BOUNDARY-ONLY INTEGRAL EQUATION APPROACH BASED ON POLYNOMIAL EXPANSION OF PLASMA CURRENT PROFILE TO SOLVE THE GRAD-SHAFRANOV EQUATION

Masafumi ITAGAKI, Jun-ichi KAMISAWADA, Shun-ichi OIKAWA
Graduate School of Engineering, Hokkaido University,
Kita 13, Nishi 8, Kita-ku, Sapporo 060-8628, JAPAN
Tel. +81-11-706-6659, Fax. +81-11-747-9366
E-mail: itagaki@qe.eng.hokudai.ac.jp

This paper contains
27 pages of text,
1 table and 9 figures.

ABSTRACT

A new type of boundary element method has been applied to solve the Grad-Shafranov equation and to give a distribution of magnetic flux function in a Tokamak nuclear fusion device. The quantity $\mu_0 r\, j_\varphi$ related to the plasma current profile is expanded into two-dimensional polynomial. Using the particular solution of the Grad-Shafranov equation with this inhomogeneous polynomial source and applying Green's second identity, the domain integral related to the plasma current is transformed into an equivalent boundary integral. Domain discretization is not required in this formulation, thus preserving all the advantages of the boundary element method. Numerical computations of all boundary integrals are only required in the initial stage of the eigenvalue iteration, so that the number of eigenvalue iterations does not hamper the total computing time. Test calculations demonstrated that the present method provides stable and accurate solutions.

Key words: tokamak, Grad-Shafranov equation, boundary element method, polynomial expansion, particular solution, boundary-only integral,

1. INTRODUCTION

The MHD equilibrium in axisymmetric plasma like a tokamak is described by the Grad-Shafranov equation **[1-4]** in terms of the magnetic flux function ψ. Analytic and numerical solutions for this equation are important in exploring the plasma configurations. The widely used numerical techniques to solve this equation is based on a 'domain' type solutions **[5, 6]**, such as finite elements and finite differences, however, some researchers recently attempted to apply the boundary element method (BEM) **[7]** to the analysis **[8-11]**. As the name implies, in the BEM the governing differential equation is transformed into a boundary integral equation that is applied over the boundary. The boundary is divided into small boundary segments (boundary elements) for the numerical integration, and then one solves a system of linear algebraic equations. The most important feature of the method is that it requires discretization of the surface only rather than of the volume. This advantage is particularly important in a series of frequent analyses; geometry data generation and modifications are easily performed. That is, the method is well suited for on-line plasma equilibrium analysis that requires efficient data preparation and computation following the change in plasma shape during the operation of an actual fusion device.

A difficulty arises when one attempts to transform the Grad-Shafranov equation into the boundary integral equation. The inhomogeneous term $\mu_0 r\, j_\varphi$ related to the plasma current j_φ still remains in the integral equation as a domain integral. If nothing is done for the domain integral, one cannot take advantage of the BEM that only the boundary discretization is required. As the plasma current is multiplied by the fundamental solution (Green's function for an infinite system) that has a singularity, to perform numerically the domain integral is quite troublesome, difficult to obtain accurate results, and furthermore, causes an immense consumption of computing time. This hampers the accomplishment of real time computing in future fusion reactor operations. As far as the authors are aware, however, in most of the attempts to apply the BEM to solve the Grad-Shafranov equation, they discretize the domain integral as it is without changing the domain integral to any boundary one **[11]**, or to apply to the equation for a vacuum region, i.e. the equation which has originally no inhomogeneous plasma current term **[8-10].**

The main purpose of the present work is to propose an elegant way of transforming the above domain integral into an equivalent boundary one. This idea is based on the polynomial expansion of the $\mu_0 r\, j_\varphi$ term in the Grad-Shafranov equation. The trick to transform the domain integral into a boundary one is to apply Green's second identity for the domain integral, as found in Section 3. Mathematical quantity which plays an important role in this transformation is a particular solution of magnetic flux function which satisfies the Grad-Shafranov equation with the above polynomial source term. The detailed form of this particular solution is derived in Section 5.

The problem to solve the Grad-Shafranov equation as a fixed boundary value problem is at the same time an eigenvalue problem, since the inhomogeneous plasma current term is a function of the unknown magnetic flux function. This type of eigenvalue search requires quite a number of iterations, however, all numerical values of boundary integrals required to assemble the matrix equation, which is the discretized form of the boundary integral equation, are calculated at the initial stage of iteration. Only the coefficients of the polynomial expansion are updated through the iteration. Another matrix to be used for this expansion coefficient determination is also invariant through the iteration. Therefore the number of iterations hardly affects the total computing time. As will be described in Section 6, one requires another type of domain integral to evaluate the above eigenvalue. However, it will be shown that this domain integral can be also transformed into a boundary one. The method proposed in the present paper does not require any computation of domain integral.

There is a possibility that the BEM shows another merit when applied to an 'inverse' problem to reproduce the profiles of magnetic flux and/or plasma current from data fixed along the plasma boundary, as the reason for this will be suggested in Section 8. The aim in the present work is, however, to propose a new formulation based on a boundary-only integral equation as the solution to the Grad-Shafranov equation, and to demonstrate its validity. Because of this, numerical examples in the present paper are limited to 'forward' problems, i.e., problems to seek the magnetic flux distribution within a plasma domain under the assumption of fixed boundary shape and with an appropriately parametrized plasma current profile as a function of magnetic flux.

2. BOUNDARY INTEGRAL EQUATION

For an axisymmetric (r, z) system the Grad-Shafranov equation is given by

$$-\Delta^{*}\psi \equiv -\left\{ r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial^{2}}{\partial z^{2}}\right\}\psi = \mu_{0}r^{2}\frac{dP}{d\psi}+\frac{d}{d\psi}\left(\frac{I^{2}}{2}\right)\equiv \mu_{0}r\, j_{\varphi} \quad (1)$$

,

where the magnetic flux function ψ is defined as $\psi = rA_{\varphi}$ with the toroidal component of vector potential A_{φ}, j_{φ} denotes the toroidal component of the plasma current, P the plasma pressure and I the poloidal current function. The aim in this section is to derive the boundary integral equation which corresponds to Eq.(1).

One here introduces the fundamental solution ψ^{*} which satisfies a subsidiary equation

$$-\Delta^{*}\psi^{*} = r\delta_{i}\,, \quad (2)$$

where δ_i is Dirac's delta function with the spike at the point i, say, the coordinates (a,b). Physically Eq.(2) describes the magnetic flux function for an arbitrary field point (r,z) caused by a unit toroidal current located at the point (a,b). The detailed form of the fundamental solution ψ^{*} is given by **[8-12]**

$$\psi^{*} = \frac{\sqrt{ar}}{\pi k}\left[\left(1-\frac{k^{2}}{2}\right)K(k)-E(k)\right] \quad (3)$$

with

$$k^{2} = \frac{4ar}{(r+a)^{2}+(z-b)^{2}}\quad , \quad (4)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and the second kind, respectively.

The second-order partial differential part $\Delta^{*}\psi$ can be reduced to

$$\Delta^{*}\psi = r\left(\nabla^{2}-\frac{1}{r^{2}}\right)\left(\frac{\psi}{r}\right). \quad (5)$$

Equation (5) includes the Laplace operator

$$\nabla^{2} = \frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^{2}}{\partial z^{2}},$$

and this is quite convenient to apply Green's second identity which will be shown as Eq.(7). Using Eq.(5), one finds the relationship

$$\int_{\Omega} \frac{1}{r^2}(\psi^* \Delta^* \psi - \psi \Delta^* \psi^*) d\Omega = \int_{\Omega} \left\{ \frac{\psi^*}{r} \nabla^2 \left(\frac{\psi}{r} \right) - \frac{\psi}{r} \nabla^2 \left(\frac{\psi^*}{r} \right) \right\} d\Omega . \tag{6}$$

Applying Green's second identity

$$\int_{\Omega} \left(\psi \nabla^2 \phi - \phi \nabla^2 \psi \right) d\Omega = \int_{\Gamma} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) d\Gamma \tag{7}$$

to the RHS of Eq.(6), one obtains

$$\text{The RHS of Eq.(6)} = \int_{\Gamma} \left\{ \frac{\psi^*}{r} \frac{\partial}{\partial n} \left(\frac{\psi}{r} \right) - \frac{\psi}{r} \frac{\partial}{\partial n} \left(\frac{\psi^*}{r} \right) \right\} d\Gamma = \int_{\Gamma} \left\{ \frac{\psi^*}{r^2} \frac{\partial \psi}{\partial n} - \frac{\psi}{r^2} \frac{\partial \psi^*}{\partial n} \right\} d\Gamma \ ,$$

where n is the outwardly directed normal direction on the boundary Γ. That is, instead of Green's second identity, one hereafter can use the following "reciprocal relationship" for the Grad-Shafranov operator:

$$\int_{\Omega} \frac{1}{r^2} \left(\psi^* \Delta^* \psi - \psi \Delta^* \psi^* \right) d\Omega = \int_{\Gamma} \frac{1}{r^2} \left(\psi^* \frac{\partial \psi}{\partial n} - \psi \frac{\partial \psi^*}{\partial n} \right) d\Gamma \tag{8}$$

Multiplying Eq.(1) and Eq.(2) by ψ^* / r^2 and ψ / r^2, respectively, subtracting, and integrating over the domain Ω, one has

$$\int_{\Omega} \frac{1}{r^2}(\psi^* \Delta^* \psi - \psi \Delta^* \psi^*) d\Omega = -\int_{\Omega} \frac{\psi^*}{r^2} (\mu_0 r \quad j_\varphi) d\Omega + \int_{\Omega} \frac{\psi}{r} \delta_i d\Omega ,$$

and next applying Eq.(8) to the LHS, one obtains

$$\int_{\Gamma} \left(\frac{\psi^*}{r^2} \frac{\partial \psi}{\partial n} - \frac{\psi}{r^2} \frac{\partial \psi^*}{\partial n} \right) d\Gamma = -\int_{\Omega} \frac{\psi^*}{r^2} (\mu_0 r \ j_\varphi) d\Omega + \int_{\Omega} \frac{\psi}{r} \delta_i d\Omega . \tag{9}$$

It should be noted that $d\Omega$ and $d\Gamma$ in Eq.(9) mean

$$d\Omega = 2\pi \, r dr dz \quad \text{and} \quad d\Gamma = 2\pi \, r \sqrt{dr^2 + dz^2} \ ,$$

respectively, i.e., Eq.(9) should be considered in a three-dimensional space. Equation (9) can then be rewritten as

$$2\pi \int_{\Gamma} \left(\frac{\psi^*}{r} \frac{\partial \psi}{\partial n} - \frac{\psi}{r} \frac{\partial \psi^*}{\partial n} \right) \sqrt{dr^2 + dz^2} = -2\pi \int_{\Omega} \frac{\psi^*}{r^2} (\mu_0 r \ j_\varphi) r dr dz + 2\pi r \psi_i \ .$$

Note that one can remove 2π from both sides of this equation. Now one denotes

$$d\Omega' = drdz \quad \text{and} \quad d\Gamma' = \sqrt{dr^2 + dz^2}$$

in the r-z plane, and hereafter one also redefines as

$$d\Omega = rdrdz \quad \text{and} \quad d\Gamma = r\sqrt{dr^2 + dz^2}\,,$$

omitting 2π for simplicity. Therefore the axisymmetric version of the integral equation can be written in the form:

$$\psi_i = \int_{\Gamma'} \left(\frac{\psi^*}{r} \frac{\partial \psi}{\partial n} - \frac{\psi}{r} \frac{\partial \psi^*}{\partial n} \right) d\Gamma' + \int_{\Omega} \frac{\psi^*}{r^2} (\mu_0 r\, j_\varphi) d\Omega\,. \tag{10}$$

The reason why $d\Omega$ on the second term of the RHS has not been changed to $d\Omega'$ is that Eq.(8) (or Green's second identity) need be again applied to the original $\Omega - \Gamma$ system to transform this domain integral into a boundary one. Equation (10) is valid for any point in the domain Ω, however, one needs to modify Eq.(10) when the point 'i' is located on the boundary Γ'. More general form of the boundary integral equation is given by

$$c_i \psi_i = \int_{\Gamma'} \left(\frac{\psi^*}{r} \frac{\partial \psi}{\partial n} - \frac{\psi}{r} \frac{\partial \psi^*}{\partial n} \right) d\Gamma' + \int_{\Omega} \frac{\psi^*}{r^2} (\mu_0 r\, j_\varphi) d\Omega\,, \tag{11}$$

where the constant c_i depends on the local boundary geometry under consideration: $c_i = 1.0$ for an internal point, while $c_i = 1/2$ on a smooth boundary. The derivation of this general form is shown in **Appendix A**.

3. BOUNDARY INTEGRAL EXPRESSION OF POLYNOMIAL SOURCE TERM

3.1 Polynomial expansion

The spatial distribution in the RHS of Eq.(1) is expressed using a polynomial with respect to ξ and η:

$$\mu_0 r\, j_\varphi \approx \sum_{\ell, m} \alpha_{\ell,m} \xi^\ell \eta^m \qquad (\ell \geq 0, \quad m \geq 0) \tag{12}$$

The absolute values of the dimensionless coordinates

$$\xi = r / L_r\,, \qquad \eta = (z - z_0) / L_z$$

do not exceed 1.0 when L_r, L_z, z_0 are taken as the outermost plasma radius in r-direction, the half length of z-directional plasma width and the z-coordinate of the plasma center, respectively.

Table 1 *Pascal's triangle for polynomial functions*

In more detail, one assumes here that the quantity $\mu_0 r j_\varphi$ is expanded up to a certain "level" of the "complete" polynomial that is specified according to Pascal's triangle shown in Table 1. For example, "level 2" means that $\mu_0 r j_\varphi$ is approximated as:

$$\mu_0 r\, j_\varphi \approx \alpha_{00} + \alpha_{10}\xi + \alpha_{01}\eta + \alpha_{20}\xi^2 + \alpha_{11}\xi\eta + \alpha_{02}\eta^2 .$$

3.2 Particular solution and the application of Eq.(8)

Using the above simple polynomial expansion, one rewrites Eq.(1) in the form:

$$\Delta^*\psi + \sum_{\ell,\, m} \alpha_{\ell,\, m}\xi^\ell \eta^m = 0 \tag{13}$$

Then the second term on the RHS of the boundary integral equation (11) is rewritten as

$$Q_i = \int_\Omega \frac{\psi^*}{r^2}(\mu_0 r\, j_\varphi) d\Omega = \int_\Omega \frac{\psi^*}{r^2}\left(\sum_{\ell, m}\alpha_{\ell, m}\xi^\ell\eta^m\right) d\Omega . \tag{14}$$

One here assumes the existence of a particular solution $\varphi^{(\ell, m)}$ which satisfies

$$\Delta^*\varphi^{(\ell,\, m)} + \xi^\ell\eta^m = 0 \tag{15}$$

.

The detailed mathematical form of $\varphi^{(\ell, m)}$ will be discussed in Section 5. Applying Eq.(15) and utilizing the reciprocal relationship given by Eq.(8), the quantity,

$$Q_i^{(\ell,\, m)} = \int_\Omega \frac{\psi^*}{r^2}(\xi^\ell \eta^m) d\Omega$$

can be arranged as follows:

$$\begin{aligned} Q_i^{(\ell,\, m)} &= -\int_\Omega \frac{\psi^*}{r^2}\Delta^*\varphi^{(\ell,\, m)} d\Omega \\ &= -\int_\Omega \frac{\varphi^{(\ell,\, m)}}{r^2}\Delta^*\psi^* d\Omega - \int_\Gamma \left\{ \left(\frac{\psi^*}{r^2}\right)\frac{\partial\varphi^{(\ell,\, m)}}{\partial n} - \left(\frac{\varphi^{(\ell,\, m)}}{r^2}\right)\frac{\partial\psi^*}{\partial n}\right\} d\Gamma \quad . \end{aligned} \tag{16}$$

Now reminding Eq.(2), Eq.(16) can be reduced to

$$Q_i^{(\ell,\,m)} = \int_\Omega \left(\frac{\varphi^{(\ell,\,m)}}{r}\right)\delta_i d\Omega - \int_\Gamma \left\{\left(\frac{\psi^*}{r^2}\right)\frac{\partial \varphi^{(\ell,\,m)}}{\partial n} - \left(\frac{\varphi^{(\ell,\,m)}}{r^2}\right)\frac{\partial \psi^*}{\partial n}\right\} d\Gamma\,. \tag{17}$$

Recalling $d\Omega = r\, d\Omega',\ d\Gamma = r\, d\Gamma'$ and introducing the singularity constant c_i, one obtains

$$Q_i^{(\ell,\,m)} = c_i\varphi_i^{(\ell,\,m)} - \int_{\Gamma'} \left\{\left(\frac{\psi^*}{r}\right)\frac{\partial \varphi^{(\ell,\,m)}}{\partial n} - \left(\frac{\varphi^{(\ell,\,m)}}{r}\right)\frac{\partial \psi^*}{\partial n}\right\} d\Gamma'\,. \tag{18}$$

That is, the domain integral related to the polynomial source is transformed into an equivalent boundary integral:

$$Q_i = \sum_{\ell,\,m}\alpha_{\ell,\,m} Q_i^{(\ell,\,m)} = c_i \sum_{\ell,\,m}\alpha_{\ell,\,m}\varphi_i^{(\ell,\,m)} - \sum_{\ell,\,m}\alpha_{\ell,\,m}\int_{\Gamma'} \left\{\left(\frac{\psi^*}{r}\right)\frac{\partial \varphi^{(\ell,\,m)}}{\partial n} - \left(\frac{\varphi^{(\ell,\,m)}}{r}\right)\frac{\partial \psi^*}{\partial n}\right\} d\Gamma'\,. \tag{19}$$

Consequently the boundary integral equation corresponding to the Grad-Shafranov equation with the polynomial source term can be given in the form:

$$c_i\psi_i - \int_{\Gamma'}\left(\frac{\psi^*}{r}\frac{\partial\psi}{\partial n} - \frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma' = \sum_{\ell,m}\alpha_{\ell,m}\left\{c_i\varphi_i^{(\ell,m)} - \int_\Gamma\left(\frac{\psi^*}{r}\frac{\partial\varphi^{(\ell,m)}}{\partial n} - \frac{\varphi^{(\ell,m)}}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma'\right\}. \tag{20}$$

It should be stressed that Eq.(20) is expressed only by curvilinear integrals. The derivation process shown here is equivalent to what is called the “Dual Reciprocity Method (DRM)” in the boundary element research field **[13]**, although the method was originally applied to solve the Poisson-type equation. The name of “dual reciprocity” comes from the fact that the reciprocity theorem (Eq.(8) or Green’s second identity) is applied to both sides of the equation to take all terms to the boundary. That is, once the theorem is applied to the LHS of the Grad-Shafranov equation as described in section 2, next it is applied to the RHS with the help of particular solutions.

3.3 Some interpretations of Eq.(20)

When all the expansion coefficients $\alpha_{\ell,m}$ take zero values, Eq.(20) is converted to the boundary integral equation for the magnetic flux function $\hat{\psi}$ in a vacuum region

$$c_i\hat{\psi}_i - \int_{\Gamma'}\left(\frac{\psi^*}{r}\frac{\partial\hat{\psi}}{\partial n} - \frac{\hat{\psi}}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma' = 0\,, \tag{21}$$

which corresponding to the homogeneous partial differential equation:

$$-\Delta^*\hat{\psi} = 0\,. \tag{22}$$

It is interesting to point out that Eq.(20) can also be derived in a simple way starting with Eqs.(21) and (22). Coupling the equation for a plasma region

$$-\Delta^*\psi = \mu_0 r\, j_\varphi = \sum_{\ell,m}\alpha_{\ell,m}\xi^\ell\eta^m$$

with the equation which the particular solutions satisfy

$$-\Delta^*\left(\sum_{\ell,m}\alpha_{\ell,m}\varphi^{(\ell,m)}\right) = \sum_{\ell,m}\alpha_{\ell,m}\xi^\ell\eta^m\,, \tag{23}$$

one knows the relationship:

$$-\Delta^*\left(\psi - \sum_{\ell,m}\alpha_{\ell,m}\varphi^{(\ell,m)}\right) = 0\,. \tag{24}$$

Note that Eqs. (22) and (24) have the same form. Then, substituting $\hat{\psi} = \psi - \sum_{\ell,m}\alpha_{\ell,m}\varphi^{(\ell,m)}$ into Eq.(21), one can easily reach Eq.(20). This derivation process is identical to the "particular solution technique" for the Poisson equation, which is described in the literature **[13]**.

Another interesting interpretation of Eq.(20) is as follows. If the process to derive Eq.(11) is again applied to the equation for a particular solution φ,

$$-\Delta^*\varphi = \mu_0 r\, j_\varphi,$$

instead of the original Grad-Shafranov equation, one also reaches the following integral equation

$$c_i\varphi_i = \int_{\Gamma'}\left(\frac{\psi^*}{r}\frac{\partial\varphi}{\partial n} - \frac{\varphi}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma' + \int_{\Omega}\frac{\psi^*}{r^2}(\mu_0 r\, j_\varphi)d\Omega\,. \tag{25}$$

Both Eq.(11) and Eq.(25) include the same term $\int_\Omega(\psi^*/r^2)(\mu_0 r\, j_\varphi)d\Omega$, then one easily find

$$c_i\psi_i - \int_{\Gamma'}\left(\frac{\psi^*}{r}\frac{\partial\psi}{\partial n} - \frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma' = c_i\varphi_i - \int_{\Gamma'}\left(\frac{\psi^*}{r}\frac{\partial\varphi}{\partial n} - \frac{\varphi}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma' . \tag{26}$$

Substituting $\varphi = \sum_{\ell,\, m}\alpha_{\ell,\, m}\varphi^{(\ell,\, m)}$ into Eq.(26), it will be completely identical to Eq.(20).

4. DISCRETIZATION － BOUNDARY ELEMENT METHOD

The following numerical schemes are basically the same as the ones in the widely used boundary element method **[7].**

4.1 Discretization using constant boundary element

For a digital computation, one simply discretizes Eq.(20) using constant boundary elements, i.e., the boundary Γ' is divided into N straight line segments. The values of ψ and $\partial\psi/\partial n$ are assumed to be constant on each element and equal to the value at the mid-node of the element. The discretized form can be written as

$$\begin{aligned} &c_i\psi_i - \sum_{j=1}^{N}\left(\frac{\partial\psi}{\partial n}\right)_j\int_{\Gamma_j}\frac{\psi^*}{r}d\Gamma' + \sum_{j=1}^{N}\psi_j\int_{\Gamma_j}\frac{1}{r}\frac{\partial\psi^*}{\partial n}d\Gamma' \\ &= \sum_{\ell,m}\alpha_{\ell,m}\left\{c_i\varphi_i^{(\ell,m)} - \sum_{j=1}^{N}\int_{\Gamma_j}\left(\frac{\psi^*}{r}\frac{\partial\varphi^{(\ell,m)}}{\partial n} - \frac{\varphi^{(\ell,m)}}{r}\frac{\partial\psi^*}{\partial n}\right)d\Gamma'\right\}. \quad (i = 1, 2, \cdots, n) \end{aligned} \tag{27}$$

Equation (27) is simplified as

$$\sum_{j=1}^{n}H_{i,j}\psi_j - \sum_{j=1}^{n}G_{i,j}q_j = Q_i \quad (i = 1, 2, \cdots, n)\,, \tag{28}$$

using the following notations:

$$q_j = (\partial\psi/\partial n)_j\,, \quad H_{i,j} = \hat{H}_{i,j} + c_i \ \ (i \neq j), \quad H_{i,j} = \hat{H}_{i,j} + c_i \ \ (i = j)$$

$$\hat{H}_{i,j} = \int_{\Gamma_j}\frac{1}{r}\frac{\partial\psi^*}{\partial n}d\Gamma', \quad G_{i,j} = \int_{\Gamma_j}\frac{\psi^*}{r}d\Gamma'.$$

It should be noted that Eq.(28) represents a set of n simultaneous algebraic equation for n unknowns, and can be written in the matrix form:

$$\mathbf{H}\psi - \mathbf{G}\mathbf{q} = \mathbf{Q} \tag{29}$$

4.2 Two stages of BEM computation

Reordering Eq.(29) in such a way that all the unknowns are on the left-hand side, Eq.(29) can be expressed as

$$\mathbf{A}\mathbf{x} = \mathbf{f}\ , \tag{30}$$

where $\mathbf{x}$ is the vector of the unknowns ψ and $\mathbf{q}$, $\mathbf{f}$ is the contributions of boundary conditions added to $\mathbf{Q}$. As the absolute value of the scalar flux function ψ is arbitrary, one can simply impose $\psi = 0$ along the plasma boundary. In this case the first stage of BEM computation is to solve

$$\mathbf{G}\mathbf{q} + \mathbf{Q} = \mathbf{0} \tag{31}$$

instead of Eq.(30).

Once all the values of ψ and $\mathbf{q}\,(=\partial\psi/\partial n)$ on the entire boundary have become known by solving Eq.(30) or Eq.(31), the values of magnetic flux at any internal point can be calculated using the discretized form of Eq.(20) with $c_i = 1.0$:

$$\psi_i = \sum_{j=1}^{n} G_{i,j} q_j - \sum_{j=1}^{n} \hat{H}_{i,j}\psi_j + Q_i , \tag{32}$$

In summary, the BEM calculation can be divided into two stages: the first calculation to seek all boundary values of ψ and $\partial\psi/\partial n$ and the second calculation for internal points, as illustrated in **Fig.1**.

Fig.1 *Two stages of boundary element calculation*

5. PARTICULAR SOLUTION FOR A MONOMIAL SOURCE TERM

The key to success to realize the boundary-only form given by Eq.(20) is to find an actual particular solution $\varphi^{(\ell,m)}$ which satisfies the Grad-Shafranov equation with a monomial source:

$$-\Delta^{*}\varphi^{(\ell,m)} = -r\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\varphi^{(\ell,m)}}{r}\right) = \xi^{\ell}\eta^{m} = \left(\frac{r}{L_r}\right)^{\ell}\left(\frac{z-z_0}{L_z}\right)^{m} . \quad (\ell \ge 0,\ m \ge 0) \tag{33}$$

The relationship

$$\Delta^*(\xi^\ell \eta^m) = \ell(\ell-2)\xi^{\ell-2}\eta^m\left(\frac{1}{L_r}\right)^2 + m(m-1)\xi^\ell \eta^{m-2}\left(\frac{1}{L_z}\right)^2 \tag{34}$$

is easily found. Substituting $m \to m+2$ into Eq.(34), one obtains the recurrence formula:

$$\xi^\ell \eta^m = \Delta^*\left(\frac{L_z^2}{(m+1)(m+2)}\xi^\ell \eta^{m+2}\right) - \frac{\ell(\ell-2)}{(m+1)(m+2)}\left(\frac{L_z}{L_r}\right)^2 \xi^{\ell-2}\eta^{m+2}$$

Next, applying this relationship itself to the second term of the LHS, one finds

$$\xi^\ell \eta^m = \Delta^*\left(\frac{L_z^2}{(m+1)(m+2)}\xi^\ell \eta^{m+2} - \frac{L_z^2}{(m+1)(m+2)}\cdot\frac{\ell(\ell-2)}{(m+3)(m+4)}\left(\frac{L_z}{L_r}\right)^2 \xi^{\ell-2}\mu^{m+4}\right)$$
$$+\frac{\ell(\ell-2)}{(m+1)(m+2)}\cdot\frac{(\ell-2)(\ell-4)}{(m+3)(m+4)}\left(\frac{L_z}{L_r}\right)^4 \xi^{\ell-4}\eta^{m+4}$$

This process can be repeated successively, and it should be noticed that the last term of the RHS vanishes before too long if ℓ is an even number. Even if ℓ is an odd number, the absolute values of the last term decreases rapidly. In this way, one finally obtains the following particular solution as an infinite series:

$$\varphi^{(\ell,m)} = -\frac{L_z^2 \xi^\ell \eta^{m+2}}{(m+1)(m+2)}\left[1. + \sum_{k=1}^{\infty}\prod_{s=1}^{k}\left\{-\frac{(\ell-2s+2)(\ell-2s)}{(m+2s+1)(m+2s+2)}\left(\frac{L_z\eta}{L_r\xi}\right)^2\right\}\right]. \tag{35}$$

Because of the geometry of usual tokamak fusion devices, one finds

$$\left|\frac{L_z\eta}{L_r\xi}\right| = \left|\frac{z-z_0}{r}\right| < 1.0$$

in Eq.(35), and this fact helps the stable convergence of the series. The derivatives of $\varphi^{(\ell,m)}$ in the r- and z-directions are given respectively by

$$\frac{\partial\varphi^{(\ell,m)}}{\partial r} = \frac{-\xi^{\ell-1}\eta^{m+2}}{(m+1)(m+2)}\left(\frac{L_z^2}{L_r}\right)\left[\ell + \sum_{k=1}^{\infty}(\ell-2k)\prod_{s=1}^{k}\left\{-\frac{(\ell-2s+2)(\ell-2s)}{(m+2s+1)(m+2s+2)}\left(\frac{L_z\eta}{L_r\xi}\right)^2\right\}\right] \tag{36a}$$

and

$$\frac{\partial\varphi^{(\ell,m)}}{\partial z} = \frac{-\xi^{\ell}\eta^{m+1}}{(m+1)(m+2)}\left(\frac{L_z^2}{L_z}\right)\left[m+2 + \sum_{k=1}^{\infty}(m+2k+2)\prod_{s=1}^{k}\left\{-\frac{(\ell-2s+2)(\ell-2s)}{(m+2s+1)(m+2s+2)}\left(\frac{L_z\eta}{L_r\xi}\right)^2\right\}\right]. \tag{36b}$$

6. EIGENVALUE ITERATION

The RHS of the Grad-Shafranov equation is often approximated in a simple form, e.g. **[8]**,

$$\mu_0 r\, j_\phi = \alpha\{r^2\beta_p + R_0^2(1-\beta_p)\}\exp(1-\gamma^2(1-X)^2), \tag{37}$$

where β_p is the poloidal beta, R_0 a characteristic radius of the machine, and $\alpha,\ \gamma$ are adjusted parameters. The normalized flux function X is defined by $X = (\psi-\psi_b)/(\psi_a-\psi_b)$,
where ψ_a is the value of ψ at the magnetic axis while ψ_b is the one on the plasma boundary.

As suggested from the original form of the Grad-Shafranov equation, Eq.(1), $\mu_0 r\, j_\varphi$ is a function of the unknown magnetic function ψ. Because of this, one needs to solve the equation iteratively as an eigenvalue problem described below.

6.1 Power iterative scheme to find eigenvalue

One here rewrites the Grad-Shafranov equation using the eigenvalue $\lambda^{(n)}$, as

$$-\Delta^*\psi^{(n)} = \lambda^{(n-1)} f(r,\psi^{(n-1)}) \equiv S^{(n)}, \qquad (n \ge 1) \tag{38}$$

where

$$f(r,\psi^{(n)}) \equiv \mu_0 r\, j_\varphi^{(n)} / \lambda^{(n)}, \tag{39}$$

and the values of $f(r,\psi^{(n)})$ can be calculated using a simple correlation formula such as Eq.(37). The iteration to seek the eigenvalue is performed in such a way that the relationship

$$\lambda^{(n-1)}\int_{\Omega'} f(r,\ \psi^{(n-1)})d\Omega' = \lambda^{(n)}\int_{\Omega'} f(r,\ \psi^{(n)})d\Omega' \tag{40}$$

is preserved through the iteration. That is, the eigenvalue is updated as follows:

$$\lambda^{(n)} = \lambda^{(n-1)} \frac{\int_{\Omega'} f(r,\psi^{(n-1)})d\Omega'}{\int_{\Omega'} f(r,\psi^{(n)})d\Omega'}\ . \tag{41}$$

A uniform source is assumed as the initial estimate of $S^{(n)}$, i.e., $S^{(1)} = const.$, then, solving Eq.(38) using the BEM, one obtains the distribution of magnetic flux function $\psi^{(1)}$ and then $f(r,\psi^{(1)})$. Sampling the values of $f(r,\psi^{(1)})$ for points in the plasma domain, one determines the expansion coefficients $\alpha_{\ell,m}$ in

Eq.(12), that will be used in the next $\psi^{(2)}$ computation. The detailed procedure to determine the coefficients is described in the next section. Thanks to this polynomial expansion, the domain integrals in Eq.(41) can be also performed using only boundary integrals, as described in Section 6.3. Once $\lambda^{(1)}$ has been calculated in this way, one again computes $\psi^{(2)}$ using the BEM scheme. The above process is repeated until a given convergence criterion, e.g.,

$$\varepsilon^{(n)} = \left| \frac{\lambda^{(n)} - \lambda^{(n-1)}}{\lambda^{(n-1)}} \right| < 10^{-5} \tag{42}$$

is satisfied. The above process is very similar to the fission source iterative scheme to find the critical eigenvalue in nuclear fission reactor analysis **[14]**.

6.2 Determination of the polynomial expansion coefficients

In every eigenvalue iteration, the polynomial expansion coefficients $\alpha_{\ell,m}$ in Eq.(12) can be determined as follows. First, one defines a rectangular domain that encloses the plasma region Ω under consideration. Next, one generates many sampling points uniformly within the rectangular domain. The points outside the domain Ω are automatically excluded with the aid of the residue theorem **[15]**. That is, whether a point $w_0 = (r_0, z_0) \equiv r_0 + i\, z_0$ resides inside or outside Ω can be determined by the result of the following complex integral:

$$\int_{\Gamma} \frac{dw}{w - w_0} \begin{cases} = 2\pi i & for \quad w_0 \in \Omega \\ = \quad 0 & for \quad w_0 \notin \Omega . \end{cases} \tag{43}$$

For all sampling points inside Ω, one calculates the values of $\mu_0 r\, j_\varphi$ according to a correlation formula, for example, Eq.(37). Based on the resultant distribution of $\mu_0 r\, j_\varphi$, the coefficients $\alpha_{\ell,m}$ in Eq.(12) are determined using the singular value decomposition (SVD) technique **[16]**.

6.3 Another boundary integral for eigenvalue calculation

It is interesting to point out that the domain integral defined by

$$\int_{\Omega'} f(r,\ \psi) d\Omega' = \int_{\Omega'} \sum_{\ell,\ m} \alpha_{\ell,\ m} \xi^{\ell} \eta^{m} d\Omega'$$

can be also transformed into a boundary one. If one finds a particular solution $\phi^{(\ell,m)}$ which satisfies

$$\nabla^2 \phi^{\ (\ell,m)} = \xi^{\ell} \eta^{m}, \tag{44}$$

one can apply Gauss' theorem to the domain integral, and then one obtains the relationship:

$$\int_{\Omega'} \sum_{\ell,\ m} \alpha_{\ell,\ m} \xi^{\ell} \eta^{m} d\Omega' = \int_{\Omega'} \sum_{\ell,\ m} \alpha_{\ell,m} \nabla^2 \phi^{\ (\ell,\ m)} d\Omega' = \int_{\Gamma'} \sum_{\ell,\ m} \alpha_{\ell,m} \frac{\partial \phi^{\ (\ell,\ m)}}{\partial \ \ n} d\Gamma'. \tag{45}$$

The particular solution is given by

$$\phi^{(\ell,m)} = \sum_{s=1}^{\left[\frac{m+2}{2}\right]} c_s \xi^{\ell+2s} \eta^{m-2s+2} + \sum_{s=1}^{\left[\frac{\ell+2}{2}\right]} d_s \xi^{\ell-2s+2} \eta^{m+2s}. \tag{46}$$

In Eq.(46), $[\ \cdot\ ]$ denotes the integer part of the argument. The coefficients c_s and d_s are evaluated using the recurrence relationship:

$$c_s = -\frac{(m-2s+4)(m-2s+3)}{(\ell+2s)(\ell+2s-1)} \cdot \frac{{L_r}^2}{{L_z}^2} \cdot c_{s-1} \quad (s \geq 2), \quad c_1 = \frac{{L_r}^2}{2(\ell+2)(\ell+1)} \tag{47a}$$

and

$$d_s = -\frac{(\ell-2s+4)(\ell-2s+3)}{(m+2s)(m+2s-1)} \cdot \frac{{L_z}^2}{{L_r}^2} \cdot d_{s-1} \quad (s \geq 2), \quad d_1 = \frac{{L_z}^2}{2(m+2)(m+1)}, \tag{47b}$$

where L_r and L_z denote the absolute lengths defined in Section 3.1.

7 NUMERICAL EXAMPLES

The following problems were solved using the present boundary-only type BEM so as to confirm its validity.

7.1 "Rectangular" Plasma

Suppose hypothetical rectangular plasma as shown in **Fig.2**. The boundary condition $\psi = 0$ is imposed along each side of the rectangle. In this case, as shown in **Appendix B**, the analytic solution exists for the equation with a monomial $r^{\ell} z^{m}$ source term

$$\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=-r^{\ell}z^{m}\qquad(\ell\geq 0,\quad m\geq 0)\,.$$

Fig.2 *Rectangular plasma*

One here assumes the sizes: $a=0.5[\mathrm{m}],\ b=0.5[\mathrm{m}],\ R=1.0[\mathrm{m}]$. Each side of the rectangle was equally divided in such a way that each constant boundary element has a length of 0.05[m]; thus a total of 80 elements were employed. Comparisons were made between the analytic and the boundary element solutions for all combinations of integers ℓ and m in the range $0\leq\ell+m\leq 8$. As an example, **Fig.3** shows the contour map of the BEM solution of ψ for a monomial source r^3z^2. Relative deviation from the analytic solution in this case is illustrated in **Fig.4**. The deviation, defined by ((BEM-Analytic)/Analytic)×100, is less than 0.01 ～0.1% in the greater part of the domain. Deviation larger than 1% is found near the edges and corners, however, the absolute values of ψ are extremely small in these places. Almost the same level of accuracy was also demonstrated for other combinations of ℓ and m.

Fig.3 *Boundary element solution of* ψ *profile for* r^3z^2
Fig.4 *Relative deviation between the BEM and the analytic solution*

7.2 Tokamak Geometry

As a more realistic test problem, one here considers a problem to model a tokamak-type device. By the courtesy of Japan Atomic Energy Research Institute (JAERI), reference data of plasma boundary, distributions of plasma current and magnetic flux function were firstly provided, which were calculated using a reliable equilibrium code, SELENE **[5]**. This equilibrium computation was made based on the current profile parametrization that has the form $\mu_0 r\, j_\varphi = c_0\{\beta_p r^2+(1-\beta_p R_0^2)\}(1-X)^{0.6}$. Here, $X=(\psi-\psi_M)/(\psi_S-\psi_M)$ in which ψ_M and ψ_S are the values of ψ on the magnetic axis and on the boundary, and β_p (= 0.60) and R_0 (= 3.32 *m*) denote the poloidal beta and the characteristic major radius, respectively.

This problem was again analyzed using the BEM as a fixed boundary problem. Only the boundary shape among the SELENE computing results was transferred to the BEM computation as input data. The boundary condition $\psi = 0$ was imposed at each nodal point along the boundary. The same current profile parametrization shown above was again assumed, and the complete polynomial of the 8-th order was adopted to approximate the $\mu_0 r\, j_\varphi$ distribution. That is, the ‘level’ defined in Section 3.1 is 8, and hence the polynomial consists of a total of 45 terms. To determine the polynomial expansion coefficients, a total of 623 sampling points were automatically generated within the domain, following the procedure described in Section 6.2. The plasma boundary is approximated by a polygon that has 80 sides, i.e., a total of 80 constant elements were employed.

A total of 8 iterations were required in the BEM analysis when the eigenvalue deviation defined by Eq.(42) was reduced to less than 10^{-5}. The CPU time consumed for this computation was ～6.9s with Alpha CPU-21164A (600MHz), while the computing time devoted for the SELENE calculation was ～0.8s with the same CPU. The present BEM computation is not always superior to the SELENE computation from the viewpoint of the computing time. The major part of the BEM computing is devoted to the boundary integrations and this numerical integration has not yet fully optimized. Since such boundary integrations can be made independently for each boundary node point, the CPU time could be drastically reduced if one adopted a parallel computing in future. Considering also the progress in computer processing capability, the authors are not pessimistic about the problem of computing time. In the BEM computation, only about 5.5% of the total computing time was consumed after the first stage of eigenvalue iteration, which was mainly devoted for boundary integral computations, and this fact shows that the number of iterations hardly affects the total computing time.

The profile of magnetic flux function thus obtained from the BEM calculation is compared with the SELENE calculation results, as shown in **Fig.5**. The plasma current profiles are also compared in **Fig.6**. In each Figure, the solid lines show the BEM solutions, while the dashed lines denote the results obtained using the SELENE code. The BEM results show good agreement with the reference data, and this demonstrates the validity of the present boundary-only integral formulation, especially of the polynomial expansion approximation of $\mu_0 r\, j_\varphi$.

The distributions of the magnetic flux functions and the plasma current along the line z=0 are shown respectively in **Fig.7** and **Fig.8**. The referenced SELENE results are plotted by dots, while the solid lines denote the BEM solutions based on the level-8 polynomial and the dashed lines are the BEM ones obtained using the level-1 to -7 polynomials.

Fig.5	*Contours of magnetic flux function*
Fig.6	*Contours of plasma current density*

Fig.7	*Results of magnetic flux function along the line z=0*
Fig.8	*Results of plasma current density along the line z=0*

8 CONCLUSION AND FURTHER REMARKS

A new type of boundary element method presented in this work does not require any computation of domain integral. The final form of the boundary integral equation has no domain integral, as shown in Eq.(20). In addition, even the process of the eigenvalue computation is also based on a boundary-only integral, as described in Section 6.3. When attempting to perform an eigenvalue iteration, one needs to distribute many sampling data points in the plasma domain for determining the coefficients of polynomial expansion of $\mu_0 r\, j_\varphi$, however, the coordinates of these sampling points are automatically generated in the computer program. Thus the data sampling never contradicts the advantage of the present method that it requires discretization of the boundary only. The program user has only to prepare boundary element data that specify the shape of the last closed magnetic surface.

The boundary values of $\partial\psi/\partial n$ and the polynomial expansion coefficients are updated in each stage of the eigenvalue iteration, however, the components in the system matrix related to the boundary integral equation and also the ones in the matrix for polynomial coefficient determination are invariant through the iteration. Thus the boundary values and the expansion coefficients are obtained only from simple multiplications of matrix and source vector, with the result that the number of iterations hardly affects the total computing time.

Test calculations indicate that the present boundary-only integral equation approach provides stable and accurate numerical solutions. The authors used constant boundary elements for the discretization in the present work, however, a new version of FORTRAN code based on isoparametric quadratic boundary elements is now under development to model the plasma boundary curvature more accurately with a smaller number of boundary nodes.

Numerical examples in the present paper are limited to fixed boundary problems; however, the application of the present method can be extended to solve a free boundary problem by adding an iterative search function to the computer code. The addition of poloidal coil current terms into the equation is also useful to expand the application. Taking advantage of the present method that requires only the boundary discretization, it is interesting to consider a problem of "moving boundary", i.e., the plasma boundary that shape is changing time-dependently.

How the boundary-only integral equation (20) can be applied to an inverse problem to reconstruct the plasma current density profile? The authors' future plan is as follows. Kurihara's Cauchy condition surface (CCS) method **[10]** is for the determination of the shape of plasma boundary, however, it can also estimate values of $\partial\psi / \partial n$ as well as the magnetic flux function ψ on the plasma boundary. This means, once the boundary shape is fixed by the method, Eq.(20) has no unknowns any more except for the polynomial expansion coefficients $\alpha_{\ell,m}$. The coefficients and then the profile of $\mu_0 r\, j_\varphi$ can be easily estimated, although one needs to add some "*a priori* information" to successfully obtain a unique solution. The followings are candidates of *a priori* information and physical constraints we can take into account.

(1) The total plasma current is known.

(2) Zero-current along the plasma boundary.

(3) Constraints derived from the equilibrium $\mathbf{J}\times\mathbf{B} = \nabla p$. (For this purpose, we can adopt the simple scalar relationship proposed by K. Kurihara **[18]** to connect the current density with the magnetic flux. This

condition requires 'iteration' to compute alternatively the magnetic flux profile and the current density distribution.)

(4) Assume that the current density (if possible) or other physical quantities closely related to the current density can be measured at a certain number of points in the plasma domain. (The authors' personal opinion is that this condition is very essential to ensure the uniqueness of the current density solution.)

All of these conditions can be described using the polynomial form, and then they should be incorporated into the algebraic equations given from Eq.(20) to determine the coefficients $\alpha_{\ell,m}$. The singular value decomposition (SVD) technique **[16]** is well suited to solve the resultant matrix equation, and in this case the Tikhonov regularization **[19]** can be also employed to stabilize the numerical ill-posedness. The detailed methodology based on the above procedure is now under development.

Appendix A: DERIVATION OF THE SINGULAR POINT PARAMETER c_i

Equation (10) is valid for any point in the domain Ω', but one must modify the equation for a point on the boundary. One divides the boundary Γ' into two parts, $\Gamma' = \Gamma'_1 + \Gamma'_2$, i.e.,

$$\int_{\Gamma'} \frac{\psi}{r}\frac{\partial \psi^*}{\partial n} \mathrm{d}\Gamma' = \lim_{\varepsilon \to 0} \int_{\Gamma'_1} \frac{\psi}{r}\frac{\partial \psi^*}{\partial n} \mathrm{d}\Gamma'_1 + \lim_{\varepsilon \to 0} \int_{\Gamma'_2} \frac{\psi}{r}\frac{\partial \psi^*}{\partial n} \mathrm{d}\Gamma'_2 \,, \tag{A1}$$

where Γ'_2 is an imaginary fan-shaped boundary with an angle of $(2\pi - \theta_0)$ as illustrated in Fig.A1. The boundary point $i\,(a,b)$ is assumed to be at the center of the circle and afterward the radius ε is reduced to zero.

Fig.A1 *Boundary point augmented by a small semicircle*

The coordinates (r, z) of an arbitrary point on Γ_2' can now be given by

$$r = a + \varepsilon \cos\theta, \quad z = b + \varepsilon \sin\theta \,,$$

then the complete elliptic integrals of the first and the second kinds are reduced to

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\sin^2\theta}}\mathrm{d}\theta \;\to\; \frac{1}{2}\ln\left(\frac{16}{1-k^2}\right)=\ln\left(\frac{4\sqrt{4a^2+4a\varepsilon\cos\theta+\varepsilon^2}}{\varepsilon}\right)$$

and

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta \;\to\; 1,$$

respectively when $\varepsilon\to 0$. The fundamental solution given by Eq.(3) can then be reduced to

$$\psi^* = \frac{\sqrt{4a^2+4a\varepsilon\cos\theta+\varepsilon^2}}{2\pi}\left[\frac{1}{2}\ln\left(\frac{4\sqrt{4a^2+4a\varepsilon\cos\theta+\varepsilon^2}}{\varepsilon}\right)-1\right].$$

Considering that the normal derivative of the fundamental solution is written in the form

$$\frac{\partial\psi^*}{\partial n}=\frac{\partial\psi^*}{\partial r}\cos\theta+\frac{\partial\psi^*}{\partial z}\sin\theta,$$

and using $\mathrm{d}\Gamma_2{}'=\varepsilon\mathrm{d}\theta$, one obtains

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Gamma'_2}\frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\mathrm{d}\Gamma'_2\\ &=\lim_{\varepsilon\to 0}\int_0^{2\pi-\theta_0}\psi\frac{1}{a+\varepsilon\cos\theta}\left(\frac{\partial\psi^*}{\partial r}\cos\theta+\frac{\partial\psi^*}{\partial z}\sin\theta\right)\varepsilon\mathrm{d}\theta=-\psi_i\left(1-\frac{\theta_0}{2\pi}\right).\end{aligned} \tag{A2}$$

Further, one knows

$$\lim_{\varepsilon\to 0}\int_{\Gamma'_1}\frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\mathrm{d}\Gamma'_1=V.P.\int_{\Gamma'}\frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\mathrm{d}\Gamma', \tag{A3}$$

where the right-hand-side is the Cauchy principal integral.

The same procedure for the term $\int_{\Gamma'}\frac{\psi^*}{r}\frac{\partial\psi}{\partial n}\mathrm{d}\Gamma'$ of Eq.(10), however, does not introduce any new term in Eq.(10).

Substituting Eqs.(A2) and (A3) into Eq.(A1), one obtains

$$\int_{\Gamma'}\frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\mathrm{d}\Gamma' \to V.P.\int_{\Gamma'}\frac{\psi}{r}\frac{\partial\psi^*}{\partial n}\mathrm{d}\Gamma'-\psi_i\left(1-\frac{\theta_0}{2\pi}\right),$$

then Eq.(10) is changed to Eq.(11) with $c_i=\theta_0/2\pi$. In Eq.(11), the symbol “$V.P.$” is omitted. The value of $c_i=1/2$ is taken on a smooth boundary ($\theta_0=\pi$), and $c_i=1$ for an internal point.

Appendix B: ANALYTIC SOLUTION FOR “RECTANGULAR PLASMA”

The following analytic solution has been originally derived by the first author for the present research. Suppose hypothetical rectangular plasma as shown in **Fig.2**. The boundary condition $\psi = 0$ is imposed along each side of the domain. In this case the analytic solution of the partial differential equation with a monomial $r^\ell z^m$,

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -r^\ell z^m , \quad (\ell \ge 0, \ \ m \ge 0) \tag{B1}$$

can be given in the form

$$\psi = -R^{\ell+2}(2b)^m \sum_{n=1}^{\infty} b_n^{(m)} g_n(s) \sin n\pi\, t \ , \tag{B2}$$

using dimensionless variables $s \equiv r/R$, $t = (z+b)/(2b)$. The expansion coefficients in Eq.(B2), $b_n^{(m)}$, is calculated as

$$b_n^{(m)} = -\frac{2}{n\pi}\sum_{j=0}^{\infty} \frac{m!}{(m-2j)!}\frac{(-1)^j}{(n\pi)^{2j}} \cdot \frac{1}{2^m}\{(-1)^n - (-1)^{m-2j}\} \tag{B3}$$

The function $g_n(s)$ in Eq.(B2) is given by

$$g_n(s) = s[C_1 e^{(x-X_1)}\sqrt{\frac{X_1}{x}}\tilde{I}_1(x) + C_2 e^{(X_2-x)}\sqrt{\frac{X_2}{x}}\tilde{K}_1(x)] + g_{np}(s) \ . \tag{B4}$$

In Eq.(B4), one denotes the quantities

$$x = B_n s \, , X_1 = B_n(1+\frac{a}{R}) \, , X_2 = B_n(1-\frac{a}{R}) \ \text{ with } \ B_n^2 = \left(\frac{R}{2b}\right)^2 (n\pi)^2 \, ,$$

while the functions $\tilde{I}_1(x) = e^{-x}\sqrt{x}\, I_1(x)$ and $\tilde{K}_1(x) = e^{x}\sqrt{x} K_1(x)$ are defined using the first order modified Bessel functions of the first and the second kinds, respectively. The particular solution in Eq.(B4) is computed as follows:

$$g_{np}(s) = -\frac{1}{B_n^{\ell+2}}\left[x^\ell + \sum_{k=1}^{\frac{\ell}{2}-1}\left(\prod_{j=1}^{k}(\ell-2j+2)(\ell-2j)\right)x^{\ell-2k}\right] \qquad (\ell \ : \text{ even integer}) \tag{B5a}$$

or

$$g_{np}(s) = -\frac{1}{B_n^{\ell+2}}\left[x^{\ell} + \sum_{k=1}^{\frac{\ell-1}{2}} \left(\prod_{j=1}^{k} (\ell - 2j + 2)(\ell - 2j) \right) x^{\ell - 2k} \right] \quad (\ell \text{ : odd integer}) \qquad \text{(B5b)}$$
$$+ \frac{1}{B_n^{\ell+2}} \left(\prod_{k=1}^{\frac{\ell-1}{2}} (\ell - 2k + 2)(\ell - 2k) \right) x \left\{ 1 + \frac{1}{2}\pi \left(L_1(x) - I_1(x) \right) \right\}.$$

One uses the first order modified Struve function **[17]**, $L_1(x)$, here in Eq.(B5b). The unknown coefficients C_1 and C_2 in Eq.(B4) can be determined in such a way that the boundary condition $\psi = 0$ is satisfied at $r = R \pm a$.

ACKNOWLEDGEMENTS

The authors wish to express their gratitude to Dr. H. Ninomiya and Dr. K. Kurihara of Japan Atomic Energy Research Institute (JAERI), Professor T. Honma and Dr. H. Igarashi of Hokkaido University for their valuable and helpful comments on this work. Further, Dr. Kurihara kindly provided them with the reference tokamak plasma data used in the numerical demonstration in Section 7.2. The authors have to mention here that the latter part of the description in Section 3.3 was kindly suggested by one of the referees in his review letter to the authors. Special thanks are also due to Dr. C.A. Brebbia of Wessex Institute of Technology, U.K., who guided the first author to the boundary element research field, for his continuous encouragement through this work.

REFERENCES

(1) SHAFRANOV, V.D., Sov. Phys. – JETP 37 (1960) 775.

(2) Mukhovatov, V.S., Shafranov, V.D., Nucl. Fusion **11** (1971) 605.

(3) Shafranov, V.D., Plasma Phys. **13** (1971) 757

(4) WESSON, J, "*Tokamaks (Second edition)*", The Oxford Engineering Series 48, Clarendon Press, Oxford (1997).

(5) AZUMI, M., KURITA, G., MATSUURA, T. et al., A Fluid Model Numerical Code System for Tokamak

Fusion Research in "Computing Methods in Applied Science and Engineering", p.335, North-Holland, Amsterdam / New York / Oxford (1980).

(6) McCLAIN, F.W., BROWN, B.B., GAQ, A Computer Program to Find and Analyze Axisymmetric MHD Plasma Equilibria, GA-A 14490, General Atomic Company (1977).

(7) BREBBIA, C.A., "*The Boundary Element Method for Engineers*", Pentech Press, London (1978).

(8) BRAAMS, B.J., Interpretation of tokamak magnetic diagnostics, report IPP 5/2, Max Plank Institute fur Plasma Physics (1985).

(9) HAKKARAINEN, S.P., FREIDBERG, J.P., Reconstruction Of Vacuum Flux Surfaces From Diagnostic Measurements In A Tokamak, report PFC/RR-87-22, MIT Plasma Fusion Center (1987).

(10) KURIHARA, K., A New Shape Reproduction Method Based on the Cauchy-Condition Surface for Real-Time Tokamak Reactor Control, *Fusion Eng. Des*., **51-52** (2000) 1049.

(11) KURIHARA, K., Tokamak Plasma Shape Identification on the Basis of Boundary Integral Equations, *Nuclear Fusion*, **33[3]** (1993) 399.

(12) TAKEDA, T., TOKUDA, S., Computation of MHD equilibrium of tokamak plasma, Journal of Computational Physics, **93[1]** (1991) 1-107.

(13) PARTRIDGE, P.W., BREBBIA, C.A., WROBEL, L.C., The Dual Reciprocity Boundary Element Method, Computational Mechanics Publications, Southampton / Boston, Co-published with Elsevier Applied Science, London / New York (1992).

(14) ITAGAKI, M., Boundary Element Methods Applied to Two-Dimensional Neutron Diffusion Problems, *J. Nucl. Sci. Tech*., **22[6]** (1985) 565.

(15) GIPSON, G.S., Use of the residue theorem in locating points within an arbitrary multiply-connected region, *Advances in Engineering Software*, **8[2]** (1986) 73.

(16) PRESS, W.H., FLANNERY, B.P., TEUKOLSKY, S.A., VETTERLING, W.T., "*Numerical Recipes - The Art of Scientific Computing*", Cambridge University Press, Cambridge (1986).

(17) ABRAMOWITZ, M., STEGUN, I.A.: "*Handbook of Mathematical Functions*", Dover Publications, New York (1965).

(18) KURIHARA, K., Current Profile Reproduction Study on the Basis of a New Expansion Method with the Eigenfunctions Defined in the Tokamak Plasma Interior, *Fusion Technology*, **34** (1998) 548-552.

(19) HANSEN, P.C., *"Rank-Deficient and Discrete Ill-Posed Problems – Numerical Aspects of Linear Inversion"*, SIAM, Philadelphia (1998).

List of Tables

List of Figures

Table 1 *Pascal's triangle for polynomial functions*

Level	0					1				
	1				ξ		η			
	2			ξ^2		$\xi\eta$		η^2		
	3		ξ^3		$\xi^2\eta$		$\xi\eta^2$		η^3	
	4	ξ^4		$\xi^3\eta$		$\xi^2\eta^2$		$\xi\eta^3$		η^4

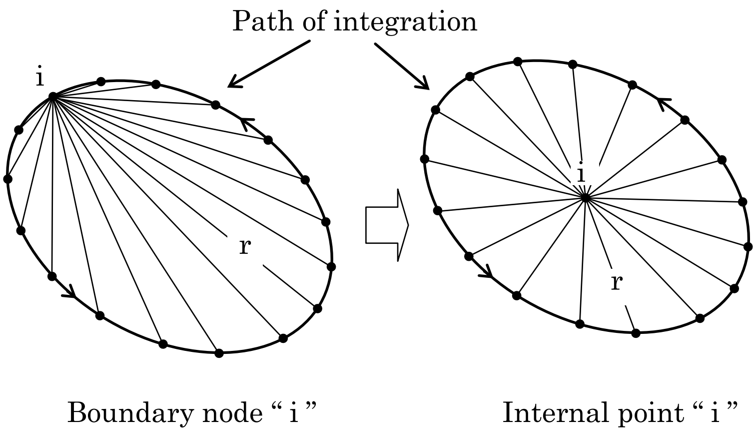

Fig.1　Two stages of boundary element calculation

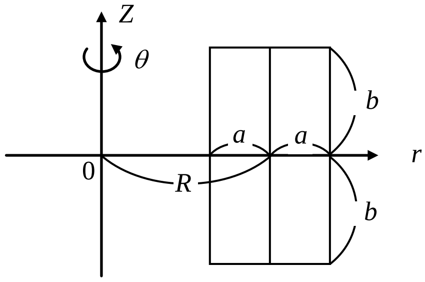

Fig.2　Rectangular plasma

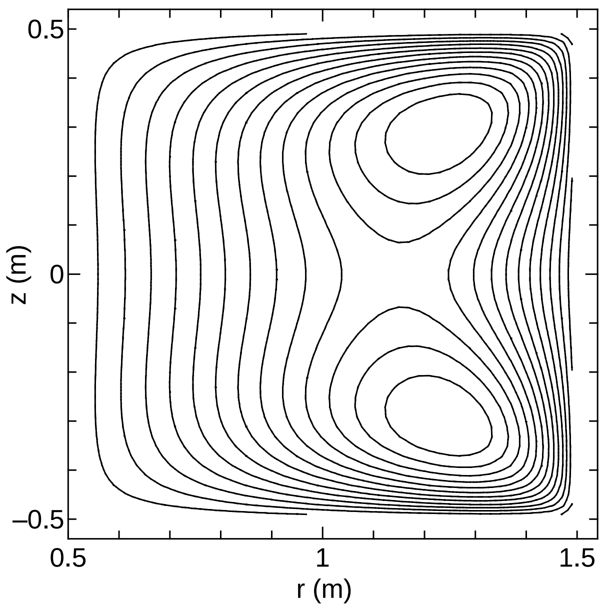

Fig.3　Boundary element solution of ψ profile for $r^3 z^2$

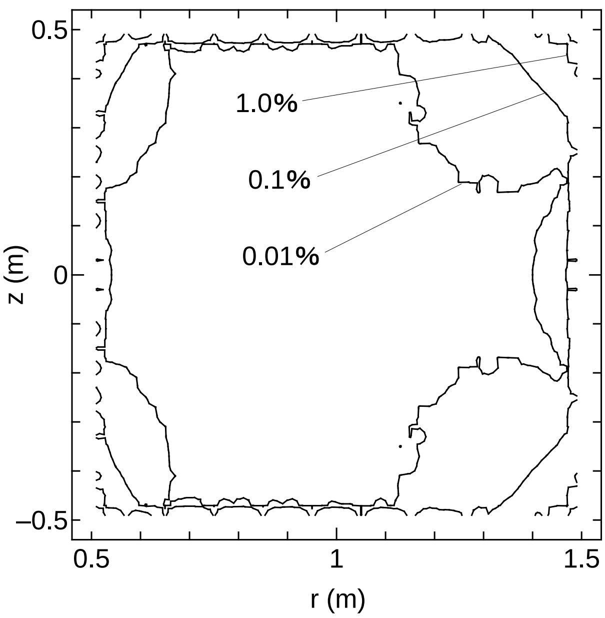

Fig.4　Relative deviation between the BEM and the analytic solution

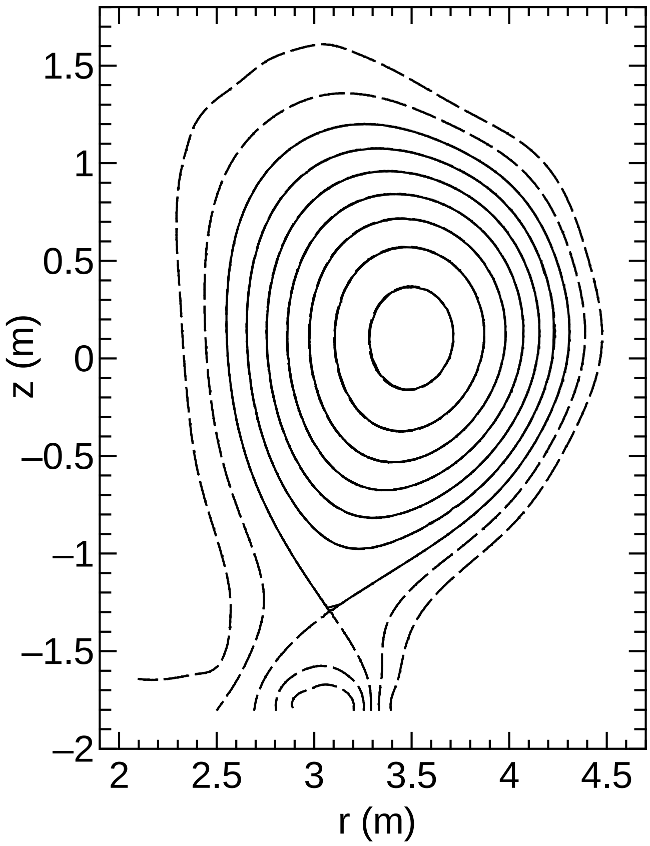

Fig.5　Contours of magnetic flux function

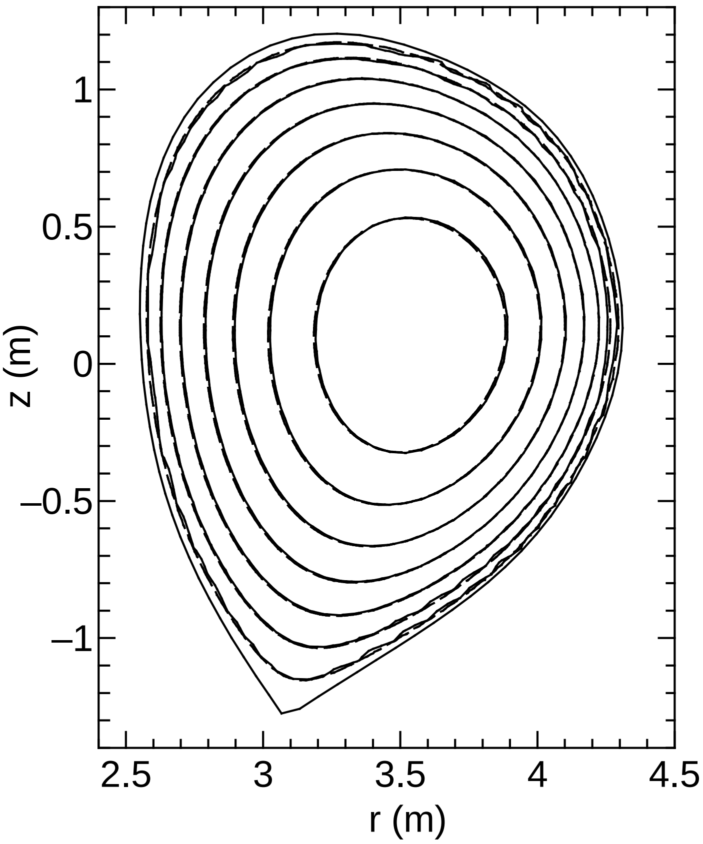

Fig.6 Contours of plasma current density

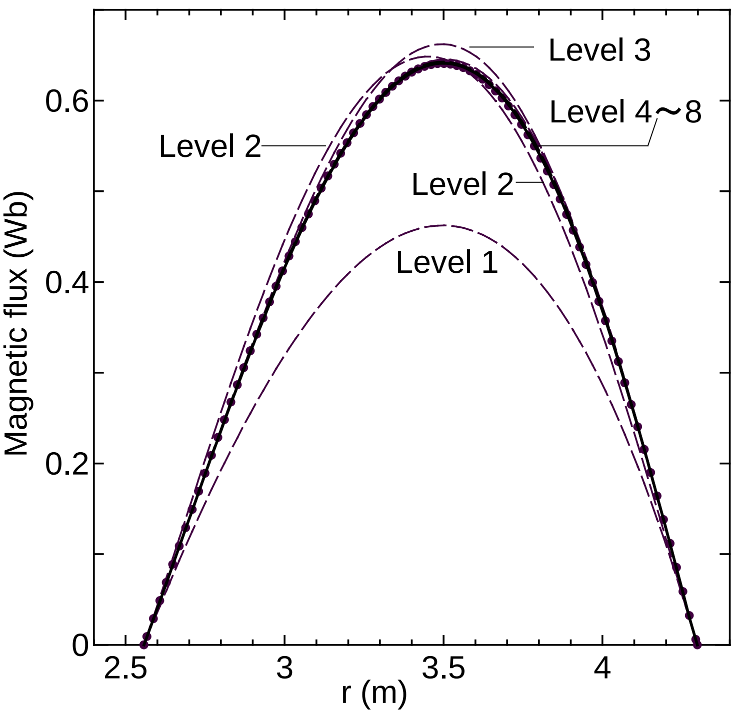

Fig.7　Results of magnetic flux function along the line z=0

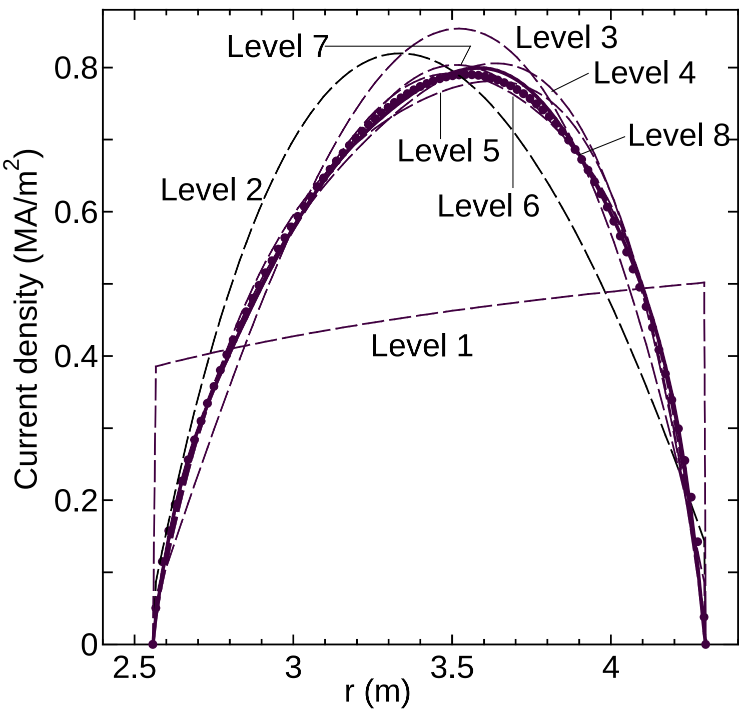

Fig.8　Results of plasma current density along the line z=0

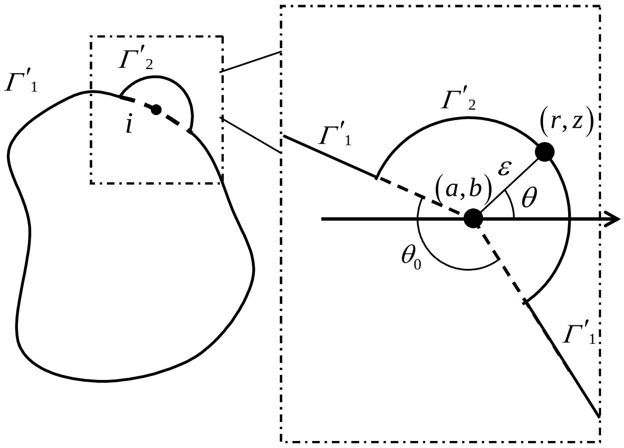

Fig.A1 Boundary point augmented by a small semicircle