ON THE COMPLETE CONTINUITY OF OPERATORS IN AN INTERPOLATION THEOREM

By

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1. In this paper, a Banach space $(\boldsymbol{X}, \|\cdot\|)$, whose elements are complex valued Lebesgue measurable functions over the interval $(0, 1)$, will be called a *Banach function space*, if it satisfies the following conditions:

(1.1) $|g| \leqslant |f|$[1], $f \in \boldsymbol{X}$ *implies* $g \in \boldsymbol{X}$ *and* $\|g\| \leqslant \|f\|$;

(1.2) $0 \leqslant f_n \uparrow$[2], $\|f_n\| \leqslant M$ $(n=1,2,\cdots)$ *implies*

$$\bigcup_{n=1}^{\infty} f_n = f \in \boldsymbol{X} \text{ and } \|f\| = \sup_{n\geq 1} \|f_n\| .$$

From (1.2) it follows that the norm $\|\cdot\|$ on a Banach function space is semicontinuous, i.e., $0 \leqslant f_n \uparrow f$, $f, f_n \in \boldsymbol{X}$ implies $\|f\| = \sup_{n\geq 1} \|f_n\|$. The space $(\boldsymbol{X}, \|\cdot\|)$ is called *rearrangement invariant*, if $0 \leqslant f \in \boldsymbol{X}$ implies $g \in \boldsymbol{X}$ and $\|f\| = \|g\|$ for each function g, equimeasurable with f. Let $\boldsymbol{L}^1$, $\boldsymbol{L}^\infty$ be the Lebesgue spaces over $(0, 1)$, and let $\boldsymbol{B}(\boldsymbol{L}^1; \boldsymbol{L}^\infty)$ be the set of all bounded linear operators from each of the spaces $\boldsymbol{L}^1$, $\boldsymbol{L}^\infty$ into itself. By $\|T\|_i$ $(i=1$, or $=\infty)$ we denote the norm of an operator $T \in \boldsymbol{B}(\boldsymbol{L}^1; \boldsymbol{L}^\infty)$ on the corresponding spaces. For each $a>0$, f_a is the function given by $f_a(x)=f(ax)$, if $ax<1$, $f_a(x)=0$, if $ax>1$. We write also

(1.3) $$\sigma_a f = f_a ;$$

it is easy to see that σ_a is a bounded linear operator on $\boldsymbol{X}$, if $\boldsymbol{X}$ is rearrangement invariant.

The following theorem was proved in [4][3]:

Theorem A. *Let $\boldsymbol{X}$ be a rearrangement invariant Banach function space. Then, for every $T \in \boldsymbol{B}(\boldsymbol{L}^1; \boldsymbol{L}^\infty)$, T is a bounded linear operator from*

1) $|f|$ denotes the function defined by $|f|(x) = |f(x)|$, $x \in (0, 1)$. $f \leqslant g$ means that $f(x) \leqslant g(x)$ holds almost everywhere.

2) We write $0 \leqslant f_n \uparrow$, if $0 \leqslant f_1 \leqslant f_2 \leqslant \cdots$. If $0 \leqslant f_n \uparrow$ and $\bigcup_{n=1}^{\infty} f_n = f$, we write $0 \leqslant f_n \uparrow f$ simply.

3) Theorem A was first proved by W. Orlicz for Orlicz spaces [7]. A. P. Calderón gave the theorem in full generarity for quasi-linear operators in [1]. In [4] Theorem A is stated for Lipschitz operators.

X *into itself, and*

$$\|T\|_X \leqslant \|T\|_\infty \cdot \|\sigma_a\|_X \text{ [4]} \tag{1.4}$$

holds, where $a = \|T\|_\infty \cdot \|T\|_1^{-1}$.

In this paper we deal with the complete continuity of T on $\boldsymbol{X}$, when T is completely continuous on either $\boldsymbol{L}^1$ or $\boldsymbol{L}^\infty$, and we give a necessary and sufficient condition in order that every $T \in \boldsymbol{B}(\boldsymbol{L}^1; \boldsymbol{L}^\infty)$ which is completely continuous on $\boldsymbol{L}^1$ (or $\boldsymbol{L}^\infty$) is also completely continuous on $\boldsymbol{X}$ (Theorems 1 and 2).

2. In the sequel, we assume that $\boldsymbol{X}$ is a Banach function space which is also rearrangement invariant. Since $\boldsymbol{X}$ is rearrangement invariant, $\boldsymbol{X}$ is contained in $\boldsymbol{L}^1$ [5]. We need the following lemma:

Lemma 1. *If* $0<a<1$, $1 \leqslant \|\sigma_a\|_X \leqslant a^{-1}$ *holds.*

Proof. The inequality: $1 \leqslant \|\sigma_a\|_X$ is evident. Let $0 \leqslant f \in \boldsymbol{X}$, and let $\alpha = n \cdot m^{-1}$, where m and n are natural numbers with $n<m$. Since $\sigma_\alpha f = \sigma_\alpha (f\chi_{(0,\alpha)})$[5], we may assume without loss of generarity that $f = f\chi_{(0,\alpha)}$. Now we define g_i by $g_i = \tau_{b_i}\sigma_n f$, $1 \leqslant i \leqslant m$, where $b_i = (i-1) \cdot m^{-1}$ and τ_{b_i} is a translation operator defined by b_i: $(\tau_{b_i}h)(x) = h(x-b_i)$, if $0<x-b_i<1$; $(\tau_{b_i}h)(x) = 0$ otherwise. Then $g_i \sim g_j$[6], $i, j = 1, 2, \cdots, m$, and $g_i g_j = 0$, if $i \neq j$. Put $h_j = \sum_{k=j}^{j+n-1} g_k$, $1 \leqslant j \leqslant m$, where we put $g_k = g_{k-m}$, if $k>m$. Obviously it follows that $f \sim h_i \sim h_j$ for all i, j, and

$$\sum_{j=1}^{m} h_j = n \sum_{i=1}^{m} g_i \sim n\sigma_{n \cdot m^{-1}} f,$$

since $\sigma_{n \cdot m^{-1}} f = \sigma_{m^{-1}}(\sigma_n f) = \sigma_{m^{-1}} g_1 \sim \sum_{i=1}^{m} g_i$. This implies

$$n \|\sigma_{n \cdot m^{-1}} f\| = \left\| \sum_{j=1}^{m} h_j \right\| \leqslant m \|h_1\| = m \|f\|,$$

because $\boldsymbol{X}$ is rearrangement invariant. Therefore $\|\sigma_{n \cdot m^{-1}} f\| \leqslant m \cdot n^{-1} \|f\|$ holds. For an arbitrary real $\alpha>0$, take natural numbers n, m such that $n \cdot m^{-1} < \alpha < 1$. We have then $\|\sigma_\alpha f^*\|$[7] $\leqslant \|\sigma_{n \cdot m^{-1}} f^*\|$, hence $\|\sigma_\alpha f^*\| \leqslant m \cdot n^{-1} \|f^*\| = m \cdot n^{-1} \|f\|$. Letting $n \cdot m^{-1} \uparrow \alpha$, we obtain $\|\sigma_\alpha f\| \leqslant \|\sigma_\alpha f^*\| \leqslant \alpha^{-1} \|f\|$, which proves Lemma 1.

Now we consider the following conditions on $\boldsymbol{X}$:

$$\|\sigma_a\|_X < 1 \qquad \textit{for some } a>1; \tag{2.1}$$

4) $\|T\|_X$ denotes the norm of T on X.

5) $\chi_{\mathbf{e}}$ denotes the characteristic function of the set $\mathbf{e}$.

6) We write $f \sim g$, if f is equimeasurable with g.

7) f^* denotes the decreasing rearrangement of $|f|$.

(2.2) $$\|\sigma_a\|_X < a^{-1} \quad \textit{for some } 0<a<1.$$

Since $\sigma_{a^2}=\sigma_a\cdot\sigma_a$ holds for every $a>0$, and $\|\sigma_a\|_X\leqslant\|\sigma_b\|_X$ if $a>b>0$, (2.1) and (2.2) are equivalent to the following conditions respectively:

(2.3) $$\lim_{a\to\infty}\|\sigma_a\|_X=0\ ;$$

(2.4) $$\lim_{a\to 0} a\,\|\sigma_a\|_X=0\ .$$

Lemma 2. *If* $\boldsymbol{X}$ *satisfies the condition* (2.1), *then* $\lim_{n\to\infty}\|T_n\|_X=0$ *holds for every sequence* $\{T_n\}$ *of* $\boldsymbol{B}(\boldsymbol{L}^1;\ \boldsymbol{L}^\infty)$ *such that* $\lim_{n\to\infty}\|T_n\|_1=0$ *and* $\sup_{n\geq 1}\|T_n\|_\infty<\infty$.

Proof. By Theorem A each T_n is a bounded linear operator on $\boldsymbol{X}$. For any $\varepsilon>0$ we can find an $\eta>1$ such that $a>\eta$ implies $\|\sigma_a\|_X<\varepsilon\cdot K^{-1}$, where $K=\sup_{n\geq 1}\|T_n\|_\infty$, since (2.1) is equivalent to (2.3). For such $\eta>0$, we choose an n_0 so large that $K\|T_n\|_1^{-1}>\eta$ holds for each $n\geq n_0$. Then, by (1.4) we get for $n\geq n_0$

$$\|T_n\|_X\leqslant\|T_n\|_\infty\|\sigma_{a_n}\|_X\,,\qquad a_n=\|T_n\|_\infty\|T_n\|_1^{-1}\,,$$

Since $\sigma_{a_n}=\sigma_{b_n}\sigma_{c_n}$ holds with $b_n=\|T_n\|_\infty K^{-1}\leqslant 1$ and $c_n=K\cdot\|T_n\|_1^{-1}$, it follows from Lemma 1 that $\|\sigma_{a_n}\|_X\leqslant\|\sigma_{b_n}\|_X\|\sigma_{c_n}\|_X\leqslant b_n^{-1}\|\sigma_{c_n}\|_X$. Therefore we obtain for $n\geq n_0$

$$\|T_n\|_X\leqslant K\|\sigma_{c_n}\|_X\leqslant\varepsilon\,,$$

which completes the proof.

An operator A on $\boldsymbol{L}^1$ is called an *averaging operator*, if A is defined by

(2.5) $$Af=A_{(\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n)}f=\sum_{i=1}^{n}d(\mathbf{e}_i)^{-1}\left(\int_{\mathbf{e}_i}f(x)\,dx\right)\chi_{\mathbf{e}_i}\,,$$

where $\mathbf{e}_i\cap\mathbf{e}_j=\phi$, if $i\neq j$, $\bigcup_{i=1}^{n}\mathbf{e}_i\subset(0,1)$, and $n=1,2,\cdots$. The averaging operators belong to $\boldsymbol{B}(\boldsymbol{L}^1;\ \boldsymbol{L}^\infty)$ clearly. When $\boldsymbol{X}$ is rearrangement invariant, both A and $I-A$ are always contractions on $\boldsymbol{X}$, because $f\succ Af$[8] and $f\succ f-Af$ hold. Moreover A is completely continuous on $\boldsymbol{X}$. As is well known, there exists a sequence of averaging operators $\{A_n\}$ such that A_n converges strongly to I, the identity operator, on $\boldsymbol{L}^1$. Now we can prove the following theorem:

Theorem 1. *Let* $\boldsymbol{X}$ *be rearrangement invariant. In order that every* $T\in\boldsymbol{B}(\boldsymbol{L}^1;\ \boldsymbol{L}^\infty)$ *which is completely continuous on* $\boldsymbol{L}^1$ *be also completely continuous on* $\boldsymbol{X}$, *it is necessary and sufficient that* $\boldsymbol{X}$ *satisfies* (2.1).

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8) $f\succ g$ means that $\int_0^t f^*(x)\,dx\geqq\int_0^t g^*(x)\,dx$ for all $0<t<1$. $f\succ g$ implies $\|f\|\geqq\|g\|$, if X is a rearrangement invariant Banach function space.

Proof. Sufficiency. Let V_1 be a unit ball of $\boldsymbol{L}^1$. By the assumption TV_1 is contained in a compact set of $\boldsymbol{L}^1$. Hence, for a sequence of averaging operators $\{A_n\}$, we have $\lim_{n\to\infty}\{\sup_{f\in V_1}\|(I-A_n)Tf\|_1\}=0$, that is, $\lim_{n\to\infty}\|(I-A_n)T\|_1=0$ together with $\|(I-A_n)T\|_\infty\leqslant\|T\|_\infty$, $n\geq 1$. It follows from Lemma 2 that $\lim_{n\to\infty}\|(I-A_n)T\|_X=0$. Since each A_nT, $n\geq 1$ is completely continuous on $\boldsymbol{X}$, T is also completely continuous on $\boldsymbol{X}$.

Necessity. Suppose that $\|\sigma_a\|_X=1$ for all $a>1$. Then for $a_n=2^{2n}$ we can find an $g_n\in\boldsymbol{X}$, $g_n\geq 0$, and $\|g_n\|=1$ such that $\|\sigma_{a_n}g_n\|>\frac{1}{2}$. Putting $g'_n=\sigma_{2^n}g_n$, we have

$$\|\sigma_{2^n}g'_n\|>\tfrac{1}{2}\ \text{and}\ \|g'_n\|\leqslant 1\,,\qquad n=1,2,\cdots$$

Since $\boldsymbol{X}$ is rearrangement invariant, we may assume without loss of generarity that $g'_n=g'_n\chi_n$, where χ_n is the characteristic function of the interval: $I_n=(2^{-n}, 2^{-n+1})$. Moreover, by the semicontinuity of the norm $\|\cdot\|$ we may assume that g'_n is a simple function for every $n\geq 1$. Now let $g'_n=\sum_{\nu=1}^{m_n}a_{n,\nu}\chi_{(c_{n,\nu-1},c_{n,\nu})}$, where $a_{n,\nu}\geq 0$ and $2^{-n}=c_{n,0}<c_{n,1}<\cdots<c_{n,m_n}\leqslant 2^{-n+1}$. A_n denotes the averaging operator defined by the intervals $I_{n,\nu}=(c_{n,\nu-1}, c_{n,\nu})$, $\nu=1,2,\cdots,m_n$, that is,

$$A_nf=\sum_{\nu=1}^{m_n}(c_{n,\nu}-c_{n,\nu-1})^{-1}\left(\int_{c_{n,\nu-1}}^{c_{n,\nu}}f(x)\,dx\right)\chi_{(c_{n,\nu-1},c_{n,\nu})}\,,\qquad f\in\boldsymbol{L}^1\,.$$

Putting $T_n=\sigma_{2^n}A_n$, we have for every $n\geq 1$ a linear operator T_n belonging to $\boldsymbol{B}(\boldsymbol{L}^1;\boldsymbol{L}^\infty)$ with $\|T_n\|_1=2^{-n}$ and $\|T_n\|_\infty=1$. Since $T_nf=T_n(f\chi_n)$ and $T_nf=(T_nf)\chi_{J_n}$, $J_n=(2^{-2n}, 2^{-2n+1})$ hold for all $n\geq 1$, the operator $T=\sum_{n=1}^{\infty}T_n$ is defined on $\boldsymbol{L}^\infty$ also and $\|T\|_\infty=1$. On the other hand, as $\|T\|_1\leqslant\sum_{n=1}^{\infty}\|T_n\|_1=1$, T acts also from $\boldsymbol{L}^1$ into itself. Furthermore as an operator on $\boldsymbol{L}^1$, T is completely continuous, as is easily seen. The operator T thus defined, however, is not completely continuous as an operator on $\boldsymbol{X}$. In fact, for each $n\geq 1$, $Tg'_n=T_ng'_n=\sigma_{2^n}A_ng'_n=\sigma_{2^n}g'_n$, hence $\|Tg'_n\|>\frac{1}{2}$. If the sequence $\{Tg'_n\}$ contains a subsequence which converges in the norm $\|\cdot\|$ to an element of $\boldsymbol{X}$, the limit must be 0, since Tg'_n converges to 0 almost everywhere by virtue of (1.2). This is a contradiction. Thus the necessity of the condition (2.1) is proved.

If $T\in\boldsymbol{B}(\boldsymbol{L}^1;\boldsymbol{L}^\infty)$ is completely continuous on $\boldsymbol{L}^\infty$, the set TV_∞ is contained in a compact set of $\boldsymbol{L}^\infty$, hence it is separable, where V_∞ is a unit ball of $\boldsymbol{L}^\infty$. Then, as is well known, there exists a sequence of averaging operators $\{A_n\}$ such that A_n converges to I strongly on TV_∞. As similarly as Lemma

2 we can prove that both $\lim_{n\to\infty}\|T_n\|_\infty=0$ and $\sup_{n\geq 1}\|T_n\|_1<\infty$ imply $\lim_{n\to\infty}\|T_n\|_X=0$ provided that $\boldsymbol{X}$ satisfies (2.2). On the other hand, if $\boldsymbol{X}$ violates (2.2), we can construct an operator T of $\boldsymbol{B}(\boldsymbol{L}^1;\ \boldsymbol{L}^\infty)$ which is completely continuous on $\boldsymbol{L}^\infty$, but not on $\boldsymbol{X}$. Such an operator can be constructed in a similar way as in Theorem 1. Thus we get[9)]

Theorem 2. *Let $\boldsymbol{X}$ be rearrangement invariant. In order that every $T\in\boldsymbol{B}(\boldsymbol{L}^1;\ \boldsymbol{L}^\infty)$ which is completely continuous on $\boldsymbol{L}^\infty$ be also completely continuous on $\boldsymbol{X}$, it is necessary and sufficient that $\boldsymbol{X}$ satisfies* (2.2).

3. In this section we give a simple condition equivalent with (2.1) or (2.2), when $\boldsymbol{X}$ is one of some concrete spaces: Orlicz spaces, Lorentz spaces $\Lambda(\varphi)$, and $M(\varphi)$ [2]. In [6] it is shown that the condition (2.2) is equivalent to the property that $\boldsymbol{X}\in$HLP, i.e., $f\in\boldsymbol{X}$ implies $\theta f\in\boldsymbol{X}$, where θf is the *Hardy-Littlewood majorant* of f. A necessary and sufficient condition for the condition (2.2) is also given in [3, 6] for Orlicz spaces, or spaces $\Lambda(\varphi)$. For a Banach function space $\boldsymbol{X}$ we denote by $\bar{\boldsymbol{X}}$ the *conjugate space of* $\boldsymbol{X}$, the set of all Lebesgue measurable functions g such that $\int_0^1|f(t)g(t)|\,dt<\infty$ for all $f\in\boldsymbol{X}$. The conjugate norm is defined by $\|g\|=\sup\left\{\int_0^1|f(t)g(t)|\,dt;\ f\in\boldsymbol{X},\ \|f\|\leqslant 1\right\}$ $(g\in\bar{\boldsymbol{X}})$. $\bar{\boldsymbol{X}}$ is rearrangement invariant, if $\boldsymbol{X}$ is so. The conditions (2.1) and (2.2) are mutually dual for the pair $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$, since $a^{-1}\sigma_{a^{-1}}$ is the conjugate of the operator σ_a. As $\bar{\boldsymbol{L}}_M=\boldsymbol{L}_N$, where N is the complementary function of M, we obtain by [3; Theorem 4, or 6; Theorem 3]

Theorem 3. i) *$\boldsymbol{L}_M$ satisfies* (2.1) *if and only if M satisfies the Δ_2-condition, i.e., there exist $u_0\geq 0$ and $\gamma>0$ such that $M(2u)\leqslant\gamma M(u)$ for all $u\geq u_0$.*

ii) *$\boldsymbol{L}_M$ satisfies* (2.2) *if and only if N satisfies the Δ_2-condition.*

For the spaces $\Lambda(\varphi)$, Put $\Phi(x)=\int_0^x\varphi(t)\,dt$, $0<x<1$. $\Phi(x)$ is a positive, nondecreasing concave function on (0, 1). In [3, 6] it is shown that (2.2) is equivalent to

(3.1) $$\limsup_{u\to 0}\Phi(2u)\Phi(u)^{-1}<2\,.$$

On the other hand, we can prove that (2.1) is equivalent to

(3.2) $$\liminf_{u\to 0}\Phi(2u)\Phi(u)^{-1}>1\,.$$

In fact, if (3.2) is true, then $\Phi(2u)\Phi(u)^{-1}\geqq 1+\delta$, $u\leqslant u_0<1$ for some $\delta>0$ and

9) Making use of the fact that σ_a and $a^{-1}\sigma_{a^{-1}}$ are mutually conjugate, we can also prove Theorem 2.

$u_0>0$. Put $\alpha=2\cdot u_0^{-1}$. Then for any a with $0<a<1$ we have[10)]

$$\|\sigma_\alpha \chi_{(0,a)}\|_\Lambda = \|\chi_{(0,2^{-1}u_0 a)}\|_\Lambda \leqslant (1+\delta)^{-1}\|\chi_{(0,u_0 a)}\|_\Lambda \leqslant (1+\delta)^{-1}\|\chi_{(0,a)}\|_\Lambda .$$

This implies $\|\sigma_\alpha f\|_\Lambda \leqslant (1+\delta)^{-1}\|f\|_\Lambda$ for all $f\in\Lambda(\varphi)$, which shows $\|\sigma_\alpha\|_\Lambda\leqslant(1+\delta)^{-1}<1$. Hence (2.1) holds.

Conversely, if (3.2) does not hold, we can find sequences of positive numbers $\{a_n\}$ and $\{\varepsilon_n\}$ such that $a_n<2^{-n}$, $a_n\downarrow$, $\Phi(2a_n)\Phi(a_n)^{-1}\leq 1+\varepsilon_n$, and $\varepsilon_n=(n2^n)^{-1}$ for every $n\geq 1$. Let $b_n=2^n a_n$ and $\chi_n=\chi_{(0,b_n)}$. Since Φ is a concave function, $\Phi(2^n a_n)\leqslant(1+2^n\varepsilon_n)\Phi(a_n)$, $n\geq 1$ holds. It follows from this that $\|\chi_n\|_\Lambda\|\sigma_{2^n}\chi_n\|_\Lambda^{-1}=\Phi(2^n a_n)\Phi(a_n)^{-1}\leqslant(1+2^n\varepsilon_n)=1+n^{-1}$. Hence $\|\sigma_{2^n}\|_\Lambda\geq(1+n^{-1})^{-1}$, $n\geq 1$, and $\limsup_{a\to\infty}\|\sigma_a\|_\Lambda\geq 1$. This is, however, inconsistent with (2.3). Therefore we have

Theorem 4. i) *$\Lambda(\varphi)$ satisfies* (2.1) *if and only if Φ satisfies* (3.2).

ii) *$\Lambda(\varphi)$ satisfies* (2.2) *if and only if Φ satisfies* (3.1).

Since the spaces $\Lambda(\varphi)$ and $M(\varphi)$ are mutually conjugate [2], we obtain immediately from Theorem 4

Theorem 5. i) *$M(\varphi)$ satisfies* (2.1) *if and only if Φ satisfies* (3.1).

ii) *$M(\varphi)$ satisfies* (2.2) *if and only if Φ satisfies* (3.2)[11)].

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10) The norm $\|\cdot\|_\Lambda$ of the space $\Lambda(\varphi)$ is defined by $\|f\|_\Lambda=\int_0^1 f^*(x)\varphi(x)dx$. In particular, $\|\chi_{(0,a)}\|_\Lambda=\Phi(a)$.

11) In [3] a condition equivalent to (2.2) is given by the condition that $\int_0^a \Phi(x)x^{-1}dx\leqslant A\Phi(a)$, $0<a<1$, for some fixed constant $A>0$.