SOME INTEGRAL FORMULAS FOR CLOSED SUBMANIFOLDS IN A RIEMANN SPACE

By

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Introduction. Let F be an ovaloid in 3-dimensional Euclidean space E^3. If we denote by H and K the mean curvature and the Gauss curvature at a point P of F respectively, then as well-known formula of Minkowski we have

$$\iint_F (Kp+H)dA = 0 ,$$

where p denotes the oriented distance from a fixed point O in E^3 to the tangent space of F at P and dA is the area element of F at P. The generalization of this formula for a closed orientable hypersurface in an n-dimensional Euclidean space E^n was established by C. C. Hsiung [1][1)]. Recently, Y. Katsurada [2] generalized this problem in an n-dimensional Riemann space R^n and gave the generalized Minkowski formulas for a closed orientable hypersurface in R^n.

The purpose of the present paper is to investigate the analogous problem for an m-dimensional submanifold V^m in an n-dimensional Riemann space R^n. In §**1**, the generalized Minkowski formulas for V^m in R^n are expressed. The Minkowski formulas concerning some special transformations of R^n are given in §**2**. Making use of those integral formulas in §**1** and §**2**, we shall show in §**3** some properties of closed orientable submanifolds in a Riemann space with constant Riemann curvature.

§ 1. Generalized Minkowski formulas for a submanifold. We consider an n-dimensional Riemann space R^n ($n \geqq 3$) of class C^r ($r \geqq 3$) which admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau , \tag{1. 1}$$

where x^i are local coordinates in R^n and ξ^i are the components of a contravariant vector ξ. If the vector ξ is a Killing vector, a homothetic Killing, a conformal Killing etc. ([3], p. 32), then the group G is called isometric,

1) Numbers in brackets refer to the references at the end of the paper.

homothetic, conformal etc. respectively.

Let us denote by V^m an m-dimensional closed orientable submanifold of class C^3 imbedded in R^n, locally given by

$$x^i = x^i(u^\alpha)^{2)}.$$

We suppose that the paths of transformations belonging to G cover R^n simply and the submanifold V^m does not pass through any singular point of a tangent vector field of the paths.

The first fundamental tensor $g_{\alpha\beta}$ of V^m is given by

$$g_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}$$

and $g^{\alpha\beta}$ are defined by $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$, where g_{ij} denotes the first fundamental tensor of R^n.

We shall indicate by $\underset{P}{n}{}^i$ $(P = m+1, m+2, \cdots, n)$ the contravariant unit vectors normal to V^m and suppose that they are mutually orthogonal.

Let us consider a differential form of $m-1$ degree at a point of V^m, defined by

$$\begin{aligned}(1.2)\qquad ((\underset{m+1}{n}, \cdots, \underset{n}{n}, \xi, \underbrace{dx, \cdots, dx}_{m-1})) &\overset{\text{def.}}{=} \sqrt{g}\,(\underset{m+1}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx) \\ &= \sqrt{g}\left(\underset{m+1}{n}, \cdots, \underset{n}{n}, \xi, \frac{\partial x}{\partial u^{\alpha_1}}, \cdots, \frac{\partial x}{\partial u^{\alpha_{m-1}}}\right) du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_{m-1}},\end{aligned}$$

where dx^i is a displacement along the submanifold V^m, i.e., $dx^i = \dfrac{\partial x^i}{\partial u^\alpha} du^\alpha$ and g denotes the determinant of the metric tensor g_{ij} of R^n. Then the exterior differential of the differential form (1.2) divided by $m!$ becomes as follows:

$$\begin{aligned}\frac{1}{m!} d((\underset{m+1}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) &= \frac{1}{m!}((\delta \underset{m+1}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) \\ (1.3)\qquad &+ \frac{1}{m!}((\underset{m+1}{n}, \delta\underset{m+2}{n}, \underset{m+3}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) + \cdots \\ &+ \frac{1}{m!}((\underset{m+1}{n}, \cdots, \underset{n-1}{n}, \delta \underset{n}{n}, \xi, dx, \cdots, dx)) + \frac{1}{m!}((\underset{m+1}{n}, \cdots, \underset{n}{n}, \delta\xi, dx, \cdots, dx)),\end{aligned}$$

where δv means $v_{;\alpha} du^\alpha$. Denoting by " ; " the operation of D-symbol due to van der Waerden-Bortolotti ([4] p. 254), we have

2) Throughout the present paper the Latin indices run from 1 to n and the Greek indices from 1 to m $(m \leqq n-1)$.

$$\underset{P}{n}^i{}_{;\alpha} = C^i_{j;k}\underset{P}{n}^j \frac{\partial x^k}{\partial u^\alpha}$$
$$= -\sum_{\lambda=1}^{m}\left(\underset{\lambda}{i}_{j;k}\underset{P}{n}^j \frac{\partial x^k}{\partial u^\alpha}\right)\underset{\lambda}{i}^i ,$$

where $C^i_j = \sum_{P=m+1}^{n} \underset{P}{n}^i \underset{P}{n}_j$ and $\underset{\lambda}{i}$ $(\lambda=1,2,\cdots,m)$ are mutually orthogonal unit tangent vectors of V^m. Then we may put

$$\underset{P}{n}^i{}_{;\alpha} = \underset{P}{\gamma}^\gamma_\alpha \frac{\partial x^i}{\partial u^\gamma} .$$

Since we have

$$g_{ij}\left(\frac{\partial x^i}{\partial u^\delta}\right)_{;\alpha}\underset{P}{n}^j = -g_{ij}\frac{\partial x^i}{\partial u^\delta}\underset{P}{n}^j{}_{;\alpha} ,$$

we obtain

$$\underset{P}{n}^i{}_{;\alpha} = -\underset{P}{b}_\alpha{}^\gamma \frac{\partial x^i}{\partial u^\gamma} \qquad (P=m+1,\ m+2, \cdots, n),$$

where $\underset{P}{b}_\alpha{}^\gamma = g^{\beta\gamma}\underset{P}{b}_{\alpha\beta}$ and $\underset{P}{b}_{\alpha\beta} = \left(\frac{\partial x^i}{\partial u^\alpha}\right)_{;\beta}\underset{P}{n}_i$. Then we have

$$\delta \underset{P}{n}^i = -\underset{P}{b}_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma}du^\alpha \qquad (P=m+1,\ m+2, \cdots, n). \tag{1.4}$$

By means of (1.4), we get

$$\frac{1}{m!}((\delta \underset{m+1}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) = (-1)^{(n-m)(n-1)}\underset{m+1}{H_1}\underset{m+1}{n_i}\xi^i dA ,$$

where dA is the area element of V^m and $\underset{m+1}{H_1}$ is the first mean curvature of V^m for the normal direction $\underset{m+1}{n}^i$. Similarly, for every integer P satisfying $m+1 \leqq P \leqq n$ we have

$$\frac{1}{m!}((\underset{m+1}{n}, \cdots, \delta \underset{P}{n}, \cdots, n, \xi, dx, \cdots, dx)) = (-1)^{(n-m)(n-1)}\underset{P}{H_1}\underset{P}{n_i}\xi^i dA , \tag{1.5}$$

where $\underset{P}{H_1}$ is the first mean curvature of V^m for the normal direction $\underset{P}{n}^i$. By means of (1.5) it follows that

$$\begin{aligned} &\frac{1}{m!}((\delta \underset{m+1}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) + \cdots + \frac{1}{m!}((\underset{m+1}{n}, \cdots, \delta \underset{n}{n}, \xi, dx, \cdots, dx)) \\ &\quad = (-1)^{(n-m)(n-1)}\left(\sum_{P=m+1}^{n}\underset{P}{H_1}\underset{P}{n_i}\right)\xi^i dA . \end{aligned} \tag{1.6}$$

Let $\underset{E}{n}^i$ be Euler-Schouten unit vector, that is, the unit vector of the same direction to the vector $g^{\alpha\beta}\left(\frac{\partial x^i}{\partial u^\alpha}\right)_{;\beta}$. Then we have

$$\underset{E}{n}^i = \sum_{P=m+1}^{n} \cos\underset{P}{\theta}\cdot \underset{P}{n}^i , \tag{1.7}$$

where $\underset{P}{\theta}$ is the angle between $\underset{E}{n}$ and $\underset{P}{n}$. Denoting by H_1 the mean curvature of V^m, we have

$$H_1 = \frac{1}{m} g^{\alpha\beta}\left(\frac{\partial x^i}{\partial u^\alpha}\right)_{;\beta}\underset{E}{n}_i . \tag{1.8}$$

Since we have

$$\underset{P}{H}_1 = \frac{1}{m} g^{\alpha\beta}\left(\frac{\partial x^i}{\partial u^\alpha}\right)_{;\beta}\underset{P}{n}_i \qquad (P=m+1,\ m+2, \cdots, n) \tag{1.9}$$

by means of (1.7), (1.8) and (1.9), it follows that

$$\underset{P}{H}_1 = H_1 \cos\underset{P}{\theta} \qquad (P=m+1,\ m+2, \cdots, n).$$

Then (1.6) is rewritten as follows:

$$\begin{aligned}&\frac{1}{m!}((\delta\underset{m+1}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) + \cdots + \frac{1}{m!}((\underset{m+1}{n}, \cdots, \delta\underset{n}{n}, \xi, dx, \cdots, dx))\\ &= (-1)^{(n-m)(n-1)} H_1 \underset{E}{n}_i \xi^i dA .\end{aligned} \tag{1.10}$$

On the other hand, we have

$$\begin{aligned}&\frac{1}{m!}((\underset{m+1}{n}, \cdots, \underset{n}{n}, \delta\xi, dx, \cdots, dx))\\ &= (-1)^{(n-m)(n-1)} \frac{1}{2m} (\underset{\xi}{\mathfrak{L}} g_{ij}) \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA ,\end{aligned} \tag{1.11}$$

where $\underset{\xi}{\mathfrak{L}} g_{ij}$ is the Lie derivative of g_{ij} with respect to the infinitesimal transformation (1. 1) ([3], p. 5).

By virtue of (1.3), (1.10) and (1.11), it follows that

$$\begin{aligned}&\frac{1}{m!} d((\underset{m+1}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx))\\ &= (-1)^{(n-m)(n-1)} \left\{ H_1 \underset{E}{n}_i \xi^i dA + \frac{1}{2m}(\underset{\xi}{\mathfrak{L}} g_{ij}) \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA \right\} .\end{aligned} \tag{1.12}$$

Integrating both sides of (1.12) over the whole submanifold and applying Stokes' theorem, we obtain

$$\frac{1}{m!}\int_{\partial V^m}\!\!\cdots\!\int((\underset{m+1}{n},\cdots,\underset{n}{n},\xi,dx,\cdots,dx))$$

$$=(-1)^{(n-m)(n-1)}\left\{\int_{V^m}\!\!\cdots\!\int H_1\underset{E}{n_i}\xi^i dA+\frac{1}{2m}\int_{V^m}\!\!\cdots\!\int g^{*ij}\underset{\xi}{\pounds}g_{ij}dA\right\},$$

where ∂V^m means the boundary of V^m and $g^{*ij}=g^{\alpha\beta}\dfrac{\partial x^i}{\partial u^\alpha}\dfrac{\partial x^j}{\partial u^\beta}$. Making use of the fact that V^m is closed, we have

$$(\mathrm{I}')\qquad \int_{V^m}\!\!\cdots\!\int H_1\underset{E}{n_i}\xi^i dA+\frac{1}{2m}\int_{V^m}\!\!\cdots\!\int g^{*ij}\underset{\xi}{\pounds}g_{ij}dA=0\,.$$

Now, we shall consider $\underset{E}{n^i}$ as one of the unit normal vectors of V^m, that is, $\underset{m+1}{n^i}=\underset{E}{n^i}$ and assume that at each point on V^m the contravariant vector ξ^i is contained in the vector space spanned by $m+1$ independent vectors $\dfrac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1,2,\cdots,m)$ and $\underset{E}{n^i}$. This assumption for ξ is evidently satisfied for the case $m=n-1$, that is, V^m is a hypersurface in R^n. Then we may put

$$(1.13)\qquad \xi^i=\varphi^\gamma\frac{\partial x^i}{\partial u^\gamma}+\rho\underset{E}{n^i}\,.$$

Now, we suppose that R^n is a constant Riemann curvature space and consider the following differential form of $m-1$ degree:

$$(1.14)\qquad \begin{aligned}&((\underset{E}{n},\underset{m+2}{n},\cdots,\underset{n}{n},\xi,\underbrace{\delta\underset{E}{n},\cdots,\delta\underset{E}{n}}_{\nu},\underbrace{dx,\cdots,dx}_{m-\nu-1}))\\ &\overset{\text{def.}}{=}\sqrt{g}\,(\underset{E}{n},\underset{m+2}{n},\cdots,\underset{n}{n},\xi,\delta\underset{E}{n},\cdots,\delta\underset{E}{n},dx,\cdots,dx)\,,\end{aligned}$$

for a fixed integer ν satisfying $1\leqq\nu\leqq m-1$.

As well-known, a submanifold V^m in R^n has the following property:

$$\underset{E}{b_{\alpha\delta;\beta}}-\underset{E}{b_{\alpha\beta;\delta}}=-R_{ijkl}\underset{E}{n^i}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\delta}\frac{\partial x^l}{\partial u^\beta}\qquad ([4],\ \text{p. } 266),$$

where R_{ijkl} is the curvature tensor of R^n. Then for R^n with constant Riemann curvature we have

$$((\underset{E}{n},\underset{m+2}{n},\cdots,\underset{n}{n},\xi,\delta(\delta\underset{E}{n}),\delta\underset{E}{n},\cdots,\delta\underset{E}{n},dx,\cdots,dx))=0.$$

Accordingly, by exterior differentiation of the differential form (1.14) we have

$$
\begin{aligned}
&d((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx))\\
&\quad = ((\underset{E}{\delta n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx))\\
&\qquad + ((\underset{E}{n}, \delta \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx))\\
&\qquad + \cdots + ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \delta \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx))\\
&\qquad + ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \delta\xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) .
\end{aligned}
\tag{1.15}
$$

On substituting $\underset{E}{\delta n^i} = -\underset{E}{b_\alpha^{\ \beta}} \dfrac{\partial x^i}{\partial u^\beta} du^\alpha$ into the first term of the right-hand member of (1.15), we have

$$
\begin{aligned}
&((\underset{E}{\delta n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx))\\
&\quad = m!(-1)^{(n-m)(n-1)+\nu} \underset{E}{H_{\nu+1}} \underset{E}{n_i} \xi^i dA ,
\end{aligned}
\tag{1.16}
$$

where $\underset{E}{H_{\nu+1}}$ denotes the $(\nu+1)$-th mean curvature of V^m for the normal direction $\underset{E}{n^i}$ and if we indicate by $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m}$ the principal curvatures of V^m for the normal vector $\underset{E}{n^i}$, $\underset{E}{H_{\nu+1}}$ is defined to be the $(\nu+1)$-th elementary symmetric function of $\underset{E}{k_\alpha}$ $(\alpha=1,2,\cdots,m)$ divided by the number of terms, that is,

$$
\binom{m}{\nu+1} \underset{E}{H_{\nu+1}} = \sum_{\alpha_1<\alpha_2<\cdots<\alpha_{\nu+1}} \underset{E}{k_{\alpha_1}} \underset{E}{k_{\alpha_2}} \cdots \underset{E}{k_{\alpha_{\nu+1}}} .
$$

Also, by virtue of (1.4) we can see that the vectors

$$
\underset{E}{n} \times \delta \underset{m+2}{n} \times \underset{m+3}{n} \times \cdots \times \underset{n}{n} \times \underbrace{\underset{E}{\delta n} \times \cdots \times \underset{E}{\delta n}}_{\nu} \times \underbrace{dx \times \cdots \times dx}_{m-\nu-1} ,
$$

$$
\underset{E}{n} \times \underset{m+2}{n} \times \delta \underset{m+3}{n} \times \cdots \times \underset{n}{n} \times \underset{E}{\delta n} \times \cdots \times \underset{E}{\delta n} \times dx \times \cdots \times dx ,
$$

....

and

$$
\underset{E}{n} \times \underset{m+2}{n} \times \cdots \times \underset{n-1}{n} \times \delta \underset{n}{n} \times \underset{E}{\delta n} \times \cdots \times \underset{E}{\delta n} \times dx \times \cdots \times dx
$$

have the same direction to the covariant vectors $\underset{m+2}{n}, \underset{m+3}{n}, \cdots,$ and $\underset{n}{n}$ respectively. Then, by means of (1.13) we obtain

$$
\begin{aligned}
&((\underset{E}{n}, \delta \underset{m+2}{n}, \underset{m+3}{n}, \cdots, \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) = 0 ,\\
&((\underset{E}{n}, \underset{m+2}{n}, \delta \underset{m+3}{n}, \cdots, \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) = 0 ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n-1}{n}, \delta \underset{n}{n}, \xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) = 0 .
\end{aligned}
\tag{1.17}
$$

Since the vector

$$\underset{E}{n} \times \underset{m+2}{n} \times \cdots \times \underset{n}{n} \times \delta \underset{E}{n} \times \cdots \times \delta \underset{E}{n} \times dx \times \cdots \times dx$$

is orthogonal to the normal vectors $\underset{E}{n}, \underset{m+2}{n}, \cdots$, and $\underset{n}{n}$ and $\delta \underset{E}{n}{}^i = -\underset{E}{b}{}_\alpha^\beta \dfrac{\partial x^i}{\partial u^\beta} du^\alpha$, the last term of the right hand member of (1.15) becomes as follows:

$$
\begin{aligned}
(1.18)\qquad &((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \delta\xi, \delta \underset{E}{n}, \cdots, \delta \underset{E}{n}, dx, \cdots, dx)) \\
&= m!(-1)^{(n-m)(n-1)+\nu} \frac{1}{2m} \underset{E}{H}{}_\nu^{\alpha\beta} \underset{\xi}{\pounds}\, g_{\alpha\beta} dA ,
\end{aligned}
$$

where $\underset{\xi}{\pounds}\, g_{\alpha\beta} = (\underset{\xi}{\pounds}\, g_{ij}) \dfrac{\partial x^i}{\partial u^\alpha} \dfrac{\partial x^j}{\partial u^\beta}$ and

$$\underset{E}{H}{}_\nu^{\alpha\beta} = \frac{1}{(m-1)!} \varepsilon^{\alpha\alpha_1\cdots\alpha_{m-1}} \varepsilon^{\beta\beta_1\cdots\beta_{m-1}} \underset{E}{b}{}_{\alpha_1\beta_1} \cdots \underset{E}{b}{}_{\alpha_\nu\beta_\nu} g_{\alpha_{\nu+1}\beta_{\nu+1}} \cdots g_{\alpha_{m-1}\beta_{m-1}} ,$$

and $\varepsilon^{\alpha\alpha_1\cdots\alpha_{m-1}}$ denotes the ε-symbol of the submanifold V^m. Accordingly, by means of (1.15), (1.16), (1.17) and (1.18) it follows that

$$
\begin{aligned}
(1.19)\qquad &\frac{1}{m!} d((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \delta \underset{E}{n}, \cdots, \delta \underset{E}{n}, dx, \cdots, dx)) \\
&= (-1)^{(n-m)(n-1)+\nu} \left\{ \underset{E}{H}{}_{\nu+1} \underset{E}{n}{}_i \xi^i dA + \frac{1}{2m} \underset{E}{H}{}_\nu^{\alpha\beta} \underset{\xi}{\pounds}\, g_{\alpha\beta} dA \right\} .
\end{aligned}
$$

Integrating both sides of (1.19) over the whole submanifold V^m and applying Stokes' theorem, we have

$$
\begin{aligned}
&\frac{1}{m!} \int_{\partial V^m}\!\!\cdots\!\int ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \delta \underset{E}{n}, \cdots, \delta \underset{E}{n}, dx, \cdots, dx)) \\
&= (-1)^{(n-m)(n-1)+\nu} \left\{ \int_{V^m}\!\!\cdots\!\int \underset{E}{H}{}_{\nu+1} \underset{E}{n}{}_i \xi^i dA + \frac{1}{2m} \int_{V^m}\!\!\cdots\!\int \underset{E}{H}{}_\nu^{\alpha\beta} \underset{\xi}{\pounds} g_{\alpha\beta} dA \right\} .
\end{aligned}
$$

Thus, for a closed orientable submanifold V^m we have

$$(\mathrm{II}')\qquad \int_{V^m}\!\!\cdots\!\int \underset{E}{H}{}_{\nu+1} \underset{E}{n}{}_i \xi^i dA + \frac{1}{2m} \int_{V^m}\!\!\cdots\!\int \underset{E}{H}{}_\nu^{\alpha\beta} \underset{\xi}{\pounds} g_{\alpha\beta} dA = 0 .$$

If $m = n-1$, that is, V^m is the hypersurface of R^n, the formulas (I′) and (II′) are coincide with the formulas given in the previous paper [2]. Then these formulas (I′) and (II′) are certain formulas of Minkowski-type generalized for the closed orientable submanifold V^m in R^n.

§2. The Minkowski formulas concerning some special transformations. In this section, we shall discuss the formulas (I′) and (II′) for a special infinitesimal transformation.

Let the group G be conformal. Then we have

$$g^{*ij}\underset{\xi}{\pounds}g_{ij}=2m\Phi\,,\qquad \underset{E}{H_\nu}{}^{\alpha\beta}\underset{\xi}{\pounds}g_{\alpha\beta}=2m\Phi\underset{E}{H_\nu}\,.$$

Therefore (I′) and (II′) are rewritten in the following forms:

$$(\mathrm{I}')_{\mathrm{c}}\qquad \int_{V^m}\!\!\cdots\!\int H_1\underset{E}{n_i}\xi^i dA+\int_{V^m}\!\!\cdots\!\int \Phi\, dA=0\,,$$

$$(\mathrm{II}')_{\mathrm{c}}\qquad \int_{V^m}\!\!\cdots\!\int \underset{E}{H_{\nu+1}}\underset{E}{n_i}\xi^i dA+\int_{V^m}\!\!\cdots\!\int \Phi\underset{E}{H_\nu}dA=0\qquad(1\leqq\nu\leqq m-1)\,.$$

Let the group G be homothetic, that is, $\Phi\equiv c$ ($=$const.). Then we have

$$(\mathrm{I}')_{\mathrm{h}}\qquad \int_{V^m}\!\!\cdots\!\int H_1\underset{E}{n_i}\xi^i dA+c\int_{V^m}\!\!\cdots\!\int dA=0\,,$$

$$(\mathrm{II}')_{\mathrm{h}}\qquad \int_{V^m}\!\!\cdots\!\int \underset{E}{H_{\nu+1}}\underset{E}{n_i}\xi^i dA+c\int_{V^m}\!\!\cdots\!\int \underset{E}{H_\nu}dA=0\qquad(1\leqq\nu\leqq m-1)\,^{3)}.$$

If the group G is isometric, that is, $\Phi\equiv 0$, then we have

$$(\mathrm{I}')_{\mathrm{i}}\qquad \int_{V^m}\!\!\cdots\!\int H_1\underset{E}{n_i}\xi^i dA=0\,,$$

$$(\mathrm{II}')_{\mathrm{i}}\qquad \int_{V^m}\!\!\cdots\!\int \underset{E}{H_{\nu+1}}\underset{E}{n_i}\xi^i dA=0\qquad(1\leqq\nu\leqq m-1)\,.$$

Especially if our space R^n is an Euclidean space E^n and if the paths of the infinitesimal transformations are the straight lines which pass through a fixed point 0, x^i being the coordinate system with the point 0 as its origin, let the position vector x^i takes as the vector ξ^i, then we have

$$\underset{\xi}{\pounds}g_{ij}=2g_{ij}\,.$$

Accordingly, from $(\mathrm{I}')_{\mathrm{h}}$ and $(\mathrm{II}')_{\mathrm{h}}$ we have

$$\int_{V^m}\!\!\cdots\!\int H_1 p\, dA+\int_{V^m}\!\!\cdots\!\int dA=0\,,$$

3) In this case, R^n becomes an Euclidean space E^n, because if R^n with constant Riemann curvature admits an one-parameter group G of homothetic transformations, then either R^n is E^n or the group G is isometric.

$$\int_{V^m}\!\!\cdots\!\int H_{\nu+1}\underset{E}{}\, p\, dA+\int_{V^m}\!\!\cdots\!\int \underset{E}{H_\nu}\, dA=0\,,$$

where $p=\underset{E}{n_i}x^i$. This means that the formulas (I′) and (II′) are generalization of those formulas given by C. C. Hsiung [1] for a closed orientable hypersurface in an n-dimensional Euclidean space E^n.

§ 3. Some properties of a closed orientable submanifold. In this section we suppose that there exists a continuous one-parameter group G of conformal transformations generated by a vector ξ^i of R^n, where the vector ξ is contained in the vector space spanned by $m+1$ vectors $\dfrac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1, 2, \cdots, m)$ and $\underset{E}{n^i}$. Then we shall prove the following four theorems for a closed orientable submanifold V^m in a Riemann space R^n with constant Riemann curvature.

Theorem 3.1. *If in R^n, there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m and if H_1 is constant, then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$, where ρ denotes $\underset{E}{n_i}\xi^i$.*

Proof. Multiplying the formula $(\mathrm{I}')_c$ by $H_1=$const., we have

$$\int_{V^m}\!\!\cdots\!\int H_1^2\rho\, dA+\int_{V^m}\!\!\cdots\!\int \Phi H_1\, dA=0\,.$$

On the other hand, for $\nu=1$ we have from $(\mathrm{II}')_c$

$$\int_{V^m}\!\!\cdots\!\int \underset{E}{H_2}\rho\, dA+\int_{V^m}\!\!\cdots\!\int \Phi H_1\, dA=0\,.$$

Consequently it follows that

$$\int_{V^m}\!\!\cdots\!\int (H_1^2-\underset{E}{H_2})\rho\, dA=0\,.$$

From our assumption about ρ, this holds if and only if $H_1^2-\underset{E}{H_2}=0$, since

$$H_1^2-\underset{E}{H_2}=\frac{1}{m^2(m-1)}\sum_{\alpha<\beta}(\underset{E}{k_\alpha}-\underset{E}{k_\beta})^2\geqq 0\,.$$

Therefore at each point of V^m we obtain

$$\underset{E}{k_1}=\underset{E}{k_2}=\cdots=\underset{E}{k_m}\,.$$

Accordingly every point of V^m is umbilic with respect to $\underset{E}{n}$.

Theorem 3.2. *If in R^n, there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m, and if the principal curvatures $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m}$ at each point of V^m are positive and $\underset{E}{H_\nu}$ is constant for any $\nu (1 \leqq \nu \leqq m-1)$, then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$,* where ρ denotes $\underset{E}{n_i}\xi^i$.

Proof. Multiplying the formula $(\mathrm{I}')_c$ by $\underset{E}{H_\nu} = \text{const.}$, we obtain

$$(3.1) \qquad \int_{V^m}\!\cdots\!\int H_1 \underset{E}{H_\nu} \rho\, dA + \int_{V^m}\!\cdots\!\int \Phi \underset{E}{H_\nu} dA = 0 .$$

By means of $(\mathrm{II}')_c$ and (3.1), we have

$$\int_{V^m}\!\cdots\!\int (H_1 \underset{E}{H_\nu} - \underset{E}{H_{\nu+1}}) \rho\, dA = 0 .$$

From our assumptions, this holds if and only if $H_1 \underset{E}{H_\nu} - \underset{E}{H_{\nu+1}} = 0$, since

$$H_1 \underset{E}{H_\nu} - \underset{E}{H_{\nu+1}} = \frac{\nu!(m-\nu-1)!}{mm!} \sum \underset{E}{k_{\alpha_1}} \cdots \underset{E}{k_{\alpha_{\nu-1}}} (\underset{E}{k_\nu} - \underset{E}{k_{\nu+1}})^2 \geqq 0 .$$

Then at each point of V^m, we obtain

$$\underset{E}{k_1} = \underset{E}{k_2} = \cdots = \underset{E}{k_m}$$

Accordingly every point of V^m is umbilic with respect to $\underset{E}{n}$.

Theorem 3.3. *If in R^n, there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m, for which $H_1 \rho + \Phi \geqq 0$ (or $\leqq 0$) at all points of V^m, then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$,* where ρ denotes $\underset{E}{n_i}\xi^i$.

Proof. If we express the formula $(\mathrm{I}')_c$ as follows:

$$\int_{V^m}\!\cdots\!\int (H_1 \rho + \Phi) dA = 0 ,$$

then from our assumption we must have

$$(3.2) \qquad \Phi = -H_1 \rho .$$

Substituting (3.2) into $(\mathrm{II}')_c$ for $\nu = 1$, we obtain

$$\int_{V^m}\cdots\int (H_1^2 - \underset{E}{H_2})\rho\, dA = 0\,.$$

Consequently we have the conclusion.

Theorem 3.4. *If H_1 is positive (or negative) at all points of V^m and if in R^n there exists such a group G of conformal transformations as Φ is positive (or negative), for which either $\rho \geqq \frac{-\Phi}{H_1}$ or $\rho \leqq \frac{-\Phi}{H_1}$ at all points of V^m, then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$,* where ρ denotes $\underset{E}{n_i}\xi^i$.

Proof. The formula $(\mathrm{I}')_c$ is rewritten as follows

$$\int_{V^m}\cdots\int H_1\left(\rho + \frac{\Phi}{H_1}\right) dA = 0\,.$$

Then, by virtue of our assumptions $H_1>0$ (or <0) and $\rho + \frac{\Phi}{H_1} \geqq 0$ (or $\leqq 0$) at all points of V^m, we must have

$$\rho = -\frac{\Phi}{H_1}\,. \tag{3.3}$$

Substituting (3.3) into $(\mathrm{II}')_c$ for $\nu=1$, we obtain

$$\int_{V^m}\cdots\int \frac{\Phi}{H_1}(H_1^2 - \underset{E}{H_2})\, dA = 0\,.$$

From our assumptions, this holds if and only if $H_1^2 - \underset{E}{H_2} = 0$. Thus we obtain the conclusion.

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