The Tits alternative for groups defined by periodic paired relations

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Abstract

The class of groups defined by periodic paired relations, as introduced by Vinberg, includes the generalized triangle groups, the generalized tetrahedron groups, and the generalized Coxeter groups. We observe that any group defined by periodic paired relations Γ can be realized as a so-called 'Pride group'. Using results of Howie and Kopteva we give necessary and sufficient conditions for this Pride group to be non-spherical. Under such conditions we show that Γ satisfies the Tits alternative.

1 Introduction

A *group defined by periodic paired relations* is a group with presentation of the form

$$\Gamma = \langle\ x_1, \ldots, x_n \mid x_i^{q_i} = 1\ (1 \leq i \leq n),\ w_{ij}(x_i, x_j)^{q_{ij}} = 1\ (1 \leq i < j \leq n)\ \rangle \tag{1}$$

where $n \geq 2$, each $q_i, q_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$, and each $w_{ij}(x_i, x_j)$ is a cyclically reduced word in the free product $\langle\ x_i \mid x_i^{q_i}\ \rangle * \langle\ x_j \mid x_j^{q_j}\ \rangle$ involving both x_i and x_j. This class of groups was introduced by Vinberg [12] and includes the generalized triangle groups ($n = 2$), the generalized tetrahedron groups ($n = 3$), Coxeter groups (each $q_i = 2$), and the generalized Coxeter groups ($w_{ij} = x_i^{\alpha_{ij}} x_j^{\beta_{ij}}$) considered by Tsaranov [11]. A group is said to satisfy the *Tits alternative* if it either contains a non-abelian free subgroup or has a soluble subgroup of finite index. Coxeter groups are known to satisfy the Tits alternative [8], and it has been conjectured that generalized triangle and generalized tetrahedron groups do the same [5, 6]. We consider the corresponding question for groups defined by periodic paired relations.

Conjecture *Every group defined by periodic paired relations satisfies the Tits alternative.*

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Vinberg [12] has shown that if each $q_i, q_{ij} < \infty$ and

$$\sum_{1\leq i\leq n} \frac{1}{q_i} + \sum_{1\leq i<j\leq n} \frac{1}{q_{ij}} < (n-1)$$

then Γ has a finite index subgroup which maps onto the free group of rank 2; in particular, Γ contains a non-abelian free subgroup. In this note we observe that any group defined by periodic paired relations Γ (with $n \geq 3$) can be realized as a so-called Pride group [9]. By appealing to results of Howie and Kopteva [6], we give necessary and sufficient conditions for such a Pride group to be non-spherical and show that under these conditions the Tits alternative holds for Γ.

We will say that two presentations P_1, P_2 of the form (1) are *equivalent* if P_1 and P_2 have the same number of generators and P_2 can be obtained from P_1 by a sequence of operations of the following type:

1. apply a permutation to the set of generators $\{x_i \mid 1 \leq i \leq n\}$;
2. if $v_{ij}(x_i, x_j)$ is a cyclically reduced conjugate of $w_{ij}(x_i, x_j)$ in the free group on x_i and x_j, then replace the relator $w_{ij}(x_i, x_j)^{q_{ij}}$ with the relator $v_{ij}(x_i, x_j)^{q_{ij}}$.

Clearly two equivalent presentations define the same group. (For generalized tetrahedron groups, a stronger definition of equivalence is given in [3], but the above is sufficient for our purposes.)

To each relator $w_{ij}(x_i, x_j)^{q_{ij}}$ in the presentation (1) we will associate a number ℓ_{ij}. If $q_{ij} < \infty$ we define ℓ_{ij} to be equal to the length of $w_{ij}(x_i, x_j)$ in the free product $\langle\, x_i \mid x_i^{q_i} \,\rangle * \langle\, x_j \mid x_j^{q_j} \,\rangle$; if $q_{ij} = \infty$ we set $\ell_{ij} = \infty$. We prove the following result (where, as throughout this paper, $1/\infty$ is understood to mean 0):

Theorem 1 *Let Γ be as defined in (1) with $3 \leq n$. If for each $1 \leq i < j < k \leq n$*

$$\frac{1}{q_{ij}\ell_{ij}} + \frac{1}{q_{ik}\ell_{ik}} + \frac{1}{q_{jk}\ell_{jk}} \leq \frac{1}{2}$$

then Γ satisfies the Tits alternative. In particular, Γ contains a non-abelian free subgroup unless Γ is equivalent to one of the following:

$$\Gamma_1 = \langle\, x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_1x_2)^3, (x_1x_3)^3, (x_2x_3)^3 \,\rangle,$$
$$\Gamma_2 = \langle\, x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_1x_2)^2, (x_1x_3)^4, (x_2x_3)^4 \,\rangle,$$
$$\Gamma_3 = \langle\, x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_1x_2)^2, (x_1x_3)^3, (x_2x_3)^6 \,\rangle,$$
$$\Gamma_4 = \langle\, x_1, x_2, x_3 \mid x_1^{q_1}, x_2^2, x_3^2, (x_1x_2)^2, (x_1x_3)^2 \,\rangle,$$
$$\Gamma_5 = \langle\, x_1, x_2, x_3, x_4 \mid x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_1x_3)^2, (x_2x_4)^2, (x_3x_4)^2 \,\rangle,$$

in which case Γ is infinite and virtually abelian.

In the case $n = 3$ (generalized tetrahedron groups) the result is essentially due to Howie and Kopteva [6]. Their definition of generalized tetrahedron group however, requires that each of the exponents $q_1, q_2, q_3, q_{12}, q_{23}, q_{13}$ is finite. We first extend their result to include the possibility that the exponents are infinite.

2 Proof of Theorem 1 for $n = 3$

Under the hypotheses of Theorem 1, Γ is realized by a non-spherical triangle of groups with vertex groups

$$\begin{aligned}
G_{12} &= \langle\, x_1, x_2 \mid x_1^{q_1}, x_2^{q_2}, w_{12}(x_1,x_2)^{q_{12}} \,\rangle,\\
G_{13} &= \langle\, x_1, x_3 \mid x_1^{q_1}, x_3^{q_3}, w_{13}(x_1,x_2)^{q_{13}} \,\rangle,\\
G_{23} &= \langle\, x_2, x_3 \mid x_2^{q_2}, x_3^{q_3}, w_{23}(x_2,x_3)^{q_{23}} \,\rangle
\end{aligned}$$

(see [6, Theorem 1]). If $q_1, q_2, q_3, q_{12}, q_{23}, q_{13}$ are all finite, then Theorem 1 is due to Howie and Kopteva [6]; if the inequality in the statement of the theorem is strict then Γ is the amalgamated sum of a negatively curved triangle of groups, so contains a non-abelian free subgroup by [6, Proposition 2.2]; and if G_{12}, G_{23}, or G_{13} contains a non-abelian free subgroup then since (by [10]) the vertex groups embed in Γ, Γ contains a non-abelian free subgroup. Thus we may assume that

$$\frac{1}{q_{12}\ell_{12}} + \frac{1}{q_{13}\ell_{13}} + \frac{1}{q_{23}\ell_{23}} = \frac{1}{2}, \tag{2}$$

that $q_1, q_2, q_3, q_{12}, q_{23}, q_{13}$ are not all finite, and that none of G_{12}, G_{13}, G_{23} contains a non-abelian free subgroup.

Suppose first that one or more of $\{q_1, q_2, q_3\}$ is equal to infinity. Without loss of generality we may assume $q_1 = \infty$. Then G_{12} (respectively G_{13}) contains a non-abelian free subgroup if $q_{12} \geq 3$ (respectively $q_{13} \geq 3$) by [4], and contains a non-abelian free subgroup if $q_2 \geq 3$ (respectively $q_3 \geq 3$) by [5, Theorem 5]. Thus we may assume that $q_2 = q_3 = q_{12} = q_{13} = 2$; the condition (2) then implies that $q_{23} = \infty$ and that $\ell_{12} = \ell_{13} = 2$. By [5, Theorem 4] G_{12} (respectively G_{13}) contains a non-abelian free subgroup unless (up to equivalence) $w_{12} = x_1x_2$ or $x_1^2x_2$ (respectively $w_{13} = x_1x_3$ or $x_1^2x_3$). This means that (up to equivalence) Γ has one of the following presentations:

$$\begin{aligned}
H_1 &= \langle\, x_1, x_2, x_3 \mid x_2^2, x_3^2, (x_1^2x_2)^2, (x_1^2x_3)^2 \,\rangle,\\
H_2 &= \langle\, x_1, x_2, x_3 \mid x_2^2, x_3^2, (x_1x_2)^2, (x_1^2x_3)^2 \,\rangle,\\
H_3 &= \langle\, x_1, x_2, x_3 \mid x_2^2, x_3^2, (x_1x_2)^2, (x_1x_3)^2 \,\rangle.
\end{aligned}$$

Adjoining the relator x_1^2 to H_1 shows that H_1 maps homomorphically onto

$$\langle\, x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2 \,\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$$

so contains a non-abelian free subgroup. Adjoining the relator x_1^2 to H_2 shows that H_2 maps homomorphically onto

$$\langle\, x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_1x_2)^2 \,\rangle \cong D_4 * \mathbb{Z}_2$$

so contains a non-abelian free subgroup.

For the group H_3, consider the epimorphism $\phi : H_3 \to \langle\ \alpha \mid \alpha^2\ \rangle \cong \mathbb{Z}_2$ given by $\phi(x_1) = 1, \phi(x_2) = \phi(x_3) = \alpha$. The Reidemeister-Schreier process provides the following presentation for $\ker(\phi)$:

$$\ker(\phi) = \langle\ a, b \mid aba^{-1}b^{-1}\ \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

Hence H_3 is infinite and virtually abelian. (Note that H_3 is of the form Γ_4 with $q_1 = \infty$).

Suppose then that q_1, q_2, q_3 are all finite, and that at least one of q_{12}, q_{23}, q_{13} is equal to infinity. We may assume $q_{23} = \infty$. The condition (2) implies that $q_{12} = q_{13} = 2$ and that $\ell_{12} = \ell_{13} = 2$. If either $q_2 \geq 3$ or $q_3 \geq 3$ then $G_{23} \cong \mathbb{Z}_{q_2} * \mathbb{Z}_{q_3}$ contains a non-abelian free subgroup. Thus we may assume $q_2 = q_3 = 2$. The groups G_{12}, G_{13} then have the presentations

$$G_{12} = \langle\ x_1, x_2 \mid x_1^{q_1}, x_2^2, (x_1^\theta x_2)^2\ \rangle,$$
$$G_{13} = \langle\ x_1, x_3 \mid x_1^{q_1}, x_3^2, (x_1^\eta x_3)^2\ \rangle,$$

where we may assume that θ, η are positive. If $\theta \geq 3$ (respectively $\eta \geq 3$) then by [5, Theorem 6] G_{12} (respectively G_{13}) contains a non-abelian free subgroup. Thus we may assume $\{\theta, \eta\} = \{1, 1\}, \{1, 2\}$, or $\{2, 2\}$.

If $\theta = \eta = 1$ then Γ is a homomorphic image of H_3, so is virtually abelian (and is of the form Γ_4). If $\theta = \eta = 2$ we can write Γ as an amalgamated free product

$$\langle\ x_1, x_2 \mid x_1^{q_1}, x_2^2, (x_1^2 x_2)^2\ \rangle \underset{\langle\ x_1 \mid x_1^{q_1}\ \rangle}{*} \langle\ x_1, x_3 \mid x_1^{q_1}, x_3^2, (x_1^2 x_3)^2\ \rangle$$

where by [5, Theorem 6] each factor is infinite. The amalgamated subgroup is then of infinite index in each of the factors, so Γ contains a non-abelian free subgroup. Suppose then that $\{\theta, \eta\} = \{1, 2\}$ and (without loss of generality) take $\theta = 1, \eta = 2$. The kernel of the epimorphism $\rho : \Gamma \to \langle\ \alpha \mid \alpha^2\ \rangle \cong \mathbb{Z}_2$ given by $\rho(x_1) = \rho(x_3) = 1$, $\rho(x_2) = \alpha$ has a presentation

$$\ker(\rho) = \langle\ a, b, c \mid a^{q_1}, b^2, c^2, (a^2 b)^2, (a^2 c)^2\ \rangle$$

and we have already shown that such a group contains a non-abelian free subgroup (see the case $\theta = \eta = 2$, above). Hence Γ contains a non-abelian free subgroup.

3 Groups defined by periodic paired relations as Pride groups

We consider a class of groups known as Pride groups, introduced in [9]; our notation and terminology is essentially that used by Meier [7]. Let $\mathcal{G}$ be a finite simplicial graph with vertex set $I = I(\mathcal{G})$, and edge set $E(\mathcal{G})$. Further, let there be non-trivial groups G_i (with fixed finite presentations) associated to each vertex $i \in I(\mathcal{G})$ and in addition, for each edge $\{i,j\} \in E(\mathcal{G})$ let $R_{\{i,j\}}$ be a finite, non-empty collection of cyclically reduced words. We assume each word in $R_{\{i,j\}}$ is of free product length greater than or equal to 2 in $G_i * G_j$. The *Pride group* based on the graph $\mathcal{G}$ with groups G_i assigned to the vertices and with edge relations $R = \cup_{\{i,j\}\in E(\mathcal{G})} R_{\{i,j\}}$ is the group $G := *_{i\in I(\mathcal{G})} G_i/N$ where N is the normal closure of R in $*_{i\in I(\mathcal{G})} G_i$.

We refer to the groups G_i as *vertex groups* and we define the *edge groups* to be $G_{\{i,j\}} = \{G_i * G_j\}/N_{\{i,j\}}$ where $\{i,j\} \in E(\mathcal{G})$ and where $N_{\{i,j\}}$ is the normal closure of $R_{\{i,j\}}$ in $G_i * G_j$. More generally, if $\mathcal{F}$ is any full subgraph of $\mathcal{G}$ generated by a set of vertices $I(\mathcal{F}) \subseteq I(\mathcal{G})$ then the *subgraph group* $G_{\mathcal{F}}$ is $\{*_{i\in I(\mathcal{F})} G_i\}/\{N_{\{i,j\}} \mid \{i,j\} \in E(\mathcal{F})\}$. In particular, $G_{\mathcal{G}} = G$. A Pride group in which the subgraph groups embed is said to be *developable.*

For each $i,j \in I$, the natural homomorphisms $G_i \to G_{\{i,j\}}$, $G_j \to G_{\{i,j\}}$ determine a homomorphism $G_i * G_j \to G_{\{i,j\}}$. Let m_{ij} denote the length of a shortest non-trivial element in its kernel (in the usual length function on the free product), or put $m_{ij} = \infty$ if the kernel is trivial. Note that either $m_{ij} = 1$ (in which case one of the natural maps $G_i \to G_{\{i,j\}}$, $G_j \to G_{\{i,j\}}$ is not injective), or m_{ij} is even.

We will say that a Pride group is *non-spherical* if for every $i,j,k \in I$

$$\frac{1}{m_{ij}} + \frac{1}{m_{ik}} + \frac{1}{m_{jk}} \le \frac{1}{2}.$$

Corson has proved the following:

Theorem 2 ([2]) *A non-spherical Pride group is developable.*

Any group defined by periodic paired relations (1) can be realized as a Pride group with the following configuration:

- $\mathcal{G}$ is the complete graph with vertex set $I = \{1,\ldots,n\}$;
- $G_i = \langle\, x_i \mid x_i^{q_i} \,\rangle$ for each $i \in I$;
- $R_{\{i,j\}} = \{w_{ij}(x_i,x_j)^{q_{ij}}\}$ for each $i,j \in I$.

The values of the m_{ij} are determined by the induced maps

$$\phi : \langle\, x_i \mid x_i^{q_i} \,\rangle * \langle\, x_j \mid x_j^{q_j} \,\rangle \to \langle\, x_i, x_j \mid x_i^{q_i}, x_j^{q_j}, w_{ij}(x_i,x_j)^{q_{ij}} \,\rangle.$$

If $q_{ij} = \infty$ then ϕ is injective, so $m_{ij} = \infty$. If $q_{ij} < \infty$ then, by the Spelling Theorem for generalized triangle groups [6, Theorem 3.2], $m_{ij} = q_{ij}\ell_{ij}$. Thus a group defined by periodic paired relations (1) can be realized as a non-spherical Pride group if and only if for all $1 \le i < j < k \le n$

$$\frac{1}{q_{ij}\ell_{ij}} + \frac{1}{q_{ik}\ell_{ik}} + \frac{1}{q_{jk}\ell_{jk}} \le \frac{1}{2}.$$

4 Proof of Theorem 1 for $n \geq 4$

Let Γ be as in the statement of the theorem and let $\mathcal{G}$ be the corresponding labelled graph, as described in Section 3. The labelled graphs corresponding to the groups $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are given by $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ in Figure 1. By Theorem 2, Γ is a developable Pride group.

Suppose $n = 4$. We may assume that every labelled full subgraph $\mathcal{F}$ of $\mathcal{G}$ with 3 vertices is of the form of one of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$, for otherwise, by Section 2, $G_{\mathcal{F}}$ (and hence Γ) contains a non-abelian free subgroup. The only possibilities for $\mathcal{G}$ are then the labelled graphs $\mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8$ in Figure 1. The corresponding groups have the presentations:

$$\Gamma_5 = \langle\ x_1, x_2, x_3, x_4 \mid x_1^{q_1}, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_1x_3)^2, (x_2x_4)^2, (x_3x_4)^2\ \rangle,$$
$$\Gamma_6 = \langle\ x_1, x_2, x_3, x_4 \mid x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^3, (x_1x_3)^3, (x_1x_4)^3, (x_2x_3)^3, (x_2x_4)^3, (x_3x_4)^3\ \rangle,$$
$$\Gamma_7 = \langle\ x_1, x_2, x_3, x_4 \mid x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_1x_3)^4, (x_1x_4)^4, (x_2x_3)^4, (x_2x_4)^4, (x_3x_4)^2\ \rangle,$$
$$\Gamma_8 = \langle\ x_1, x_2, x_3, x_4 \mid x_1^2, x_2^2, x_3^2, x_4^2, (x_1x_2)^2, (x_1x_3)^3, (x_1x_4)^6, (x_2x_3)^6, (x_2x_4)^3, (x_3x_4)^2\ \rangle.$$

The group Γ_5 can be expressed as an amalgamated free product

$$\langle\ x_1, x_2, x_4 \mid x_1^{q_1}, x_2^2, x_4^2, (x_1x_2)^2, (x_2x_4)^2\ \rangle \underset{\langle\ x_1, x_4\ \mid\ x_1^{q_1}, x_4^2\ \rangle}{*} \langle\ x_1, x_3, x_4 \mid x_1^{q_1}, x_3^2, x_4^2, (x_1x_3)^2, (x_3x_4)^2\ \rangle$$

If $q_1 \geq 3$ then the amalgamated subgroup, and hence Γ_5, contains a non-abelian free subgroup. If $q_1 = 2$ then the kernel of the epimorphism $\psi : \Gamma_5 \to \langle\ \alpha, \beta \mid \alpha^2, \beta^2, \alpha\beta\alpha^{-1}\beta^{-1}\ \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\psi(x_1) = \psi(x_4) = \alpha$, $\psi(x_2) = \psi(x_3) = \beta$ has a presentation

$$\ker(\psi) = \langle\ a, b \mid aba^{-1}b^{-1}\ \rangle \cong \mathbb{Z} \times \mathbb{Z},$$

so Γ_5 is infinite and virtually abelian.

The kernel of the epimorphism $\phi : \Gamma_6 \to \langle\ \alpha, \beta \mid \alpha^2, \beta^2, (\alpha\beta)^3\ \rangle \cong D_6$ given by $\phi(x_1) = \alpha$, $\phi(x_2) = \beta$, $\phi(x_3) = \phi(x_4) = \alpha\beta\alpha$ has a presentation:

$$\ker(\phi) = \langle\ a, b, c, d \mid (ad)^3, (bc)^3, (abcd)^3\ \rangle.$$

Adjoining the relations $ad = 1, bc = 1$ shows that $\ker(\phi)$ is mapped onto the free group of rank 2. Hence Γ_6 contains a non-abelian free subgroup. (This fact is observed in [1], where an alternative proof is also given.)

The kernel of the epimorphism $\sigma : \Gamma_7 \to \langle\ \alpha \mid \alpha^2\ \rangle \cong \mathbb{Z}_2$ given by $\sigma(x_1) = \sigma(x_2) = 1$, $\sigma(x_3) = \sigma(x_4) = \alpha$ has a presentation

$$\ker(\sigma) = \langle\ a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^2, (ab)^2, (ad)^2, (be)^2, (de)^2, (bcec)^2, (acdc)^2\ \rangle.$$

Adjoining the relations $a = 1, e = 1$ shows that $\ker(\sigma)$ maps onto the group $\langle\ b, c, d \mid b^2, c^2, d^2\ \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, which contains a non-abelian free subgroup. Hence Γ_7 contains a non-abelian free subgroup.

The kernel of the epimorphism $\rho : \Gamma_8 \to \langle\ \alpha, \beta \mid \alpha^2, \beta^2, (\alpha\beta)^2\ \rangle \cong D_4$ given by $\rho(x_1) = \rho(x_3) = \alpha$, $\rho(x_2) = \rho(x_4) = \beta$ has a presentation

$$\ker(\rho) = \langle\ a, b, c, d \mid a^3, b^3, c^3, d^3, (ad)^3, (bc)^3, ab^{-1}dc^{-1}\ \rangle.$$

Adjoining the relation $d = 1$ shows that $\ker(\rho)$ maps onto the generalized triangle group $\langle\ a, b \mid a^3, b^3, (ab^{-1})^3\ \rangle$, which contains a non-abelian free subgroup by [5, Theorem 6]. Hence Γ_8 contains a non-abelian free subgroup.

This completes the analysis for the case $n = 4$.

Now suppose $n \geq 5$. We may assume that every labelled full subgraph $\mathcal{F}$ of $\mathcal{G}$ with 4 vertices is of the form of $\mathcal{F}_5$ with $q = 2$, for otherwise $G_{\mathcal{F}}$ (and hence Γ) contains a non-abelian free subgroup. But, under the hypotheses of the theorem, $\mathcal{G}$ cannot satisfy this condition, so the proof is complete.

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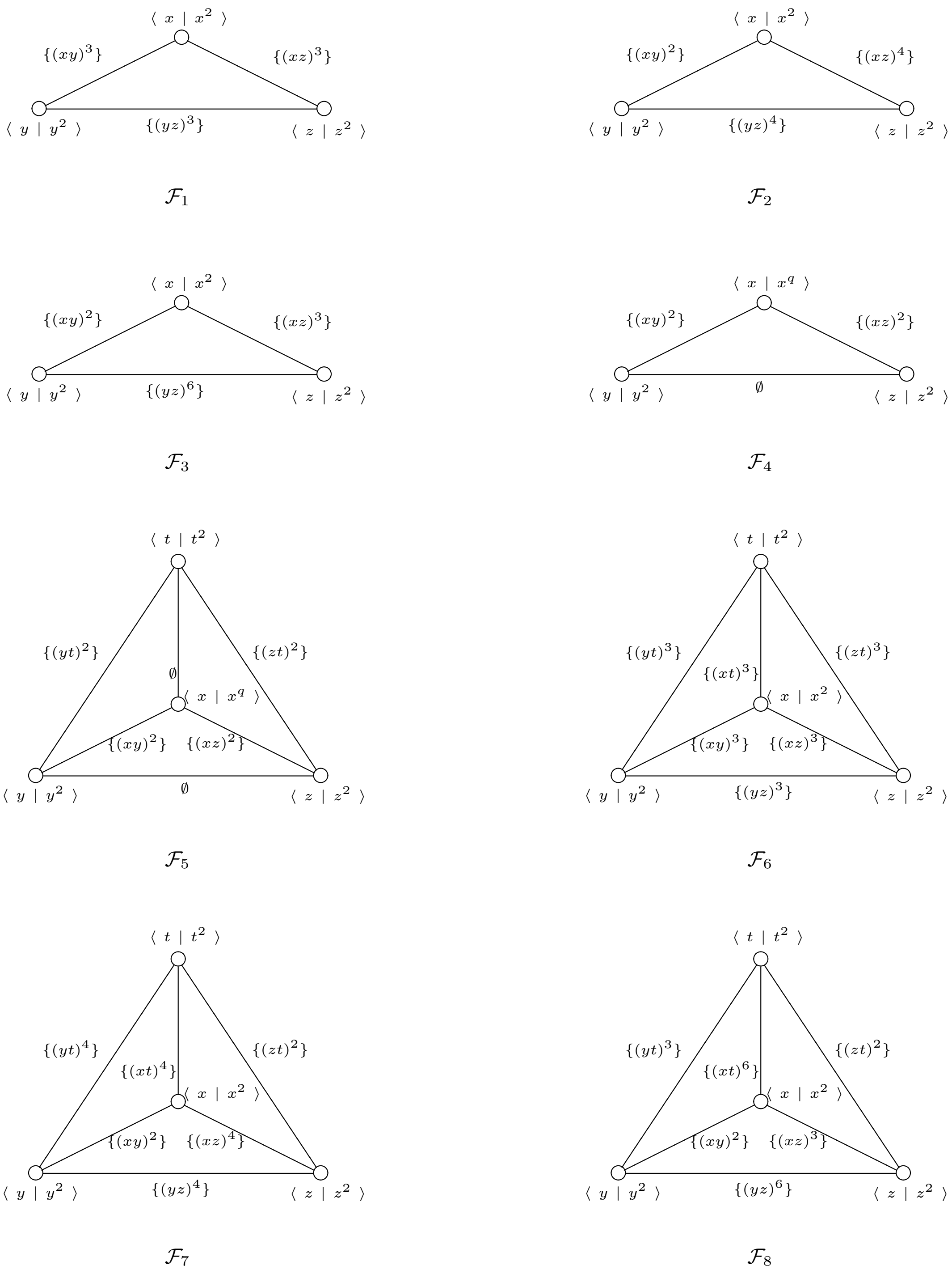

$\langle x \mid x^2 \rangle$
$\{(xy)^3\}$
$\{(xz)^3\}$
$\langle y \mid y^2 \rangle$
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$\langle z \mid z^2 \rangle$
$\mathcal{F}_1$
$\langle x \mid x^2 \rangle$
$\{(xy)^2\}$
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$\{(yz)^4\}$
$\langle z \mid z^2 \rangle$
$\mathcal{F}_2$
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$\{(xz)^3\}$
$\langle y \mid y^2 \rangle$
$\{(yz)^6\}$
$\langle z \mid z^2 \rangle$
$\mathcal{F}_8$

Figure 1:

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