A remark on doubly transitive groups

To Professor Yoshie Katsurada on the occasion of her 60th birthday

By Hiroshi KIMURA

1. This note is a continuation of [12]. We shall use the same notation. The purpose of this note is to prove the following.

THEOREM. *Let $\mathfrak{G}$ be a doubly transitive permutation group of odd degree satisfying the following conditions.*

(1) $\mathfrak{G}_{1,2}$ *is of even order,*

(2) *All Sylow subgroups of* $\mathfrak{G}_{1,2}$ *are cyclic,*

(3) $\chi(\tau)$ *contains a regular normal subgroup,*

(4) $\mathfrak{G}$ *has one class of involutions,*

(5) $\mathfrak{G}_{1,2}$ *has unique involution.*

Then $\mathfrak{G}$ *contains a regular normal subgroup.*

From this and [12, Theorem] we obtain the following.

COROLLARY. *Let $\mathfrak{G}$ be a doubly transitive permutation group of odd degree satisfying the above conditions* (1), (2) *and* (3). *Then $\mathfrak{G}$ contains a regular normal subgroup or it is isomorphic to one of the groups S_5 with $n=5$ and* PSL (2, 11) *with $n=11$.*

2. Assume $\mathfrak{G}$ does not contain a regular normal subgroup. By [12, Theorem 1] we may assume that $|\mathfrak{K}|>2$ and $\mathfrak{K}_0=\langle\tau\rangle$. Thus $d/2$ is odd. From the condition (4) a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is also a Sylow 2-subgroup of $\mathfrak{G}$.

LEMMA 1. *A Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is not metacyclic.*

PROOF. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ containing $\langle\mathfrak{K}, I\rangle$ and let $\mathfrak{S}'$ be a cyclic normal subgroup of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{S}'$ is cyclic. If $|\mathfrak{S}/\mathfrak{S}'|>2$, then $\mathfrak{G}$ is solvable by [11]. Therefore $\mathfrak{S}=\langle I, \mathfrak{S}'\rangle$. Since $\mathfrak{K}\neq\langle\tau\rangle$, $|\mathfrak{S}'|>2$. If $\mathfrak{S}$ is abelian, then $\mathfrak{G}$ is solvable by the Burnside's splitting theorem. If $\mathfrak{S}$ is dihedral or semi-dihedral, then $\mathfrak{K}_0\neq\langle\tau\rangle$, which is a contradiction. If $S^I=S\tau$ for a generator S of $\mathfrak{S}'$, $\mathfrak{G}$ is solvable by [13]. Thus $\mathfrak{S}$ is not metacylic.

LEMMA 2. *$\chi_1(\tau)$ is contained in $C_{\mathfrak{G}}(I)$.*

PROOF. Assume that there exists a Sylow q-subgroup $\mathfrak{H}_q'$ of $\chi_1(\tau)$ such that $\langle\mathfrak{H}_q', I\rangle$ is dihedral. Let $\mathfrak{S}'$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\mathfrak{H}_q')$ containing

τ and $\mathfrak{S}$ a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{H}_q')$ containing $\mathfrak{S}'$. By the Frattini argument it may be assumed that $\mathfrak{S}$ is a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$. Since Aut $(\mathfrak{H}_q')$ is cyclic, so is $\mathfrak{S}/\mathfrak{S}'$. Assume $\mathfrak{S}'$ is not cyclic. If $\mathfrak{S}'$ contains an involution $\eta(\neq\tau)$, then take η instead of I, and $d=2d(\eta)$ is not divisible by q, which is a contrasiction. Thus $\mathfrak{S}'$ is a generalized quaternion. Let ζ and ξ be elements of $\mathfrak{S}'$ such that $\zeta^2=\xi^2=\tau$ and $\xi^{\zeta}=\xi^{-1}$. If $\alpha(\xi)\geqq 2$, then let a and b be two points in $\mathfrak{J}(\xi)$ such that ζ has the cycle structure $(a, b)\cdots$. a and b are contained in $\mathfrak{J}(\tau)$. Let I' be an involution with cycle structure $(a, b)\cdots$. Then ζ is an element in $I'\mathfrak{G}_{a,b}$. Therefore $\langle\zeta,\xi\rangle$ is a subgroup of $\langle I', \mathfrak{G}_{a,b}\rangle$. Since a Sylow 2-subgroup of $\langle I', \mathfrak{G}_{a,b}\rangle$ is conjugate to $\langle\mathfrak{K}, I\rangle$, $\langle\zeta,\xi\rangle$ is not a quaternion group, which is a contradiction. If $\alpha(\zeta)\geqq 2$ and $\alpha(\xi)=1$, then ζ acts on the set of transpositions which appear in the cycle structure of ξ. Thus there exist two points a and b in $\mathfrak{J}(\zeta)$ such that ξ has the cycle structure $(a, b)\cdots$. Therefore ξ is an element of $I'\mathfrak{G}_{a,b}$, where I' is an involution with the cycle structure $(a, b),\cdots$. Again we have a contradiction. Next assume $\mathfrak{J}(\zeta)=\mathfrak{J}(\xi)=\mathfrak{J}(\zeta\xi)=\{a\}$. a is a point of $\mathfrak{J}(\tau)$. Consider a homorphism ρ of $\langle\zeta,\xi\rangle$ into $\chi(\tau)_a=C_{\mathfrak{G}_a}(\tau)/\chi_1(\tau)$. Then ker $\rho=\langle\tau\rangle$. Since $\chi(\tau)$ contains a regular normal subgroup, $\langle\zeta,\xi\rangle/\langle\tau\rangle\cong\langle\zeta,\xi\rangle\chi_1(\tau)/\chi_1(\tau)$ is a Frobenius complement. Thus it must be cyclic or a (generalized) quaternion, which is a contradiction. Hence $\mathfrak{S}'$ must be cyclic and $\mathfrak{S}$ is metacyclic. This contradicts Lemma 1. This proves the lemma.

LEMMA 3. *If $\mathfrak{K}$ is not contained in $\chi_1(\tau)$, then p is prime to d and d–1.*

PROOF. Since $\chi_1(\tau)$ does not contain $\mathfrak{K}$, $\chi(\tau)$ satisfies the conditions (1), (2) and (3) in Theorem. $\chi(\tau)$ has two classes of involutions since $\mathfrak{K}$ is cyclic. Let K be an element of $\mathfrak{K}$ not contained in $\chi_1(\tau)$ such that K^2 is contained in $\chi_1(\tau)$. Apply [12, Lemma 8] to $\chi(\tau)$. If $\langle I, \mathfrak{H}_q\rangle\,\chi_1(\tau)$ and $\langle K, \mathfrak{H}_q\rangle\,\chi_1(\tau)$ is dihedral, then $\alpha(\langle\mathfrak{H}_q, \tau\rangle)$ is even and $p=q$. On the other hand, since $\chi(\tau)$ contains a regular normal subgroup, $\alpha(\langle\mathfrak{H}_q, \tau\rangle)$ must be equal to a power of p, which is a contradiction. Thus if $\langle I, \mathfrak{H}_q\rangle\,\chi_1(\tau)$ is dihedral, then $\langle K, \mathfrak{H}_q\rangle\,\chi_1(\tau)$ is abelian and if $\langle K, \mathfrak{H}_q\rangle\,\chi_1(\tau)$ is dihedral, then $\langle I, \mathfrak{H}_q\rangle\,\chi_1(\tau)$ is abelian. Hence $i=i'(\beta'(i'-1)/\gamma'+1)$, where $i'=\alpha(K)$ and $d/2=\beta'/\gamma'$. Since $i=p^m$, $d/2-1$ is divisible by p and hence p is prime to d and $d-1$.

LEMMA 4. *If $\mathfrak{K}$ is contained in $\chi_1(\tau)$ and $d\neq 2$, then d is a factor of $i-1$.*

PROOF. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $N_{\mathfrak{G}}(\mathfrak{K})$ containing I. Since $\bar{\mathfrak{S}}=\mathfrak{S}/\mathfrak{K}\cong\mathfrak{S}\chi_1(\mathfrak{K})/\chi_1(\mathfrak{K})$ is a Frobenius complement, it is cyclic or a (generalized) quaternion group. Let q be a prime factor of $d/2$ which is prime to $i-1$. By Lemma 2 and the Frattini argument $\langle\mathfrak{H}_q, \mathrm{K}\rangle$ is abelian. As in the proof of [8, Lemma 3. 9] we may prove that $\bar{\mathfrak{S}}$ is cyclic. That is, $\mathfrak{S}$

is metacyclic. This contradicts Lemma 1. This proves the lemma.

COROLLARY 5. *d is prime to p.*

PROOF. If d is divisible by p, them $\mathfrak{K}$ is not contained in $\chi_1(\tau)$ by Lemma 4. This contradicts Lemma 3.

By Corollary 5 if $\mathfrak{H}_p \neq 1$. then $\langle \mathfrak{H}_p, I \rangle$ is abelian.

LEMMA 6. *$(n, |\mathfrak{H}|)$ is a power of p.*

PROOF. Assume $(n, |\mathfrak{H}|)$ is not a power of p. Let q be a prime factor $(\neq p)$ of $(n, |\mathfrak{H}|)$. Assume that $|\chi_1(\tau)|$ is divisible by q. Let $\mathfrak{H}_q'$ be a Sylow q-subgroup of $\chi_1(\tau)$ contained in $\mathfrak{H}_q$. If $\mathfrak{J}(\tau)$ is proper subset of $\mathfrak{J}(\mathfrak{H}_q')$, then $\alpha(\mathfrak{H}_q') = i\ (\beta'(i-1)+1)$, where β' is some integer. By inductive hypothesis $\chi(\mathfrak{H}_q')$ contains a regular normal subgroup. In particular $\alpha(\mathfrak{H}_q')$ is a power of p. Since q is a factor of $n-\alpha(\mathfrak{H}_q')$, $q=p$. If $\alpha(\tau)=\alpha(\mathfrak{H}_q')$, then $q=p$ since q is a factor of $n-i$, which is a contradiction. Thus $\mathfrak{H}_q'=1$. Set $\mathfrak{X}=\langle \tau, \mathfrak{H}_q \rangle$. Then $\mathfrak{J}(\mathfrak{X})$ is a proper subset of $\mathfrak{J}(\tau)$. If $\mathfrak{J}(\mathfrak{X})=\mathfrak{J}(\mathfrak{H}_q)$, then $q=p$ since $n-i$ is divisible by q. If $\mathfrak{J}(\mathfrak{X})$ is proper subset of $\mathfrak{J}(\mathfrak{H}_q)$, then, as above, $\alpha(\mathfrak{H}_q)=\alpha(\mathfrak{X})\ (\beta'(\alpha(\mathfrak{X})-1)+1)$ and $\alpha(\mathfrak{H}_q)$ is a power of $\alpha(\mathfrak{X})$. Since $\chi(\tau)$ contains a regular normal subgroup, $\alpha(\mathfrak{X})$ is a power of p and so is $\alpha(\mathfrak{H}_q)$. Since $n-\alpha(\mathfrak{H}_q)$ is divisible by q, $p=q$, which is a contradiction. This proves the lemma.

As in [8, 4–2] we may prove that $\mathfrak{H}_p=1$. Similarly by using Lemma 6 we may prove that if $\mathfrak{H}_p=1$, then d has a prime factor which is prime to $i-1$ and $d-1$ is divisible by p (see [8, 4–1]). By Lemma 3 $\mathfrak{K}$ is contained in $\chi_1(\tau)$. This contradicts Lemma 4.

This complete a proof of Theorem.

Department of Mathematics,
Hokkaido University

References

[1] G. GLAUBERMAN: Central elements in core-free groups, J. Alg., 4 (1966), 403–420.

[2] D. GORENSTEIN: Finite groups, Harper and Row, New York, 1968.

[3] D. GORENSTEIN and J. H. WALTER: The characterization of finite groups with dihedral Sylow 2-subgroups, I, II, III, J. Alg., 2 (1965), 85–151, 218–270, 334–393.

[4] N. ITO: On doubly transitive groups of degree n and order $2(n-1)n$, Nagoya Math. J., 27 (1966), 409–417.

[5] W. M. KANTOR, M. E. O'NAN and G. M. SEITZ: 2-transitive groups in which the stabilizer of two points is cyclic (to appear).

[6] H. KIMURA: On some doubly transitive permutation groups of degree n and

order $2^l(n-1)n$, J. Math. Soc. Japan, 22 (1970), 263–277.

[7] H. KIMURA: On doubly transitive permutaion groups of degree n and order $2p(n-1)n$, Osaka. J. Math., 7 (1970), 275–290.

[8] H. KIMURA: On some doubly transitive groups such that the stabilizer of two symbols is cyclic, J. Faculty of Science, Hokkaido University (to appear).

[9] H. LÜNEBURG: Charakterisierungen der endlichen desargusschen projektiven Ebenen, Math. Zeit., 85 (1964), 419–450.

[10] H. WIELANDT: Finite permutaion groups, Academic Press, New York, 1964.

[11] A. R. CAMINA and T. M. GAGEN: Groups with metacyclic Sylow 2-subgroups, Canad. J., 21 (1969), 1234–1237.

[12] H. KIMURA: A characterization of PSL (2, 11) and S_5 (to appear).

[13] W. J. WONG: On finite groups whose 2-Sylow subgroups has cyclic subgroups of index 2, J. Austral. Math. Soc., 4 (1964), 90–112.

(Received on April 15, 1971)