A REMARK ON A STAR-SHAPED HYPERSURFACE WITH CONSTANT REDUCED MEAN CURVATURE

By

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The purpose of the present paper is to give another proof of the following proposition proved by A. Aeppli:

Proposition. *A star-shaped hypersurface F (with respect to a fixed point O) in an $(n+1)$-dimensional Euclidean space R^{n+1} which has constant reduced mean curvature rH_1 (with respect to O) is a hypersphere around O, where $r=r(p)$ is the distance between O and $p\in F$, and $H_1=H_1(p)$ is the first mean curvature of F.*

Concerning the problem whether a closed hypersurface with $rH_1=\text{const}$ becomes a hypersphere around O or not, Aeppli gave the affirmative answer in the cases

(1) F is star-shaped (cf [1][1)] Theorem 4 and Footnote 11)),

(2) $rH_1=1$ (cf [2] Proposition 1),

(3) F is simple (i. e., without self-intersections) and O lies in the interior of F (cf [2] Proposition 1″).

The author showed already that under a certain condition (which he called "radial convexity") F cannot possess constant reduced mean curvature (cf [5]). In general, the problem of characterizing the hypersphere around O by constant reduced curvatures is not solved. The method of the proof used in this paper is based on "reflection and sliding" due to A. D. Alexandrov (cf [3] and [4]).

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1. Notations and Lemmas. Let F be a connected simple closed hypersurface of class C^2 in an $(n+1)$-dimensional Euclidean space R^{n+1}, $n\geqq 2$, and O be a fixed point in R^{n+1}. Let F be star-shaped with respect to O, i. e., there exists a bijective differentiable central projection from F to the unit hypersphere around O without critical points. Therefore O lies in the

1) Numbers in brackets refer to the references at the end of the paper.

interior of F, and $\mathfrak{n}(p)\cdot\mathfrak{x}(p)<0$, where $\mathfrak{n}(p)$ is the inner normal of F at p and $\mathfrak{x}(p)$ is the radius vector from O to p, and any ray issued from O meets F only once (cf [1] Footnote 10) and [4] §2). In [2], Aeppli uses the word "star-shaped" in a slightly weaker sense.

Let $H_\nu = H_\nu(p)$ $(p\in F)$ be the ν-th mean curvature $H_\nu = ({}_nC_\nu)^{-1}\sum k_1\cdots k_\nu$, where k_i's are the principal curvatures. The ν-the reduced mean curvature with respect to O is $r^\nu H_\nu$. In [2], Aeppli showed the following two lemmas:

Lemma 1. *(In a local expression of* F: $z=x^{n+1}=x^{n+1}(x^1, \cdots, x^n)$*) (a)* $rH_1 = c$ *(*c*=const) is an everywhere absolutely elliptic (partial) differential equation. (b) If* $k_i>0$ *for all* $i=1, \cdots, n$*, then* $r^\nu H_\nu = c$ *is an absolutely elliptic differential equation.*

Lemma 2. *Let* F_1 *and* F_2 *be two (regular) surfaces in contact at* p *both of which are solutions of the absolutely elliptic (partial) differential equation (of second order)* $\Phi=0$ *(*F_1, F_2 *of class* C^2; $\Phi=\Phi(x, z, z_i, z_{ij})$ *of class* C^0 *in all variables* $x^1, x^2, \cdots, z_{nn}$ *and of class* C^1 *in* $z, z_1, \cdots, z_{nn}$*). Then the intersection of* F_1 *and* F_2 *consists of a set* N *(*$p\in N$*) such that the contact at* p *between* F_1 *and* F_2 *is not semi-proper, or else* F_1 *and* F_2 *coincide in a neighbourhood of* p.

Let R_P and E_t be two mappings from R^{n+1} onto itself defined as follows:

(1) R_P is the reflection with respect to a hyperplane P which passes through O,

(2) $\mathfrak{x}(E_t(q))$ is equal to the radius vector $t\,\mathfrak{x}(q)$ for all $q\in R^{n+1}$, that is, E_t is the t-times homothetical extention with respect to O, where t is a positive number.

It is clear that $r^\nu H_\nu$ is invariant under R_P and E_t for all P and t, if we take always the inner normals.

2. Another proof of the proposition. Let P be a hyperplane which passes through O. For sufficiently large positive number t, $(E_t\circ R_P)(F)\cap F=\phi$ and F lies in the interior of the hypersurface $(E_t\circ R_P)(F)$. Let t decrease until $(E_t\circ R_P)(F)$ touches F for the first time (consider a continuous function $\rho(p)$ on the compact set F, such that $\rho(p)=r(q)/r(p)$, where q is the unique point on $(E_t\circ R_P)(F)$ which lies on the ray issued from O through p, then the point at which ρ takes the minimum is a first common point such that $(E_t\circ R_P)(F)$ touches F). As we always consider inner normals, and because of the star-shapedness of $(E_t\circ R_P)(F)$ and F, the contact of F and $(E_t\circ R_P)(F)$ is positive and proper. Therefore by Lemma 1(a) and Lemma 2, F and $(E_t\circ R_P)(F)$ coincide in a neighbourhood of a point of contact, i. e., the set of the points of contact is open. By continuity it is also closed. Therefore

$F=(E_t\circ R_P)(F)$[2].

Let q be a point in $F\cap P$. Since $R_P(q)=q$, q and $(E_t\circ R_P)(q)$ lie on the same ray issued from O. The star-shapedness of F yields $q=(E_t\circ R_P)(q)$ because of $F=(E_t\circ R_P)(F)$, i.e., $q=E_t(q)$ and t must coincide with 1, and $E_t=E_1=$identity. Therefore $F=R_P(F)$, in other words, F is symmetric with respect to P.

Since we can take P in any direction, F becomes a hypersphere around O. Q.E.D.

The proposition holds good for the ν-th reduced mean curvature $r^\nu H_\nu$, if $k_i>0$ for all $i=1, \cdots, n$, by virtue of Lemma 1(b):

Let F be the same as in Proposition (except $rH_1=c$). If $k_i>0$ for all $i=1, \cdots, n$ and $r^\nu H_\nu=c$ for a fixed ν, then F is a hypersphere around O.

Remark 1. In the proof above we stated that a closed (star-shaped) hypersurface F which is symmetric with respect to any hyperplane through O is a hypersphere around O. For $n=2$ the proof of the anologuos assertion is seen in [3]. For higher dimensions we can verify it without difficulty by induction (it is sufficient if we show that the intersection of F and the hyperplane through O is an $(n\text{-}1)$-sphere around O with constant radius).

Remark 2. The more generalization of the proposition is given by A. D. Alexandrov in [4] §2 (Theorem 2). There he showed first that $c=1$ and concluded that F is a hypersphere.

References

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2) The coincidence of F and $(E_t\circ R_P)(F)$ is also obtained by the method based on the integral formula used in [1].