Braid groups in complex Grassmannians

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Abstract

We describe the fundamental group and second homotopy group of ordered $k-$point sets in $Gr(k,n)$ generating a subspace of fixed dimension.

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1 Introduction

Let M be a manifold and Σ_h be the symmetric group on h elements.
The *ordered* and *unordered configuration spaces* of h distinct points in M, $\mathcal{F}_h(M) = \{(x_1,\ldots,x_h) \in M^h | x_i \neq x_j,\ i \neq j\}$ and $\mathcal{C}_h(M) = \mathcal{F}_h(M)/\Sigma_h$, have been widely studied. In recent papers ([BP, MPS, MS]), new configuration spaces were introduced when M is, respectively, the projective space $\mathbb{CP}^n$, the affine space $\mathbb{C}^n$ and the Grassmannian manifold $Gr(k,n)$ of k-dimensional subspaces of $\mathbb{C}^n$, by stratifying the configuration spaces $\mathcal{F}_h(M)$ (resp. $\mathcal{C}_h(M)$) with complex submanifolds $\mathcal{F}_h^i(M)$ (resp. $\mathcal{C}_h^i(M)$) defined as the ordered (resp. unordered) configuration spaces of all h points in M

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generating a subspace of dimension i. The homotopy groups of those configuration spaces are interesting as they are strongly related to the homotopy groups of the Grassmannian manifolds, i.e. of spheres.
In [BP] (resp. [MPS]), the fundamental groups $\pi_1(\mathcal{F}_h^i(\mathbb{CP}^n))$ and $\pi_1(\mathcal{C}_h^i(\mathbb{CP}^n))$ (resp. $\pi_1(\mathcal{F}_h^i(\mathbb{C}^n))$ and $\pi_1(\mathcal{C}_h^i(\mathbb{C}^n)))$ are computed, proving that the former are trivial and the latter are isomorphic to the symmetric group Σ_h except when $i = 1$ (resp. $i = 1$ and $i = n = h - 1$) providing, in this last case, a presentation for both $\pi_1(\mathcal{F}_h^1(\mathbb{CP}^n))$ and $\pi_1(\mathcal{C}_h^1(\mathbb{CP}^n))$ (resp. $\pi_1(\mathcal{F}_h^i(\mathbb{C}^n))$ and $\pi_1(\mathcal{C}_h^i(\mathbb{C}^n)))$ which is similar to those of the braid groups of the sphere.
In this paper we generalize the results obtained in [BP] when M is the projective space $\mathbb{CP}^{n-1} = Gr(1, n)$, to the case of Grassmannian manifold $Gr(k, n)$ of k-dimensional subspaces of $\mathbb{C}^n$. We prove that if $\mathcal{F}_h^i(k, n)$ is the i-th ordered configuration space of all distinct points $H_1, \dots, H_h$ in the Grassmannian manifold $Gr(k, n)$ whose sum is a subspace of dimension i, then the following result holds.

Theorem 1.1. *The non-empty ordered configuration spaces $\mathcal{F}_h^i(k, n)$ are all simply connected if $k > 1$.*

From this, we immediately obtain that the fundamental group of the i-th unordered configuration space $\mathcal{F}_h^i(k, n)/\Sigma_h$ is isomorphic to Σ_h.
These results are stated in Section 2. In Section 3 we compute the second homotopy group of the i-th configuration spaces in two special cases: the case in which the subspaces are in direct sum and the case of two subspaces.

Theorem 1.2. *If $hk < n$, $\pi_2(\mathcal{F}_h^{hk}(k, n)) = \mathbb{Z}^h$, while $\pi_2(\mathcal{F}_h^{hk}(k, hk)) = \mathbb{Z}^{h-1}$. If $k < i < n$, $\pi_2(\mathcal{F}_2^i(k, n)) = \mathbb{Z}^3$, while $\pi_2(\mathcal{F}_2^n(k, n)) = \mathbb{Z}^2$.*

2 The first homotopy group of $\mathcal{F}_h^i(k, n)$

Let $Gr(k, n)$ be the Grassmannian manifold parametrizing k-dimensional subspaces of $\mathbb{C}^n$, $0 < k < n$. In [MS] authors define the space $\mathcal{F}_h^i(k, n)$ as the ordered configuration space of all h distinct points $H_1, \dots, H_h$ in $Gr(k, n)$ such that the dimension of the sum $\dim(H_1 + \cdots + H_h)$ equals i.

Remark 2.1. *The following easy facts hold:*

1. *if $h = 1$, $\mathcal{F}_h^i(k, n)$ is empty except for $i = k$ and $\mathcal{F}_1^k(k, n) = Gr(k, n)$;*

2. *if* $i=1$, $\mathcal{F}_h^i(k,n)$ *is empty except for* $k,h=1$ *and* $\mathcal{F}_1^1(1,n)=Gr(1,n)=\mathbb{CP}^{n-1}$*;*

3. *if* $h\geq 2$ *and* $k=n-1$ *then* $\mathcal{F}_h^i(k,n)$ *is empty except for* $i=n$*, and, since the sum of two (different) hyperplanes is* $\mathbb{C}^n$*,* $\mathcal{F}_h^n(n-1,n)=\mathcal{F}_h(Gr(n-1,n))=\mathcal{F}_h(\mathbb{CP}^{n-1})$*;*

4. *if* $h\geq 2$ *then* $\mathcal{F}_h^i(k,n)\neq\emptyset$ *if and only if* $k+1\leq i\leq\min(kh,n)$*;*

5. *if* $h\geq 2$ *then* $\mathcal{F}_h(Gr(k,n))=\coprod_{i=k+1}^{\min(hk,n)}\mathcal{F}_h^i(k,n)$*, with the open stratum given by the case of maximum dimension* $i=$*min*(hk,n)*;*

6. *if* $h\geq 2$ *then the adjacency of the non-empty strata is given by*

$$\overline{\mathcal{F}_h^i(k,n)}=\mathcal{F}_h^{k+1}(k,n)\coprod\dots\coprod\mathcal{F}_h^i(k,n).$$

As the case $k=1$ has been treated in [BP] and, by the above remarks, the case $h=1$ is trivial, in this paper we will consider $h,k>1$ (and hence $i>k$).

In [MS], authors proved that $\mathcal{F}_h^i(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^h$ of dimension $i(n-i)+hk(i-k)$, and that if $i=\min(n,hk)$ and $n\neq hk$ then the open strata $\mathcal{F}_h^i(k,n)$ are simply connected except for $n=2$ (and $k=1$), i.e.

$$\pi_1(\mathcal{F}_h^{\min(n,kh)}(k,n))=\begin{cases}0 & \text{if } n\neq hk\\ \mathcal{PB}_h(S^2) & \text{if } n=2,\ k=1\end{cases}\tag{1}$$

where $\mathcal{PB}_h(S^2)$ is the pure braid group on h strings of the sphere S^2.

In order to complete this result and compute fundamental groups in all cases we need two Lemmas.

Lemma 2.2. *Let* $V=(H_1,\dots,H_h)$ *be an element in the space* $\mathcal{F}_h^i(k,n)$ *and denote the sum* $H_1+\dots+H_h\in Gr(i,n)$ *by* $\gamma(V)$*, then the map*

$$\gamma:\mathcal{F}_h^i(k,n)\to Gr(i,n)\tag{2}$$

is a locally trivial fibration with fiber $\mathcal{F}_h^i(k,i)$*.*

Proof. Let V_0 be an element in the Grassmannian manifold $Gr(i,n)$. Fix $L_0 \in Gr(n-i,n)$ such that $L_0 \cap V_0 = \{0\}$ and let $\varphi : \mathbb{C}^n \to V_0$ be the linear projection on V_0 given by the direct sum decomposition $L_0 + V_0 = \mathbb{C}^n$. If $\mathcal{F}_h^i(k,V_0)$ is the ordered configuration space of h distinct k-dimensional spaces in V_0 whose sum is an i-dimensional subspace, then $\mathcal{F}_h^i(k,V_0)$ coincides with $\mathcal{F}_h^i(k,i)$ when a basis in V_0 is fixed.

Let $\mathcal{U}_{L_0}$ be the open neighborhood of V_0 in $Gr(i,n)$ defined as

$$\mathcal{U}_{L_0} = \{V \in Gr(i,n) | \ L_0 \cap V = \{0\}\}.$$

The restriction of the projection φ to an element V in $\mathcal{U}_{L_0}$ is a linear isomorphism $\varphi_V : V \to V_0$ and a local trivialization for γ is given by the homeomorphism

$$\begin{aligned} f : \gamma^{-1}(\mathcal{U}_{L_0}) &\to \mathcal{U}_{L_0} \times \mathcal{F}_h^i(k,V_0) \\ y = (H_1,\ldots,H_h) &\mapsto \big(\gamma(y), (\varphi_{\gamma(y)}(H_1),\ldots,\varphi_{\gamma(y)}(H_h))\big) \end{aligned}$$

which makes the following diagram commute.

$$\begin{array}{ccc} \gamma^{-1}(\mathcal{U}_{L_0}) & \xrightarrow{\quad f \quad} & \mathcal{U}_{L_0} \times \mathcal{F}_h^i(k,i) \\ & {}_{\gamma}\searrow \quad \swarrow_{pr_1} & \\ & \mathcal{U}_{L_0} & \end{array}$$

This completes the proof. ☐

Lemma 2.3. *The projection map on the first $h-1$ entries*

$$\begin{aligned} pr : \mathcal{F}_h^{kh}(k,n) &\to \mathcal{F}_{h-1}^{k(h-1)}(k,n) \\ (H_1,\ldots,H_h) &\mapsto (H_1,\ldots,H_{h-1}) \end{aligned} \tag{3}$$

is a locally trivial fibration for any $n \geq kh$. Moreover, if $n = kh$, the fiber is $\mathbb{C}^{k(kh-k)}$.

Proof. Let V_0 be an element in $\mathcal{F}_{h-1}^{k(h-1)}(k,n)$. Fix $L_0 \in Gr(n-k(h-1),n)$ such that $L_0 \cap \gamma(V_0) = \{0\}$ and let $\varphi : \mathbb{C}^n \to \gamma(V_0)$ be the linear projection

on $\gamma(V_0)$ given by the direct sum decomposition $L_0 + \gamma(V_0) = \mathbb{C}^n$.
The fiber of the projection map pr over V_0 is the open set

$$U_{\gamma(V_0)} = \{H \in Gr(k,n) | H \cap \gamma(V_0) = \{0\}\}.$$

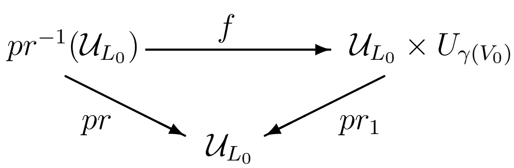
Let $\mathcal{U}_{L_0}$ be the open neighborhood of V_0 in $\mathcal{F}_{h-1}^{k(h-1)}(k,n)$ defined as

$$\mathcal{U}_{L_0} = \{V \in \mathcal{F}_{h-1}^{k(h-1)}(k,n) |\ L_0 \cap \gamma(V) = \{0\}\}.$$

If V is a point in $\mathcal{U}_{L_0}$, the restriction of the map φ to $\gamma(V)$ is a linear isomorphism $\tilde{\varphi}_V : \gamma(V) \to \gamma(V_0)$ that can be extended to an isomorphism φ_V of $\mathbb{C}^n$ by requiring it to be the identity on L_0.
A local trivialization for the projection pr is given by the homeomorphism

$$\begin{aligned} f : pr^{-1}(\mathcal{U}_{L_0}) &\to \mathcal{U}_{L_0} \times U_{\gamma(V_0)} \\ y = (H_1, \ldots, H_h) &\mapsto \big(pr(y), \varphi_{\gamma(pr(y))}(H_h)\big) \end{aligned}$$

which makes the following diagram commute.



Remark that if $n = kh$, then $U_{\gamma(V_0)} = \{H \in Gr(k,n) | H + \gamma(V_0) = \mathbb{C}^n\}$ is a single coordinate chart of the Grassmannian manifold $Gr(k,kh)$, that is it is homeomorphic to $\mathbb{C}^{k(kh-k)}$. This completes the proof. $\square$

Let us remark that if $V = (H_1, \ldots, H_h)$ is a point in the space $\mathcal{F}_h^{kh}(k,n)$, then the h subspaces $H_1, \ldots, H_h$ are in direct sum and the map

$$\begin{aligned} pr : \mathcal{F}_h^{kh}(k,n) &\to \mathcal{F}_{h-1}^{k(h-1)}(k,n) \\ (H_1, \ldots, H_h) &\mapsto (H_1, \ldots, H_{h-1}) \end{aligned}$$

is well defined.

We have, from the homotopy long exact sequence of the fibration pr with $n = kh$, that

$$\pi_j(\mathcal{F}_h^{kh}(k,kh)) = \pi_j(\mathcal{F}_{h-1}^{k(h-1)}(k,kh)) \tag{4}$$

for all j and, by equation (1), that

$$\pi_1(\mathcal{F}_h^{kh}(k,kh)) = \pi_1(\mathcal{F}_{h-1}^{k(h-1)}(k,kh)) = 0.$$

It follows that the open stratum $\mathcal{F}_h^{kh}(k,kh)$ is simply connected, hence all open strata are simply connected.
Moreover, from the homotopy long exact sequence of the fibration γ, we have that

$$\pi_1(\mathcal{F}_h^i(k,i)) \to \pi_1(\mathcal{F}_h^i(k,n)) \to \pi_1(Gr(i,n)) = 0.$$

As $\mathcal{F}_h^i(k,i)$ is an open stratum, it is simply connected and hence $\pi_1(\mathcal{F}_h^i(k,n)) = 0$.

That is, all our configuration spaces are simply connected and Theorem 1.1 is proved.

3 The second homotopy group

In this section we compute the second homotopy group $\pi_2(\mathcal{F}_h^i(k,n))$ when $i = hk$, i.e. subspaces in direct sum, and when $h = 2$, i.e. the case of two subspaces. In order to compute those homotopy groups, we need to know that the third homotopy group for Grassmannian manifolds is trivial if $k > 1$. Even if it should be a classical result we didn't find references and we decided to give a proof here.

Let $V_{k,n}$ be the space parametrizing the (ordered) k-uples of orthonormal vectors in $\mathbb{C}^n$, $1 \leq k \leq n$. It is an easy remark that $V_{1,n} = S^{2n-1}$ and $V_{n,n} = U(n)$. It's well known that the function that maps an element of $V_{k,n}$ to the subspace generated by its entries is a locally trivial fibration:

$$V_{k,k} \hookrightarrow V_{k,n} \to Gr(k,n) \quad (k < n), \tag{5}$$

while the projection on the last entry is the locally trivial fibration:

$$V_{k-1,n-1} \hookrightarrow V_{k,n} \to S^{2n-1} \quad (k > 1). \tag{6}$$

Using the long exact sequence in homotopy induced by fibration (6), it's easy to see (crf. [St]) that $\pi_1(V_{k,n}) = \pi_2(V_{k,n}) = \pi_3(V_{k,n}) = 0$, except for $\pi_1(V_{n,n}) = \pi_3(V_{n,n}) = \pi_3(V_{n-1,n}) = \mathbb{Z}$.

The exact sequence of homotopy groups associated to fibration (5) for $k < n-1$ then becomes

$$\mathbb{Z} \to 0 \to \pi_3(Gr(k,n)) \to 0 \to 0 \to \pi_2(Gr(k,n)) \to \\ \to \mathbb{Z} \to 0 \to \pi_1(Gr(k,n)) \to 0 \quad ,$$

that is $\pi_1(Gr(k,n)) = 0$, $\pi_2(Gr(k,n)) = \mathbb{Z}$ and $\pi_3(Gr(k,n)) = 0$ if $k < n-1$. If $k = n-1$ then $Gr(n-1,n) = \mathbb{P}^{n-1}$ and $\pi_3(Gr(n-1,n)) = 0$ except if $n = 2$ in which case $Gr(1,2) = S^2$ and $\pi_3(Gr(1,2)) = \mathbb{Z}$. That is the third homotopy group of the Grasmannian manifold $Gr(k,n)$ is trivial if $k > 1$.

Since the third homotopy group of the Grasmannian manifold $Gr(k,n)$ is trivial if $k > 1$ then for $i < n$ the homotopy long exact sequence of the fibration γ defined in equation (2) gives :

$$0 = \pi_3(Gr(i,n)) \to \pi_2(\mathcal{F}_h^i(k,i)) \to \pi_2(\mathcal{F}_h^i(k,n)) \to \mathbb{Z} = \pi_2(Gr(i,n)) \to 0.$$

As the second homotopy groups are abelian and the above short exact sequence splits, we have

$$\pi_2(\mathcal{F}_h^i(k,n)) = \pi_2(\mathcal{F}_h^i(k,i)) \times \mathbb{Z}.$$

The case $i = hk$. If $i = hk$, by equation (4), $\pi_2(\mathcal{F}_h^{hk}(k,hk)) = \pi_2(\mathcal{F}_{h-1}^{k(h-1)}(k,hk))$ and the following equalities hold:

$$\begin{aligned}
\pi_2(\mathcal{F}_h^{hk}(k,hk)) &= \pi_2(\mathcal{F}_{h-1}^{k(h-1)}(k,k(h-1))) \times \mathbb{Z} = \\
&= \pi_2(\mathcal{F}_{h-2}^{k(h-2)}(k,k(h-1))) \times \mathbb{Z} = \\
&= \pi_2(\mathcal{F}_{h-2}^{k(h-2)}(k,k(h-2))) \times \mathbb{Z}^2 = \\
&= \pi_2(\mathcal{F}_2^{2k}(k,2k)) \times \mathbb{Z}^{h-2} = \\
&= \pi_2(\mathcal{F}_1^{k}(k,2k)) \times \mathbb{Z}^{h-2} = \\
&= \pi_2(Gr(k,2k)) \times \mathbb{Z}^{h-2} = \\
&= \mathbb{Z}^{h-1}
\end{aligned}$$

while, if $hk < n$, $\pi_2(\mathcal{F}_h^{hk}(k,n)) = \mathbb{Z}^h$.

The case $h = 2$. If $h = 2$ a point (H_1, H_2) is in the space $\mathcal{F}_2^i(k,n)$ if and only if the dimension of intersection $\dim(H_1 \cap H_2) = 2k - i$. If $i = 2k$ (which includes the cases $k = 1$ and $n = 2$) H_1 and H_2 are in direct sum otherwise the following Lemma holds.

Lemma 3.1. *If $k < i < 2k$, the map*

$$\begin{aligned} \eta : \mathcal{F}_2^i(k,n) &\to Gr(2k-i,n) \\ (H_1, H_2) &\mapsto H_1 \cap H_2 \end{aligned}$$

is a locally trivial fibration with fiber $\mathcal{F}_2^{2i-2k}(i-k, n-2k+i)$.

Proof. Let V_0 be a point in the Grassmannian manifold $Gr(2k - i, n)$. Fix $L_0 \in Gr(n - 2k + i, n)$ such that $L_0 \cap V_0 = \{0\}$ and let $\varphi : \mathbb{C}^n \to V_0$ be the linear projection given by the direct sum decomposition $L_0 + V_0 = \mathbb{C}^n$.
The fiber $\eta^{-1}(V_0)$ is the set of all pairs (H_1, H_2) of k-dimensional subspaces of $\mathbb{C}^n$ such that $H_1 \cap H_2 = V_0$. That is, a pair (H_1, H_2) is in $\eta^{-1}(V_0)$ if and only if it corresponds to a pair of $(i-k)$-dimensional subspaces of $\mathbb{C}^n/V_0$ are in direct sum, i.e. a point in $\mathcal{F}_2^{2(i-k)}(i-k, n-2k+i)$.

Let $\mathcal{U}_{L_0}$ be the open neighborhood of V_0 in $Gr(2k-i, n)$, defined as

$$\mathcal{U}_{L_0} = \{V \in Gr(2k-i,n) |\ L_0 \cap V = \{0\}\}.$$

If V is a point in $\mathcal{U}_{L_0}$, the restriction of φ to $\gamma(V)$ is a linear isomorphism $\tilde{\varphi}_V : V \to V_0$ that can be extended to an isomorphism φ_V of $\mathbb{C}^n$ by requiring it to be the identity on L_0.

A local trivialization for η is the homeomorphism

$$\begin{aligned} f : \eta^{-1}(\mathcal{U}_{L_0}) &\to \mathcal{U}_{L_0} \times \eta^{-1}(V_0) \\ (H_1, H_2) &\mapsto \big(\eta(y), (\varphi_{\eta(y)}(H_1), \varphi_{\eta(y)}(H_2))\big) \end{aligned}$$

This completes the proof. □

By the homotopy long exact sequence of the map η, we get:

$$0 \to \pi_2(\mathcal{F}_2^{2i-2k}(i-k, n-2k+i)) \to \pi_2(\mathcal{F}_2^i(k,n)) \to \mathbb{Z} \to 0$$

and hence $\pi_2(\mathcal{F}_2^i(k,n)) = \mathbb{Z} \times \pi_2(\mathcal{F}_2^{2(i-k)}(i-k, n-2k+i))$. By the previous case, $\pi_2(\mathcal{F}_2^{2(i-k)}(i-k, n-2k+i))$ is equal to $\mathbb{Z}$ if $2(i-k) = n-2k+i$, that is if $i = n$, and is equal to $\mathbb{Z}^2$ otherwise. So, we get $\pi_2(\mathcal{F}_2^n(k,n)) = \mathbb{Z}^2$ and $\pi_2(\mathcal{F}_2^i(k,n)) = \mathbb{Z}^3$ if $i < n$.

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