SIMPLE-ROOT BASES FOR SHI ARRANGEMENTS

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Abstract. In his affirmative answer to the Edelman-Reiner conjecture, Yoshinaga proved that the logarithmic derivation modules of the cones of the extended Shi arrangements are free modules. However, we know very little about the bases themselves except their existence. In this article, we prove the unique existence of two distinguished bases which we call the simple-root basis plus (SRB+) and the simple-root basis minus (SRB−). They are characterized by nice divisibility properties relative to the simple roots.

1. Introduction

Let V be an ℓ-dimensional Euclidean space. Let Φ be an **irreducible (crystallographic) root system** in the dual space V^*. Fix a set Φ^+ of **positive roots** throughout this article. For any $\alpha \in \Phi^+$ and $j \in \mathbb{Z}$, the affine hyperplane

$$H_{\alpha,j} := \{x \in V \mid \alpha(x) = j\}$$

is a parallel translation of $H_\alpha := H_{\alpha,0}$. The arrangement $\mathcal{A}(\Phi) := \{H_\alpha \mid \alpha \in \Phi^+\}$ is called the **crystallographic arrangement** of the type Φ.

Definition 1.1. *Let $k \in \mathbb{Z}_{>0}$. An* ***extended Shi arrangement*** *Shi^k of the type Φ is an affine arrangement defined by*

$$\mathit{Shi}^k := \mathit{Shi}^k(\Phi^+) := \{H_{\alpha,j} \ \mid \ \alpha \in \Phi^+, \ j \in \mathbb{Z}, -k+1 \le j \le k\}.$$

The extended Shi arrangements for $k = 1$ were introduced by J.-Y. Shi [12, 13] in his study of the Kazhdan-Lusztig representation theory of the affine Weyl groups. For $k \geq 1$, they were studied in [14, 6] among others. Recall that the **cone** [10, Definition 1.15]

$$\mathcal{S}^k := \mathcal{S}^k(\Phi^+) := \mathbf{c}\mathit{Shi}^k$$

over Shi^k is a central arrangement in an $(\ell+1)$-dimensional Euclidean space $E := \mathbb{R}^{\ell+1}$. (Let z be the last coordinate of E and embed V into E as the affine hyperplane defined by $z = 1$.) Let $S(E^*)$ be the symmetric algebra of the dual space E^* of E. Let $\mathrm{Der}(S(E^*))$ denote the $S(E^*)$-module of derivations of $S(E^*)$ to itself:

$$\mathrm{Der}(S(E^*)) := \{\theta : S(E^*) \to S(E^*) \mid \theta \text{ is } \mathbb{R}\text{-linear and } \theta(fg) = f\theta(g) + g\theta(f) \text{ for any } f, g \in S(E^*)\}.$$

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A derivation $\theta \in \mathrm{Der}(S(E^*))$ is said to be **homogeneous of degree** d if $\theta(\alpha)$ is a homogeneous polynomial of degree d for any $\alpha \in E^*$ unless $\theta(\alpha) = 0$. Choose $\alpha_H \in E^*$ with $H = \ker(\alpha_H) \in \mathcal{S}^k$. Define

$$D_0(\mathcal{S}^k) := \{\theta \in \mathrm{Der}(S(E^*)) \mid \theta(\alpha_H) \in \alpha_H S(E^*) \text{ for each } H \in \mathcal{S}^k, \theta(z) = 0\}.$$

In [8], Edelman and Reiner conjectured that $D_0(\mathcal{S}^k)$ is a free $S(E^*)$-module for each $k \geq 1$. Yoshinaga verified this conjecture in [20]. Thus the module $D_0(\mathcal{S}^k)$ has bases over $S(E^*)$. However, we know very little about the bases other than their existence and degrees. In this article, we prove the unique existence of two distinguished homogeneous bases **SRB**+ and **SRB**− for $D_0(\mathcal{S}^k)$ with the following nice divisibility properties relative to the simple system $\Delta := \{\alpha_1, \dots, \alpha_\ell\} \subset V^*$.

Theorem 1.2. *Let h be the Coxeter number of Φ.*
(1) There exists a homogeneous basis $\varphi_1^+, \dots, \varphi_\ell^+$ of the same degree kh for $D_0(\mathcal{S}^k)$ such that $\varphi_i^+(\alpha_j + kz)$ is divisible by $\alpha_j + kz$ whenever $i \neq j$. (This basis is called a **simple-root basis plus** *(SRB+).)*
(2) There exists a homogeneous basis $\varphi_1^-, \dots, \varphi_\ell^-$ of the same degree kh for $D_0(\mathcal{S}^k)$ such that φ_i^- is divisible by $\alpha_i - kz$ for any i. (This basis is called a **simple-root basis minus** *(SRB−).)*

This is an existence theorem. Actually we do not have a uniform method to construct simple-root bases at this writing. The following is the uniqueness theorem for the simple-root bases:

Theorem 1.3. *(1) Suppose that $\varphi_1^+, \dots, \varphi_\ell^+$ form a simple-root basis plus. If derivations $\phi_1^+, \dots, \phi_\ell^+$ are a homogeneous basis for $D_0(\mathcal{S}^k)$ such that each $\phi_i^+(\alpha_j + kz)$ is divisible by $\alpha_j + kz$ whenever $i \neq j$, then there exist nonzero constants $c_1^+, \dots, c_\ell^+$ satisfying $\phi_i^+ = c_i^+ \varphi_i^+$ for any i.*
(2) Suppose that $\varphi_1^-, \dots, \varphi_\ell^-$ form a simple-root basis minus. If derivations $\phi_1^-, \dots, \phi_\ell^-$ are a homogeneous basis for $D_0(\mathcal{S}^k)$ such that each derivation ϕ_i^- is divisible by $\alpha_i - kz$ for any i, then there exist nonzero constants $c_1^-, \dots, c_\ell^-$ satisfying $\phi_i^- = c_i^- \varphi_i^-$ for any i.

Only for $k = 1$, case-by-case constructions of the SRB− are given when the root system is either of the type $A_\ell, B_\ell, C_\ell, D_\ell$ or G_2 in [16, 15, 9].

The organization of this article is as follows. In Section 2, we review a recent refinement [5] of Yoshinaga's freeness criterion [20] before proving the two key results (Propositions 2.4 and 2.6) which we apply in Section 3 when we prove Theorems 1.2 and 1.3. We also characterize the simple roots in terms of the freeness of deleted/added Shi arrangements in Theorem 3.6. In Section 4, we will describe a unique W-invariant derivation (**k-Euler derivation**) related to the Catalan arrangement in terms of the SRB+. The 0-Euler derivation coincides with the classical Euler derivation.

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2. Freeness criteria

In the rest of the article we use [10] as a general reference.

Let $\mathcal{C}$ be a central arrangement in an $(\ell+1)$-dimensional vector space E. Choose $\alpha_H \in E^*$ with $\ker \alpha_H = H \in \mathcal{C}$. Let $\mathbf{m}$ be a **multiplicity** of $\mathcal{C}$:

$$\mathbf{m} : \mathcal{C} \to \mathbb{Z}_{>0}.$$

Let $S(E^*)$ be the ring of polynomial functions on E. Define an $S(E^*)$-module

$$D(\mathcal{C}, \mathbf{m}) := \{\theta \in \mathrm{Der}(S(E^*)) \mid \theta(\alpha_H) \in \alpha_H^{\mathbf{m}(H)} S(E^*) \text{ for each } H \in \mathcal{C}\}.$$

When $D(\mathcal{C}, \mathbf{m})$ is a free $S(E^*)$-module, we say that the multiarrangement $(\mathcal{C}, \mathbf{m})$ is **free**. We say that $\mathcal{C}$ is a **free arrangement** if $(\mathcal{C}, \mathbf{1})$ is free. Here $\mathbf{1}$ indicates the constant multiplicity whose value is equal to one. When $(\mathcal{C}, \mathbf{m})$ is free, $\exp(\mathcal{C}, \mathbf{m})$ of **exponents** denotes the set of degrees of homogeneous basis for $D(\mathcal{C}, \mathbf{m})$. We simply write $\exp(\mathcal{C})$ instead of $\exp(\mathcal{C}, \mathbf{1})$ if $\mathcal{C}$ is a free arrangement.

For a fixed hyperplane $H_0 \in \mathcal{C}$, define a multiarrangement $(\mathcal{C}'', \mathbf{z})$, which we call the **Ziegler restriction** [21], by

$$\mathcal{C}'' := \{H_0 \cap K \mid K \in \mathcal{C}' := \mathcal{C} \setminus \{H_0\}\}, \ \mathbf{z}(X) := |\{K \in \mathcal{C}' \mid X = K \cap H_0\}|,$$

where $\mathcal{C}''$ is an arrangement in H_0 and $X \in \mathcal{C}''$. For the intersection lattice $L(\mathcal{C})$ [10, Definition 2.1] of $\mathcal{C}$ and any $Y \in L(\mathcal{C})$ define the **localization** $\mathcal{C}_Y$ of $\mathcal{C}$ at Y by $\mathcal{C}_Y := \{H \in \mathcal{C} \mid Y \subseteq H\}$. Let us present a recent refinement of Yoshinaga's freeness criterion in [20]:

Theorem 2.1 ([5])**.** *Suppose $\ell + 1 > 3$. For a central arrangement $\mathcal{C}$ and an arbitrary hyperplane $H_0 \in \mathcal{C}$, the following two conditions are equivalent:*
(1) $\mathcal{C}$ is a free arrangement,
(2) (2-i) the Ziegler restriction $(\mathcal{C}'', \mathbf{z})$ is free and (2-ii) $\mathcal{C}_Y$ is free for any $Y \in L(\mathcal{C})$ such that $Y \subset H_0$ with $\mathrm{codim}_{H_0} Y = 2$.

For a fixed hyperplane $H_0 \in \mathcal{C}$, we may choose a basis $x_1, x_2, \ldots, x_\ell, z$ for E^* so that the hyperplane H_0 is defined by the equation $z = 0$. Then the Ziegler restriction $(\mathcal{C}'', \mathbf{z})$ is a multiarrangement in H_0. Let

$$D_0(\mathcal{C}) := \{\theta \in D(\mathcal{C}) \mid \theta(z) = 0\}.$$

Then

$$D(\mathcal{C}) = S\theta_E \oplus D_0(\mathcal{C}),$$

where θ_E is the Euler derivation. Note that $D_0(\mathcal{C})$ is a free $S(E^*)$-module if and only if $\mathcal{C}$ is a free arrangement. When $\mathcal{C}$ is a free arrangement, let $\exp_0(\mathcal{C})$ denote the set of degrees of homogeneous basis for $D_0(\mathcal{C})$. Note that the set $\exp_0(\mathcal{C})$ does not depend upon the choice of H_0. When we describe $\exp_0(\mathcal{C})$, we will use the notation a^n instead of listing $a, \ldots, a$ (n times). Let $D_0(\mathcal{C})_p$ denote the vector space consisting of the homogeneous derivations in $D_0(\mathcal{C})$ of degree p (and the zero derivation).

Theorem 2.2 (Ziegler [21])**.** *The Ziegler restriction map*

$$\mathbf{res} : D_0(\mathcal{C}) \to D(\mathcal{C}'', \mathbf{z})$$

defined by setting $z = 0$ is surjective if $\mathcal{C}$ is a free arrangement.

The following theorem was proved in [4] using the shift isomorphism of Coxeter multiarrangements:

Theorem 2.3 ([4], Corollary 12)**.** *Let $\mathcal{A}$ be a Coxeter arrangement in an ℓ-dimensional Euclidean space. For a $\{0,1\}$-valued multiplicity $\mathbf{m} : \mathcal{A} \to \{0,1\}$ and an integer $k > 0$, the following conditions are equivalent:*

(1) a multiarrangement $(\mathcal{A}, \mathbf{m})$ is free with exponents $(e_1, \dots, e_\ell)$.

(2) a multiarrangement $(\mathcal{A}, 2k+\mathbf{m})$ is free with exponents $(kh+e_1, \dots, kh+e_\ell)$.

(3) a multiarrangement $(\mathcal{A}, 2k-\mathbf{m})$ is free with exponents $(kh-e_1, \dots, kh-e_\ell)$.

Now we go back to the situation in Section 1: let Φ, Φ^+ and Δ be an irreducible root system, the set of postive roots, and the set of simple roots respectively. The following two Propositions 2.4 and 2.6 are keys to our proofs of Theorems 1.2 and 1.3. They are dual to each other.

Proposition 2.4. *For any subset Γ of the simple system Δ, the arrangement*

$$\mathcal{B}_\Gamma^+ := \mathcal{B}_\Gamma^+(\Phi^+) := \mathcal{S}^k \cup \{\mathbf{c}H_{\alpha,-k} \mid \alpha \in \Gamma\}$$

is a free arrangement with

$$\exp_0(\mathcal{B}_\Gamma^+) = ((kh+1)^{|\Gamma|}, (kh)^{\ell-|\Gamma|}).$$

Proof. Case 1. When $\ell = 2$, Φ is of the type either A_2, B_2 or G_2. Then $\exp_0(\mathcal{S}^k) = ((kh)^2) = (kh, kh)$ and $\Delta = \{\alpha_1, \alpha_2\}$. For an affine 2-arrangement $\mathcal{A}$ and an affine line H_0, define

$$\mathcal{A} \cap H_0 := \{K \cap H_0 \mid K \in \mathcal{A}, K \neq H_0\}.$$

Then, by directly counting intersection points, we get the following equalities:

$$|Shi^k \cap H_{\alpha,-k}| = kh \ (\alpha \in \Delta),$$

$$|(Shi^k \cup \{H_{\alpha_1,-k}\}) \cap H_{\alpha_2,-k}| = kh + 1.$$

Thus we may verify the statement by applying the addition theorem [17] [10, Theorem 4.49] to $\mathcal{S}^k$ for the types of A_2, B_2 and G_2.

Case 2. Suppose that $\ell \geq 3$. We will apply Theorem 2.1 by verifying the two conditions (2-i) and (2-ii).

(2-i) Note that the Ziegler restriction of $\mathcal{B}_\Gamma^+$ to the hyperplane H_∞ at infinity coincides with $(\mathcal{A}(\Phi), \mathbf{z}_\Gamma^+)$, where

$$\mathbf{z}_\Gamma^+(H_\alpha) = |\{j \mid \mathbf{c}H_{\alpha,j} \in \mathcal{B}_\Gamma^+\}| = \begin{cases} 2k+1 & \text{if } \alpha \in \Gamma \\ 2k & \text{otherwise} \end{cases} \quad (\alpha \in \Phi^+).$$

If Γ is empty, then $\mathbf{z}_\Gamma^+ = \mathbf{z}_\emptyset^+ \equiv 2k$. Note that Γ is linearly independent because it is a set consisting of simple roots. Thus the arrangement

$$\mathcal{A}(\Gamma) := \{H_\alpha \mid \alpha \in \Gamma\}$$

is a free (Boolean) subarrangement of $\mathcal{A}(\Phi)$. Let χ_Γ be the characteristic function of $\mathcal{A}(\Gamma)$ in $\mathcal{A}(\Phi)$:

$$\chi_\Gamma(H_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Gamma \\ 0 & \text{otherwise} \end{cases} \quad (\alpha \in \Phi^+).$$

Since $\mathbf{z}_\Gamma^+ = \mathbf{z}_\emptyset^+ + \chi_\Gamma$, we may apply Theorem 2.3 to conclude that $(\mathcal{A}(\Phi), \mathbf{z}_\Gamma)$ is a free multiarrangement with exponents $((kh+1)^{|\Gamma|}, (kh)^{\ell-|\Gamma|})$.

(2-ii) We will prove that $(\mathcal{B}_\Gamma^+)_Y$ is free for any $Y \in L(\mathcal{B}_\Gamma^+)$ such that $Y \subset H_\infty$ with $\operatorname{codim}_{H_\infty} Y = 2$. Define X to be the unique subspace of V such that $X \in L(\mathcal{A}(\Phi))$ and $\mathbf{c}X \cap H_\infty = Y$. Let $X^\perp := \{\alpha \in V^* \mid \alpha|_X \equiv 0\}$. Then $\Phi_X := \Phi \cap X^\perp$ is also a (not necessarily irreducible) root system in $X^{\perp}$. The set of positive roots of Φ_X is induced from Φ^+: $\Phi_X^+ = \Phi^+ \cap \Phi_X$. It is not hard to see (e.g. [3, Lemma 3.1]) that

$$(\mathcal{S}^k)_Y = \mathbf{c}\left(Shi^k(\Phi_X^+) \times \emptyset_X\right),$$

where $\emptyset_X$ denotes the empty arrangement in X. Since $\dim X^{\perp} = 2$, Φ_X is either of the type $A_1 \times A_1$, A_2, B_2 or G_2.

Case 2.1. When Φ_X is of the type $A_1 \times A_1$, the arrangement $Shi^k(\Phi_X^+)$ is a product of two affine 1-arrangements. Thus any subarrangement of $(\mathcal{S}^k)_Y$ is a free arrangement. In particular, $(\mathcal{B}_\Gamma^+)_Y$ is a free arrangement.

Case 2.2. Suppose that Φ_X is of the type either A_2, B_2 or G_2. Suppose that $\alpha \in \Phi_X$ is a simple root of Φ. Then α is also a simple root of Φ_X because it cannot be expressed as a sum of two positive roots of Φ_X. Thus $\Phi_X \cap \Gamma$ consists of simple roots of Φ_X. Therefore

$$\begin{aligned}(\mathcal{B}_\Gamma^+)_Y &= (\mathcal{S}^k)_Y \cup \{\mathbf{c}H_{\alpha,k} \mid \alpha \in \Phi_X \cap \Gamma\} \\ &= \mathbf{c}\left(\left(Shi^k(\Phi_X^+) \cup \{H_{\alpha,k} \mid \alpha \in \Phi_X \cap \Gamma\}\right) \times \emptyset_X\right)\end{aligned}$$

is a free arrangement because of *Case 1.*

Now we apply Theorem 2.1 to complete the proof. □

Corollary 2.5. *The vector space $D_0(\mathcal{B}_\Gamma^+)_{kh}$ is $(\ell - |\Gamma|)$-dimensional.*

Proposition 2.6. *For any subset Γ of the simple system Δ, the arrangement*

$$\mathcal{B}_\Gamma^- := \mathcal{B}_\Gamma^-(\Phi^+) := \mathcal{S}^k \setminus \{\mathbf{c}H_{\alpha,k} \mid \alpha \in \Gamma\}$$

is a free arrangement with

$$\exp_0(\mathcal{B}_\Gamma^-) = ((kh-1)^{|\Gamma|}, (kh)^{\ell-|\Gamma|}).$$

Proof. Case 1. When $\ell = 2$, Φ is of the type either A_2, B_2 or G_2. Let $\exp_0(\mathcal{S}^k) = ((kh)^2)$ and $\Delta = \{\alpha_1, \alpha_2\}$. Then, by directly counting intersection points, we get the following equalities:

$$\begin{aligned}|Shi^k \cap H_{\alpha,k}| &= kh \ (\alpha \in \Delta), \\ |(Shi^k \setminus \{H_{\alpha_1,k}\}) \cap H_{\alpha_2,k}| &= kh - 1.\end{aligned}$$

Thus we may verify the statement by applying the deletion theorem [17] [10, Theorem 4.49] to $\mathcal{S}^k$ for the types of A_2, B_2 and G_2.

The rest is exactly the same as the proof of Proposition 2.4 if one replaces $\mathcal{B}_\Gamma^+$, $kh+1$, $H_{\bullet,-k}$, $2k+1$, $\cup$, $\mathbf{z}_\Gamma^+$, $\mathbf{z}_\Gamma^+ + \chi_\Gamma$ with $\mathcal{B}_\Gamma^-$, $kh-1$, $H_{\bullet,k}$, $2k-1$, $\setminus$, $\mathbf{z}_\Gamma^-$, $\mathbf{z}_\Gamma^- - \chi_\Gamma$ respectively. □

Corollary 2.7. *The vector space $D_0(\mathcal{B}_\Gamma^-)_{kh-1}$ is $|\Gamma|$-dimensional.*

3. Proof of main results

We will prove Theorems 1.2 and 1.3 in this section. Fix $k \in \mathbb{Z}_{>0}$ throughout in the rest of this article. Consider

$$D(\mathcal{A}(\Phi), 2k) = \{\theta \in \mathrm{Der}(S(V^*)) \mid \theta(\alpha) \in \alpha^{2k} S(V^*) \text{ for each } \alpha \in \Phi^+\}.$$

In [20], Yoshinaga proved the Edelman-Reiner conjecture [8] by verifying the surjectivity of the Ziegler restriction

$$\mathbf{res} : D_0(\mathcal{S}^k) \longrightarrow D(\mathcal{A}(\Phi), 2k)$$

defined by setting $z = 0$. Since the multiarrangement $(\mathcal{A}(\Phi), 2k)$ is free with exponents $(kh, kh, \ldots, kh)$ by [18], the central arrangement $\mathcal{S}^k$ is free with exponents $(1, kh, kh, \ldots, kh)$. Thus the homogeneous part

$$\mathbf{res} : D_0(\mathcal{S}^k)_{kh} \longrightarrow D(\mathcal{A}(\Phi), 2k)_{kh}$$

of degree kh of the Ziegler restriction map is a linear isomorphism.

We will first see that V and $D(\mathcal{A}(\Phi), 2k)_{kh}$ are W-isomorphic. Let F be the field of quotients of $S = S(V^*) = \mathbb{R}[x_1, \ldots, x_\ell]$. Recall a primitive derivation $D \in \mathrm{Der}(F)$ associated with $\mathcal{A}(\Phi)$: D satisfies

$$D(P_j) = \begin{cases} c \in \mathbb{R}^\times & \text{if } j = \ell \ , \\ 0 & \text{if } 1 \le j \le \ell - 1 \ . \end{cases}$$

Here $P_1, \ldots, P_\ell$ are basic invariants of the invariant subring S^W with

$$2 = \deg P_1 < \deg P_2 \le \cdots \le \deg P_{\ell-1} < \deg P_\ell = h.$$

Then the choice of D has the ambiguity of nonzero constant multiples.

Consider the Levi-Civita (or Riemannian) connection with respect to the standard and W-invariant inner product I (e.g., see [7, 3.6]):

$$\nabla : \mathrm{Der}(F) \times \mathrm{Der}(F) \longrightarrow \mathrm{Der}(F), \quad (\xi, \eta) \mapsto \nabla_\xi \eta.$$

Note that

$$\nabla_\xi \eta = \nabla_\xi \sum_{i=1}^{\ell} f_i (\partial/\partial x_i) = \sum_{i=1}^{\ell} \xi(f_i)(\partial/\partial x_i)$$

for $\xi \in \mathrm{Der}(F)$ and $\eta := \sum_{i=1}^{\ell} f_i(\partial/\partial x_i) \in \mathrm{Der}(F)$ because $\nabla_\xi(\partial/\partial x_i) = 0$ for $1 \le i \le \ell$. Consider $T := \mathbb{R}[P_1, \ldots, P_{\ell-1}]$-linear covariant derivative $\nabla_D : \mathrm{Der}(F) \to \mathrm{Der}(F)$. By [2] it induces a T-linear bijection

$$\nabla_D : D(\mathcal{A}(\Phi), 2k+1)^W \xrightarrow{\sim} D(\mathcal{A}(\Phi), 2k-1)^W \ (k > 0).$$

The covariant derivative ∇_D was introduced by K. Saito (e.g. [11]) to study the flat structure (or the Frobenius manifold structure) of the orbit space V/W. Let $\theta_E := \sum_{i=1}^{\ell} x_i(\partial/\partial x_i)$ denote the Euler derivation. Since $\theta_E \in D(\mathcal{A}(\Phi), 1)^W$, one has

$$\nabla_D^{-k} \theta_E \in D(\mathcal{A}(\Phi), 2k+1)^W$$

which plays a principal role in this section.

For any $v \in V$, there exists a unique derivation $\partial_v \in \mathrm{Der}(S)_0$ of degree zero such that

$$\partial_v(\alpha) := \langle \alpha, v \rangle \quad (\alpha \in V^*).$$

Thus we may identify V with $\mathrm{Der}(S)_0$ by the W-isomorphism

$$V \longrightarrow \mathrm{Der}(S)_0$$

defined by $v \mapsto \partial_v$.

Let $\Omega(S)$ be the S-module of regular one-forms:

$$\Omega(S) = S(dx_1) \oplus \cdots \oplus S(dx_\ell).$$

Then $\Omega(S) = \bigoplus_{p \ge 0} \Omega(S)_p$ is a graded S-module where

$$\Omega(S)_p := \left\{ \sum_{i=1}^{\ell} f_i \ dx_i \mid f_i \in S_p \text{ for } 1 \le i \le \ell \right\}.$$

We may identify V^* with $\Omega(S)_0$ as W-modules by the bijection $\alpha \mapsto d\alpha$.

Recall the W-invariant dual inner product $I^* : V^* \times V^* \to \mathbb{R}$. Define a W-isomorphism

$$I^* : \Omega(S)_0 \to \mathrm{Der}(S)_0$$

by $(I^*(d\alpha))(\beta) := I^*(d\alpha, d\beta) \quad (\alpha, \beta \in V^*)$.

When a W-isomorphism $\Xi : \mathrm{Der}(S)_0 \to D(\mathcal{A}(\Phi), 2k)_{kh}$ is given, the new map Θ^* is defined by the diagram:

$$\begin{tikzcd}
 & D_0(\mathcal{S}^k)_{kh} \arrow[d, "\mathbf{res}", "\wr"'] & \\
 & D(\mathcal{A}(\Phi), 2k)_{kh} & \\
V^* = \Omega(S)_0 \arrow[uur, "\Theta^*"] \arrow[ur, "\Xi^*", "\wr\ W\text{-iso.}"'] \arrow[rr, "I^*"'] & & \mathrm{Der}(S)_0 = V. \arrow[uul, "\Theta"'] \arrow[ul, "\Xi"', "\wr\ W\text{-iso.}"]
\end{tikzcd}$$

Proposition 3.1. *(1) For any primitive derivation D, define*

$$\Xi_D : \mathrm{Der}(S)_0 \longrightarrow D(\mathcal{A}(\Phi), 2k)_{kh}$$

by

$$\Xi_D(\partial_v) := \nabla_{\partial_v} \nabla_D^{-k} \theta_E \quad (v \in V).$$

Then Ξ_D is a W-isomorphism.
(2) Conversely, for any W-isomorphism $\Xi : \mathrm{Der}(S)_0 \xrightarrow{\sim} D(\mathcal{A}(\Phi), 2k)_{kh}$, there exists a unique primitive derivation such that $\Xi = \Xi_D$.

Proof. (1) was proved by Yoshinaga in [19]. (See [18] also.) (2) follows from (1) and Schur's lemma. □

Remark 3.2. *Note that the arrangement $\mathcal{S}^k$ is not W-stable. Therefore the ℓ-dimensional vector space $D_0(\mathcal{S}^k)_{kh}$ is not naturally a W-module, while the ℓ-dimensional vector spaces V and $D(\mathcal{A}(\Phi), 2k)_{kh}$ are both W-modules.*

Proof of Theorem 1.2. Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be the set of simple roots. Then Δ is a basis for V^*. Let $\{\alpha_1^*, \dots, \alpha_\ell^*\}$ be the basis for V which is dual to Δ: $\langle \alpha_i, \alpha_j^* \rangle = \delta_{ij}$ (Kronecker's delta). Then $\partial_{\alpha_i^*}(\alpha_j) = \delta_{ij}$. Fix a primitive derivation D and let $\Xi = \Xi_D$.

(1) Define

$$\varphi_i^+ := \Theta(\partial_{\alpha_i^*}) \ (1 \le i \le \ell). \tag{3.1}$$

Then we have, for $i \neq j$,

$$\begin{aligned}\left[\mathbf{res}(\varphi_i^+)\right](\alpha_j) &= \left[\mathbf{res}(\Theta(\partial_{\alpha_i^*}))\right](\alpha_j) = \left[\Xi(\partial_{\alpha_i^*})\right](\alpha_j) = (\nabla_{\partial_{\alpha_i^*}}\nabla_D^{-k}\theta_E)(\alpha_j) \\ &= \partial_{\alpha_i^*}\left(\left(\nabla_D^{-k}\theta_E\right)(\alpha_j)\right) \in \alpha_j^{2k+1}S\end{aligned}$$

because $\partial_{\alpha_i^*}(\alpha_j) = 0$. Define $\Gamma_i^+ := \Delta \setminus \{\alpha_i\}$. Then we have the following commutative diagram

$$\begin{array}{ccc} D_0(\mathcal{S}^k)_{kh} & \xrightarrow[\sim]{\mathbf{res}} & D(\mathcal{A}(\Phi), 2k)_{kh} \\ \bigcup & & \bigcup \\ D_0(\mathcal{B}_{\Gamma_i^+}^+)_{kh} & \xrightarrow[\sim]{\mathbf{res}} & D(\mathcal{A}(\Phi), \mathbf{z}_{\Gamma_i^+}^+)_{kh} \end{array}$$

because of Proposition 2.4, Theorem 2.2 and the horizontal linear isomorphisms in the diagram above. Since $\varphi_i^+ \in D_0(S^k)_{kh}$ and $\mathbf{res}\left(\varphi_i^+\right) \in D(\mathcal{A}(\Phi), \mathbf{z}_{\Gamma_i^+}^+)_{kh}$, we conclude that $\varphi_i^+ \in D_0(\mathcal{B}_{\Gamma_i^+}^+)_{kh}$ by chasing the diagram above. Therefore the derivations $\varphi_1^+, \dots, \varphi_\ell^+$ form an SRB+ because they form a basis for $D_0(\mathcal{S}^k)_{kh}$ and each $\varphi_i^+(\alpha_j + kz)$ is divisible by $\alpha_j + kz$ whenever $i \neq j$.

(2) Define

$$\varphi_i^- := \Theta^*(d\alpha_i)) \quad (1 \le i \le \ell). \tag{3.2}$$

Then we have

$$\mathbf{res}(\varphi_i^-) = \mathbf{res}\left(\Theta^*(d\alpha_i)\right) = \Xi(I^*(d\alpha_i)) = \nabla_{I^*(d\alpha_i)}\nabla_D^{-k}\theta_E.$$

Let s_i denote the orthogonal reflection with respect to α_i. Since $s_i(d\alpha_i) = -d\alpha_i$, we have

$$s_i(\mathbf{res}(\varphi_i^-)) = -\mathbf{res}(\varphi_i^-).$$

Express

$$\mathbf{res}(\varphi_i^-) = \sum_{p=1}^{\ell} f_p \partial_{\alpha_p^*} \quad (f_p \in S).$$

Recall $s_i(\partial_{\alpha_p^*}) = \partial_{\alpha_p^*}$ whenever $p \neq i$. Thus we have $s_i(f_p) = -f_p$ whenever $p \neq i$. Therefore f_p is divisible by α_i whenever $p \neq i$. We also know that

$$f_i = \left[\mathbf{res}(\varphi_i^-)\right](\alpha_i) = (I^*(d\alpha_i))\left((\nabla_D^{-k}\theta_E)(\alpha_i)\right)$$

is divisible by α_i^{2k} because $\nabla_D^{-k}\theta_E(\alpha_i)$ is divisible by α_i^{2k+1}. Therefore we conclude that φ_i^- is divisible by α_i for any i. Define $\Gamma_i^- := \{\alpha_i\}$. Then we have the following commutative diagram

$$\begin{array}{ccc} D_0(\mathcal{S}^k)_{kh} & \xrightarrow[\sim]{\mathbf{res}} & D(\mathcal{A}(\Phi), 2k)_{kh} \\ \bigcup & & \bigcup \\ (\alpha_i - kz) \cdot D_0(\mathcal{B}_{\Gamma_i^-}^-)_{kh-1} & \xrightarrow[\sim]{\mathbf{res}} & \alpha_i \cdot D(\mathcal{A}(\Phi), \mathbf{z}_{\Gamma_i^-}^-)_{kh-1} \end{array}$$

because of Proposition 2.6, Theorem 2.2 and the horizontal linear isomorphisms in the diagram above. Since $\varphi_i^- \in D_0(S^k)_{kh}$ and

$$\mathbf{res}\left(\varphi_i^-\right) \in \alpha_i \cdot D(\mathcal{A}(\Phi), \mathbf{z}_{\Gamma_i^-}^-)_{kh-1},$$

we may conclude $\varphi_i^- \in (\alpha_i - kz) \cdot D_0(\mathcal{B}^-_{\Gamma_i^-})_{kh-1}$ by chasing the diagram above. Therefore the derivations $\varphi_1^-, \dots, \varphi_\ell^-$ form an SRB$-$ because they form a basis for $D_0(\mathcal{S}_k^-)_{kh}$ and each φ_i^- is divisible by $\alpha_i - kz$. □

Proof of Theorem 1.3. (1) Assume that $\varphi_1^+, \dots, \varphi_\ell^+$ form an SRB+. Then

$$\varphi_i^+ \in D_0(\mathcal{B}^+_{\Gamma_i^+})_{kh}.$$

Thus (1) follows from

$$\dim D_0(\mathcal{B}^+_{\Gamma_i^+})_{kh} = 1$$

which is a consequence of Corollary 2.5 because $|\Gamma_i^+| = \ell - 1$.

(2) Assume that $\varphi_1^-, \dots, \varphi_\ell^-$ form an SRB$-$. Then

$$\varphi_i^- \in (\alpha_i - kz) D_0(\mathcal{B}^-_{\Gamma_i^-})_{kh-1}.$$

Thus (2) follows from the equality

$$\dim D_0(\mathcal{B}^-_{\Gamma_i^-})_{kh-1} = 1$$

which is a consequence from Corollary 2.7 because $|\Gamma_i^-| = 1$. □

Remark 3.3. *We have just proved the existence of simple root bases. It is, however, not easy to find an SRB± because the Ziegler restriction isomorphism*

$$\mathbf{res} : D_0(\mathcal{S}^k)_{kh} \xrightarrow{\sim} D(\mathcal{A}(\Phi), 2k)_{kh}$$

is hard to invert.

Example 3.4. *Let Φ be a root system of the type A_2. Let $\Delta := \{\alpha_1, \alpha_2\}$ be a simple system. The Coxeter number h is equal to* 3. *Let $k = 1$. In this case,*

$$\begin{aligned}
\varphi_1^- &:= -(1/2)(\alpha_1 - z)\left\{\alpha_1(\alpha_1 + 2\alpha_2 - z)\partial_{\alpha_1^*} + \alpha_2(\alpha_2 - z)\partial_{\alpha_2^*}\right\},\\
\varphi_2^- &:= (1/2)(\alpha_2 - z)\left\{\alpha_1(\alpha_1 - z)\partial_{\alpha_1^*} + \alpha_2(2\alpha_1 + \alpha_2 - z)\partial_{\alpha_2^*}\right\}
\end{aligned}$$

form an SBR- for $D_0(\mathcal{S}^1)$. They are the inverse images of

$$\begin{aligned}
-(1/2)\alpha_1\left\{\alpha_1(\alpha_1 + 2\alpha_2)\partial_{\alpha_1^*} + \alpha_2^2\ \partial_{\alpha_2^*}\right\},\\
(1/2)\alpha_2\left\{\alpha_1^2\ \partial_{\alpha_1^*} + \alpha_2(2\alpha_1 + \alpha_2)\partial_{\alpha_2^*}\right\}
\end{aligned}$$

respectively by the Ziegler restriction isomorphism

$$\mathbf{res} : D_0(\mathcal{S}^1)_3 \xrightarrow{\sim} D(\mathcal{A}(\Phi), 2)_3.$$

For $k \geq 2$, the SRB± for the type A_2 are explicitly obtained in [1]. *For $\ell \geq 2$ and $k = 1$, we need the Bernoulli polynomials to explicitly describe the SRB± for the type A_ℓ* [16].

The derivations $\varphi_1^+, \dots, \varphi_\ell^+$ (or $\varphi_1^-, \dots, \varphi_\ell^-$) defined by (3.1) (resp. (3.2)) are called the **standard simple-root basis plus (SSRB+)** (resp. **standard simple-root basis minus (SSRB$-$)**). Both SSRB+ and SSRB$-$ have no ambiguity at all when a primitive derivation D is fixed. The following proposition gives a relationship between the SSRB+ and the SSRB$-$:

Proposition 3.5. *Fix a primitive derivation D. Suppose that $\varphi_1^+, \dots, \varphi_\ell^+$ and $\varphi_1^-, \dots, \varphi_\ell^-$ are the SSRB+ and the SSRB− respectively. Then*

$$\varphi_i^- = \sum_{p=1}^{\ell} I^*(\alpha_i, \alpha_p)\varphi_p^+ \quad (1 \le i \le \ell).$$

Proof. For each i, one has

$$\begin{aligned}\varphi_i^- = \Theta^*(d\alpha_i) = \Theta(I^*(d\alpha_i)) &= \Theta(\sum_{p=1}^{\ell} I^*(d\alpha_i, d\alpha_p)\partial_{\alpha_p^*}) \\ &= \sum_{p=1}^{\ell} I^*(d\alpha_i, d\alpha_p)\Theta(\partial_{\alpha_p^*}) = \sum_{p=1}^{\ell} I^*(\alpha_i, \alpha_p)\varphi_p^+.\end{aligned}$$

□

The following proposition asserts that the simple roots can be characterized by the freeness of an added/deleted Shi arrangement:

Theorem 3.6. *Let $\alpha \in \Phi^+$. Then*
(1) the arrangement $\mathcal{S}^k \cup \{\mathbf{c}H_{\alpha,-k}\}$ is a free arrangement if and only if α is a simple root, and
(2) the arrangement $\mathcal{S}^k \setminus \{\mathbf{c}H_{\alpha,k}\}$ is a free arrangement if and only if α is a simple root.

Proof. (1) By Proposition 2.4, the "if part" is already proved. Assume that $\alpha \in \Phi^+$ is a non-simple root. We will prove that $\mathcal{S}^k \cup \{\mathbf{c}H_{\alpha,-k}\}$ is not free. We may express $\alpha = \beta_1 + \beta_2$ with $\beta_i \in \Phi^+$ $(i = 1, 2)$. Let H_i be the hyperplane defined by $\beta_i = 0$ $(i = 1, 2)$. Then $X := H_1 \cap H_2$ is of codimension two in V because of basic properties of the root systems. As in the proof of Proposition 2.4, consider the two-dimensional root system $\Phi_X = \Phi \cap X^\perp$. Note that Φ_X is not of the type $A_1 \times A_1$ because $\{\alpha, \beta_1, \beta_2\} \subseteq \Phi_X$. Recall that the localization $\left(\mathcal{S}^k \cup \{\mathbf{c}H_{\alpha,-k}\}\right)_Y$ is free if $\mathcal{S}^k \cup \{\mathbf{c}H_{\alpha,-k}\}$ is free. In the root system Φ_X, α cannot be a simple root since α is a sum of two positive roots β_1 and β_2. Thus we may assume that Φ is a two-dimensional root system without loss of generality. In this case the arrangement $\mathcal{S}^k \cup \{\mathbf{c}H_{\alpha,-k}\}$ is not free because of the addition theorem and the equality

$$|Shi^k \cap H_{\alpha,-k}| = kh + 1 \ (\alpha \notin \Delta),$$

which can be verified by directly counting intersection points.

(2) Exactly the same as the proof of (1) if one replaces Proposition 2.4, $\cup$, $kh+1$, $H_{\bullet,-k}$, addition theorem with Proposition 2.6, $\setminus$, $kh - 1$, $H_{\bullet,k}$, deletion theorem respectively. □

4. The k-Euler derivations

Let $k \in \mathbb{Z}_{\ge 0}$. An **extended Catalan arrangement** Cat^k of the type Φ is an affine arrangement defined by

$$Cat^k := Cat^k(\Phi^+) := \{H_{\alpha,j} \ \mid \ \alpha \in \Phi^+, \ j \in \mathbb{Z}, -k \le j \le k\}.$$

Its cone

$$\mathcal{C}^k := \mathcal{C}^k(\Phi^+) := \mathbf{c}\,Cat^k$$

was proved to be a free arrangement with

$$\exp_0(\mathcal{C}^k) = (kh + d_1, kh + d_2, \ldots, kh + d_\ell)$$

by Yoshinaga [20]. Here the integers $d_1, d_2, \ldots, d_\ell$ are the exponents of the root system Φ with

$$d_1 \leq d_2 \leq \cdots \leq d_\ell.$$

Since $1 = d_1 < d_2$, one has $kh + 1 < kh + d_2$. Thus we know that $D_0(\mathcal{C}^k)_{kh+1}$ is a one-dimensional vector space.

Definition 4.1. *We say that η^k is a k-***Euler derivation** *if η^k is a basis for the vector space $D_0(\mathcal{C}^k)_{kh+1}$.*

Note that the ordinary Euler derivation $\theta_E = \sum_{i=1}^{\ell} x_i(\partial/\partial x_i)$ is a 0-Euler derivation because $Cat^0 = \mathcal{A}(\Phi)$. The choice of a k-Euler derivation has the ambiguity of nonzero constant multiples.

Proposition 4.2. *A k-Euler derivation is W-invariant, where the group W acts trivially on the variable z.*

Proof. Note that $\mathcal{C}^k$ is stable under the action of W unlike $\mathcal{S}^k$. Since the W-invariant derivation $\nabla_D^{-k}\theta_E$ is a basis for $D(\mathcal{A}(\Phi), 2k+1)_{kh+1}$, we obatin

$$D(\mathcal{A}(\Phi), 2k+1)_{kh+1} = D(\mathcal{A}(\Phi), 2k+1)^W_{kh+1}.$$

Since the Ziegler restriction map

$$D_0(\mathcal{C}^k)_{kh+1} \xrightarrow[\sim]{\mathbf{res}} D(\mathcal{A}(\Phi), 2k+1)_{kh+1}$$

is a W-isomorphism, we obtain

$$D_0(\mathcal{C}^k)_{kh+1} = D_0(\mathcal{C}^k)^W_{kh+1}.$$

$\square$

We may describe a k-Euler derivation as follows:

Theorem 4.3. *Let $\varphi_1^+, \ldots, \varphi_\ell^+$ be an SSRB+. The derivation*

$$\eta^k := \sum_{i=1}^{\ell} (\alpha_i + kz)\varphi_i^+$$

is a k-Euler derivation.

Proof. Note that $\mathcal{B}_\Delta^+$ is a subarrangement of $\mathcal{C}^k$. Consider a commutative diagram

$$\begin{array}{ccc} D_0(\mathcal{B}_\Delta^+)_{kh+1} & \xrightarrow[\sim]{\mathbf{res}} & D(\mathcal{A}(\Phi), \mathbf{z}_\Delta^+)_{kh+1} \\ \bigcup & & \bigcup \\ D_0(\mathcal{C}^k)_{kh+1} & \xrightarrow[\sim]{\mathbf{res}} & D(\mathcal{A}(\Phi), 2k+1)_{kh+1}. \end{array}$$

Since $(\alpha_i + kz)\varphi_i^+ \in D_0(\mathcal{B}_\Delta^+)_{kh+1}$ for any i by Theorem 1.2 (1), we have $\eta^k \in D_0(\mathcal{B}_\Delta^+)_{kh+1}$. We also have

$$\begin{aligned}\mathbf{res}\left(\eta^k\right) &= \mathbf{res}\left(\sum_{i=1}^{\ell}(\alpha_i + kz)\varphi_i^+\right)\\ &= \sum_{i=1}^{\ell}\alpha_i \Xi(\partial_{\alpha_i^*}) = \sum_{i=1}^{\ell}\alpha_i \nabla_{\partial_{\alpha_i^*}}\nabla_D^{-k}\theta_E = \nabla_{\theta_E}\nabla_D^{-k}\theta_E\\ &= (kh+1)\nabla_D^{-k}\theta_E \in D(\mathcal{A}(\Phi), 2k+1)_{kh+1}.\end{aligned}$$

Thus we may conclude that $\eta^k \in D_0(\mathcal{C}^k)_{kh+1}$ by chasing the diagram above. $\square$

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