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CLOSED SUBMANIFOLDS WITH CONSTANT m -TH MEAN CURVATURE RELATED WITH A VECTOR FIELD IN A RIEMANNIAN MANIFOLD

By

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Introduction. H. Liebmann (1900) [1]¹⁾, proved the following theorem:
The only ovaloids with constant mean curvature H in an Euclidean space E^3 are the spheres.

Extension of this theorem to a convex hypersurface in an n -dimensional Euclidean space E^n has been given by W. Süss (1929) [2], (cf. also [3], p. 118, and [4]). Then H. Hope (1951) [5], and A. D. Alexandrov (1958) [6], have shown the results that the convexity is not necessary for the validity of the Liebmann–Süss theorem.

Recently the analogous problem for closed hypersurfaces in an n -dimensional Riemannian manifold R^n has been discussed by the present author [8], [9], [10], A. D. Alexandrov [7], K. Yano [13], T. Ôtsuki [15], M. Tani [16], and K. Nomizu [17], [18]. And also for a submanifold of codimension 2 in an odd dimensional sphere, M. Okumura has treated the analogue [19].

In the previous papers [11], [12], which are common works by T. Nagai, H. Kôjyo and the present author, we have given a certain extension of this problem to an m -dimensional closed submanifold V^m ($1 \leq m \leq n-1$) in the n -dimensional Riemannian manifold R^n admitting a vector field ξ . However we have given there a restriction such that at each point on V^m , the vector ξ lies in the vector space spanned by the tangent space of V^m and the Euler–Schouten vector n .

The purpose of this paper is to give more general results except this restriction.

§ 1. Some integral formulas for a submanifold. We suppose an n -dimensional Riemannian manifold R^n ($n \geq 3$) of class C^r ($r \geq 3$) which admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$(1.1) \quad \tilde{x}^i = x^i + \xi^i(x) \delta\tau$$

1) Numbers in brackets refer to the references at the end of the paper.

(where x^i are local coordinates in R^n and ξ^i are the components of a contravariant vector ξ). If ξ is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. ([14], p. 32), then the group G is called isometric, homothetic, conformal, etc.

In R^n , we consider a domain M . If the domain M is simply covered by the orbits of the transformations generated by ξ , and ξ is everywhere of class C^r and $\neq 0$ in M ; then we call M a regular domain with respect to the vector field ξ .

Let us denote by V^m an m -dimensional closed orientable submanifold of class C^3 imbedded in a regular domain M with respect to the vector field ξ , locally given by

$$(1.2) \quad x^i = x^i(u^\alpha) \quad \begin{matrix} i = 1, \dots, n \\ \alpha = 1, \dots, m, \end{matrix}$$

where u^α are local coordinates of V^m . Throughout the present paper Latin indices run from 1 to n and Greek indices from 1 to m . We assume that at any point on V^m the vector ξ is not on its tangent space.

We shall indicate by n^i ($p=m+1, \dots, n$) the contravariant unit vectors normal V^m and suppose that they are mutually orthogonal. Let n be in the vector space spanned by $m+1$ independent vectors $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha=1, \dots, m$) and ξ and be the unit vector normal V^m . Then, we may consider n as one of the unit normal vectors of V^m , that is, $n^i = n^i$.

Let us consider a differential form of $m-1$ degree at a point of V^m , defined by

$$(1.3) \quad \begin{aligned} ((n, n, \dots, n, \xi, \underbrace{dx, \dots, dx}_{m-1})) &\stackrel{\text{def.}}{=} \sqrt{g} (n, \dots, n, \xi, dx, \dots, dx) \\ &= \sqrt{g} \left((n, \dots, n, \xi, \frac{\partial x}{\partial u^{\alpha_1}}, \dots, \frac{\partial x}{\partial u^{\alpha_{m-1}}}) du^{\alpha_1} \wedge \dots \wedge du^{\alpha_{m-1}}, \right) \end{aligned}$$

where the symbol $()$ means a determinant of order n whose columns are the components of respective vectors, dx is a displacement along V^m , g is the determinant of the metric tensor g_{ij} of R^n . Then, the exterior differential of the differential form (1.3) divided by $m!$ becomes as follows

$$(1.4) \quad \begin{aligned} \frac{1}{m!} d ((n, n, \dots, n, \xi, dx, \dots, dx)) &= \frac{1}{m!} \{ ((\delta n, n, \dots, n, \xi, dx, \dots, dx)) \\ &+ ((n, \delta n, \dots, n, \xi, dx, \dots, dx)) + \dots \\ &+ ((n, \dots, n, \delta n, \xi, dx, \dots, dx)) + ((n, \dots, n, \delta \xi, dx, \dots, dx)) \} \end{aligned}$$

where δv means $v_{;\alpha} du^\alpha$ and the symbol “;” the operation of D -symbol due to van der Waerden–Bortolotti ([20] p. 254).

Let C_j^i be $\sum_{p=m+1}^n n_p^i n_j^p$ ($n = n$) and i_λ ($\lambda=1, \dots, m$) mutually orthogonal unit tangent vectors of V^m . Then we have

$$n_{p;\alpha}^i = C_{j;\alpha}^i n_p^j \frac{\partial x^k}{\partial u^\alpha} = - \sum_{\lambda=1}^m \left(i_{\lambda;j;k} n_p^j \frac{\partial x^k}{\partial u^\alpha} \right) i_\lambda^i.$$

Therefore we may put

$$n_{p;\alpha}^i = \gamma_\alpha^r \frac{\partial x^i}{\partial u^r}.$$

Since we have

$$g_{ij} \left(\frac{\partial x^i}{\partial u^\beta} \right)_{;\alpha p} n^j = - g_{ij} \frac{\partial x^i}{\partial u^\beta} n_{\alpha p}^j,$$

we obtain

$$(1.5) \quad n_{p;\alpha}^i = - b_{\alpha p}^r \frac{\partial x^i}{\partial u^r} \quad (p = \xi, m+2, \dots, n)$$

where $b_{\alpha p}^r$ means $g^{r\beta} b_{\alpha\beta}^p$ and $b_{\alpha\beta}^p \equiv \left(\frac{\partial x^i}{\partial u^\alpha} \right)_{;\beta p} n_i$, and $g^{r\beta}$ is the contravariant metric tensor of V^m .

From (1.5) the first term of the right-hand member of (1.4) becomes

$$(1.6) \quad \frac{1}{m!} ((\delta n, n, \dots, n, \xi, dx, \dots, dx)) = (-1)^{(n-m)(n-1)} H_1 n_\xi^i \xi^i dA,$$

where dA is the area element of V^m and H_1 means the first mean curvature of V^m with respect to the normal direction n_ξ^i . Similarly, for every integer p satisfying $m+2 \leq p \leq n$ we have

$$(1.7) \quad \frac{1}{m!} ((n, \dots, \delta n, \dots, n, \xi, dx, \dots, dx)) = (-1)^{(n-m)(n-1)} H_1 n_\xi^i \xi^i dA = 0$$

because ξ lies in the vector space spanned by $m+1$ independent vectors $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha=1, \dots, m$) and n_ξ .

On the other hand the last term of the right-hand member of 1.4 becomes

$$(1.8) \quad \frac{1}{m!} ((n, \dots, n, \delta \xi, dx, \dots, dx)) = (-1)^{(n-m)(n-1)} \frac{1}{2m} (L g_{ij}) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA,$$

where $L_{\xi} g_{ij}$ is the Lie derivative of g_{ij} with respect to ξ ([14], p. 5).

From (1.6), (1.7) and (1.8), (1.4) is rewritten as follows

$$(1.9) \quad \frac{1}{m!} d \left(\binom{n}{\xi} \binom{\dots}{\xi} \binom{n}{n} \xi, dx, \dots, dx \right) = (-1)^{(n-m)(n-1)} \left\{ H_1 n_i \xi^i dA + \frac{1}{2m} (L_{\xi} g_{ij}) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA \right\}.$$

Integrating both members of (1.9) over the whole submanifold and applying Stokes' theorem, we obtain

$$\frac{1}{m!} \int_{\partial V^m} \left(\binom{n}{\xi} \binom{n}{m+2} \binom{\dots}{n} \xi, dx, \dots, dx \right) = (-1)^{(n-m)(n-1)} \left\{ \int_{V^m} H_1 n_i \xi^i dA + \frac{1}{2m} \int_{V^m} g^{*ij} L_{\xi} g_{ij} dA \right\},$$

where ∂V^m means the boundary of V^m and g^{*ij} is $\frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta}$. Making use of the fact that V^m is closed, we have

$$(I'') \quad \int_{V^m} H_1 n_i \xi^i dA + \frac{1}{2m} \int_{V^m} g^{*ij} L_{\xi} g_{ij} dA = 0.$$

If the manifold R^n assumes of constant Riemann curvature which includes an Euclidean space, then we consider the following differential form of $m-1$ degree

$$(1.10) \quad \left(\binom{n}{\xi} \binom{n}{m+2} \binom{\dots}{n} \xi, \underbrace{\delta n, \dots, \delta n}_{\nu}, \underbrace{dx, \dots, dx}_{m-\nu-1} \right) = \overset{\text{def.}}{\sqrt{g}} \left(\binom{n}{\xi} \binom{n}{m+2} \binom{\dots}{n} \xi, \delta n, \dots, \delta n, dx, \dots, dx \right)$$

for a fixed integer ν satisfying $m-1 \geq \nu \geq 1$.

As well-known, a submanifold V^m in R^n has the following property:

$$b_{\alpha\beta;\gamma} - b_{\alpha\gamma;\beta} = -R_{\iota j k \iota} n_{\xi}^{\iota} \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} \frac{\partial x^{\iota}}{\partial u^{\gamma}} \quad ([20], \text{ p. } 226),$$

where $R_{\iota j k \iota}$ is the curvature tensor of R^n . Since R^n is of constant Riemann curvature, we have

$$(1.11) \quad n_{\xi}^{\iota, \alpha; \beta} - n_{\xi}^{\iota; \beta; \alpha} = 0.$$

Consequently differentiating exteriorly the differential form (1.10), we have

$$\begin{aligned}
 & d((n, n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 &= ((\delta n, n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 (1.12) \quad &+ ((n, \delta n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 &+ \dots + ((n, n, \dots, \delta n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 &+ ((n, n, \dots, n, \delta \xi, \delta n, \dots, \delta n, dx, \dots, dx)),
 \end{aligned}$$

because $((n, n, \dots, n, \xi, \delta \delta n, \delta n, \dots, \delta n, dx, \dots, dx)) = 0$ from (1.11).

On substituting $n^i_{;\alpha} = -b^{\beta}_{\alpha} \frac{\partial x^i}{\partial u^{\beta}}$ into the first term of the right-hand member of (1.12), we get

$$\begin{aligned}
 (1.13) \quad & ((\delta n, n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 &= m! (-1)^{(n-m)(n-1)-\nu} H_{\nu+1} n_i \xi^i dA,
 \end{aligned}$$

where $H_{\nu+1}$ denotes the $\nu+1$ -th mean curvature of V^m with respect to the normal direction n^i and if we indicate by k_1, k_2, \dots, k_m the principal curvatures of V^m for the normal vector n , $H_{\nu+1}$ is defined to be the $\nu+1$ -th elementary symmetric function of k_{α} ($\alpha=1, \dots, m$) divided by the number of terms, that is,

$$\binom{m}{\nu+1} H_{\nu+1} = \sum_{\alpha_1 < \alpha_2 < \dots < \alpha_{\nu+1}} k_{\alpha_1} k_{\alpha_2} \dots k_{\alpha_{\nu+1}}.$$

Also, by virtue of (1.5) we can see that the vectors

$$\begin{aligned}
 & n \times \delta n \times n \times \dots \times n \times \underbrace{\delta n \times \dots \times \delta n}_{\nu} \times \underbrace{dx \times \dots \times dx}_{m-\nu-1}, \\
 & n \times n \times \delta n \times \dots \times n \times \delta n \times \dots \times \delta n \times dx \times \dots \times dx, \\
 & \dots
 \end{aligned}$$

and

$$n \times n \times \dots \times n \times \delta n \times \delta n \times \dots \times \delta n \times dx \times \dots \times dx$$

have the same direction to the covariant vectors n, n, \dots and n respectively.

Thus we obtain

$$((n, \delta n, n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) = 0,$$

$$\begin{aligned}
 & ((n, n, \delta n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) = 0, \\
 (1.14) \quad & \dots, \\
 & ((n, n, \dots, n, \delta n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) = 0,
 \end{aligned}$$

because ξ lies in the vector space spanned by $m + 1$ independent vectors $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha = 1, \dots, m$) and n .

From that the vector $n \times n \times \dots \times n \times \delta n \times \dots \times \delta n \times dx \times \dots \times dx$ is orthogonal to the normal vectors n, n, \dots and n , and $\delta n^\xi = -b_\alpha^\xi \frac{\partial x^i}{\partial u^\alpha} du^\alpha$, the last term of the right-hand member of (1.12) becomes as follows

$$\begin{aligned}
 (1.15) \quad & ((n, n, \dots, n, \delta \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 & = m! (-1)^{(n-m)(n-1)-\nu} \frac{1}{2m} H_\xi^{\alpha\beta} L g_{\alpha\beta} dA,
 \end{aligned}$$

where $L g_{\alpha\beta} = (L g_{ij}) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}$ and

$$H_\xi^{\alpha\beta} = \frac{1}{(m-1)!} \epsilon^{\alpha\alpha_1 \dots \alpha_{m-1}} \epsilon^{\beta\beta_1 \dots \beta_{m-1}} b_{\alpha_1 \beta_1} \dots b_{\alpha_{m-1} \beta_{m-1}} g_{\alpha_{\nu+1} \beta_{\nu+1}} \dots g_{\alpha_{m-1} \beta_{m-1}},$$

and $\epsilon^{\alpha\alpha_1 \dots \alpha_{m-1}}$ denotes the ϵ -symbol of the submanifold V^m . Accordingly we have

$$\begin{aligned}
 (1.16) \quad & \frac{1}{m!} d((n, n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 & = (-1)^{(n-m)(n-1)-\nu} \left\{ H_{\nu+1} n_i \xi^i dA + \frac{1}{2m} H_\xi^{\alpha\beta} L g_{\alpha\beta} dA \right\}.
 \end{aligned}$$

Integrating both members of (1.16) over the whole submanifold V^m and applying Stokes' theorem, we have

$$\begin{aligned}
 & \frac{1}{m!} \int_{\partial V^m} ((n, n, \dots, n, \xi, \delta n, \dots, \delta n, dx, \dots, dx)) \\
 & = (-1)^{(n-m)(n-1)-\nu} \left\{ \int_{V^m} H_{\nu+1} n_i \xi^i dA + \frac{1}{2m} \int_{V^m} H_\xi^{\alpha\beta} L g_{\alpha\beta} dA \right\}.
 \end{aligned}$$

Thus, for a closed orientable submanifold V^m we obtain

$$(II'') \quad \int_{V^m} H_{\nu+1} n_i \xi^i dA + \frac{1}{2m} \int_{V^m} H_\xi^{\alpha\beta} L g_{\alpha\beta} dA = 0.$$

If $m = n - 1$, that is, V^m is the hypersurface in R^n , the formulas (I'') and

(II'') are coincide with the formulas (I) and (II) given in the previous paper [8]. Especially if the vector n is coincide with the Euler-Schouten unit vector n at each point on V^m , then the formulas (I'') and (II'') become the formulas (I') and (II') given in the previous paper [12].

§ 2. The integral formulas concerning some special transformations.

In this section, we shall discuss the formulas (I'') and (II'') for a special infinitesimal transformation. Let the group G of transformations be conformal, that is, ξ^i satisfies an equation: $Lg_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ ([14], p. 32). Then we obtain

$$g^{*ij} Lg_{ij} = 2m\phi, \quad H_\nu^{\alpha\beta} Lg_{\alpha\beta} = 2m\phi H_\nu.$$

Therefore (I'') and (II'') are rewritten in the following forms:

$$(I'')_c \quad \int_{V^m} H_1 n_i \xi^i dA + \int_{V^m} \phi dA = 0,$$

$$(II'')_c \quad \int_{V^m} H_{\nu+1} n_i \xi^i dA + \int_{V^m} \phi H_\nu dA = 0 \quad (1 \leq \nu \leq m-1)$$

and we can see

$$(I'')_h \quad \int_{V^m} H_1 n_i \xi^i dA + c \int_{V^m} dA = 0,$$

$$(II'')_h \quad \int_{V^m} H_{\nu+1} n_i \xi^i dA + c \int_{V^m} H_\nu dA = 0 \quad (1 \leq \nu \leq m-1)^2)$$

in case of $\phi = \text{constant} (\equiv c)$ (G being homothetic), and

$$(I'')_i \quad \int_{V^m} H_1 n_i \xi^i dA = 0,$$

$$(II'')_i \quad \int_{V^m} H_{\nu+1} n_i \xi^i dA = 0 \quad (1 \leq \nu \leq m-1)$$

in case of $\phi = 0$ (G being isometric).

Especially if our manifold R^n is an Euclidean space E^n and if ξ is the homothetic Killing vector field on E^n with components $\xi^i = x^i$, x^i being rectangular coordinates with a point in the interior of V^m as origin in the space E^n , then the orbits of the transformations generated by ξ are the lines through

2) In this case, R^n becomes an Euclidean space, because if R^n with constant Riemann curvature admits an one-parameter group G of homothetic transformations, then either R^n is E^n or the group G is isometric.

the origin and we have

$$L_{\xi} g_{ij} = 2g_{ij}.$$

Consequently, from $(I'')_h$ and $(II'')_h$ we obtain

$$(I^*) \quad \int_{V^m} H_1 \rho dA + \int_{V^m} dA = 0,$$

$$(II^*) \quad \int_{V^m} H_{v+1} \rho dA + \int_{V^m} H_v dA = 0,$$

where $\rho = n_i x^i$. This means that the formulas (I^*) and (II^*) are generalization of those formulas given by C. C. Hsiung [4] for a closed orientable hypersurface in an n -dimensional Euclidean space E^n .

§ 3. Some properties of a closed orientable submanifold related with a vector field. In this section we suppose again that the group G is conformal. Then we shall prove the following four theorems for a closed orientable submanifold V^m in a Riemannian manifold R^n with constant Riemann curvature.

Theorem 3.1. *If in R^n , there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m and if H_1 is constant, then every point of V^m is umbilic with respect to the normal vector n , where ρ denotes $n_i x^i$.*

Proof. Multiplying the formula $(I'')_c$ by $H_1 = \text{const.}$, we have

$$\int_{V^m} H_1^2 \rho dA + \int_{V^m} \phi H_1 dA = 0.$$

On the other hand, from $(II'')_c$ we have

$$\int_{V^m} H_2 \rho dA + \int_{V^m} \phi H_1 dA = 0.$$

Consequently it follows that

$$\int_{V^m} (H_1^2 - H_2) \rho dA = 0.$$

From our assumption about ρ , this holds if and only if $H_1^2 - H_2 = 0$, since

$$H_1^2 - H_2 = \frac{1}{m^2(m-1)} \sum_{\alpha < \beta} (k_\alpha - k_\beta)^2 \geq 0.$$

Therefore at each point of V^m we obtain

$$k_{\xi 1} = k_{\xi 2} = \dots = k_{\xi m}.$$

Accordingly every point of V^m is umbilic with respect to n_{ξ} .

Theorem 3.2. *If in R^n , there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m , and if the principal curvatures $k_{\xi 1}, k_{\xi 2}, \dots, k_{\xi m}$ at each point of V^m are positive and H_{ν} is constant for any ν ($1 < \nu \leq m-1$), then every point of V^m is umbilic with respect to the normal vector n_{ξ} .*

Proof. Multiplying the formula $(I'')_c$ by $H_{\nu} = \text{const.}$, we obtain

$$(3.1) \quad \int_{V^m} H_1 H_{\nu} \rho dA + \int_{V^m} \phi H_{\nu} dA = 0.$$

By value of $(II'')_c$ and (3.1), we have

$$\int_{V^m} (H_1 H_{\nu} - H_{\nu+1}) \rho dA = 0.$$

From our assumption, this holds if and only if $H_1 H_{\nu} - H_{\nu+1} = 0$, since

$$H_1 H_{\nu} - H_{\nu+1} = \frac{\nu!(m-\nu-1)!}{m m!} \sum_{\xi} k_{\xi \alpha_1} \dots k_{\xi \alpha_{\nu-1}} (k_{\xi \alpha_{\nu}} - k_{\xi \alpha_{\nu+1}})^2 \geq 0.$$

Then at each point of V^m , we obtain

$$k_{\xi 1} = k_{\xi 2} = \dots = k_{\xi m}.$$

Accordingly every point of V^m is umbilic with respect to n_{ξ} .

Theorem 3.3. *If in R^n , there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m , for which $H_1 \rho + \phi \geq 0$ (or ≤ 0) at all points of V^m , then every point of V^m is umbilic with respect to n_{ξ} .*

Proof. If we express the formula $(I'')_c$ as follows

$$\int_{V^m} (H_1 \rho + \phi) dA = 0,$$

then from our assumption we have the relation:

$$(3.2) \quad \phi = -H_1 \rho.$$

Substituting (3.2) into $(II'')_c$ for $\nu=1$, we have

$$\int_{V^m} (H_1^2 - H_2) \rho dA = 0.$$

Thus, we can see the conclusion.

Theorem 3.4. *If H_1 is positive (or negative) at all points of V^m and if R^n admits such a group G of conformal transformations as ϕ is positive (or negative), for which either $\rho \geq \frac{-\phi}{H_1}$ or $\rho \leq \frac{-\phi}{H_1}$ at all points of V^m , then every point of V^m is umbilic with respect to n .*

Proof. The formula $(I'')_c$ is rewritten as follows

$$\int_{V^m} H_1 \left(\rho + \frac{\phi}{H_1} \right) dA = 0.$$

By virtue of our assumptions $H_1 > 0$ (or < 0) and $\rho + \frac{\phi}{H_1} \geq 0$ (or ≤ 0) at all points of V^m , we have the following relation

$$(3.3) \quad \rho = -\frac{\phi}{H_1}.$$

Substituting (3.3) into $(II'')_c$ for $\nu=1$, we obtain

$$\int_{V^m} \frac{\phi}{H_1} (H_1^2 - H_2) dA = 0,$$

which holds if and only if $H_1^2 - H_2 = 0$. Thus we obtain the conclusion.

Remark I. If V^m is the hypersurface in R^n , these four theorems are coincide with the theorems given in the previous paper [8]. Especially if the vector n is coincide with the Euler-Schouten unit vector n at each point of V^m , then these four theorems become those theorems given in the previous paper [12].

Remark II. In all these sections we have treated the normal unit vector n with respect to the vector field ξ and the mean curvature H_ν . These are the notions due to R. E. Stong [21].

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