ON VALUATIONS OF POLYNOMIAL RINGS OF MANY VARIABLES

Part two

By

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I published a paper on the same theme as this in Jour. Science. Hokkaido. Univ. Vol. 21, No. 1, in 1970. The following is its continuation. So I want to use in this paper the same symbols that I used in the preceding one as much as possible.

K is a field with a discrete non-archimedean valuation V_{00} and W is a given extension of it to a polynomial ring $K[x, y]$. Then $V_{10}=[V_{00}, V_{10}x = Wx=\mu_1]$ is a valuation of the ring $K[x]$, namely, of its quotient field K_x and in the same way $V_{01}=[V_{00}, V_{01}y=Wy=\nu_1]$ is a valuation of the ring $K[y]$, consequently, of its quotient field K_y. And $V_{11}=[V_{00}, V_{11}x=\mu_1, V_{11}y =\nu_1]$ is a valuation of the ring $K[x, y]$ and the last simply augmented valuation U which appears in the series of augmented inductive valuations binding V_{11} and W is decided uniquely by W and the series of continuous xy-doubly augmented inductive valuations binding U and W is decided uniquely by W. I proved all of them in Part one.

In this paper, first in §11, I make series of continuous simply augmented inductive valuations binding V_{11} and U. Next I expect to investigate structures of key-polynomials which produce simply augmented valuations or doubly augmented ones and structures of the residue class rings of $K[x, y]$ by these valuations. But for the sake of it, I found a fact that we can inspect these structures naturally after we can solve the next problem which I name "Principal Problem" in this paper.

Principal Problem

V_{p0} is a given discrete non-archimedean valuation of $K[x]$ and V_{0q} is a given one of $K[y]$ and both V_{p0} and V_{0q} *induce the same valuation V_{00} in K.*

In this case, the problem is how we should make such valuations W of $K[x, y]$ that induce V_{p0} in $K[x]$ and V_{0q} in $K[y]$.

To make such valuations is my chief study in this paper.

§ 11. Series of continuous simply augmented inductive valuations

Among these series, I presented the two simplest ones in Part one; the first one in §8,

$$V_{11} \overset{xs}{<} V_{21} \overset{xs}{<} \cdots \overset{xs}{<} V_{p1} \overset{ys}{<} V_{p2} \overset{ys}{<} V_{p3} \overset{ys}{<} \cdots \overset{ys}{<} V_{pq} = U$$

and the second one in §9

$$V_{11} \overset{ys}{<} V_{12} \overset{ys}{<} \cdots \overset{ys}{<} V_{1q'} \overset{xs}{<} V_{2q'} \overset{xs}{<} \cdots \overset{xs}{<} V_{p'q'} = U \quad .$$

In addition to these two, there are many other series of continuous simply augmented inductive valuations binding V_{11} and U. A method to make one of them is as follows;

If we want to make a continuous x-simply augmented valuation of V_{11}, first of all we must inspect whether such a polynomial $f(x)$ of x for which $Wf(x) > V_{11}f(x)$ exists. If such a polynomial $f(x)$ does not exist, then no x-simply augmented valuation of V_{11} exists. If such polynomials $f(x)$ exist, then out of them we select such polynomials $g(x)$ whose degrees are minimum and next out of these $g(x)$ we pick up such a polynomial $\phi_2(x)$ for which $W\phi_2(x) - V_{11}\phi_2(x)$ is minimum and whose leading coefficient is 1.

Then $\phi_2(x)$ becomes an x-key polynomial of V_{11} which is a valuation of $K_y[x]$ whose coefficient field K_y has the valuation V_{01}. We make V_{21} as follows;

$$V_{21} = [V_{11},\ V_{21}\phi_2 = W\phi_2 = \mu_2] .$$

V_{21} is a continuous x-simply augmented valuation of V_{11}.

In the same way as above, we continue such processes and we obtain a series of continuous x-simply augmented inductive valuations

$$V_{11} \overset{xs}{<} V_{21} \overset{xs}{<} V_{31} \overset{xs}{<} \cdots \overset{xs}{<} V_{s1} .$$

Next, if we want to make a continuous y-simply augmented valuation of V_{s1}, first of all, we must inspect whether such a polynomial $h(y)$ of y for which $Wh(y) > V_{s1}h(y)$ exists. If such a polynomial does not exist, no y-simply augmented valuation of V_{s1} exists.

If such a polynomial exists, we make a valuation V_{s0} which V_{s1} induces in $K[x]$. And we select such polynomials $l(x, y)$ whose degrees with respect to y are minimum, out of such polynomials $h(x, y)$ for which $Wh(x, y) > V_{s1}h(x, y)$. And moreover out of these $l(x, y)$ we pick up a polynomial $\psi_2(x, y)$ for which $W\psi_2(x, y) - V_{s1}\psi_2(x, y)$ is minimum and whose coefficient of the term of the highest power of y is 1. Then $\psi_2(x, y)$ is a y-key polynomial of V_{s1} which is a valuation of $K_x[y]$ whose coefficient field K_x has the valuation V_{s0}. Let be an expansion of an arbitrary polynomial $k(x, y)$ by $\phi_2(x, y)=\phi_2$

$$k(x, y) = \sum_{i=0}^{n} k_i(x, y)\phi_2^i$$

, where $\deg_y k_i(x, y) < \deg_y \phi_2$ for $\mathrm{i}=0, 1, 2, \cdots, \mathrm{n}$.

Then the function V_{s2} defined as follows is a continuous y-simply augmented valuation of V_{s1} of the ring $K_x[y]$;

$$V_{s2}k(x, y) = \operatorname*{Min}_i \left[V_{s1}k_i(x, y) + iW\phi_2 \right] .$$

V_{s2} is denoted simply as

$$V_{s2} = [V_{s1},\ V_{s2}\phi_2 = W\phi_2 = \nu_2] .$$

By repeating such processes we obtain a series of continuous y-simply augmented inductive valuations

$$V_{s1} \overset{ys}{<} V_{s2} \overset{ys}{<} \cdots \overset{ys}{<} V_{st} .$$

Next, if we want to make a continuous x-simply augmented valuation of V_{st}, we must inspect whether such a polynomial $f(x)$ of x for which $Wf(x) > V_{st}f(x)$ exists. If such a polynomial does not exist, then no x-simply augmented valuation of V_{st} exists. If such a polynomial $f(x)$ exists, we can make an x-simply augmented valuation of V_{st}. But for the sake of it, as I explained in §4, we must transform the valuation V_{st} of the ring $K_x[y]$ into a valuation $V_{s't'}$ of the ring $K_y[x]$. Namely, first we must make a valuation $V_{0t'}$ which V_{st} induces in $K[y]$. In this case each y-key polynomial which appears at each stage on the way while we make a series of continuous y-augmented inductive valuations from V_{01} to $V_{0t'}$, may be generally different from the y-key polynomials which produced respectively $V_{s1}, V_{s2}, \cdots, V_{st}$ and t is not always equal to t', as I showed in Example 10. 6. Next we make the valuation V_{st} as the last valuation $V_{s't'}$ of a series of continuous x-simply augmented inductive valuations of the polynomial ring $K_y[x]$ whose coefficient field K_y has the valuation $V_{0t'}$.

$$V_{st} = V_{s't'} = [V_{0t'},\ V_{1t'}x = Wx,\ V_{2t'}\phi_2' = W\phi_2',\ \cdots,\ V_{s't'}\phi_{s'}' = W\phi_{s'}'] .$$

Herein $\phi_2', \phi_3', \cdots, \phi_{s'}'$ are generally polynomials of x and y and may generally be respectively different from the polynomials $\phi_2, \phi_3, \cdots, \phi_s$ which produce $V_{21}, V_{31}, \cdots, V_{s1}$ respectively and s' is not always equal to s.

Namely V_{st} is a valuation of the polynomial ring $K_x[y]$ whose coefficient field K_x has the valuation V_{s0}, but only after we transform V_{st} into a valuation $V_{s't'}$ of the polynomial ring of $K_y[x]$ whose coefficient field K_y has the valuation $V_{0t'}$, we can make an x-augmented valuation of $V_{s't'}$.

After we make $V_{s't'}$, out of such polynomial $l(x, y)$ for which $Wl(x, y) > V_{s't'}l(x, y)$ we pick up such a polynomial $\phi_{s'+1}$ whose degree with respect to x is minimum and whose difference between $W\phi_{s'+1} - V_{s't'}\phi_{s'+1}$ is minimum and whose coefficient of the term of the highest power of x is 1. And with $\phi_{s'+1}$ we make a continuous x-simply augmented valuation $V_{s'+1,t'}$ of $V_{s't'}$

$$V_{s'+1,t'} = [V_{s't'},\ V_{s'+1,t'}\phi_{s'+1} = W\phi_{s'+1}]\,.$$

Thus, as last we can attain U, because U is a discrete valuation. Therefore many series of continuous simply augmented inductive valuations binding V_{11} and U exist.

Now, I want to prepare for solution of Principal Problem, but for the sake of it I must use many notions which MacLane used. So I explain these notions as simply as possible.

§ 12. Homogeneous inductive valuations

MacLane introduced a notion of "*homogeneous inductive valuations*" as "*normal forms*" of valuations in M §16.

We first choose in K a complete set K' of "representatives" with respect to V_0, such that each element of K is equivalent in V_0 to one and only one representative.

A series of augmented inductive valuations of $K[x]$

$$V_1 < V_2 < \cdots < V_{i-1} < V_i < \cdots < V_k$$

is called that of *homogeneous inductive valuations* when they satisfy the following conditions;

let be $\qquad V_i = \left[V_{i-1},\ V_i\phi_i(x) = \mu_i\right] \qquad$ for $\quad i = 1, 2, \cdots, k,$

and $\qquad \phi_i = \sum_j a_j \phi_1^{m_{1j}} \phi_2^{m_{2j}} \cdots \phi_{i-1}^{m_{i-1,j}} \qquad (12.1)$

then $\qquad m_{pj} < \dfrac{deg\ \phi_{p+1}}{deg\ \phi_p} \qquad$ for all terms in (12. 1)

and a_j belongs to K' and

$$V_{i-1}\phi_i = V_{i-1}a_j\phi_1^{m_{1j}} \cdots \phi_{i-1}^{m_{i-1,j}} \qquad \text{for all terms in (12. 1).}$$

According to M. Theorem 16.3, any inductive constructed valuation from a discrete valuation V_0 of K is equal to a homogeneous inductive valuation and two equal homogeneous valuations of $K[x]$ are identical by M. Theorem 16.4. (See §27).

Namely, if V_k and W_t are two homogeneous valuations of $K[x]$ and for any polynomial $f(x)$

$$V_k f(x) = W_t f(x)$$

and
$$V_k = [V_0,\ V_1 x = \mu_1, \cdots, V_k\phi_k = \mu_k]$$
$$W_t = [W_0,\ W_1 x = \nu_1, \cdots, W_t\psi_t = \nu_t]$$

and
$$1 < deg\ \phi_2 < deg\ \phi_3 < \cdots < deg\ \phi_k$$
$$1 < deg\ \psi_2 < deg\ \psi_3 < \cdots < deg\ \psi_t$$

then
$$V_0 = W_0 \quad \text{and} \quad k = t$$

and
$$\phi_i = \psi_i \qquad (i = 1, 2, \cdots, k)$$
$$\mu_i = \nu_i \qquad (i = 1, 2, \cdots, k).$$

In the same way a series of continuous x-augmented homogeneous inductive valuations from V_{10} to W is decided uniquely by W according to Theorem 2. 5. and every key polynomial $\phi_k(x)$ and its value $\mu_k = V_k\phi_k$ are decided uniquely by W.

Hence I let be all valuations homogeneous in this paper. For example, in this paper I write the key polynomial $\phi_{k+1}(x)$ as follows;

$$\phi_{k+1} = \phi_k^{m\tau_k} + \cdots + a_i(x)\phi_k^{i\tau_k} + \cdots + a_0(x) \qquad (12, 2).$$

In (12,2) I dropped all such terms $a_j(x)\phi_k^{j\tau_k}$ which appear in the expansion of $\phi_{k+1} = f(x)$ by M. Theorem 9. 4. and for which $V_k a_j(x)\phi_k^{j\tau_k} > V_k\phi_{k+1}$.

So, for all terms that appear in (12, 2) the followings hold;

$$V_k\phi_{k+1} = V_k a_i(x)\phi_k^{i\tau_k} = V_k\phi_k^{m\tau_k} = V_k a_0(x)$$

and
$$deg\ a_i(x) < deg\ \phi_k.$$

§ 13. Approximants

MacLane established a paper "A construction for prime ideals as absolute values of an algebraic field" in "Duke Jour. Vol. 2, No. 3." in 1936. In this paper I will often quote this paper of his. For example, D. Theorem 5. 2. is Theorem 5. 2. in this paper.

In it, MacLane studies to seek such series of augmented inductive valuations of $K[x]$.

$$V_1 < V_2 < \cdots < V_{k-1} < V_k \qquad (13. 1).$$

Let be
$$V_1 = [V_0,\ V_1 x = \mu_1]$$
$$V_k = [V_{k-1},\ V_k\phi_k(x) = \mu_k] \qquad \text{for} \quad k = 2, 3, \cdots$$

and for a given polynomial $G(x)$ which is irreducible in $K[x]$, the following relations hold

$$V_1G(x) < V_2G(x) < \cdots < V_{k-1}G(x) < V_kG(x)$$

and moreover $V_kG(x)$ must be ∞ or $\lim_{k\to\infty} V_kG(x)$ must be ∞.

$$G(x) = g_m(x)\phi_k^m + \cdots + g_i(x)\phi_k^i + \cdots + g_0(x)$$

where $\qquad deg\, g_i(x) < deg\, \phi_k \qquad$ for $\quad i = 0, 1, 2, \cdots, m$.

Among the exponents j for which $V_kG(x) = V_k(g_j\phi_k^j)$, let α be the largest and β the smallest. The difference $\alpha - \beta$, which depends on both K and $G(x)$, is called the project of V_k(*Symbol*: *Proj* V_k) (D §3). And in (13.1), if such a valuation V_{k+1} for which $V_kG(x) < V_{k+1}G(x)$ exists, then this case is said that the "*terminating*" case does not occur by the k-th stage, and if $V_kG(x) = \infty$, then the case is said that the "*terminating*" case occurs by the k-th stage.

A necessary and sufficient condition that the "*terminating*" case does not occur by the k-stage is that Proj V_k is positive. V_k is called a k-th *approximant* to $G(x)$. (D. Definition 3. 3.)

When $G(x)$ is given, generally there are many series of approximants of $G(x)$, and MacLane gives the two following theorems;

D. Theorem 5. 2.

If the "terminating" case does not occur by the k-th stage, there is a finite number of k-th approximants, such that

$$\sum (Proj\ \ V_k) \left(deg\, \phi(V_k)\right) = deg\, G(x)\,,$$

where $\phi(V)$ represents the last key of V and the sum is taken over all such augmented valuations V_k.

D. Theorem 5. 3.

If there is a non-finite homogeneous inductive value V_k with $V_kG(x) = \infty$, then for $i < k$ the value V_i from which V_k is obtained is the only i-th approximant.

If $G(x)$ can be decomposed in V_{k-1} as in D §5 (4),

$$G(x) \sim e(x)\phi_{k-1}(x)^{n_0}\psi_1(x)^{n_1}\cdots\psi_t(x)^{n_t} \qquad \text{in} \quad V_{k-1}\,,$$

namely one homogeneous key polynomial $\psi_1(x)$ appears in this decomposition, by giving a suitable value $V_k\psi_1(x)$ greater than $V_{k-1}\psi_1(x)$ to $\psi_1(x)$, we can make as follows;

$$V_{k-1}G(x) < V_kG(x) \neq \infty\,.$$

Therefore, when $V_k G(x)=\infty$, $G(x)$ itself must be a homogeneous key polynomial in V_{k-1}.

Definition 13. 1.

$$V_1 < V_2 < \cdots < V_k \qquad (\alpha)$$
$$V_1' < V_2' < \cdots < V_t' \qquad (\beta)$$

Both (α) *and* (β) *are series of augmented inductive valuations which are extensions of* V_0 *to* $K[x]$ *and when we make both* (α) *and* (β) *into series of continuous augmented inductive valuations by making dense the ways of the two series as in* § *3, the two series of continuous augmented inductive valuations completely coincide with each other. In this case we say that* (α) *and* (β) *are on the same course.*

Then by D. Theorem 5. 2. we obtain the following corollary.

Corollary 13. 2.

If $G(x)$ *is irreducible in* $K[x]$ *and* (α) *and* (β), *the two following series of augmented inductive valuations which are extensions of* V_0 *to* $K[x]$

$$V_1 < V_2 < \cdots < V_{k-1} < V_k \qquad (\alpha)$$
$$V_1' < V_2' < \cdots < V_{k-1}' < V_k' \qquad (\beta)$$

satisfy the following conditions

$$V_1 G(x) < V_2 G(x) < \cdots < V_{k-1} G(x) < V_k G(x)$$
$$V_1' G(x) < V_2' G(x) < \cdots < V_{k-1}' G(x) < V_k' G(x)$$

and $V_k = [V_{k-1}, V_k G(x) = \mu_k]$, *that is,* $G(x)$ *is the last key polynomial that produces* V_k *and* $V_k G(x) = V_k' G(x)$, *then* (α) *and* (β) *are on the same course and* $V_k = V_k'$.

In this corollary $V_k \phi_k = \mu_k$ *may be finite.*

Theorem 13. 3.

If

$$V_1 < V_2 < \cdots < V_{k-1} < V_k \qquad (\gamma)$$

is a series of augmented inductive valuations which are extensions of V_0 *to* $K[x]$ *and*

$$V_k = [V_{k-1}, V_k \phi_k = \mu_k],$$

and

$$V_1' < V_2' < \cdots < V_{t-1}' < V_t' \qquad (\delta)$$

is another series of augmented inductive valuations which are extensions of V_0 *to* $K[x]$ and

$$V_1' \phi_k < V_2' \phi_k < \cdots < V_{t-1}' \phi_k < V_t' \phi_k = \mu_k,$$

then $V_k = V'_t$ *or* V'_t *is an augmented valuation of* V_k.

Proof.

ϕ_k is the last key polynomial, so ϕ_k is irreducible and $V_1\phi_k < V_2\phi_k < \cdots < V_{k-1}\phi_k < V_k\phi_k = \mu_k$.

We can make both (γ) and (δ) into series of continuous augmented inductive valuation by making dense the ways of (γ) and (δ) as in §3, then according to Corollary 13. 2. and Theorem 2. 6, V_k equals to V'_t, or V_k is equal to a valuation between V'_{t-1} and V'_t.

§ 14. Decompositions of the last key polynomial in V_2

Theorem 14. 1.

If
$$V_1 < V_2 < \cdots < V_{p-1} < V_p \tag{14. 1}$$
is a series of homogeneous augmented inductive valuations which are extensions of V_0 *to* $K[x]$, *then*
$$\phi_p(x) \sim \phi_i(x)^{b_i} \quad \textit{in} \quad V_{i-1} \quad \textit{for} \quad i = 2, 3, \cdots, P,$$
where
$$b_i = deg\, \phi_p / deg\, \phi_i \quad \textit{for} \quad i = 2, 3, \cdots, P.$$
Therefore especially $\phi_p \sim \phi_2^{b_2}$ *in* V_1.

In order to prove this theorem, I will prove the two following lemmas before it.

Lemma 14. 2.

In the series (14. 1), *if*
$$V_i = [V_{i-2},\ V_{i-1}\phi_{i-1} = \mu_{i-1},\ V_i\phi_i = \mu_i]$$
and $h(x)$ *is a homogeneous polynomial in* V_{i-1}

namely $$h(x) = g_n(x)\phi_{i-1}^n + \cdots + g_k(x)\phi_{i-1}^k + \cdots + g_0(x), \tag{14. 2}$$

where $deg\, g_k(x) < deg\, \phi_{i-1}$ *for all terms in* (14. 2)

and $V_{i-1}h(x) = V_{i-1}g_k\phi_{i-1}^k$ *for all terms in* (14. 2)

, *then* $h(x) \sim g_n\phi_{i-1}^n$ *in* V_{i-2}

and $V_{i-1}h(x) - V_{i-2}h(x) = n[V_{i-1}\phi_{i-1} - V_{i-2}\phi_{i-1}]$.

Proof.
$$deg\, g_k < deg\, \phi_{i-1}$$
so $$V_{i-1}g_k = V_{i-2}g_k \quad \text{for all } g_k \text{ in (14. 2.)}$$

So, when $n > k$

$$V_{i-1}g_n\phi_{i-1}^n - V_{i-2}g_n\phi_{i-1}^n = n(V_{i-1}\phi_{i-1} - V_{i-2}\phi_{i-1})$$
$$> k(V_{i-1}\phi_{i-1} - V_{i-2}\phi_{i-1})$$
$$= V_{i-1}g_k\phi_{i-1}^k - V_{i-2}g_k\phi_{i-1}^k .$$

while $$V_{i-1}h(x) = V_{i-1}g_n\phi_{i-1}^n = V_{i-1}g_k\phi_{i-1}^k ,$$

then $$\operatorname*{Min}_k [V_{i-2}g_k\phi_{i-1}^k] = V_{i-2}g_n\phi_{i-1}^n .$$

Therefore $$h(x) \sim g_n\phi_{i-1}^n \quad \text{in} \quad V_{i-2} .$$

$$V_{i-2}h(x) = V_{i-2}g_n\phi_{i-1}^n$$

and $$V_{i-1}h(x) - V_{i-2}h(x) = n[V_{i-1}\phi_{i-1} - V_{i-2}\phi_{i-1}] .$$

Lemma 14. 3.

In Lemma 14.2 *if*

$$deg\ h(x) < deg\ \phi_i$$

then $$V_{i-1}h(x) - V_{i-2}h(x) < V_{i-1}\phi_i - V_{i-2}\phi_i .$$

Proof.

Let be $$\phi_i = \phi_{i-1}^C + \cdots + g_k'\phi_{i-1}^k + \cdots + g_0'(x) \qquad (14.3).$$

ϕ_i is homogeneous in V_{i-1}, so

$$V_{i-1}\phi_i = V_{i-1}g_k'\phi_{i-1}^k \qquad \text{for all terms in (14. 3).}$$

$$h(x) = g_n\phi_{i-1}^n + \cdots + g_0(x)$$

$$deg\ h(x) < deg\ \phi_i$$

$$n < C .$$

So $$V_{i-1}\phi_i - V_{i-2}\phi_i$$
$$= C[V_{i-1}\phi_{i-1} - V_{i-2}\phi_{i-1}] \qquad \text{owing to Lemma 14. 2.}$$
$$> n[V_{i-1}\phi_{i-1} - V_{i-2}\phi_{i-1}] = V_{i-1}h(x) - V_{i-2}h(x) .$$

It is evident that both lemmas 14.2. and 14.3. hold when i moves from 3 to p in the series (14.1).

Now I can prove Theorem 14. 1.

Proof of Theorem 14. 1.

$$\phi_p = \phi_{p-1}^{b_{p-1}} + \cdots + h_k(x)\phi_{p-1}^k + \cdots + h_0(x) \qquad (14.4)$$

Owing to Lemma 14. 2.

$$\phi_p \sim \phi_{p-1}^{b_{p-1}} \qquad in \quad V_{p-2}$$

and $$V_{p-2}\phi_{p-1}^{b_{p-1}} < V_{p-2}h_k\phi_{p-1}^k \qquad \text{for all terms in (14. 4)} \qquad (14.5)$$

Let be $$\phi_{p-1}=\phi_{p-2}^{C}+\cdots+h'(x)$$

then in the same way

$$\phi_{p-1}\sim\phi_{p-2}^{C} \quad in \quad V_{p-3}.$$

$$V_{p-2}\phi_{p-1}^{b_{p-1}}-V_{p-3}\phi_{p-1}^{b_{p-1}}\geqq(k+1)[V_{p-2}\phi_{p-1}-V_{p-3}\phi_{p-1}]$$

$$\because \quad b_{p-1}>k \quad and \quad V_{p-2}\phi_{p-1}>V_{p-3}\phi_{p-1}$$

$$=(V_{p-2}\phi_{p-1}-V_{p-3}\phi_{p-1})+k(V_{p-2}\phi_{p-1}-V_{p-3}\phi_{p-1})$$

$$>(V_{p-2}h_k-V_{p-3}h_k)+k(V_{p-2}\phi_{p-1}-V_{p-3}\phi_{p-1})$$

$$\because \quad deg\, h_k<deg\, \phi_{p-1}, \quad \text{so Lemma 14. 2 holds.}$$

$$=V_{p-2}h_k\phi_{p-1}^{k}-V_{p-3}h_k\phi_{p-1}^{k}.$$

Namely $$V_{p-2}\phi_{p-1}^{b_{p-1}}-V_{p-3}\phi_{p-1}^{b_{p-1}}>V_{p-2}h_k\phi_{p-1}^{k}-V_{p-3}h_k\phi_{p-1}^{k}$$

$$V_{p-3}h_k\phi_{p-1}^{k}>V_{p-3}\phi_{p-1}^{b_{p-1}}+(V_{p-2}h_k\phi_{p-1}^{k}-V_{p-2}\phi_{p-1}^{b_{p-1}})$$

$$>V_{p-3}\phi_{p-1}^{b_{p-1}} \quad by\ (14.5).$$

So $$\operatorname*{Min}_{k}[V_{p-3}h_k\phi_{p-1}^{k}]=V_{p-3}\phi_{p-1}^{b_{p-1}}.$$

Therefore $$\phi_p\sim\phi_{p-1}^{b_{p-1}} \quad in \quad V_{p-3}. \qquad \phi_{p-1}\sim\phi_{p-2}^{C} \quad in \quad V_{p-3}$$

then $$\phi_p\sim\phi_{p-2}^{Cb_{p-1}} \quad in \quad V_{p-3}.$$

$$deg\,\phi_p=b_{p-1}\,deg\,\phi_{p-1}=Cb_{p-1}\,deg\,\phi_{p-2},$$

so $$Cb_{p-1}=deg\,\phi_p/deg\,\phi_{p-2}=b_{p-2}$$

$$\therefore \quad \phi_p\sim\phi_{p-2}^{b_{p-2}} \quad \text{in} \quad V_{p-3}. \qquad deg\,\phi_p=deg\,\phi_{p-2}^{b_{p-2}}.$$

Let be an expansion of ϕ_p by ϕ_{p-2}

$$\phi_p=\phi_{p-2}^{b_{p-2}}+\cdots+l_k(x)\phi_{p-2}^{k}+\cdots+l_0(x) \tag{14. 6}$$

$$V_{p-3}l_k\phi_{p-2}^{k}>V_{p-3}\phi_{p-2}^{b_{p-2}} \quad \text{for all terms in (14.6),}$$

because $$\phi_p\sim\phi_{p-2}^{b_{p-2}} \quad in \quad V_{p-3}.$$

Then we can prove that $\phi_p\sim\phi_{p-3}^{b_{p-3}}$ in V_{p-4} in the same way as when I proved that $\phi_p\sim\phi_{p-2}^{b_{p-2}}$ in V_{p-3}.

Thus at last we can prove that $\phi_p\sim\phi_2^{b_2}$ in V_1.

§ 15. Effective degrees

There is a famous theorem called "Hensel's Lemma" in theory of valuation. In order to solve the Principal Problem in my own way, I want to make an extension of this "Hensel's Lemma" to V_k in $K[x]$. For the sake of it I must arrange some lemmas.

$$V_k=[V_{k-1},\ V_k\phi_k=\mu_k].$$

And the following is an expansion of an arbitrary polynomial $f(x)$;

$$f(x)=f_n(x)\phi_k^n+f_{n-1}(x)\phi_k^{n-1}+\cdots+f_j(x)\phi_k^j+\cdots+f_0(x) \qquad (15.\ 1)$$

$$deg\, f_j(x)<deg\, \phi_k \qquad \text{for all terms in(15. 1).}$$

Then, the largest exponent α for which $V_kf(x)=V_k(f_\alpha\phi_k^\alpha)$ is called the effective degree of $f(x)$ in ϕ_k and is denoted by $D_{\phi_k}f$ or simply by $D_\phi f$. (D §4)

If $h(x)\sim l(x)$ in V_k, then $D_\phi h(x)=D_\phi l(x)$ and for any two polynomials $f(x)$ and $g(x)$

$$D_\phi(f\,g)=D_\phi f+D_\phi g.$$

If $D_\phi e(x)=0$, then the polynomial $e(x)$ is called an equivalent unit in V_k or simply a unit.

Lemma 15.

Every unit is equivalent to a unit whose degree is less than that of ϕ_k *and a set of all equivalent-units in* V_k *is a group with respect to multiplication.* (D §4)

§ 16. Canon polynomials

Γ_{k-1} is a group of all values $V_{k-1}f(x)$, where $f(x)\in k[x]$ and τ_k is the smallest integer such that $\tau_k\mu_k\in\Gamma_{k-1}$. Then there exists such a polynomial $Q(x)$ that

$$V_kQ(x)\phi_k^{\tau_k}=0 \quad \text{and} \quad deg\, Q(x)<deg\, \phi_k.$$

In this paper, hence I write this polynomial $Q(x)\phi_k^{\tau_k}$ as Φ. Then Δ_k, the residue class ring of $K[x]$ by V_k is $F_k[X]$, where F_k is an algebraic extension of F_0 which is the residue class field of K by V_0 and $X=H_k\Phi$ is a transcendental element with respect to F_k, herein H_k is a natural homomorphism by which Φ corresponds to X. (M. Theorem 12. 1)

Then a set $\{e(x)|D_\phi e(x)=0 \text{ and } V_ke(x)=0\}$

is a subgroup of a group of all units in V_k.

Lemma 16. 1.

If $\phi_k\nmid l(x)$ *in* V_{k-1}, then $l(x)$ is a unit in V_k.

Proof.

By M. Lemma 9. 2. $\qquad V_kl(x)=V_{k-1}l(x)$, so according to M. Lemma 9. 1, there exists such a polynomial $l'(x)$ that

$$l(x)\,l'(x)\sim 1 \qquad in \quad V_k .$$

Therefore

$$0 = D_\phi 1 = D_\phi(ll') = D_\phi l + D_\phi l'$$

$$\therefore \qquad D_\phi l(x) = 0 .$$

Lemma 16. 2.

If $V_k f(x)=0$ *and* $\dfrac{D_\phi f(x)}{\tau_k}=m>0$, *then there exists such a polynomial* $f^*(x)$ *that*

$$f(x)\sim f^*(x) = a'_m(x)(\Phi^m + \cdots + b_i(x)\Phi^i + \cdots + b_q(x)\Phi^q) \qquad (16.1)$$

in V_k

and

$$H_k f(x) = H_k^* f(x)$$
$$= \bar{a}'_m(X^m + \cdots + \bar{b}_i X^i + \cdots + \bar{b}_q X^q)$$

where $\bar{a}'_m = H_k a'_m(x) \neq 0$ *and* $\bar{b}_i = H_k b_i(x) \neq 0$

and $deg\, a'_m(x) < deg\, \phi_k$

$deg\, b_i(x) < deg\, \phi_k$ *for all terms in* (16. 1) .

Proof.

Let be $$f(x) = \sum_{i=0}^{n} a_i(x)\phi_k^i \qquad (16.2),$$

where $deg\, a_i(x) < deg\, \phi_k$ for all terms in (16. 2).

First we drop all such terms $a_i(x)\phi_k^i$ for which

$$V_k a_i \phi_k^i > V_k f(x), \qquad \text{from } 16.2.$$

Then we obtain

$$f(x)\sim a_{m'}(x)\phi_k^{m'} \cdots + a_{i'}(x)\phi_k^{i'} + \cdots + a_{q'}(x)\phi_k^{q'} \qquad (16.3)$$

in V_k.

$$0 = V_k f(x) = V_k(a_{i'} f_k^{i'}) \qquad \text{for all terms in (16.3)}$$

$$\therefore \quad \tau_k \mid i' \qquad \text{so} \quad i' = i\tau_k .$$

Namely $$f(x)\sim a_{m'}(\phi_k^{\tau_k})^m + \cdots + a_{i'}(\phi_k^{\tau_k})^i + \cdots + a_{q'}(\phi_k^{\tau_k})^q \qquad (16.4)$$

in V_k.

Owing to M. Lemma 9.1, there exists such a polynomial $Q'(x)$ that

$$Q(x)Q'(x)\sim 1 \quad \text{in } V_k \quad \text{and} \quad V_kQ'(x)=V_{k-1}Q'(x).$$

Then

$$f(x)\sim a_{m'}Q'^m(Q\phi_k^{\tau_k})^m+\cdots+a_{i'}Q'^i(Q\phi_k^{\tau_k})^i+\cdots+a_{q'}Q'^q(Q\phi_k^{\tau_k})^q \quad \text{in } V_k,$$

because values of all terms in (16.4) are zero.

$$f(x)\sim a_{m'}Q'^m\left[\Phi^m+\cdots+\frac{a_{i'}Q'^i}{a_{m'}Q'^m}\Phi^i+\cdots+\frac{a_{q'}}{a_{m'}Q'^m}\Phi^q\right] \quad (16.5) \quad \text{in } V_k.$$

According to Lemma 15, every coefficient in (16.5) is equivalent to a unit whose degree is less than that of ϕ_k.

$$f(x)\sim a'_m(\Phi^m+\cdots+b_i\Phi^i+\cdots+b_q\Phi^q) \quad \text{in } V_k.$$

So, $$H_kf(x)=\bar{a}'_m(X^m+\cdots+\bar{b}_iX^i+\cdots+\bar{b}_qX^q)$$

where all coefficients $\bar{a}'_m$ and $\bar{b}_i$ belong to F_k and none of them is zero.

Lemma 16. 3.

If $V_kg(x)=\alpha>0$, then there exists such a polynomial $c(x)\phi_k^lg^*(x)$ that

$$g(x)\sim c(x)\phi_k^lg^*(x) \quad \textit{in } V_k$$

where $$V_kc(x)\phi_k^l=\alpha$$

and $$g^*(x)=\Phi^m+\cdots+b_i(x)\Phi^i+\cdots+b_0(x) \quad (16.6)$$

and $$\deg c(x)<\deg\phi_k \quad \textit{and} \quad \deg b_i(x)<\deg\phi_k$$

$$0\neq\bar{b}_i=H_kb_i(x)\in F_k \quad \textit{for all terms in } (16.6).$$

Proof.

Let be an expansion of $g(x)$ by ϕ_k as follows;

$$g(x)=g_s(x)\phi_k^s+\cdots+g_j(x)\phi_k^j+\cdots+g_0(x) \quad (16.7).$$

We drop all such terms $g_j(x)\phi_k^j$ for which

$V_kg(x)<V_kg_j\phi_k^j$, from (16.7) and we obtain

$$g(x)\sim g_p(x)\phi_k^p+\cdots+g_j(x)\phi_k^j+\cdots+g_l(x)\phi_k^l \quad (16.8) \quad \text{in } V_k$$

and $$V_kg(x)=V_kg_j\phi_k^j=\alpha \quad \text{for all terms in (16.8).}$$

$$g(x)\sim g_l\phi_k^l\left(\frac{g_p}{g_l}\phi_k^{p-l}+\cdots+\frac{g_j}{g_l}\phi_k^{j-l}+\cdots+1\right) \quad \text{in } V_k$$

$$V_k\left(\frac{g_p}{g_l}\phi_k^{p-l}+\cdots+1\right)=0 \quad \therefore \quad \tau_k\,|\,p-l.$$

Let be $$p-l=m\tau_k,$$

then owing to Lemma 16.2

$$\frac{g_p}{g_l}\phi_k^{p-l}+\cdots+1\sim a_m(x)\left(\Phi^m+\cdots+b_i(x)\Phi^i+\cdots+b_0(x)\right) \quad \textit{in} \quad V_k\,.$$

So, consequently

$$g(x)\sim g_l a_m(x)\phi_k^l\left(\Phi^m+\cdots+b_i(x)\Phi^i+\cdots+b_0(x)\right) \quad \text{in } V_k\,.$$

$$\deg g_l<\deg \phi_k \quad \textit{and} \quad \deg a_m(x)<\deg \phi_k\,,$$

then owing to Lemma 15,

$$g_l a_m(x)\sim c(x) \quad \text{in } V_k \quad \text{and} \quad \deg c(x)<\deg \phi_k\,.$$

$$\therefore \quad g(x)\sim c(x)\phi_k^l\left(\Phi^m+\cdots+b_i(x)\Phi^i+\cdots+b_0(x)\right) \quad \text{in } V_k$$

$$V_k c(x)\phi_k^l=V_k g(x)=\alpha\,.$$

Corollary 16.4. *In Lemma* 16.3, *we can make l less than τ_k.*

Proof. When $l\geqq\tau_k$, $l=h\tau_k+l'$,

where $$0\leqq l'\leqq\tau_k-1\,.$$

$$c(x)\phi_k^l=c(x)\phi_k^{l'}(\phi_k^{\tau_k})^h$$
$$\sim c(x)\left(Q'(x)\right)^h\phi_k^{l'}\left(Q(x)\phi_k^{\tau_k}\right)^h\sim c'(x)\phi_k^{l'}\Phi^h \quad \text{in } V_k\,,$$

where $c'(x)\sim c(x)\left(Q'(x)\right)^h$ in V_k and $\deg c'(x)<\deg \phi_k$.

Therefore

$$g(x)\sim c'(x)\phi_k^{l'}\left(\Phi^{m+h}+\cdots+b_i(x)\Phi^{i+h}+\cdots+b_0(x)\Phi^h\right) \quad \textit{in} \quad V_k$$

where $$0\leqq l'\leqq\tau_k-1$$

and $$V_k c'(x)\phi_k^{l'}=V_k g(x)=\alpha\,.$$

Definition 16.5. *A polynomial $e(x)$ for which $D_\phi e(x)=0$ and $V_k e(x)=0$ is called a V_k-unit and $H_k e(x)=\overline{e(x)}$ is called a F_k-unit.*

Definition 16.6. *If $f(x)=\Phi^m+\cdots+b_i(x)\Phi^i+\cdots+b_q(x)\Phi^q$, ,where all $b_i(x)$ are V_k-units and $m\geqq1$, then $f(x)$ is called a canon polynomial in V_k*

and $$H_k f(x)=X^m+\cdots+\overline{b_i(x)}X^i+\cdots+\overline{b_q(x)}X^q$$

where all $\overline{b_i(x)}$ are F_k-units, is called a canon class of $f(x)$.

Definition 16.7. *$f(x)$ is a canon polynomial* *and* $$f(x)=\left(Q(x)\phi_k^{\tau_k}\right)^m+\cdots+b_i(x)\left(Q(x)\phi_k^{\tau_k}\right)^i+\cdots+b_0(x)$$

and $$f(x)\sim Q(x)^m\left[\phi_k^{m\tau_k}+\cdots+c_i(x)\phi_k^{i\tau_k}+\cdots+c_0(x)\right],\qquad in\ V_k,$$

where $$deg\ c_i(x)<deg\ \phi_k$$

$\phi_k^{m\tau_k}+\cdots+c_i(x)\phi^{i\tau_k}+\cdots+c_0(x)$ *is called a key part of* $f(x)$.

Then we have the following theorem.

Theorem 16.8. *A necessary and sufficient condition that a polynomial is a homogeneous key polynomial in* V_k *is that it is a key part of a canon polynomial whose canon class is irreducible in* $F_k[X]$. (M. Lemma 11.2. and M. Theorem 9.4.)

§ 17. Prime each other in V_k

Theorem 17.1. *If both* $g(x)$ *and* $h(x)$ *are canon polynomials and there exist such polynomials* $l(x)$ *and* $m(x)$ *that*

$$g(x)l(x)+h(x)m(x)\sim 1\qquad in\ V_k,$$

and $$V_k l(x)\geqq 0\quad and\quad V_k m(x)\geqq 0,$$

then $$V_k l(x)=V_k m(x)=0$$

and $$\left(H_k g(x)\right)\left(H_k l(x)\right)+\left(H_k h(x)\right)\left(H_k m(x)\right)=1\qquad in\ F_k[X].$$

Theorem 17.2. *If all* $G(X)$, $H(X)$, $L(X)$ *and* $M(X)$ *are residue classes in* $F_k[X]$ *and*

$$G(X)L(X)+H(X)M(X)=1,$$

then there are such polynomials $g(x)$, $h(x)$, $l(x)$ *and* $m(x)$ *whose values by* V_k *are all zero and whose residue classes in* $F_k[X]$ *are respectively* $G(X)$, $H(X)$, $L(X)$ *and* $M(X)$.

Definition 17.3. *In Theorem* 17.1 *we say that* $g(x)$ *and* $h(x)$ *are prime each other in* V_k. *In the same way we say that many canon polynomials are prime each other in* V_k, *when their residue classes in* $F_k[X]$ *are prime each other in* $F_k[X]$.

Proof of Theorem 17.1 *and Theorem* 17.2. Assuming that

$$V_k l(x)>0\quad \text{and}\quad V_k m(x)>0,$$

$$0=V_k 1=V_k(gl+hm)\geqq \text{Min}\,[V_k gl,\ V_k hm]>0.$$

This is a contradiction.

Next, assuming that $V_k l(x)>0$ and $V_k m(x)=0$,

$$V_k gl > 0 = V_k hm = V_k 1$$

so $$h(x)m(x) \sim 1 \qquad \text{in } V_k .$$

But $D_\phi h(x) > 1$, so this is also a contradiction.

When we assume that $V_k l(x) = 0$ and $V_k m(x) > 0$, in the same way a contradiction takes place.

Therefore $$V_k l(x) = V_k m(x) = 0 .$$

Then we can prove both Therem 17.1 and 17.2 naturally, because H_k is a homomorphism which makes every polynomial $f(x)$ of x whose value by V_k is non-negative, correspond to its residue class $H_k f(x)$ in $F_k[X]$.

Definition 17.4. *If $V_k(f(x) - l(x)) - V_k f(x) > \omega > 0$, then $f(x)$ and $l(x)$ are said to be equivalent higher than ω and are denoted as*

$$f(x) \overset{\omega}{\approx} l(x) \qquad \textit{in } V_k .$$

§ 18. Extension of Hensel's Lemma

Theorem 18.1. *Both $g_0(x)$ and $h_0(x)$ are canon polynomials which are prime each other in V_k and $f(x)$ is a polynomial and ω is an arbitrary given positive number.*

If $$f(x) \sim g_0(x) h_0(x) \qquad \textit{in } V_k ,$$

then there are such two polynomials $g(x)$ and $h(x)$ that

$$f(x) \overset{\omega}{\approx} g(x)\, h(x) \qquad \textit{in } V_k ,$$

where $$g(x) \sim g_0(x) \quad \textit{and} \quad h(x) \sim h_0(x) \qquad \textit{in } V_k$$

and $$\deg g(x) = \deg g_0(x) .$$

Proof. $f(x) \sim g_0 h_0$ in V_k.

Let be $$V_k(f - g_0 h_0) = \alpha > 0 , \quad \text{then owing to Corollary 16.4}$$

$$f - g_0 h_0 \sim c(x) \phi_k^l f^*(\Phi) \qquad \text{in } V_k$$

where $$\deg c(x) < \deg \phi_k$$

and $$0 \leqq l \leqq \tau_k - 1$$

and $$V_k c(x) \phi_k^l = \alpha$$

and $$f^*(\Phi) \quad \text{is a canon polynomial.}$$

Let be $$f = g_0 h_0 + c(x) \phi_k^l f^* + B_1 \tag{18.1}$$

then $$V_k B_1 > V_k(f - g_0 h_0) = \alpha .$$

g_0 and h_0 are prime each other in V_k, then there are two polynomials $l(x)$ and $m(x)$ such that

$$g_0 l + h_0 \mathrm{m} \sim 1 \qquad in \quad V_k ,$$

where $\qquad V_k l = V_k m = 0 \qquad$ by Theorem 17. 1.

$$g_0 l f^* + h_0 m f^* \sim f^* \quad \text{in } V_k , \quad \text{where } f^* = f^*(\Phi) .$$

In $F_k[X]$, let be $H_k(\mathrm{m} f^*) = (H_k q(x))(H_k g_0(x)) + H_k r(x)$,

namely $\qquad \bar{m} \bar{f}^* = \bar{q} \bar{g}_0 + \bar{r}$,

where $\qquad deg_x \bar{r} = deg_x H_k\big(r(x)\big) < deg_x H_k\big(g_0(x)\big) = deg\, \bar{g}_0$.

$$g_0 l f^* + h_0 m f^* \sim f^* \qquad in \quad V_k .$$

So in $F_k[X]$, when we write $H_k l(x)$ as $\bar{l}$,

$$\bar{f}^* = \bar{g}_0 \bar{l} \bar{f}^* + \bar{h}_0 \bar{m} \bar{f}^*$$
$$= \bar{g}_0 \bar{l} \bar{f}^* + \bar{h}_0 (\bar{q} \bar{g}_0 + \bar{r}) = \bar{g}_0 (\bar{l} \bar{f}^* + \bar{h}_0 \bar{q}) + \bar{h}_0 \bar{r}$$

Let be $\qquad \bar{l} \bar{f}^* + \bar{h} \bar{q} = \bar{l}'$, then $\quad \bar{f}^* = \bar{g}_0 \bar{l}' + \bar{h}_0 \bar{r}$

so $\qquad f^* \sim g_0 l'(x) + h_0 r(x) \qquad$ in $V_k \qquad$ (18. 2)

and $\qquad D_\phi r(x) < D_\phi g_0(x) \qquad$.

Let be $\qquad F(x) = f - \big(g_0 + c(x) \phi_k^l r(x)\big) \big(h_0 + c(x) \phi_k^l l'(x)\big)$

, then $\quad F(x) = (f - g_0 h_0) - c(x) \phi_k^l \big(g_0 l'(x) + h_0 r(x)\big) - \big(c(x) \phi_k^l\big)^2 r(x) l'(x)$.

(By 18. 1) $\qquad F(x) = \mathrm{B}_1 - c(x) \phi_k^l (g_0 l' + h_0 r - f^*) - \big(c(x) \phi_k^l\big)^2 r l'$.

By (18. 2) $\qquad V_k(g_0 l' + h_0 r - f^*) > 0$,

so $\qquad V_k c(x) \phi_k^l (g_0 l' + h_0 r - f^*) > \alpha$,

and $\qquad V_k \big(c(x) \phi_k^l\big)^2 r l' = 2\alpha + V_k r l' = 2\alpha > \alpha$.

$$V_k B_1 > \alpha . \qquad \text{Therefore} \quad V_k F(x) > \alpha .$$

When $\qquad deg\, c(x) \phi_k^l r(x) \geqq deg\, g_0(x)$,

$$l \leqq \tau_k - 1 \quad \text{and} \quad deg_x H_k r(x) < deg_x H_k g_0(x) ,$$

so we can select such a polynomial $R(x)$

that $\qquad R(x) \sim c(x) \phi_k^l r(x) \qquad$ in V_k

and $\qquad deg\, R(x) < deg\, g_0(x)$.

So $\qquad R(x) = c(x) \phi_k^l r(x) + B_2$, $\qquad$ where $\quad V_k B_2 > \alpha$.

And we make $F'(x)$ as follows;

$$\begin{aligned} F'(x) &= f-\left(g_0+R(x)\right)\left(h_0+c(x)\phi_k^l l'(x)\right) \\ &= f-\left(g_0+c(x)\phi_k^l r(x)+B_2\right)\left(h_0+c(x)\phi_k^l l'(x)\right) \\ &= F(x)-B_2\left(h_0+c(x)\phi_k^l l'(x)\right). \end{aligned}$$

$$V_k\left(B_2\left(h_0+c(x)\phi_k^l l'(x)\right)\right) = V_k B_2+V_k\left(h_0+c(x)\phi_k^l l'(x)\right) > \alpha .$$

And $\quad V_k F(x)>\alpha$, $\quad$ so consequently $\quad V_k F'(x)>\alpha$.

Then let be $\quad g_0(x)+R(x)=g_1(x)$

and $\quad h_0+c(x)\phi_k^l l'(x)=h_1(x)$

$$V_k\left(f-g_1(x)\,h_1(x)\right)>\alpha .$$

and $\quad g_1(x)\sim g_0(x) \quad$ in V_k $\quad$ (18. 3)

When $\quad deg\ g_0>deg\ c(x)\phi_k^l r(x)$, $\quad$ let be $\quad g_1(x)=g_0(x)+c(x)\phi_k^l r(x)$.

In this case, also $g(x)\sim g_0(x)$ in V_k.

In the same way

$$h_1(x)\sim h_0(x) \qquad in \quad V_k \tag{18. 4}$$

$$g_0 l+h_0 m\sim 1 \qquad in \quad V_k ,$$

so by (18.3) and (18.4)

$$g_1 l+h_1 m\sim 1 \qquad in \quad V_k .$$

And $\quad deg\ g_1(x)=deg\ g_0(x)$.

Therefore both $g_1(x)$ and $h_1(x)$ satisfy all conditions given to $g_0(x)$ and $h_0(x)$ and moreover

$$V_k(f-g_1h_1)>V_k(f-g_0h_0) .$$

So we can repeat the same processes as above and obtain two sequences of canon polynomials $(g_1, g_2, \cdots, g_p, \cdots)$ and $(h_1, h_2, \cdots, h_p, \cdots)$,

for which $\quad g_p\sim g_0 \quad$ in V_k $\quad$ for $p=1, 2, \cdots$.

$\quad h_p\sim h_0 \quad$ in V_k $\quad$ for $p=1, 2, \cdots$.

and $\quad g_p l+h_p m\sim 1 \quad$ in V_k

and $\quad deg\ g_p=deg\ g_0 \quad$ for $p=1, 2, \cdots$.

$V_k(f-g_p h_p)$ $\quad$ increases, as p increases.

V_k is a discrete valuation, then at last $V_k(f-g_ph_p)$ can exceed ω. So

$$f \overset{\omega}{\approx} g_p(x)h_p(x) \qquad \text{in} \quad V_k\,.$$

On extensions of Hensel's Lemma I will again relate in §23.

Corollary 18.2.

All $\phi_1(x), \phi_2(x), \cdots, \phi_t(x)$ are canon polynomials that are prime each other in V_k and $f(x)$ is a polynomial and ω is an arbitrary given positive number.

If

$$f(x) \sim \phi_1\phi_2\cdots\phi_t \qquad \text{in} \quad V_k\,,$$

then there exist such t polynomials $\Psi_1(x), \Psi_2(x), \cdots, \Psi_t(x)$ that

$$f(x) \overset{\omega}{\approx} \Psi_1\Psi_2\cdots\Psi_t \qquad \text{in} \quad V_k\,,$$

where

$$\Psi_i \sim \phi_i \qquad \text{in} \quad V_k \qquad \text{for} \quad i=1,2,\cdots,t$$

and

$$\deg \Psi_i = \deg \phi_i \qquad \text{for} \quad i=1,2,\cdots,t-1\,.$$

Proof. ϕ_1 and $(\phi_2\phi_3\cdots\phi_t)$ are prime each other in V_k, so owing to Theorem 18.1, there exist two polynomials $\Psi_1(x)$ and $h_2(x)$ that

$$f \overset{\omega}{\approx} \Psi_1 h_2 \qquad \text{in} \quad V_k\,,$$

where

$$\Psi_1 \sim \phi_1 \qquad \text{in} \quad V_k$$

$$h_2 \sim \phi_2\phi_3\cdots\phi_t \qquad \text{in} \quad V_k$$

and

$$\deg \Psi_1 = \deg \phi_1\,.$$

$$V_k\Psi_1 = V_k\phi_1 = 0\,.$$

ϕ_2 and $(\phi_3\phi_4\cdots\phi_t)$ are prime each other in V_k, so again by Theorem 18.1, there exist two polynomials $\Psi_2(x)$ and $h_3(x)$ that

$$h_2 \overset{\omega}{\approx} \Psi_2 h_3 \quad \text{in} \quad V_k \qquad \text{and} \qquad \Psi_2 \sim \phi_2 \quad \text{in} \quad V_k$$

$$h_3 \sim \phi_3\phi_4\cdots\phi_t \qquad \text{in} \quad V_k$$

and

$$\deg \Psi_2 = \deg \phi_2\,.$$

$$\begin{aligned} V_k(f-\Psi_1\Psi_2h_3) &= V_k\Big[(f-\Psi_1h_2)+\Psi_1(h_2-\Psi_2h_3)\Big] \\ &\geqq \operatorname{Min}\Big[V_k(f-\Psi_1h_2),\; V_k\Psi_1+V_k(h_2-\Psi_2h_3)\Big] \\ &> \omega\,. \end{aligned}$$

Because $V_k\Psi_1 = 0$.

Therefore $$f \overset{\omega}{\approx} \Psi_1 \Psi_2 h_3 \quad \text{in } V_k .$$

Thus, by repeating the same processes as above, we can prove this corollary.

Generally by D. §5 (4) a polynomial $f(x)$ can be decomposed as follows;

$$f(x) \sim e(x) \phi_k^{n_0} \psi_1^{n_1} \cdots \psi_t^{n_t} \quad \text{in } V_k . \tag{18.5}$$

Let be $$n_0 = q\tau_k + l \quad \text{and} \quad 0 \leqq l \leqq \tau_k - 1 .$$

Then, by Theorem 16.8, we can change (18.5) as follows;

$$e'(x) \phi_k^{\tau_k - l} f(x) \sim \psi_0' \psi_1' \psi_2' \cdots \psi_t \quad \text{in } V_k$$

where $$deg\, e'(x) < deg\, \phi_k \quad and \quad 1 \leqq \tau_k - l \leqq \tau_k$$

and $\psi_0', \psi_1', \cdots, \psi_t'$ are canon polynomials whose key parts are respectively $\phi_k^{(q+1)\tau_k}, \psi_1^{n_1}, \cdots, \psi_t^{n_t}$.

Then owing to Corollary 18.2 and Theorem 16.8

$$e'(x) \phi_k^{\tau_k - l} f(x) \overset{\omega}{\approx} \Psi_0 \Psi_1 \cdots \Psi_t \quad \text{in } V_k , \tag{18.6}$$

where $$\Psi_0 \sim \psi_0' \sim \phi_k^{(q+1)\tau_k} e_0(x) \quad \text{in } V_k$$

$$\Psi_i \sim \psi_i' \sim e_i(x) \psi_i^{n_i} \quad \text{in } V_k \quad \text{for } i = 1, 2, \cdots, t,$$

and $$deg\, e_i(x) < deg\, \phi_k \quad \text{for } i = 0, 1, 2, \cdots, t .$$

§ 19. Application of Hensel's Iemma

When $$f(x) \sim e(x) \phi_k^{n_0} \psi_1^{n_1} \cdots \psi_t^{n_t} \quad \text{in } V_k ,$$

if we want to make an augmented valuation V_{k+1} of V_k for which $V_k f(x) < V_{k+1} f(x)$, we must select one polynomial ψ_i out of $\psi_1, \psi_2, \cdots, \psi_t$ and must increase a value of ψ_i. And $\psi_1, \psi_2, \cdots, \psi_t$ are prime each other in V_k, so if

$$V_k \psi_1 < V_{k+1} \psi_1 , \quad \text{then}$$
$$V_k \psi_i = V_{k+1} \psi_i \quad \text{for } i = 2, 3, \cdots, t .$$

Consequently, when we make an augmented valuation V_{k+l} of V_k for which $V_k f(x) < V_{k+l}(x)$, by (18.6)

if $$V_k \Psi_1 < V_{k+1} \Psi_1, \quad \text{then}$$
$$V_k \Psi_i = V_{k+1} \Psi_i \quad \text{for } i = 2, 3, \cdots, t . \tag{18.7}$$

Let be $$V_k f < V_{k+1} f < \cdots < V_s f \quad \text{and}$$
$$V_s f - V_k f = \omega . \tag{18.8}$$

$$V_s(e'(x) \phi_k^{\tau_k - l} f - \Psi_0 \Psi_1 \cdots \Psi_t)$$
$$\geqq V_k(e' \phi_k^{\tau_k - l} f - \Psi_0 \Psi_1 \cdots \Psi_t)$$

$$\because \quad V_s \text{ is an augmented valuation of } V_k$$

$$> \omega + V_k(e'\phi_k^{\tau_k - l} f) \qquad \text{by (18.6)}$$

$$= \omega + V_k f + V_k e'\phi_k^{\tau_k - l} = V_s f + V_k e'\phi_k^{\tau_k - l} \qquad \text{(by 18.8)}$$

$$= V_s(e'\phi_k^{\tau_k - l} f)$$

$$\because \quad V_k e'\phi_k^{\tau_k - l} = V_s e'\phi_k^{\tau_k - l}.$$

So
$$e'\phi^{\tau_k - l} f \sim \Psi_0 \Psi_1 \cdots \Psi_t \qquad \text{in } V_s.$$

$$V_s(e'\phi_k^{\tau_k - l} f) = V_s(\Psi_0 \Psi_1 \cdots \Psi_t).$$

By (18.6)
$$V_k(e'\phi_k^{\tau_k - l} f) = V_k(\Psi_0 \Psi_1 \cdots \Psi_t).$$

$$\therefore \quad V_s(e'\phi_k^{\tau_k - l} f) - V_k(e'\phi_k^{\tau_k - l} f) = V_s(\Psi_0 \cdots \Psi_t) - V_k(\Psi_0 \cdots \Psi_t),$$

by (18. 7)
$$V_s f - V_k f = V_s \Psi_1 - V_k \Psi_1.$$

$$V_s f - V_k f = \omega \tag{18. 8}$$

Therefore, if we want to make a series of augmented inductive valuations $V_k < V_{k+1} < \cdots < V_s$ for which

$$V_k f < V_{k+1} f < \cdots < V_s f = \omega + V_k f,$$

we may make such a series of augmented inductive valuations $V_k < V_{k+1} < \cdots < V_s$ for which

$$V_k \Psi_1 < V_{k+1} \Psi_1 < \cdots < V_s \Psi_1 = \omega + V_k \Psi_1,$$

where Ψ_1 is one factor in (18.6).

§ 20. Residue class ring of $K_y[x]$ with respect to V_{1q}.

V_{0q} is a valuation of $K[y]$ and is the last valuation of a series of augmented inductive valuations which are extensions of V_{00} to $K[y]$.

$$V_{01} < V_{02} < \cdots < V_{0,q-1} < V_{0q}$$

$$V_{01} = [V_{00},\ V_{01} y = \nu_1]$$

$$V_{0i} = [V_{0,i-1},\ V_{0i} \zeta_i(y) = \nu_i] \qquad \text{for } i = 2, 3, \cdots, q.$$

$$V_{0q} = [V_{0,q-1},\ V_{0q} \zeta_q(y) = \nu_q]$$

Then the residue class ring of $K[y]$ by V_{0q} is made as follows by M. Theorem 12.1.

$\Gamma_{0,q-1}$ is a set of values of all polynomials $f(y)$ by $V_{0,q-1}$, and σ is the smallest integer such that $\sigma\nu_q \in \Gamma_{0,q-1}$. $a(y)$ is such a polynomial of y for which

$$V_{0q}a(y)\zeta_q^{\sigma}=0\,.$$

H_{0q} is the homomorphism which makes a polynomial $f(y)$ whose value by V_{0q} is non-negative, correspond to its residue class $H_{0q}f(y)$ in $F_{0q}[Y]$, where $H_{0q}(a(y)\zeta_q^{\sigma})=Y$ is a transcendental element with respect to F_{0q} and F_{0q} is an algebraic extension of the residue class field F_{00} of K by V_{00}.
V_{1q} is a valuation of a ring of polynomials of x $K_y[x]$ whose coefficient field K_y has the valuation V_{0q}.

Let be $$V_{1q}=[V_{0q},\ V_{1q}x=\mu_1]\,,$$

namely, if $$f(x,\,y)=\sum_{i=0}^{n}f_i(y)x^i\,,$$

then $$V_{1q}f(x,\,y)=\underset{i}{\mathrm{Min}}\,[V_{0q}f_i(y)+i\mu_1]\,.$$

Now I will make the residue class ring of $K_y[x]$ by V_{1q}.
β is the smallest integer such that $\beta\mu_1\in\Gamma_{0q}$,
where Γ_{0q} is a set of values of all polynomials of y by V_{0q}. Then $b(y)\zeta_q^{\pi}$ is such a polynomial of y that

$$V_{1q}b(y)\zeta_q^{\pi}x^{\beta}=0\,,$$

where $$deg\,b(y)<deg\,\zeta_q(y)\quad\text{and}\quad 0\leqq\pi<\sigma\,.$$

H_{1q} is the homomorphism that makes polynomials $f(x,\,y)$ of y and x whose values by V_{1q} are non-negative, correspond to their residue classes $H_{1q}f(x,\,y)$. Let be $H_{1q}b(y)\zeta_q^{\pi}x^{\beta}=X_1$, which is a transcendental element with respect to the residue class field $F_{0q}(Y)$ of the coefficient field K_y by V_{0q}. Then the residue class ring of $K_y[x]$ by V_{1q} is $F_1^*[X_1]$, where $F_1^*=F_{0q}(Y)$, owing to M. Theorem 10.2. But, here I will specially seek the residue class of $g(x,\,y)$ which is a polynomial of x and y and that of $\phi(x)$ which is a polynomial of x, because we shall use them, when we solve the "Principal Problem".

Let be $$V_{1q}g(x,\,y)=0\,.$$

$$g(x,\,y)\sim\sum_i g_i(y)x^i\qquad\text{in }V_{1q}\,,$$

and $$g_i(y)\sim\sum_j l_{ij}(y)\zeta_q^j\qquad\text{in }V_{1q}\,,$$

where $$deg_y\,l_{ij}(y)<deg_y\,\zeta_q(y)\,.$$

Then $$g(x,\,y)\sim\sum_{i,j}l_{ij}\zeta_q^jx^i\qquad\text{in }V_{1q}\,.$$

Let be $b'(y)$ and $a'(y)$ such polynomials that

$$a(y)a'(y)\sim1\quad\text{and}\quad b(y)b'(y)\sim1\qquad in\ V_{0q}\,.$$

and $$deg\, b'(y) < deg\, \zeta_q(y)\,.$$

$$\beta \mid i\,, \text{ because } V_{1q}\, l_{ij}\zeta_q^j x^i = 0 \qquad \therefore \quad i = \beta\gamma\,.$$

$$l_{ij}\zeta_q^j x^i \sim \frac{(b(y)\zeta_q^\pi x^\beta)^\gamma}{(a(y)\zeta_q^\sigma)^\theta} \frac{(b'(y))^\gamma l_{ij}}{(a'(y))^\theta} \qquad \text{in } V_{1q},$$

where $$j = \pi r - \sigma\theta\,. \tag{20. 1}$$

$$\therefore \quad \mathrm{H}_{1q}(l_{ij}\zeta_q^j x^i) = \frac{X_1^\gamma}{Y^\theta} \bar{d}_{\gamma\theta}\,,$$

where $$\bar{d}_{\gamma\theta} = \mathrm{H}_{1q} \frac{(b'(y))^\gamma l_{ij}}{(a'(y))^\theta} \in F_{0q}\,.$$

From (20. 1) $$\pi\gamma - \sigma\theta \geqq 0\,, \qquad \text{so} \quad \gamma \geqq \frac{\sigma}{\pi}\theta\,,$$

where $\frac{\sigma}{\pi}$ is a constant number.

So $$H_{1q}g(x, y) = \sum_{\gamma\theta} \bar{d}_{\gamma\theta} \frac{X_1^\gamma}{Y^\theta}\,, \tag{20. 2}$$

where $$\bar{d}_{\gamma\theta} \in F_{0q} \quad \text{and} \quad \gamma \geqq \frac{\sigma}{\pi}\theta\,.$$

§ 21. Residue class of $\phi(x)$

Definition 21. 1. *When $f(x, y)$ is a polynomial of x and y and an expansion of $f(x, y)$ by $\zeta_q(y)$ and x is*

$$f(x, y) = \sum_{i,j} a_{ij}(y)\zeta_q^j x^i\,, \tag{21. 1}$$

$$deg_y\, a_{ij}(y) < deg_y\, \zeta_q(y) \qquad \textit{for all terms in } (21.1)$$

, if $$V_{1q} f(x, y) = V_{1q}\, a_{ij}(y)\zeta_q^j x^i \qquad \textit{for all terms in } (21.1)$$

then $f(x, y)$ is said to be homogeneous in V_{1q}.

Lemma 21. 2. *If both $f(x) = \sum_i f_i x^i$ and $g(x) = \sum_i g_i x^i$ are homogeneous in V_{10} and*

$$f(x) \sim g(x) \qquad in \;\; V_{10}\,,$$

then $$f_i \sim g_i \qquad in \;\; V_{00} \qquad \textit{for all terms in } f(x) \textit{ and } g(x).$$

Proof. This is evident from its definition.

Lemma 21. 3. *If both* $$f(x, y) = \sum_{i,j} f_{ij}(y)\zeta_q^j x^i \tag{21. 2}$$

and $g(x, y) = \sum_{i,j} g_{ij}(y)\zeta_q^j x^i$ (21.3) *are homogeneous in* V_{1q},

where $\deg f_{ij}(y) < \deg \zeta_q(y)$

and $\deg g_{ij}(y) < \deg \zeta_q(y)$, *for all terms in* (21.2) *and* (21.3),

then $g_{ij}(y) \sim f_{ij}(y)$ *in* V_{1q} *for all terms in* (21.2) *and* (21.3).

Proof. Let be $f(x, y) = \sum_i (\sum_j f_{ij}(y)\zeta_q^j)x^i$

and $g(x, y) = \sum_i (\sum_j g_{ij}(y)\zeta_q^j)x^i$

$$f(x, y) \sim g(x, y) \qquad \text{in } V_{1q}.$$

Then by Lemma 21.2

$$\sum_j f_{ij}(y)\zeta_q^j \sim \sum_j g_{ij}(y)\zeta_q^j \qquad \text{in } V_{0q}$$

Then, owing to M. Lemma 16.2

$$f_{ij}(y) \sim g_{ij}(y) \qquad \text{for all terms in (21.2) and (21.3).}$$

Now I will seek the residue class of $\phi(x)$ in the residue class ring of $K_y[x]$ by V_{1q}.

When Γ_{00} is a set of values of all elements in K by V_{00}, let be τ the smallest integer such that $\tau\mu_1 \in \Gamma_{00}$ and λ the smallest integer that $\lambda\mu_1 \in \Gamma_{0,q-1}$, where $\Gamma_{0,q-1}$ is a set of all values of $f(y)$ by $V_{0,q-1}$.

$V_{1q}x^\lambda \in \Gamma_{0,q-1}$, then there exists such a polynomial $c(y)$ that $V_{1,q-1}(c(y)x^\lambda) = 0$,

where $\deg c(y) < \deg \zeta_q(y)$.

$$0 = V_{1,q-1}\left(c(y)x^\lambda\right) = V_{1q}\left(c(y)x^\lambda\right).$$

$$H_{1q}\left(c(y)x^\lambda\right) \in F_1^*[X_1]. \quad \text{Then by (20.2)}$$

$$H_{1q}\left(c(y)x^\lambda\right) = \sum_{\gamma\theta} \bar{d}_{\gamma\theta} \frac{X_1^\gamma}{Y^\theta}.$$

$$c(y)x^\lambda \sim \sum_{\gamma,\theta} d_{\gamma\theta}(y) \frac{(b(y)\zeta_q^\pi x^\beta)^\gamma}{(a(y)\zeta_q^\sigma)^\theta} \qquad \text{in } V_{1q} \qquad (21.4)$$

We multiply the both sides by the least common multiple of all denominators in (21.4), then

$$c(y)\left(a(y)\right)^n x^\lambda \sim \sum_{\gamma,\theta} d''_{\gamma\theta}(y)\left(b(y)\right)^\gamma \zeta_q^{\pi\gamma} x^{\beta\gamma} / \zeta_q^{\sigma\theta} \qquad \text{in } V_{1q} \qquad (21.5)$$

We decrease degrees of all coefficients in (21.5) less than that of $\zeta_q(y)$.

$$c'(y)x^{\lambda} \sim \sum d'(y)\zeta_q^{\pi\gamma} x^{\beta\gamma}/\zeta_q^{\sigma\theta} \quad \text{in } V_{1q}. \tag{21.6}$$

Then, owing ot Lemma 21.3, $c'(y)x^{\lambda}$ is equivalent to only one term $d'(y)\zeta_q^{\pi}x^{\beta\gamma}/\zeta_q^{\sigma\theta}$ in the left side of (21.6).

So $$c'(y)x^{\lambda} \sim d'(y)\zeta_q^{\pi\gamma}x^{\beta\gamma}/\zeta_q^{\sigma\theta} \quad \text{in } V_{1q},$$

where $$\sigma n = \pi\gamma \quad \text{and} \quad \lambda = \beta\gamma$$

and $$c'(y) \sim d'(y) \quad \text{in } V_{0q}.$$

$$\therefore \quad H_{1q}\left(c(y)x^{\lambda}\right) = \bar{d}\frac{X_1^{\gamma}}{Y^n},$$

where γ is a constant number and $n = \dfrac{\pi}{\sigma}\gamma$ is also a constant number, and $\bar{d} \in F_{0q}$.

Now we can seek the residue class of a polynomial $\phi(x)$ which is a polynomial of x, in the residue class ring of $K_y[x]$ by V_{1q}. We select such a number k in K that

$$V_{1q}k\phi(x) = V_{10}k\phi(x) = 0.$$

And let the homogeneous part of $k\phi(x)$ be $\sum_i a_i x^i$. (M. Lemma 16.2)

$$V_{10}(a_i x^i) = 0 \qquad \therefore \quad \lambda \mid i, \qquad so \quad i = \lambda i'.$$

$$a_i x^i = a_i x^{\lambda i'} \sim \left(c(y)x^{\lambda}\right)^{i'} \left(c'(y)\right)^{\lambda i'} a_i \qquad \text{in } V_{1q},$$

where $$c'(y)c(y) \sim 1 \quad \text{in } V_{1q} \quad \text{and} \quad deg\, c'(y) < deg\, \zeta_q(y).$$

$$a_i x^i \sim \left(c(y)x^{\lambda}\right)^{i'} b_i(y) \qquad \text{in } V_{1q},$$

where $b_i(y)$ is a V_{1q}-unit.

So $$H_{1q}(a_i x^i) = \bar{b}_{i'}\left(\frac{X_1^{r}}{Y^n}\right)^{i'}$$

$$H_{1q}\left(k\phi(x)\right) = \sum_{i'} \bar{b}_{i'}\left(\frac{X_1^{r}}{Y^n}\right)^{i'} \quad \text{where} \quad \bar{b}_{i'} \in F_{0q} \tag{21.7}$$

and both γ and n are constant numbers.

Therefore, if we substitute z for $\dfrac{X_1^{r}}{Y^n}$, then

$$H_{1q}\left(k\phi(x)\right) = \sum_{i'} \bar{b}_{i'} z^{i'}, \tag{21.8}$$

where $$z = H_{1q}\left(c(y)x^{\lambda}\right).$$

§ 22. Solution of the "Principal Problem"

I finished making preparations for solution of the Prinicipal Problem, now I will begin the solution.

Principal Problem

V_{0q} is a given valuation of $K[y]$ and V_{p0} is a given one of $K[x]$ and V_{0q} and V_{p0} induce the same valuation V_{00} in K. In the case, this problem is how we should make such valuations W of $K[x, y]$ that induce V_{0q} in $K[y]$ and V_{p0} in $K[x]$.

Solution

Let be $$V_{0q}=\left[V_{00},\ V_{01}y=\nu_1, \cdots, V_{0q}\zeta_q(y)=\nu_q\right]$$

and $$V_{p0}=\left[V_{00},\ V_{10}x=\mu_1, \cdots, V_{p0}\phi_p(x)=\mu_p\right]. \tag{22.1}$$

Then this problem is that we make such extensions W of the valuation V_{0q} to the ring $K_y[x]$ that induce in $K[x]$ the valuation V_{p0}.
Let be $W_1=[V_{0q}, W_1x=V_{10}x=\mu_1]$ an extension of V_{0q} to $K_y[x]$ and the shortest series of x-simply augmented inductive valuations of $K_y[x]$ binding W_1 and W, as follows; (§ 3)

$$W_1<W_2<\cdots<W_{t-1}<W_t=W. \tag{22.2}$$

A series of augmented inductive valuations which the valuations in (22.2) induce in $K[x]$ is

$$V_{10}<V_2'<\cdots<V_{t-1}'<V_t'.$$

When we bind V_{10} and V_t' with a series of continuous augmented inductive valuations and obtain the following series;

$$V_{10}<V_2''<V_3''<\cdots V_{s-1}''<V_t'. \tag{22.3}$$

Then $$W_1\phi_p<W_2\phi_p<\cdots<W_{t-1}\phi_p<W_t\phi_p=W\phi_p=\mu_p,$$

and $$V_{10}\phi_p<V_2'\phi_p<\cdots<V_{t-1}'\phi_p<V_t'\phi_p=\mu_p.$$

and $$V_{10}\phi_p<V_2''\phi_p<\cdots<V_{s-1}''\phi_p<V_t'\phi_p=\mu_p=V_{p0}\phi_p. \tag{22.4}$$

Then, from (22.1) and (22.4), owing to Theorem (13.2) we see that the two series (22.1) and (22.3) are on the same course and $V_t'=V_{p0}$.

Namely in this problem we must find such a series (22.2) that satisfies all conditions that I related above. But when we practically make a series of x-simply augmented inductive valuations binding W_1 and W, we need not make the shortest series, but we may make a series of x-simply augmented inductive valuations which appear naturally at that time.

I am afraid that you, readers of this paper, may think that my explanation here is very complex. The reason why I explained as above is as follows; if I make a series (α) of augmented inductive valuations of $K_y[x]$ binding W_1 and W and make a series (β) of augmented inductive valuations which (α) induces in $K[x]$, it may happen that (β) is not a series of augmented inductive valuations as I shall explain late in this paper.

Now we want to make such an augmented valuation W_2 of W_1 of $K_y[x]$ that $W_1\phi_p < W_2\phi_p$. For the sake of it, we must find a decomposition of $\phi_p(x)$ in W_1 as in D. §5 (4). W_1 induces V_{10} in $K[x]$ and according to Theorem 14.1

$$\phi_p(x) \sim \phi_2(x)^{b_2} \quad \text{in } V_{10} \quad \text{and} \quad deg\, \phi_p(x) = deg\, \phi_2(x)^{b_2}.$$

Namely $$\phi_p(x) \sim \phi_2(x)^{b_2} \quad \text{in } W_1,$$

where $\phi_2(x)$ is the key polynomial which produces

$$V_{20} = \left[V_{10},\ V_{20}\phi_2(x) = \mu_2\right].$$

Therefore in this case we may find a decomposition of $\phi_2(x)^{b_2}$ in W_1 of $K_y[x]$.

$$\phi_2(x) = x^{m\tau} + \cdots + a_i x^{i\tau} + \cdots + a_0, \tag{22.4}$$

where $$a_i \in K \quad \text{and} \quad V_{10}\phi_2 = W_1\phi_2 = V_{10}a_i x^{i\tau}$$

for all terms in (22.4), because ϕ_2 is homogeneous in V_{10}, namely in W_1 and τ is the smallest integer such that $\tau\mu_1 = V_{10}x^\tau \in \Gamma_{00}$ which is a set of values of all elements in K. Let be k such a number in K that $W_1(k\phi_2(x)) = 0$. In order to find a decomposition of $(k\phi_2(x))^{b_2}$ in W_1, we must find factorization of the residue class $H(k\phi_2)^{b_2}$ in the residue class ring $F_1^*[X_1]$ of the ring $K_y[x]$ by W_1, where H is the natural homomorphism which makes every polynomial $f(x, y)$ of x and y whose value by W_1 is non-negative, correspond to its residue class $Hf(x, y)$. By (21.7) and (21.8) and from the fact that $\phi_p \sim \phi_2^{b_2}$ in W_1,

$$H(k^{b_2}\phi_p) = H(k\phi_2)^{b_2} = \left(\sum_i \bar{b}_i \left(\frac{X_1^\gamma}{Y^n}\right)^i\right)^{b_2}$$

$$= (\sum_i \bar{b}_i z^i)^{b_2} \qquad \text{where} \quad \bar{b}_i \in F_{0q}. \tag{22.5}$$

Both γ and n are constants, so factorization of $\left(\sum_i \bar{b}_i\left(\frac{X_1^\gamma}{Y^n}\right)^i\right)$ in $F_1^*[X_1]$, where $F_1^* = F_{0q}(Y)$, is the same as that of $(\sum_i \bar{b}_i z^i)$ in $F_{0q}[z]$.

So $$\bar{b}(\sum_i \bar{b}_i z^i)^{b_2} = \overline{l_1(z)}^{s_1 b_2} \cdots \overline{l_t(z)}^{s_t b_2}, \tag{22.6}$$

where $\bar{b}$ is a suitable F_{0q}-unit and every $\overline{l_i(z)}$ is an irreducible canon class. In (22.6), z does not appear as a factor, because the last term of $\sum_i \bar{b}_i z^i$, namely the constant term, is not zero.

Let be $$c(y)x^{\lambda} = \Phi \quad \text{and} \quad W_1\Phi = 0$$

then $$b(y)k^{b_2}\phi_p \sim b(y)\,(k\phi_2)^{b_2} \sim l_1(\Phi)^{s_1 b_2}\cdots l_t(\Phi)^{s_t b_2} \qquad in \quad W_1,$$

where $$H\big(b(y)\big) = \bar{b} \quad \text{and} \quad deg\, b(y) < deg\, \zeta_q(y)$$

and every $l_i(\Phi)$ is an irreducible canon polynomial in W_1 whose irreducible canon class is $\overline{l_i(z)}$ in $F_1^*[X_1]$ and $z = \dfrac{X_1^{\gamma}}{Y^n}$

and $$z = \frac{X_1^{\gamma}}{Y^n} = H\left(c(y)x^{\lambda}\right) = H(\Phi).$$

Then according to Theorem 18.1

$$b(y)k^{b_2}\phi_p(x) \overset{\omega}{\approx} \Psi_1\Psi_2\cdots\Psi_t \qquad \text{in} \quad W_1,$$

where $$\Psi_i \sim l_i(\Phi)^{s_i b_2} \quad \text{in} \quad W_1 \qquad \text{for} \quad i = 1, 2, \cdots, t.$$

According to explanation in § 19, herein we decide a value of ω so that $\omega = V_{p0}\phi_p(x) - W_1\phi_p(x)$.

And we must make such a series of x-augmented inductive valuations that

$$W_1 < W_2 < \cdots < W_t$$

and $$W_1\Psi_1 < W_2\Psi_1 < \cdots < W_t\Psi_1 = \omega + W_1\Psi_1.$$

We may select any Ψ_i out of $\Psi_1, \cdots, \Psi_t$, so herein we selected Ψ_1.

$$\Psi_1 \sim l_1(\Phi)^{s_1 b_2} \qquad \text{in} \quad W_1.$$

Let the key part of $l_1(\Phi)$ be $\varphi(x) = \sum_i l_i(y)x^{i\lambda}$,

then $$d(y)\Psi_1 \sim \left(\varphi(x)\right)^{s_1 b_2} \qquad \text{in} \quad W_1,$$

where $\varphi(x)$ is a homogeneous key polynomial in W_1, and

$$\deg d(y) < deg\, \zeta_q(y) \quad \text{and} \quad \deg l_i(y) < deg\, \zeta_q(y).$$

Let $\sum_{i=0}^{m} h_i(y, x)(\varphi(x))^i$ be an expansion of $d(y)\Psi_1$ by $\varphi(x)$, then $m \geqq s_1 b_2$ and $h_i(y, x)$ are polynomials of x and y, because the coefficient of the term of the highest power of φ is 1. And degrees of $h_i(y, x)$ with respect to x are less then that of $\varphi(x)$. We write it as follows; $deg_x\, h_i(x, y) < deg_x\, \varphi(x)$.

Now we must make such an augmented valuation W_2 of W_1 that

$W_2\Psi_1 > W_1\Psi_1$.

Here the three following cases occur.

α-case is the one when $d(y)\Psi_1 = (\varphi(x))^{s_1 b_2}$.
β-case is the one when $W_1 h_0(y, x) < \omega + W_1 d(y)\Psi_1$.
γ-case is the one when $W_1 h_0(y, x) \geqq \omega + W_1 d(y)\Psi_1$.

When the α-case occurs, we may decide a value of $\varphi(x)$ by W_2 so that $W_2(\varphi(x))^{s_1 b_2} = \omega + W_1\Psi_1 + W_1 d(y)$,

namely $$W_2\varphi(x) = \frac{1}{s_1 b_2}\Big(W_1\Psi_1 + \omega + W_1 d(y)\Big).$$

$$d(y)\Psi_1 \sim \big(\varphi(x)\big)^{s_1 b_2} \qquad \text{in } W_1$$

so $$W_1 d(y) + W_1\Psi_1 = s_1 b_2 W_1\varphi(x)$$

$$W_2\varphi(x) = \frac{1}{s_1 b_2}\big(s_1 b_2 W_1\varphi(x) + \omega\big)$$

$$W_2\varphi(x) > W_1\varphi(x).$$

Then as I explained in § 19, where I used $f(x)$ for $\phi_p(x)$,

$$W_2\phi_p(x) = V_{p0}\phi_p(x) = \mu_p.$$

So, when the α-case occurs, thus the problem is solved completely. When the β-case occurs, we must decide the value $W_2\varphi(x)$ so that $d(y)\Psi_1$ can be decomposed in W_2 as follows;

$$d(y)\Psi_1 \sim \varphi(x)^{n_0}\rho_1(x)^{n_1}\cdots\rho_r(x)^{n_r} \qquad \text{in } W_2, \tag{22. 7}$$

where every $\rho_i(x)$ is a homogeneous key polynomial in W_2 and at least such one $\rho_i(x)$ must appear in this decomposition.

For the sake of it, plot the points $P_i = (s_1 b_2 - i,\ W_1(h_i(y, x)\varphi(x)^i))$ in a cartesian plane, where i moves from zero to $s_1 b_2$ and make the Newton polygon of these points. (D § 5)

Let one side of the polygon be $P_a P_c$, where $a > c$.

Namely $$P_a = \big(s_1 b_2 - a,\ W_1(h_a\varphi^a)\big)$$

$$P_c = \big(s_1 b_2 - c,\ W_1(h_c\varphi^c)\big).$$

Now we decide the value $W_2\varphi$ so that $W_2(h_a\varphi^a) = W_2(h_c\varphi^c)$.
Then $W_2 h_a = W_1 h_a$ and $W_2 h_c = W_1 h_c$, so

$$W_2\varphi(x) = \frac{1}{a-c}(W_1 h_c - W_1 h_a).$$

The slope of P_aP_c is $\dfrac{W_1(h_c\varphi^c)-W_1(h_a\varphi^a)}{a-c}$ which is equal to $W_2\varphi(x)-W_1\varphi(x)$.

The ordinate of $P_{s_1b_2}$ is the lowest, so the slope of P_aP_c is positive.

Then $$W_2\varphi(x)>W_1\varphi(x).$$

$\varphi(x)$ is a homogeneous key polynomial in W_1. So W_2 is an augmented valuation of W_1 such that $W_2\varphi(x)>W_1\varphi(x)$.

And all other points of $P_0, P_1, P_2, \cdots, P_{s_1b_2}$ except P_a and P_c are above the straight line P_aP_c or are on it. From this fact, through simple calculation of analytical geometry we find that

$$W_2d(y)\Psi_1=\operatorname{Min}_i[W_2h_i\varphi^i]=W_2h_a\varphi^a=W_2h_c\varphi^c$$

and $$d(y)\Psi_1\sim h_a\varphi^a+\cdots+h_c\varphi^c \qquad \text{in } W_2$$

and we obtain the decomposition (22.7). Namely $n_0=a-c$ in (22.7).
When $m>s_1b_2$, by simple calculation we find that $W_2(h_m\varphi^m)>\operatorname{Min}_i[W_2h_i\varphi^i]=W_2d\Psi_1$, so in order to find the decomposition of $d\Psi_1$ in W_2, such terms $h_m\varphi^m$, where $m>s_1b_2$, are not necessary.

The Newton Polygon which I present here is pretty different from that which MacLane presented in D §5. But I use this polygon in this paper.

The first reason is that a number of sides of this polygon is much less than that of the one which MacLane presented in his paper. The second reason is that, when we make an augmented valuation W_2 of W_1, we elevate each point P_i and we must make at least one straight line binding these two points be parallel to x-axis so that projection of W_2 of $d(y)\Psi_1$ becomes positive.

On my polygon, the position of $P_{s_1b_2}$ is the lowest, because $d(y)\Psi_1\sim\varphi^{s_1b_2}$ in W_1. And as the valuations increase from W_1 to W_2, we can inspect geometrically elevating movement of value of each term $h_i\varphi^i$, namely elevating movement of each point P_i. And $W_2\varphi(x)$ is always a sum of $W_1\varphi(x)$ and the slope of P_aP_b.

I call this Newton Polygon "*Neo-Newton Polygon*".
Again, from (22.7), by Theorem 18.1 we obtain the following decomposition;

$$e(y,x)\Psi_1\overset{\omega}{\approx}\Psi_{*0}\Psi_{*1}\cdots\Psi_{*r} \qquad in\ W_2,$$

where $$e_0(y,x)\Psi_{*0}\sim\varphi(x)^{n_0} \qquad in\ W_2$$

$$e_i(y,x)\Psi_{*i}\sim\rho_i(x)^{n_i} \qquad in\ W_2 \qquad \text{for } i=1,2,\cdots,r,$$

and $$deg_x\, e(y,x)<deg_x\,\varphi(x)$$

and $$deg_x\, e_i(y,x)<deg_x\,\varphi(x) \qquad \text{for } i=0,1,2,\cdots,r.$$

$\varphi(x)$ and $\rho_i(x)$ may include y, but they are x-key polynomials, so purposely I wrote like this.

Now we must make such an augmented valuations W_3 of W_2 that increases value of one Ψ_{*i} and only Ψ_{*i} out of $\Psi_{*i}, \cdots, \Psi_{*r}$,

$$e_1(y, x)\Psi_{*1} \sim \rho_1(x)^{n_1} \qquad \text{in} \quad W_2 .$$

So in the same way, the three cases occur.
If the α-case occurs, then this problem can be solved as I did.
If the β-case occurs, then we repeat the same process that I did and we obtain such an augmented valuation W_3 of W_2 that

$$W_3\Psi_{*1} > W_2\Psi_{*1} .$$

Thus we get the following series of augmented inductive valuations

$$W_1 < W_2 < W_3 < \cdots ,$$

, where

$$W_1\Psi_1 < W_2\Psi_1 < W_3\Psi_1 < \cdots$$

and

$$W_i\phi_p(x) - W_{i-1}\phi_p(x) = W_i\Psi_1 - W_{i-1}\Psi_1 \quad \text{for} \quad i = 2, 3, 4, \cdots .$$

And $\mu_p = V_{p0}\phi_p(x)$ is a finite number and all augmented valuations that appear in this problem are discrete. So at last $W_k\phi_p$ arrives at μ_p. As long as the β-cases occur, $W_k\phi_p$ can not arrive at μ_p. So, at last the r-case or the α-case must occur.
If the r-case occurs, then we can solve this problem at once as follows.
Let be

$$d(y, x)\Psi \sim \sigma(x)^s \qquad \text{in} \quad W_t$$

where $\sigma(x)$ is a homogeneous key polynomial in W_t which is an augmented valuation of W_1 and $deg_x d(y, x) < deg_x \phi'(x)$, where $\phi'(x)$ is the key polynomial that produces W_t. Such value $W_{t+1}d\Psi = \Omega$ that makes $W_{t+1}\phi_p(x) = \mu_p$, can be calculated.
Let an expansion of $d\Psi$ by $\sigma(x)$ be

$$\sum_{i=0}^{m} l_i(y, x)\left(\sigma(x)\right)^i ,$$

where

$$deg_x l_i < deg_x \sigma(x) \qquad \text{for every term}$$

and

$$m \geqq s \quad \text{and} \quad l_s(y, x) = 1 .$$

The r-case occurs, namely $W_t l_0(y, x) \geqq \Omega$.

Plot the points $P_i = (s-i, W_t(l_i(y, x)\sigma^i))$ in a cartesian plane. But this time, i moves from 1 to s. We do not plot P_0 and instead of P_0 we plot the point $Q = (s, \Omega)$. Here $\sigma = \sigma(x)$ and $l_i = l_i(x, y)$.

Next we make the Neo-Newton Polygon of $Q, P_1, P_2, \cdots, P_s$. Let one side of the polygon which passes through Q be QP_c.

$$P_c = \left(s-c,\ W_t(l_c\sigma^c)\right).$$

Then assign the value $W_{t+1}\sigma(x)$ so that

$$W_{t+1}\left(l_c(y, x)\left(\sigma(x)\right)^c\right) = \Omega .$$

$$deg_x\, l_c(y, x) < deg_x\, \sigma(x), \qquad \text{so} \quad W_{t+1}l_c(y, x) = W_t l_c(y, x) .$$

So
$$W_{t+1}\sigma(x) = \frac{1}{c}\left(\Omega - W_t l_c(y, x)\right).$$

$$\Omega > W_t\left(l_c(y, x)\left(\sigma(x)\right)^c\right) \quad \text{and} \quad \frac{1}{c}(\Omega - W_t l_c) > W_t\sigma .$$

Therefore
$$W_{t+1}\sigma(x) > W_t\sigma(x) .$$

$\sigma(x)$ is a homogeneous key polynomial in W_t, so W_{t+1} is an augmented valuation of W_t. All other points of $Q, P_1, \cdots, P_s$ except Q and P_c are above or on the straight line QP_c and from this fact, by simple calculation of analytical geometry, we can find that

$$W_{t+1}d\Psi = \operatorname*{Min}_i\left[W_{t+1}l_i(y, x)\left(\sigma(x)\right)^i\right] = \Omega .$$

Thus at last $W_{t+1}\phi_p(x)$ can arrive at μ_p. And the Principal Problem is completely solved in every case. Because when $s < t+1$, $W_s\phi_p < W_{t+1}\phi_p = V_{p0}\phi_p(x)$. Therefore by Theorem 13.2 W_{t+1} induces the valuation V_{p0} in $K[x]$.

§ 23. Generalization of Hensel's lemma

Theorem 18.1 is pretty different from Hensel's Lemma. It is convenient for me to solve the Principal Problem with it. But in this paragraph I will try to make it approach the primitive type of Hensel's lemma as nearly as possible.

For the sake of it, we must make a complete ring of $K[x]$ with respect to the valuation V_k.

As I related in § 12, MacLane introduced notion of "homogeneous" in M § 16. First he made a set of representatives of elements of K and defined homogeneous polynomials and homogeneous inductive valuations and in M. Lemma 16, 2, he related that every polynomial $f(x)$ is equivalent in V_k to one and only one homogeneous polynomial $h(x)$. The $h(x)$ is called the "*homogeneous part*" of $f(x)$. Then an arbitrary polynomial $f(x)$ is equivalent to its homogeneous part $f_1(x)$.

$$f(x) \sim f_1(x) \qquad \text{in } V_k.$$

Let be
$$f(x) - f_1(x) = g_2(x),$$
$$V_k g_2(x) > V_k f(x) = V_k f_1(x).$$

$g_2(x)$ is equivalent to its homogeneous part $f_2(x)$.

$$f_2(x) \sim g_2(x) \qquad \text{in } V_k$$

so, let be
$$f(x) - f_1(x) - f_2(x) = g_3(x), \quad \text{then}$$
$$V_3 g_3(x) > V_k f_2(x) = V_k g_2(x) > V_k f_1(x).$$

When we repeat such processes, then $g_{n+1}(x)$ becomes zero or $\lim_{n\to\infty} V_k g_{n+1}(x) = \infty$, where

$$g_{n+1}(x) = f(x) - f_1(x) - f_2(x) - \cdots - f_n(x),$$

because V_k is a discrete valuation.

In this case we let $f(x)$ be equal to

$$f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots.$$

Even if a number of terms which are included in $f(x)$ is finite, it happens often that this sequence lasts endlessly. For example, when we select 0, 1, 2 as representatives of residue classes by mod 3 in 3-adic field, $\frac{1}{2}$ is a sum of endless sequence as follows;

$$\frac{1}{2} = 2 - 2\cdot 3 + 2\cdot 3^2 + \cdots + (-1)^n 2\cdot 3^n + \cdots \quad .$$

We consider all such sequences that satisfy the following conditions;

$$h_1(x),\ h_2(x),\ \cdots,\ h_n(x),\ \cdots$$

are all homogeneous polynomials and a number of terms which are included in each $h_n(x)$ is finite and

$$V_k h_1(x) < V_k h_2(x) < \cdots < V_k h_n(x) < V_k h_{n+1}(x) < \cdots$$

Then a set M of all sums of these polynomials is a complete ring with respect to V_k.

$$M = \left\{ h_1(x) + h_2(x) + \cdots + h_n(x) + h_{n+1}(x) + \cdots \right\}$$

Let be
$$h(x) = h_1(x) + \cdots + h_n(x) + \cdots$$

and $$g(x)=g_1(x)+\cdots+g_n(x)+\cdots,$$

then a necessary and sufficient condition that $h(x)=g(x)$ is that $h_n(x)=g_n(x)$ for $n=1,2,\cdots$. And we let $V_k h(x)$ be $V_k h_1(x)$.

It may happen that some tails of these series consist of zero only.

Lemma 23. 1.

If $K[x]$ is a complete ring with respect to a homogeneous valuation V_k which a key polynomial ϕ_k produces and $V_k a(x)=0$ and $deg\ a(x)<deg\ \phi_k$, then in $K[x]$ there exists such a polynomial $\alpha(x)$ that $a(x)\alpha(x)=1$.

Proof.

ϕ_k is irreducible in $K[x]$ and $deg\ a(x)<deg\ \phi_k$, so in $K[x]$ there are such two polynomials $\alpha'(x)$ and $q(x)$ that

$$a(x)\alpha'(x)+q(x)\phi_k=1$$

where $$deg\ \alpha'<deg\ \phi_k \quad \text{and} \quad deg\ q<deg\ a<deg\ \phi_k.$$

By M. Lemma 4. 3

$$V_k q\phi_k>V_{k-1}q\phi_k\geqq V_{k-1}1=0=V_k a\alpha'.$$

$$\therefore \quad \frac{1}{a(x)}=\alpha'(x)+\frac{1}{a(x)}q\phi_k. \tag{23. 1}$$

Now in (23. 1) we substitute $\alpha'(x)+\frac{1}{a(x)}q\phi_k$ for $\frac{1}{a(x)}$ repeatedly and we obtain the following sequence

$$\frac{1}{a(x)}=\alpha'(x)+\alpha'(x)q\phi_k+\alpha'(x)(q\phi_k)^2+\cdots+\alpha'(x)(q\phi_k)^n\frac{1}{a(x)} \tag{23. 2}$$

$$V_k(q\phi_k)>V_k(a\alpha')=0.$$

So we can transform the right side of (23.2) into such a polynomial $\alpha(x)$ in $K[x]$ that $a(x)\alpha(x)=1$, because $K[x]$ is complete with respect to V_k.

Theorem 23. 2.

If $K[x]$ is a ring in Lemma 23.1 and in $K[x]$ there are such five polynomials $f(x)$, $g_0(x)$, $h_0(x)$, $l(x)$, $m(x)$ that

$$f(x)\sim g_0(x)h_0(x) \qquad in\ V_k$$

$$g_0(x)l(x)+h_0(x)m(x)\sim 1 \qquad in\ V_k$$

and $$V_k l(x)\geqq 0=V_k g_0(x)=V_k h_0(x)$$

and $$D_{\phi_k}g_0(x)>0 \quad and \quad D_{\phi_k}h_0(x)>0,$$

then in $K[x]$ *there exist such two polynomial* $g(x)$ *and* $h(x)$ *that* $f(x) = g(x)h(x)$

and $g(x) \sim g_0(x)$ *in* V_k *and* $h(x) \sim h_0(x)$ *in* V_k.

Proof.

$$V_k g_0(x) = 0 \quad \text{and} \quad D_{\phi_k} g_0(x) > 0$$
$$V_k h_0(x) = 0 \quad \text{and} \quad D_{\phi_k} h_0(x) > 0$$

so by Lemma 16.2 there are such two canon polynomials $g'(x)$ and $h'(x)$ in V_k that

$$g_0(x) \sim a(x)g'(x)$$
$$h_0(x) \sim b(x)h'(x) \qquad \text{in } V_k,$$

where $$V_k a(x) = V_k b(x) = 0$$

and $$deg\, a(x) < deg\, \phi_k \quad \text{and} \quad deg\, b(x) < deg\, \phi_k .$$

According to Lemma 23.1

let be $$a(x)\alpha(x) = b(x)\beta(x) = 1 .$$
$$f \sim g_0 h_0 \sim a g' b h' \qquad \text{in } V_k$$

so $$\alpha(x)\beta(x)f(x) \sim g'(x)h'(x) \qquad \text{in } V_k .$$

Owing to Theorem 18.1 there exist such polynomials $g_n(x)$ and $h_n(x)$ that

$$\alpha(x)\beta(x)f(x) \overset{\omega}{\approx} g_n(x)h_n(x) \qquad \text{in } V_k$$

where $$g_n(x) \sim g'(x) \quad \text{and} \quad h_n(x) \sim h'(x) \qquad \text{in } V_k .$$

We repeat this process endlessly, then

$$\lim_{n\to\infty} \omega = \infty$$

and $$\alpha(x)\beta(x)f(x) = g^*(x)h^*(x)$$

where $$g^*(x) = \lim_{n\to\infty} g_n(x) \in K[x]$$
$$h^*(x) = \lim_{n\to\infty} h_n(x) \in K[x]$$

because $K[x]$ is complete with respect to V_k.

$$f(x) = a(x)\alpha(x)b(x)\beta(x)f(x) = a(x)g^*(x)b(x)h^*(x) .$$

Now let be $g(x) = a(x)g^*(x)$ *and* $h(x) = b(x)h^*(x)$

then $$f(x) = g(x)h(x)$$

and
$$g(x)=a(x)g^*(x)\sim a(x)g'(x)\sim g_0(x)$$
$$h(x)=b(x)h^*(x)\sim b(x)h'(x)\sim h_0(x) \qquad \text{in } V_k.$$

§ 24. Some examples of Principal Problem.

Example 24. 1

Let V_{p0} be an extension of V_{00} to $K[x]$ and V_{0p} an extension of V_{00} to $K[y]$ and

$$V_{p0}=\left[V_{00},\ V_{10}x=\mu_1,\ V_{20}\phi_2(x)=\mu_2, \cdots, V_{p0}\phi_p(x)=\mu_p\right]$$

$$V_{0p}=\left[V_{00},\ V_{01}y=\mu_1,\ V_{02}\phi_2(y)=\mu_2, \cdots, V_{p0}\phi_p(y)=\mu_p\right].$$

Namely the key polynomial $\phi_i(y)$ which produces V_{0i} of $K[y]$ is equal to the polynomial which is made by substitute y for x in the key polynomial $\phi_i(x)$ which produces V_{i0}. And their key values are equal to each other.

In this case, we can very easily make a valuation W of $K[x, y]$ that induces V_{p0} in $K[x]$ and V_{0p} in $K[y]$.
Let $W_1=[V_{0p}, W_1x=\mu_1]$ be an extension of V_{0p} to $K_y[x]$.
By the remainder theorem

$$\phi_p(x)-\phi_p(y)=(x-y)Q(x, y) \quad . \tag{24. 1}$$

Now we want to make such an augmented valuation W_2 of W_1 of $K_y[x]$ that $W_2\phi_p(x)=\mu_p$.
By M. Corollary 13. 2 $x-y$ is a key polynomial in W_1.

$$W_1\phi_p(y)>W_1\phi_p(x), \quad \text{because}$$

$$W_1\phi_p(y)=V_{0p}\phi_p(y)=\mu_p=V_{p0}\phi_p(x)>V_{10}\phi_p(x)=W_1\phi_p(x).$$

So
$$\mu_p=W_1\phi_p(y)>W_1\phi_p(x)=W_1\left[(x-y)Q(x, y)\right].$$
$$W_1(x-y)=\text{Min}\,[W_1x,\ W_1y]=\mu_1.$$

Let be
$$\mu_p-W_1\left[(x-y)\,Q(x, y)\right]=\omega \quad . \tag{24. 2}$$

Then we may decide so that $W_2(x-y)=\mu_1+\omega+1$, and

$$W_2=\left[W_1,\ W_2(x-y)=\mu_1+\omega+1\right] \quad .$$

W_2 is an augmented valuation of W_1 of $K_y[x]$ and induces V_{p0} in $K[x]$.

Because
$$\begin{aligned} W_2\left[(x-y)\,Q(x, y)\right] &= W_2(x-y)+W_2Q(x, y) \\ &\geqq \mu_1+\omega+1+W_1Q(x, y) \\ &=\mu_p+1 \qquad \because \text{ by } (24. 2) \\ &>\mu_p=V_{0p}\phi_{0p}(y)=W_2\phi_{0p}(y). \end{aligned}$$

So $$W_2\left[(x-y)Q(x, y)\right] > W_2\phi_{0p}(y) = \mu_p = W_2\phi_p(x) \quad \text{by (24. 1)}.$$

If $V_{10} < V_{20} < \cdots < V_{p0}$ is the shortest series of augmented inductive valuations, when we make V_{p0} from V_{00} in $K[x]$; we must pass at least P stages by M. Theorem 15. 3 and M. Theorem 16. 3.
But in this case, we can make such an augmented valuation W_2 of W_1 of $K_y[x]$ that induces V_{p0} in $K[x]$, only by one stage.

Example 24. 2.

K is a set of all rational numbers and V_{00} is 3-adic valuation.

$$V_{10} = [V_{00}, V_{10}x = 0] \quad \text{is a valuation of } K[x].$$

x^2+1 is a key polynomial in V_{10} of $K[x]$ which was explained in Example 10. 1. Then $V_{20}=[V_{10}, V_{20}(x^2+1)=1]$ is an augmented valuation of V_{10} of $K[x]$.

$$\phi(x) = (x^2+1)^3 - 9(x^2+1) + 27 \quad \text{is a key polynomial in } V_{20}.$$

The reason is as follows.
F_1, the residue class field of K by V_{00} is $(-1, 0, 1)$.
$V_{10}x=0$, so $H_1x=X_1$, then the residue class ring of $K[x]$ by V_{10} is $F_1[X_1]$, where H_1 is the natural homomorphism which makes the polynomials $f(x)$ whose values by V_{10} are non-negative, correspond to their residue classes $H_1f(x)$ in $F_1[X_1]$.

$$x^2+1 \quad \text{is a key polynomial which produces } V_{20}.$$

$$H_1(x^2+1) = X_1^2 + 1 = 0.$$

One root of this equation is $i=\sqrt{-1}$.

$$V_{20}\left(\frac{x^2+1}{3}\right) = 0 \quad \text{and} \quad H_2\left(\frac{x^2+1}{3}\right) = X_2.$$

$$F_2 = F_1(i). \quad H_2 \text{ is the natural homomorphism of } V_2.$$

Then the residue class ring of $K[x]$ by V_{20} is $F_2[X_2]$.

$$H_2\left(\frac{(x^2+1)^3 - 9(x^2+1) + 27}{27}\right) = X_2^3 - X_2 + 1.$$

If $X_2^3-X_2+1$ is irreducible in $F_2[X_2]$, then $(x^2+1)^3-9(x^2+1)+27$ is equivalent irreducible in V_{20}.

If $X_2^3-X_2+1$ is reducible in $F_2[X_2]$, one of its factors must be an expression of X_2 of one degree. Let the factor be $X_2-(a+bi)$. Here both a and b are elements of F_1, because $F_2=F_1(i)$. We substitute $a+bi$ for X_2 in $X_2^3-X_2+1$, then

$$(a+bi)^3-(a+bi)+1$$
$$=a^3-b^3i-a-bi+1 \qquad \because \quad V_{00} \text{ is 3-adic valuation.}$$
$$=a(a-1)(a+1)+1-ib(b^2+1)$$

If this expression is zero, then

$$a(a-1)(a+1)+1=0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (24.3)$$

and

$$b(b^2+1)=0 \qquad .$$

$$F_1=(-1,\ 0,\ 1), \qquad \text{so (24.3) never happens.}$$

Then $\quad X_2^3-X_2+1 \quad$ is irreducible in $F_2[X_2]$ and

$$(x^2+1)^3-9(x^2+1)+27 \qquad \text{is equivalent irreducible in } V_{20}.$$

This expression is a key polynomial in V_{20} by M. Theorem 9.4.

$$V_{20}\phi(x)=V_{20}\left[(x^2+1)^3-9(x^2+1)+27\right]=\text{Min }[3,\ 3,\ 3]=3\,.$$

So $V_{30}=[V_{20},\ V_{30}\phi(x)=4]$ is an augmented valuation of V_{20} of $K[x]$.
Next we make a valuation of $K[y]$. Let be

$$V_{01}=[V_{00},\ V_{01}y=0] \quad \text{and} \quad V_{02}=\left[V_{01},\ V_{02}(y^2+1)=\frac{1}{2}\right].$$

Now we shall make such an extension W' of V_{30} to $K_x[y]$ that induces V_{02} in $K[y]$.
Let be $W_1'=[V_{30}, W_1'y=0]$ which is an extension of V_{30} to $K_x[y]$.

$$y^2+1\sim y^2-x^2 \qquad \text{in } W_1',$$

because

$$W_1'(y^2+1)=\text{Min }[W_1'y^2,\ W_1'1]=0$$

$$W_1'\left[(y^2+1)-(y^2-x^2)\right]=W_1'(x^2+1)=V_{30}(x^2+1)=1>0\,.$$

$$y^2+1\sim(y-x)(y+x) \qquad \text{in } W_1'\,.$$

So, in order to make such an augmented valuation W_2' of W_1' in $K_x[y]$ that $W_2'(y^2+1)>W_1'(y^2+1)$, we must make such a valuation W_2' that increases a value of $y-x$ or that of $y+x$.
By M. Corollary 13.2, both $y-x$ and $y+x$ can be key polynomials in W_1'.

Let be

$$W_2'(y-x)=\frac{1}{2}>0=W_1'(y-x)\,.$$

Then

$$y^2+1=(y-x)^2+2x(y-x)+(x^2+1)$$

$$\therefore \quad W_2'(y^2+1)=\text{Min}\left[1,\ \frac{1}{2},\ 1\right]=\frac{1}{2}=V_{02}(y^2+1)\,.$$

Therefore W_2' induces V_{02} by Theorem 13. 2.
Next let us try to make an extension W of V_{02} to $K_y[x]$ that induces V_{30} in $K[x]$.

$$V_{20} = \left[V_{00},\ V_{10}x=0,\ V_{20}(x^2+1)=1\right]$$

$$V_{30} = \left[V_{20},\ V_{30}\left\{(x^2+1)^3-9(x^2+1)+27\right\}=4\right].$$

$$V_{02} = \left[V_{00},\ V_{01}y=0,\ V_{02}(y^2+1)=\frac{1}{2}\right].$$

First we decide that $W_1=[V_{02},\ W_1x=0]$.
W_1 is an extension of V_{02} to $K_y[x]$.
Next we try to make such an augmented valuation W_2 of W_1 that

$$W_2(x^2+1)=1>0=W_1(x^2+1).$$

$$x^2+1\sim x^2-y^2 \qquad \text{in } W_1.$$

Because $$W_1\left[(x^2+1)-(x^2-y^2)\right]=V_{02}(y^2+1)=\frac{1}{2}>W_1(x^2+1)$$

$$x^2-y^2\sim(x-y)(x+y) \qquad \text{in } W_1.$$

So, again by M. Corollary 13. 2, both $x-y$ and $x+y$ can be key polynomials in W_1.

Let be $$W_2(x-y)=\frac{1}{2}>0=W_1(x-y).$$

Then $W_2=\left[W_1,\ W_2(x-y)=\frac{1}{2}\right]$ is an augmented valuation of W_1 in $K_y[x]$.

$$x^2+1=(x-y)^2+2y(x-y)+(y^2+1)$$

$$W_2(x^2+1)=\text{Min}\left[W_2(x-y)^2,\ W_2 2x(x-y),\ W_2(y^2+1)\right]$$

$$W_2(x^2+1)=\text{Min}\left[1,\frac{1}{2},\frac{1}{2}\right]=\frac{1}{2}.$$

$$\frac{1}{2}=W_2(x^2+1)<1=V_{02}(x^2+1).$$

But this is a natural result from the fact that

$$W_1\left[(x^2+1)-(x-y)(x+y)\right]=\frac{1}{2},\qquad W_1(x^2+1)=0$$

and $$x^2+1\sim x^2-y^2 \qquad \text{in } W_1.$$

In order to make such an augmented valuation W_3 of W_2 in $K_y[x]$ that $W_3(x^2+1) > W_2(x^2+1)$, there are two ways.

One way is that we make a decomposition of x^2+1 in W_2 and pick one x-key polynomial $\phi_3(x)$ out of factors of x^2+1 and make such an x-augmented valuation W_3 of W_2 that $W_3\phi_3(x) > W_2\phi_3(x)$.

The other way is that by Theorem 18.1 we find such two polynomials $g(x)$ and $h(x)$ that

$$x^2+1 \overset{1}{\approx} g(x)\,h(x) \qquad \text{in } W_1,$$

where
$$g(x) \sim x-y \qquad \text{in } W_1.$$

First, let us take the former way.

$$W_2(x-y)^2 = 1 = W_2(x^2+1)$$

So
$$x^2+1 \sim 2y(x-y)+y^2+1 \qquad \text{in } W_2$$

$$x^2+1 \sim 2y\left[(x-y)+\frac{y^2+1}{2y}\right] \qquad \text{in } W_2.$$

Here we can simplify

$$x-y+\frac{y^2+1}{2y} \qquad \text{as follows.}$$

Because
$$W_2(x-y) = W_2\frac{y^2+1}{2y}$$

and
$$2 \sim -1 \quad \text{and} \quad \frac{1}{-y} \sim y \qquad \text{in } W_2$$

so
$$(x-y)+\frac{y^2+1}{2y} \sim (x-y)+y(y^2+1) \qquad \text{in } W_2$$

$$x^2+1 \sim 2y\left[(x-y)+y(y^2+1)\right] = (x+y^3)\,2y \qquad \text{in } W_2.$$

This is a decomposition of x^2+1 in W_2 and $(x-y)+(y^3+y)=x+y^3$ is a homogeneous key polynomial $\phi_3(x)$ in W_2, because

$$W_1(x-y) = \frac{1}{2} = W_1 y(y^2+1) = \frac{1}{2}.$$

So we assign a value of $W_3(x+y^3)$ equal to 1 which is greater than

$$\frac{1}{2} = W_2(x+y^3) = \text{Min}\left[W_2(x-y),\ W_1(y^3+y)\right].$$

Then W_3 is an x-augmented valuation of W_2 and $W_3(x^2+1)=1$, because;

$$x^2+1=(x+y^3)^2-2y^3(x+y^3)+(y^6+1) \tag{24.4}$$

$$y^6+1=(y^2+1)^3-3(y^2+1)^2+3(y^2+1)$$

$$\therefore \quad W_3(y^6+1)=\mathrm{Min}\left[\frac{3}{2},\ 2,\ \frac{3}{2}\right]=V_{02}(y^6+1)=\frac{3}{2}$$

$$W_3(x^2+1)=\mathrm{Min}\left[W_3(x+y^3)^2,\ W_3 2y^3(x+y^3),\ W_3(y^6+1)\right]$$

$$=\mathrm{Min}\left[2,\ 1,\ \frac{3}{2}\right]=1 \quad . \tag{24.5}$$

Next, let us take the latter way.

$$x^2+1\sim(x-y)(x+y) \qquad \text{in } W_1.$$

Because

$$W_1=[V_{02},\ W_1 x=0]$$

$$W_1(x-y)=0=W_1(x+y)$$

$$(x^2+1)-(x-y)(x+y)=y^2+1$$

$$W_1\left[(x^2+1)-(x-y)(x+y)\right]=W_1(y^2+1)=\frac{1}{2}.$$

Therefore, along the proof of Theorem 18.1 we must find such two polynomials l and m of x and y that

$$W_1\left[(x^2+1)-\{(x-y)+(y^2+1)m\}\{(x+y)+(y^2+1)l\}\right]>\frac{1}{2}. \tag{24.6}$$

$$W_1(y^2+1)+W_1\left[1-(x-y)l-(x+y)m-(y^2+1)lm\right]>\frac{1}{2}$$

so

$$W_1\left[1-(x-y)l-(x+y)m-(y^2+1)lm\right]>0$$

$$W_1(y^2+1)lm\geqq W_1(y^2+1)=\frac{1}{2}.$$

If we can find such two polynomial l and m that

$$1-(x-y)l-(x+y)m=1+y^2$$

then $W_1\left[1-(x-y)l-(x+y)m\right]=W_1(1+y^2)>0$ and (24.6) holds.

Assumed that $1-x(l+m)+y(l-m)=1+y^2$,

then $l+m=0$ and $l-m=y$

namely $$l=\frac{y}{2} \qquad m=-\frac{y}{2}.$$

$$x-y+(y^2+1)m=x-y-\frac{y}{2}(y^2+1)$$

$$\sim x-y+y(y^2+1)$$

$$x-y+(y^2+1)m \sim x+y^3 \qquad \text{in } W_1$$

because $$1\sim\frac{1}{-2} \qquad \text{in } W_1 \quad .$$

$$x+y+(y^2+1)l=x+y+(y^2+1)\frac{y}{2}$$

$$\sim x+y-y(y^2+1)$$

$$x+y+(y^2+1)l\sim x-y^3 \qquad \text{in } W_1 \quad .$$

So, let be $$g(x)=x+y^3 \quad \text{and} \quad h(x)=x-y^3$$

then $$g(x)\sim x-y \quad \text{and} \quad h(x)\sim x+y \qquad \text{in } W_1$$

and $$W_1\left[(x^2+1)-g(x)h(x)\right]=W_1(y^6+1)=\frac{3}{2}=V_{02}(y^6+1).$$

So, $$x^2+1\overset{1}{\approx} g(x)h(x) \qquad \text{in } W_1$$

and $$g(x)\sim x-y$$

$$h(x)\sim x+y \qquad \text{in } W_1.$$

$$W_1\left[(x^2+1)-g(x)h(x)\right]=\frac{3}{2}>1,$$

so let be $$W^*g(x)=W^*(x+y^3)=1, \qquad \text{then}$$

$$W^*(x^2+1)=1, \qquad \text{for} \quad W^*g(x)h(x)=1.$$

So $$W^*=W_3.$$

Let be $$\phi=(x^2+1)^3-9(x^2+1)+27.$$

And we must make such an augmented valuation W_4 that $W_4\phi>W_3\phi$.

But, here we meet a problem which we must solve immediately. We consider about it in § 25.

§ 25. Greater valuation

$$W_1<W_2<W_3$$

This is a series of x-simply augmented inductive valuations which are extensions of V_{02} to the ring $K_y[x]$.

$$W_1 = [V_{02},\ W_1 x = 0]$$

$$W_2 = \left[W_1,\ W_2(x-y) = \frac{1}{2}\right]$$

$$W_3 = \left[W_2,\ W_3(x+y^3) = 1\right]$$

$$W_1(x^2+1) = 0,\ \ W_2(x^2+1) = \frac{1}{2},\ \ W_3(x^2+1) = 1\,.$$

so $$W_1(x^2+1) < W_2(x^2+1) < W_3(x^2+1)\,.$$

Let the valuations which W_1, W_2, and W_3 induce in $K[x]$, be respectively W_1', W_2' and W_3'.

$$W_1' = [V_{00},\ W_1 x = 0 = W_1' x]$$

$$W_2' = \left[W_1',\ W_2'(x^2+1) = \frac{1}{2}\right]$$

$$W_3' = \left[W_2',\ W_3'(x^2+1) = 1\right]$$

Because x^2+1 is a key polynomial in W_1' of $K[x]$ which I proved in Example 10.1. But here $W_3'(x^2+1) > W_2'(x^2+1)$.

Here I give newly some definitions.

Definition 25. 1.

If U_1 and U_2 are valuations of $K[x]$ and $U_1 \neq U_2$ and for every polynomial $f(x)$

$$U_1 f(x) \leqq U_2 f(x)\,,$$

then U_2 is called a greater valuation of U_1.

Definition 25. 2.

If U_2 is a greater valuation of U_1, but U_2 does not change a value of the key polynomial $\phi_1(x)$ which produces U_1,

namely $$U_1\phi_1(x) = U_2\phi_1(x)\,,$$

then U_2 is called an augmented valuation of U_1.

These definitions never contradict those which MacLane gave. And the definition of an augmented valuation of a valuation which I give here, coincides completely with what MacLane gave.

Here W_3' is a greater valuation of W_2', but it is not an augmented valuation of W_2'. In this case, if we make newly such a valuation W_3^*

$$W_3^* = \left[W_1', \ W_3^*(x^2+1) = 1\right]$$

then W_3^* is an augmented valuation of W_1' and $W_3^* = W_3'$. This means that $W_3^* f(x) = W_3' f(x)$ for every polynomial $f(x)$.

When we discuss about valuations, the most important valuation is the last valuation of a series of augmented inductive valuations.

In Example 24.2, of course, the valuation that we want to make is the last valuation W which is an extension of V_{02} to $K_y[x]$ and induces V_{p0} in $K[x]$.

If we make such a valuation $\widetilde{W}$ of $K_y[x]$ that

$$\widetilde{W} = \left[W_1, \ \widetilde{W}(x+y^3) = 1 = W_3(x+y^3)\right]$$

then by M. Lemma 15.1 $\widetilde{W} = W_3$ and $\widetilde{W}$ is an augmented valuation of W_1 and $\widetilde{W}$ induces W_3^* in $K[x]$.

So
$$W_1 < \widetilde{W} = W_3 .$$

Thus, from a series of augmented inductive valuations, we can drop excessive valuations not to change the last valuation of the series.

Like this, if we drop W_2 from a series,

$$W_1 < W_2 < W_3 = \widetilde{W}$$

then we obtain a new series which induces in $K[x]$ a new series $W' < W_3^* = W_3'$, which does not change the last valuation W_3'.

So in Example 24.2, we drop W_2 and we continue our study about the new series $W_1 < W_3$. So hence $W_3 = \widetilde{W}$.

§ 26. Continuation of soluation of Example 24.2

Again we consider about $\phi = (x^2+1)^3 - 9(x^2+1) + 27$. From (24.4) and (24.5)

$$x^2+1 = (x+y^3)^2 - 2y^3(x+y^3) + (y^6+1)$$

$$W_3(x+y^3)^2 = 2, \quad W_3(y^6+1) = \frac{3}{2} \quad \text{and} \quad W_3\left[-2y^3(x+y^3)\right] = 1 .$$

Now let be $\quad (x+y^3)^2 + (y^6+1) = \alpha \quad$ and $\quad W_3\alpha = \dfrac{3}{2}$

and $\quad -2y^3(x+y^3) = \xi \quad$ and $\quad W_3\xi = 1 .$

$$\begin{aligned} \phi(x) = \phi &= (x^2+1)^3 - 9(x^2+1) + 27 \\ &= (\alpha+\xi)^3 - 9(\alpha+\xi) + 27 \\ &= (\alpha^3 + 3\alpha^2\xi + 3\alpha\xi^2) - 9\alpha + (\xi^3 - 9\xi + 27) . \qquad (26.1) \end{aligned}$$

$$W_3\alpha^3 = 4.5, \quad W_3(3\alpha^2\xi) = 5, \quad W_3(3\alpha\xi^2) = 4.5 \quad .$$

Therefore
$$W_3(\alpha^3 + 3\alpha^2\xi + 3\alpha\xi^2) \geqq 4.5 \tag{26. 2}$$

$$W(-9\alpha) = 3.5 \tag{26. 3}$$

$$W_3(\xi^3 - 9\xi + 27) = W_3\left[\left\{-2y^3(x+y^3)\right\}^3 - 9\left\{-2y^3(x+y^3)\right\} + 27\right]$$
$$= \mathrm{Min}\,[3,\ 3,\ 3] = 3 \quad . \tag{26. 4}$$

From (26. 1), (26. 2), (26. 3), and (26. 4)

$$W_3\phi = W_3(\xi^3 - 9\xi + 27) = 3 \quad .$$

In this example, we must make such an augmented valuation W that $W\phi = 4 = V_{30}\phi$.

Let be
$$\xi^3 - 9\xi + 27 = P.$$

$$P = -8y^9(x+y^3)^3 + 18y^3(x+y^3) + 27$$

$$y^2 \sim -1, \ \frac{1}{y} \sim -y \quad \text{and} \quad 2 \sim -1 \qquad \text{in } W_3$$

$$\frac{P}{3^3} \sim y\left[\left(\frac{x+y^3}{3}\right)^3 + \frac{x+y^3}{3} - y\right] \qquad \text{in } W_3$$

$$\left(\frac{x+y^3}{3^3}\right)^3 + \frac{x+y^3}{3} - y = \frac{1}{3^3}\left[(x+y^3)^3 + 9(x+y^3) - 27y\right] \quad .$$

By M. Theorem 9. 4, if we prove only that $(x+y^3)^3 + 9(x+y^3) - 27y$ is equivalent irreducible in W_3, then this expression is a homogeneous key polynomial in W_3.

For the sake of it, we may prove that $H_3^*\left(\dfrac{P}{3^3}\right)$ is irreducible in the residue class ring of $K_y[x]$ by W_3, where H_3^* is a natural homomorphism of W_3.

$$V_{01} = [V_{00},\ V_{01}\,y = 0] \qquad \text{is a valuation of } K[y].$$

So $H_1K[y] = F_1[Y_1]$, where H_1 is a natural homomorphism of V_{01} and $H_1K[y]$ is the residue class ring of $K[y]$ by V_{01} and $H_1[y] = Y_1$.

$$V_{02} = \left[V_{01},\ V_{02}(y^2+1) = \frac{1}{2}\right].$$

$$H_1(y^2+1) = Y_1^2 + 1 = 0 \qquad \therefore \quad Y_1 = i = \sqrt{-1}$$

$$V_{02}\frac{(y^2+1)^2}{3} = 0 \quad \text{and} \quad H_2\frac{(y^2+1)^2}{3} = Y_2 \quad .$$

Then $\quad H_2K[y]=F_2[Y_2]\,, \qquad$ where $\quad F_2=F_1(i)$

and H_2 is a natural homomorphism of V_{02}, and $H_2K[y]$ is the residue class ring of $K[y]$ by V_{02}.

$$W_1=[V_{02},\ W_1x=0]$$

W_1 is a valuation of $K_y[x]$.

$$H_1^*x=X_1 \quad \text{and} \quad H_1^*K_y[x]=F_2^*[X_1]$$

where $F_2^*=F_2(Y_2)$ and H_1^* is a natural homomorpnism of $K_y[x]$ by W_1.

$$W_3=\left[W_1,\ W_3(x+y^3)=1\right].$$

$$\begin{aligned} H_1^*(x+y^3) &= H_1^*x+(H_1^*y)^3 \\ &= X_1+Y_1^3 \\ &= X_1+i^3=X-i\,. \end{aligned}$$

$$X_1-i=0 \qquad X_1=i$$

$$H_3^*\frac{x+y^3}{3}=X_3\,, \qquad \text{then}$$

$$H_3^*K_y[x]=F_3^*[X_3], \qquad \text{where} \quad F_3^*=F_2(Y_2)$$

because $\qquad F_2(i)=F_2\,.$

$$H_3^*\left(\frac{P}{3^3}\right)=i(X_3^3+X_3-i)\,, \qquad \text{because} \quad H_3^*y=i\,.$$

Assumed that $\qquad X_3^3+X_3-i \qquad$ can be factorized in $F_3^*[X_3]$,

one of its factors is $X_3-(a+bi)$, where both a and b are 0, 1, *or* -1 in F_1, for Y_2 and X_3 are algebrically independent with respect to F_2.

We subsitute $a+bi$ for X_3, then

$$(a+bi)^3+a+bi-i=0$$

$$a^3-b^3i+a+bi-i=0 \qquad \because \quad H_3^*3=0$$

$$b^3-b+1=b(b+1)(b-1)+1=0 \qquad .$$

Such b does not exist in F_1. So b^3-b+1 can not equal zero.

And $\qquad X_3^*+X_3-i \qquad$ is irreducible in $F_3^*[X_3]$.

Then $\qquad (x+y^3)^3+9(x+y^3)-27y \qquad$ is a key polynomial in W_3.

Now we decide an augmented valuation W_4 of W_3 as

$$W_4\left[(x+y^3)^3+9(x+y^3)-27y\right]=3.5,$$

where $$3.5 > W_3\left[(x+y^3)^3+9(x+y^3)-27y\right] = 3\,.$$

Then $$W_4 P = 3.5\,.$$

Let be $(x+y^3)^3+9(x+y^3)-27y = \phi_4$, then

$$W_4\phi_4 = 3.5\,.$$

In (26. 3) $$-9\alpha = -(9x+y^3)^2-9(y^6+1)$$

$$W_4\left[-9(x+y^3)^2\right] = W_3\left[-9(x+y^3)^2\right] = 4$$

and $$W_4\left[-9(y^6+1)\right] = W_1\left[-9(y^6+1)\right] = 3.5$$

$$deg_x\left[-9(x+y^3)^2\right] < deg_x\,\phi_4\,.$$

So $$W_4\phi = \mathrm{Min}\left[W_4\left\{-9(x+y^3)^2\right\},\ W_4\left\{-9(y^6+1)\right\},\ W_4 P\right]$$
$$= \mathrm{Min}\,[4,\ 3.5,\ 3.5] \qquad \because \quad W_3(\alpha^3+3\alpha^2\xi+3\alpha\xi^2) \geqq 4.5$$
$$= 3.5 > W_3\phi = 3\,.$$

And $$P-9(y^6+1) \sim \phi \qquad \text{in } W_4\,. \tag{26. 5}$$

Therefore, if we make such an augmented valuation W_5 of W_4 that $W_5[P-9(y^6+1)]=4$, then

$$W_5\phi = 4 \qquad \text{by (26. 5)}\,.$$

And this example 24. 2 is solved completely.

$$P-9(y^6+1) = -8y^9(x+y^3)^3+18y^3(x+y^3)+27-9(y^6+1)\,.$$

$$-2\sim 1\,,\ y^2\sim -1 \quad \text{and} \quad -\frac{1}{y}\sim y \qquad \text{in } W_4$$

$$P-9(y^6+1)\sim y\left[(x+y^3)^3+9(x+y^3)-27y+9y(y^6+1)\right]$$

$$P-9(y^6+1)\sim y\left[\phi_4+9y(y^6+1)\right] \qquad \text{in } W_4\,. \tag{26. 6}$$

$$W_4\left[9y(y^6+1)\right] = 3.5 = W_4\phi_4\,.$$

By M. Corollary 13. 2, $\phi_4+9y(y^6+1)$ is a key polynomial in W_4.

Then we decide an augmented valuation W_5 of W_4 that

$$W_5\left[\phi_4+9y(y^6+1)\right] = 4 > W_4\left[\phi_4+9y(y^6+1)\right].$$

By (26. 6) $$W_5\left[P-9(y^6+1)\right] = W_5\, y\left[\phi_4+9y(y^6+1)\right]$$
$$= 4\,.$$

And $$W_5\phi = 4 = V_{30}\phi \qquad \text{by (26.5)}.$$

In (26.1) $$W_5(\alpha^3+3\alpha^2\xi+3\alpha\xi^2) \geqq W_3(\alpha^3+3\alpha^2\xi+3\alpha\xi^2) \geqq 4.5\,.$$

Therefore when we make such an augmented valuation W_5, of W_3 that $W_5\phi=4>3=W_3\phi$, we may neglet this term. Thus, example 24.2 is solved completely.

§ 27. Homogeneous valuations

Lemma 27. 1.

If $W=[V, W\phi(x)=\mu]$ is an augmented valuation of a homogeneous valuation V of K $[x]$ and we can insert such a valuation U between them that $V<U<W$, then $\phi(x)$ is not a homogeneous polynomial in V.

Proof. Let be $U=[V, U\psi(x)=\nu]$, and $\psi(x)$ homogeneous in V.

For every polynomial $f(x)$ whose degree with respect to x is less than that of $\phi(x)$

$$Wf(x) = Vf(x)\,.$$

$$W\psi(x) = U\psi(x) > V\psi(x)\,. \tag{27.1}$$

So $$deg\,\psi(x) = deg\,\phi(x)\,, \qquad \text{for } \deg\psi(x) \geqq \deg\phi(x) \quad \text{by (27.1)}.$$

Let be $$\phi(x) = \psi(x) + r(x)$$

then $$deg\,r(x) < deg\,\phi(x) = deg\,\psi(x)\,.$$

$\phi(x)=\psi(x)+r(x)$ is a key polynomial in U which the key polynomial $\psi(x)$ produces from V, so by M. Theorem 9.4. $Ur(x)=U\psi(x)>V\psi(x)$

while $$Vr(x) = Ur(x) \qquad \because\ deg\,r(x) < deg\,\psi(x)\,.$$

Namely $$Vr(x) > V\psi(x)\,.$$

So $\phi(x)$ is not a homogeneous polynomials in V. For ψ is homogeneous in V.

Therefore, from this Lemma we obtain immediately the following theorem.

Theorem 27. 2.

Every series of augmented inductive homogeneous valuations of $K[x]$

$$V_1 < V_2 < \cdots < V_n$$

is a series of continuous augmented inductive valuations of $K[x]$, where all $V_1, V_2, \cdots, V_n$ are extensions of V_0 wihch is a valuation of K.

Example

$$V_2=\left[V_0,\ V_1x=0,\ V_2(x^2+1)=1\right]$$

V_2 is the valuation in Example 10.1 in which I substitute x for y.

Let be $$W=\left[V_2,\ W\left\{(x^2+1)+3+3^2\right\}=3\right].$$

Then we can insert U so that

$$V_2<U<W$$

where $$U=\left[V_2,\ U\left\{(x^2+1)+3\right\}=2\right].$$

$$(x^2+1)+3+3^2\sim(x^2+1)+3 \qquad \text{in } V_2 \quad .$$

$(x^2+1)+3+3^2$ is not a homogeneous polynomial in V_2, for $12\sim3$ in V_0, so 12 is not a representative.

But let be $$(x^2+1)+3=\psi$$

and $$(x^2+1)+3+3^2=\phi\,.$$

$$\phi=\psi+3^2 \quad \text{and} \quad U\phi=U\psi=U3^2$$

namely ϕ is a homogeneous polynomial in U.
So in a series of continuous inductive valuations

$$V<U<W$$

every key polynomial is homogeneous.

In this paper, I often use the two following lemmas which are related as Remark 3 and Remark 4, but I omit their proofs, for they are very easy.

Remark 3. If $a(x)\sim b(x)+c(x)$ and $b(x)\sim b'(x)$ in V

and $$Va(x)=Vb(x)=Vc(x)\,,$$

then $$a(x)\sim b'(x)+c(x) \qquad \text{in } V.$$

Remark 4.

If $\phi(x)$ is a key polynomial which produces an augmented valuation W of a valuation V of $K[x]$ and $a(x)\sim\phi(x)$ in V and $\deg\left(a(x)-\phi(x)\right)<\deg\phi(x)$

and $$V\left(a(x)-\phi(x)\right)\geqq\delta+V\phi(x)$$

and $$W\phi(x)=\delta+V\phi(x)\,,$$

then $$Wa(x)=W\phi(x)\,.$$

I want to relate what I can not do in this paper, in Part three of the same theme which will be published in future.

References

(1) Hiroshi INOUE: On valuations of polynomial rings of many variables, Part one, Jour. Math. Hokkaido Univ. 21 (1970) 46–74.

(2) Saunder MACLANE: A construction for absolute values in polynomial rings, Trans. Amer. Math. Soc. 40 (1936) 363–395.

(3) Saunder MACLANE: A construction for prime ideals as absolute values of an algebraic field, Duke. Math. Jour. 2 (1936) 492–510.

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