The Tits alternative for generalized triangle groups of type $(3, 4, 2)$

James Howie and Gerald Williams

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Abstract

A generalized triangle group is a group that can be presented in the form $G = \langle\ x, y \mid x^p = y^q = w(x, y)^r = 1\ \rangle$ where $p, q, r \geq 2$ and $w(x, y)$ is a cyclically reduced word of length at least 2 in the free product $\mathbb{Z}_p * \mathbb{Z}_q = \langle\ x, y \mid x^p = y^q = 1\ \rangle$. Rosenberger has conjectured that every generalized triangle group G satisfies the Tits alternative. It is known that the conjecture holds except possibly when the triple (p, q, r) is one of $(2, 3, 2)$, $(2, 4, 2)$, $(2, 5, 2)$, $(3, 3, 2)$, $(3, 4, 2)$, or $(3, 5, 2)$. In this paper we show that the Tits alternative holds in the case $(p, q, r) = (3, 4, 2)$.

Keywords: Generalized triangle group, Tits alternative, free subgroup.
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1 Introduction

A *generalized triangle group* is a group that can be presented in the form

$$G = \langle\ x, y \mid x^p = y^q = w(x, y)^r = 1\ \rangle$$

where $p, q, r \geq 2$ and $w(x, y)$ is a cyclically reduced word of length at least 2 in the free product $\mathbb{Z}_p * \mathbb{Z}_q = \langle\ x, y \mid x^p = y^q = 1\ \rangle$ that is not a proper power. It was conjectured by Rosenberger [15] that every generalized triangle group G satisfies the Tits alternative. That is, G either contains a non-abelian free subgroup or has a soluble subgroup of finite index.

If $1/p + 1/q + 1/r < 1$ then G contains a non-abelian free subgroup [2]; if $r \geq 3$ then the Tits alternative holds, and in most cases G contains a non-abelian free subgroup [8]. (These results are also described in the survey article [9] and in [10].) The cases $r = 2$, $1/p + 1/q + 1/r \geq 1$ have had to be treated on a case by case basis. The Tits alternative was shown to hold for the cases $(3, 6, 2), (4, 4, 2)$ in [13], and for the cases $(2, q, 2)$ $(q \geq 6)$ in [1],[3],[4],[6],[14]. Thus the open cases of the conjecture are $(p, q, r) = (2, 3, 2), (2, 4, 2), (2, 5, 2), (3, 3, 2), (3, 4, 2)$, and $(3, 5, 2)$. In this paper show that the conjecture holds for the case $(3, 4, 2)$:

Main Theorem. *Let* $\Gamma = \langle\, x, y \mid x^3 = y^4 = w(x,y)^2 = 1 \,\rangle$ *where* $w(x,y) = x^{\alpha_1}y^{\beta_1}\ldots x^{\alpha_k}y^{\beta_k}$, $1 \le \alpha_i \le 2$, $1 \le \beta_i \le 3$ *for each* $1 \le i \le k$ *where* $k \ge 1$. *Then the Tits alternative holds for* Γ.

Benyash-Krivets and Barkovich [5],[6] have proved this result when k is even, and for this reason we focus on the case when k is odd.

2 Preliminaries

We first recall some definitions and well-known facts concerning generalized triangle groups; further details are available in (for example) [9].

Let $G = \langle\, x, y \mid x^\ell = y^m = w(x,y)^2 = 1 \,\rangle$ where $w(x,y) = x^{\alpha_1}y^{\beta_1}\ldots x^{\alpha_k}y^{\beta_k}$, $1 \le \alpha_i < \ell$, $1 \le \beta_i < m$ for each $1 \le i \le k$ where $k \ge 1$. A homomorphism $\rho : G \to H$ (for some group H) is said to be *essential* if $\rho(x), \rho(y), \rho(w)$ are of orders $\ell, m, 2$ respectively. By [2] G admits an essential representation into $PSL(2,\mathbb{C})$.

A projective matrix $A \in PSL(2,\mathbb{C})$ is of order n if and only if $\mathrm{tr}(A) = 2\cos(q\pi/n)$ for some $(q,n) = 1$. Note that in $PSL(2,\mathbb{C})$ traces are only defined up to sign. A subgroup of $PSL(2,\mathbb{C})$ is said to be *elementary* if it has a soluble subgroup of finite index, and is said to be *non-elementary* otherwise.

Let $\rho : \langle\, x, y \mid x^\ell = y^m = 1 \,\rangle \to PSL(2,\mathbb{C})$ be given by $x \mapsto X,\ y \mapsto Y$ where X, Y have orders ℓ, m, respectively. Then $w(x,y) \mapsto w(X,Y)$. By Horowitz [12] $\mathrm{tr}w(X,Y)$ is a polynomial with integer coefficients in $\mathrm{tr}X, \mathrm{tr}Y, \mathrm{tr}XY$, of degree k in $\mathrm{tr}XY$. Since X, Y have orders ℓ, m, respectively, we may assume (by composing ρ with an automorphism of $\langle\, x, y \mid x^\ell = y^m = 1 \,\rangle$, if necessary), that $\mathrm{tr}X = 2\cos(\pi/\ell)$, $\mathrm{tr}Y = 2\cos(\pi/m)$. Moreover (again by [12]) X and Y can be any elements of $PSL(2,\mathbb{C})$ with these traces. We refer to $\mathrm{tr}w(X,Y)$ as the *trace polynomial* of G. The representation ρ induces an essential representation $G \to PSL(2,\mathbb{C})$ if and only if $\mathrm{tr}\rho(w) = 0$; that is, if and only if $\mathrm{tr}XY$ is a root of $\mathrm{tr}w(X,Y)$. By [12] the leading coefficient of $\mathrm{tr}w(X,Y)$ is given by

$$c = \prod_{i=1}^{k} \frac{\sin(\alpha_i\pi/\ell)\sin(\beta_i\pi/m)}{\sin(\pi/\ell)\sin(\pi/m)}. \tag{1}$$

Now if X, Y generate a non-elementary subgroup of $PSL(2,\mathbb{C})$ then $\rho(G)$ (and hence G) contains a non-abelian free subgroup. Thus in proving that G contains a non-abelian free subgroup we may assume that X, Y generate an elementary subgroup of $PSL(2,\mathbb{C})$. By Corollary 2.4 of [15] there are then three possibilities: (i) X, Y generate a finite subgroup of $PSL(2,\mathbb{C})$; (ii) $\mathrm{tr}[X,Y] = 2$; or (iii) $\mathrm{tr}XY = 0$. The finite subgroups of $PSL(2,\mathbb{C})$ are the alternating groups A_4 and A_5, the symmetric group S_4, cyclic and dihedral groups (see for example [7]). The Fricke identity

$$\mathrm{tr}[X,Y] = (\mathrm{tr}X)^2 + (\mathrm{tr}Y)^2 + (\mathrm{tr}XY)^2 - (\mathrm{tr}X)(\mathrm{tr}Y)(\mathrm{tr}XY) - 2$$

implies that (ii) is equivalent to $\mathrm{tr}XY = 2\cos(\pi/\ell \pm \pi/m)$. These values occur as roots of $\mathrm{tr}w(X,Y)$ if and only if G admits an essential cyclic representation. Such a representation can be realized as $x \mapsto A, y \mapsto B$ where

$$A = \begin{pmatrix} e^{i\pi/\ell} & 0 \\ 0 & e^{-i\pi/\ell} \end{pmatrix}, \quad B = \begin{pmatrix} e^{\pm i\pi/m} & 0 \\ 0 & e^{\mp i\pi/m} \end{pmatrix}.$$

We summarize the above as

Lemma 1 *Let $G = \langle\ x, y \mid x^\ell = y^m = w(x,y)^2 = 1\ \rangle$. Suppose $G \to PSL(2, \mathbb{C})$ is an essential representation given by $x \mapsto X, y \mapsto Y$, where $\mathrm{tr}X = 2\cos(\pi/\ell)$, $\mathrm{tr}Y = 2\cos(\pi/m)$. If G does not contain a non-abelian free subgroup then one of the following occurs:*

1. *X, Y generate A_4, S_4, A_5 or a finite dihedral group;*
2. *$\mathrm{tr}XY = 2\cos(\pi/\ell \pm \pi/m)$;*
3. *$\mathrm{tr}XY = 0$.*

Case (2) occurs if and only if G admits an essential cyclic representation.

3 Proof of Main Theorem

Throughout this section Γ will be the group defined in the Main Theorem.

Lemma 2 *If Γ admits an essential cyclic representation then Γ contains a non-abelian free subgroup.*

Proof
Let $\rho : \Gamma \to \mathbb{Z}_{12}$ be an essential representation. Then $K = \mathrm{ker}\rho$ has a deficiency zero presentation with generators

$$\begin{array}{lll} a_1 = yxy^{-1}x^{-1}, & a_2 = y^2xy^{-2}x^{-1}, & a_3 = y^3xy^{-3}x^{-1}, \\ a_4 = xyxy^{-1}x^{-2}, & a_5 = xy^2xy^{-2}x^{-2}, & a_6 = xy^3xy^{-3}x^{-2}, \end{array}$$

and with relators

$$W'(a_i, \ldots, a_6, a_1, \ldots, a_{i-1})W'(y^2a_iy^2, \ldots, y^2a_6y^2, y^2a_1y^2, \ldots, y^2a_{i-1}y^2) \quad (1 \le i \le 6)$$

where W' is a rewrite of W.

Let $S = \{\, [a_i, a_j],\ a_i(y^2a_iy^2)\ (1 \le i, j \le 6)\,\}$, and let L, N respectively be the normal closures of S and $S \cup \{a_6\}$ in K. Noting that

$$\begin{array}{lll} y^2a_1y^2 = a_3a_2^{-1}, & y^2a_2y^2 = a_2^{-1}, & y^2a_3y^2 = a_1a_2^{-1}, \\ y^2a_4y^2 = a_2a_6a_5^{-1}a_2^{-1}, & y^2a_5y^2 = a_2a_5^{-1}a_2^{-1}, & y^2a_6y^2 = a_2a_4a_5^{-1}a_2^{-1}, \end{array}$$

we have that $K/L \cong \mathbb{Z}^4$ and $K/N \cong \mathbb{Z}^3$, and hence that $N/N' \neq 0$.

Let $\phi : K \to K$ be given by $a_i \mapsto y^2a_iy^2$ $(1 \le i \le 6)$. It is clear from the presentation of K that ϕ is an automorphism of K; furthermore $\phi(N) = N$. In the abelian group K/N, $\phi(a_i) = y^2 a_i y^2 = a_i^{-1}$ $(1 \leq i \leq 6)$. That is, ϕ induces the antipodal automorphism $\alpha \mapsto -\alpha$ on K/N. By Corollary 3.2 of [13], K contains a non-abelian free subgroup. $\square$

We will write the trace polynomial of Γ as $\tau(\lambda) = \mathrm{tr} w(X,Y)$, where $\mathrm{tr}(X) = 1$, $\mathrm{tr}(Y) = \sqrt{2}$, $\lambda = \mathrm{tr}(XY)$. By Lemmas 1 and 2 we may assume that $\mathrm{tr}XY = 0$ or X, Y generate A_4, S_4, or A_5. But Y has order 4 so X, Y cannot generate A_4 or A_5. If X, Y generate S_4 then the product XY has order 2 or 4 so $\mathrm{tr}XY = 0, \pm\sqrt{2}$. Suppose $\mathrm{tr}XY = -\sqrt{2}$. It follows from the identity

$$\mathrm{tr}XY + \mathrm{tr}X^{-1}Y = (\mathrm{tr}X)(\mathrm{tr}Y)$$

that $\mathrm{tr}X^{-1}Y = 2\sqrt{2}$. Replacing X by X^{-1} in Lemma 1 shows that Γ contains a non-abelian free subgroup. Thus we may assume that the only roots $\lambda = \mathrm{tr}XY$ of τ are $\lambda = 0, \sqrt{2}$. Using (1) the leading coefficient of τ is given by $c = \pm(\sqrt{2})^{\kappa}$ where κ denotes the number of values of i for which $\beta_i = 2$. Hence $\tau(\lambda)$ takes the form

$$\tau(\lambda) = (\sqrt{2})^{\kappa} \lambda^s (\lambda - \sqrt{2})^{k-s} \tag{2}$$

where $s \geq 0$. Moreover, Theorem 2 of [6] implies that the Main Theorem holds when k is even, so we may assume that k is odd.

Let

$$A = \begin{pmatrix} e^{i\pi/3} & 0 \\ 1 & e^{-i\pi/3} \end{pmatrix}, \quad B = \begin{pmatrix} e^{i\pi/4} & z \\ 0 & e^{-i\pi/4} \end{pmatrix}.$$

Then $\mathrm{tr}A = 1$, $\mathrm{tr}B = \sqrt{2}$, $\mathrm{tr}AB = z - (\sqrt{6} - \sqrt{2})/2$. Consider the representation $\rho : \langle\ x, y \mid x^3 = y^4 = 1\ \rangle \to PSL(2, \mathbb{C})$ given by $x \mapsto A, y \mapsto B$. Then $\mathrm{tr}\rho(x^{\alpha_1} y^{\beta_1} \ldots x^{\alpha_k} y^{\beta_k}) = \tau(z - (\sqrt{6} - \sqrt{2})/2)$ whose constant term (by (2)) is

$$\pm(\sqrt{2})^{\kappa}((\sqrt{6} - \sqrt{2})/2)^s((\sqrt{6} + \sqrt{2})/2)^{k-s}$$

which simplifies to

$$\pm(\sqrt{2})^{\kappa}((\sqrt{6} + \sqrt{2})/2)^{k-2s}.$$

Now the constant term in $\mathrm{tr}(A^{\alpha_1} B^{\beta_1} \ldots A^{\alpha_k} B^{\beta_k})$ is equal to

$$2\cos\left(\frac{(4\sum_{i=1}^{k} \alpha_i + 3\sum_{i=1}^{k} \beta_i)\pi}{12}\right).$$

Thus $(\sqrt{2})^{\kappa}((\sqrt{6}+\sqrt{2})/2))^{k-2s} = 2\cos\left(\frac{(4\sum_{i=1}^{k} \alpha_i + 3\sum_{i=1}^{k} \beta_i)\pi}{12}\right)$ and since k is odd, this only happens if $\kappa = 0$ and $k - 2s = \pm 1$. It follows that

$$4\sum_{i=1}^{k} \alpha_i + 3\sum_{i=1}^{k} \beta_i = 1, 5, 7, 11 \bmod 12. \tag{3}$$

Since $\kappa = 0$ there is no value of i for which $\beta_i = 2$ and hence Γ maps homomorphically onto the group

$$\bar{\Gamma} = \langle\ x, y \mid x^3 = y^2 = \bar{w}(x, y)^2 = 1\ \rangle \tag{4}$$

where $\bar{w}(x,y) = x^{\alpha_1}y \dots x^{\alpha_k}y$. If $\bar{w}$ is a proper power then $\bar{\Gamma}$ contains a non-abelian free subgroup by [2]. Thus we may assume that $\bar{w}$ is not a proper power, and so (4) is a presentation of $\bar{\Gamma}$ as a generalized triangle group.

We will write the trace polynomial of $\bar{\Gamma}$ as $\sigma(\mu) = \mathrm{tr}\bar{w}(\bar{X},\bar{Y})$, where $\mathrm{tr}(\bar{X}) = 1$, $\mathrm{tr}(\bar{Y}) = 0$, $\mu = \mathrm{tr}(\bar{X}\bar{Y})$. It follows from (3) that $\sum_{i=1}^{k}\alpha_i \neq 0 \bmod 3$ so $\bar{\Gamma}$ admits no essential cyclic representation. By Lemma 1 we may assume that $\mu = 0$ or $\bar{X},\bar{Y}$ generate A_4, S_4, A_5 or a finite dihedral group, in which case $\bar{X}\bar{Y}$ has order $2,3,4$, or 5 and hence $\mu = 0, \pm 1, \pm\sqrt{2}, (\pm 1 \pm \sqrt{5})/2$. Moreover $\bar{X}$ is of order 4 in $SL(2,\mathbb{C})$ so $\bar{X}^{-1} = -\bar{X}$ and thus $\mathrm{tr}(\bar{X}^{-1}\bar{Y}) = -\mu$ and $\mathrm{tr}\bar{w}(\bar{X},\bar{Y}) = (-1)^k\mathrm{tr}\bar{w}(\bar{X}^{-1},\bar{Y})$, so $\sigma_w(\mu) = \pm\sigma_w(-\mu)$. Thus μ and $-\mu$ occur as roots of σ with equal multiplicity. By (1) the leading coefficient of σ is ± 1 so

$$\sigma(\mu) = \pm\mu^{u_1}(\mu^2-1)^{u_2}(\mu^2-2)^{u_3}(\mu^2-(3+\sqrt{5})/2)^{u_4}(\mu^2-(3-\sqrt{5})/2)^{u_5}$$

where $u_1, u_2, u_3, u_4, u_5 \geq 0$ and $u_1 + 2u_2 + 2u_3 + 2u_4 + 2u_5 = k$. Since $\mathrm{tr}(\bar{X}\bar{Y})$ is a polynomial with integer coefficients in $\mathrm{tr}\bar{X} = 1, \mathrm{tr}\bar{Y} = 0$ we have that $u_5 = u_4$ so

$$\sigma(\mu) = \pm\mu^{u_1}(\mu^2-1)^{u_2}(\mu^2-2)^{u_3}(\mu^4-3\mu^2+1)^{u_4} \tag{5}$$

and $u_1 + 2u_2 + 2u_3 + 4u_4 = k$. Let

$$\tilde{A} = \begin{pmatrix} e^{i\pi/3} & 0 \\ 1 & e^{-i\pi/3} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} i & z \\ 0 & -i \end{pmatrix}.$$

Then $\mathrm{tr}\tilde{A} = 1$, $\mathrm{tr}\tilde{B} = 0$, $\mathrm{tr}\tilde{A}\tilde{B} = z - \sqrt{3}$. Now the constant term in $\sigma(z-\sqrt{3})$ is $(-\sqrt{3})^{u_1} \cdot 2^{u_2}$. But the constant term in $\mathrm{tr}(\tilde{A}^{\alpha_1}\tilde{B}\dots\tilde{A}^{\alpha_k}\tilde{B})$ is $2\cos((2\sum_{i=1}^{k}\alpha_i + 3k)\pi/3) = \pm\sqrt{3}$ so $u_1 = 1, u_2 = 0$ and thus $k = 1 + 2u_3 + 4u_4$.

Lemma 3 *If $\sqrt{2}$ is a repeated root of $\sigma(\mu)$ then Γ contains a non-abelian free subgroup.*

Proof
Let $q : \Gamma \to \bar{\Gamma}$ denote the canonical epimorphism. By hypothesis, there is an essential representation $\rho : \bar{\Gamma} \to PSL(2, \mathbb{C}[\mu]/(\mu-\sqrt{2})^2)$. Indeed, we can construct ρ explicitly via:

$$\rho(x) = \begin{pmatrix} e^{i\pi/3} & \mu \\ 0 & e^{-i\pi/3} \end{pmatrix}, \qquad \rho(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Composing this with the canonical epimorphism

$$\psi : PSL(2, \mathbb{C}[\mu]/(\mu-\sqrt{2})^2) \to PSL(2, \mathbb{C}[\mu]/(\mu-\sqrt{2})) \cong PSL(2,\mathbb{C})$$

gives an essential representation $\tilde{\rho} = \psi \circ \rho : \bar{\Gamma} \to PSL(2,\mathbb{C})$ with image S_4, corresponding to the root $\sqrt{2}$ of the trace polynomial.

Let $\bar{K}$ denote the kernel of $\tilde{\rho}$, V the kernel of ψ, and K the kernel of the composite map $\tilde{\rho} \circ q : \Gamma \to PSL(2,\mathbb{C})$. Then V is a complex vector space, since its elements have the form $\pm(I + (\mu-\sqrt{2})A)$ for various 2×2 matrices A, with multiplication

$$[\pm(I + (\mu-\sqrt{2})A)][\pm(I + (\mu-\sqrt{2})B)] = \pm(I + (\mu-\sqrt{2})(A+B)).$$

Now $\bar{K}$ is generated by conjugates of $(xy)^4$ and $\rho((xy)^4) = -I + (\mu - \sqrt{2})M$ where $M = \begin{pmatrix} 2\sqrt{2} & -2(1+i\sqrt{3}) \\ 2(1-i\sqrt{3}) & -2\sqrt{2} \end{pmatrix}$. Since M is non-zero, $\bar{K}$ (and hence K) maps onto the free abelian group of rank 1. Let N be a normal subgroup of K such that $K/N \cong \mathbb{Z}$.

Note that K is the fundamental group of a 2-dimensional CW-complex X arising from the given presentation of Γ. This complex X has 24 cells of dimension 0, 48 cells of dimension 1, and $24(\frac{1}{4} + \frac{1}{3} + \frac{1}{2}) = 26$ cells of dimension 2. Here, $24/4 = 6$ of the 2-cells (call them $\alpha_1, \ldots, \alpha_6$, say) arise from the relator y^4, $24/3 = 8$ ($\alpha_7, \ldots, \alpha_{14}$, say) arise from the relator x^3, and $24/2 = 12$ ($\alpha_{15}, \ldots, \alpha_{26}$, say) arise from the relator $w(x,y)^2$. Moreover, $\alpha_1, \ldots, \alpha_6$ are attached by maps which are 2nd powers. Let $\widehat{X}$ be the regular covering complex of X corresponding to the normal subgroup N of K and let $\widehat{\alpha}_i$ denote a lift of the 2-cell α_i. Then each of $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_6$ is a 2-cell attached by a map which is a 2nd power.

Let GF_2 denote the field of 2 elements. Now $H_2(\widehat{X}, GF_2)$ is a subgroup of the 2-chain group $C_2(\widehat{X}, GF_2)$ and since K/N freely permutes the cells of $\widehat{X}$, $C_2(\widehat{X}, GF_2)$ is a free $GF_2(K/N)$-module on the basis $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{26}$. Let Q be the free $GF_2(K/N)$-submodule of $C_2(\widehat{X}, GF_2)$ of rank 6 generated by $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_6$. Since these 2-cells are attached by maps which are 2nd powers, their boundaries in the 1-chain group $C_1(\widehat{X}, GF_2)$ are zero. Thus Q is a subgroup of $H_2(\widehat{X}, GF_2)$. Since the rank of Q is greater than $\chi(X) = 2$, Theorem A of [13] implies that K, and hence Γ, contains a non-abelian free subgroup $\square$

Lemma 4 *If* $(1+\sqrt{5})/2$ *is a repeated root of* $\sigma(\mu)$ *then* Γ *contains a non-abelian free subgroup.*

Proof

The proof is similar to that of Lemma 3. In this case $\tilde{\rho}$ has image A_5, corresponding to the root $(1+\sqrt{5})/2$. The complex X has 60 0-cells, 120 1-cells, and $60(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}) = 65$ 2-cells (so $\chi(X) = 5$). Moreover, $60/4 = 15$ of the 2-cells (call them $\alpha_1, \ldots, \alpha_{15}$, say) are attached by maps which are 2nd powers. As before, the free $GF_2(K/N)$-submodule, Q, of $C_2(\widehat{X}, GF_2)$ of rank 15 generated by $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{15}$ is a subgroup of $H_2(\widehat{X}, GF_2)$. Since the rank of Q is greater than $\chi(X)$, Theorem A of [13] again implies that K contains a non-abelian free subgroup. $\square$

By Lemmas 3 and 4 we may assume $u_3, u_4 \leq 1$ so $k \leq 7$. A computer search reveals that if $k = 3$ or 7 then there is no word $w(x,y)$ such that $\tau(\lambda)$ is of the form (2). If $k = 5$ then (up to cyclic permutation, inversion, and automorphisms of $\langle\ x \mid x^3\ \rangle$ and $\langle\ y \mid y^4\ \rangle$) the only word $w(x,y)$ with $\tau(\lambda)$ of the form (2) is $w = xyxyx^2y^3x^2yxy^3$. In this case, a computer search using GAP [11] shows that Γ contains a subgroup of index 4 which maps onto the free group of rank 2. If $k = 1$ then either $\Gamma = \langle\ x, y \mid x^3 = y^4 = (xy)^2 = 1\ \rangle$ or $\Gamma = \langle\ x, y \mid x^3 = y^4 = (xy^2)^2 = 1\ \rangle$. In the first case $\Gamma \cong S_4$, and in the second Γ can be written as an amalgamated free product

$$\Gamma = \langle\ x, y^2 \mid x^3 = y^4 = (xy^2)^2 = 1\ \rangle \underset{\langle\ y^2\ \mid\ y^4\ \rangle}{*} \langle\ y \mid y^4\ \rangle$$

in which the amalgamated subgroup has index 3 in the first factor and index 2 in the second, and thus Γ contains a non-abelian free subgroup. This completes the proof of the Main Theorem.

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Author addresses:

James Howie
School of Mathematical and Computer Sciences
Heriot-Watt University
Edinburgh EH14 4AS
`J.Howie@hw.ac.uk`

Gerald Williams (corresponding author)
Institute of Mathematics, Statistics and Actuarial Science
University of Kent
Canterbury
Kent CT2 7NF
`g.williams@kent.ac.uk`