ON THE CONNECTION PARAMETERS IN A NON-HOLONOMIC SPACE OF LINE-ELEMENTS

By

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§ 0. Introduction.

The theory of a non-holonomic space of line-elements or non-holonomic system depending on line-elements was treated already by T. HOSOKAWA(1), H. HOMBU(2), T. SUGURI(3), V. WAGNER(4) and the present author(5).

In the present papers we shall introduce many properties appearing only in a special non-holonomic space of line-elements defined in the previous paper(6) (N. S. F.). In § 1, we shall study the connection parameters, torsion tensors and curvature tensors in a non-holonomic space of line-elements and in § 2 find them belonging to the groups of non-holonomic transformations under which the non-holonomic subspaces of line-elements: X_n^m and X_n^{n-m} are invariant. § 3 is devoted to determine the structure of the same quantities in a non-holonomic EUCLIDEAN space of line-elements. The second fundamental tensors of non-holonomic subspace and geodesic non-holonomic subspace are introduced in § 4.

The present author wishes to express to Prof. A. KAWAGUCHI her very sincere appreciation for his helpful guidance and his careful criticisms.

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(1) T. HOSOKAWA: Über nicht-holonome Übertragung in allgemeiner Mannigfaltigkeit T_n, Jour. Fac. Sci. Hokkaido Imper. Univ., Series 1, Vol. 2, No. 1-2 (1934), 1-11.

(2) H. HOMBU: Die Krümmungstheorie in Finslershen Raume, Jour. Fac. Sci. Hokkaido Imper. Univ., Series 1, Vol. 5 (1936), 67-94.

(3) T. SUGURI: On the non-holonomic FINSLER space (not yet printed).

(4) V. WAGNER: The inner geometry of non-linear non-holonomic manifolds, Rec. Math. N. S., Vol. 13 (1943), 135-167.

(5) Y. KATSURADA: On the theory in a non-holonomic system in a FINSLER space (in printing).

(6) This will be in the present paper referred with N.S.F..

§ 1. A non-holonomic space of line-elements.

1. The definition of a non-holonomic space of line-elements. Corresponding to a manifold X_n referred to a coordinate system (x^α) $(\alpha = 1, 2, \cdots\cdots, n)$, let us consider a space in which the displacement of a point is defined as follows

$$ds^a = A^a(x, dx^\alpha) \qquad \begin{array}{l} a = 1, 2, \cdots n^{(7)} \\ \alpha = 1, 2, \cdots n \end{array} \tag{1.1}$$

where A^a are n mutually independent and homogeneous functions of degree one in the dx and analytic in the x and dx.

By homogeneity of the functions A^a, (1. 1) can be written in the form

$$ds^a = \frac{\partial A^a}{\partial(dx^\alpha)} dx^\alpha = \lambda^a_\alpha \, dx^\alpha = A^a \tag{1.2}$$

putting $\partial A^a/\partial(dx^\alpha) = \lambda^a_\alpha$, where λ^a_α are evidently homogeneous of degree zero in the dx. Then it follows that

$$\lambda^a_\alpha(x, dx^\alpha) = \lambda^a_\alpha\left(x, \frac{dx}{dt}\right) = \partial A^a\left(x, \frac{dx}{dt}\right) \Big/ \partial\left(\frac{dx^\alpha}{dt}\right) \tag{1.3}$$

for the displacement dx along a curve $x^\alpha = x^\alpha(t)$ and these values are finite.

Now we consider the manifold of the line-elements given by x and x' and $\lambda^a_\alpha(x, x')$ as functions of the line-element. Here we shall have a field of a family of n mutually independent covariant vectors $\lambda^a_\alpha(x, x')$ that are reduced from the functions A^a.

Let us define *a non-holonomic space of line-elements* in which the displacement of a point and the direction of a line-elements are related respectively with that in a holonomic space as follows

$$ds^a = \lambda^a_\alpha(x, x')\, dx^\alpha \text{ and } \frac{ds^a}{dt} = \lambda^a_\alpha(x, x') \frac{dx^\alpha}{dt} \left(\text{where}: x' = \frac{dx}{dt}\right). \tag{1.4}$$

If a displacement of $x^\alpha : dx^\alpha$ lies in the direction of the line-elements (x, x'), then it follows

$$ds^a = \lambda^a_\alpha \, dx^\alpha = A^a$$

(7) We use Greek indices in holonomic spaces and Latin indices in non-holonomic spaces.

but $ds^a = \lambda^a_\alpha\, dx^\alpha \neq A^a$ for $dx^\alpha \neq \rho \dfrac{dx^\alpha}{dt}$.

Next we shall induce the reciprocal contravariant vectors λ^α_b of λ^a_α. In fact, let us solve the equations (1.1) for dx^α, then the solutions may be of the form

$$(1.5)\qquad dx^\alpha = B^\alpha(x, ds^a)$$

where B^α are homogeneous of degree one in the ds^a. Further if we put $\partial B^\alpha / \partial (ds^a) = {}^*\lambda^\alpha_a(x, ds)$, (1.5) are written in

$$(1.6)\qquad dx^\alpha = \frac{\partial B^\alpha}{\partial (ds^a)} \cdot ds^a = \lambda^\alpha_a(x, ds)\, ds^a = B^\alpha .$$

Replacing the ds^a by the last equation of (1.2), we get

$$(1.7)\qquad dx^\alpha = \left\{\frac{\partial B^\alpha}{\partial (ds^a)}\right\}_{ds^a \equiv A^a} \frac{\partial A^a}{\partial (dx^\beta)} dx^\beta = \lambda^\alpha_a(x, dx)\, \lambda^a_\beta(x, dx)\, dx^\beta .$$

Differentiating the above equations with respect to the dx^β, we have

$$(1.8)\qquad \left\{\frac{\partial B^\alpha}{\partial (ds^a)}\right\}_{ds^a \equiv A^a} \frac{\partial A^a}{\partial (dx^\beta)} = \lambda^\alpha_a(x, x')\, \lambda^a_\beta(x, x') = \delta^\alpha_\beta$$

for a direction of line-elements having the same direction as the dx^α, because of the homogeneity of degree one in the dx and ds of the functions A^a and B^α. The second equations of (1.8) hold for any x and x'. From this fact it follows that

$$(1.9)\qquad \lambda^a_\alpha(x, x')\, \lambda^\alpha_b(x, x') = \delta^a_b .$$

The relation (1.8) and (1.9) are written as follows also:

$$(1.10)\qquad {}^*\lambda^\alpha_a(x, s')\, {}^*\lambda^a_\beta(x, s') = \delta^\alpha_\beta, \quad {}^*\lambda^a_\alpha(x, s')\, {}^*\lambda^\alpha_b(x, s') = \delta^a_b .$$

The quantities ${}^*\lambda^\alpha_a(x, s') = \lambda^\alpha_a(x, x')$ are considered components of the *reciprocal contravariant vectors* of ${}^*\lambda^a_\alpha(x, s') = \lambda^a_\alpha(x, x')$. Using the symbol $*$ upon a function $f(x, x')$, we denote the functional form in the x and s' of the $f(x, x')$ as ${}^*f(x, s') = f(x, x')$. This symbol plays an important rôle for the derivatives of f with respect to the x and x' or s', but it is not necessary when only a functional value is referred.

Inversely the equations (1.4) may be solved for dx^α and $\dfrac{dx^\alpha}{dt}$, and the solutions are then of the form

$$(1.11)\qquad dx^{\alpha} = {}^*\lambda^{\alpha}_{a}\left(x, \frac{ds}{dt}\right) ds^{a} \quad\text{and}\quad \frac{dx^{\alpha}}{dt} = {}^*\lambda^{\alpha}_{a}\left(x, \frac{ds}{dt}\right)\frac{ds^{a}}{dt}.$$

We shall introduce the fundamental differential operations in the space. Let us at first define the partial derivatives of ${}^*f(x, s')$ with respect to the s^a as follows

$$(1.12)\qquad \frac{\partial\, {}^*f}{\partial s^{a}} = \frac{\partial\, {}^*f}{\partial x^{\alpha}}\, {}^*\lambda^{\alpha}_{a} = \left(\frac{\partial f}{\partial x^{\alpha}} + \frac{\partial f}{\partial x'^{\beta}}\, \frac{\partial\, {}^*\lambda^{\beta}_{b}}{\partial x^{\alpha}}\, s'^{b}\right) {}^*\lambda^{\alpha}_{a}\, ,$$

then we obtain

$$\frac{\partial f}{\partial x^{\alpha}} = \frac{\partial\, {}^*f}{\partial x^{\alpha}} + \frac{\partial\, {}^*f}{\partial s'^{a}}\, \frac{\partial \lambda^{a}_{\beta}}{\partial x^{\alpha}}\, x'^{\beta}\, .$$

From (1.12), we have

$$\frac{\partial^2\, {}^*f}{\partial s^{a}\partial s^{b}} - \frac{\partial^2\, {}^*f}{\partial s^{b}\partial s^{a}} = -\,\omega^{c}_{ab}\,\frac{\partial\, {}^*f}{\partial s^{c}}\, ,$$

where

$$(1.13)\qquad \omega^{c}_{ab} = \left(\frac{\partial\, {}^*\lambda^{c}_{\alpha}}{\partial x^{\beta}} - \frac{\partial\, {}^*\lambda^{c}_{\beta}}{\partial x^{\alpha}}\right) {}^*\lambda^{\alpha}_{a}\, {}^*\lambda^{\beta}_{b}\, .$$

Moreover differentiating ${}^*f$ with respect to the s'^a, we get

$$(1.14)\qquad \frac{\partial\, {}^*f(x, s')}{\partial s'^{a}} = \frac{\partial f(x, x')}{\partial x'^{\alpha}}\, \lambda^{\alpha}_{a}\, ,$$

$$\frac{\partial^2\, {}^*f(x, s')}{\partial s'^{a}\partial s'^{b}} = \frac{\partial^2 f(x, x')}{\partial x'^{\alpha}\partial x'^{\beta}}\, {}^*\lambda^{\alpha}_{a}\, {}^*\lambda^{\beta}_{b} + \frac{\partial f}{\partial x'^{\alpha}}\, \frac{\partial\, {}^*\lambda^{\alpha}_{a}}{\partial s'^{b}}\, ,$$

accordingly

$$\frac{\partial^2\, {}^*f(x, s')}{\partial s'^{a}\partial s'^{b}} - \frac{\partial^2 f(x, x')}{\partial x'^{\alpha}\partial x'^{\beta}}\, {}^*\lambda^{\alpha}_{a}\, {}^*\lambda^{\beta}_{b} = \Omega^{c}_{ab}\,\frac{\partial\, {}^*f}{\partial s'^{c}}$$

where

$$(1.15)\qquad \Omega^{c}_{ab} = \frac{\partial\, {}^*\lambda^{\alpha}_{a}}{\partial s'^{b}}\, {}^*\lambda^{c}_{\alpha}\, .$$

Thus we have these new important quantities ω^{c}_{ab} and Ω^{c}_{ab} which will give fundamental properties in the non-holonomic space.

2. Non-holonomic transformation of coordinate. We shall consider the relation between any two sets of the function A^a, $\bar{A}^i$ inducing a non-holonomic space as follows

$$(1.16) \qquad A^a = C^a(x, \bar{A}^i) \qquad \begin{matrix} a = 1, 2, \cdots, n \\ i = 1, 2, \cdots, n \end{matrix}$$

where C^a are n mutually independent and homogeneous functions of degree one in the $\bar{A}^i$ and analytic in the x and $\bar{A}^i$. Therefore (1.16) are written in

$$(1.17) \qquad A^a = \frac{\partial C^a}{\partial (\bar{A}^i)} \bar{A}^i = C^a_i \bar{A}^i \qquad \left(\text{where : } \frac{\partial C^a}{\partial (\bar{A}^i)} = C^a_i\right)$$

where $C^a_i(x, d\bar{s}) = C^a_i\left(x, \frac{d\bar{s}}{dt}\right)$ for the displacement: $d\bar{s}$ along a curve, that follows from their homogeneity of degree zero in the $\bar{A}^i$.

Let us define the change of the displacement of a point: ds^a and the elements $\frac{ds^a}{dt}$ in the non-holonomic space of line-elements given by (1.4) as follows

$$(1.18) \qquad ds^a = C^a_i\left(x, \frac{d\bar{s}}{dt}\right) d\bar{s}^i \quad \text{and} \quad \frac{ds^a}{dt} = C^a_i\left(x, \frac{d\bar{s}}{dt}\right) \frac{d\bar{s}^i}{dt}$$

respectively and these transformations are called *the non-holonomic transformations of coordinates.* Inversely, the equations (1.16) may be solved for $\bar{A}^i$, and the solutions are then of the form $\bar{A}^i = \bar{C}^i(x, A^a)$, where $\bar{C}^i$ are homogeneous of degree one in the A^a. By using the method by which (1.11) has been derived from (1.5), we obtain the following results

$$(1.19) \qquad d\bar{s}^i = \bar{C}^i_a \, ds^a \,, \quad \frac{d\bar{s}^i}{dt} = \bar{C}^i_a \frac{ds^a}{dt} \qquad \left(\text{where: } \bar{C}^i_a = \frac{\partial \bar{C}^i(x, s')}{\partial s'^a}\right)$$

and

$$(1.20) \qquad \bar{C}^i_a C^b_i = \delta^b_a \,, \qquad \bar{C}^i_a C^a_j = \delta^i_j \,,$$

from which it follows easily that

$$(1.21) \qquad \bar{\lambda}^\alpha_i = C^a_i \lambda^\alpha_a \,, \qquad \bar{\lambda}^i_\alpha = \bar{C}^i_a \lambda^a_\alpha \,, \qquad \lambda^\alpha_a = \bar{C}^j_a \bar{\lambda}^\alpha_j \,, \qquad \lambda^a_\alpha = C^a_i \bar{\lambda}^i_\alpha$$

between two families: $(\lambda^\alpha_a, \lambda^a_\beta)$ and $(\bar{\lambda}^\alpha_i, \bar{\lambda}^i_\beta)$ corresponding to A^a and $\bar{A}^i$ respectively.

Let us define a vector or a tensor in a non-holonomic space of line-elements as the quantities changing under transformations of coordinates as follows

$$(1.22)\qquad v^a = C^a_i \bar{v}^i\,,\qquad T^{a_1\cdots a_p}{}_{b_1\cdots b_q} = C^{a_1}_{i_1}\cdots C^{a_p}_{i_p}\bar{C}^{j_1}_{b_1}\cdots\bar{C}^{j_q}_{b_q}\bar{T}^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

where v^a or $T^{a_1\cdots a_p}{}_{b_1\cdots b_q}$ are analytic function in the x, $\dfrac{ds^a}{dt}$. Then the quantities λ^a_α give us n contravariant vectors in the non-holonomic space and n covariant vectors in the holonomic space, and the λ^α_a are covariant vectors in the non-holonomic space and contravariant ones in the holonomic space.

If we put $v^a = \lambda^a_\alpha v^\alpha$, v^a are then a contravariant vector in the non-holonomic space, where v^α is a vector in the holonomic space. Likely the quantities

$$(1.23)\qquad T^{a_1\cdots a_p}{}_{b_1\cdots b_q} = \lambda^{a_1}_{\alpha_1}\cdots\lambda^{a_p}_{\alpha_p}\bar{\lambda}^{\beta_1}_{b_1}\cdots\bar{\lambda}^{\beta_q}_{b_q}\,T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

are a tensor in the non-holonomic space, when $T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$ is a tensor in the holonomic space.

Next, we shall study the change of the quantities in the non-holonomic space under non-holonomic transformations of coordinates. At first the quantities ω^a_{bc} and Ω^a_{bc} will go into

$$(1.24)\qquad \begin{cases} \bar{\Omega}^k_{ij} = \dfrac{\partial C^a_i}{\partial \bar{s}'^j}\bar{C}^k_a + C^a_i C^b_j \bar{C}^k_c \Omega^c_{ab}\,, \\[2ex] \bar{\omega}^k_{ij} = C^a_i C^b_j\left\{\left(\dfrac{\partial\, {}^*\bar{C}^k_a}{\partial s^b} - \dfrac{\partial\, {}^*\bar{C}^k_b}{\partial s^a}\right) + C^k_c\,\omega^c_{ab}\right\} \\[2ex] \qquad\qquad + \left(\dfrac{\partial C^d_i}{\partial \bar{s}^j}C^a_i - \dfrac{\partial C^d_l}{\partial \bar{s}^i}C^a_j\right)\bar{s}'^l\left(\dfrac{\partial \bar{\bar{C}}^k_a}{\partial s'^d} - \bar{C}^k_c\,\Omega^c_{ad}\right) \end{cases}$$

under non-holonomic transformations of coordinates (See N.H.F.).

Let us consider quantities $B^a_b = \lambda^a_\alpha \dfrac{\partial \lambda^\alpha_c}{\partial s^b} s'^c$ in the non-holonomic space, then we can know, according to the first equation of (1.21), the relation $\dfrac{\partial \lambda^\alpha_a}{\partial s'^b} s'^a = 0$ and equations (1.24), that the quantities B^a_b and $\omega^a_{bc} + 2\Omega^a_{[b|e|}B^e_{c]}$ are changed as follows

$$(1.25)\qquad \begin{cases} \bar{B}^i_j = \bar{C}^i_a \dfrac{\partial C^a_k}{\partial \bar{s}^j}\bar{s}'^k + \bar{C}^i_a C^b_j B^a_b \\ \bar{\omega}^i_{jk} + 2\bar{\Omega}^i_{[j|l|}\bar{B}^l_{k]} = C^b_j C^c_k \left\{\left(\dfrac{\partial\, {}^*\bar{C}^i_b}{\partial s^c} - \dfrac{\partial\, {}^*\bar{C}^i_c}{\partial s^b}\right)\right. \\ \qquad -\left(\dfrac{\partial \bar{C}^i_b}{\partial s'^d} B^d_c - \dfrac{\partial \bar{C}^i_c}{\partial s'^d} B^d_b\right) + \left.\bar{C}^i_a\left(\omega^a_{bc} + 2\Omega^a_{[b|e|}B^e_{c]}\right)\right\} \end{cases}$$

by non-holonomic transformations of coordinates. Such quantities B^a_b and $\omega^a_{bc} + \Omega^a_{be}B^e_c - \Omega^a_{ce}B^e_b$ play an important rôle to determine connection parameters which we shall see later.

At last we shall proceed to find the change of the connection parameters under these transformations. We suppose now a contravariant vector v^a in the non-holonomic space and introduce its covariant differential in the form

$$(1.26)\qquad \delta v^a = dv^a + \Gamma^{*a}_{bc} v^b ds^c + C^{*a}_{bc} v^b \delta s'^c ,$$

then the definition (1. 22) demands that

$$(1.27)\qquad \delta\bar{v}^i = \bar{C}^i_a \delta v^a = \bar{C}^i_a (dv^a + \Gamma^{*a}_{bc} v^b ds^c + C^{*a}_{bc} v^b \delta s'^c) .$$

Let the base connections be

$$(1.28)\qquad \delta s'^c = d s'^c + G^c_a d s^a ,$$

then we obtain after some calculations

$$(1.29)\qquad \begin{cases} \bar{\Gamma}^{*i}_{jk} = \left\{\Gamma^{*a}_{bc} \bar{C}^i_a - \dfrac{\partial\, {}^*\bar{C}^i_b}{\partial s^c} + \dfrac{\partial \bar{C}^i_b}{\partial s'^d} G^d_c\right\} C^b_j C^c_k , \\ \bar{G}^i_j = \bar{C}^i_a \dfrac{\partial C^a_k}{\partial \bar{s}^j}\bar{s}'^k + \bar{C}^i_a C^b_j G^a_b \end{cases} \qquad \text{(N.H.F.)}$$

and

$$(1.30)\qquad \bar{C}^{*i}_{jk} = \left(\bar{C}^i_a C^{*a}_{bc} - \frac{\partial \bar{C}^i_b}{\partial s'^c}\right) C^b_j C^c_k .$$

3. Some connection parameters of the non-holonomic space. We shall go to find some admissible connections parameters in the non-holonomic space of line-elements given by (1. 4). For this purpose, putting the torsion tensor $''T^a_{bc}$ and the curvature tensor $'L^a_{bc}$ given by the second equations of (4. 8) and (4. 17) respectively in N.H.F. to be equal to zero, we have the following connection parameters

$$(1.31) \qquad \Gamma^{*e}_{bc} = G^{e}_{c;b} - \Omega^{f}_{cb} G^{e}_{f} , \qquad C^{*a}_{bc} = \Omega^{a}_{bc} .$$

Since we can take the quantities B^a_b as base connection parameters, as the firts equation of (1.25) shows, we can obtain

$$(1.32) \qquad C^{*a}_{bc} = \Omega^{a}_{bc} , \qquad \Gamma^{*e}_{bc} = B^{e}_{c;b} - \Omega^{f}_{cb} B^{e}_{f} , \qquad B^{e}_{c}$$

for some admissible connection parameters in the non-holonomic space of line-elements given by (1.4). Further we can see that the quantities B^a_b are equal to a base connection parameter $\Gamma^{*e}_{bc} s'^b$ obtained from the second equation of (1.32). In fact,

$$\Omega^{a}_{bc} s'^b = 0 , \qquad \Gamma^{*e}_{bc} s'^b = (B^{e}_{c;b} - \Omega^{f}_{cb} B^{e}_{f}) s'^b = B^{e}_{c;b} s'^b = B^{e}_{c}$$

where $\quad B^{e}_{c;b} = \dfrac{\partial B^{e}_{c}}{\partial s'^b} .$

4. Torsion tensors and curvature tensors in the space. We obtain torsion tensors in the non-holonomic space

$$(1.33) \qquad \begin{cases} 'T^{a}_{bc} = \omega^{a}_{bc} + 2\Omega^{a}_{[b\ f|} B^{f}_{c]} + 2\Gamma^{*a}_{[bc]} = \omega^{a}_{bc} + 2\Omega^{a}_{[b\,|f|} B^{f}_{c]} + 2B^{a}_{[c;b]} , \\ ''T^{a}_{bc} = 0 \end{cases}$$

from (4.8) in N.H.F., and three curvature tensors different from zero:

$$(1.34) \qquad \begin{cases} K^{a}_{dbc} = \Gamma^{*a}_{d[b:c]} + \Gamma^{*e}_{d[c} \Gamma^{*a}_{|e|b]} - \Omega^{a}_{df} {'K}^{f}_{bc} , \\ 'K^{f}_{bc} = - B^{f}_{b,c} + B^{f}_{[b|g|} B^{g}_{c]} + B^{e}_{c;b} B^{f}_{e} \qquad \left(\text{where}: B^{f}_{b,c} = \dfrac{\partial\, {}^{*}B^{f}_{b}}{\partial s^c}\right), \end{cases}$$

where

$$\Gamma^{*a}_{db:c} = \Gamma^{*a}_{db,c} - \Gamma^{*a}_{db;f} B^{f}_{c} + \Gamma^{*a}_{fc} \Gamma^{*f}_{db} - \Gamma^{*e}_{bc} \Gamma^{*a}_{de} - \Gamma^{*e}_{dc} \Gamma^{*a}_{eb} ,$$

given by (4.15) in N.H.F. and

$$(1.35) \qquad L^{a}_{dbc} = \Omega^{a}_{db,c} - \Gamma^{*a}_{dc;b} + \Omega^{e}_{bc} \Gamma^{*a}_{de} ,$$

where

$$\Omega^{a}_{db.c} = \Omega^{a}_{db,c} - \Omega^{a}_{db;f} B^{f}_{c} + \Gamma^{*a}_{fc} \Omega^{f}_{db} - \Gamma^{*e}_{bc} \Omega^{a}_{de} - \Gamma^{*e}_{dc} \Omega^{a}_{eb} ,$$

reduced from (4.17) in N.H.F.. At last it follows that the curvature tensors S^{a}_{ebc}, P^{a}_{ebc} and R^{a}_{ebc} in N.H.F. become in this case

$$S^a_{\iota bc} = 0 \ , \qquad P^a_{\iota bc} = L^a_{\iota bc} \ , \qquad R^a_{\iota bc} = K^a_{\iota bc} \ .$$

Hence the space has a torsion tensor $'T^a_{bc}$ and three curvature tensors K^a_{dbc}, $'K^\iota_{bc}$, $L^a_{\iota bc}$ different from zero.

§ 2. A non-holonomic space of line-elements belonging to a non-holonomic transformation group having invariant non-holonomic subspaces X^m_n and $X^{n-m}_{n.}$.

5. Definition of a non-holonomic subspace of line-elements. We shall henceforth confine ourselves to that the indices $a, b, \cdots\cdots$, etc. and $a', b', c' \cdots\cdots$, etc. run from 1 to m and from $m+1$ to n respectively and to use capital letters $A, B, C \cdots\cdots$ etc. for indices which run from 1 to n. Let us define a non-holonomic subspace X^m_n in the non-holonomic space of line-elements given by (1. 4) by the following equations

$$(2.1) \qquad ds^{a'} = \lambda^{a'}_\alpha dx^\alpha = 0 \ , \qquad \frac{ds^{a'}}{dt} = \lambda^{a'}_\alpha \frac{dx^\alpha}{dt} = 0$$

$$(a', b' \cdots = m+1, \cdots, n) \ .$$

Then it follows from the second equation of (2. 1) that a direction of line-elements in our subspace depends on only $\frac{ds^a}{dt}$ $(a = 1, 2, \cdots, m)$.

For the non-holonomic transformation of coordinates under which the subspace X^m_n is invariant, we must have

$$(2.2) \qquad \bar{\bar{C}}^{c'}_a = 0$$

by reason that if $ds^{a'} = 0$ and $\frac{ds^{a'}}{dt} = 0$, then $d\bar{s}^{\iota'} = 0$ and $\frac{d\bar{s}^{\iota'}}{dt} = 0$. Therefore the function $\bar{\bar{C}}^{\iota'}$ of the x and $\frac{ds^A}{dt}$ can not depend on the $\frac{ds^a}{dt}$ $(a = 1, \cdots\cdots, m)$. Similarly, the necessary and sufficient conditions for invariance of a non-holonomic subspace X^{n-m}_n given by

$$(2.3) \qquad ds^a = \lambda^a_\alpha dx^\alpha = 0 \ , \qquad \frac{ds^a}{dt} = \lambda^a_\alpha \frac{dx^\alpha}{dt} = 0 \qquad (a = 1, \cdots, m)$$

are that

$$(2.4) \qquad \bar{\bar{C}}^\iota_{a'} = 0$$

i.e. the functions $\bar{\bar{C}}^\iota$ of the x and $\frac{ds^A}{dt}$ can not depend on the $\frac{ds^{a'}}{dt}$

$(a' = m+1, \cdots, n)$. It can be then concluded $C^a_{i'} = C^{a'}_i = 0$. From the above results the group of our non-holonomic transformations is given by

$$(2.5)\quad \begin{cases} d\bar{s}^i = \bar{C}^i_a\, ds^a\,, \qquad d\bar{s}^{i'} = \bar{C}^{i'}_{a'}\, ds^{a'}\,, \\ \dfrac{d\bar{s}^i}{dt} = \bar{C}^i_a \dfrac{ds^a}{dt}\,, \qquad \dfrac{d\bar{s}^{i'}}{dt} = \bar{C}^{i'}_{a'} \dfrac{ds^{a'}}{dt}\,. \end{cases}$$

This group is evidently the subgroup of (1.18) or (1.19).

6. Invariants of the space belonging to the subgroup of non-holonomic transformations. Under a transformation (2.5), we get the results:

$$(2.6)\quad \begin{cases} \bar{\Gamma}^{*i}_{jk} = \left\{\Gamma^{*a}_{bc}\,\bar{C}^i_a - \dfrac{\partial\, {}^*\bar{C}^i_b}{\partial s^c} + \dfrac{\partial \bar{C}^i_b}{\partial s'^d}\, G^d_c\right\} C^b_j C^c_k\,, \\ \bar{G}^i_j = \bar{C}^i_a \dfrac{\partial C^a_l}{\partial \bar{s}^j}\,\bar{s}'^l + \bar{C}^i_a\, C^b_j\, G^a_b\,, \qquad \bar{C}^{*i}_{jk} = \left(\bar{C}^i_a\, C^{*a}_{bc} - \dfrac{\partial \bar{C}^i_b}{\partial s'^c}\right) C^b_j C^c_k\,, \end{cases}$$

$$(2.7)\quad \begin{cases} \bar{\Gamma}^{*i}_{jk'} = \left\{\Gamma^{*a}_{bc'}\,\bar{C}^i_a - \dfrac{\partial\, {}^*\bar{C}^i_b}{\partial s^{c'}} + \dfrac{\partial \bar{C}^i_b}{\partial s'^d}\, G^d_{c'}\right\} C^b_j C^{c'}_{k'}\,, \\ \bar{G}^i_{k'} = \bar{C}^i_a \dfrac{\partial C^a_l}{\partial \bar{s}^{k'}}\,\bar{s}'^l + \bar{C}^i_a\, C^{b'}_{k'}\, G^a_{b'}\,, \qquad \bar{C}^{*i}_{jk'} = \bar{C}^i_a\, C^b_j\, C^{c'}_{k'}\, C^{*a}_{bc'} \end{cases}$$

$$(2.8)\quad \bar{\Gamma}^{*i}_{j'k} = \Gamma^{*a}_{b'c}\,\bar{C}^i_a\, C^{b'}_{j'}\, C^c_k\,, \qquad \bar{C}^{*i}_{j'k} = \bar{C}^i_a\, C^{b'}_{j'}\, C^c_k\, C^{*a}_{b'c}\,,$$

$$(2.9)\quad \bar{\Gamma}^{*i}_{j'k'} = \Gamma^{*a}_{b'c'}\,\bar{C}^i_a\, C^{b'}_{j'}\, C^{c'}_{k'}\,, \qquad \bar{C}^{*i}_{j'k'} = \bar{C}^i_a\, C^{b'}_{j'}\, C^{c'}_{k'}\, C^{*a}_{b'c'}$$

and same ones exchanged the indices $a, b, c, \cdots; i, j, k, \cdots\cdots$ etc. and $a', b', c' \cdots; i', j', k' \cdots$ etc. with one another in the equations (2.6), (2.7), (2.8) and (2.9). The above results show us that the parts of the connection parameters and the base connection parameters in the non-holonomic space:

$$(\Gamma^{*a}_{bc}\,,\ G^a_b)\,,\ C^{*a}_{bc}\,,\ (\Gamma^{*a}_{bc'}\,,\ G^a_{c'})\,,\ C^{*a}_{bc'}\,,\ \Gamma^{*a}_{b'c}\,,\ C^{*a}_{b'c}\,,\ \Gamma^{*a}_{b'c'}\,,\ C^{*a}_{b'c'}\ \text{etc.}$$

are all the geometrical objects, *i.e.* the quantities Γ^{*a}_{bc}, G^a_b and C^{*a}_{bc} are the connection parameters that give a absolute differential in the non-holonomic subspace X^m_n. The quantities $\Gamma^{*a}_{bc'}$, $G^a_{c'}$ and $C^{*a}_{bc'}$ are the connection parameters that give components on X^m_n of a absolute differential of a vector in X^m_n along the non-holonomic subspace X^{n-m}_n

etc. We can apply the same treatment to the other quantities.

Further, under the transformation, we have

$$(2.10)\quad \begin{cases} \overline{\Omega}^{i}_{jk} = \left(\overline{C}^{i}_{a}\,\Omega^{a}_{bc} - \dfrac{\partial C^{i}_{b}}{\partial s'^{c}}\right) C^{b}_{j} C^{c}_{k}\,, & \overline{\Omega}^{i}_{jk'} = \overline{C}^{i}_{a}\, C^{b}_{j}\, C^{c'}_{k'}\, \Omega^{a}_{bc'}\,, \\ \overline{\Omega}^{i}_{j'k} = \overline{C}^{i}_{a}\, C^{b'}_{j'}\, C^{c}_{k}\, \Omega^{a}_{b'c}\,, & \overline{\Omega}^{i}_{j'k'} = \overline{C}^{i}_{a}\, C^{b'}_{j'}\, C^{c'}_{k'}\, \Omega^{a}_{b'c'}\,, \end{cases}$$

$$(2.11)\quad \begin{cases} \overline{\omega}^{i'}_{jk} + 2\overline{\Omega}^{i'}_{[j|T|}\, \overline{B}^{T}_{k]} = C^{b}_{j}\, C^{c}_{k}\, \overline{C}^{i'}_{a'}\, (\omega^{a'}_{bc} + 2\Omega^{a'}_{[b|E|}\, B^{E}_{c]})\,, \\ \overline{\omega}^{i}_{jk'} + 2\overline{\Omega}^{i}_{[j|T|}\, \overline{B}^{T}_{k']} = C^{b}_{j}\, C^{c'}_{k'} \left\{ \dfrac{\partial\, {}^{*}\overline{C}^{i}_{b}}{\partial s^{c'}} - \dfrac{\partial\, \overline{C}^{i}_{b}}{\partial s'^{d}}\, B^{d}_{c'} \right. \\ \qquad\qquad \left. + \overline{C}^{i}_{a}\, (\omega^{a}_{bc'} + 2\Omega^{a}_{[b|E|}\, B^{E}_{c']}) \right\} \end{cases}$$

and same ones exchanged the indices $a, b, c, \cdots i, j, k \cdots$ etc. and $a', b', c', \cdots i', j', k' \cdots$ etc. with one another in the equations (2.10) and (2.11).

In § 1 it has been shown that the space has the following connection parameters

$$(2.12)\qquad \Gamma^{*A}_{BC} = B^{A}_{C;B} - \Omega^{S}_{BC}\, B^{A}_{S}\,, \qquad G^{A}_{B} = B^{A}_{B}\,, \qquad C^{*A}_{BC} = \Omega^{A}_{BC}$$

but since the quantities $-(\omega^{a}_{bc'} + 2\Omega^{a}_{[b|E|}\, B^{E}_{c']})$ and $-(\omega^{a'}_{bc} + 2\Omega^{a'}_{[b|E|}\, B^{E}_{c]})$ change as likely as the $\Gamma^{*a}_{bc'}$ and $\Gamma^{*a'}_{bc}$ respectively under the transformations, we may take for connection parameters those which are replaced the parts in the quantities Γ^{*A}_{BC} of (2.12): $\Gamma^{*a}_{bc'}$, $\Gamma^{*a'}_{b'c}$, $\Gamma^{*a'}_{bc}$ and $\Gamma^{*a}_{b'c'}$ by the quantities

$$-(\omega^{a}_{bc'} + 2\,\Omega^{a}_{[b|E|}\, B^{E}_{c']})\,, \qquad -(\omega^{a'}_{b'c} + 2\,\Omega^{a'}_{[b'|E|}\, B^{E}_{c]})\,,$$

$$-(\omega^{a'}_{bc} + 2\,\Omega^{a'}_{[b|E|}\, B^{E}_{c]})\,, \qquad \text{and} \qquad -(\omega^{a}_{b'c'} + 2\,\Omega^{a}_{[b'|E|}\, B^{E}_{c']})\,,$$

i.e.

$$(2.13)\quad \begin{cases} \Gamma^{*a}_{bc} = B^{a}_{c;b} - \Omega^{E}_{cb}\, B^{a}_{E}\,, & \Gamma^{*a}_{bc'} = -(\omega^{a}_{bc'} + 2\,\Omega^{a}_{[b|E|}\, B^{E}_{c']})\,, \\ \Gamma^{*a}_{b'c} = B^{a}_{c;b'} - \Omega^{E}_{cb'}\, B^{a}_{E}\,, & \Gamma^{*a}_{b'c'} = -(\omega^{a}_{b'c'} + 2\,\Omega^{a}_{[b'|E|}\, B^{E}_{c']})\,, \\ \Gamma^{*a'}_{b'c'} = B^{a'}_{c';b'} - \Omega^{E}_{c'b'}\, B^{a'}_{E}\,, & \Gamma^{*a'}_{b'c} = -(\omega^{a'}_{b'c} + 2\,\Omega^{a'}_{[b'|E|}\, B^{E}_{c]})\,, \\ \Gamma^{*a'}_{bc'} = B^{a'}_{c';b} - \Omega^{E}_{c'b}\, B^{a'}_{E}\,, & \Gamma^{*a'}_{bc} = -(\omega^{a'}_{bc} + 2\,\Omega^{a'}_{[b|E|}\, B^{E}_{c]})\,, \\ G^{A}_{B} = B^{A}_{B}\,, \qquad C^{*A}_{BC} = \Omega^{A}_{BC}\,. \end{cases}$$

Here we must remark that the base connection parameters reduced from the connection parameters (2. 13) do not coincide with B^A_B. By this reason, we shall determine the base connection parameters so as

$$(2.14)\qquad \Gamma^{*A}_{BC}\, s'^B = {'B}^A_C \,.$$

For this purpose, calculating the quantities $\Gamma^{*A}_{BC}\, s'^B$, we have

$$(2.15)\qquad \begin{cases} \Gamma^{*a}_{Ac}\, s'^A = (B^a_{c;b} - \Omega^E_{cb}\, B^a_E)\, s'^b + (B^a_{c;b'} - \Omega^E_{cb'}\, B^a_E)\, s'^{b'} \\ \qquad\quad = B^a_{c;b}\, s'^b + B^a_{c;b'}\, s'^{b'} - (\Omega^E_{cb}\, s'^b + \Omega^E_{cb'}\, s'^{b'})\, B^a_E \\ \qquad\quad = B^a_c \qquad\qquad (\text{because } \Omega^E_{cb}\, s'^b + \Omega^E_{cb'}\, s'^{b'} = 0) \end{cases}$$

and likely

$$(2.16)\qquad \Gamma^{*a'}_{Ac'}\, s'^A = B^{a'}_{c'} \,,$$

accordingly we can adopt ${'B}^a_c = B^a_c$ and ${'B}^{a'}_{c'} = B^{a'}_{c'}$ as the parts of the base connection parameters which satisfy the equation (2. 14). Further, it follows that

$$(2.17)\qquad \begin{cases} \Gamma^{*a}_{Ac'}\, s'^A = \Gamma^{*a}_{bc'}\, s'^b + \Gamma^{*a}_{b'c'}\, s'^{b'} \\ \qquad\quad = -\,\omega^a_{Ec'}\, s'^E + \Omega^a_{c'b}(B^b_d\, s'^d + B^b_{d'}\, s'^{d'}) + \Omega^a_{c'b'}(B^{b'}_d\, s'^d + B^{b'}_{d'}\, s'^{d'}), \end{cases}$$

so that it is enough to modify other parts of B^A_B such that holds (2. 14), because, for instance the parts of the base connection parameters: $B^a_{c'}$, are changed into the corresponding part as (2. 7) shows.

Let such quantities be ${'B}^a_{c'}$ and ${'B}^{a'}_c$, then the equation $\Gamma^{*a}_{Ac'}\, s'^A = {'B}^a_{c'}$ are written in the form

$$(2.18)\qquad \omega^a_{Ec'}\, s'^E - \Omega^a_{c'b}\, B^b_d\, s'^d - \Omega^a_{c'b'}\, B^{b'}_{d'}\, s'^{d'} = (\Omega^a_{c'b}\, s'^{d'} - \delta^a_b\, \delta^{d'}_{c'})\, {'B}^b_{d'} + \Omega^a_{c'b'}\, s'^d\, {'B}^{b'}_d,$$

similarly it follows from the equations $\Gamma^{*a'}_{Ac}\, s'^A = {'B}^{a'}_c$ that

$$(2.19)\qquad \omega^{a'}_{Ec}\, s'^E - \Omega^{a'}_{cb'}\, B^{b'}_{d'}\, s'^{d'} - \Omega^{a'}_{cb}\, B^b_d\, s'^d = (\Omega^{a'}_{cb'}\, s'^d - \delta^{a'}_{b'}\, \delta^d_c)\, {'B}^{b'}_d + \Omega^{a'}_{cb}\, s'^{d'}\, {'B}^b_{d'} \,.$$

Putting now

$$\begin{cases} \omega^a_{Ec'}\, s'^E - \Omega^a_{c'b}\, B^b_d\, s'^d - \Omega^a_{c'b'}\, B^{b'}_{d'}\, s'^{d'} = D^a_{c'} \,, \\ \omega^{a'}_{Ec}\, s'^E - \Omega^{a'}_{cb'}\, B^{b'}_{d'}\, s'^{d'} - \Omega^{a'}_{cb}\, B^b_d\, s'^d = D^{a'}_c \,, \\ \Omega^a_{c'b}\, s'^{d'} - \delta^a_b\, \delta^{d'}_{c'} = X^{a\ d'}_{b\ c'} \,, \qquad \Omega^{a'}_{cb}\, s'^{d'} = Y^{a'\ d'}_{c\ b} \end{cases}$$

and using the functions ${'X}^{a\ d'}_{b\ c'}$ which, since $|X| \neq 0$, are determined uniquely by

$$X^{a\ d'}_{b\ c'}\ 'X^{b\ e'}_{c\ d'} = \delta^a_c\, \delta^{e'}_{c'} \ ,$$

the equations (2.18) and (2.19) may be solved for $'B^{b'}_d$ and $'B^b_{d'}$, and the solutions are then of the form

$$(2.20) \qquad 'B^{b'}_d = (D^{a'}_c - Y^{a'd'}_{c\ b}\ 'X^{bc'}_{ad'}\, D^a_{c'})\ 'W^{b'c}_{a'd},\ 'B^b_{d'} = (D^a_{c'} - Y^{ad}_{c'b'}\ 'X^{b'c}_{a'd}\, D^{a'}_c)\ 'W^{bc'}_{ad'}$$

where the functions $'W^{b'c}_{a'd}$ are determined uniquely by $W^{a'd}_{b'c}\ 'W^{c'e}_{a'e} = \delta^{c'}_{b'}\, \delta^d_e$ when $|W| \neq 0$, putting

$$X^{a'\ d}_{b'\ e} - Y^{a'\ f'}_{c\ \ e}\ 'X^{e\ c'}_{a\ f'}\, \Omega^a_{c'b'}\, s'^d = W^{a'\ d}_{b'\ c} \ .$$

Thus if in the beginning we adopt the quantities in the right hand members of (2.15), (2.16) and (2.20) as the base connection parameters in the space: $'B^A_B$, these coincide with the base connection parameters which are reduced from the connection parameters (2.13) made of $'B^A_B$.

In order that an absolute differential of a contravariant vector v^h in the X^m_n lies in the X^m_n, it must be

$$\Gamma^{*h'}_{kj} = \Gamma^{*h'}_{kj'} = C^{*h'}_{kj} = C^{*h'}_{kj'} = 0 \ ,$$

as $v^{h'} = 0$ and $\delta v^{h'} = 0$.

Similarly in the X^{n-m}_n, we get also

$$\Gamma^{*h}_{k'j'} = \Gamma^{*h}_{k'j} = C^{*h}_{k'j'} = C^{*h}_{k'j} = 0 \ .$$

The connection parameters (2.13) become then as follows

$$(2.21)\ \begin{cases} \Gamma^{*a}_{bc} = B^a_{c;b} - \Omega^E_{cb}\, B^a_E,\ \Gamma^{*a}_{bc'} = -(\omega^a_{bc'} + \Omega^a_{bE}\, B^E_{c'} - \Omega^a_{c'E}\, B^E_b),\ \Gamma^{*a}_{b'c} = \Gamma^{*a}_{b'c'} = 0, \\ \Gamma^{*a'}_{b'c'} = B^{a'}_{c';b'} - \Omega^E_{c'b'} B^{a'}_E,\ \Gamma^{*a'}_{b'c} = -(\omega^{a'}_{b'c} + \Omega^{a'}_{b'E} B^E_c - \Omega^{a'}_{cE} B^E_{b'}),\ \Gamma^{*a'}_{bc'} = \Gamma^{*a'}_{bc} = 0, \\ C^{*a}_{bc} = \Omega^a_{bc}\ ,\quad C^{*a}_{bc'} = \Omega^a_{bc'}\ ,\quad C^{*a}_{b'c} = C^{*a}_{b'c'} = 0\ , \\ C^{*a'}_{b'c'} = \Omega^{a'}_{b'c'}\ ,\quad C^{*a'}_{b'c} = \Omega^{a'}_{b'c}\ ,\quad C^{*a'}_{bc'} = C^{*a'}_{bc} = 0\ , \end{cases}$$

where it should be remarked that the base connection parameters reduced from the connection parameters (2.21) do not coincide with B^A_B.

7. Torsion tensors and curvature tensors. We shall consider torsion tensors for the displacement in X^m_n: $d_1 s^a$ $(d_1 s^{a'} = 0)$ and for that in X^{n-m}_n: $d_2 s^{b'}$ $(d_2 s^b = 0)$, in the considered space with the connection parameters (2.21), where the line-elements may be chosen arbitrarily and not necessarily in the subspaces.

Denoting vectors of the natural reference of the line-elements with $e_a(x, x')$ $a = 1, 2, \cdots\cdots, n$, then the displacement of a point M (x) is given by $dM = dx^a e_a$, consequently

$$dM = ds^A \lambda_A^a e_a \,. \tag{2.22}$$

In account of $\lambda_A^a e_a = e_A$, (2.22) is written in

$$dM = ds^A e_A \,. \tag{2.23}$$

Hence n vectors e_A $(A = 1, \cdots\cdots, n)$ are vectors of the natural reference in the non-holonomic space. (4.6), (4.7) and (4.8) of N.H.F. show us

$$d_2 d_1 M - d_1 d_2 M = \Omega^A e_A, \quad \Omega^A = 'T^A_{BC} d_{[1} s^B d_{2]} s^C + 2''T^A_{BC} d_{[1} s^B \delta_{2]} s'^C \tag{2.24}$$

where

$$'T^A_{BC} = \omega^A_{BC} + 2\Omega^A_{[B|F|} B^F_{C]} + 2\Gamma^{*A}_{[BC]}, \qquad ''T^A_{BC} = C^{*A}_{BC} - \Omega^A_{BC} \,. \tag{2.25}$$

In our case, we have

$$\Omega^a = 'T^a_{bc'} d_1 s^b d_2 s^{c'} - ''T^a_{Jd'} \delta_1 s'^J d_2 s^{d'}$$

and $\quad \Omega^{a'} = 'T^{a'}_{bc'} d_1 s^b d_2 s^{c'} + ''T^{a'}_{Jc} \delta_2 s'^J d_1 s^c$

where $\quad 'T^a_{bc'} = 'T^{a'}_{bc'} = 0, \quad ''T^a_{Jd'} = -\Omega^a_{Jd'}, \quad ''T^{a'}_{Jc} = -\Omega^{a'}_{Jc}$

from (2.21) and (2.25).

Consider three infinitesimally near line-elements l_P, l_Q and l_R with the origins $P(x^a)$, $Q(x^a + d_1 x^a)$ and $R(x^a + d_2 x^a)$ in the original holonomic space, where we assume that the direction $\overrightarrow{PQ}$ ($d_1 s^A$ in the non-holonomic system) lies in X_n^m and $\overrightarrow{PR}$ ($d_2 s^A$) in X_n^{n-m}, then the vector Ω^A in (2.24) represents the displacement TS, when $\overrightarrow{RS}$ and $\overrightarrow{QT}$ are the parallel displacements of $d_1 s^A$ and $d_2 s^A$ from l_P to l_R and from l_P to l_Q in the non-holonomic space of line-elements respectively.

Similarly, if the line-element lies in X_n^m *i.e.* $\dfrac{ds^{a'}}{dt} = 0$, for the torsion tensors with respect to two displacements $d_1 s^a$ $(d_1 s^{a'} = 0)$ and $d_2 s^a$ $(d_2 s^{a'} = 0)$ in X_n^m we have

$$\begin{cases} \Omega^a = (\omega^a_{bc} + 2\Omega^a_{[b|M|} B^M_{c]} + 2\Gamma^{*a}_{[bc]}) d_{[1} s^b d_{2]} s^c \,, \\ \Omega^{a'} = (\omega^{a'}_{bc} + 2\Omega^{a'}_{[b|M|} B^M_{c]}) d_{[1} s^b d_{2]} s^c - \Omega^{a'}_{bE} \delta_{[2} s'^E d_{1]} s^b \,, \end{cases}$$

we obtain thus three torsion tensors as follows

$$\omega^{a}_{bc} + 2\,\Omega^{a}_{[b|M|}\,B^{M}_{c]} + 2\,\Gamma^{*a}_{[bc]}\ , \qquad \omega^{a'}_{bc} + 2\,\Omega^{a'}_{[b|M|}\,B^{M}_{c]}\ , \qquad \Omega^{a'}_{bE}\ ,$$

where we must remark $\delta_1 s'^{b'}$, $\delta_2 s'^{b} \neq 0$ by reason that the base connection parameters reduced from the connection parameter (2. 21) do not coincide with B^{A}_{B}. Further we can have many curvature tensors as same as ones in N.H.F..

§ 3. A non-holonomic Euclidean space of line-elements.

8. A non-holonomic Euclidean space of line-elements. In the present and following Chapters, we confine ourselves to consider only a FINSLER space. A FINSLER space means such a space that the length of a curve given by analytic functions $x^i = x^i(t)$ of the parameter t is defined by $s = \int L(x, x')\,dt$ where $L(x, x')$ has continuous partial derivatives of at least the forth order in the x and x' and homogeneous of degree one in the x'.

In such a space of line-elements, a metric tensor has been given by

$$\frac{\partial^2 (\frac{1}{2} L^2)}{\partial x'^{\alpha}\,\partial x'^{\beta}} = g_{\alpha\beta}(x, x') \tag{3. 1}$$

and its contravariant components $g^{\alpha\beta}$ are defined by the equations

$$g^{\alpha\beta}\,g_{\alpha\gamma} = \delta^{\beta}_{\gamma}\ . \tag{3. 2}$$

Let us assume that there are functions $A^B(x, dx^\alpha)$ which satisfy the conditions

$$g^{\alpha\beta}\,\lambda^{A}_{\alpha}\,\lambda^{B}_{\beta} = \delta^{AB}\ , \qquad \lambda^{A}_{\alpha} = \frac{\partial A^A}{\partial (dx^\alpha)} \tag{3. 3}$$

and call a non-holonomic space of line-elements given by (1. 4) corresponding to these functions A^B *non-holonomic Euclidean space with respect to* s'. Then we get the following results

$$g_{\alpha\beta} = \sum_{A} \lambda^{A}_{\alpha}\,\lambda^{A}_{\beta}\ , \qquad g^{\alpha\beta} = \sum_{A} \lambda^{\alpha}_{A}\,\lambda^{\beta}_{A} \tag{3. 4}$$

from the first equations of (3. 3) and (1. 9), (3. 2) and (1. 8) respectively.

In this case, the connection parameters in the non-holonomic space induced from those in the original FINSLER space (in means of Cartan) become as follows

$$\Gamma^{*A}_{BC} = \left(\lambda^A_\alpha \Gamma^{*\alpha}_{\beta\gamma} - \frac{\partial \lambda^A_\beta}{\partial x^\gamma} + \frac{\partial \lambda^A_\beta}{\partial x'^\delta}\frac{\partial G^\delta}{\partial x'^\gamma}\right)\lambda^\beta_B \lambda^\gamma_C , \quad = -\lambda^A_{\beta\cdot\gamma}\lambda^\beta_B \lambda^\gamma_C \tag{3.5}$$

(N.H.F.)

where

$$\lambda^A_{\beta\cdot\gamma} = \frac{\partial \lambda^A_\beta}{\partial x^\gamma} - \frac{\partial \lambda^A_\beta}{\partial x'^\delta}\frac{\partial G^\delta}{\partial x'^\gamma} - \Gamma^{*\alpha}_{\beta\gamma}\lambda^A_\alpha .$$

Then the index A of λ^A_α denotes individual vectors in the holonomic space and the index α components of the vectors, accordingly the right hand member of the above equation gives the covariant derivatives of the covariant vectors: λ^A_β in the holonomic space with respect to x^γ. Then the quantities Γ^{*A}_{BC} may be considered as the coefficients of rotation of the orthogonal ennuple λ^A_β.

Considering the covariant derivatives of the following equation

$$g^{\alpha\beta}\lambda^A_\alpha \lambda^B_\beta = \delta^{AB} \tag{3.6}$$

with respect to x^γ and using the second equations of (3.4), we have the equations

$$\lambda^\alpha_B \lambda^A_{\alpha\cdot\gamma} + \lambda^\beta_A \lambda^B_{\beta\cdot\gamma} = 0 ,$$

contracting the both sides of the last equation with λ^γ_C, we get

$$\lambda^\gamma_C(\lambda^\alpha_B \lambda^A_{\alpha\cdot\gamma} + \lambda^\beta_A \lambda^B_{\beta\cdot\gamma}) = 0 ,$$

then it follows that

$$\Gamma^{*A}_{BC} = -\Gamma^{*B}_{AC} . \tag{3.7}$$

On the other hand, deforming the right hand member of the first equations of (3.5), we get the following results

$$\begin{cases} \Gamma^{*A}_{BC} = \left\{\lambda^A_\alpha \Gamma^{*\alpha}_{\beta\gamma} - \dfrac{\partial^* \lambda^A_\beta}{\partial x^\gamma}\right\}\lambda^\beta_B \lambda^\gamma_C - \Omega^A_{BD} G^D_C , \\ 2\Gamma^{*A}_{[BC]} = 2\lambda^A_\alpha \Gamma^{*\alpha}_{[\beta\gamma]}\lambda^\beta_{[B}\lambda^\gamma_{C]} - (\omega^A_{BC} + 2\Omega^A_{[B|D|}G^D_{C]}) , \end{cases} \tag{3.8}$$

consequently

$$\Gamma^{*A}_{BC} - \Gamma^{*A}_{CB} = -(\omega^A_{BC} + 2\Omega^A_{[B|D|}G^D_{C]}) \tag{3.9}$$

in virtue of $\Gamma^{*\alpha}_{[\beta\gamma]} = 0$. Putting

$$\omega^A_{BC} + 2\Omega^A_{[B|D|}G^D_{C]} = \theta^A_{BC} \tag{3.10}$$

and using (3.7) and (3.9), we shall obtain the results

$$\Gamma^{*A}_{BC} = -\frac{1}{2}(\theta^A_{BC} + \theta^B_{CA} + \theta^C_{BA}) \ . \tag{3.11}$$

In this case, the quantities C_{ABC} given by (2.7) in N.H.F. become

$$C_{ABC} = \frac{1}{2}\left(\frac{\partial \delta_{AB}}{\partial s'^C} - \Omega^D_{AC}\delta_{DB} - \Omega^D_{BC}\delta_{DA}\right) = \frac{-1}{2}(\Omega^B_{AC} + \Omega^A_{BC}) \tag{3.12}$$

and $C_{A[BC]}=0$ from the second equations of (2.9) in N.H.F.. It follows then that $\Omega^B_{CA}=\Omega^C_{BA}$, (3.12) are denoted with $C_{ABC}=-\Omega^B_{AC}$. Consequently we have that

$$C^{*A}_{BC} = \delta^{AE}C_{EBC} + \Omega^A_{BC} = -\Omega^B_{AC} + \Omega^A_{BC} = 0 \ . \tag{3.13}$$

This fact indicates that such a special non-holonomic space of line-elements (ds^A, s'^A) is *Euclidean* with respect to $\frac{ds^A}{dt}$, *i.e.* the absolute differential is always independent of $\delta s'^A$, but not with respect to ds^A.

Moreover we shall try to represent the base connection parameters G^A_B in terms of the quantities ω^A_{BC} and Ω^A_{BC}. From (3.10) and (3.11), we have

$$\Gamma^{*A}_{BC} = -\frac{1}{2}\left\{\omega^A_{BC}+\omega^B_{CA}+\omega^C_{BA}+(-\Omega^A_{CD}G^D_B+\Omega^C_{BD}G^D_A-\Omega^C_{AD}G^D_B+\Omega^B_{CD}G^D_A)\right\}, \tag{3.14}$$

further

$$G^A_C=\Gamma^{*A}_{BC}s'^B=\frac{-1}{2}\left\{(\omega^A_{BC}+\omega^B_{CA}+\omega^C_{BA})s'^B+(-2\Omega^A_{CD}s'^B+2\Omega^C_{ED}s'^E\delta^B_A)G^D_B\right\}, \tag{3.15}$$

consequently

$$G^D_B\left\{\delta^A_D\delta^B_C - \Omega^A_{CD}s'^B + \Omega^C_{ED}s'^E\delta^B_A\right\} = \frac{-1}{2}(\omega^A_{BC} + \omega^B_{CA} + \omega^C_{BA})s'^B \ , \tag{3.16}$$

putting

$$\delta^A_D\delta^B_C - \Omega^A_{CD}s'^B + \Omega^C_{ED}s'^E\delta^B_A = Z^{AB}_{CD}$$

and using the quantities $\tilde{Z}^{AB}_{CD}$ given by the equations

$$Z^{AB}_{CD}\tilde{Z}^{EC}_{AF} = \delta^E_D\delta^B_F \tag{3.17}$$

when $|Z^{AB}_{CD}| \neq 0$, we have

$$G^D_B = -\frac{1}{2}(\omega^A_{EC} + \omega^E_{CA} + \omega^C_{EA})s'^E\tilde{Z}^{DC}_{AB} \ . \tag{3.18}$$

§ 4. The second fundamental tensor of the non-holonomic Euclidean subspace X_n^m and the geodesic non-holonomic Euclidean subspace.

9. The fundamental quantities. In this Chapter, we shall consider a subgroup of the non-holonomic Euclidean transformations under which non-holonomic subspaces X_n^m and X_n^{n-m} given by (2. 1) and (2. 3) respectively are invariant, so that we must have the conditions (2. 2) and (2. 4), and also the relations as follows

$$\sum_{a} \bar{C}_b^h \bar{C}_c^h = \delta_{bc} \, , \qquad \sum_{a'} \bar{C}_{b'}^{h} \bar{C}_{c'}^{h} = \delta_{b'c'} \, , \tag{4. 1}$$

in order that an orthogonal ennuple goes into same one too. Thus we shall study a non-holonomic space which belongs to subgroups of non-holonomic transformations (2. 5) holding the relation (4. 1).

Under such a transformation, the quantities $\theta_{bc'}^a \, (= \omega_{bc'}^a + 2\Omega_{[b|D|}^a G_{c']}^D)$ are changed as

$$\bar{\theta}_{jk'}^i = C_j^b C_{k'}^{c'} \left\{ \frac{\partial \, {}^*\bar{C}_b^i}{\partial s^{c'}} - \frac{\partial \bar{C}_b^i}{\partial s'^d} G_{c'}^d + \bar{C}_a^i \theta_{bc'}^a \right\} , \tag{4. 2}$$

the above equation may be written in

$$\frac{\partial \, {}^*\bar{C}_b^h}{\partial s^{c'}} - \frac{\partial \, {}^*\bar{C}_b^h}{\partial s'^d} G_c^d = \bar{\theta}_{jk'}^h C_b^j C_{c'}^{k'} - \bar{C}_a^h \theta_{bc'}^a \, , \tag{4. 3}$$

then using the first relation of (4. 1), we obtain the results

$$(\bar{\theta}_{jk}^h + \bar{\theta}_{hk'}^j) \bar{C}_c^h \bar{C}_b^j \bar{C}_{c'}^{k'} = \theta_{bc'}^c + \theta_{cc'}^b \, . \tag{4. 4}$$

Consequently the quantities $\theta_{bc'}^c + \theta_{cc'}^b$ are tensors in the non-holonomic space. In such a space we can also get the connection parameters for which absolute differential of a contravariant vector v^h in the X_n^m lies in the X_n^m, as follows

$$\begin{cases} \Gamma^{*a}_{bc} = -\dfrac{1}{2} (\theta_{bc}^a + \theta_{ca}^b + \theta_{ba}^c) \, , \quad \Gamma^{*a}_{bc'} = -\theta_{bc'}^a \, , \quad \Gamma^{*a}_{b'c} = \Gamma^{*a}_{b'c'} = 0 \, , \\ \Gamma^{*a'}_{b'c'} = -\dfrac{1}{2} (\theta_{b'c'}^{a'} + \theta_{c'a'}^{b'} + \theta_{b'a'}^{c'}) \, , \quad \Gamma^{*a'}_{b'c} = -\theta_{b'c}^{a'} \, , \quad \Gamma^{*a'}_{bc'} = \Gamma^{*a'}_{bc} = 0 \, , \\ C^{*A}_{BC} = 0 \, . \end{cases} \tag{4. 5}$$

or

$$(4.6)\quad \begin{cases} \tilde{\Gamma}^{*a}_{bc} = -\dfrac{1}{2}(\theta^a_{bc} + \theta^b_{ca} + \theta^c_{ba}), \quad \tilde{\Gamma}^{*a}_{bc'} = -\dfrac{1}{2}(\theta^a_{bc'} + \theta^{c'}_{ba} + \theta^b_{c'a}), \quad \tilde{\Gamma}^{*a}_{b'c} = \tilde{\Gamma}^{*a}_{b'c'} = 0, \\ \tilde{\Gamma}^{*a'}_{b'c'} = -\dfrac{1}{2}(\theta^{a'}_{b'c'} + \theta^{b'}_{c'a'} + \theta^{c'}_{b'a'}), \quad \tilde{\Gamma}^{*a'}_{b'c} = -\dfrac{1}{2}(\theta^{a'}_{b'c} + \theta^c_{b'a} + \theta^{b'}_{ca}), \quad \tilde{\Gamma}^{*a'}_{bc'} = \tilde{\Gamma}^{*a'}_{bc} = 0, \\ C^{*A}_{BC} = 0 \ . \end{cases}$$

Here the base connection parameters in the function θ^A_{BC} of (4.5) and (4.6) coincide with neither the quantities $\Gamma^{*A}_{BC}s'^B$ induced from (4.5) nor the $\tilde{\Gamma}^A_{BC}s'^B$ induced from (4.6). Then the difference between (4.5) and (4.6) is nothing but the tensor

$$\begin{cases} \Gamma^{*a}_{bc'} - \tilde{\Gamma}^{*a}_{bc'} = -\theta^a_{bc'} + \dfrac{1}{2}(\theta^a_{bc'} + \theta^{c'}_{ba} + \theta^b_{c'a}) \\ \qquad = -\dfrac{1}{2}(\theta^a_{bc'} + \theta^b_{ac'} - \theta^{c'}_{ba}) \ . \end{cases}$$

Thus we have tensors of four kinds in this space as follows

$$\theta^{a'}_{bc} \ , \quad \theta^a_{b'c'} \ , \quad \theta^a_{bc'} + \theta^b_{ac'} \ , \quad \theta^{a'}_{b'c} + \theta^{b'}_{a'c} \ .$$

10. The second fundamental tensor. Next, we shall find the geometrical meaning of the tensor $\theta^c_{ab'} + \theta^b_{cb'}$ in the non-holonomic subspace X^m_n with the metric

$$(4.7)\qquad ds^2 = \sum_{a=1}^{m} (ds^a)^2 = \sum_a \lambda^a_\alpha \lambda^a_\beta \, dx^\alpha dx^\beta \ .$$

At first, we consider the variation of such a metric (4.7) in the direction which is normal to X^m_n.

Putting for the variational displacement of the line-elements

$$(4.8)\qquad d_2 x^\alpha = \lambda^\alpha_{c'} \varepsilon^{c'} \ , \qquad d_2 s'^\lambda = - G^\alpha_{c'} \varepsilon^{c'}$$

where the quantities $\lambda^\alpha_{c'}$ are the functions of the x^α and s'^c and the quantities $\varepsilon^{c'}$ represent the certain infinitesimal constants. The displacement $d\sigma^a$ in the subspace $\overline{X}^m_n$ corresponding to the ds^a in the X^m_n can be represented as follows

$$(4.9)\qquad d\sigma^a = {}^*\lambda^a_\alpha (x^\alpha + d_2 x^\alpha, \ s'^a + d_2 s'^a) \ d(x^\alpha + d_2 x^\alpha) \ ,$$

using the next calculation

$$\begin{cases} {}^*\lambda^a_\alpha(x^\alpha + d_2x^\alpha, s'^a + d_2 s'^a) = {}^*\lambda^a_\alpha(x, s') + \dfrac{\partial\, {}^*\lambda^a_\alpha}{\partial x^\beta}\lambda^\beta_{c'}\varepsilon^{c'} - \dfrac{\partial\, {}^*\lambda^a_\alpha}{\partial s'^b} G^b_{c'}\varepsilon^{c'} + [\quad], \\ d(x^\alpha + d_2 x^\alpha) = dx^\alpha + \left(\dfrac{\partial\, {}^*\lambda^\alpha_{c'}}{\partial x^\beta}dx^\beta - \dfrac{\partial\, {}^*\lambda^\alpha_{c'}}{\partial s'^d} G^d_b ds^b + \dfrac{\partial\, {}^*\lambda^\alpha_{c'}}{\partial s'^d}\delta s'^d\right)\varepsilon^{c'} + [\quad], \end{cases}$$

where the symbol [ ] denotes the higher order term than degree two in the $\varepsilon^{c'}$, it follows that

$$d\sigma^a = ds^a + \{(\omega^a_{bc'} + 2\Omega^a_{[b|E|} G^E_{c'}])\, ds^b + \Omega^a_{c'd}\,\delta s'^d\}\,\varepsilon^{c'} + [\quad],$$

consequently we have

$$\sum_a (d\sigma^a)^2 - \sum_a (ds^a)^2 = \{(\theta^a_{bc'} + \theta^b_{ac'})\, ds^a ds^b + 2\Omega^a_{c'd}\,\delta s'^d ds^a\}\,\varepsilon^{c'} + [\quad].$$

Thus if we consider the variation of the metric (4.7) for the variational displacement (4.8), we have

(4.10) $\quad (\theta^a_{bc'} + \theta^b_{ac'})\, ds^a ds^b + 2\Omega^a_{c'd}\,\delta s'^d ds^a$

as the first variation. Let us suppose that the element s'^d be displaced parallelly in the subspace X^m_n *i.e.* $\delta s'^d = 0$, then the quantities (4.10) are written in

(4.11) $\quad (\theta^a_{bc'} + \theta^b_{ac'})\, ds^a ds^b$.

Since the above quantities in an Euclidean or Riemannian space give the second fundamental form of X^m_n, we shall call the quantities

(4.12) $\quad \theta^b_{ac'} + \theta^a_{bc'}$

the second fundamental tensor of X^m_n.

11. The geodesic non-holonomic sudspace. In such a Euclidean non-holonomic space in which either the subspace X^m_n or X^{n-m}_n is invariant, if we adopt the connection parameters of the space as follows

(4.13) $\quad \Gamma^{*A}_{BC} = -\dfrac{1}{2}(\theta^A_{BC} + \theta^B_{CA} + \theta^C_{BA})$

and in the non-holonomic subspace X^m_n as

$$\Gamma^{*a}_{bc} = -\frac{1}{2}(\theta^a_{bc} + \theta^b_{ca} + \theta^c_{ba}),$$

a geodesic curve in the X_n^m is then given by the equation

$$\frac{d^2 s^a}{d s^2} = \Gamma^{*a}_{bc} \frac{ds^b}{ds} \frac{ds^c}{ds} . \tag{4.14}$$

In order that the curve is geodesic also in the space, it must be

$$\frac{d}{ds}\left(\frac{ds^{a'}}{ds}\right) - \Gamma^{*a'}_{bc} \frac{ds^b}{ds} \frac{ds^c}{ds} = 0 . \tag{4.15}$$

Here, by reason of $\frac{ds^{a'}}{ds} = 0$, (4.15) must be $\Gamma^{*a'}_{bc} \frac{ds^b}{ds} \frac{ds^c}{ds} = 0$. Consequently, from (4.13) and $\theta^A_{(BC)} = 0$, we have $\theta^c_{ba'} + \theta^b_{ca'} = 0$. Thus we have the theorem:

A neseccary and sufficient condition that such the subspace X_n^m be geodesic in the space with the connection parameters (4.13), is that

$$\theta^b_{ac'} + \theta^a_{bc'} = 0 . \tag{4.16}$$

(August, 1949).