ON CERTAIN PROPERTIES OF MODULAR CONVERGENCE

By

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Let R be a universally continuous semi-ordered linear space. A functional $m(a)$ $(a \in R)$ is said to be a modular on R if it satisfies the following modular conditions:

(1) $0 \leqq m(a) \leqq +\infty$ for all $a \in R$;

(2) if $m(\xi a)=0$ for all $\xi \geqq 0$, then $a=0$;

(3) for any $a \in R$ there exists $\alpha > 0$ such that $m(\alpha a) < +\infty$;

(4) for every $a \in R$, $m(\xi a)$ is a convex function of ξ;

(5) $|a| \leqq |b|$ implies $m(a) \leqq m(b)$;

(6) $a \frown b = 0$ implies $m(a+b) = m(a) + m(b)$;

(7) $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $m(a) = \sup_{\lambda \in \Lambda} m(a_\lambda)$.

Throughout the paper we use the notations and terminologies used in [2]. Here $|w|\text{-}\lim_{\nu\to\infty} a_\nu = a$ or $w\text{-}\lim_{\nu\to\infty} a_\nu = a$ for $a, a_\nu \in R$ $(\nu = 1, 2, 3, \cdots)$ means $\lim_{\nu\to\infty} |\bar{a}|(|a_\nu - a|) = 0$ or $\lim_{\nu\to\infty} \bar{a}(a_\nu - a) = 0$ respectively for any $\bar{a} \in \bar{R}^m$ [1].

If $\Phi(u)$ is a real convex function, defined for $u \geqq 0$, such that $\Phi(0)=0$ and $\Phi(u) \geqq 0$ for $u > 0$, but $\Phi(u)$ not identically zero or infinity for $u > 0$, then $\Phi(u)$ is called a YOUNG function.

We assume that Δ is a point set, and that a countably additive non-negative measure $\mu(E)(E \in \Lambda)$ is defined for the σ-ring Λ of subsets of Δ. We suppose furthermore that the measure μ is complete (i.e. $\mu(E_1)=0$, $E_2 \subset E_1$ implies $E_2 \in \Lambda$, so $\mu(E_2)=0$), totally σ-finite (i.e. Δ is a countable union of sets of finite measure) and $\mu(\Delta) > 0$.

If $f(x)$ is an arbitrary real-valued μ-measurable function on Δ, and $\Phi(u)$ is a YOUNG function, the space

$$L_\Phi = \left\{ f: \int_\Delta \Phi(\alpha |f(x)|) d\mu(x) < +\infty \quad \text{for some} \quad \alpha > 0 \right\}$$

is called ORLICZ space.

1) $\bar{R}^m$ be the modular conjugate space of R, i.e. $\bar{R}^m$ is the space of the modular bounded universally continuous linear functionals on R. The conjugate modular $\bar{m}$ of m is defined as $\bar{m}(\bar{a}) = \sup_{x \in R} \{\bar{a}(x) - m(x)\}$ for every $\bar{a} \in \bar{R}^m$.

If $f(x)$, $g(x)$ are arbitrary real-valued μ-measurable functions on Δ, we define $f \geqq g$ to mean that we have $f(x) \geqq g(x)$ almost everywhere on Δ. For a Young function $\Phi(u)$, putting

$$m_{\Phi}(f) = \int_{\Delta} \Phi(|f(x)|) d\mu(x) ,$$

we obtain a modular $m_{\Phi}(f)$ on L_{Φ}, where considering L_{Φ} as a semi-ordered linear space. Then L_{Φ} is a modulared semi-ordered linear space.[2]

W. A. J. Luxemburg and A. C. Zaanen (cf. [1] §8) have proved about the Orlicz space L_{Φ}

Theorem A$_1$. *w–$\lim_{\nu\to\infty} a_\nu = a$ for $a, a_\nu \in L_{\Phi}$ $(\nu = 1, 2, 3, \cdots)$, implies $\underline{\lim}_{\nu\to\infty} m_{\Phi}(\xi a_\nu) \geqq m_{\Phi}(\xi a)$ for all $\xi \geqq 0$.*

Theorem A$_2$. *For $a, a_\nu \in L_{\Phi}(\nu=1,2,3,\cdots)$, if $a_\nu(x)$ converges in measure on every set of finite measure to $a(x)$, and*

$$\lim_{\nu\to\infty} m_{\Phi}(a_\nu) = m_{\Phi}(a) < +\infty ,$$

then $\lim_{\nu\to\infty} m_{\Phi}\left(\frac{1}{2}(a_\nu - a)\right) = 0$.

Theorem A$_3$. *For $a, a_\nu \in L_{\Phi}$ $(\nu=1,2,3,\cdots)$, if $a_\nu(x)$ converges in measure to $a(x)$ on every set of finite measure, and m_{Φ} is simple[3] and*

$$\lim_{\nu\to\infty} m_{\Phi}(\alpha a_\nu) = m_{\Phi}(\alpha a) < +\infty \text{ for some } \alpha > 0,$$

then
$$|w|\text{–}\lim_{\nu\to\infty} a_\nu = a .$$

On the other hand, H. Nakano (cf. [2] §47) had proved:

Theorem B. *If R is semi-regular[4], then w–$\lim_{\nu\to\infty} a_\nu = a$, $m(a) < \infty$ implies $\underline{\lim}_{\nu\to\infty} m(\alpha a_\nu) \geqq m(\alpha a)$ for $0 \leqq \alpha < 1$.*

But this fact can be generalized as follows:

Theorem B$_1$. *If R is semi-regular, then w–$\lim_{\nu\to\infty} a_\nu = a$ implies $\underline{\lim}_{\nu\to\infty} m(\xi a_\nu) \geqq m(\xi a)$ for all $\xi \geqq 0$.*

Proof. By the formula (0) in §38 [2] we have

$$\bar{a}(\xi a_\nu) - \overline{m}(\bar{a}) \leqq m(\xi a_\nu)\ (\nu=1,2,3,\cdots) \qquad \text{for any } \bar{a} \in \bar{R}^m, \xi \geqq 0.$$

We obtain hence

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2) The Orlicz space L_{Φ} is the Banach space having the modular norm as its norm. L_{Φ} is an example of the modulared semi-ordered linear space with a constant modular.

3) A modular m is said to be simple, if $m(a)=0$ implies $a=0$.

4) R is said to be semi-regular, if $\bar{a}[p]=0$ for all $\bar{a} \in \bar{R}$ implies $p=0$.

$$\bar{a}(\xi a)-\overline{m}(\bar{a})\leqq\varliminf_{\nu\to\infty} m(\xi a_\nu)$$

by assumption. Since $\sup_{\bar{a}\in\bar{R}^m}\{\bar{a}(\xi a)-\overline{m}(\bar{a})\}=m(\xi a)$ by Theorem 39.3 [2], we conclude that $m(\xi a)\leqq\varliminf_{\nu\to\infty} m(\xi a_\nu)$. Q.E.D.

The above Theorem B_1 is an extension of Theorem A_1 to the modulared semi-ordered linear space.

Theorem B_2. *If R is semi-regular, then $|w|$-$\lim_{\nu\to\infty} a_\nu=a$, $\lim_{\nu\to\infty} m(a_\nu)=m(a)$ for a domestic*[5] *$a\in R$ implies $\lim_{\nu\to\infty} m\left(\frac{1}{2}(a_\nu-a)\right)=0$.*

We can replace the domesticness of a by $m(a)<+\infty$, as is seen in Corollary 1 of Theorem D.

By this Theorem B_2 we obtain immediately:

Theorem B_3 ([2] Theorem 47.9). *If R is semi-regular, then $|w|$-$\lim_{\nu\to\infty} a_\nu=a$, $\lim_{\nu\to\infty} m(\xi a_\nu)=m(\xi a)<+\infty$ for all $\xi\geqq 0$ implies m-$\lim_{\nu\to\infty} a_\nu=a$*[6].

This Theorem B_3 is an extension of Corollary 1 in [1] to the modulared semi-ordered linear space.

The main purpose of this paper is to give the extensions of Theorems A_2 and A_3 to the modulared semi-ordered linear space, and to consider its relations to Theorems B_2 and B_3.

KANTOROVITCH [4] introduced *star convergence*, i.e., we write s-$\lim_{\nu\to\infty} a_\nu=a$ if every partial sequence from $a_\nu\in R$ $(\nu=1,2,3,\cdots)$ contains a partial sequence which is order convergent to a.

We write *ind*-$\varlimsup_{\nu\to\infty} a_\nu=a$, if $\varlimsup_{\nu\to\infty}(a_\nu\frown p)=a\frown p$ for all $p\in R$; and we write *ind*-$\varliminf_{\nu\to\infty} a_\nu=a$, if $\varliminf_{\nu\to\infty}(a_\nu\smile p)=a\smile p$ for all $p\in R$.

If *ind*-$\varlimsup_{\nu\to\infty} a_\nu=$*ind*-$\varliminf_{\nu\to\infty} a_\nu=a$, then a_ν $(\nu=1,2,3,\cdots)$ is said to be *individually convergent* to the individual limit a, and we write *ind*-$\lim_{\nu\to\infty} a_\nu=a$.

We define that a sequence $a_\nu\in R$ $(\nu=1,2,3,\cdots)$ is *star individually convergent* to a and write s-*ind*-$\lim_{\nu\to\infty} a_\nu=a$, if, for any partial sequence of a_ν $(\nu=1,2,3,\cdots)$, we can select a partial sequence which is individually convergent to a (cf. [2], p. 112).

Lemma 1. (cf. [2] Theorem 27.10) *If R is semi-regular and super-*

5) An element $a\in R$ is said to be domestic, if $m(\alpha a)<+\infty$ for some $\alpha>1$.

6) m-$\lim_{\nu\to\infty} a_\nu=a$ for $a, a_\nu\in R$ $(\nu=1,2,3,\cdots)$ means $\lim_{\nu\to\infty} m(\xi(a_\nu-a))=0$ for all $\xi\geqq 0$.

universally continuous[7], *then*

$$|w|\text{-}\lim_{\nu\to\infty} a_\nu = a \quad \textit{implies} \quad s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a .$$

Theorem C. *If $\varDelta$ is a countable union of sets of finite measure, then in order that $a_\nu \in L_\Phi$ $(\nu=1,2,3,\cdots)$ converges in measure to $a\in L_\Phi$ on every set of finite measure, it is necessary and sufficient that a_ν $(\nu=1,2,3,\cdots)$ is star individually convergent to a, considering L_Φ as a modulared semi-ordered linear space, i.e., s-ind-$\lim_{\nu\to\infty} a_\nu = a$.*

Proof. We prove first: in order that $a_\nu(x)$ $(\nu=1,2,3,\cdots)$ converges in measure to $a(x)$ on every set of finite measure, it is necessary and sufficient that any subsequence of $\{a_\nu(x)\}$ contains a subsequence which converges to $a(x)$ almost everywhere.

For some measurable sets E_ρ $(\rho=1,2,3,\cdots)$ of finite measure, if $\varDelta = \bigcup_{\rho=1}^{\infty} E_\rho$, and $a_\nu(x)$ $(\nu=1,2,3,\cdots)$ converges in measure to $a(x)$ on E_ρ $(\rho=1,2,3,\cdots)$, then, by induction we can find integers $\nu(\kappa,\rho)$ $(\kappa,\rho=1,2,3,\cdots)$ such that

$$\mu\left(\left\{x:\ |a_\nu(x)-a(x)| \geqq \frac{1}{2^\kappa}\right\} \cap E_\rho\right) < \frac{1}{2^\kappa} \qquad \text{for} \quad \nu \geqq \nu(\kappa,\rho).$$

For such a double sequence $\nu(\kappa,\rho)$ $(\kappa,\rho=1,2,3,\cdots)$ we obtain another double sequence $\nu_\kappa(\rho)$ $(\kappa,\rho=1,2,3,\cdots)$ such that $\nu_\kappa(\rho+1)$ $(\kappa=1,2,3,\cdots)$ is a subsequence of $\nu_\kappa(\rho)$ $(\kappa=1,2,3,\cdots)$ for every ρ and $\nu_\kappa(\rho) < \nu_{\kappa+1}(\rho)$ $(\kappa=1,2,3,\cdots)$ for every ρ.

For such a sequence $\nu_\kappa(\rho)$ $(\kappa,\rho=1,2,3,\cdots)$ we obtain

$$\lim_{\kappa\to\infty} a_{\nu_\kappa(\rho)}(x) = a(x) \qquad \text{almost everywhere on } E_\rho \quad (\rho=1,2,3,\cdots),$$

because, putting $F_{\kappa,\rho} = \left\{x:\ |a_{\nu_\kappa(\rho)}(x)-a(x)| \geqq \frac{1}{2^\kappa}\right\} \cap E_\rho$ $(\kappa,\rho=1,2,3,\cdots)$, for $\kappa \leqq \xi$ and $x \in E_\rho - \bigcup_{\gamma=\kappa}^{\infty} F_{\gamma,\rho}$ $(\kappa,\rho=1,2,3,\cdots)$, we have

$$|a_{\nu_\xi(\rho)}(x)-a(x)| < \frac{1}{2^\xi} \qquad (\rho,\xi=1,2,3,\cdots).$$

Since

$$\mu\left(\bigcup_{\gamma=\kappa}^{\infty} F_{\gamma,\rho}\right) \leqq \sum_{\gamma=\kappa}^{\infty} \mu(F_{\gamma,\rho}) < \frac{1}{2^{\kappa-1}} \qquad (\kappa,\rho=1,2,3,\cdots),$$

7) R is said to be superuniversally continuous, if for any system $a_\lambda \geqq 0$ $(\lambda\in\varLambda)$ there exist countable a_{λ_ν}, $\lambda_\nu \in \varLambda$ $(\nu=1,2,3,\cdots)$ for which $\bigcap_{\nu=1}^{\infty} a_{\lambda_\nu} = \bigcap_{\lambda\in\varLambda} a_\lambda$.

putting $$F_\rho = \bigcap_{\kappa=1}^{\infty} \bigcup_{\gamma=\kappa}^{\infty} F_{\gamma,\rho} \qquad (\rho=1,2,3,\cdots),$$
we obtain $$\mu(F_\rho) = 0 \qquad (\rho=1,2,3,\cdots).$$

Moreover we obtain by diagonal process

$$\lim_{\kappa\to\infty} a_{\nu_{\kappa(\kappa)}}(x) = a(x) \qquad \text{almost everywhere.}$$

Conversely, if $\{a_\nu(x)\}$ is not convergent in measure to $a(x)$ on some set E of finite measure, then there exist $\varepsilon>0$ and a partial sequence $\{a_{\nu_\kappa}(x)\}$ of $\{a_\nu(x)\}$ such that

$$\mu\left(\left\{x: \ |a_{\nu_\kappa}(x)-a(x)| \geqq \varepsilon\right\} \cap E\right) > 0 \qquad (\kappa=1,2,3,\cdots).$$

Thus we can not select any partial sequence from $\{a_\nu(x)\}$ which converges to $a(x)$ almost everywhere on E. Contradicting the assumption. Thus we obtain our conclusion.

Therefore we need only prove, by definition of star individually convergence, the following fact: in order that $a_\nu(x)\geqq 0$ $(\nu=1,2,3,\cdots)$ converges to 0 almost everywhere, it is necessary and sufficient that a_ν $(\nu=1,2,3,\cdots)$ is individually convergent to 0, considering L_Φ as a semi-ordered linear space.

If $a_\nu(x)$ $(\nu=1,2,3,\cdots)$ converges to 0 almost everywhere, then, putting $b_\nu(x)=\inf(a_\nu(x), p(x))$ $(\nu=1,2,3,\cdots)$ for any $0\leqq p\in L_\Phi$, we have

$$0 \leqq b_\nu(x) \leqq p(x) \qquad \text{everywhere,}$$

and hence there exist $$\sup_{\nu\geqq 1} b_\nu \in L_\Phi .$$

Since $$\lim_{\nu\to\infty} b_\nu(x) = 0 \quad \text{almost everywhere, we have then}$$
$$\inf_{\mu\geqq 1} \sup_{\nu\geqq\mu} b_\nu(x) = 0 \quad \text{almost everywhere.}$$

Therefore we obtain, considering L_Φ as a semi-ordered linear space,

$$\bigcap_{\mu\geqq 1} \bigcup_{\nu\geqq\mu} b_\nu = 0. \quad \text{Thus we have } \lim_{\nu\to\infty} b_\nu = 0 \text{ for any } 0\leqq p\in L_\Phi .$$

Therefore we conclude $ind\text{-}\lim_{\nu\to\infty} a_\nu = 0$ by definition.

Conversely, we prove that

$$ind\text{-}\lim_{\nu\to\infty} a_\nu = 0, \quad 0\leqq a_\nu \in L_\Phi \quad (\nu=1,2,3,\cdots) \text{ implies } \lim_{\nu\to\infty} a_\nu(x) = 0$$

almost everywhere.

Let χ_E be the characteristic function of E, where E is an arbitrary set of finite measure.

Since $\chi_E \in L_\Phi$, we have by assumption $\lim_{\nu\to\infty}(a_\nu \frown \chi_E)=0$.

Therefore we conclude easily

$$\lim_{\nu\to\infty} a_\nu(x) = 0 \qquad \text{almost everywhere on } E.$$

Since the measure μ is totally σ-finite, we obtain

$$\lim_{\nu\to\infty} a_\nu(x) = 0 \qquad \text{almost everywhere.}$$ Q.E.D.

Lemma 2. *s-ind-$\lim_{\nu\to\infty} a_\nu = a$ implies $\underline{\lim}_{\nu\to\infty} m(\xi a_\nu) \geqq m(\xi a)$ for all $\xi \geqq 0$.*

Proof. If there exists a partial sequence a_{μ_ν} $(\nu=1,2,3,\cdots)$ such that $\lim_{\nu\to\infty} m(\xi a_{\mu_\nu}) < m(\xi a)$ for $\xi>0$, *ind*-$\lim_{\nu\to\infty} a_{\mu_\nu}=a$, then putting $b_\nu = \bigcap_{\rho\geqq\nu} |a_{\mu_\rho}|$, we have

$b_\nu \frown p = \bigcap_{\rho\geqq\nu}(|a_{\mu_\rho}| \frown p) \uparrow_{\nu=1}^{\infty} |a| \frown p$ for any $p \geqq 0$, since *ind*-$\lim_{\nu\to\infty} |a_{\mu_\nu}| = |a|$

by Theorem 15.4 [2]. Therefore, putting $p=|a|$ we have

$$m(\xi a) = \lim_{\nu\to\infty} m(\xi(b_\nu \frown |a|)) \leqq \lim_{\nu\to\infty} m(\xi a_{\mu_\nu})$$

contradicting the assumption. Q.E.D.

Theorem D. *s-ind-$\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty$ for $a, a_\nu \in R$ $(\nu=1,2,3,\cdots)$ implies $\lim_{\nu\to\infty} m(\alpha(a_\nu - a)) = 0$ for $0 \leqq \alpha \leqq \frac{1}{2}$.*

Proof. We may assume that $a \neq 0$ is a simple element and *ind*-$\lim_{\nu\to\infty} a_\nu = a$.

For, there exists a projector $[p]$[8] such that $[p]|a|$ is simple and $m((1-[p])a)=0$ by Theorem 35.4 [2]. We obtain by Lemma 2

$$\underline{\lim}_{\nu\to\infty} m([p]a_\nu) \geqq m([p]a), \quad \underline{\lim}_{\nu\to\infty} m((1-[p])a_\nu) \geqq m((1-[p])a).$$

But we have by assumption

$$\lim_{\nu\to\infty} \left\{ m([p]a_\nu) + m((1-[p])a_\nu) \right\} = m(a) = m([p]a) + m((1-[p])a).$$

Consequently we obtain

$$\lim_{\nu\to\infty} m([p]a_\nu) = m([p]a), \quad \lim_{\nu\to\infty} m((1-[p])a_\nu) = m((1-[p])a).$$

8) $[p]$ is a projection operator to the normal manifold generated by p: $[p]a = \bigcup_{\nu=1}^{\infty}(a \frown \nu|p|)$ for $0 \leqq a \in R$.

Since $m((1-[p])a)=0$ and

$$m\left(\frac{1}{2}(1-[p])(a_\nu-a)\right)\leqq\frac{1}{2}m((1-[p])a_\nu)+\frac{1}{2}m((1-[p])a),$$

we have thus $$\lim_{\nu\to\infty} m\left(\frac{1}{2}(1-[p])(a_\nu-a)\right)=0\,.$$

Now let a be simple. Since $ind\text{-}\lim_{\nu\to\infty} a_\nu=a$ by assumption, there exists by Theorem 15.3 [2] $[s_\rho]\uparrow_{\rho=1}^\infty[a]$ such that $\lim_{\nu\to\infty}[s_\rho]|a_\nu-a|=0(\rho=1,2,3,\cdots)$.

Putting $b_{\rho,\kappa}=\bigcup_{\nu\geqq\kappa}[s_\rho]|a_\nu-a|$ $(\rho,\kappa=1,2,3,\cdots)$, we obtain by definition

$$b_{\rho,\kappa}\downarrow_{\kappa=1}^\infty 0 \qquad (\rho=1,2,3,\cdots)\,.$$

Since a normal manifold $[a]R$ is totally continuous[9] as a space by Theorem 36.2 [2], there exist $[q_{\rho,\lambda}]\uparrow_{\lambda=1}^\infty[a]$ $(\rho=1,2,3,\cdots)$ and positive numbers $\varepsilon_{\rho,\lambda,\kappa}\downarrow_{\kappa=1}^\infty 0$ $(\rho,\lambda=1,2,3,\cdots)$ such that

$$[q_{\rho,\lambda}]b_{\rho,\kappa}\leqq\varepsilon_{\rho,\lambda,\kappa}|a| \qquad (\rho,\lambda,\kappa=1,2,3,\cdots) \text{ by Theorem 14.2 [2].}$$

Therefore there exists a sequence $[q_\mu]\uparrow_{\mu=1}^\infty[a]$ and $\lambda_{\rho,\mu}$ $(\rho,\mu=1,2,3,\cdots)$ such that $[q_\mu]\leqq[q_{\rho,\lambda_{\rho,\mu}}]$ $(\rho,\mu=1,2,3,\cdots)$ by definition of total continuity. We have then

$$[q_\mu][s_\mu]|a_\kappa-a|\leqq[q_{\mu,\lambda_{\mu,\mu}}][s_\mu]|a_\kappa-a|\leqq[q_{\mu,\lambda_{\mu,\mu}}]b_{\mu,\kappa}\leqq\varepsilon_{\mu,\lambda_{\mu,\mu},\kappa}|a| \qquad (\mu,\kappa=1,2,3,\cdots)\,.$$

Therefore, putting $[p_\mu]=[q_\mu][s_\mu]$ $(\mu=1,2,3,\cdots)$, there exists κ_0 such that $[p_\mu]|a_\kappa-a|\leqq|a|$ for every $\kappa\geqq\kappa_0$, and we obtain

$$[p_\mu]\uparrow_{\mu=1}^\infty[a]\,,\quad \lim_{\kappa\to\infty}[p_\mu]|a_\kappa-a|=0 \qquad (\mu=1,2,3,\cdots).$$

We have then by Theorem 35.1 [2]

$$\lim_{\nu\to\infty} m([p_\mu](a_\nu-a))=0 \qquad (\mu=1,2,3,\cdots).$$

If for some $\varepsilon>0$ we have

$$\overline{\lim_{\nu\to\infty}}\, m(([a]-[p_\mu])a_\nu)>\varepsilon \qquad (\mu=1,2,3,\cdots),$$

then, since $\overline{\lim_{\nu\to\infty}}\, m([a]a_\nu)\geqq\overline{\lim_{\nu\to\infty}}\, m(([a]-[p_\mu])a_\nu)+\underline{\lim_{\nu\to\infty}}\, m([p_\mu]a_\nu)$

9) R is said to be totally continuous, if for any double sequence of projectors $[p_{\nu,\mu}]\uparrow_{\mu=1}^\infty[p]$ $(\nu=1,2,3,\cdots)$ there exist a sequence $[p_\rho]\uparrow_{\rho=1}^\infty[p]$ and $\mu_{\nu,\rho}$ $(\nu,\rho=1,2,3,\cdots)$ such that $[p_\rho]\leqq[p_{\nu,\mu_{\nu,\rho}}]$ $(\nu,\rho=1,2,3,\cdots)$.

we have hence $\overline{\lim_{\nu\to\infty}} m([a]a_\nu) > \underline{\lim_{\nu\to\infty}} m([p_\mu]a_\nu) + \varepsilon$

and hence, as $\lim_{\nu\to\infty} m(a_\nu) = m(a)$ by assumption,

$$m(a) > m([p_\mu]a) + \varepsilon \qquad (\mu = 1, 2, 3, \cdots)$$

by lemma 2, contradicting $m([p_\mu]a) \uparrow_{\mu=1}^{\infty} m(a)$.

Therefore for any $\varepsilon > 0$ there exists μ_0 for which

$$\lim_{\nu\to\infty} m(([a]-[p_\mu])a_\nu) \leqq \varepsilon \qquad \text{for} \quad \mu \geqq \mu_0.$$

Since we have obviously

$$m\left(\frac{1}{2}(a_\nu - a)\right) \leqq m\left(\frac{1}{2}[p_\mu](a_\nu - a)\right) + \frac{1}{2} m(([a]-[p_\mu])a)$$
$$+ \frac{1}{2} m(([a]-[p_\mu])a_\nu),$$

we obtain hence for $\mu \geqq \mu_0$

$$\overline{\lim_{\nu\to\infty}} m\left(\frac{1}{2}(a_\nu - a)\right) \leqq \varepsilon + \frac{1}{2} m(([a]-[p_\mu])a)$$

and consequently for $\mu \to \infty$

$$\overline{\lim_{\nu\to\infty}} m\left(\frac{1}{2}(a_\nu - a)\right) \leqq \varepsilon \qquad \text{for any} \quad \varepsilon > 0,$$

that is,

$$\lim_{\nu\to\infty} m\left(\frac{1}{2}(a_\nu - a)\right) = 0. \qquad \text{Q.E.D.}$$

Remark 1. The above Theorem D is not true if $\alpha > \frac{1}{2}$ [10].

For example, consider the modulared space l^∞ [11].

Putting $a = (1, 1, 1, \cdots)$, $a_\nu = (1, 1, \cdots, \overset{\nu}{-1}, 1, 1, \cdots)$,

we have $\lim_{\nu\to\infty} a_\nu = a$, $m(a) = \lim_{\nu\to\infty} m(a_\nu) = 0$,

and $\lim_{\nu\to\infty} m(\alpha(a - a_\nu)) = +\infty \qquad \text{for} \quad \alpha > \frac{1}{2}$.

10) Mr. T. SHIMOGAKI remarked this fact.

11) We define the modular by the formula

$$m(a) = \begin{cases} 0 & \text{if} \quad \sup_{\nu \geqq 1} |\xi_\nu| \leqq 1 \\ +\infty & \text{if} \quad \sup_{\nu \geqq 1} |\xi_\nu| > 1 \end{cases} \qquad \text{for} \quad a = (\xi_\nu).$$

Remark 2. Since for $0 \leqq a, b \in R$ we have (cf. [2], Theorem 36.8)

$$m(a)+m(b) \geqq m(a-b),$$

we obtain in the same manner used in the proof of the above Theorem D:

$$s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty \quad \text{for} \quad 0 \leqq a, a_\nu \in R$$

$(\nu = 1, 2, 3, \cdots)$ implies $\lim_{\nu\to\infty} m(a_\nu - a) = 0$.

Remark 3. If $m(\xi a) < +\infty$ for every $\xi > 0$, then

$$s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} m(a_\nu) = m(a) \quad \text{for} \quad a, a_\nu \in R \ (\nu = 1, 2, 3, \cdots)$$

implies $\lim_{\nu\to\infty} m(\alpha(a_\nu - a)) = 0$ for $0 \leqq \alpha < 1$ [10].

Because, for any projectors $[p_\mu] \uparrow_{\mu=1}^{\infty} [a]$, we have obviously

$$m(\alpha(a_\nu - a)) \leqq \alpha m([p_\mu](a_\nu - a)) + \alpha m(([a]-[p_\mu])a_\nu) + (1-\alpha) m\left(\frac{\alpha}{1-\alpha}([a]-[p_\mu])a\right)$$

$(\mu = 1, 2, 3, \cdots)$ for $0 \leqq \alpha < 1$.

Therefore we conclude easily in the same manner we proved the above Theorem D

$$\lim_{\nu\to\infty} m(\alpha(a_\nu - a)) = 0 \quad \text{for} \quad 0 \leqq \alpha < 1.$$

The above Theorem D is an extension of Theorem A_2 to the modulared semi-ordered linear space.

In the following we shall explain the relations of the above Theorem D to Theorems B_2 and B_3.

Corollary 1 of Theorem D. *If R is semi-regular, then*

$$|w|\text{-}\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty$$

implies $\lim_{\nu\to\infty} m(\alpha(a_\nu - a)) = 0$ *for* $0 \leqq \alpha \leqq \frac{1}{2}$.

Proof. From $|w|\text{-}\lim_{\nu\to\infty} a_\nu = a$ we obtain by Theorem 27.7 [2]

$$|w|\text{-}\lim_{\nu\to\infty} [p]a_\nu = [p]a, \quad |w|\text{-}\lim_{\nu\to\infty} (1-[p])a_\nu = (1-[p])a \quad \text{for any } p \in R,$$

and hence by Theorem B_1

$$\varliminf_{\nu\to\infty} m([p]a_\nu) \geqq m([p]a), \quad \varliminf_{\nu\to\infty} m((1-[p])a_\nu) \geqq m((1-[p])a).$$

Therefore by the method applied to Theorem D we need only prove the case where $a \neq 0$ is simple.

If a is simple, then we have by Theorem 35.2 [2] the normal manifold $[a]R$ is superuniversally continuous as a space, and hence by Theorem D and Lemma 1 we obtain immediately our conclusion. Q.E.D.

Remark. The following fact may be found in the proof of Theorem 47.8 [2]. If R is semi-regular, then

$$|w|\text{-}\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty \quad \text{for } 0 \leqq a, a_\nu \in R \ (\nu = 1,2,3,\cdots)$$

implies
$$\lim_{\nu\to\infty} m(a_\nu - a) = 0 .$$

The above Corollary 1 is the generalization of Theorm B_2.

Corollary 2 of Theorem D. *If R is semi-regular, then*

$$|w|\text{-}\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} m(\xi a_\nu) = m(\xi a) < +\infty \quad \textit{for all } \xi \geqq 0$$

implies
$$s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a .$$

Proof. By assumption and Corollary 1 we have

$$m\text{-}\lim_{\nu\to\infty} a_\nu = a.$$

Therefore we obtain by Theorem 47.4 [2]

n-$\lim_{\nu\to\infty} a_\nu = a$ for the first (or second) norm[12] by m. Since the first (or second) norm is semi-continuous by definition, we obtain immediately our conclusion by Theorem 33.1 [2]. Q.E.D.

By Theorem D (or Corollary 2 of Theorem D) and Theorem 48.1 [2] we have obviously:

Corollary 3 of Theorem D. *If m is uniformly simple*[13], *then*

$$s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a \quad (\textit{or } |w|\text{-}\lim_{\nu\to\infty} a_\nu = a) \quad \textit{and} \quad \lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty$$

implies
$$m\text{-}\lim_{\nu\to\infty} a_\nu = a .$$

Corollary 4 of Theorem D. *If m is finite*[14] *and monotone complete*[15],

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12) On R we can define two norms as follows (cf. [3], p. 213 and p. 218):
$$\|x\| = \inf_{\xi>0} \frac{1+m(\xi x)}{\xi}, \quad |||x||| = \inf_{m(\xi x)\leqq 1} \frac{1}{|\xi|} \qquad (x \in R).$$
$\|x\|$ is said to be the first norm by m and $|||x|||$ is said to be the second norm or the modular norm by m.

13) A modular m is said to be uniformly simple, if
$$\inf_{m(x)\geqq 1} m(\xi x) > 0 \qquad \text{for all } \xi > 0 .$$

14) An element $a \in R$ is said to be finite, if
$m(\xi a) < +\infty$ for all $\xi \geqq 0$. A modular m on R is said to be finite, if all elements are finite.

15) A modular m on R is said to be monotone complete, if for
$$0 \leqq a_\lambda \uparrow_{\lambda\in\Lambda}, \ \sup_{\lambda\in\Lambda} m(a_\lambda) < +\infty \qquad \text{there exists } a \in R \text{ for which } a_\lambda \uparrow_{\lambda\in\Lambda} a .$$

then $s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a,\ \lim_{\nu\to\infty} m(a_\nu) = m(a)$ *for* $a, a_\nu \in R$ $(\nu = 1, 2, 3, \cdots)$

implies $$m\text{-}\lim_{\nu\to\infty} a_\nu = a .$$

Proof. Since by Theorem D

$$\lim_{\nu\to\infty} m\left(\frac{1}{2}(a_\nu - a)\right) = 0 ,$$

there exists a subsequence a_{ν_μ} $(\mu=1,2,3,\cdots)$ of a_ν $(\nu=1,2,3,\cdots)$ such that

$$m\left(\frac{1}{2}(a_{\nu_\mu} - a)\right) \leqq \frac{1}{2^\mu} \qquad (\mu = 1, 2, 3, \cdots).$$

Putting $$b_\kappa = \frac{1}{2}\bigcup_{\mu=1}^{\kappa} |a_{\nu_\mu} - a| \qquad (\kappa = 1, 2, 3, \cdots),$$ we have

$$0 \leqq b_\kappa \uparrow_{\kappa=1}^{\infty}, \quad \sup_{\kappa\geqq 1} m(b_\kappa) \leqq \sup_{\kappa \geqq 1} \sum_{\mu=1}^{\kappa} m\left(\frac{1}{2}(a_{\nu_\mu} - a)\right) \leqq 1 .$$

Since m is monoton complete, there exists $b \in R$ for which $b = \bigcup_{\mu=1}^{\infty} |a_{\nu_\mu} - a|$. Thus we obtain $|a_{\nu_\mu} - a| \leqq b$ $(\mu = 1, 2, 3, \cdots)$, and hence $|a_{\nu_\mu}| \leqq |a| + b$ $(\mu = 1, 2, 3, \cdots)$.

Therefore we have by assumption

$$\lim_{\mu\to\infty} |a_{\nu_\mu} - a| = 0 ,$$

and hence $$\lim_{\mu\to\infty} \xi\, |a_{\nu_\mu} - a| = 0 \qquad \text{for any} \quad \xi > 0 .$$

Since m is finite by assumption, we obtain by Theorem 35.1 [2]

$$\lim_{\mu\to\infty} m(\xi(a_{\nu_\mu} - a)) = 0 \qquad \text{for any} \quad \xi > 0 ,$$

and hence $$\lim_{\nu\to\infty} m(\xi(a_\nu - a)) = 0 \qquad \text{for any} \quad \xi > 0 ,$$

that is, $$m\text{-}\lim_{\nu\to\infty} a_\nu = a . \qquad \text{Q.E.D.}$$

Theorem E. *If* R *is semi-regular, then*

$$s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty$$

implies $$|w|\text{-}\lim_{\nu\to\infty} a_\nu = a .$$

Proof. We need only prove that

$$ind\text{-}\lim_{\nu\to\infty} a_\nu = a , \quad \lim_{\nu\to\infty} m(a_\nu) = m(a) < +\infty$$

implies $$|w|\text{-}\lim_{\nu\to\infty} a_\nu = a .$$

For any $0 \leqq p \in R$ there exist $[p_\mu] \uparrow_{\mu=1}^{\infty} [p]$ such that $\lim_{\nu\to\infty} [p_\mu] |a_\nu - a| = 0$ $(\mu = 1, 2, 3, \cdots)$ by Theorem 15.3 [2]. We have then obviously

$$\lim_{\nu\to\infty} |\bar{a}|([p_\mu]|a_\nu - a|) = 0 \quad (\mu = 1, 2, 3, \cdots) \text{ for any } \bar{a} \in \bar{R}^m.$$

By the formula (3) in §38 [2] there exists $\alpha > 0$ such that $\bar{m}(\alpha\bar{a}) < +\infty$ for any $\bar{a} \in \bar{R}^m$.

We obtain by the formula (0) in §38 [2]

$$\frac{\alpha}{2} |\bar{a}|(([p]-[p_\mu])|a_\nu - a|) \leqq \bar{m}(\alpha|\bar{a}|([p]-[p_\mu])) + \mathrm{m}\left(\frac{1}{2}(a_\nu - a)\right)$$

$$(\nu, \mu = 1, 2, 3, \cdots) \text{ for any } \bar{a} \in \bar{R}^m.$$

By assumption, Theorem D and Theorem 35.1 [2], for any $\varepsilon > 0$, there exist ν_0 and μ_0 for which

$$m\left(\frac{1}{2}(a_\nu - a)\right) \leqq \frac{\varepsilon}{2} \qquad \text{for } \nu \geqq \nu_0,$$

$$\bar{m}(\alpha|\bar{a}|([p]-[p_\mu])) \leqq \frac{\varepsilon}{2} \qquad \text{for } \mu \geqq \mu_0,$$

and hence $\quad |\bar{a}|(([p]-[p_\mu])|a_\nu - a|) \leqq \varepsilon \quad$ for $\nu \geqq \nu_0$ and $\mu \geqq \mu_0$.

Therefore we obtain $\lim_{\nu\to\infty} |\bar{a}|([p]|a_\nu - a|) = 0$ for any $0 \leqq p \in R$ and $\bar{a} \in \bar{R}^m$.

On the other hand, we have by the formula (0) in §38 [2]

$$\frac{\alpha}{2} |\bar{a}|((1-[p])|a_\nu - a|) \leqq \bar{m}(\alpha|\bar{a}|(1-[p])) + m\left(\frac{1}{2}(a_\nu - a)\right)$$

$$(\nu = 1, 2, 3, \cdots)$$

for any $p \in R$ and $\bar{a} \in \bar{R}^m$.

Since $(1-[p]) \downarrow_{p \in R} 0$ by Theorem 5.35 [2], we obtain in the same manner proved above

$$\lim_{\nu\to\infty} |\bar{a}|(|a_\nu - a|) = 0 \qquad \text{for all } \bar{a} \in \bar{R}^m, \text{ that is,}$$

$$|w|\text{-}\lim_{\nu\to\infty} a_\nu = a. \qquad \text{Q.E.D.}$$

Remark. The above Theorem E is the extension of Theorem A_3 to the modulared semi-ordered linear space without any condition of simpleness.

By Theorem D we obtain immediately:

Corollary 5 of Theorem D. *If* $s\text{-}ind\text{-}\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} m(\xi a_\nu) = m(\xi a) < +\infty$ *for all* $\xi \geqq 0$, *then we have* $m\text{-}\lim_{\nu\to\infty} a_\nu = a$.

Remark. If R is semi-regular, then we obtain immediately by Theorem E and Corollary 2 that the assumptions of Theorem B_3 and that of Corollary 5 of Theorem D are equivalent.

By Corollary 5 of Theorem D (or Theorem B_3) and Theorem 47.4 [2] we obtain immediately:

Corollary 6 of Theorem D. *If* s–*ind*–$\lim_{\nu\to\infty} a_\nu = a$ (*or* R *is semi-regular and* $|w|$–$\lim_{\nu\to\infty} a_\nu = a$), $\lim_{\nu\to\infty} m(\xi a_\nu) = m(\xi a) < +\infty$ *for all* $\xi \geqq 0$ *implies* n–$\lim_{\nu\to\infty} a_\nu = a$ *for the first* (*or second*) *norm by* m.

Corollary 6 is an extension of Corollary 2 in §8 [1] (i.e., if $|w|$–$\lim_{\nu\to\infty} a_\nu = a$ for $a, a_\nu \in L_\Phi$ $(\nu = 1, 2, \cdots)$, and if $\lim_{\nu\to\infty} m_\Phi(\xi a_\nu) = m_\Phi(\xi a) < +\infty$ for all $\xi \geqq 0$, then n–$\lim_{\nu\to\infty} a_\nu = a$ for the norms by m_Φ) to the modulared semi-ordered linear space.

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