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Pricing Cumulative Loss Derivatives Under Additive Models via Malliavin Calculus

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ABSTRACT: We show that the integration by parts formula based on Malliavin-Skorohod calculus techniques for additive processes helps us to compute quantities like $\mathbb{E}(L_T h(L_T))$, or more generally $\mathbb{E}(H(L_T))$, for different suitable functions h or H and different models for the cumulative loss process L. These quantities are important in Insurance and Finance. For example they appear in computing expected shortfall risk measures or prices of stop-loss contracts. The formulas given in the present paper generalize the formulas given in a recent paper by Hillairet, Jiao and Réveillac (HJR). In the HJR paper, despite the use of advanced models, including the Cox process, the treatment of the formulas is based only on Malliavin calculus techniques for the standard Poisson process, a particular case of additive process. In the present paper, Malliavin calculus techniques for additive processes are used, more general results are obtained and proofs appears to be shorter.

Key Words: Insurance derivatives, Risk measures, Loss process, Malliavin-Skorohod calculus, Additive processes, Lévy processes.

Contents

1. Introduction

The cumulative loss process is the main process in insurance modeling. It is described as a process $L := \{L_t, t \geq 0\}$ such that $L_0 = 0$ a.s. and

$$L_t := \sum_{i=1}^{N_t} Y_i,$$

where N is a counting process that describes the arrival of claims and Y_i, for $i \geq 1$, are positive random variables that describe the size or the amount of claims. So, cumulative loss processes are pure jump processes, null at the origin, and with increasing trajectories. General references for the importance of the cumulative loss process in Insurance are for example [13], [8], [12], [3] and [2].

The most simple case of cumulative loss process is the so-called Cramér-Lundberg model, where L is assumed to be a time homogeneous Compound Poisson process that corresponds with the case where N is a standard Poisson process with intensity $\lambda > 0$ and the random variables Y_i are independent and identically distributed with a certain probability law defined on $(0, \infty)$ and independent of process N.

Of course, these hypotheses can be generalized. In the present paper we mainly consider two generalizations. First of all, it is usual to assume N is a time inhomogeneous Poisson process with non constant

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intensity given by a positive function λ defined on $[0, \infty)$. But more generally, we can assume N is a Cox process, that corresponds with the assumption that the intensity is random and given by a stochastic process with positive trajectories $\lambda = \{\lambda_s, s \geq 0\}$. On other hand, the basic Cramér-Lundberg model assumes that claim sizes are independent of the claim arrivals; here we will include possible dependencies between the intensity process, the jump times and the jump sizes.

The different models of cumulative loss process considered in the present paper will be included in the class of pure jump additive processes or in the class of conditionally pure jump additive processes. As we will see, integration by parts formulas for these type of processes can be useful to compute expectations related with the cumulative loss process under different frameworks.

Many contracts in insurance and reinsurance are written on the cumulative loss process. In general their payoff can be described as $H(L_T)$ where T denotes the maturity time of the contract and H is a positive measurable function. So, the current price of the contract is given by $e^{-rT}\mathbb{E}(H(L_T))$ under a certain probability measure and assuming a constant interest rate $r \geq 0$.

Following [10], we describe some examples of payoffs of type $H(L_T)$. The stop-loss contract, for example, is an important tool in the risk management of an insurance company. It gives protection against losses which are larger than a certain quantity thanks to a re-insurer that plays the role of counterpart. For example, in a typical contract the re-insurer pays an amount of money if the loss process exceeds a certain quantity K but with a maximum quantity $M > K$, that is,

$$H(L_T) = (L_T - K)\mathbb{1}_{(K,M]}(L_T) + (M - K)\mathbb{1}_{(M,\infty)}(L_T).$$

In this case, the computation of the expectation reduces to

$$\mathbb{E}(H(L_T)) = \mathbb{E}[L_T \mathbb{1}_{(K,M]}(L_T)] - K\mathbb{P}(L_T \in (K, M]) + (M - K)\mathbb{P}(L_T > M).$$

Here, the main problem is to compute the term

$$\mathbb{E}[L_T \mathbb{1}_{(K,M]}(L_T)]$$

that is a particular case of $\mathbb{E}(L_T h(L_T))$ for a certain positive function h.

Similar computations appear, for example, in the treatment of collateralized debt obligations (CDOs), where tranches are defined with different K_i and M_i.

Another source of examples is risk measures. The most famous risk measure is Value at Risk V_α. It is defined as the α-quantile of $-L_T$ for some prescribed level $\alpha \in (0, 1)$ with a change of sign. That is, $V_\alpha(-L_T) := -\inf\{x : F(x) > \alpha\}$ where F is the cumulative probability function of $-L_T$. Note that being L_T a positive quantity this implies that $V_\alpha(-L_T)$ is also a positive quantity.

A very useful risk measure in risk control is the so called expected shortfall. See for example [9]. It is defined as

$$ES_\alpha(-L_T) := -\mathbb{E}[-L_T \mid -L_T \leq -V_\alpha(-L_T)].$$

Note that we can write

$$ES_\alpha(-L_T) = \frac{\mathbb{E}[L_T \mathbb{1}_{\{L_T \geq \beta\}}]}{\mathbb{P}(L_T \geq \beta)},$$

where $\beta = V_\alpha(-L_T)$. So, another time, the computation of a quantity like $\mathbb{E}[L_T \mathbb{1}_{\{L_T \geq \beta\}}]$ becomes crucial.

In general we can say that the computation of a quantity as $\mathbb{E}(L_T h(L_T))$ for a certain positive function h is of crucial importance in managing risk in Finance and Insurance.

As it is explained in [10], in many cases, a more general situation is of interest. For example, we can be interested in the problem to compute $\mathbb{E}(\hat{L}_T h(L_T))$, where $\hat{L}_T := \sum_{i=1}^{N_T} \hat{Y}_i$ and the quantities $\hat{Y}_i$ are different from the quantities Y_i. This is what happens when L_T determines the activation of the contract but the true claim is given by the quantities $\hat{Y}_i$. In the CDOs, for example, the recovery rate is not necessary equal to the real loss. Frequently, $\hat{Y}_i$ is a deterministic function of Y_i, but not always. To cover this type of problems in full generality we consider, in the present paper, pure jump additive processes taking values in $\mathbb{R}^d$, despite in the majority of applications, the case $d = 1$ will be enough.

The purpose of this paper is to find integration by parts formulas, based on Malliavin-Skorohod calculus techniques, that help us to compute quantities like $\mathbb{E}(H(L_T))$ for different suitable functions H and different models for L_T. This has been done in [10] for models with arrivals described by a Cox process, using the Malliavin-Skorohod calculus framework for the standard Poisson process. In our paper we use the Malliavin-Skorohod calculus framework for additive processes developed in [5] and we obtain more general results with shorter proofs.

In Section 2 we describe a general additive model for the cumulative loss process L. Following [5], in Section 3, we recall the Malliavin-Skorohod framework suitable for this type of processes. In Section 4 we give formulas for computing $\mathbb{E}(L_T h(L_T))$ or $\mathbb{E}(H(L_T))$ under different scenarios. Concretely, Theorem 4.7 in the present paper generalizes Theorem 3.6 in [10]. Finally Section 5 is devoted to conclusions.

2. Models for the cumulative loss process

2.1. Pure jump additive processes

In this subsection we recall the basic elements of the theory of additive processes taking values in $\mathbb{R}^d$. In the majority of possible applications $d = 1$ would be enough, but, as we have commented in the Introduction, to treat some problems, we have to consider different jump amplitudes for any jump instant to describe different amounts of money for the same claim. General results about additive processes can be found, for example, in [4] and [15].

Let $A := \{A_t, t \geq 0\}$ be a stochastic process taking values in $\mathbb{R}^d$ and defined in a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Recall that $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ is the completed natural filtration associated to process A. Denote by $\mathbb{E}$ and $\mathbb{V}$ respectively, the expectation and the variance associated to $\mathbb{P}$.

Set $\mathbb{R}^d_0 := \mathbb{R}^d - \{0\}$. Denote by $||\cdot||$ the Euclidean norm in $\mathbb{R}^d$ and by $|\cdot|$ the corresponding norm for the case $d = 1$. For any $\epsilon > 0$ define the sets $S_\epsilon := \{x \in \mathbb{R}^d : ||x|| > \epsilon\}$. Let us denote by $\mathcal{B}$ and $\mathcal{B}_0$ the $\sigma-$algebras of Borel sets of $\mathbb{R}^d$ and $\mathbb{R}^d_0$ respectively.

It is said that process A is an additive process if it satisfies the following conditions:

- It is null at the origin, that is, $A_0 = 0$ a.s.
- It has independent increments, that is, for any n and any $0 \leq t_1 \leq \cdots \leq t_n$, the random vectors $A_{t_i} - A_{t_{i-1}}$ are independent.
- It has right continuous trajectories with left limits, a.s.
- It is stochastically continuous, that is, for any $c > 0$ and $t \geq 0$ fixed, and for $\delta \in \mathbb{R}$,

$$\lim_{\delta \to 0} \mathbb{P}(||A_{t+\delta} - A_t|| > c) = 0.$$

It is well-known that any additive process can be characterized by the triplet $(\Gamma_t, \Sigma^2_t, \nu_t)$ where

- Γ is a continuous function null at the origin taking valued in $\mathbb{R}^d$.
- Σ^2_t is a continuous function null at the origin taking values in the space of symmetric and non-negative definite matrices of order d.
- $\{\nu_t, t \geq 0\}$ is a set of Lévy measures on $\mathbb{R}^d$ such that for any set $B \in \mathcal{B}_0$ such that $B \subseteq S_\epsilon$ for a certain ϵ, $\nu_\cdot(B)$ is a continuous and increasing function null at the origin. Recall that ν_t is a Lévy measure if it is a positive measure null at the origin and $\int_{\mathbb{R}^d}(1 \wedge ||x||^2)\nu_t(dx) < \infty$.

If we assume, in addition, stationarity of the increments, A becomes a Lévy process, and the functions of the triplet become linear, that is, we have the triplet $(\gamma_L t, \Sigma^2_L t, \nu_L t)$ for a certain triplet $(\gamma_L, \Sigma^2_L, \nu_L)$ that fully characterize process A. In this case, γ_L is a real vector, Σ_L is a symmetric and non-negative definite matrix and ν_L is a Lévy measure on $\mathbb{R}^d$.

Define $\Theta := [0,\infty) \times \mathbb{R}^d$. Let us denote by $\theta = (s,x)$ the elements of Θ. For any $T \geq 0$ and $\epsilon > 0$ we introduce the sets $\Theta_{T,\epsilon} := [0,T] \times S_\epsilon$ with its corresponding Borel $\sigma-$algebras $\mathcal{B}_{T,\epsilon}$. Note that $\Theta_{\infty,0} := [0,\infty) \times \mathbb{R}_0^d$ and $\Theta = \Theta_{\infty,0} \cup ([0,\infty) \times \{0\})$. When necessary, we identify $[0,\infty)$ with $[0,\infty) \times \{0\}$.

We introduce a measure ν on $\Theta_{\infty,0}$ such that for any $B \in \mathcal{B}_0$ we define $\nu([0,t] \times B) = \nu_t(B)$. The hypotheses on ν_t guarantee that ν is $\sigma-$finite and continuous, that is, $\nu(\{t\} \times B) = 0$ for any $t \geq 0$ and $B \in \mathcal{B}_0$. In particular, note that ν is diffuse on $\Theta_{\infty,0}$ and a Radon measure, that is, it is finite on compacts subsets of $\Theta_{\infty,0}$. Moreover, the Lévy character of measures ν_t guarantee that for any $\delta > 0$, $\nu([0,t] \times \{||x|| > \delta\}) < \infty$.

Given $C \in \mathcal{B}_{\infty,0}$ we introduce the jump measure N associated to A defined as

$$N(C) := c\{t : (t, A_t - A_{t-}) \in C\}$$

where c denotes the cardinal.

It is well-known that N is a Poisson random measure on $\mathcal{B}_{\infty,0}$ with

$$\mathbb{E}(N(C)) = \mathbb{V}(N(C)) = \nu(C).$$

Moreover we define the compensated Poisson measure $\tilde{N}(dt,dx) := N(dt,dx) - \nu(dt,dx)$.

The Lévy-Itô decomposition allows us to write

$$A_t = \Gamma_t + G_t + J_t$$

where Γ is a continuous function null at the origin and G and J are two independent additive processes with triplets $(0, \Sigma_t^2, 0)$ and $(0,0,\nu_t)$ respectively. That is, G is a centered Gaussian process with continuous trajectories and covariance function $\Sigma_{s\wedge t}^2$ and J is a pure jump additive process that can be represented as

$$J_t = \int_{\Theta_{t,0}-\Theta_{t,1}} x\tilde{N}(ds,dx) + \int_{\Theta_{t,1}} xN(ds,dx),$$

where the first integral has to be understood as the almost surely limit

$$\int_{\Theta_{t,0}-\Theta_{t,1}} x\tilde{N}(ds,dx) = \lim_{\epsilon\downarrow 0} \int_{\Theta_{t,\epsilon}-\Theta_{t,1}} x\tilde{N}(ds,dx).$$

Recall that the limit is uniform with respect to t on every bounded interval.

In order to model cumulative loss processes we restrict our analysis to the family of additive processes with piecewise constant trajectories. This is equivalent to assume that $\Sigma \equiv 0$,

$$\int_{\Theta_{t,0}-\Theta_{t,1}} ||x||\nu(ds,dx) < \infty$$

and $\Gamma_t = \int_{\Theta_{t,0}-\Theta_{t,1}} x\nu(ds,dx)$.

In this case,

$$J_t = \int_{\Theta_{t,0}} xN(ds,dx)$$

is a pure jump additive process, with piecewise constant trajectories, defined on $\Theta_{\infty,0} := [0,\infty) \times \mathbb{R}_0^d$, taking values in $\mathbb{R}^d$ and determined measure ν. This process J will be the main object of interest in the rest of the paper. Note that it is a pure jump additive process with finite or infinite activity, but with finite variation trajectories.

Recall also that

$$\mathbb{E}(J_t) = \mathbb{E}(\int_0^t \int_{\mathbb{R}_0^d} xN(ds,dx)) = \int_0^t \int_{(0,\infty)^d} x\nu(ds,dx)$$

and

$$\mathbb{V}(J_t) = \mathbb{V}(\int_0^t \int_{\mathbb{R}_0^d} xN(ds,dx)) = \int_0^t \int_{(0,\infty)^d} x^2 \nu(ds,dx),$$

provided the right hand side integrals are well defined.

These formulas describes clearly the role of measure ν.

A particular and very relevant case is the finite activity case, that is, when measures ν_t are finite for every t. In this case, if we define $\Lambda_t := \nu([0,t] \times (0, \mathbb{R}_0^d)$, that is finite, we can write, for any $B \in \mathcal{B}_0$,

$$\nu([0,t] \times B) = \Lambda_t Q_t(B)$$

with

$$Q_t(B) := \frac{\nu([0,t] \times B)}{\nu([0,t] \times (0, \mathbb{R}_0^d)}.$$

Note that $N_{\Lambda_t}(\omega) := N([0,t] \times \mathbb{R}_0^d, \omega)$ computes the number of jumps in $[0,t]$ and it is a Poisson process with cumulative intensity Λ_t. A more particular case is the case $Q_t = Q$ for any $t \geq 0$. In this case the process is a time-inhomogeneous compound Poisson process with cumulative intensity Λ and jump sizes given by the law Q.

In Insurance, typically, losses are positive and the cumulative loss process L has increasing trajectories. This can be described assuming ν is concentrated in $[0,\infty) \times (0,\infty)^d$. Therefore, J takes values in $[0,\infty)^d$. To restrict to this case we simply have to change $\mathbb{R}_0^d$ by $(0,\infty)^d$ everywhere in the previous explanations. Pure jump additive processes with finite variation increasing trajectories are also called subordinators in the literature. In all the examples of the present paper we will assume J is a subordinator.

2.2. Poisson integral processes

In fact, pure jump additive processes J introduced above, can be put in a slightly more general framework. I call them Poisson integral processes. See chapters 7 and 8 of [12] or Chapter 2 of [4] for more information about this point of view.

Consider E a finite-dimensional euclidean space with the corresponding Borel $\sigma-$algebra $\mathcal{B}(E)$. Typically, in our case, $E \subseteq \mathbb{R}^k$ for a certain $k \geq 1$. Consider a diffuse and Radon measure μ on E. Note that the measure ν introduced in Subsection 2.1 is a diffuse and Radon measure on $[0,\infty) \times \mathbb{R}_0^d$. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measure μ with the above conditions we can define a Poisson random measure N on $\Omega \times \mathcal{B}(E)$ that is an integer-valued measure such that for any $C \in \mathcal{B}(E)$ with $\mu(C) < \infty$, $N(C)$ is a Poisson random variable with mean $\mu(C)$ and for any $C_1, C_2, \ldots, C_m$ disjoint sets, the random variables $N(C_1), \ldots, N(C_m)$ are mutually independent.

Given a Poisson random measure N with mean measure μ and a measurable function g we can define $\int_E g dN$ provided suitable integrability conditions. In the case $E = [0,\infty) \times \mathbb{R}_0^d$ we can define the Poisson integral process:

$$J_t(g) := \int_0^t \int_{\mathbb{R}_0^d} g(s,x) N(ds,dx).$$

It is well-known that this integral is finite a.s. if and only if

$$\int_0^t \int_{\mathbb{R}_0^d} (|g(s,x)| \wedge 1) \mu(ds,dx) < \infty.$$

A particular important case is the case $g \equiv 1$. In this case the following generalization of the Poisson process:

$$N_t = J_t(1) := \int_0^t \int_{\mathbb{R}_0^d} N(ds,dx)$$

that is the number of jumps of any size in $[0,t]$ that is well defined provided

$$\int_0^t \int_{\mathbb{R}_0^d} \mu(ds,dx) = \mu([0,t]\times\mathbb{R}_0^d) < \infty.$$

Note that if

$$\int_0^t \int_{\mathbb{R}_0^d} |g(s,x)|\mu(ds,dx) < \infty,$$

then,

$$\mathbb{E}(J_t(g)) = \int_0^t \int_{\mathbb{R}_0^d} g(s,x)\mu(ds,dx).$$

And if

$$\int_0^t \int_{\mathbb{R}_0^d} \max\{|g(s,x)|^2, |g(s,x)|\}\mu(ds,dx) < \infty,$$

then,

$$\mathbb{V}(J_t(g)) = \int_0^t \int_{\mathbb{R}_0^d} g(s,x)^2\mu(ds,dx).$$

In particular, if $\mu(E) < \infty$, we can write

$$J(g) := \int_E g(s,x)N(ds,dx) = \sum_{i=1}^M Z_i$$

where M is a Poisson random variable with intensity $\mu(E)$ and Z_i are independent and identically distributed random variables with law

$$Q(Z \in B) = \frac{\mu\{(s,x) : g(s,x) \in B\}}{\mu(E)},\ B \in \mathcal{B}(\mathbb{R}).$$

The case $E = [0,T]$ and $\mu = \ell$ where ℓ is the Lebesgue measure, is the case of the standard Poisson process. And the case $E = [0,T]\times\mathbb{R}_0$ with $\mu = \ell\times\rho$ where ρ is a finite measure on on $\mathbb{R}_0$ is the case of an homogeneous compound Poisson process with intensity $T\rho(\mathbb{R}_0)$ and jumps of law Q.

Note also that we can generalize the idea to the case $\mu([0,t]\times\mathbb{R}_0^d)$ finite for any $t \geq 0$. In this case M and Q depend on t. M is a Poisson random process with intensity $\mu([0,t]\times\mathbb{R}_0^d)$ and

$$Q_t(Z \in B) = \frac{\mu\{(s,x) : s \leq t, g(s,x) \in B\}}{\mu([0,t]\times\mathbb{R}_0^d)},\ B \in \mathcal{B}(\mathbb{R}).$$

The particular cases seen before correspond now with $E = [0,\infty)$ and $E = [0,\infty)\times\mathbb{R}_0$.

2.3. Modeling the cumulative loss process

In this subsection we show how pure jump additive process with finite variation increasing trajectories, or more generally, Poisson integral processes

$$J_t(g) := \int_0^t \int_{(0,\infty)^d} g(s,x)N(ds,dx)$$

are a suitable framework for modeling a cumulative loss processes.

Recall that N is a Poisson random measure with mean measure given by a diffuse Radon measure ν. Assume moreover that $\nu([0,t]\times(\delta,\infty)) < \infty$ for any $t \geq 0$, $\delta > 0$ and g a non-negative function.

Consider a time inhomogeneous cumulative loss processes whose claim arrivals follow an inhomogeneous Poisson process with deterministic cumulative intensity Λ and whose claim amounts are independent and identically distributed random variables X_i with law Q. Recall that this is the basic model in Insurance as presented for example in chapters 2 and 7 of [12]. Take $d = 1$ for simplicity. We can write

$$L_t = \sum_{i=1}^{N_{\Lambda_t}} X_i = \int_0^t \int_0^\infty x N(ds, dx)$$

where in this case, $\nu(ds, dx) = d\Lambda_t \times Q(dx)$.

A particular but quite general case is the case

$$\Lambda_t := \int_0^t \lambda(s) ds$$

where the intensity process λ is assumed to be an a.e. strictly positive and locally integrable measurable function defined on $[0, \infty)$. If moreover Q has a density q we can write

$$\nu(ds, dx) = \lambda(s) q(x) ds dx$$

on $[0, \infty) \times (0, \infty)$.

Concretely, the classical Cramér-Lundberg model corresponds with the case $d = 1$ and $\lambda(s) = \lambda$ for a constant $\lambda > 0$.

A more general and useful situation, also described in Chapter 1 of [12], is

$$L_t = \sum_{i=1}^{N_{\Lambda_t}} f(T_i, X_i) = \int_0^t \int_{(0,\infty)} f(s, x) N(ds, dx)$$

where f is a deterministic measurable function from $(0, \infty)^2$ to $[0, \infty)$. A typical and simple example where claim amounts depend on jump times is the discounted cumulative loss process where $f(T_i, X_i) = e^{-rT_i} X_i$. Another typical case is shot noise, see also Chapter 1 of [12], where

$$S_t := \sum_{i=1}^{N_t} e^{-\theta(t-T_i)} X_i = e^{-\theta t} \sum_{i=1}^{N_t} e^{\theta T_i} X_i = e^{-\theta t} \int_0^t \int_0^\infty e^{\theta s} x N(ds, dx)$$

where N is a standard Poisson process with intensity λ and $\nu(ds) = \lambda ds Q(dx)$. Here $f(T_i, X_i) = e^{\theta T_i} X_i$.

So, in general, naturally, we will consider as a model of a cumulative loss process the integral

$$L_t = \int_0^t \int_{(0,\infty)^d} f(s, x) N(ds, dx)$$

with N a Poisson random measure with mean measure ν under the established conditions previously. Note that this model is very general and includes dependency between jumps times and jump sizes.

Of course, a more general situation, that will be considered later, is to describe the cumulative intensity function Λ by a stochastic process. This is the case where N become a Cox process, also called double stochastic Poisson process. In this case, it is possible to see the cumulative loss process L as a conditional pure jump additive process or a conditional Poisson integral process, conditioned to the cumulative intensity process.

3. Integration by parts formula for pure jump additive processes

Malliavin-Skorohod calculus for processes with jumps, concretely for Lévy processes, was introduced for the first time in [11]. A good reference for Malliavin-Skorohod calculus for Lévy processes is [6]. The development of this theory for additive processes has been introduced in [18]. In this subsection we follow closely the point of view established in [5].

Consider a pure jump additive process

$$J_t = \int_0^t \int_{\mathbb{R}_0^d} xN(ds, dx)$$

with intensity ν as defined before. We can identify canonically the trajectories of J as elements ω of the set Ω^J, introduced in [16], and defined as the set of finite or infinite sequences of pairs $\theta_i = (s_i, x_i) \in (0, \infty) \times \mathbb{R}_0^d$ such that for any m only a finite number of them belong to $\Theta_m := [0, m] \times S_{\frac{1}{m}}$. This includes, in particular, the no jump trajectory, that we denote by α. Recall that J is defined on its natural filtered probability space $(\Omega^J, \mathcal{F}^J, \mathbb{F}^J, \mathbb{P}^J)$. A detailed construction can be found in [5]. Recall in particular that the sets of $\mathcal{F}^J$ are the anti-images of the canonical projections on sets $\Theta_{T,\epsilon}$ of symmetric in time sets of $\mathcal{B}(\Theta_{T,\epsilon})$.

We can define two families of transformations on Ω^J. They coincide with the transformations introduced in [14]. A creation transformation

$$\varsigma_\theta^+ \omega := ((s, x), (s_1, x_1,), \ldots, (s_n, x_n), \ldots),$$

that adds a jump $\theta = (s, x)$ to a trajectory ω and an annihilation transformation

$$\varsigma_\theta^- \omega := ((s_1, x_1), (s_s, x_2), \ldots) - \{(s, x)\}$$

that takes away a jump $\theta = (s, x)$ from a trajectory ω provided it is in the trajectory.

These two transformations are well defined. Note that ς^+ is well defined except on the set $\{(\theta, \omega) : \theta \in \omega\}$ that has null $\nu \otimes \mathbb{P}$ measure. On this set we define $\varsigma^+\omega = \omega$. In the case of ς^-, this operator is the identity except on the same set $\{(\theta, \omega) : \theta \in \omega\}$.

Consider $L^0(\Omega^J)$ the set of random variables on Ω^J and $L^0(\Theta_{\infty,0} \times \Omega^J)$ the set of measurable processes indexed by $\Theta_{\infty,0}$. Following [5] we introduce the following operators $\mathcal{T}$ and $\mathcal{S}$.

Definition 3.1. *Given $F \in L^0(\Omega^J)$ we define*

$$\mathcal{T} : L^0(\Omega^J) \longrightarrow L^0(\Theta_{\infty,0} \times \Omega^J)$$

such that

$$(\mathcal{T}F)(\theta, \omega) := F(\varsigma_\theta^+ \omega).$$

Operator $\mathcal{T}$ is a closed linear operator defined on the entire $L^0(\Omega^J)$. Note that $F = 0$ implies $\mathcal{T}F = 0$.

Definition 3.2. *Given a process $u \in L^0(\Theta_{\infty,0} \times \Omega^J)$ we define*

$$\mathcal{S} : Dom\mathcal{S} \subseteq L^0(\Theta_{\infty,0} \times \Omega^J) \longrightarrow L^0(\Omega^J)$$

such that

$$(\mathcal{S}u)(\omega) := \int_{\Theta_{\infty,0}} u_\theta(\varsigma_\theta^- \omega) N(d\theta, \omega) = \sum_i u_{\theta_i}(\varsigma_{\theta_i} \omega)$$

and

$$(\mathcal{S}u)(\alpha) = 0.$$

The domain of the operator $\mathcal{S}$, $Dom\mathcal{S}$, is the set of processes $u \in L^0(\Theta_{\infty,0} \times \Omega^J)$ such that $\sum_i |u_{\theta_i}(\varsigma_{\theta_i} \omega)| < \infty$.

Recall that $L^1(\Theta_{\infty,0})$, $L^1(\Omega^J)$ and $L^1(\Theta_{\infty,0} \times \Omega^J)$ denote respectively the spaces of integrable functions with respect measures ν, $\mathbb{P}$ and $\nu \times \mathbb{P}$.

It can be seen that $L^1(\Theta_{\infty,0} \times \Omega^J) \subseteq Dom\mathcal{S}$. In fact, it is proved in [5] that $\mathcal{S}$ is a well defined closed operator from $L^1(\Theta_{\infty,0} \times \Omega^J)$ to $L^1(\Omega^J)$. We also have, for $u \in L^1(\Theta_{\infty,0} \times \Omega^J)$,

$$\mathbb{E}(\mathcal{S}u) = \mathbb{E} \int_{\Theta_{\infty,0}} u_\theta(\omega) \nu(d\theta).$$

Moreover, if u is a predictable process, we have

$$(\mathcal{S}u)(\omega) = \int_{\Theta_{\infty,0}} u_\theta(\omega) N(d\theta, \omega).$$

The main result in relation with operators $\mathcal{T}$ and $\mathcal{S}$ is the following duality relationship or integration by parts formula.

Theorem 3.3. *Assume $F \in L^0(\Omega^J)$ and $u \in Dom\mathcal{S}$. Then $F \cdot \mathcal{S}u \in L^1(\Omega^J)$ if and only if $\mathcal{T}F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ and in this case*

$$\mathbb{E}[F \cdot \mathcal{S}u] = \mathbb{E} \int_{\Theta_{\infty,0}} \mathcal{T}_\theta F \cdot u_\theta \, \nu(d\theta).$$

Proof. This Theorem is a straightforward extension, to the $d-$dimensional case, of Theorem 5.6 in [5]. But being a key result in this paper, for the sake of completeness, we repeat here the proof.

Let Ω_m^J be the set of sequences of pairs in Θ_m. Note that all these sequences are finite. Denote by $\omega = (\theta_1, \ldots, \theta_n)$, any of them. Let $\theta \in \Theta_m$. Denote $c_m := e^{-\nu(\Theta_m)}$. We have

$$\begin{aligned}
\mathbb{E}[F \cdot \mathcal{S}u \cdot \mathbb{1}_{\Omega_m^J}] &= \sum_{n=1}^{\infty} \frac{c_m}{n!} \int_{\Theta_m^n} F(\theta_1, \ldots, \theta_n)(\mathcal{S}u)(\theta_1, \ldots, \theta_n)\nu(d\theta_1)\cdots\nu(d\theta_n) \\
&= \sum_{n=1}^{\infty} \frac{c_m}{n!} \int_{\Theta_m^n} F(\theta_1, \ldots, \theta_n) \sum_{i=1}^{n} u_{\theta_i}(\varsigma_{\theta_i}^{-}\omega)\nu(d\theta_1)\cdots\nu(d\theta_n) \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{c_m}{n!} \int_{\Theta_m^n} \mathcal{T}_{\theta_i} F(\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_n) u_{\theta_i}(\varsigma_{\theta_i}^{-}\omega)\nu(d\theta_1)\cdots\nu(d\theta_n) \\
&= \sum_{n=1}^{\infty} n \frac{c_m}{n!} \int_{\Theta_m^{n-1}} \int_{\Theta_m} \mathcal{T}_{\theta_n} F(\theta_1, \ldots, \theta_{n-1}) u_\theta(\varsigma_{\theta_n}^{-}\omega)\nu(d\theta_1)\cdots\nu(d\theta_n) \\
&= \mathbb{E}(\mathbb{1}_{\Omega_m^J} \int_{\Theta_m} \mathcal{T}_\theta F u_\theta \nu(d\theta)).
\end{aligned}$$

Using dominated convergence we extend the result to Ω^J. □

Remark 3.4. *Note that we have only assumed that measure ν is a diffuse and Radon measure on $\Theta_{\infty,0}$ such that it is finite on any Θ_m for $m \geq 1$. Therefore any Poisson integral process J with an intensity satisfying these conditions satisfies the integration by parts formula proved above. Recall that to be a good model for a cumulative loss process we have to assume moreover finite variation trajectories that means that for a certain $\delta > 0$,*

$$\int_0^t \int_{||x|| \leq \delta} ||x|| \nu(ds, dx) < \infty, \, \forall t \geq 0.$$

In the case $d = 1$, if we consider only positive losses the condition becomes

$$\int_0^t \int_0^\delta x \nu(ds, dx) < \infty, \, \forall t \geq 0.$$

4. Main Results: pricing formulas for cumulative loss derivatives

4.1. Claim arrivals with deterministic intensity

Consider a cumulative loss process $\{L_t, t \geq 0\}$ such that claims arrive independently with a cumulative deterministic intensity Λ_t and positive claim sizes. This process can be seen as a particular case of a pure jump additive process J as we have seen in Subsection 2.3.

Fix $T>0$. Let $f\in L^1(\Theta_{T,0})$. As we have seen in Subsection 2.3, we can write

$$L_T := \sum_{i=1}^{N_{\Lambda_T}} f(s_i,x_i) = \int_0^T \int_{(0,\infty)^d} f(s,x)N(ds,dx).$$

The following theorem gives a formula to compute $\mathbb{E}[L_T h(L_T)]$ where h is a positive function such that $L_T h(L_T)\in L^1(\Omega^J)$. Of course this formula is valid for $\mathbb{E}[H(L_T)]$ for any positive function H such that $\mathbb{E}[H(L_T)]<\infty$, defining $h(x):=\frac{H(x)}{x}$.

Theorem 4.1. *Consider $f\in L^1(\Theta_{T,0})$. We have*

$$\mathbb{E}[L_T h(L_T)] = \int_0^T \int_{(0,\infty)^d} \mathbb{E}[h(L_T+f(s,x))]f(s,x)\nu(ds,dx) \tag{4.1}$$

Proof. Write $L_T=\mathcal{S}f$. Note that being f a deterministic function, it is in particular, predictable. Applying the duality formula in Theorem 3.3 and the definition of $\mathcal{T}$ in Definition 3.1 we have

$$\begin{aligned}
\mathbb{E}[L_T h(L_T)] &= \mathbb{E}\int_0^T \int_{(0,\infty)^d} \mathcal{T}_{s,x}h(L_T)f(s,x)\nu(ds,dx) \\
&= \mathbb{E}\int_0^T \int_{(0,\infty)^d} h(\mathcal{T}_{s,x}L_T)f(s,x)\nu(ds,dx) \\
&= \mathbb{E}\int_0^T \int_{(0,\infty)^d} h(L_T+f(s,x))f(s,x)\nu(ds,dx) \\
&= \int_0^T \int_{(0,\infty)^d} \mathbb{E}[h(L_T+f(s,x))]f(s,x)\nu(ds,dx).
\end{aligned}$$

□

Frequently, h is an indicator function of a set $A\subseteq[0,\infty)$. In this case we have

$$\mathbb{E}[h(L_T+f(s,x))] = \mathbb{P}(L_T+f(s,x)\in A) = \mathbb{P}(L_T\in A-f(s,x))$$

that can be seen as a function of (s,x).

Consider now the more general case of different claim sizes, commented in the Introduction and in [10]. We want to compute $\mathbb{E}[\hat{L}_T h(L_T)]$, where

$$\hat{L}_T := \sum_{i=1}^{N_{\Lambda_T}} g(s_i,x_i) = \int_0^T \int_{(0,\infty)^d} g(s,x)N(ds,dx)$$

with $g\in L^1(\Theta_{\infty,0})$ and assuming of course that $\hat{L}_T h(L_T)\in L^1(\Omega^J)$. The following corollary it is straightforward:

Corollary 4.2. *We have,*

$$\mathbb{E}[\hat{L}_T h(L_T)] = \int_0^T \int_{(0,\infty)^d} \mathbb{E}[h(L_T+f(s,x))]g(s,x)\nu(ds,dx). \tag{4.2}$$

Let us see some concrete examples.

Example 4.3. *A first quite general example is the Cramér-Lundberg model with inhomogeneous intensity. Consider $d = 1$ and $\nu(ds, dx) = \lambda(s)dsQ(dx)$ where λ is an a.e. strictly positive and locally integrable function that models the intensity of the claim arrivals and Q is a probability law on $(0, \infty)$ with finite expectation, that describes the amount of claims that are assumed to be independent and identically distributed. Take $f(s, x) = x$. So, we have*

$$\mathbb{E}[L_T h(L_T)] = \int_0^T \int_0^\infty \mathbb{E}[h(L_T + x)]xQ(dx)\lambda(s)ds = \Lambda_T \int_0^\infty \mathbb{E}[h(L_T + x)]xQ(dx)$$

where of course $\Lambda_T = \int_0^T \lambda(s)ds$ is the cumulative intensity.

If h is the indicator of an interval $[K, M] \subseteq [0, \infty)$ we have

$$\mathbb{E}[L_T h(L_T)] = \Lambda_T \int_0^\infty \mathbb{P}(L_T \in [K - x, M - x])xQ(dx). \tag{4.3}$$

Example 4.4. *Another example cited in the Introduction is the discounted cumulative loss process. In this case $f(s, x) = e^{-rs}x$. Consider the much more general case $f(s, x) = a(s)b(x)$. In this case*

$$\mathbb{E}[L_T h(L_T)] = \int_0^T a(s)\lambda(s)(\int_0^\infty \mathbb{E}[h(L_T + a(s)b(x)]b(x)Q(dx))ds,$$

under the integrability condition

$$\int_0^T \int_0^\infty a(s)b(x)\lambda(s)dsQ(dx) = (\int_0^T a(s)\lambda(s)ds)(\int_0^\infty b(x)Q(dx)) < \infty.$$

This formula covers for example the case described in [10] where $f(t, x) = \sqrt{\frac{\Lambda_t}{t}}x$.

Example 4.5. *A more complicated example, also described in [10], is the following. Assume $d = 2$. Assume we have two cumulative loss processes L and $\hat{L}$ where the amounts of L are given by $f(t, x, y)$ and the amounts of $\hat{L}$ are given by $g(t, x, y)$, with $(x, y) \in (0, \infty)^2$. Then,*

$$\mathbb{E}[\hat{L}_T h(L_T)] = \int_0^T \int_{(0,\infty)^2} \mathbb{E}[h(L_T + f(s, x, y))]g(s, x, y)\nu(ds, dx, dy).$$

Many particular cases are covered for different choices of f, g and ν. For example the case treated in [10] given by $\nu(ds, dx, dy) = Q(dx, dy)\lambda(s)ds$.

Example 4.6. *Not only cases on $[0, \infty) \times (0, \infty)^d$ are covered by previous formulas. Also cases on $[0, \infty)^k \times (0, \infty)^2$ or $\mathbb{R}^k \times \mathbb{R}^d_0$. For example the case every claim is marked by a vector (T_i, D_i, X_i) where D is the positive time between the event and its declaration or the event and its payment. Or in other models, T_i is an arrival time and D_i a service time. If T, D and X are independent and Q_D is the law of D we have $\nu(ds, dr, dx) = Q_D(dr)Q(dx)\lambda(s)ds$. In the general case, we can consider a general measure ν. See [12], chapters 7 and 8, for concrete examples of these types.*

4.2. Claim arrivals with random intensity

In Insurance, it is frequently interesting to assume random intensity in jump arrivals. So we need to extend the previous formulas to conditionally additive processes. This is what we do in this section. We follow some ideas of [17]. In relation with [10] we establish here a different probability space.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space that will be precised below. Assume we have a family $\{\nu(\omega, \cdot, \cdot), \omega \in \Omega\}$ of random diffuse Radon measures on $\Theta_{\infty,0}$ under the conditions of Remark 3.4. As we are modeling cumulative loss processes we consider here $\Theta_{\infty,0} = [0, \infty) \times (0, \infty)^d$. For any $B \in \mathcal{B}_{\infty,0}$, $\nu(\cdot, B)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Of course we can consider $\mathcal{F}^\nu_\infty$, the complete $\sigma-$algebra generated by these random variables, and $\mathbb{F}^\nu$ the filtration defined by the $\sigma-$algebras $\mathcal{F}^\nu_t$ generated for every t by the family $\{\nu(\cdot, B), B \in \mathcal{B}_{t,0}\}$.

Given $\nu(\omega,\cdot,\cdot)$ we can construct a Poisson random measure $N(\omega,\cdot,\cdot)$ such that

$$J_t(\omega)=\int_0^t\int_{(0,\infty)^d} xN(\omega,ds,dx)$$

is a pure jump conditional additive process with respect the $\sigma-$algebra $\mathcal{F}_\infty^\nu$.

Moreover, for any $f\in L^1(\Omega\times\Theta_{\infty,0},\mathcal{F}_\infty^\nu\otimes\mathcal{B}_{\infty,0},\nu)$ we can consider the $\mathcal{F}_\infty^\nu-$ measurable random variable

$$\int_0^\infty\int_{(0,\infty)^d} f(\omega,s,x)\nu(\omega,ds,dx)$$

and the Poisson integral process

$$J_t(f)=\int_0^t\int_{(0,\infty)^d} f(\omega,s,x)N(\omega,ds,dx)$$

such that a.s.,

$$\mathbb{E}[\int_0^t\int_{(0,\infty)^d} f(\omega,s,x)N(\omega,ds,dx)|\mathcal{F}_\infty^\nu]=\int_0^t\int_{(0,\infty)^d} f(\omega,s,x)\nu(\omega,ds,dx).$$

Moreover, taking expectations another time,

$$\mathbb{E}[\int_0^t\int_{(0,\infty)^d} f(\omega,s,x)N(\omega,ds,dx)]=\mathbb{E}[\int_0^t\int_{(0,\infty)^d} f(\omega,s,x)\nu(\omega,ds,dx)].$$

Denote by $\mathbb{F}^J$ the complete natural filtration generated by process J. And define $\mathbb{F}$ the filtration generated by the $\sigma-$algebras $\mathcal{F}_t:=\mathcal{F}_t^J\vee\mathcal{F}_\infty^\nu$. Of course J is adapted to the filtration $\mathbb{F}:=\{\mathcal{F}_t^J\times\mathcal{F}_\infty^\nu,t\geq 0\}$. Assume moreover $\mathcal{F}:=\mathcal{F}_\infty$.

All the previous statements are also true if we consider all previous objects defined on $\Theta_{T,0}$ and measurable with respect to $\mathcal{F}_T^\nu$ for a fixed $T>0$.

For such a process J, if we assume $\nu(\omega,\cdot,\cdot)$ are finite measures on any $\Theta_{t,0}$, we can consider the cumulative intensity process $\Lambda_t=\nu(\omega,[0,t]\times(0,\infty)^d)$ and the associated Poisson process $N_{\Lambda_t(\omega)}(\omega)$ that is a Cox process or a conditional inhomogeneous Poisson process.

So, we can extend naturally all the previous theory of pure jump additive processes to conditionally pure jump additive processes, and all the previous integration by parts formulas, to formulas conditioned to $\mathcal{F}_T^\nu$ for a fixed horizon $T>0$. Concretely, the following theorem gives a conditional version of formula (4.2). Formula (4.1) is a particular case.

Theorem 4.7. *Given*

$$L_T(\omega):=\int_{\Theta_{T,0}} f(\omega,s,x)N(\omega,ds,dx)=\sum_{i=1}^{N_{\Lambda_T(\omega)}(\omega)} f(\omega,s_i,x_i)$$

and

$$\hat{L}_T(\omega):=\int_{\Theta_{T,0}} g(\omega,s,x)N(\omega,ds,dx)=\sum_{i=1}^{N_{\Lambda_T(\omega)}(\omega)} g(\omega,s_i,x_i)$$

with f,g *and* ν, $\mathcal{F}_T^\nu-$*measurable, the conditional version of formula (4.2) is given by*

$$\mathbb{E}[\hat{L}_T h(L_T)|\mathcal{F}_T^\nu]=\int_{\Theta_{T,0}}\mathbb{E}[h(L_T(\omega)+f(\omega,s,x))|\mathcal{F}_T^\nu]g(\omega,s,x)\nu(\omega,ds,dx)$$

Taking expectations another time, we have

$$\mathbb{E}[\hat{L}_T h(L_T)]=\mathbb{E}[\int_{\Theta_{T,0}} h(L_T(\omega)+f(\omega,s,x))g(\omega,s,x)\nu(\omega,ds,dx)]. \tag{4.4}$$

Proof. Similarly as in the proof of Theorem 4.1, we write $\hat{L}_T = \mathcal{S}g$ and we have

$$\begin{aligned}
\mathbb{E}[\hat{L}_T h(L_T)|\mathcal{F}_T^\nu] &= \mathbb{E}[\int_{\Theta_{T,0}} \mathcal{T}_{s,x} h(L_T(\omega))g(\omega,s,x)\nu(\omega,ds,dx)|\mathcal{F}_T^\nu] \\
&= \mathbb{E}[\int_{\Theta_{T,0}} h(\mathcal{T}_{s,x} L_T(\omega))g(\omega,s,x)\nu(\omega,ds,dx)|\mathcal{F}_T^\nu] \\
&= \mathbb{E}[\int_{\Theta_{T,0}} h(L_T(\omega)+f(\omega,s,x))g(\omega,s,x)\nu(\omega,ds,dx)|\mathcal{F}_T^\nu] \\
&= \int_{\Theta_{T,0}} \mathbb{E}[h(L_T(\omega)+f(\omega,s,x))|\mathcal{F}_T^\nu]g(\omega,s,x)\nu(\omega,ds,dx).
\end{aligned}$$

□

The following example shows the interest of this Theorem in Insurance:

Example 4.8. *Assume $d=2$ and*

$$\nu(\omega,ds,dx,dy) = \lambda_s(\omega)Q(dx,dy)ds$$

where Q is concentrated in $(0,\infty)^2$. Then, we have

$$\begin{aligned}
\mathbb{E}[\hat{L}_T h(L_T)] &= \int_0^T\int_0^\infty\int_0^\infty \mathbb{E}[h(L_T(\omega)+f(\omega,s,x,y))g(\omega,s,x,y)\lambda_s(\omega)]Q(dx,dy)ds \\
&= \mathbb{E}\int_0^T\int_0^\infty\int_0^\infty \varphi_h(\omega,f(\omega,s,x,y))g(\omega,s,x,y)\lambda_s(\omega)Q(dx,dy)ds
\end{aligned}$$

where

$$\varphi_h(\omega,z) = \mathbb{E}[h(L_T(\omega)+z)|\mathcal{F}_T^\nu],$$

for any $z\in\mathbb{R}$. This coincides with Theorem 3.5 in [10].

Therefore,

$$\mathbb{E}[\hat{L}_T h(L_T)] = \mathbb{E}\int_0^T\int_0^\infty\int_0^\infty \mathbb{E}[(h(L_T(\omega)+f(\omega,s,x,y)))|\mathcal{F}_T^\nu]g(\omega,s,x,y)\lambda_s(\omega)Q(dx,dy)ds.$$

If h is an indicator function of a set A, as in the case of a stop-loss contract ($A=(K,M]$ with $0<K<M<\infty$) we have

$$\mathbb{E}[\hat{L}_T h(L_T)] = \mathbb{E}\int_0^T\int_0^\infty\int_0^\infty \mathbb{P}[L_T(\omega)+f(\omega,s,x,y)\in A|\mathcal{F}_T^\nu]g(\omega,s,x,y)\lambda_s(\omega)Q(dx,dy)ds.$$

In the computation of an expected shortfall ($A=[0,\beta]$ with $\beta>0$) and $\hat{L}_T = L_T$ and the formula reduces to

$$\mathbb{E}[L_T \mathbb{1}_A(L_T)] = \mathbb{E}\int_0^T\int_0^\infty \mathbb{P}[L_T(\omega)+f(\omega,s,x)\in A|\mathcal{F}_T^\nu]f(\omega,s,x)\lambda_s(\omega)Q(dx)ds.$$

In the Cramér-Lundberg case, but now with random intensity, $f(\omega,s,x)=x$ and the formula becomes

$$\mathbb{E}[L_T \mathbb{1}_A(L_T)] = \mathbb{E}\int_0^T\int_0^\infty \mathbb{P}[L_T(\omega)\in A-x|\mathcal{F}_T^\nu]xQ(dx)\lambda_s(\omega)ds.$$

Of course, all these formulas have practical applications when we assume concrete laws for the cumulative intensity process, the time between jumps and the jump amplitudes.

Example 4.9. *Moreover, note that we could consider objects like*

$$J_t(\omega) = \int_0^t \int_{\mathbb{R}_0^d} x \mathbb{1}_{[0,\tau(\omega)]}(s) N(\omega, ds, dx)$$

with τ a stopping time with respect the filtration $\mathbb{F}^\nu$. This would allow to model random expiry dates determined by ν and so, to apply Theorem 4.7 to this situation.

Example 4.10. *Note that in [10] a typical example of L_T has jump sizes $f(\Lambda_{s_i}, s_i, x_i)$ where f is a deterministic function and assuming independence between x_i and Λ_{s_i} (Hypothesis 3.1). In the present paper a typical jump size is $f(\omega, s_i, x_i)$, that is, $f(\omega, s_i, x_i)$ can depend on all the trajectory ω until T. A possible applied example is the following. Given $\nu(\omega, \cdot, \cdot)$ we can consider the process $\{\nu(\omega, [0,t] \times B), t \geq 0\}$ as the history of a client or group of clients related with certain aspects of interest given by B. Note that if $B = (0, \infty)^d$, $\{\nu(\omega, [0,t] \times (0,\infty)^d), t \geq 0\}$ is the process $\{\Lambda_t(\omega), t \geq 0\}$, that can represent the whole history of a client or a group of clients. Therefore, we are allowing L_T to depend on the relevant history of the client or group of clients in a very general way, that is, random variables $f(\cdot, s_i, x_i)$ are simply $\mathcal{F}_T^\nu-$measurables.*

Example 4.11. *Assume the measures $\{\nu(\omega, B), B \in \mathcal{B}_{t,0}\}$ are finite for any $t \geq 0$. Define $\Lambda_t(\omega) = \nu(\omega, [0,t] \times (0,\infty))$ and for any Borel set C of $(0,\infty)$ define*

$$Q_t(C) = \frac{\nu(\omega, [0,t] \times C)}{\nu(\omega, [0,t] \times (0,\infty))}.$$

Therefore,

$$\nu(\omega, [0,t] \times C) = \Lambda_t(\omega) Q_t(\omega, C).$$

This means that in this model the jump arrivals are chosen by a Cox process of random intensity Λ_t and the jump sizes x_i are chosen according the law Q_{s_i} for any i. This means the jump size depends on time and on the random intensity process Λ. A model with jump sizes depending on time is considered for example in [7].

Remark 4.12. *Note that conditions on process Λ that determine the Cox process are very general. Any process with increasing trajectories such that $\lim_{t\uparrow\infty} \Lambda_t = \infty$, a.s. is included. This includes the case Λ is an additive subordinator as considered for example in [1].*

5. Conclusions

In this paper we have proved that given a cumulative loss process L with random intensity and jump sizes given by f, and a function h, we have the very general formula

$$\mathbb{E}[L_T h(L_T)] = \mathbb{E}[\int_0^T \int_{\mathbb{R}_0^d} h(L_T(\omega) + f(\omega, s, x)) f(\omega, s, x) \nu(\omega, ds, dx)] \tag{5.1}$$

that is a relevant quantity in Insurance and Finance. Here, claim amounts can depend on claim arrivals, and both can be dependent on the random intensity process. The formula is a consequence of the Malliavin-Skorohod duality formula given in Theorem 3.3, and it is a generalization of formulas in [10] in the sense that Assumption 3.1 in [10] is unnecessary. The methodology applied in this paper shows the mathematical power of Malliavin-Skorohod calculus for additive processes developed in [5] to obtain results with interest in Finance and Insurance.

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