SOME CHARACTERIZATIONS OF A SUBMANIFOLD WHICH IS ISOMETRIC TO A SPHERE

By

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Introduction. H. Liebmann (1900) [1][1], proved the following.

Theorem 0.1. *The only ovaloids with constant mean curvature H in an Euclidean space E^3 are spheres.*

Extension of this theorem to a convex hypersurface in an n-dimensional Euclidean space E^n has been given by W. Süss (1929) [2], (cf. also [3], p. 118, and [4]). Then H. Hopf (1951) [5], and A. D. Alexandrov (1958) [6], have shown the results that the convexity is not necessary for the validity of Liebmann-Süss theorem.

Recently the analogous problem for closed hypersurfaces in an n-dimensional Einstein space R^n admitting a conformal Killing vector field ξ has been discussed by the present author [8], [9], A. D. Alexandrov [7], K. Nomizu [14], [15]. Afterward T. Nagai, H. Kôjyo and the present author [10], [11], [13] have given a certain extension of this problem to an m-dimensional closed submanifold V^m $(1 \leqq m \leqq n-1)$ in R^n. However we have given there a restriction such that at each point on V^m, the vector ξ lies in the vector space spanned by the tangent space of V^m and the Euler-Schouten vector $\underset{E}{n}$.

In the previous paper [12], the present author has proved the following two theorems except this restriction for a closed orientable submanifold V^m in an orientable Riemannian manifold R^n with constant Riemann curvature.

Theorem 0.2. *If in R^n, there exists such a group G of conformal transformations generated by ξ as ρ is positive (or negative) at each point of V^m and if $\underset{\xi}{H}_1$ is constant, then every point of V^m is umbilic with respect to the normal vector $\underset{\xi}{n}$, where ρ denotes $\underset{\xi}{n}_i \xi^i$, $\underset{\xi}{n}$ is the unit vector normal V^m in the vector space spanned by the tangent space of V^m and ξ and $\underset{\xi}{H}_\nu$ means the ν-th mean curvature of V^m with respect to $\underset{\xi}{n}$.*

Thorem 0. 3. *If in R^n, there exists such a grop G of conformal transformations generated by ξ as ρ is positive (or negative) at each point of V^m,*

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1) Numbers in brackets refer to the references at the end of the paper.

and if the principal curvatures with respect to $\underset{\xi}{n}$, $\underset{\xi}{k_1}, \cdots, \underset{\xi}{k_m}$ *at each point of* V^m, *are positive and* $\underset{\xi}{H_\nu}$ *is constant for any* ν $(1<\nu\leqq m-1)$, *then every point of* V^m *is umbilic with respect to the normal vector* $\underset{\xi}{n}$.

The purpose of this paper is to induce a characteristic property such that V^m is isometric to a sphere from these theorems.

§ 1. Some characterizations of a submanifold to be isometric to a sphere. We suppose an n-dimensional Einstein space R^n $(n\geqq 3)$ of class C^r $(r\geqq 3)$ which has local coordinates x^i and admits a proper conformal Killing vector field ξ^i, that is, ξ^i satisfies an equation:

$$\underset{\xi}{\pounds} g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij} \tag{1.1}$$ ([16], p. 32)

where g_{ij} is the metric tensor of R^n, $\underset{\xi}{\pounds} g_{ij}$ means the Lie derivative of g_{ij} with respect to ξ and the symbol " ; " the operation of D-symbol due to van der Waerden-Bortlotti ([18], p. 254). Then the Lie derivative of the Riemann curvature tensor R^h_{ijk} of R^n with respect to the conformal Killing vector field ξ is given by

$$\underset{\xi}{\pounds} R^h_{ijk} = \delta^h_j \phi_{i;k} - \delta^h_k \phi_{i;j} + g_{ik}\phi^h_{;j} - g_{ij}\phi^h_{;k} \tag{1.2}$$ ([16], p. 160)

where ϕ_i means $\phi_{;i}$ and δ^h_j is the Kronecker delta.

Since R^n is an Einstein s pace, we have

$$R_{ij} = \frac{R}{n} g_{ij} \tag{1.3}$$ $(R=\text{Constant})$

where R_{ij} and R are the Ricci tensor and the curvature scalar of R^n respectively.

Making use of (1.2) and (1.3), after some calculations we obtain the following result

$$\phi_{i;j} = -\frac{R}{n(n-1)} \phi g_{ij}. \tag{1.4}$$

From (1.4) we can see that the Einstein space R^n admitting the conformal Killing vector field ξ with the non-constant scalar field ϕ must always admit the vector field ϕ^i $(\phi^i = g^{ij}\phi_j)$, which is the special conformal Killing vector field.

In R^n, we consider a domain M. If the domain M is simply covered by the orbits of transformations generated by ϕ^i, and ϕ^i is everywhere of class C^r and $\neq 0$ in M; then we call M a regular domain with respect to the vector field ϕ^i.

Let us denote by V^m an m-dimensional closed orientable submanifold of class C^3 imbedded in a regular domain M with respect to the vector field ϕ^i, locally given by

$$x^i = x^i(u^\alpha) \qquad i=1, \cdots, n\,;\ \alpha=1, \cdots, m \tag{1.5}$$

where u^α are local coordinates of V^m. Throghout the present paper Latin indices run from 1 to n and Greek indices from 1 to m. We assume that at any point on V^m the vector ϕ^i is not on its tangent space.

We shall indicate by $\underset{p}{n}^i$ $(p=m+1, \cdots, n)$ the contravariont unit vector normal V^m and suppose that they are mutually orthogonal. Let $\underset{\phi}{n}$ be in the vector space spanned by $m+1$ independent vector $\frac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1, \cdots, m)$ and ϕ^i and be the unit vector normal V^m. Then we may consider $\underset{\phi}{n}$ as one of the unit normal vectors of V^m, that is, $\underset{m+1}{n}^i = \underset{\phi}{n}^i$.

If we denote by $\underset{\phi}{k}_1, \cdots, \underset{\phi}{k}_m$ the principal curvatures of V^m with respect to $\underset{\phi}{n}^i$ and by $\underset{\phi}{H}_\nu$ the ν-th mean curvature, the υ-th elementary symmetric function of $\underset{\phi}{k}_1, \cdots, \underset{\phi}{k}_m$ divided by the number of terms, *i.e.*,

$$\binom{m}{\upsilon}\underset{\phi}{H}_\nu = \sum_{\alpha_1<\cdots<\alpha_\nu} \underset{\phi}{k}_{\alpha_1}\cdots \underset{\phi}{k}_{\alpha_\nu} \qquad (1\leqq\upsilon\leqq m),$$

then as a special case of Theorem 0.2. and of Theorem 0.3., we have the following two theorems.

Theorem 1.1. *Let ξ^i be a praper conformal Killing vector field in R^n with constant Riemann curvature* $(i, e., \xi_{i;j}+\xi_{j;i}=2\phi g_{ij})$. *Then every point of a closed orientable submanifold V^m such that H_1=constant and $\phi_i \underset{\phi}{n}^i>0$ (or<0) for all points of V^m is umbilic* with respect to $\underset{\phi}{n}$.

Remark. We remark here that $\phi_i \underset{\phi}{n}^i>0$ (or<0) for all points of V^m is equivalent that the vector ϕ^i is not on the tangent space of V^m at any point fo V^m.

Theorem 1.2. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature* $(i.e., \xi_{i;j}+\xi_{j;i}=2\phi g_{ij})$. *Then every point of a closed orientable submanifold V^m such that $\underset{\phi}{H}_\nu$=constant, $\underset{\phi}{k}_1>0, \cdots, \underset{\phi}{k}_m>0$ and $\phi_i \underset{\phi}{n}^i>0$ (or<0) for all points of V^m is umbilic* with respect to $\underset{\phi}{n}$.

From the above theorems and the following theorem due to M. Obata [19], we obtain Theorem 1.3 and Theorem 1.4.

Theorem (*Obata*). *Let R^n $(n \geqq 2)$ be a complete Riemann manifold which admits a non-null function φ such that $\varphi_{;i;j} = -C^2 \varphi g_{ij}$ $(C = constant)$. Then R^n is isometric to a sphere of radius $1/C$.*

Theorem 1.3. *If ξ^i is a proper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$), then a closed orientable submanifold V^m such that $\underset{\phi}{H_1} = constant$ and $\phi_i \underset{\phi}{n}^i > 0$ $(or < 0)$ for all points of V^m is isometric to a sphere.*

Proof. Let $C^i{}_j$ be $\sum_{p=m+1}^{n} \underset{p}{n}^i \underset{p}{n}_j$ $(\underset{m+1}{n} = \underset{\phi}{n})$ and $\underset{\lambda}{i}$ $(\lambda = 1, \cdots, m)$ mutually orthogonal unit tangent vectors of V^m. Then putting $B^i{}_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$, we have

$$\underset{p}{n}^i{}_{;\alpha} = C^i{}_{j;k} \underset{p}{n}^j B^k{}_\alpha = -\sum_{\lambda=1}^{m} (\underset{\lambda}{i}_{j;k} \underset{p}{n}^j B^k{}_\alpha) \underset{\lambda}{i}^i .$$

Therefore we may write as follows

$$\underset{p}{n}^i{}_{;\alpha} = \underset{p}{\gamma}_\alpha{}^\gamma B^i{}_\gamma .$$

Since we have

$$g_{ij} B^i_{\beta;\alpha} \underset{p}{n}^j = -g_{ij} B^i_\beta \underset{p}{n}^j{}_{;\alpha} ,$$

we obtain

$$\underset{p}{n}^i{}_{;\alpha} = -\underset{p}{b}_\alpha{}^\gamma B^i_\gamma \qquad (p = \phi,\ m+2, \cdots, n) \tag{1.6}$$

where $\underset{p}{b}_\alpha^\gamma$ means $g^{\gamma\beta} \underset{p}{b}_{\alpha\beta}$ and $\underset{p}{b}_{\alpha\beta} \overset{\text{def.}}{=} B^i_{\alpha;\beta} \underset{p}{n}_i$, and $g^{\gamma\beta}$ is the contravariant metric tensor of V^m.

As well-known on the theory of a submanifold in a Riemannian manifold, we have

$$B^i_{\alpha;\beta} = \frac{\partial B^i_\alpha}{\partial u^\beta} + \Gamma^i{}_{jk} B^j_\alpha B^k_\beta - \Gamma'^\gamma_{\alpha\beta} B^i_\gamma ,$$

$$\underset{p}{n}^i{}_{;\alpha} = \frac{\partial \underset{p}{n}^i}{\partial u^\alpha} + \Gamma^i{}_{jk} \underset{p}{n}^j B^k_\alpha - \Gamma''^Q_{p\alpha} \underset{Q}{n}^i ,$$

where Γ^i_{jk} and $\Gamma'^\gamma_{\alpha\beta}$ are the Christoffel symbols with respect to g_{ij} and $g_{\alpha\beta}$ respectively, and

$$\Gamma''^Q_{p\alpha} \overset{\text{def.}}{=} \frac{\partial \underset{p}{n}^i}{\partial u^\alpha} \underset{Q}{n}_i + \Gamma^i{}_{jk} \underset{p}{n}^j B^k_\alpha \underset{Q}{n}_i .$$

Since $\underset{p}{n}^i \underset{Q}{n}_i = \delta_{PQ}$ (δ_{PQ} means the Kronecker delta, $\Gamma''^Q_{P\alpha}$ is anti-symmetric with respect to the indices P and Q. Consequently we have

$$\Gamma''^{P}_{P\alpha} = 0 \qquad \text{for any } P \quad (P=\phi,\ m+2,\cdots,n). \tag{1.7}$$

We now consider the scalar function on V^m given by

$$\varphi = \phi_i \underset{\phi}{n}^i ,$$

and we assume, moreover, that $\varphi \neq$ constant along V^m, then we have

$$\varphi_\alpha = \phi_{i;j} \underset{\phi}{n}^i B^j_\alpha + \phi_i (\underset{\phi}{n}^i{}_{;\alpha} + \Gamma''^{P}_{\phi\alpha} \underset{P}{n}^i).$$

On the other hand the vector ϕ_i is expressed as follows

$$\phi_i = \varphi \underset{\phi}{n}_i + C_\gamma B^\gamma{}_i , \tag{1.8}$$

where $B_i^\gamma \overset{\text{def.}}{=} g_{ji} B^j{}_\beta g^{\beta\gamma}$.

From (1.7) and (1.8), we have

$$\phi_i \Gamma''^{P}_{\phi\alpha} \underset{P}{n}^i = 0 .$$

Since R^n with constant Riemann curvature is an Einstein space, we get

$$\phi_{i;j} \underset{\phi}{n}^i B_\alpha{}^j = -\frac{R}{n(n-1)} \phi g_{ij} \underset{\phi}{n}^i B^j_\alpha = 0 ,$$

and also from (1.6), we have

$$\varphi_\alpha = -\underset{\phi}{b}_\alpha{}^\gamma B^i_\gamma \phi_i .$$

Furthermore by virtue of Theorem 1.1, every point of V^m is umbilic with respect to $\underset{\phi}{n}^i$, that is,

$$\underset{\phi}{b}_{\alpha\beta} = \underset{\phi}{H}_1 g_{\alpha\beta} . \tag{1.9}$$

So, it follows that

$$\underset{\phi}{b}_\alpha{}^\beta = \underset{\phi}{H}_1 \delta_\alpha{}^\beta .$$

Thus we obtain

$$\varphi_\alpha = -\underset{\phi}{H}_1 B^i_\alpha \phi_i$$

and

$$\varphi = -\underset{\phi}{H}_1 \phi + C \qquad (C = \text{Constant}). \tag{1.10}$$

Since $\underset{\phi}{H}_1 = \text{constant}$, we get

(1.11) $$\varphi_{\alpha;\beta}=-\underset{\phi}{H_1}(\phi_{i;j}B^i_\alpha B^j_\beta+\phi_i B^i_{\alpha;\beta}).$$

Consequently from (1.4) and (1.8), because of the relation $B_i^{\ \gamma}B^i_{\alpha;\beta}=0$, we have

$$\varphi_{\alpha;\beta}=-\underset{\phi}{H_1}\left(-\frac{R}{n(n-1)}\phi g_{\alpha\beta}+\varphi \underset{\phi}{b_{\alpha\beta}}\right),$$

and from (1.9) and (1.10), (1.11) becomes as follows

(1.12) $$\varphi_{\alpha;\beta}=\left\{-\left(\underset{\phi}{H_1^2}+\frac{R}{n(n-1)}\right)\varphi+\frac{R}{n(n-1)}C\right\}g_{\alpha\beta}$$

If $\underset{\varphi}{H_1^2}+\frac{R}{n(n-1)}=0$, then (1.12) becomes

$$\varphi_{\alpha;\beta}=\frac{R}{n(n-1)}Cg_{\alpha\beta},$$

from which $\Delta\varphi=\frac{mR}{n(n-1)}C$ (Δ is the Laplacian operator on V^m), which is impossible unless $\varphi=$constant. Therefore $\underset{\phi}{H_1^2}+\frac{R}{n(n-1)}\neq 0$, then we have

$$\varphi_{\alpha;\beta}=-\left(\underset{\phi}{H_1^2}+\frac{R}{n(n-1)}\right)\left(\varphi-\frac{\frac{R}{n(n-1)}C}{\underset{\phi}{H_1^2}+\frac{R}{n(n-1)}}\right)g_{\alpha\beta}.$$

putting

$$\sigma=\varphi-\frac{\frac{R}{n(n-1)}C}{\underset{\phi}{H_1^2}+\frac{R}{n(n-1)}},$$

we obtain

$$\sigma_{\alpha;\beta}=-\left(\underset{\phi}{H_1^2}+\frac{R}{n(n-1)}\right)\sigma g_{\alpha\beta}$$

and

$$\Delta\sigma=-m\left(\underset{\phi}{H_1^2}+\frac{R}{n(n-1)}\right)\sigma,$$

consequently it follows that $H_1^2+\frac{R}{n(n-1)}>0$ [17]. Thus V^m is isometric to a sphere according to Obata's theorem.

Theorem 1.4. *If ξ^i is a proper conformal Killing vector field in R^n with constant Riemann curvature (i.e., $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$), then a closed orientable submanifold V^m such that $\underset{\phi}{H}_\nu=$ const. for any υ $(1<\upsilon\leqq m-1)$, $\underset{\phi}{k}_1>0,\cdots,\underset{\phi}{k}_m>0$ and $\phi_i\underset{\phi}{n}^i>0$ (or <0) for all points of V^m is isometric to a sphere.*

Proof. From Theorem 1.2, every point of V^m is umbilic with respect to $\underset{\phi}{n}^i$, that is,

$$\underset{\phi}{k}_1=\cdots=\underset{\phi}{k}_m .$$

Accordingly we have

$$\underset{\phi}{H}_\nu=(\underset{\phi}{k}_1)^\nu=(\underset{\phi}{H}_1)^\nu=\text{const.}$$

Thus we get

$$\underset{\phi}{H}_1=\text{const.}$$

Consquently by virtue of Theorem 1.3, V^m is isometric to a sphere.

§ 2. Some specializations of the preceding theorems. In this section, we shall discuss on some special cases of Theorem 1.3 and Theorem 1.4.

Let R^n be a a Riemannian manifold with constant Riemann curvature and a vector field ξ^i be a vector field given by a concircular scalar field η, that is ξ^i satisfies an equation

$$\xi_{i;j}=\phi g_{ij}$$

where ϕ is not constant, we call such the vector field ξ^i a proper concircular Killing vector field. Then we have

$$\xi_{i;j;k}=\phi_k g_{ij}$$

and

$$\xi_{i;j;k}-\xi_{i;k;j}=\phi_k g_{ij}-\phi_j g_{ik} .$$

On the other hand, by means of Ricci identity it follows that

$$\xi_{i;j;k}-\xi_{i;k;j}=-R^l_{ijk}\xi^l .$$

Since R^n is of constant Rimann curvature, we obtain

$$R_{lijk}=\kappa(g_{lj}g_{ik}-g_{lk}g_{ij})$$

and so we have

$$R_{lijk}\xi^l=\phi_j g_{ik}-\phi_k g_{ij} ,$$

$$\kappa(\xi_j g_{ik} - \xi_k g_{ij}) = \phi_j g_{ik} - \phi_k g_{ij}$$

and

$$\kappa \xi_j = \phi_j \,. \tag{2.1}$$

In this case we can see easily that the unit normal vector $\underset{\xi}{n}^i$ with respect to ξ^i is coincide with the unit normal vector $\underset{\phi}{n}^i$ with respect to ϕ^i, that is,

$$\underset{\xi}{n}^i = \underset{\phi}{n}^i \tag{2.2}$$

Hence we have the following relations

$$\underset{\xi}{k}_\nu = \underset{\phi}{k}_\nu, \qquad \underset{\xi}{H}_\nu = \underset{\phi}{H}_\nu$$

for any υ $(1 \leqq \upsilon \leqq m)$. Consequently we can give

Theorem 2.1. *If ξ^i is a proper concircular Killing vector field in R^n with constant Riemann curvature, then a closed orientable submanifold V^m such that $\underset{\xi}{H}_1 = const.$ and $\xi_i \underset{\xi}{n}^i > 0$ (or < 0) for all points of V^m is isometric to a sphere.*

Theorem 2.2. *If ξ^i is a proper concircular Killing vector field in R^n with constant Riemann curvature, then a closed orientable submanifold V^m such that $\underset{\xi}{H}_\nu = const.$ for any υ $(1 < \upsilon \leqq m-1)$, $\underset{\xi}{k}_1 > 0, \cdots, \underset{\xi}{k}_m > 0$ and $\xi_i \underset{\xi}{n}^i > 0$ (or < 0) for all points of V^m is isometric to a sphere.*

If ξ^i is a proper conformal Killing vector field which satisfies $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ in R^n with constant Riemann curvature and if we put a restriction such that at each point on V^m the vector ϕ^i lies in the vector space spanned by the tangent space of V^m and the Eular-Schouten unit vector $\underset{E}{n}^i$ of V^m, then we find that the unit normal vector $\underset{\phi}{n}^i$ with respect to ϕ^i is coincide with the Euler-Schouten unit vector $\underset{E}{n}^i$, that is,

$$\underset{\phi}{n}^i = \underset{E}{n}^i \,, \tag{2.3}$$

Consequently we have

$$\underset{\phi}{k}_\nu = \underset{E}{k}_\nu \,, \qquad \underset{\phi}{H}_\nu = \underset{E}{H}_\nu$$

for any υ $(1 \leqq \upsilon \leqq m)$, where $\underset{E}{H}_1$ means the first mean curvature H_1 of V^m. Therefore we have

Theorem 2.3. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ in R^n with constant Riemann curvature and V^m a closed* *orientable submanifold such that*

(i) H_1 — *constant,*

(ii) $\phi_i \underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ϕ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

Theorem 2.4. *Let* ξ^i *be a proper conformal Killing vector field such that* $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ *in* R^n *with constant Riemann curvature and* V^m *a closed orientable submanifold such that*

(i) $\underset{E}{H_\nu}=$ *constant for any* ν $(1<\nu\leqq m-1)$,

(ii) $\underset{E}{k_1}>0, \cdots, \underset{E}{k_m}>0$ *and* $\phi_i \underset{E}{n^i}>0$ (*or* <0) *for all points of* V^m,

(iii) ϕ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

From these four theorems, we have the following corollaries.

Corollary 2.1. *Let* ξ^i *be a proper concircular Killing vector field in* R^n *with constant Riemann curvature and* V^m *a closed orientable submanifold such that*

(i) $H_1=$ *constant,*

(ii) $\xi_i \underset{E}{n^i}$ *has fixed sign on* V^m,

(iii) ξ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

Corollary 2.2. *Let* ξ^i *be a proper concircular Killing vector field in* R^n *with constant Riemann curvature and* V^m *a closed orientable submanifold such that*

(i) $\underset{E}{H_\nu}=$ *constant for any* ν $(1<\nu\leqq m-1)$,

(ii) $\underset{E}{k_1}>0, \cdots, \underset{E}{k}>0$ *and* $\xi_i \underset{E}{n^i}>0$ (*or* <0) *for all poinis of* V^m,

(iii) ξ^i *lies in the vector space spanned by the tangent space of* V^m *and* $\underset{E}{n^i}$ *at each point of* V^m.

Then V^m *is isometric to a sphere.*

§ 3. Characteristic properties of a hypersurface to be isometric to

a sphere In this section, we consider a special case such that $m=n-1$, that is, V^m is a hypersurface V^{n-1}.

If n^i, k_ν $(\nu=1, \cdots, n-1)$ and H_ν are the normal unit vector, the principal curvatures and the ν-th mean curvature of V^{n-1} respectively, then we can easily find the following relations

$$\underset{\xi}{n^i} = \underset{\phi}{n^i} = \underset{E}{n^i} = n^i\,,$$
$$\underset{\xi}{k_\nu} = \underset{\phi}{k_\nu} = \underset{E}{k_\nu} = k_\nu\,,$$
$$\underset{\xi}{H_\nu} = \underset{\phi}{H_\nu} = \underset{E}{H_\nu} = H_\nu$$

for any ν $(1 \leqq \nu \leqq m-1)$. Therefore we have

Theorem 3.1. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

(i) $H_1=$ *constant,*

(ii) $\phi_i n^i$ *has fixed sign on* V^{n-1}.

Then V^{n-1} is isometric to a sphere.

Theorem 3.2. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

(i) $H_\nu=$ *constant for any* ν $(1<\nu\leqq n-2)$,

(ii) $k_1>0, \cdots, k_{n-1}>0$ *and* $\phi_i n^i>0$ (*or* <0) *for all points of* V^{n-1}.

Then V^{n-1} is isometric to a sphere.

Moreover from Theorem 3.1 and Theorem 3.2 in the previous paper [8] given by the present author, making use of the Gauss equation for a hypersurface in R^n with constant Riemann curvature, we also have without any difficulty the following theorems.

Theorem 3.3. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that $H_1=$constant and $\xi_i n^i$ has fixed sign on V^{n-1}. Then V^{n-1} is isometric to a shpere.*

Theorem 3.4. *Let ξ^i be a proper conformal Killing vector field in R^n with constant Riemann curvature and V^{n-1} a closed orientable hypersurface such that*

(i) $H_\nu=$ *constant for any* ν $(1<\nu\leqq n-2)$

(ii) $k_1>0, \cdots, k_{n-1}>0$ *and* $\xi_i n^i>0$ (*or* <0) *for all points of* V^{n-1}.

Then V^{n-1} is isometric to a sphere.

In an Einstein space R^n, the present author gave already the following theorem ([9], p. 170).

Theorem (*Katsurada*). *Let R^n be an Einstein space which admits a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ and V^{n-1} a closed orientable hypersurface in R^n such that $H_1=$ constant and $\xi_i n^i>0$ (or <0) for all points of V^{n-1}. Then every point of V^{n-1} is umbilic.*

From the above theorem, we then obtain the following two theorems.

Theorem 3.5. *Let ξ^i be a proper conformal Killing vector field such that $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ in an Einstein space R^n and V^{n-1} a closed orientable hypersurface such that $H_1=$constant and $\phi_i n^i>0$ (or <0) for all points of V^{n-1}. Then V^{n-1} is isometric to r sphere.*

Theorem 3.6. *Let ξ^i be a proper conformal Killing vector field in an Einstein space R^n and V^{n-1} a closed orientable hypersurface such that $H_1=$ constant and $\xi_i n^i>0$ (or <0) for all points of V^{n-1}. Then V^{n-1} is isometric to a sphere.*

These theorems are proved almost similarly to the proof of Theorem 1.3.

References

[1] H. Liebmann: *Ueber die Verbiegung der geschlossenen Flächen positive Krümmung*, Math. Ann. 53 (1900), 91–112.

[2] W. Süss: *Zur relativen Differentialgeometrie V.*, Tôhoku Math. J., 30 (1929), 202–209.

[3] T. Bonnesen and W. Fenchel: *Theorie der konvexen Körper*, (Springer, Berlin 1934).

[4] C.C. Hsiung: *Some integral formulas for closed hypersurfaces*, Marth. Scand. 2 (1954), 286–294).

[5] H. Hopf: *Unber Flächen mit einer Relation zwischen der Hauptkrümmugen*, Math. Nachr. 4 (1951), 232–249.

[6] A. D. Alexandrov: *Uniqueness theorems for surfaces in the large*, V. Vestnik Leningrad University 13 (1958), 5–8. (Russian, with English summary).

[7] A. D.Alexandrov: *A characteristic property of spheres*, Ann. di Mat. p. appl., 58 (1962), 303–315.

[8] Y. Katsurada: *Generalized Minkowski formulas for closed hypersurfaces in Riemann space*, Ann. di Mat. p. appl., 57 (1962), 283–293.

[9] Y. Katsurada: *On a certain property of closed hypersurfaces in an Einstein space*, Comment. Math. Helv., 38 (1964), 165–171.

[10] Y. Katsurada and T. Nagai: *On some properties of a submanifold with constant mean curvature in a Riemann space*, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 79–89.

[11] Y. Katsurada and H. Kôjyo: *Some integral formulas for closed submanifolds in a Riemann space*, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 90-100.

[12] Y. Katsurada: *Closed submanifolds with constant ν-th maen curvature related with a vector field in a Riemann manifold*, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 171-181.

[13] T. Nagai: *On certain conditions for a submanifold in a Riemann space to be isometric to a sphere*, Jour. Fac. Sci. Hokkaido Univ. 20 (1968), 135-159.

[14] K. Nomizu: *I. Hypersurfaces with constant mean curvature in S^n*, to appear.

[15] K. Nomizu: *II. Surfaces with constant mean curvature in S^3*, to appear.

[16] K. Yano: *The theory of Lie derivative and its applications*, (North-Holland, Amsterdam, 1957).

[17] K. Yano: *Differentia geometry on complex and almost complex space*, (Pergamon, 49, 1965).

[18] J. A. Schouten: *Ricci-Calculas*, (Second edition) (Springer, Berlin 1954).

[19] M.Obata: *Certain conditons for a Riemannian manifold to be isometric with a sphere*, Jour. Math. Soc. Japan, 14 (1962), 333-340.

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(Received Sep. 2, 1969)