Unimodular integer circulants associated with trinomials

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Abstract

The $n \times n$ circulant matrix associated with the polynomial $f(t) = \sum_{i=0}^{d} a_i t^i$ (with $d < n$) is the one with first row $(a_0 \dots a_d\ 0 \dots 0)$. The problem as to when such circulants are unimodular arises in the theory of cyclically presented groups and leads to the following question, previously studied by Odoni and Cremona: when is $\mathrm{Res}(f(t), t^n - 1) = \pm 1$? We give a complete answer to this question for trinomials $f(t) = t^m \pm t^k \pm 1$. Our main result was conjectured by the author in an earlier paper and (with two exceptions) implies the classification of the finite Cavicchioli-Hegenbarth-Repovš generalized Fibonacci groups, thus giving an almost complete answer to a question of Bardakov and Vesnin.

1 Introduction

The $n \times n$ circulant matrix $M_n(f)$ associated with the polynomial $f(t) = \sum_{i=0}^{d} a_i t^i$ where $d < n$ and $a_i \in \mathbb{Z}$ is the one whose first row is $(a_0 \dots a_d\ 0 \dots 0)$. Well known properties of circulants and resultants give that $\det(M_n) = \mathrm{Res}(f, t^n - 1)$. The question as to when M_n is unimodular arises in the theory of cyclically presented groups and has been considered by Odoni [7] and Cremona [3].

For $n \geq 1$ define

$$R_n(f) = \prod_{\theta^n = 1} f(\theta).$$

Our approach, as in [3],[7], is to work with $R_n(f)$ rather than with $M_n(f)$. It was shown in [3],[7] that, for $n > d$, $\det(M_n) = R_n(f)$ so it is enough to consider when $R_n(f) = \pm 1$. We note that $R_n(f)$ is defined for all $n \geq 1$ whereas $M_n(f)$ is only defined for $n > d$.

Briefly, the connection with cyclically presented groups is as follows. Fix a word $w(x_0, \dots, x_{n-1})$ in generators $x_0, \dots, x_{n-1}$ and let $\Gamma_n(w)$ be the group defined by the presentation with these n generators and the n relators

$$w(x_0, x_1, \dots, x_{n-2}, x_{n-1}), w(x_1, x_2, \dots, x_{n-1}, x_0), \dots, w(x_{n-1}, x_0, \dots, x_{n-3}, x_{n-2}).$$

If a_i is the exponent sum of x_i in $w(x_0, \dots, x_{n-1})$ then $\Gamma_n(w)$ has infinite abelianization if and only if $R_n(f) = 0$ and is perfect if and only if $R_n(f) = \pm 1$ [5],[7]. Indeed $\Gamma_n(w)^{\mathrm{ab}}$ has order $|R_n(f)|$ ([5, page 77]).

In this paper we consider trinomials $f(t) = t^m \pm t^k \pm 1$. When both signs are '+' it is easy to deduce that $R_n(f) \neq \pm 1$. In the other three cases we can reduce to a polynomial of the form $t^m - t^k + 1$; moreover we may assume $(n, m, k) = 1$ (see Section 3). We note that Lemma 5 of [8] and Lemma 2.3 of [4] determine when $R_n(t^m \pm t^k \pm 1) = 0$.

The Cavicchioli-Hegenbarth-Repovš generalized Fibonacci groups $G_n(m, k)$ are the cyclically presented groups with generators $x_1, \ldots, x_n$ and relators $x_i x_{i+m} x_{i+k}^{-1}$ and these are our primary motivation for considering trinomials $f(t) = t^m - t^k + 1$. Our main result is

Main Theorem *Let $n \geq 1$ and $f(t) = t^m - t^k + 1$ where $m, k \in \mathbb{Z}$ and $(n, m, k) = 1$. Then $R_n(f) = \pm 1$ if and only if ($(n, 6) = 1$ and $m = 2k$ mod n) or $k = 0$ mod n or $k = m$ mod n.*

This was conjectured (in group theoretic terms) in [8] and is a natural generalization of a theorem of Odoni [7] which deals with the case $k = 1$. With the exception of two groups, the Main Theorem implies the classification of the finite groups $G_n(m, k)$ (see [8]), thus giving an almost complete answer to a problem posed by Bardakov and Vesnin ([1, Question 1]).

2 Preliminaries

A number of equivalent characterizations of $R_n(f) = \pm 1$ were given in [3],[7]. We only need some of them:

Lemma 2.1 ([3, 7]) *For $f \in \mathbb{Z}[t]$ and $n \geq 1$ the following are equivalent:*

- *(a)* $R_n(f) = \pm 1$;
- *(b)* $f(\zeta_d)$ *is a unit in the ring* $\mathbb{Z}[\zeta_d]$ *for all* $d|n$ *where* ζ_d *denotes a primitive dth root of unity;*
- *(c)* $\mathrm{Res}(f, t^n - 1) = \pm 1$.

We record some properties of R_n; those in Proposition 2.2 follow directly from its definition.

Proposition 2.2 *Let $f, g \in \mathbb{Z}[t]$ and let $n \geq 1$. Then the following hold:*

- *(a)* $R_n(fg) = R_n(f)R_n(g)$;
- *(b)* $R_n(t) = (-1)^{n+1}$;
- *(c)* *If* $m|n$ *then* $R_m(f)|R_n(f)$.

Proposition 2.3 ([7]) *Let* $f(t) = c\prod_{j=1}^{k}(t-\beta_j)$. *Then*

$$R_n(f) = \Big((-1)^k c\Big)^n \prod_{j=1}^{k}(\beta_j^n - 1).$$

In [3] the expression $c^n \prod_{j=1}^{k}(\beta_j^n - 1)$ was denoted $B(f,n)$ and so $R_n(f) = \pm B(f,n)$.

Proposition 2.4 *Let* $f, F \in \mathbb{Z}[t]$ *be polynomials such that* $f(t) = F(t^\alpha)$ *for some* $\alpha \in \mathbb{N}$. *Then*

$$R_n(f) = \big(R_{n/(n,\alpha)}(F)\big)^{(n,\alpha)}.$$

In particular $R_n(t^m \pm t^k \pm 1) = \big(R_N(t^M \pm t^K \pm 1)\big)^{(n,m,k)}$ *where* $N = n/(n,m,k)$, $M = m/(m,k)$, $K = k/(m,k)$.

Proof
Let $d = (n,\alpha)$. Then we have

$$R_n(f) = \prod_{\theta^n=1} F(\theta^\alpha) = \prod_{q=0}^{n-1} F(e^{2\pi i q\alpha/n}) = \prod_{q=0}^{n-1} F(e^{2\pi i q(\alpha/d)/(n/d)})$$

which is equal to

$$\left(\prod_{q=0}^{n/d-1} F(e^{2\pi i q(\alpha/d)/(n/d)})\right)^d.$$

so $R_n(f) = \big(R_{n/d}(F)\big)^d$. Now since $(\alpha/d, n/d) = 1$, for each $q = 0, \ldots, (n/d-1)$ there exists a unique $Q = 0, \ldots, (n/d-1)$ such that $q(\alpha/d) = Q \bmod n/d$. Hence

$$\prod_{q=0}^{n/d-1} F(e^{2\pi i q(\alpha/d)/(n/d)}) = \prod_{Q=0}^{n/d-1} F(e^{2\pi i Q/(n/d)}) = \prod_{\phi^{n/d}=1} F(\phi) = R_{n/d}(F)$$

so $R_n(f) = \big(R_{n/d}(F)\big)^d$.

To prove the last claim let $f(t) = t^m \pm t^k \pm 1$ and $F(t) = t^M \pm t^K \pm 1$. □

Since $(N,M,K) = 1$, in considering when $R_n(t^m \pm t^k \pm 1) = \pm 1$ Proposition 2.4 allows us to assume that $(n,m,k) = 1$.

3 Properties of $R_n(t^m \pm t^k \pm 1)$

We have that $R_1(t^m + t^k + 1) = 3$ so by Proposition 2.2(c) $R_n(t^m + t^k + 1) \neq \pm 1$ for all n. Thus we may assume that at least one of the signs is a '$-$'.

Proposition 3.1 *(a)* $|R_n(t^m - t^k - 1)| = |R_n(t^k - t^m + 1)|$;

(b) $|R_n(t^m + t^k - 1)| = |R_n(t^{k-m} - t^k + 1)|$;

(c) $|R_n(t^m - t^k + 1)| = |R_n(t^m - t^{m-k} + 1)|$.

Proof
(a) $t^m - t^k - 1 = -(t^k - t^m + 1)$ so $|R_n(t^m - t^k - 1)| = |R_n(t^k - t^m + 1)|$.
(b) $t^m + t^k - 1 = t^k(t^{m-k} - t^{-k} + 1)$ so

$$\begin{aligned} R_n(t^m + t^k - 1) &= R_n(t^k)R_n(t^{m-k} - t^{-k} + 1) \\ &= (R_n(t))^k R_n(t^{k-m} - t^k + 1) \\ &= \pm R_n(t^{k-m} - t^k + 1). \end{aligned}$$

(c) $t^m - t^k + 1 = t^m(t^{-m} - t^{k-m} + 1)$ so

$$\begin{aligned} R_n(t^m - t^k + 1) &= R_n(t^m)R_n(t^{-m} - t^{k-m} + 1) \\ &= (R_n(t))^m R_n(t^m - t^{m-k} + 1) \\ &= \pm R_n(t^m - t^{m-k} + 1). \end{aligned}$$

□

Other similar identities can be established. For example [2, Theorem 2] implies that if n, m, k, m', k' are integers such that $(n, m, k) = 1$, $(n, m', k') = 1$, $(n, k') = 1$ and $m'(m - k) = mk'$ mod n then $R_n(t^m - t^k + 1) = \pm R_n(t^{m'} - t^{k'} + 1)$.

Parts (a) and (b) of Proposition 3.1 show that $R_n(t^m - t^k - 1) = \pm R_n(t^{m'} - t^{k'} + 1)$ (for some m', k') and $R_n(t^m + t^k - 1) = \pm R_n(t^{m'} - t^{k'} + 1)$ (for some m', k'), so we only need consider $R_n(f)$ for $f(t) = t^m - t^k + 1$. Moreover, by Proposition 2.4 we may assume that $(n, m, k) = 1$. Proposition 3.1(c) shows that the roles of k and $(m - k)$ can be interchanged.

The next result was prompted by [1, Lemma 1.3].

Proposition 3.2 *(a) If $(k, n) = 1$ then $R_n(t^m - t^k + 1) = R_n(t^{m\ell} - t + 1)$ where $\ell = k^{-1}$ mod n;*

(b) if $(m - k, n) = 1$ then $R_n(t^m - t^k + 1) = R_n(t^{m\ell} - t + 1)$ where $\ell = (m - k)^{-1}$ mod n;

(c) if $(m, n) = 1$ then $R_n(t^m - t^k + 1) = R_n(t - t^{k\ell} + 1)$ where $\ell = m^{-1}$ mod n.

Proof
(a) Let $\phi = \theta^k$, then $\theta = \phi^\ell$ so

$$R_n(t^m - t^k + 1) = \prod_{\theta^n=1} \theta^m - \theta^k + 1 = \prod_{\phi^n=1} (\phi^\ell)^m - \phi + 1 = R_n(t^{m\ell} - t + 1).$$

(b) This follows from (a) by interchanging the roles of k and $(m-k)$.
(c) Similar to (a). □

Parts (a),(b) of Proposition 3.2 show that it is sometimes enough to consider the polynomials considered by Odoni [7] (that is, polynomials of the form $t^m - t + 1$).

When $k = 0$ mod n or $k = m$ mod n it is clear that $R_n(t^m - t^k + 1) = \pm 1$. We can obtain the value of R_n in some other cases; for example by Proposition 2.3 and Proposition 2.4 we have that $R_n(t^0 - t^k + 1) = \left(2^{n/(n,k)} - 1\right)^{(n,k)}$. By [8, Lemma 3] we also have

Lemma 3.3 *Suppose that n is even, $(m,k) = 1$ and either $k = n/2$ mod n or $(m-k) = n/2$ mod n. Then $|R_n(t^m - t^k + 1)| = 2^{n/2} - (-1)^{m+n/2}$.*

4 Proof of Main Theorem

Odoni proved the Main Theorem in the case $k = 1$: we summarize this result ([7, Theorem 2(ii),(iii)]) as

Theorem 4.1 ([7]) *Let $n \geq 1$ and $f(t) = t^m - t + 1$ where $m \in \mathbb{Z}$. Then $R_n(f) = \pm 1$ if and only if ($(n,6) = 1$ and $m = 2$ mod n) or $m = 1$ mod n.*

Corollary 4.2 *Let $n \geq 1$ and $f(t) = t^m - t^k + 1$ where $m,k \in \mathbb{Z}$, $(n,m,k) = 1$ and suppose that either $(k,n) = 1$ or $(m-k,n) = 1$. Then $R_n(f) = \pm 1$ if and only if ($(n,6) = 1$ and $m = 2k$ mod n) or $k = 0$ mod n or $k = m$ mod n.*

Proof
By interchanging the roles of k and $(m-k)$ we may assume that $(k,n) = 1$. By Proposition 3.2(a) $R_n(f) = R_n(t^{m\ell} - t + 1)$, where $\ell = k^{-1}$ mod n. Now $m\ell = 1, 2$ mod n if and only if $m = k, 2k$ mod n, so the result follows from Theorem 4.1. □

The following corollary generalizes [7, Lemma 3.2] to our setting.

Corollary 4.3 *Let $n = p^u$ where $p = 2$ or 3, $u \geq 1$, and $f(t) = t^m - t^k + 1$ where $m,k \in \mathbb{Z}$, $(n,m,k) = 1$. Then $R_n(f) = \pm 1$ if and only if $k = 0$ mod n or $k = m$ mod n.*

Proof
The hypotheses imply that either $(k,n) = 1$ or $(m-k,n) = 1$ and so the result follows from Corollary 4.2. □

The 'if' direction of the Main Theorem is straightforward to prove (see [8, Lemma 5]) so from now on we focus on the 'only if' direction.

Lemma 4.4 *Let $n = 2^r 3^s \geq 1$ and $f(t) = t^m - t^k + 1$ where $m, k \in \mathbb{Z}$, $(n, m, k) = 1$. If $R_n(f) = \pm 1$ then $k = 0$ mod n or $k = m$ mod n.*

Proof
By Corollary 4.3 we may assume $r \geq 1, s \geq 1$. Now $R_n(f) = \pm 1$ implies $R_{2^r}(f) = \pm 1$ and so by Corollary 4.3 we have $k = 0 \bmod 2^r$ or $(m - k) = 0 \bmod 2^r$. By interchanging the roles of k and $(m - k)$ we may assume that the first of these holds. We also have $R_{3^s}(f) = \pm 1$ so $k = 0 \bmod 3^s$ or $k = m \bmod 3^s$. In the first case we have $k = 0 \bmod n$, so assume the second.

Let $d = 2 \cdot 3^s$. Now $k \neq m \bmod d$, for otherwise $2|(n, m, k) = 1$; thus $k = m + d/2 \bmod d$. It follows that $(m \bmod d, k \bmod d) = 1$ so Lemma 3.3 implies that $R_d(f) \neq \pm 1$ and hence $R_n(f) \neq \pm 1$. □

Our next lemma generalizes [7, Lemma 3.3] to our setting. We use ideas from the proof of that result.

Lemma 4.5 *Let $n = pq$ where $q = 2$ or 3 and $p \geq 5$ is prime, $f(t) = t^m - t^k + 1$ where $m, k \in \mathbb{Z}$, $(n, m, k) = 1$. If $R_n(f) = \pm 1$ then $k = 0$ mod n or $k = m$ mod n.*

Proof
By Corollary 4.2 we may assume $(k, n) > 1$, $(m - k, n) > 1$ so (since $(n, m, k) = 1$) either ($q|k$ and $p|(m - k)$) or ($p|k$ and $q|(m - k)$). By interchanging the roles of k and $(m - k)$ we may assume that the first case occurs, i.e. $k = 0 \bmod q$ and $(m - k) = 0 \bmod p$. Moreover we may assume $k \neq 0 \bmod p$, $(m - k) \neq 0 \bmod q$ for otherwise $k = 0 \bmod n$ or $(m - k) = 0 \bmod n$.

If either p or q divides m then we get a contradiction to $(n, m, k) = 1$ so $(m, n) = 1$. Now by Proposition 3.2 $R_n(f) = R_n(g)$ where $g(t) = t^{m'} - t^{k'} + 1$ where $m' = 1$, $k' = km^{-1}$. The conditions on m, k imply $k' = 0 \bmod q$, $(m' - k') = 0 \bmod p$, $(m' - k') \neq 0 \bmod q$. When $q = 2$ we have that $(m' - k') = n/2 \bmod n$ and since $(m', k') = 1$ Lemma 3.3 implies $R_n(g) \neq \pm 1$.

Suppose then that $q = 3$ and $R_n(g) = \pm 1$. We have that $k' = 1 \bmod p$, $k' = 0 \bmod 3$. Now $R_{3p}(g) = \pm 1$ so, writing ζ_d for a primitive dth root of unity, Lemma 2.1 implies that $g(\theta)$ is a unit in $\mathbb{Z}[\zeta_{3p}]$ whenever θ is a primitive $(3p)$th root of unity. In particular $g(\zeta_p\zeta_3) = \zeta_p(\zeta_3 - 1) + 1$ and $g(\zeta_p\zeta_3^2) = \zeta_p(\zeta_3^2 - 1) + 1$ are units in $\mathbb{Z}[\zeta_{3p}]$ and hence so is their product $3\zeta_p^2 - 3\zeta_p + 1$, which must therefore also be a unit in $\mathbb{Z}[\zeta_p]$. Let $h(x) = 3x^2 - 3x + 1$. Then $h(1) = 1$ and $h(\zeta_p)$ are units in $\mathbb{Z}[\zeta_p]$ so by Lemma 2.1 we have that $R_p(h) = \pm 1$. Now Proposition 2.3 implies that $R_p(h) = 3^p(\beta_1^p - 1)(\beta_2^p - 1)$ where $\beta_1, \beta_2 = 3^{-1/2}e^{\pm i\pi/6}$ are the roots of h. But

$$3^p(\beta_1^p - 1)(\beta_2^p - 1) = 3^p + 1 \pm 3^{(p+1)/2} \neq \pm 1$$

and we have a contradiction. □

Our next result (Lemma 4.8) deals with the case $(n, 6) = 1$. It generalizes [7, Lemma 3.1] to our setting and its proof is essentially a re-run of the proof of that result. We will require the following theorem of Kronecker, a proof of which can be found on page 46 of [6].

Lemma 4.6 *Let $\beta = \beta_1$ be a non-zero algebraic integer and let $\beta_1, \ldots, \beta_k$ be the conjugates of β over $\mathbb{Q}$. If $\max_j |\beta_j| \leq 1$ then β is a root of unity.*

We will also need the following:

Lemma 4.7 *If $\sum_{i=1}^{\ell} w_i^j = \sum_{i=1}^{\ell} z_i^j$ for all $j = 1, \ldots, \ell$ then the multisets $\{w_1, \ldots, w_\ell\}$ and $\{z_1, \ldots, z_\ell\}$ are equal.*

The proof is a standard application of the Newton-Girard formula and so is omitted.

Lemma 4.8 *Let $n \geq 1$ and $f(t) = t^m - t^k + 1$ where $m, k \in \mathbb{Z}$, $(n, m, k) = 1$ and suppose $(n, 6) = 1$. If $R_n(f) = \pm 1$ then $m = 2k$ mod n or $k = 0$ mod n or $k = m$ mod n.*

Proof
By Lemma 2.1 $\lambda = f(\zeta)$ is a unit in the ring $\mathbb{Z}[\zeta]$ for some primitive nth root of unity ζ, and therefore so is $\sigma(\lambda)$ for any $\sigma \in \Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. Let $\mu = \overline{\lambda}\lambda^{-1}$. Then, since Γ is abelian, we have

$$|\sigma(\mu)|^2 = \sigma(\mu)\overline{\sigma(\mu)} = \sigma(\mu\overline{\mu}) = \sigma(1) = 1.$$

Lemma 4.6 implies that μ is a root of unity in $\mathbb{Q}(\zeta)$, and thus $\mu = s\zeta^j$ for some $j \in \mathbb{Z}$, $s = \pm 1$. Since $\mu = \overline{\lambda}\lambda^{-1} = \overline{f(\zeta)}f(\zeta)^{-1} = f(\zeta^{-1})f(\zeta)^{-1}$ it follows that

$$s\zeta^j(\zeta^m - \zeta^k + 1) = \zeta^{-m} - \zeta^{-k} + 1. \tag{4.1}$$

Case 1: $s = -1$. Let $w_1 = \zeta^{-m}$, $w_2 = 1$, $w_3 = \zeta^{m+j}$, $w_4 = \zeta^j$, $z_1 = \zeta^{-k}$, $z_2 = \zeta^{j+k}$, $z_3 = z_4 = 0$. Then (4.1) is equivalent to

$$w_1 + w_2 + w_3 + w_4 = z_1 + z_2 + z_3 + z_4. \tag{4.2}$$

Since $(n, 6) = 1$ we have that $(r, n) = 1$ for $r = 1, 2, 3, 4$. Thus the maps $\zeta \mapsto \zeta^r$ $(r = 1, 2, 3, 4)$ are automorphisms of $\mathbb{Q}(\zeta)$. Applying these to (4.2) we get

$$\sum_{i=1}^{4} w_i^r = \sum_{i=1}^{4} z_i^r \quad (r = 1, 2, 3, 4). \tag{4.3}$$

By Lemma 4.7 we have that $\{w_1, w_2, w_3, w_4\} = \{z_1, z_2, z_3, z_4\}$, but $z_3 = 0 \notin \{w_1, w_2, w_3, w_4\}$ which gives a contradiction.

Case 2: $s = +1$. Let $w_1 = \zeta^{-m}$, $w_2 = 1$, $w_3 = \zeta^{k+j}$, $z_1 = \zeta^{-k}$, $z_2 = \zeta^{j+m}$, $z_3 = \zeta^{j}$. Then (4.1) is equivalent to

$$w_1 + w_2 + w_3 = z_1 + z_2 + z_3. \qquad (4.4)$$

As in Case 1, the maps $\zeta \mapsto \zeta^r$ $(r = 1, 2, 3)$ are automorphisms of $\mathbb{Q}(\zeta)$ and applying them to (4.4) gives $\{w_1, w_2, w_3\} = \{z_1, z_2, z_3\}$. If $(z_1, z_2, z_3) = (w_3, w_1, w_2)$ then $\zeta^{2k} = \zeta^{2m} = 1$ so $k = 0$ or $n/2$ mod n and $m = 0$ or $n/2$ mod n and so $k = 0$ or m mod n or $m = 2k$ mod n. If $(z_1, z_2, z_3) = (w_1, w_2, w_3), (w_1, w_3, w_2)$, or (w_2, w_3, w_1) then $\zeta^{m-k} = 1$ and hence $k = m$ mod n. If $(z_1, z_2, z_3) = (w_3, w_2, w_1)$ then $2k = m$ mod n. If $(z_1, z_2, z_3) = (w_2, w_1, w_3)$ then $k = 0$ mod n. □

Proof of Main Theorem

The 'if' direction was proved in [8, Lemma 5] so suppose that $R_n(f) = \pm 1$. By Lemmas 4.4 and 4.8 we may assume $n = ab$ where $a = 2^r 3^s > 1$, $(b, 6) = 1$, $b > 1$. Now $R_a(f) = \pm 1$ implies (by Lemma 4.4) that $k = 0$ mod a or $(m - k) = 0$ mod a. By interchanging the roles of k and $(m - k)$ we may assume that $k = 0$ mod a. Also, $R_b(f) = \pm 1$ implies (by Lemma 4.8) that $k = 0$ mod b or $m = 2k$ mod b or $k = m$ mod b. If $k = 0$ mod b then $k = 0$ mod n so assume otherwise.

Suppose $m = 2k$ mod b. Then no prime divisor of n divides $(m - k)$ for otherwise it would also divide $(n, m, k) = 1$. Therefore $(m - k, n) = 1$ and the result follows from Corollary 4.2. Suppose then that $k = m$ mod b and let $p \geq 5$ be a prime divisor of b and let $q = 2$ if $r \geq 1$ or $q = 3$ otherwise. Now $k \neq m$ mod pq and $k \neq 0$ mod pq for otherwise $q|(n, m, k) = 1$ or $p|(n, m, k) = 1$ (respectively) and so Lemma 4.5 implies $R_{pq}(f) \neq \pm 1$ so $R_n(f) \neq \pm 1$. □

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