. The operator functions

$$\gamma(z) := (\Gamma_2^B|_{\mathfrak{N}_z(B^*)})^{-1} \quad \text{and} \quad M(z) := \Gamma_1^B(\Gamma_2^B|_{\mathfrak{N}_z(B^*)})^{-1}, \qquad z \in \rho(A_2),$$

are called the γ*-field* and the *Weyl function*, respectively, of the dual pair (A, B), corresponding to the boundary triple $\Pi = (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$.

Clearly, the operator functions

$$\gamma^T(z) := (\Gamma_1^A|_{\mathfrak{N}_z(A^*)})^{-1} \quad \text{and} \quad M^T(z) := \Gamma_2^A(\Gamma_1^A|_{\mathfrak{N}_z(A^*)})^{-1}, \qquad z \in \rho(B_1),$$

are the γ-field and the Weyl function, respectively, of the dual pair (B, A), corresponding to the transposed boundary triple $(\mathcal{H}_2 \times \mathcal{H}_1, (\Gamma^B)^T, (\Gamma^A)^T)$. Notice that

$$M^T(z) = M(\bar{z})^*, \qquad z \in \rho(B_1) = \overline{\rho(A_2)}.$$

Let Θ be a linear relation from $\mathcal{H}_1$ to $\mathcal{H}_2$, i.e., a subspace of $\mathcal{H}_1 \times \mathcal{H}_2$, see, e.g., Arens (1961). Consider the restriction A_Θ of B^* to the subspace

$$\text{dom } A_\Theta = \{f \in \text{dom } B^* : \Gamma^B f \in \Theta\}.$$

The following statement describes some spectral properties of the extension A_Θ.

Lemma 2.4. *Let (A, B) be a dual pair in a Hilbert space $\mathfrak{H}$, let $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ be a boundary triple for the dual pair (A, B), let M be the corresponding Weyl function, let Θ be a linear relation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let $z \in \rho(A_2)$. Then the following statements hold:*

(i) *A^*_Θ is the restriction of A^* to*

$$\mathrm{dom}\ A^*_\Theta = \{f \in \mathrm{dom}\ A^* : \Gamma^A f \in \Theta^*\}.$$

(ii) *$z \in \sigma_p(A_\Theta)$ if and only if $0 \in \sigma_p(I_{\mathcal{H}_2} - \Theta M(z))$. In this case*

$$\ker(A_\Theta - zI) = \gamma(z)\ker(I_{\mathcal{H}_2} - \Theta M(z)).$$

(iii) *$z \in \rho(A_\Theta)$ if and only if $0 \in \rho(I_{\mathcal{H}_2} - \Theta M(z))$.*

2.2 Coupling of dual pairs

Theorem 2.5. *Let (A_+, B_+) and (A_-, B_-) be dual pairs in Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, respectively, let $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be a boundary triple for the dual pair $(A_\pm, B_\pm)$, and let $M_\pm$ be the corresponding Weyl function. Denote by A^* and B^* the restrictions of the operators $A^*_+ \oplus A^*_-$ and $B^*_+ \oplus B^*_-$ to the domains*

$$\mathrm{dom}\ A^* = \{g_+ \oplus g_- : g_\pm \in \mathrm{dom}\ A^*_\pm,\ \Gamma_1^{A+} g_+ = \Gamma_1^{A-} g_-\} \tag{2.7}$$

and

$$\mathrm{dom}\ B^* = \{f_+ \oplus f_- : f_\pm \in \mathrm{dom}\ B^*_\pm,\ \Gamma_2^{B+} f_+ = \Gamma_2^{B-} f_-\}, \tag{2.8}$$

respectively. Then the following statements hold:

(i) *The operators $A := (A^*)^*$ and $B := (B^*)^*$ are restrictions of the operators B^* and A^*, respectively, to the domains*

$$\mathrm{dom}\ A = \{f_+ \oplus f_- : f_\pm \in \mathrm{dom}\ B^*_\pm, \Gamma_2^{B+} f_+ = \Gamma_2^{B-} f_- = \Gamma_1^{B+} f_+ + \Gamma_1^{B-} f_- = 0\}, \tag{2.9}$$

$$\mathrm{dom}\ B = \{g_+ \oplus g_- : g_\pm \in \mathrm{dom}\ A^*_\pm, \Gamma_1^{A+} g_+ = \Gamma_1^{A-} g_- = \Gamma_2^{A+} g_+ + \Gamma_2^{A-} g_- = 0\}, \tag{2.10}$$

and (A, B) is a dual pair in $\mathfrak{H}_+ \oplus \mathfrak{H}_-$.

(ii) *The triple $\Pi = (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ with*

$$\Gamma^A g = \begin{pmatrix} \Gamma_1^{A+} g_+ \\ \Gamma_2^{A+} g_+ + \Gamma_2^{A-} g_- \end{pmatrix} \quad \text{and} \quad \Gamma^B f = \begin{pmatrix} \Gamma_1^{B+} f_+ + \Gamma_1^{B-} f_- \\ \Gamma_2^{B+} f_+ \end{pmatrix}, \qquad \begin{matrix} f \in \mathrm{dom}\ B^*, \\ g \in \mathrm{dom}\ A^*, \end{matrix}$$

is a boundary triple for the dual pair (A, B).

(iii) *The Weyl function $M(z)$ corresponding to the boundary triple $\Pi = (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ is given by*

$$M(z) = M_+(z) + M_-(z), \qquad z \in \rho(A_2), \tag{2.11}$$

where A_2 is defined by (2.5).

Proof. The proof of this theorem consists of three parts: (i) and (ii) are established in (a) and (b), and (iii) is proven in (c).

(a) Let $f = f_+ \oplus f_- \in \mathrm{dom}\,(B_+^* \oplus B_-^*)$, $g = g_+ \oplus g_- \in \mathrm{dom}\,(A_+^* \oplus A_-^*)$. Then it follows from the equalities

$$\langle B_+^* f_+, g_+\rangle - \langle f_+, A_+^* g_+\rangle = \langle \Gamma_1^{B+} f_+, \Gamma_1^{A+} g_+\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+\rangle_{\mathcal{H}_2},$$
$$\langle B_-^* f_-, g_-\rangle - \langle f_-, A_-^* g_-\rangle = \langle \Gamma_1^{B-} f_-, \Gamma_1^{A-} g_-\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B-} f_-, \Gamma_2^{A-} g_-\rangle_{\mathcal{H}_2},$$

that

$$\begin{aligned}\langle (B_+^* \oplus B_-^*) f, g\rangle - \langle f, (A_+^* \oplus A_-^*) g\rangle &= \langle \Gamma_1^{B+} f_+, \Gamma_1^{A+} g_+\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+\rangle_{\mathcal{H}_2} \\ &+ \langle \Gamma_1^{B-} f_-, \Gamma_1^{A-} g_-\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B-} f_-, \Gamma_2^{A-} g_-\rangle_{\mathcal{H}_2}.\end{aligned} \tag{2.12}$$

The equality (2.9) follows from (2.12) since the mappings $\Gamma^{A_\pm} : \mathrm{dom}\ A_\pm^* \to \mathcal{H}_1 \times \mathcal{H}_2$ are surjective. Similarly, (2.10) follows from (2.12) since the mappings $\Gamma^{B_\pm} : \mathrm{dom}\ B_\pm^* \to \mathcal{H}_1 \times \mathcal{H}_2$ are surjective.

(b) Next, for $f \in \mathrm{dom}\ B^*$ and $g \in \mathrm{dom}\ A^*$ the equation (2.12) takes the form

$$\langle B^* f, g\rangle - \langle f, (A^*) g\rangle = \langle \Gamma_1^{B+} f_+ + \Gamma_1^{B-} f_-, \Gamma_1^{A+} g_+\rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+ + \Gamma_2^{A-} g_-\rangle_{\mathcal{H}_2}.$$

This proves that (A, B) is a dual pair in $\mathfrak{H}_+ \oplus \mathfrak{H}_-$ and that (ii) holds.

(c) It follows from (2.8) that the γ-field of (A, B) corresponding to the boundary triple Π takes the form

$$\gamma(z) = \gamma_+(z) \oplus \gamma_-(z),$$

where $\gamma_\pm(z)$ are γ-fields of $(A_\pm, B_\pm)$ corresponding to the boundary triples $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$. Now formula (2.11) follows from the definition of the Weyl function, see Definition 2.3. □

Definition 2.6. The dual pair (A, B) constructed in (2.9) and (2.10) is called the *coupling of the dual pairs* (A_+, B_+) *and* (A_-, B_-) relative to the triples

$$(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A_+}, \Gamma^{B_+}) \quad \text{and} \quad (\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^{A_-}, \Gamma^{B_-}).$$

2.3 Real dual pairs and real boundary triples

Let $\mathcal{T}$ be a *conjugation* (time reversal) operator in a Hilbert space $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$, i.e., $\mathcal{T}$ is antilinear, $\mathcal{T}^2 = I_{\mathfrak{H}}$, and

$$\langle \mathcal{T} f, \mathcal{T} g\rangle = \langle g, f\rangle \quad \text{for all} \quad f, g \in \mathfrak{H}.$$

In what follows, we suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide: $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$.

Definition 2.7. Let $\mathcal{T}$ and $j_{\mathcal{H}}$ be conjugations in $\mathfrak{H}$ and $\mathcal{H}$, respectively. A dual pair (A, B) in $\mathfrak{H}$ is called $\mathcal{T}$-*real* if

$$\mathcal{T}\,\mathrm{dom}\ A = \mathrm{dom}\ B \quad \text{and} \quad \mathcal{T} A = B \mathcal{T}. \tag{2.13}$$

A boundary triple $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ for (A, B) is called $(j_{\mathcal{H}}, \mathcal{T})$-*real* if

$$j_{\mathcal{H}} \Gamma_1^B = \Gamma_2^A \mathcal{T} \quad \text{and} \quad j_{\mathcal{H}} \Gamma_2^B = \Gamma_1^A \mathcal{T}.$$

Observe that the conditions (2.13) are clearly equivalent to

$$\mathcal{T} \operatorname{dom} A^* = \operatorname{dom} B^* \quad \text{and} \quad \mathcal{T} A^* = B^* \mathcal{T}.$$

Lemma 2.8. *Let (A, B) be a $\mathcal{T}$-real dual pair and let $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ be a $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triple for (A, B). Then the corresponding Weyl function $M(z)$ satisfies the condition*

$$M(z) = j_{\mathcal{H}} M(z)^* j_{\mathcal{H}}, \qquad z \in \rho(A_2).$$

In what follows we consider a Hilbert space $\mathfrak{H}$ decomposed into an orthogonal sum

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \tag{2.14}$$

of two subspaces $\mathfrak{H}_\pm$ with conjugations $\mathcal{T}_\pm \in \mathcal{L}(\mathfrak{H}_\pm)$. Then the orthogonal sum

$$\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_- \tag{2.15}$$

is a conjugation in $\mathfrak{H}$.

Theorem 2.9. *Let a Hilbert space $\mathfrak{H}$ and a conjugation $\mathcal{T}$ in $\mathfrak{H}$ be such that* (2.14) *and* (2.15) *hold. Moreover, let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in the Hilbert spaces $\mathfrak{H}_\pm$. Finally, with $j_{\mathcal{H}}$ a conjugation in $\mathcal{H}$, let $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triples for $(A_\pm, B_\pm)$, and let*

$$A_0 := A_+ \oplus A_- \quad \textit{and} \qquad B_0 := B_+ \oplus B_-.$$

Then the following statements hold:

(i) *The dual pair (A_0, B_0) is $\mathcal{T}$-real and the boundary triple $((\mathcal{H} \oplus \mathcal{H})^2, \Gamma^{A_0}, \Gamma^{B_0})$ with*

$$\Gamma^{A_0} = \Gamma^{A_+} \oplus \Gamma^{A_-} \quad \textit{and} \quad \Gamma^{B_0} = \Gamma^{B_+} \oplus \Gamma^{B_-}$$

is $(j_{\mathcal{H}\oplus\mathcal{H}}, \mathcal{T})$-real, where $j_{\mathcal{H}\oplus\mathcal{H}} := j_{\mathcal{H}} \oplus j_{\mathcal{H}}$.

(ii) *The coupling (A, B) of the dual pairs (A_+, B_+) and (A_-, B_-), constructed in* (2.9) *and* (2.10) *is $\mathcal{T}$-real.*

(iii) *The boundary triple $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ from Theorem* 2.5 *is $(j_{\mathcal{H}}, \mathcal{T})$-real.*

2.4 Parity and $\mathcal{P}$-selfadjoint operators

Definition 2.10. Let $\mathcal{H}_\pm$ be Hilbert spaces and $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. An operator $\mathcal{P} \in \mathcal{L}(\mathfrak{H})$ will be called an (abstract) *parity* operator if

$$\mathcal{P} = \mathcal{P}^*, \quad \mathcal{P}^2 = I_{\mathfrak{H}}, \quad \text{and} \quad \mathcal{P}\mathfrak{H}_\pm = \mathfrak{H}_\mp.$$

Now consider a Hilbert space $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ with a parity operator $\mathcal{P}$ and a conjugation $\mathcal{T} \in \mathcal{L}(\mathfrak{H})$, such that

$$\mathcal{T}\mathcal{P} = \mathcal{P}\mathcal{T} \quad \text{and} \quad \mathcal{T}\mathfrak{H}_\pm = \mathfrak{H}_\pm. \tag{2.16}$$

The conditions (2.16) mean that the operator $\mathcal{T}$ admits the representation as an orthogonal sum $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$ of two conjugations $\mathcal{T}_+$ and $\mathcal{T}_-$ in Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, respectively.

Lemma 2.11. *Let $\mathcal{P}$ be a parity operator in $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ and let $\mathcal{T}$ be a conjugation in $\mathfrak{H}$ such that* (2.16) *holds. Let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in the Hilbert spaces $\mathfrak{H}_\pm$, such that*

$$\mathcal{P}A_+ = B_-\mathcal{P} \quad \textit{and} \quad \mathcal{P}B_+ = A_-\mathcal{P}. \tag{2.17}$$

Then the following statements hold:

(i) $\mathcal{P}\mathcal{T}\,\mathrm{dom}\ A_+ = \mathrm{dom}\ A_-$, $\mathcal{P}\mathcal{T}\,\mathrm{dom}\ B_+ = \mathrm{dom}\ B_-$, *and*

$$\mathcal{P}\mathcal{T}A_+ = A_-\mathcal{P}\mathcal{T}, \qquad \mathcal{P}\mathcal{T}B_+ = B_-\mathcal{P}\mathcal{T}; \tag{2.18}$$

(ii) $\mathcal{P}\,\mathrm{dom}\ A_+^* = \mathrm{dom}\ B_-^*$, $\mathcal{P}\,\mathrm{dom}\ B_+^* = \mathrm{dom}\ A_-^*$, *and*

$$\mathcal{P}A_+^* = B_-^*\mathcal{P}, \qquad \mathcal{P}B_+^* = A_-^*\mathcal{P}.$$

Proof. (i) Since the dual pairs $(A_\pm, B_\pm)$ are real with respect to $\mathcal{T}_\pm$, one has

$$\mathcal{T}_+A_+ = B_+\mathcal{T}_+, \qquad \mathcal{T}_-A_- = B_-\mathcal{T}_-. \tag{2.19}$$

Let $f_+ \in \mathrm{dom}\ A_+$. Then by (2.19) $\mathcal{T}f_+ \in \mathrm{dom}\ B_+$ and $B_+\mathcal{T}f_+ = \mathcal{T}A_+f_+$. Next by (2.17)

$$\mathcal{P}\mathcal{T}f_+ \in \mathrm{dom}\ A_- \quad \text{and} \quad A_-\mathcal{P}\mathcal{T}f_+ = \mathcal{P}B_+\mathcal{T}f_+ = \mathcal{P}\mathcal{T}A_+f_+.$$

The proofs of the inclusion $\mathcal{P}\mathcal{T}\,\mathrm{dom}\ A_- \subseteq \mathrm{dom}\ A_+$ and of the second equality in (2.18) are similar.

(ii) Applying $\mathcal{P}$ to the left and right of the equalities in (2.17) and using the identity $\mathcal{P}^2 = I_{\mathfrak{H}}$ yields $A_+\mathcal{P} = \mathcal{P}B_-$ and $B_+\mathcal{P} = \mathcal{P}A_-$. From these identities the assertions in (ii) are immediate. □

Definition 2.12. A closed linear operator A in $\mathfrak{H}$ is said to be $\mathcal{P}\mathcal{T}$*-symmetric* if for all $f \in \mathrm{dom}\ A$ we have

$$\mathcal{P}\mathcal{T}f \in \mathrm{dom}\ A \quad \text{and} \quad \mathcal{P}\mathcal{T}Af = A\mathcal{P}\mathcal{T}f.$$

Consider the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ with an indefinite inner product given by

$$[f, g] := \langle \mathcal{P}f, g\rangle_{\mathfrak{H}}. \tag{2.20}$$

For the definition of a Kreĭn space we refer to the books of Azizov & Iokhvidov (1989) and Bognar (1974). Recall that a densely defined linear operator A in $\mathfrak{H}$ is called $\mathcal{P}$*-symmetric* if

$$[Af, g] = [f, Ag] \qquad \text{for all } f,\ g \in \mathrm{dom}\ A.$$

Denote by A^+ the adjoint operator in $(\mathfrak{H}, [\cdot,\cdot])$, i.e., $A^+ = \mathcal{P}A^*\mathcal{P}$. For a $\mathcal{P}$-symmetric operator A one has $A \subseteq A^+$. The operator A is called $\mathcal{P}$*-selfadjoint* if $A = A^+$. The following definition of a boundary triple for the $\mathcal{P}$-symmetric operator A was presented in Derkach (1995).

Definition 2.13. Let $\mathcal{H}$ be an auxiliary Hilbert space and let Γ_1, Γ_2 be linear operators from $\mathrm{dom}\ A^+$ to $\mathcal{H}$. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ is called a *boundary triple for the $\mathcal{P}$-symmetric operator A* if the following conditions are satisfied:

(i) the mapping $\Gamma := \begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}$ from $\mathrm{dom}\ A^+$ to $\mathcal{H}^2$ is surjective;

(ii) the following identity holds for every $f, g \in \mathrm{dom}\ A^+$

$$[A^+f, g] - [f, A^+g] = \langle \Gamma_1 f, \Gamma_2 g\rangle_{\mathcal{H}} - \langle \Gamma_2 f, \Gamma_1 g\rangle_{\mathcal{H}}.$$

In the next theorem we show that the coupling operator A is $\mathcal{P}$-symmetric and $\mathcal{PT}$-symmetric, and describe the set of all $\mathcal{P}$-selfadjoint and $\mathcal{PT}$-symmetric extensions of the operator A.

Theorem 2.14. *Let $\mathcal{P}$ be a parity operator in $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, let $\mathcal{T}$ be a conjugation in $\mathfrak{H}$ such that* (2.16) *holds, and let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in the Hilbert spaces $\mathfrak{H}_\pm$ such that* (2.17) *holds. With $j_{\mathcal{H}}$ a conjugation in $\mathcal{H}$, let $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triples for $(A_\pm, B_\pm)$, such that*

$$\begin{pmatrix}\Gamma_1^{B_+}\\ \Gamma_2^{B_+}\end{pmatrix} f_+ = \begin{pmatrix}\Gamma_2^{A_-}\\ \Gamma_1^{A_-}\end{pmatrix} \mathcal{P}f_+ \quad \text{and} \quad \begin{pmatrix}\Gamma_1^{B_-}\\ \Gamma_2^{B_-}\end{pmatrix} f_- = \begin{pmatrix}\Gamma_2^{A_+}\\ \Gamma_1^{A_+}\end{pmatrix} \mathcal{P}f_-, \quad f_\pm \in \mathrm{dom}\ B_\pm^*. \tag{2.21}$$

Moreover, let (A, B) be the coupling of the dual pairs (A_+, B_+) and (A_-, B_-) given by (2.9), (2.10)*, and let Θ be a linear relation in $\mathcal{H}$. Then the following statements hold:*

(i) *The operator A is $\mathcal{PT}$-symmetric, $\mathcal{P}$-symmetric, and $A^+ = B^*$.*

(ii) *The triple $(\mathcal{H}, \Gamma_1^B, \Gamma_2^B)$ is a boundary triple for the $\mathcal{P}$-symmetric operator A.*

(iii) *The extension A_Θ of the operator A, given by*

$$\mathrm{dom}\ A_\Theta = \left\{ f \in \mathrm{dom}\ B^* : \begin{pmatrix}\Gamma_1 f\\ \Gamma_2 f\end{pmatrix} \in \Theta \right\}, \quad A_\Theta = B^*|_{\mathrm{dom}\ A_\Theta},$$

is $\mathcal{P}$-selfadjoint if and only if $\Theta = \Theta^*$.

(iv) *A_Θ is $\mathcal{PT}$-symmetric if and only if $\Theta = j_{\mathcal{H}}\Theta j_{\mathcal{H}}$.*

3 $\mathcal{PT}$-symmetric Hamiltonians

Here we return to the investigation of the non-Hermitian $\mathcal{PT}$-invariant Hamiltonians presented in the introduction, that is, we study equation (1.1) on the wedge shaped contour Γ, cf. (1.2). By substituting $z(x) := xe^{i\phi\,\mathrm{sgn}\,x}$ into (1.1) one obtains the two differential expressions given by (1.3) and (1.4). Assume that the differential expressions $\mathfrak{a}_\pm$ in (1.4) are in the limit point case at $\pm\infty$. As presented in Section 1, this is the case if and only if the angle ϕ of the wedge satisfies

$$\phi \neq -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \quad \text{for } k = 0, \dots, N+3. \tag{3.1}$$

Then by Leben & Trunk (2019: Lemma 1) the differential expressions $\mathfrak{b}_\pm$ in (1.6) are also in the limit point case at $\pm\infty$. Define the operators $A_\pm$ and $B_\pm$ associated with $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ in $L^2(\mathbb{R}_\pm)$ as

$$A_\pm f_\pm := \mathfrak{a}_\pm[f_\pm] \quad \text{and} \quad B_\pm g_\pm := \mathfrak{b}_\pm[g_\pm] \quad \text{for } f_\pm \in \mathrm{dom}\ A_\pm,\ g_\pm \in \mathrm{dom}\ B_\pm,$$

respectively, with the domains

$$\mathrm{dom}\ A_\pm := \{u_\pm \in L^2(\mathbb{R}_\pm) :\ \mathfrak{a}_\pm[u_\pm] \in L^2(\mathbb{R}_\pm),\ u'_\pm \in AC_{loc}(\mathbb{R}_\pm),\ u_\pm(0_\pm) = u'_\pm(0_\pm) = 0\},$$

$$\mathrm{dom}\ B_\pm := \{v_\pm \in L^2(\mathbb{R}_\pm) :\ \mathfrak{b}_\pm[v_\pm] \in L^2(\mathbb{R}_\pm),\ v'_\pm \in AC_{loc}(\mathbb{R}_\pm),\ v_\pm(0_\pm) = v'_\pm(0_\pm) = 0\}.$$

These operators are in some sense the minimal operators. It follows from Leben & Trunk (2019: Proposition 1 & Theorem 3) that the (maximal) operators $A_\pm^*$ and $B_\pm^*$ are generated by differential expressions in $L^2(\mathbb{R}_\pm)$ where the roles of $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ are switched in the sense that the differential expressions $\mathfrak{a}_\pm$ are now related to $B_\pm^*$ and the differential expressions $\mathfrak{b}_\pm$ are related to $A_\pm^*$:

$$B_\pm^* f_\pm := \mathfrak{a}_\pm[f_\pm] \quad \text{and} \quad A_\pm^* g_\pm := \mathfrak{b}_\pm[g_\pm] \quad \text{for } f_\pm \in \mathrm{dom}\ B_\pm^*,\ g_\pm \in \mathrm{dom}\ A_\pm^*,$$

with

$$\mathrm{dom}\ B_\pm^* := \{u_\pm \in L^2(\mathbb{R}_\pm) :\ \mathfrak{a}_\pm[u_\pm] \in L^2(\mathbb{R}_\pm),\ u'_\pm \in AC_{loc}(\mathbb{R}_\pm)\},$$
$$\mathrm{dom}\ A_\pm^* := \{v_\pm \in L^2(\mathbb{R}_\pm) :\ \mathfrak{b}_\pm[v_\pm] \in L^2(\mathbb{R}_\pm),\ v'_\pm \in AC_{loc}(\mathbb{R}_\pm)\}.$$

Theorem 3.1. *The pairs* (A_-, B_-) *and* (A_+, B_+) *are dual pairs. The triple* $(\mathbb{C}^2, \Gamma^{A_+}, \Gamma^{B_+})$,

$$\Gamma^{B_+} u_+ = \begin{pmatrix} e^{-2i\phi} u'_+(0) \\ u_+(0) \end{pmatrix} \quad \textit{and} \quad \Gamma^{A_+} v_+ = \begin{pmatrix} v_+(0) \\ e^{2i\phi} v'_+(0) \end{pmatrix}, \qquad \begin{matrix} u_+ \in \mathrm{dom}\ B_+^*, \\ v_+ \in \mathrm{dom}\ A_+^*, \end{matrix}$$

is a boundary triple for the dual pair (A_+, B_+). *The triple* $(\mathbb{C}^2, \Gamma^{A_-}, \Gamma^{B_-})$,

$$\Gamma^{B_-} u_- = \begin{pmatrix} -e^{2i\phi} u'_-(0) \\ u_-(0) \end{pmatrix} \quad \textit{and} \quad \Gamma^{A_-} v_- = \begin{pmatrix} v_-(0) \\ -e^{-2i\phi} v'_-(0) \end{pmatrix}, \qquad \begin{matrix} u_- \in \mathrm{dom}\ B_-^*, \\ v_- \in \mathrm{dom}\ A_-^*, \end{matrix}$$

is a boundary triple for the dual pair (A_-, B_-).

Proof. Integration by parts and (Leben & Trunk, 2019: Proposition 1) show

$$\langle A_\pm u_\pm, v_\pm \rangle = \langle u_\pm, B_\pm v_\pm \rangle, \qquad u_\pm \in \mathrm{dom}\ A_\pm, v_\pm \in \mathrm{dom}\ B_\pm.$$

This proves the first statement. It follows from (Leben & Trunk, 2019: Proposition 1) that for $u_+ \in \mathrm{dom}\ B_+^*$ and $v_+ \in \mathrm{dom}\ A_+^*$

$$\begin{aligned} \langle B_+^* u_+, v_+ \rangle - \langle u_+, A_+^* v_+ \rangle &= -e^{-2i\phi} \int_0^\infty u''_+(x) \overline{v_+(x)}\, dx + e^{-2i\phi} \int_0^\infty u_+(x) \overline{v''_+(x)}\, dx \\ &= e^{-2i\phi} (u'_+(0) \overline{v_+(0)} - u_+(0) \overline{v'_+(0)}). \end{aligned}$$

Hence, $(\mathbb{C}^2, \Gamma^{A_+}, \Gamma^{B_+})$ is a boundary triple for the dual pair (A_+, B_+). The statement for the dual pair (A_-, B_-) is shown in the same way. □

Recall that the coupling (A, B) of the dual pairs (A_+, B_+) and (A_-, B_-) consists of a pair of operators $A = (B_+^* \oplus B_-^*)|_{\mathrm{dom}\ A}$ and $B = (A_+^* \oplus A_-^*)|_{\mathrm{dom}\ B}$ with the domains

$$\mathrm{dom}\ A = \{u_+ \oplus u_- : u_\pm \in \mathrm{dom}\ B_\pm^*,\ u_+(0) = u_-(0) = e^{-2i\phi} u'_+(0) - e^{2i\phi} u'_-(0) = 0\}, \tag{3.2}$$
$$\mathrm{dom}\ B = \{u_+ \oplus u_- : u_\pm \in \mathrm{dom}\ A_\pm^*,\ u_+(0) = u_-(0) = e^{2i\phi} u'_+(0) - e^{-2i\phi} u'_-(0) = 0\}, \tag{3.3}$$

see Theorem 2.5.

We define the parity $\mathcal{P}$ and time reversal $\mathcal{T}$ as in (1.5). The parity $\mathcal{P}$ gives rise to a new inner product $[\cdot,\cdot] = \langle \mathcal{P}\cdot,\cdot\rangle$ (see also (2.20)), which was considered in many papers, we mention only (Mostafazadeh, 2010). It is easy to see that the parity $\mathcal{P}$ and the time reversal $\mathcal{T}$ satisfy (2.16), where $\mathfrak{H}_\pm := L^2(\mathbb{R}_\pm)$. Due to Theorem 2.14, the operator A is $\mathcal{PT}$-symmetric and $\mathcal{P}$-symmetric in the Kreĭn space $(L^2(\mathbb{R}),[\cdot,\cdot]) = (L^2(\mathbb{R}_-)\oplus L^2(\mathbb{R}_+),[\cdot,\cdot])$. The (Kreĭn space) adjoint A^+ of A coincides with $B^* = (B_+^* \oplus B_-^*)|_{\mathrm{dom}\, B^*}$, where

$$\mathrm{dom}\ B^* = \{u_+ \oplus u_- : u_\pm \in \mathrm{dom}\ B_\pm^*,\ u_+(0) = u_-(0)\}.$$

An application of Theorem 2.14 gives a one-parameter family $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ of $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A in the Kreĭn space $(L^2(\mathbb{R}),[\cdot,\cdot])$. This is the main result of this note.

Theorem 3.2. *Let the angle ϕ satisfies* (3.1) *and let A be the coupling operator constructed in* (3.2)*. Then the following statements are true:*

(i) *A boundary triple $(\mathbb{C},\Gamma_1,\Gamma_2)$ for the $\mathcal{P}$-symmetric operator A is given by*

$$\Gamma_1 u = e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) \quad \textit{and} \quad \Gamma_2 u = u_+(0), \quad u = u_+\oplus u_- \in \mathrm{dom}\ B^*.$$

(ii) *The extension H_α of the operator A, defined as a restriction of A^+ to the domain*

$$\mathrm{dom}\ H_\alpha = \left\{u_+\oplus u_- \in \mathrm{dom}\ B^* : e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = \alpha u_+(0)\right\},$$

is $\mathcal{P}$-selfadjoint if and only if $\alpha\in\mathbb{R}$.

(iii) *H_α is $\mathcal{PT}$-symmetric if and only if $\alpha\in\mathbb{R}$.*

Proof. By construction the dual pairs (A_+,B_+) and (A_-,B_-) are $\mathcal{T}_\pm$-real and the parity operator $\mathcal{P}$ intertwines the operators A_+, B_- and A_-, B_+, that is, (2.17) holds. Moreover, the boundary triples $(\mathbb{C}^2,\Gamma^{A_+},\Gamma^{B_+})$ and $(\mathbb{C}^2,\Gamma^{A_-},\Gamma^{B_-})$ are also $(j_{\mathbb{C}},\mathcal{T})$-real and satisfy the condition (2.21). Here $j_{\mathbb{C}}$ stands for the usual complex conjugation in $\mathbb{C}$. Hence, all assumptions in Theorem 2.14 are satisfied and the statements in Theorem 3.2 follow directly from Theorem 2.14. □

In Leben & Trunk (2019) only the extension for the parameter value $\alpha = 0$ was considered. More precisely, there it was shown that H_0 is an extension of A with domain

$$\mathrm{dom}\ H_0 = \left\{u_+\oplus u_- : u_\pm \in \mathrm{dom}\ B_\pm^*,\ u_+(0)-u_-(0) = e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = 0\right\}$$

which is $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint. The family H_α, $\alpha\in\mathbb{R}$, of extensions obtained in Theorem 3.2 is in some sense an analog of the δ-interaction for the differential operation $\mathfrak{a}$.

Acknowledgements: The research of V.D. was supported by the German Research Foundation (DFG), grant TR 903/22-1. The research of Ph.S. has been supported by DFG, grant TR 903/19-2.

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