On the sectional curvatures and the Euler-Poincaré characteristic of a Riemannian manifold

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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§ 1. Introduction.

In a compact oriented Riemannian manifold M of even dimension n, the generalized Gauss-Bonnet formula ([1], [2]) is expressed by the following integral:

$$\chi(M)=(2/c_n)\int_M K_n dV \tag{1}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M, c_n is the volume of the Euclidean unit n-sphere, K_n denotes the Lipschitz-Killing curvature of M, and dV is the volume element of M.

It is well known that the following is conjectured from the generalized Gauss-Bonnet formula.

CONJECTURE. *Let M be a compact oriented Riemannian manifold of even dimension $n(=2m)$. If the sectional curvatures are all non-negative, then M has the non-negative Euler-Poincaré characteristic $\chi(M)$. If the sectional curvatures are all non-positive, then* $(-1)^m\chi(M)\geqq 0$.

In 2-dimensional case, the sign of the Gaussian curvatures, i.e., of the sectional curvatures determines the sign of $\chi(M)$. In 4-dimensional case, the conjecture is resolved by J. Milnor, and its proof was provided by S. S. Chern [3]. S. S. Chern [3], J. A. Thorpe [4], and Y. K. Cheung and C. C. Hsiung [7] gave certain answers under the curvature conditions respectively for determining the sign of $\chi(M)$.

The purpose of the present paper is to give an answer to the conjecture on a Riemannian manifold with some other curvature conditions.

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§ 2. Preliminaries.

Let M be a compact oriented Riemannian manifold of even dimension n. Let $\Lambda^p(M)$ denote the bundle of p-vectors of M and let $\Lambda^p(x)$ be the fiber over $x \in M$. The inner product on the fiber $\Lambda^p(x)$ related to the inner product on the tangent space M_x of M at x is given by

$$(2) \qquad \langle u_1 \wedge \cdots \wedge u_p,\ v_1 \wedge \cdots \wedge v_p \rangle = det\Big[\langle u_i, v_j \rangle\Big] \qquad (u_i, v_j \in M_x).$$

We can consider the Grassmann bundle $G_p(M)$ of tangent p-planes of M as a subbundle of the unit sphere bundle of $\Lambda^p(M)$ by identifying $P \in G_p(M)$ with $e_1 \wedge \cdots \wedge e_p \in \Lambda^p(M)$ where $\{e_1, \cdots, e_p\}$ is any orthonormal basis for P.

Let R be the curvature operator and R_p the p-th curvature operator which is used in [6]. Explicitly, this curvature operator R_p is given by

$$(3) \qquad \langle R_p(u_1 \wedge \cdots \wedge u_p),\ v_1 \wedge \cdots \wedge v_p \rangle$$
$$\equiv \frac{1}{2^{\frac{p}{2}} p!} \mathcal{E}^{i_1 \cdots i_p} \mathcal{E}^{j_1 \cdots j_p} R(u_{i_1}, u_{i_2}, v_{j_1}, v_{j_2}) \cdots R(u_{i_{p-1}}, u_{i_p}, v_{j_{p-1}}, v_{j_p}) \qquad (u_i,\ v_j \in M_x)$$

where $\mathcal{E}^{i_1 \cdots i_p}$ denotes the sign of permutation $(i_1, \cdots, i_p)$ of $(1, \cdots, p)$. Clearly $R_2 = R$. Futhermore, for $x \in M$,

$$(4) \qquad K_n(x) = \langle R_n(e_1 \wedge \cdots \wedge e_n),\ e_1 \wedge \cdots \wedge e_n \rangle$$

where $\{e_1, \cdots, e_n\}$ is any orthonormal basis for M_x.

§ 3. The p-th sectional curvatures K_p.

Now for any positive even integer $p \leqq n$, we define a smooth function K_p on the Grassmann bundle $G_p(M)$. The function K_p, called the p-th sectional curvature of M, means the Lipschitz-Killing curvature of the geodesic p-dimensional submanifolds of M. Clearly, the p-th sectional curvature of M is the real valued function $K_p : G_p(M) \to \boldsymbol{R}$ given by

$$(5) \qquad K_p(P) = \langle R_p(P),\ P \rangle \qquad \big(P \in G_p(M)\big).$$

For $p=2$, K_p is the usual sectional curvature K of M. We can easily get the following

PROPOSITION. Let M be *4*-dimensional Einstein space with $K_4 \equiv 0$. Then M is flat.

PROOF. In *4*-dimensional case, there exsists an orthonormal basis $\{e_1, \cdots, e_4\}$ of M_x such that

$$(6) \qquad K_4(x) = \frac{1}{3}\Big\{K_{12}K_{34} + K_{13}K_{42} + K_{14}K_{23} + (R_{1234})^2 + (R_{1342})^2 + (R_{1423})^2\Big\} \qquad (x \in M)$$

where $K_{ij} = K(e_i \wedge e_j)$. It is well known, that $K_{12} = K_{34}$, $K_{13} = K_{42}$, and $K_{14} = K_{23}$ in *4*-dimensional Einstein space. Consequently M is flat $(K \equiv 0)$. q.e.d.

We consider the following *curvature condition* (C):
When the sectional curvatures are always non-negative (or, non-positive), there exists the following property

$$K(P)\cdot K_{p-2}(P^{\perp}) \leqq \langle R(P)_{\wedge} R_{p-2}(P^{\perp}),\ W\rangle$$

$$\left(\text{or, } (-1)^{\frac{p}{2}} K(P)\cdot K_{p-2}(P^{\perp}) \leqq (-1)^{\frac{p}{2}} \langle R(P)_{\wedge} R_{p-2}(P^{\perp}),\ W\rangle\right)$$

for any even $p(\leqq n)$ and all p-plane $W = P_{\wedge} P^{\perp} \in G_p(M)$ where P is 2-plane and $P^{\perp}$ is the orthogonal complement of P in W.

REMARK 1. It is clear that the *curvature condition* (C) is satisfied for the constant curvature. At this juncture, an equality is preserved.

REMARK 2. It follows from the simple calculation that the *curvature condition* (C) is satisfied for $n=4$.

§ 4. The main theorems.

We have the following Lemma in [5]:

LEMMA. *Suppose that p and q are positive even integers with $p+q \leqq n$. The form $\Theta^{(p+q)}_{i_1\cdots i_{p+q}}$ (see* [4] *and* [5]*) can be expressed as*

$$\Theta^{(p+q)}_{i_1\cdots i_{p+q}} = \frac{p!\,q!}{(p+q)!} \sum_{A} \Theta^{(p)}_{k_1\cdots k_p} \vee \Theta^{(q)}_{k_{p+1}\cdots k_{p+q}}$$

where the sum ranges over all partitions $A=(A_1, A_2)$ of $\{i_1\cdots, i_{p+q}\}$ into sets A_1 of p elements and A_2 of q elements, and where $(k_1, \cdots, k_{p+q})$ is, for each A, an even permutation of $(i_1\cdots, i_{p+q})$ such that

$$k_1, \cdots, k_p \in A_1 \quad \textit{and} \quad k_{p+1}, \cdots, k_{p+q} \in A_2\,.$$

If we reform the above lemma, we have the following

LEMMA*. *Suppose that p and q are even integers with $p+q \leqq n$. For $Q \in G_{p+q}(M)$, let $\{e_1, \cdots, e_{p+q}\}$ be an orthonormal basis for Q and let*

$$\beta = \left\{e_{i_1} {}_{\wedge} \cdots {}_{\wedge} e_{i_p} \,\middle|\, 1 \leqq i_1 < \cdots < i_p \leqq p+q\right\}.$$

Then $\beta \subset G_p(M)$ and

$$R_{p+q}(Q) = \frac{p!\,q!}{(p+q)} \sum_{P\in\beta} R_p(P)_{\wedge} R_q(P^{\perp}) \tag{7}$$

where $P^{\perp}$ is the oriented orthogonal complement of P in Q.

THEOREM 1. *Let M be a compact oriented Riemannian manifold of even dimension $n=2m$. Suppose that M has the curvature condition (C). If the sectional curvatures are all non-negative, then the Euler-Poincaré* *characteristic is non-negative, and if the sectional curvatures are non-positive, then* $(-1)^m \chi(M) \geqq 0$.

PROOF. From the equation (4) and Lemma*, for $x \in M$,

$$K_n(x) = \langle R_n(e_1 \wedge \cdots \wedge e_n),\ e_1 \wedge \cdots \wedge e_n \rangle$$
$$= \frac{2!\,(n-2)!}{n!} \sum_{P_1 \in \beta_1} \langle R(P_1) \wedge R_{n-2}(P_1{}^{\perp}),\ M_x \rangle$$

where $\{e_1, \cdots, e_n\}$ is any orthonormal basis for M_x. Here, we use the curvature condition (C) over and over again. If $K \equiv K_2 \geqq 0$,

$$K_n(x) \geqq \frac{2^m}{n!} \sum_{P_1 \in \beta_1} K(P_1) \sum_{P_2 \in \beta_2} K(P_2) \cdots \sum_{P_{m-1} \in \beta_{m-1}} K(P_{m-1}) \cdot K(P_{m-1}{}^{\perp})$$

where $\beta_1 = \{e_{i_1} \wedge e_{i_2} | 1 \leqq i_1 < i_2 \leqq n\}, \cdots, \beta_{m-1} = \{e_{i_{n-3}} \wedge e_{i_{n-2}} | 1 \leqq i_{n-3} < i_{n-2} \leqq n\}$, $P_{m-1}{}^{\perp} = e_{i_{n-1}} \wedge e_{i_n}$ and $(i_1, \cdots, i_n)$ is an even permutation of $(1, \cdots, n)$. If $K \leqq 0$,

$$(-1)^m K_n(x) \geqq (-1)^m \frac{2^m}{n!} \sum_{P_1 \in \beta_1} K(P_1) \cdots \sum_{P_{m-1} \in \beta_{m-1}} K(P_{m-1}) \cdot K(P_{m-1}{}^{\perp}).$$

Comparison with (1) completes the proof. q.e.d.

For some even p dividing n, we consider the following *curvature condition* (C^*): When the p-th sectional curvatures are always non-negative (or, non-positive), there exists the following property

$$K_p(P) \cdot K_{(k-1)p}(P^{\perp}) \leqq \langle R_p(P) \wedge R_{(k-1)p}(P^{\perp}),\ W \rangle$$
$$\left(or,\ (-1)^k K_p(P) \cdot K_{(k-1)p}(P^{\perp}) \leqq (-1)^k \langle R_p(P) \wedge R_{(k-1)p}(P^{\perp}),\ W \rangle \right)$$

for any positive integer $k (\leqq n/p)$ and all kp-plane $W = P \wedge P^{\perp} \in G_{kp}(M)$ where P is a p-plane and $P^{\perp}$ is the orthogonal complement of P in W.

We can easily obtain the following

THEOREM 2. *Let M be a compact oriented Riemannian manifold of even dimension n. Suppose that M has the curvature condition (C^*). If, for some even p dividing n, the p-th sectional curvatures are all non-negative, then $\chi(M) \geqq 0$, and if the p-th sectional curvatures are non-positive, then* $(-1)^{n/p} \chi(M) \geqq 0$.

REMARK 3. *It is clear from* [**4**], *that the curvature condition (C^*) is satisfied when the p-th sectional curvatures are constant on $G_p(M)$. At this juncture an equality is preserved.*

Theorem 2 contains the result of J. A. Thorpe (Theorem B in [**4**]).

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