A NOTE ON S^1-ACTING COBORDISMS

By

Haruo SUZUKI

Introduction

Let Y be a connected compact oriented differentiable manifold (without boundary). Suppose that S^1 acts on Y differentiably and effectively without fixed points. This means that no point of Y is left fixed by the whole group, that is $Y^{S^1}=\phi$. Let Y' be another manifold which has the properties of Y and let W be a compact differentiable manifold, such that $\partial W=Y\cup(-Y')$ and S^1-action on ∂W extends differentiably all over W. Suppose also that we have $W^{S^1}=\phi$ for the S^1-action on W. If Y and Y' satisfy the above condition, we say that they are S^1-*acting cobordant* or they belong to the same S^1-*acting cobordism class*.

Any differentiable principal S^1-bundle over a differentiable manifold is a manifold on which S^1 acts differentiably and effectively without fixed points by the multiplication of complex numbers in fibre. We shall consider the S^1-acting cobordism for Y and Y' which have the differential principal S^1-bundle structures over differentiable structures of a connected compact manifold M and have the S^1-actions stated in the above. It should be noted that if Y and Y' are S^1-acting cobordant by the manifold W, the S^1-action on W is not necessarily free, that is, Y and Y' do not necessarily belong to the same equivariant cobordism class in the usual sense. By imposing some cohomology conditions upon M, Y and Y' have unique spin structures. The purpose of this note is to show, directly by the powerful invariant ρ of M. F. Atiyah and F. Hirzebruch [2], that when W has a spin-structure which is compatible with those of Y and Y' and the dimension of M is $2k$, the principal S^1-bundle structures of Y and Y' are isomorphic. Finally we shall show several examples.

§ 1. Statement of results

Let M be a connected compact oriented differentiable manifold of dimension $2k$ such that

$$(1)\qquad H^1(M;Z_2)=0,\quad H^2(M;Z)=Z,\quad w_2(M)\neq 0\,.$$

Suppose that if g is a generator of $H^2(M;Z)$, $g^k\in H^{2k}(M;Z)$ is not zero.

We regard $S^1=SO(2)$ as the multiplicative group which consists of the complex numbers with absolute value 1. In the following, we assume complex line bundles have structural group $U(1)=SO(2)$. The differentiable S^1-bundles are in 1–1 correspondence with the differentiable complex line bundles over the same base space. A complex line bundle ξ is characterized topologically by its first Chern class $c=c_1(\xi)$ which is of the form ag where a is an integer. We assume that ξ is a differentiable complex line bundle and we denote by Y_ξ the total space of the principal S^1-bundle corresponding to ξ and denote by X_ξ the total space of the closed unit disk bundle corresponding to ξ. Y_ξ is a closed connected oriented differentiable manifold on which $S^1=\{z||z|=1\}$ acts differentiably and effectively without fixed points by the multiplication of z^m in fibre, for some integer $m\neq 0$. We call this action a *natural S^1-action* on Y_ξ for m. X_ξ is a connected compact oriented differentiable manifold with the boundary Y_ξ, on which S^1 acts also non-trivially. We notice that if ξ' is the complex line bundle having the first Chern class $c_1(\xi')=-ag$, S^1-bundles Y_ξ, $Y_{\xi'}$ are isomorphic by the bundle map over the identity of M, which reverses the orientation of fibre.

We assume that $c_1(\xi)\not\equiv 0 \mod 2$. Then from the assumption (1), it follows that

$$H^1(Y_\xi\,;Z_2)=0, \quad w_2(Y_\xi)=0,$$

and

$$H^1(X_\xi\,;Z_2)=0, \quad w_2(X_\xi)=0\,.$$

Therefore Y_ξ has (for some Riemannian metric) exactly one spin-structure up to isomorphism and X_ξ also does, by the arguments of M.F. Atiyah and R. Bott [1], p. 480. We denote by $\mathcal{S}_M$ the set of S^1-bundles Y_ξ over differentiable structures of M such that $c_1(\xi)=ag$, $a\not\equiv 0 \mod 2$ and moreover if k is even, a is positive.

Let Y and Y' be connected compact oriented differentiable manifolds such that

$$H^1(Y\,;Z_2)=H^1(Y'\,;Z_2)=0, \quad w_2(Y)=w_2(Y')=0\,.$$

And suppose that S^1 acts on them differentiably and effectively without fixed points. We say that Y and Y' are *S^1-acting spin-cobordant* or they belong to the same *S^1-acting spin-cobordism class*, if there exists a compact differentiable manifold W on which S^1 acts differentiably and effectively without fixed points and

$$H^1(W\,;Z_2)=0, \quad w_2(W)=0,$$
$$\partial W=Y\cup(-Y')\,.$$

This definition of the S^1-acting spin-cobordism is essentially due to M.F.

Atiyah and F. Hirzebruch [2].

Now we state the following main theorem.

Theorem 1. 1. *Any two S^1-bundles of $\mathscr{S}_M$ are isomorphic if and only if they belong to the same S^1-acting spin-cobordism class up to topological equivalences, with respect to some natural S^1-actions.*

Remark (1) For the S^1-equivariant oriented cobordism class, the theorem is a direct consequence of the invariance of the Chern number $(c_1)^k[M]$, but we are dealing with the S^1-acting spin-cobordism class. (2) Suppose that k is even. Let ξ and ξ' be differentiable complex line bundles such that $c_1(\xi)=ag$, $w_2(M)=ag \bmod 2$, $a<0$ and $c_1(\xi')=a'g$, $w_2(M)=a'g \bmod 2$, $a'>0$. Reversing the orientation of fibre of $Y_{\xi'}$, we obtain the differentiable S^1-bundle $Y_{\xi''}$ such that $c_1(\xi'')=(-a')g$. By the above theorem Y_ξ and $Y_{\xi''}$ are S^1-bundle isomorphic if and only if they belong to the same S^1-acting spin-cobordism class with respect to natural S^1-actions.

To prove the Theorem 1. 1, we obtain, in §2, some results about values of local spin-numbers for the S^1-action on X_ξ. We shall give a proof of the main theorem and some examples in §3.

§2. The invariant $\rho(z, Y)$

Let Y be the connected compact oriented differential manifold (without boundary) of dimension $2k+1$ such that $H^1(Y; Z_2)=0$, $w_2(Y)=0$. Suppose that S^1 acts on Y differentiably and effectively without fixed points and suppose that Y bounds a connected compact oriented differenteable S^1-manifold X such that $H^1(X; Z_2)=0$, $w_2(X)=0$. We denote by $X_\nu^{S^1}$ a component of the fixed point set for the S^1-action on X. By the definition of M.F. Atiyah and F. Hirzebruch [2] or M.F. Atiyah and I.M. Singer [3], we have

$$(2) \qquad \rho(z, Y) = \sum_\nu \operatorname{spin}(z, X_\nu^{S^1}),$$

which does not depend on the choice of X.

For the S^1-manifold X_ξ in the preceding section, it is easily verified that the fixed point set $(X_\xi)^{S^1}$ is the zero section of the complex line bundle ξ. Hence it is diffeomorphic to the manifold M of base space and is connected. The normal bundle $N((X_\xi)^{S^1})$ of $(X_\xi)^{S^1}$ in X_ξ is isomorphic to ξ and $z\in S^1$ operates as a multiplication of a complex number z^m in fibre, for the integer m. The eigenvalue of the operation of z in the fibre is z^m and for $z\in S^1$ but z not a root of unity, we have

$$(3) \qquad \operatorname{spin}(z, (X_\xi)^{S^1}) = (-1)^{(k+1)}\,\mathfrak{A}(M)\,(z^{-m/2}e^{c/2}-z^{m/2}e^{-c/2})^{-1}[M]$$

by M. F. Atiyah and F. Hirzebruch [2], where $\hat{\mathfrak{A}}$ is the multiplicative sequence with the characteristic series

$$\frac{x/2}{\sinh\ x/2}=\frac{x}{e^{x/2}-e^{-x/2}}$$

and $c=c_1(\xi)$ is the first Chern class of ξ. c can be written as ag where g is a generator of $H^2(M;Z)$ and $a\in Z$. By the definition of $\hat{\mathfrak{A}}$, we have

$$\hat{\mathfrak{A}}(M)=\sum_{r=0}^{\infty}\hat{A}_r(p_1(M),\cdots,p_r(M)) \tag{4}$$

where $\hat{A}_r(p_1(M),\cdots,p_r(M))\in H^{4r}(M;Q)$ are polynomials in the Pontrjagin classes $p_i(M)$. We notice that the right hand side of (3) is a rational function on the complex number plane C if m is even and it is a function on the double branched covering of C if m is odd.

To distinguish values of $\rho(z, Y_\xi)$, we need the following lemma.

Lemma 2. 1. *We have*

$$(z^{-m/2}e^{c/2}-z^{m/2}e^{-c/2})^{-1}=(z^{-m/2}-z^{m/2})^{-1}(1+\alpha_1c+\cdots+\alpha_kc^k), \tag{5}$$

where α_i is a polynomial of order i, in the variable

$$z'=(z^{-m/2}+z^{m/2})(z^{-m/2}-z^{m/2})^{-1}.$$

Proof. From direct computations, it follows that

$$\begin{aligned}&(z^{-m/2}e^{c/2}-z^{m/2}e^{-c/2})^{-1}\\&\quad=(z^{-m/2}-z^{m/2})^{-1}\left(1+\frac{z'}{2}c+\frac{1}{2^2 2!}c^2+\cdots+\frac{z'^{(k-2[k/2])}}{2^k k!}c^k\right)^{-1}.\end{aligned}$$

We put

$$\left(1+\frac{z'}{2}c+\frac{1}{2^2 2!}c^2+\cdots+\frac{z'^{(k-2[k/2])}}{2^k k!}c^k\right)^{-1}=1+\alpha_1c+\alpha_2c^2+\cdots+\alpha_kc^k.$$

Then α_i is determined uniquely by the following formulas:

$$\alpha_1+\frac{z'}{2}=0,$$

$$\alpha_2+\alpha_1\left(\frac{z'}{2}\right)+\frac{1}{2^2 2!}=0,$$

....

$$\alpha_i+\alpha_{i-1}\left(\frac{z'}{2}\right)+\cdots+\alpha_{i-j}\frac{z'^{(j-2[j/2])}}{2^j j!}+\cdots+\frac{z'^{(i-2[i/2])}}{2^i i!}=0,$$

....

$$\alpha_k+\alpha_{k-1}\left(\frac{z'}{2}\right)+\cdots+\frac{z'^{(k-2[k/2])}}{2^k k!}=0.$$

By an obvious induction, the term of the highest power of z' in α_i is

$$(-1)^i(z')^i/2^i.$$

Thus the lemma is proved.

Theorem 2.2. *It follows that*

$$\begin{aligned}\rho(z,\ Y_\xi) &= \mathrm{spin}(z,\ (X_\xi)^{S^1})\\ &= (-1)^{k+1}(z^{-m/2}-z^{m/2})^{-1}F,\end{aligned}$$

where F is a polynomial of order k, in $z'=(z^{-m/2}+z^{m/2})(z^{-m/2}-z^{m/2})^{-1}$. If $c=c_i(\xi)=ag$, then the coefficient of $(z')^k$ in the polynomial F is

$$(-1)^k(a^k/2^k)g^k[M].$$

Proof. From (3), (4) and the above lemma, it follows that

$$\begin{aligned}(-1)^{k+1}\rho(z,\ Y_\xi) &= \hat{\mathfrak{A}}(M)(z^{-m/2}-z^{m/2})^{-1}(1+\alpha_1 c+\cdots+\alpha_k c^k)[M]\\ &= (z^{-m/2}-z^{m/2})^{-1}(\sum_{i=0}^{\infty}\hat{A}_i(M))(1+\alpha_1 c+\cdots+\alpha_k c^k)[M]\\ &= (z^{-m/2}-z^{m/2})^{-1}(\alpha_k\hat{A}_0(M)c^k+\alpha_{(k-2)}\hat{A}_1(M)c^{(k-2)}\\ &\qquad+\cdots+\alpha^{(k-2[k/2])}\hat{A}_{[k/2]}(M)c^{(k-2[k/2])})[M]\\ &= (z^{-m/2}-z^{m/2})^{-1}(a^k\alpha_k g^k[M]+a^{(k-2)}\alpha_{(k-2)}\hat{A}_1(M)g^{(k-2)}[M]\\ &\qquad+\cdots+a^{(k-2[k/2])}\alpha_{(k-2[k/2])}\hat{A}_{[k/2]}(M)g^{(k-2[k/2])}[M]).\end{aligned}$$

The formula

$$\begin{aligned}&a^k\alpha_k g^k[M]+a^{k-2}\alpha_{(k-2)}\hat{A}_1(M)g^{k-2}[M]+\\ &\qquad+\cdots+a^{(k-2[k/2])}\alpha_{(k-2[k/2])}\hat{A}_{[k/2]}(M)g^{(k-2[k/2])}[M]\end{aligned}$$

is a polynomial in z' with rational coeffecients. The term of the highest order (kth order) of z' in the polynomial is in $a^k\alpha_k g^k[M]$ and it is

$$(-1)^k(a^k/2^k)g^k[M].$$

Thus the proof of the theorem is completed.

§3. S^1-acting cobordism of Y_ξ and examples

Using results of the previous section, we prove our main theorem stated in §1.

Proof of Theorem 1.1. Let ξ_1 and ξ_2 be differentiable complex line bundles with structural group $U(1)$, over differentiable structures of M. Y_{ξ_1} and Y_{ξ_2} denote the S^1-bundles corresponding to ξ_1 and ξ_2 respectively.

Suppose that Y_{ξ_1} and $Y_{\xi_2}\in\mathscr{S}_M$ are topologically isomorphic differentiable S^1-bundles. Natural S^1-actions on them for each m are isomorphic. If we take

$$W=Y_{\xi_1}\times \mathrm{I},$$

then we have

$$H^1(W;Z_2)=H^1(Y_{\xi_1};Z_2)=0,$$
$$\partial W=Y_{\xi_1}\cup(-Y_{\xi_1}) \text{ and } Y_{\xi_1}\cong Y_{\xi_2}.$$

Let π_1: $Y_{\xi_1}\times I\longrightarrow Y_{\xi_1}$ be the projection to the first factor. It follows that

$$\begin{aligned} w_2(W) &= w_2(Y_{\xi_1}\times I) \\ &= \pi_1^* w_2(Y_{\xi_1}) \\ &= 0. \end{aligned}$$

Thus Y_{ξ_1} and Y_{ξ_2} belong to the same S^1-acting spin-cobordism class up to topological equivalences, with respect to natural S^1-actions for the same m.

Conversely, suppose that Y_ξ and $Y_{\xi'}$ with natural S^1-actions for m_1 and m_2 respectively, belong to the same S^1-acting spin-cobordism class. Then by M. F. Atiyah and F. Hirzebruch [2], we have

$$\rho(z, Y_{\xi_1})=\rho(z, Y_{\xi_2})$$

and hence

$$\mathrm{spin}(z, Y_{\xi_1})=\mathrm{spin}(z, Y_{\xi_2}).$$

Let $c_1(\xi_1)=a_1g$, $c_1(\xi_2)=a_2g$ where a_1, $a_2>0$.
By Theorem 2. 2, we have

$$m_1=m_2 \text{ and } a_1^k=a_2^k.$$

Since a_1, a_2 are positive integers, it follows that

$$a_1=a_2.$$

Thus we have $c_1(\xi_1)=c_1(\xi_2)$ and hence

$$\xi_1\cong\xi_2.$$

The proof of Theorem 1. 1 is completed.

For the complex line bundle ξ over M with $c_1(\xi)=ag$, the manifold Y_ξ is obviously spin-cobordant to S^{2k+1} in the following sense: There exists a differentiable manifold W such that

$$H^1(W;Z_2)=0,\quad w_2(W)=0,$$
$$\partial W=Y_\xi\cup(-S^{2k+1}).$$

But from Theorem 2. 2, we obtain,

Proposition 3. 1. *If $c_1(\xi)\neq\pm g$, Y is not S^1-acting spin-cobordant to S^{2k+1} with the standard circle action.*

We consider some examples of S^1-bundles satisfying the conditions of Theorem 1. 1.

Example 1. Let M be a complex projective space CP^{2n}. We take, as $\mathscr{S}_M = \mathscr{S}_{CP^{2n}}$, the set of differentiable S^1-bundles Y_ξ corresponding to differentiable complex line bundles ξ over differentiable structures of CP^{2n} such that $c_1(\xi) = ag$, $a \equiv 1 \bmod 2$ and $a > 0$, where $g \in H^2(CP^{2n}; Z)$ is the canonical generator. The condition (1) for $M = CP^{2n}$ are easily verified. By Theorem 1. 1, any two S^1-bundles of $\mathscr{S}_{CP^{2n}}$ are isomorphic if and only if they are S^1-acting spin-cobordant upto topological equivalences with respect to natural S^1-actions.

Example 2. Let HP^n denote a quaternionic projective space of quaternionic dimension n. We can choose $M = CP^{2n} \sharp HP^n (n > 2)$ which is the connected sum of CP^{2n} and HP^n. The condition (1) for this manifold are satisfied. By similar arguments, one can also take, as M,

$$CP^{2n} \sharp HP^n \sharp \cdots \sharp HP^n,$$

$$CP^{2n} \sharp (HP^{k_1} \times HP^{k_2} \times \cdots \times HP^{k_r}) \, (\Sigma k_i = n).$$

References

[1] M. F. ATIYAH and R. BOTT: *A Lefschetz fixed point formula for elliptic complexes*, II. *Applications*, Ann. of Math. (2) 88 (1968), 451–491.

[2] M. F. ATIYAH and F. HIRZEBRUCH: *Spin-manifolds and group actions*, Essays on Topology and Related Topics (Memoires dédiés à Georges de Rham), Springer New York 1970, 29–47.

[3] M. F. ATIYAH and I. M. SINGER: *Index of elliptic operators*, III, Ann. of Math. (2) 87 (1968), 546–604.

Department of Mathematics,
Hokkaido University

(Received Oct. 30, 1970)