Algebraic methods for chromatic polynomials

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Abstract: The chromatic polynomials of certain families of graphs can be calculated by a transfer matrix method. The transfer matrix commutes with an action of the symmetric group on the colours. Using representation theory, it is shown that the matrix is equivalent to a block-diagonal matrix. The multiplicities and the sizes of the blocks are obtained.

Using a repeated inclusion-exclusion argument the entries of the blocks can be calculated. In particular, from one of the inclusion-exclusion arguments it follows that the transfer matrix can be written as a linear combination of operators which, in certain cases, form an algebra. The eigenvalues of the blocks can be inferred from this structure.

The form of the chromatic polynomials permits the use of a theorem by Beraha, Kahane and Weiss to determine the limiting behaviour of the roots. The theorem says that, apart from some isolated points, the roots approach certain curves in the complex plane. Some improvements have been made in the methods of calculating these curves.

Many examples are discussed in detail. In particular the chromatic polynomials of the family of the so-called generalized dodecahedra and four similar families of cubic graphs are obtained, and the limiting behaviour of their roots is discussed.

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For Amalia, who left too early.

Acknowledgements

First of all, I would like to thank Norman Biggs for being such a excellent supervisor. He has supported, guided and encouraged me, and he has also kept my feet on the ground when it was necessary. I am very grateful for his time and patience throughout my PhD.

Next I would like to thank the rest of the department. In particular Jan van den Heuvel, Mark Baltovic, Jackie Everid and David Scott for always having an open ear and good advice.

Thanks also to Robert Shrock, of Stony Brook University, New York, for finding the time and the financial support for me to visit and work with him.

I am indebted to the UK Engineering and Physical Sciences Research Council (EPSRC), the London School of Economics and Political Science and the Department of Mathematics for financial assistance throughout my PhD.

My parents have also been a source of moral and financial support throughout my studies. They have always given my sisters and myself unconditional support and the security and warmth of a true family. I must also thank Maria del Mar for her most wonderful support, patience and understanding over the last three years.

Finally I would like to finish by praising the European Union for providing a framework and some financial support for my studies abroad. In particular I have to thank the English university system for its openness and flexibility: recognizing my German high school degree and allowing me to study in England; covering my tuition fees; fully validating my year of studies in Italy; and all with a minimal amount of bureaucracy. I hope that this openness and flexibility becomes common practice throughout the European Union.

Statement of originality

Most of the work presented in this thesis is a continuation of work by N.L. Biggs [5], [8], [7], [9], [4] and [6]. The work appearing in this thesis is entirely my own except where stated otherwise. In particular:

- The observation that the compatibility matrix commutes with the action of the symmetric group has also been exploited by M.H. Klin and C. Pech; see [9].
- The Examples 3.7, 3.8 and 3.9 have been published in [9]. They are due to me.
- The chromatic polynomial for the generalised dodecahedron was obtained by S.C. Chang [11]. Here, the polynomial is calculated in a different way, to illustrate the compatibility matrix method.
- The Lemmas 6.12, 6.14 and Corollaries 6.13 and 6.15 are joint work with Jan van den Heuvel.

Chapter 1

Introduction

1.1 Overview

A graph B consists of two sets; a vertex set and a edge set whose members are unordered pairs of vertices. We say a pair of vertices are adjacent if they are an edge. Given a set of k "colours", usually the first k positive integers, a proper vertex k-colouring of the graph B is a function from the vertex set into the set of colours such that adjacent vertices take different "colours" under the colouring. We omit the words "proper" and "vertex", and just speak of a k-colouring of B. The chromatic polynomial $P(B, k)$ corresponding to a graph B is the polynomial function which evaluated at a positive integer k equals the number of k-colourings of B.

In theory, the standard method of deletion-and-contraction allows us to find the chromatic polynomial for any given finite graph. However this method is not very elegant in the sense that it requires exponentially many steps (in the number of edges). In general there is no efficient method.

In this thesis we are studying the chromatic polynomials for families of graphs with a cyclic symmetry using a transfer matrix method. These families of graphs consist of n copies of a "base graph" arranged in a "ring". Adjacent copies of the "base graph" have extra edges between them according to a "linking set".

Although the deletion-and-contraction method destroys the symmetry in the first step, it has been used to obtain a transfer matrix via a recursion relation by D.A. Sands (in an unpublished thesis, 1972), N.L. Biggs and G.H.J. Meredith in [1], J. Salas and A.D. Sokal in [21], and by R. Shrock and co-workers in [25], [13] and a series of other works.

Here, in this work we use and develop a slightly different transfer matrix method which enables us to utilize the symmetry to a maximum. This method was introduced by N.L. Biggs in [2], and recently used and developed in [5], [8], [7], [19], and [9].

This transfer matrix commutes with an action of the symmetric group permuting the colours. Using representation theory, it is shown that the matrix is equivalent to a block-diagonal matrix. The multiplicities and the sizes of the blocks are obtained. Using a repeated inclusion-exclusion argument the entries of the blocks can be calculated (Chapters 2 and 3).

In particular, from one of the inclusion-exclusion arguments it follows that the transfer matrix can be written as a linear combination of operators which, in certain cases, form an algebra. In Chapter 6 parts of the structure of this algebra are investigated.

In Chapter 4 many examples are discussed in detail. In particular the chromatic polynomials of the family of the so-called generalized dodecahedra and four similar families of cubic graphs are obtained.

The form of the chromatic polynomials permits the use of a theorem by Beraha, Kahane and Weiss to determine the limiting behaviour of the roots as the number of copies of the "base graph" goes to infinity. The theorem says that, apart from some isolated points, the roots approach certain curves in the complex plane. Chapter 5 contains calculations based on [4] and [6] by N.L. Biggs. The results here are by no means complete, and many phenomena observed in the limiting curves described in the examples of Chapter 5 remain to be analyzed.

These limiting curves have also been studied by R. Shrock and co-workers in [24] and in a series of works, and by J. Salas and A.D. Sokal in, for example, [21] and [15].

The chromatic polynomials for this type of graphs have also been the focus of research in statistical mechanics. This is due to the fact that the zero-temperature partition function of the k-state Potts antiferromagnet on the graph B is equal to $P(B, k)$; [13] and [22]. In particular the behaviour of the roots of $P(B, k)$ as the number of vertices goes to infinity is of paramount interest.

In future the theoretical framework introduced in Chapters 2 and 3 will hopefully be used to obtain the chromatic polynomials for more families of graphs. In particular the families of graphs with the cycle or the path on b vertices as "base graphs", and the "identity linking set" are obvious candidates for further research. In [23] A. D. Sokal finds a upper bound for the radius of a disc in the complex plane containing all the roots. This upper bound depends on the maximum degree of the graph. The hope is to be able to find a connection between the limiting curves of the roots and the type of "base graph" or the "linking set".

Chapter 2

Modules and colourings

The first part of this chapter gives a brief outline of some basic results of representation theory, in particular of the symmetric group. This is based on the books by G.D. James [16], W. Ledermann [17] and B.E. Sagan [20]. In the second part this theory is applied to the modules obtained when the symmetric group Sym_k acts on the set of k-colourings of a graph.

2.1 Some representation theory

Let G be a (finite) group written multiplicatively. We denote the identity element of G by ϵ. Let V be a vector space over $\mathbb{C}$ of dimension n. A *representation* of G on V is a group homomorphism $\rho : G \to \mathrm{Aut}(V)$ where $\mathrm{Aut}(V)$ is the group of automorphisms of V. By choosing a particular basis for V it follows that ρ assigns to every $g \in G$ a non-singular $n \times n$ matrix $A(g)$ with coefficients in $\mathbb{C}$. We say that $A(g)$, or A, is a *matrix representation* of G with *degree* n corresponding to ρ.

We denote by $\mathbb{C}G$ the *group algebra* consisting of all finite linear combinations

$$\sum_{g \in G} z_g \, g \qquad (z_g \in \mathbb{C})$$

with the componentwise addition, and multiplication given by

$$\left(\sum_{g\in G} z_g\, g\right)\left(\sum_{h\in G} z_h\, h\right) = \sum_{f\in G}\left(\sum_{gh=f} z_g z_h\right) f.$$

Denote by $\mathrm{End}(V)$ the algebra of homomorphisms on V. Then a representation of G can be extended to a representation of $\mathbb{C}G$. That is $\rho : \mathbb{C}G \to \mathrm{End}(V)$ is an algebra homomorphism defined as:

$$\rho\left(\sum_{g\in G} z_g\, g\right) = \sum_{g\in G} z_g\, \rho(g)$$

with $z_g \in \mathbb{C}$. This makes V into a $\mathbb{C}G$-module. The two notions of a representation of $\mathbb{C}G$ - $\mathbb{C}G$-module V and the algebra homomorphism $\rho : \mathbb{C}G \to \mathrm{End}(V)$ - are equivalent and we use them interchangeably. We denote by Mat_n the algebra of all $n \times n$ matrices with coefficients in $\mathbb{C}$. Then, as before, by choosing a particular basis for V it follows that $\rho : \mathbb{C}G \to \mathrm{Mat}_n$ is the corresponding matrix representation.

A subspace U of V is a *submodule* of V if U is invariant under the action of $\mathbb{C}G$. A module V is *irreducible* if its only submodules are V itself and the zero module, otherwise we call V *reducible*. We say that two matrix representations $A(x)$ and $B(x)$ are *equivalent* if there exists a non-singular matrix T such that $T^{-1}A(x)\ T = B(x)$. Let $A(x)$ be the matrix representation corresponding to ρ. Then, from the above definition of reducibility, it follows that $A(x)$ is reducible if it is equivalent to a representation of the form

$$\begin{pmatrix} D(x) & \mathbf{O} \\ E(x) & F(x) \end{pmatrix},$$

where $\mathbf{O}$ is an all-zero matrix. Otherwise $A(x)$ is irreducible.

We say that a matrix is the *direct sum* of the matrices $A_1,\ A_2, \ldots,\ A_l$ if A is the diagonal block matrix diag$(A_1,\ A_2,\ \ldots,\ A_l)$. We write:

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_l = \bigoplus_{i=1}^{l} A_i.$$

Maschke's Theorem asserts that over the field $\mathbb{C}$ every matrix representation is *completely reducible*, that is for some choice of the basis of V it follows that

$$A(x) = \mathrm{diag}(A_1(x), A_2(x), \ldots, A_l(x)) = \bigoplus_{i=1}^{l} A_i(x)$$

where the $A_i(x)$ are irreducible representations. Equivalently, the corresponding $\mathbb{C}G$-module V is the direct sum of l irreducible submodules.

Let $A = (a_{ij})$ and B be matrices of degrees n and m respectively. Then the *tensor product, direct product* or *Kronecker product* $A \otimes B$ is the $nm \times nm$ matrix obtained by replacing the entry a_{ij} in A by the matrix $a_{ij}B$. With this notation we can write every matrix representation $A(x)$ as:

$$A(x) = \bigoplus_i (I_{m_i} \otimes A_i(x))$$

where the $A_i(x)$ are now inequivalent, irreducible representations of degree n_i and *multiplicity* m_i in $A(x)$, and I_{m_i} is the identity matrix of size m_i.

Let A be a matrix representation of $\mathbb{C}G$. Then $\mathcal{C}(A)$ is the *commutant algebra* of A. This is the subalgebra of Mat_n consisting of all T satisfying $A(x)T = TA(x)$ for all $x \in \mathbb{C}G$. If A is irreducible then Schur's Lemma asserts that $\mathcal{C}(A)$ only consists of scalar multiples of the identity matrix.

If $A(x) = I_m \otimes B(x)$ where B is irreducible then $T \in \mathcal{C}(A)$ is of the form $X \otimes I_n$ where $X \in \mathrm{Mat}_m$ and n is the degree of $B(x)$. By a change of basis, that is reordering the basis vectors, it can be shown that T is equivalent to $I_n \otimes X$. In general the following lemma holds.

Lemma 2.1 *Let $A(x)$ be any matrix representation of $\mathbb{C}G$ of the form:*

$$A(x) = \bigoplus_{i=1}^{l} (I_{m_i} \otimes A_i(x))$$

where the $A_i(x)$ are inequivalent, irreducible representations of degree n_i and multiplicity m_i in $A(x)$. Then every $T \in \mathcal{C}(A)$ is equivalent to a matrix of the form:

$$\bigoplus_{i=1}^{l} (I_{n_i} \otimes X_i),$$

with $X_i \in Mat_{m_i}$.

Proof: Let $A = \bigoplus_{i=1}^{l} B_i(x)$ where $B_i(x) = (I_{m_i} \otimes A_i(x))$. If $T \in \mathcal{C}(A)$ then

$$\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_l \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1l} \\ T_{21} & T_{22} & \cdots & T_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ T_{l1} & T_{l2} & \cdots & T_{ll} \end{pmatrix}$$
$$= \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1l} \\ T_{21} & T_{22} & \cdots & T_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ T_{l1} & T_{l2} & \cdots & T_{ll} \end{pmatrix} \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_l \end{pmatrix}$$

implies that $B_i T_{ij} = T_{ij} B_j$. The matrices B_i and B_j are inequivalent by assumption. Thus from Schur's Lemma follows that T_{ij} is the zero matrix if $i \neq j$. Again Schur's Lemma and the argument preceding this lemma imply that $T_{ii} = X_i \otimes I_{n_i}$ where $X_i \in \mathrm{Mat}_{m_i}$. Rearranging the order of the basis vectors it follows that T_{ii} is equivalent to $I_{n_i} \otimes X_i$. □

2.2 The symmetric group

Let us now focus on the symmetric group and its representations. A *permutation* of a set K is a bijection from K into itself. We can assume that K is the set of numbers $\{1, 2, \ldots, k\}$. Then a permutation ω can be expressed as a product of disjoint cycles. For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 4 & 3 & 8 & 6 & 1 & 7 \end{pmatrix} = (1587)(2)(34)(6),$$

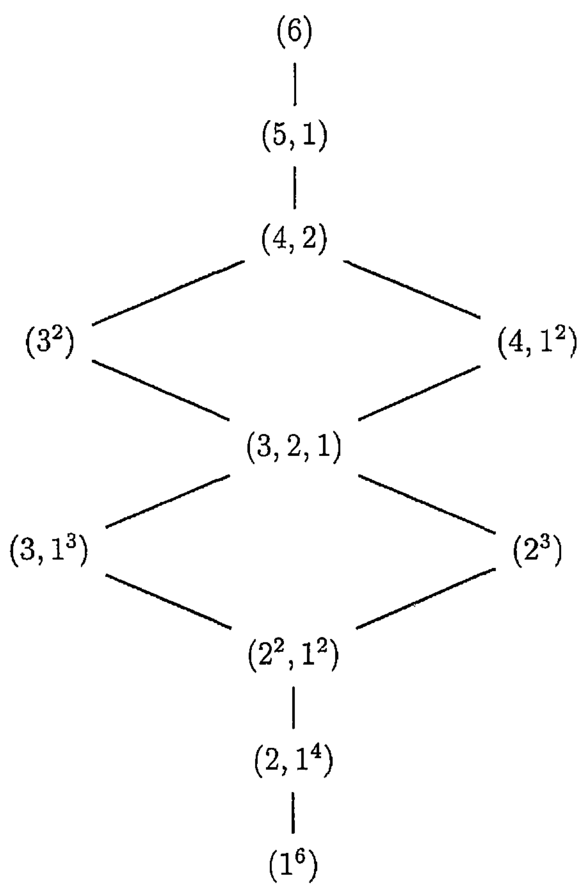
where 1-cycles are often omitted. For any two functions g and f their composition is defined as $(g \circ f)(x) = gf(x) = g(f(x))$. In particular, the composition of two permutations is a sequence of instructions read *from right to left*. For example $(12)(23) = (123)$. The set of all permutations of the set $\{1, 2, \ldots, k\}$ together with the composition of functions is the *symmetric group* Sym_k of degree k. The identity element is denoted by ϵ. In general, we denote by Sym_X the group of all permutations of a set X.

The sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a *partition* of $k \in \mathbb{N}$ if $\lambda_1, \lambda_2, \ldots \lambda_k$ are non-negative integers, with $\sum_{i=1}^{k} \lambda_i = k$. For example $(5,3,1,1,0,0,0,0,0,0)$ is a partition of 10. We usually omit the zeros and order $\lambda_1, \lambda_2, \ldots \lambda_l$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l$. For example we write $(5,3,1^2)$. If λ is a partition of k we write $\lambda \vdash k$. For two partitions λ and μ of k we say that λ *dominates* μ and write $\lambda \succeq \mu$ if for all j

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i.$$

If $\lambda \succeq \mu$ and $\lambda \neq \mu$ then we write $\lambda \succ \mu$.

Example 2.1: The partial ordering $\succeq$ of the eleven partitions of 6 is as follows:



The *diagram* $[\lambda]$ corresponding to $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash k$ where $\lambda_l \neq 0$ is the array

$$\{(i,j) \mid i,j \in \mathbb{Z}, \quad 1 \leq i \leq l, \quad 1 \leq j \leq \lambda_i\}.$$

If $(i,j) \in [\lambda]$ then (i,j) is called a *node* of $[\lambda]$. The n^{th} row (column) consists of those nodes whose first (second) coordinate is n. We can draw diagrams by replacing each node in $[\lambda]$ by a " $\times$ ". For example:

$$\begin{array}{ccccc} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & \times & & & \\ \times & & & & \end{array} \qquad \text{is the diagram } [\lambda] \text{ for } \lambda = (5, 2^2, 1).$$

Let $\lambda \vdash k$ and let X be a set. A λ-*tableau* is a function $t : [\lambda] \to X \subset \mathbb{N} \cup \{0\}$. Unless stated otherwise we assume that $X = \{1, 2, \ldots, k\}$. If a λ-tableau is a bijection we denote it by a lowercase t, if it is not a bijection we denote it by a capital T. We can construct a λ-tableau t by replacing each node in $[\lambda]$ by an integer with no repeats.

Example 2.2:

$$\begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \quad \begin{array}{cc} 2 & 1 \\ 3 & \end{array}, \quad \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \quad \begin{array}{cc} 3 & 1 \\ 2 & \end{array}, \quad \begin{array}{cc} 3 & 2 \\ 1 & \end{array} \quad \text{and} \quad \begin{array}{cc} 2 & 3 \\ 1 & \end{array}$$

are the $(2, 1)$-tableaux.

Example 2.3:

$$t_1 = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & & & \\ 8 & 9 & & & \\ 10 & & & & \end{array} \qquad t_2 = \begin{array}{ccccc} 4 & 1 & 7 & 8 & 5 \\ 6 & 3 & & & \\ 2 & 9 & & & \\ 10 & & & & \end{array}$$

are $(5, 2^2, 1)$-tableaux.

Define the action of the symmetric group Sym_k on the set of λ-tableaux by:

$$(\omega, t)(i, j) = \omega(t(i, j)) \qquad \text{for all } (i, j) \in [\lambda]$$

for any $\omega \in \mathrm{Sym}_k$ and λ-tableau t. Writing ωt instead of (ω, t) we get for example:

$$(1482)(37)t_1 = t_2$$

where t_1 and t_2 are as in Example 2.3. For a given t we denote by C_t the subgroup of Sym_k which fixes setwise the elements in each column of t. That is

$$C_t = \{\omega \in \mathrm{Sym}_k \mid \forall (i,j) \in [\lambda] \quad \exists (p,j) \in [\lambda] \text{ such that } \omega t(i,j) = t(p,j)\}.$$

We call C_t the *column-stabilizer* of t. Similarly we define the *row-stabilizer* R_t of t as:

$$R_t = \{\omega \in \mathrm{Sym}_k \mid \forall (i,j) \in [\lambda] \quad \exists (i,p) \in [\lambda] \text{ such that } \omega t(i,j) = t(i,p)\}.$$

Define the equivalence relation $\sim$ on the set of λ-tableaux by $t \sim t'$ if and only if $\omega t = t'$ for some $\omega \in R_t$. Therefore $t \sim t'$ if and only if the set of entries in row i is the same for t and t' for all i. We denote by $\{t\}$ the equivalence class of t under this relation and call it a *tabloid*. Roughly speaking $\{t\}$ is obtained from t by ignoring the order of the elements in each of the respective rows, i.e. the rows in $\{t\}$ are unordered sets. This means the λ-tabloid $\{t\}$ is a partition of the set X corresponding to λ. The parts of $\{t\}$ are its rows. We indicate a tabloid $\{t\}$ by drawing lines between the rows of t.

Example 2.4: The $(2,1)$-tabloids are:

$$\left\{ \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 1 \\ 3 & \end{array} \right\} = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}, \qquad \left\{ \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \begin{array}{cc} 3 & 1 \\ 2 & \end{array} \right\} = \begin{array}{cc} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}$$

and

$$\left\{ \begin{array}{cc} 3 & 2 \\ 1 & \end{array}, \begin{array}{cc} 2 & 3 \\ 1 & \end{array} \right\} = \begin{array}{cc} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}.$$

Example 2.5: The tabloids corresponding to the tableaux t_1 and t_2 given in Example 2.3 are:

$$\{t_1\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & & & \\ \cline{1-2} 8 & 9 & & & \\ \cline{1-2} 10 & & & & \\ \cline{1-1} \end{array} \quad \text{and} \quad \{t_2\} = \begin{array}{ccccc} \hline 4 & 1 & 7 & 8 & 5 \\ \hline 6 & 3 & & & \\ \cline{1-2} 2 & 9 & & & \\ \cline{1-2} 10 & & & & \\ \cline{1-1} \end{array} .$$

Let M^λ be the vector space over $\mathbb{C}$ spanned by the λ-tabloids. The action of Sym_k on the λ-tableaux induces an action on the λ-tabloids. For every choice of two λ-tableaux t and t' there exists a $\omega \in \mathrm{Sym}_k$ such that $t = \omega t'$. It follows that M^λ is generated by one λ-tabloid under this action of $\mathbb{C}\mathrm{Sym}_k$. This makes M^λ into a cyclic $\mathbb{C}\mathrm{Sym}_k$-module. Its dimension is

$$\dim(M^\lambda) = \frac{k!}{\lambda_1!\lambda_2!\dots\lambda_k!}.$$

For a given t define the signed column sum $\kappa_t \in \mathbb{C}\mathrm{Sym}_k$ as $\kappa_t = \sum_{\omega\in C_t} \mathrm{sign}(\omega)\ \omega$. Then the *polytabloid* $e_t \in \mathcal{M}^\lambda$ is defined as $e_t = \kappa_t\{t\}$.

Example 2.6: Let $\lambda = (3,2)$ and $t = \begin{matrix} 3 & 1 & 5 \\ 2 & 4 & \end{matrix}$ then $\kappa_t = \epsilon - (23) - (14) + (23)(14)$ and

$$e_t = \frac{\overline{3\ 1\ 5}}{\underline{2\ 4}} - \frac{\overline{2\ 1\ 5}}{\underline{3\ 4}} - \frac{\overline{3\ 4\ 5}}{\underline{2\ 1}} + \frac{\overline{2\ 4\ 5}}{\underline{3\ 1}}.$$

The vector space spanned by the polytabloids e_t for a given λ is a submodule of M^λ. We call this submodule the *Specht module* $\mathcal{S}^\lambda$ corresponding to the partition λ.

Example 2.7: Let $\lambda = (2,1)$ then:

$$e_{t_1} = \frac{\overline{1\ 2}}{\underline{3}} - \frac{\overline{3\ 2}}{\underline{1}}, \qquad e_{t_2} = \frac{\overline{2\ 1}}{\underline{3}} - \frac{\overline{3\ 1}}{\underline{2}}, \qquad e_{t_3} = \frac{\overline{1\ 3}}{\underline{2}} - \frac{\overline{2\ 3}}{\underline{1}},$$

$$e_{t_4} = \frac{\overline{3\ 1}}{\underline{2}} - \frac{\overline{2\ 1}}{\underline{3}}, \qquad e_{t_5} = \frac{\overline{2\ 3}}{\underline{1}} - \frac{\overline{1\ 3}}{\underline{2}}, \qquad e_{t_6} = \frac{\overline{3\ 2}}{\underline{1}} - \frac{\overline{1\ 2}}{\underline{3}}.$$

There are several linear relationships between these polytabloids. For example, $e_{t_1} = -e_{t_6}$, $e_{t_2} = -e_{t_4}$, $e_{t_3} = -e_{t_5}$ and $e_{t_3} = e_{t_1} - e_{t_2}$. In fact, the Specht module $\mathcal{S}^{(2,1)}$ is of dimension two and the polytabloids e_{t_1} and e_{t_2} are a basis. Acting with $\mathbb{C}\mathrm{Sym}_3$ on this basis gives:

$$(1\,2)e_{t_1} = e_{t_2} \qquad (1\,2)e_{t_2} = e_{t_1} \qquad (1\,2\,3)e_{t_1} = e_{t_2} - e_{t_1} \qquad (1\,2\,3)e_{t_2} = -e_{t_1}.$$

For a given partition λ of k, a tableau t is called a *standard tableau* if the numbers increase along the rows and down the columns of t. A tabloid $\{t\}$ is a *standard tabloid* if there is a standard tableau in the equivalence class $\{t\}$. The polytabloid e_t is a *standard polytabloid* if t is standard.

The set of polytabloids e_t, where t is standard, forms a basis for $\mathcal{S}^\lambda$ (Theorem 2.5.2 [20]). The matrix representation corresponding to $\mathcal{S}^\lambda$ with respect to this basis is called *Young's natural representation.* In Example 2.7 the standard basis consists of e_{t_1} and e_{t_3}, and Young's natural representation is generated by:

$$R^{(2,1)}(1\,2) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad R^{(2,1)}(1\,2\,3) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

It can easily be checked that for example

$$R^{(2,1)}(1\,3) = R^{(2,1)}(1\,2\,3)\ R^{(2,1)}(1\,2) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

For a given partition λ and a set $X = \{x_1, x_2, \ldots, x_l\}$ with $x_1 < x_2 < \ldots < x_l$ denote by $T : [\lambda] \to X$ a tableau of shape λ but with possible repeated entries. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash k$ be a second partition of k with $\mu_i = 0$ if $i > l$. Here the parts are not necessarily arranged in descending order and zero parts are not omitted. We say that T is a λ-tableau of *type* μ if the entry x_i appears μ_i times in T. Unless stated otherwise we assume that $X = \{1, 2, \ldots, k\}$.

Example 2.8: For $k = 9$, $\lambda = (5, 2, 2)$ and $\mu = (3, 0, 2, 4)$, two possible λ-tableaux of type μ are:

$$T_1 = \begin{matrix} 1 & 3 & 3 & 1 & 4 \\ 4 & 4 & & & \\ 1 & 4 & & & \end{matrix} \qquad T_2 = \begin{matrix} 1 & 1 & 1 & 4 & 4 \\ 3 & 3 & & & \\ 4 & 4 & & & \end{matrix}\ .$$

A tableau T is said to be *semistandard* if the entries are weakly increasing along the rows and strictly increasing along the columns of T. For example, the above tableau T_2 is semistandard whereas T_1 is not.

Theorem 2.2 *For any given $\mu \vdash k$, the Specht modules $\mathcal{S}^\lambda$ with $\lambda \succeq \mu$ are all the irreducible submodules of $\mathcal{M}^\mu$. The dimension n_λ of $\mathcal{S}^\lambda$ is equal to the number of standard λ-tableaux and its multiplicity m_λ as irreducible submodule of $\mathcal{M}^\mu$ is equal to the number of semistandard λ-tableaux of type μ.*

Proof: The statement is a combination of Theorem 4.13, Theorem 8.4 and Theorem 14.1 (Young's Rule) in [16]. □

Lemma 2.3 *Let λ be any partition of k. Then,*

$$\dim(\mathcal{S}^\lambda) = n_\lambda = k! \prod_{i=1}^{k} x_i(\lambda)$$

where,

$$x_i(\lambda) = \frac{\prod_{j=i+1}^{k} (\lambda_i - \lambda_j + j - i)}{(\lambda_i + k - i)!} \qquad \textit{for } i = 1, 2, \ldots, k-1 \textit{ and } \qquad x_k(\lambda) = \frac{1}{\lambda_k!}.$$

Proof: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of k with l non zero parts.

$$\begin{aligned}
\prod_{i=1}^{k} x_i(\lambda) &= \frac{\prod_{i=1}^{k-1} \prod_{j=i+1}^{k} (\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{k} (\lambda_i + k - i)!} \\
&= \frac{\prod_{i=1}^{l-1} \prod_{j=i+1}^{l} (\lambda_i - \lambda_j + j - i) \prod_{i=1}^{l} \prod_{j=l+1}^{k} (\lambda_i + j - i) \prod_{i=l+1}^{k-1} \prod_{j=i+1}^{k} (j - i)}{\prod_{i=1}^{l} (\lambda_i + k - i)! \prod_{i=l+1}^{k} (k - i)!} \\
&= \frac{\prod_{i=1}^{l-1} \prod_{j=i+1}^{l} (\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{l} (\lambda_i + k - i)! \prod_{i=l+1}^{k} (k - i)!} \prod_{i=1}^{l} \frac{(\lambda_i + k - i)!}{(\lambda_i + l - i)!} \prod_{i=l+1}^{k-1} (k - i)! \\
&= \frac{\prod_{i=1}^{l-1} \prod_{j=i+1}^{l} (\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{l} (\lambda_i + l - i)!}.
\end{aligned}$$

From Theorem 20.1 [16] it follows that this is equal to $\frac{n_\lambda}{k!}$. □

2.3 The module of colourings $\mathcal{V}_k(B)$

A *graph* B consists of a *vertex set* $V = \{v_1, v_2, \dots, v_b\}$ and a subset of unordered pairs of vertices called the *edge set* of B. In the following, we exclude the possibility that $\{v, v\}$ is in the edge set. That is, we deal only with loop-less graphs. We say that two vertices v and w of B are *adjacent* if $\{v, w\}$ is in the edge set of B. We usually assume that $V = \{1, 2, \dots, b\}$.

The graph with edge set consisting of all possible unordered pairs of vertices (but excluding the case $\{v, v\}$) is called a complete graph and we denote it by K_b. Its vertex set is denoted by V_b.

Let B be a graph with vertex set V. Throughout this section we regard the natural number k as fixed and we denote by $K = \{1, 2, \dots k\}$ the set of colours. A k-colouring of B is a function $\alpha : V \to \{1, 2, \dots k\}$ satisfying $\alpha(v) \neq \alpha(w)$ whenever $\{v, w\}$ is in the edge set of B. That is, adjacent vertices in B take different colours. We denote the set of all colourings of B by $\Gamma_k(B)$. In the case where B is the complete graph on b vertices, $\Gamma_k(b) = \Gamma_k(K_b)$ is the set of injections from V_b into K.

Every function $\theta : V \to K$ induces a partition $\mathcal{R} = \{R_1, R_2, \dots, R_r\}$ of V, written as $\theta \models \mathcal{R}$, by letting two vertices v and w be in the same part R_i if and only if $\theta(v) = \theta(w)$.

An *independent set* R is a subset of the vertex set V such that no pair of vertices of R are an edge of B. Note that all singletons are independent sets. A collection of disjoint non-empty independent sets whose union is V is called a *colour-partition* of V. We write colour-partitions as sets and separate the independent sets by " | ". For example, in case of the path on four vertices there are five colour-partitions:

$$\mathcal{R}_1 = \{1|2|3|4\}, \quad \mathcal{R}_2 = \{13|2|4\}, \quad \mathcal{R}_3 = \{1|24|3\},$$

$$\mathcal{R}_4 = \{14|2|3\} \quad \text{and} \quad \mathcal{R}_5 = \{13|24\}.$$

$|\mathcal{R}|$ denotes the number of independent sets in $\mathcal{R}$, and $\Pi(B)$ is the set of all colour-partitions of V for a given graph B. Note that α is a colouring of B if and only if

$\alpha \models \mathcal{R}$ for some $\mathcal{R} \in \Pi(B)$.

The symmetric group Sym_k acts on $\Gamma_k(B)$ in the obvious way. That is, for any $\omega \in \mathrm{Sym}_k$ and $\alpha \in \Gamma_k(B)$,

$$(\omega, \alpha)(v) = \omega(\alpha(v)) \qquad \text{for all } v \in V.$$

Two colourings α and β lie in the same orbit under Sym_k if and only if they induce the same colour-partition.

Denote by $\mathcal{V}_k(B)$ the vector space of complex-valued functions defined on $\Gamma_k(B)$. If $B = K_b$ we write $\mathcal{V}_k(b)$. The standard basis for $\mathcal{V}_k(B)$ consists of the functions $[\alpha]$ for every $\alpha \in \Gamma_k(B)$ defined as follows

$$[\alpha](\beta) = \begin{cases} 1 & \text{if } \alpha = \beta; \\ 0 & \text{otherwise.} \end{cases}$$

The action of Sym_k on $\Gamma_k(B)$ induces an action on $\mathcal{V}_k(B)$. This makes $\mathcal{V}_k(B)$ into a $\mathbb{C}\mathrm{Sym}_k$-module.

For any colour-partition $\mathcal{R}$ the cyclic submodule of $\mathcal{V}_k(B)$ spanned by the set $\{[\alpha] \mid \alpha \models \mathcal{R}\}$ will be denoted by $\langle \mathcal{R} \rangle$. Since for every $[\alpha] \in \mathcal{V}_k(B)$ we have that $[\alpha] \in \langle \mathcal{R} \rangle$ for exactly one $\mathcal{R} \in \Pi(B)$, it follows that $\mathcal{V}_k(B)$ is the direct sum of the $\langle \mathcal{R} \rangle$.

For any $\mathcal{R} \in \Pi(B)$, let $\lambda_{k,\mathcal{R}}$ be the partition $(k - |\mathcal{R}|, 1^{|\mathcal{R}|})$ of k. Recall that $M^{\lambda_{k,\mathcal{R}}}$ is the $\mathbb{C}\mathrm{Sym}_k$-module generated by the $\lambda_{k,\mathcal{R}}$-tabloids.

Theorem 2.4 *The $\mathbb{C}\mathrm{Sym}_k$-modules $\langle \mathcal{R} \rangle$ and $M^{\lambda_{k,\mathcal{R}}}$ are isomorphic.*

Proof: Let t be any $\lambda_{k,\mathcal{R}}$-tableau. Then t' is in the equivalence class $\{t\}$ if and only if

$$t'(i,1) = t(i,1) \qquad \text{for all } i = 2, 3, \ldots, |\mathcal{R}| + 1.$$

Let $\alpha_t \in \langle \mathcal{R} \rangle$ be such that $\alpha_t(i-1) = t(i,1)$ for all $i = 2, 3, \ldots, |\mathcal{R}|+1$. This defines a bijection between the set of $\lambda_{k,\mathcal{R}}$-tabloids and the set of colourings α satisfying

$\alpha \models \mathcal{R}$. This bijection clearly respects the action of $\mathbb{C}\mathrm{Sym}_k$. Hence follows the result. $\square$

Since $\mathcal{V}_k(B)$ is the direct sum of the $\langle\mathcal{R}\rangle$ it follows from Theorem 2.4 that:

Corollary 2.5 *The $\mathbb{C}Sym_k$-modules $\mathcal{V}_k(B)$ and $\bigoplus_{\mathcal{R}\in\Pi(B)} M^{\lambda_{k,\mathcal{R}}}$ are isomorphic.* $\square$

From Theorem 2.2 and Lemma 2.3 we know the decomposition of the $M^{\lambda_{k,\mathcal{R}}}$ in terms of irreducible submodules. This allows us to deduce the structure of $\mathcal{V}_k(B)$.

Denote by $\lambda_{k,b}$ the partition $(k-b, 1^b)$ of k, where $b = |V|$. Then $\lambda_{k,\mathcal{R}} \succeq \lambda_{k,b}$ for all $\mathcal{R} \in \Pi(B)$. From Theorem 2.2 and Corollary 2.5, it follows that every irreducible composition factor of $\mathcal{V}_k(B)$ is isomorphic to some $\mathcal{S}^\lambda$ with $\lambda \succeq \lambda_{k,b}$.

Let $0 \le \ell \le b$ and $\pi \vdash \ell$. Denote by π^k the partition $(k-\ell, \pi_1, \pi_2, \ldots, \pi_\ell)$ of k. Then $\pi^k \succeq \lambda_{k,b}$ and every $\lambda \succeq \lambda_{k,b}$ is of the form π^k for some $0 \le \ell \le b$ and $\pi \vdash \ell$.

As a result of Lemma 2.3, the dimension n_λ of $\mathcal{S}^\lambda$ is given by

$$\dim(\mathcal{S}^\lambda) = n_\lambda = k! \prod_{i=1}^{k} x_i(\lambda)$$

where

$$x_i(\lambda) = \frac{\prod_{j=i+1}^{k} (\lambda_i - \lambda_j + j - i)}{(\lambda_i + k - i)!} \qquad \text{for } i = 1, 2, \ldots, k-1 \text{ and } \qquad x_k(\lambda) = \frac{1}{\lambda_k!}.$$

Assume $b + 2 \le k$ and replace λ by π^k. If $\ell = 0$ then $n_{\pi^k} = 1$. If $\ell \ge 1$, it follows that

$$\begin{aligned}
x_1(\pi^k) &= \frac{\prod_{j=1}^{\ell}(k-\ell-\pi_j+j) \qquad \prod_{j=\ell+2}^{k}(k-\ell+j-1)}{(2k-\ell-1)!} \\
&= \frac{\prod_{j=1}^{\ell}(k-\ell-\pi_j+j)}{k!}, \\
x_i(\pi^k) &= x_{i-1}(\pi) \quad \text{for } i = 2, 3, \ldots, \ell+1 \\
\text{and} \quad x_i(\pi^k) &= 1 \qquad \text{for } i \ge \ell+2.
\end{aligned}$$

The dimension of $\mathcal{S}^{\pi^k}$ can then be written as

$$n_{\pi^k} = \frac{k!}{\ell!}\, n_\pi\, x_1(\lambda) \;=\; \frac{n_\pi}{\ell!} \prod_{i=1}^{\ell} (k - h_i(\pi)) \qquad \text{where} \quad h_i(\pi) = \pi_i + \ell - i.$$

To find the multiplicity m_{π^k} in $\mathcal{V}_k(B)$ of the submodule isomorphic to $\mathcal{S}^{\pi^k}$, we have to add up the numbers of semistandard π^k-tableaux of type $\lambda_{k,\mathcal{R}}$ for all $\mathcal{R} \in \Pi(B)$. If $\mathcal{R}$ is a given colour-partition, then any $\pi^k \succeq \lambda_{k,\mathcal{R}}$ is of the form $\pi^k = (k - \ell, \pi_1, \pi_2, \ldots, \pi_\ell)$ for some $0 \le \ell \le |\mathcal{R}|$ and $\pi \vdash \ell$. Every π^k-tableaux T of type $\lambda_{k,\mathcal{R}}$ has $k - |\mathcal{R}|$ times the entry 1 and each of the entries $2, 3, \ldots, |\mathcal{R}| + 1$ exactly once. A necessary condition for T to be semistandard is that all the entries 1 are in the first row and the first $k - |\mathcal{R}|$ columns. The entries not equal to 1 in the first row have to be in increasing order along the row. If T satisfies this necessary condition then T is semistandard if and only if the restriction of T to $[\pi]$ is a standard π tableau (assuming that $k > |\mathcal{R}|$). Hence the multiplicity of $\mathcal{S}^{\pi^k}$ in $M^{\lambda_{k,\mathcal{R}}}$ is $\binom{|\mathcal{R}|}{\ell} n_\pi$. Now, summing over the set of colour-partitions gives the multiplicity in $\mathcal{V}_k(B)$ of the submodule isomorphic to $\mathcal{S}^{\pi^k}$.

Theorem 2.6 *Every irreducible submodule of $\mathcal{V}_k(B)$ is isomorphic to some $\mathcal{S}^{\pi^k}$ with $0 \le \ell \le b$ and $\pi \vdash \ell$. If $\ell = 0$ the dimension of $\mathcal{S}^{\pi^k}$ is one. If $\ell > 0$ the dimension of $\mathcal{S}^{\pi^k}$ is*

$$n_{\pi^k} \;=\; \frac{n_\pi}{\ell!} \prod_{i=1}^{\ell} (k - h_i(\pi)) \qquad \textit{where} \quad h_i(\pi) = \pi_i + \ell - i$$

and n_π is the dimension of $\mathcal{S}^\pi$. The number of submodules in $\mathcal{V}_k(B)$ isomorphic to $\mathcal{S}^{\pi^k}$ is

$$m_{\pi^k} \;=\; \sum_{\mathcal{R} \in \Pi(B)} \binom{|\mathcal{R}|}{\ell} n_\pi \qquad \textit{with} \quad \binom{|\mathcal{R}|}{\ell} = 0 \;\; \textit{if} \;\; \ell > |\mathcal{R}|.$$

□

2.4 The irreducible submodules of $\mathcal{V}_k(B)$

In this section we investigate the irreducible submodules of the $\mathbb{C}\mathrm{Sym}_k$-module $\mathcal{V}_k(B)$. In particular we obtain a basis of the irreducible submodules.

Recall that for every $\mathcal{R} \in \Pi(B)$ the submodule $\langle \mathcal{R} \rangle$ of $\mathcal{V}_k(B)$ is generated by the set $\{[\alpha] \mid \alpha \models \mathcal{R}\}$. Since $\mathcal{V}_k(B)$ is equal to the direct sum of the $\langle \mathcal{R} \rangle$ we can decompose each of the $\langle \mathcal{R} \rangle$ separately.

Let $\mathcal{R}$ be a colour-partition with b independent sets. That is $\mathcal{R} = \{R_i\}_{i=1}^{b}$. For every $\alpha \models \mathcal{R}$ we denote by $\overline{\alpha} : \{1, 2, \ldots, b\} \to K$ the injection defined by $\overline{\alpha}(i) = \alpha(R_i)$ (see diagram).

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\text{proj.}"'] & K \\ \{1, 2, \ldots, b\} = V/\mathcal{R} \arrow[ur, "\overline{\alpha}"] & \end{tikzcd}$$

The injection $\overline{\alpha}$ is a colouring of the complete graph K_b. This induces a bijection between the colourings in $\Gamma_k(B)$ that induce the colour-partition $\mathcal{R} \in \Pi(B)$ and the colourings in $\Gamma_k(b)$. This bijection respects the action of the symmetric group, and we have:

Lemma 2.7 *Let B be a graph. For each $\mathcal{R} \in \Pi(B)$ the homomorphism*

$$\langle \mathcal{R} \rangle \to \mathcal{V}_k(|\mathcal{R}|) \qquad \textit{given by} \qquad [\,\alpha\,] \mapsto [\,\overline{\alpha}\,]$$

is a $\mathbb{C}Sym_k$-module isomorphism. □

It follows that finding the irreducible submodules of $\langle \mathcal{R} \rangle$ is equivalent to finding the irreducible submodules of $\mathcal{V}_k(|\mathcal{R}|)$.

Note that the above isomorphism depends on the labelling of the independent sets of $\mathcal{R}$. In order to avoid confusion later, let us define the following: The independent sets $\{R_i\}_{i=1}^{b}$ are labelled such that

$$\min(R_i) < \min(R_j) \qquad \text{if} \qquad i < j.$$

That is, we order the independent sets according to the smallest element contained and label them in this order consecutively.

2.4.1 The complete graph case

We are now going to find the irreducible submodules of $\mathcal{V}_k(b)$. Let $0 \leq \ell \leq b$ and $\pi \vdash \ell$ be given. **For the rest of this section let t be a fixed π^k-tableau.**

Let $T : [\pi^k] \to \{0\} \cup V_b$ be a π^k-tableau of type $\lambda_{k,b}$. That is T is a surjection with kernel of size $k-b$. Denote by $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ the set of π^k-tableaux of type $\lambda_{k,b}$. We define the action of Sym_k on $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ by

$$(\omega, T)(i,j) = T(i',j') \quad \text{where} \quad \omega\, t(i',j') = t(i,j) \qquad \text{for all} \quad (i,j) \in [\pi^k]$$

for every $\omega \in \mathrm{Sym}_k$. This agrees with the definition given in [20] Page 80, and makes $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ into a $\mathbb{C}\mathrm{Sym}_k$-module.

We are going to show that $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ and $\mathcal{V}_k(b)$ are isomorphic as $\mathbb{C}\mathrm{Sym}_k$-modules. Then we use results from [20] Section 2.9 to deduce the decomposition of $\mathcal{V}_k(b)$ in terms of irreducible submodules.

For every $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$ define $\alpha_T : V_b \to K$ as $\alpha_T(v) = t(i,j)$ where $T(i,j) = v$ for all $v \in V_b$. This is an injection and hence a colouring of K_b.

Example 2.9: Let $b = 7$, $\ell = 5$ and $\pi = (2^2, 1)$. If

$$t = \begin{array}{ccccccc} 6 & 7 & 8 & 9 & 10 & \dots & k \\ 1 & 2 \\ 3 & 4 \\ 5 \end{array} \qquad \text{and} \quad T = \begin{array}{cccccccc} 0 & 3 & 6 & 0 & 0 & \dots & 0 & 4 \\ 5 & 2 \\ 1 & 0 \\ 7 \end{array}$$

then $\alpha_T = (3, 2, 7, k, 1, 8, 5)$, that is α assigns the colour 3 to vertex 1, colour 2 to vertex 2, and so on.

Lemma 2.8 *Let $0 \leq \ell \leq b$ and $\pi \vdash \ell$. Then $T \mapsto \alpha_T$ defines a bijection between the sets $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ and $\Gamma_k(b)$.*

Proof: Let α be any colouring in $\Gamma_k(b)$. Define the π^k-tableau T by $T(i,j) = v$ if $t(i,j) = \alpha(v)$ for some $v \in V_b$ and $T(i,j) = 0$ otherwise. Since t and α are injections it follows that T is well defined. Clearly, T is of type $\lambda_{k,b}$. It follows that $\alpha_T = \alpha$, and hence α_T defines a surjection between the sets $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ and $\Gamma_k(b)$.

Suppose that $\alpha, \beta \in \Gamma_k(b)$ with $\alpha \neq \beta$. Then there exists $v \in V_b$ such that $\alpha(v) \neq \beta(v)$ From the definition of α_T it follows that $\alpha(v) = t(i,j)$ with $T(i,j) = v$ and $\beta(v) = t(i',j')$ with $T'(i',j') = v$. Since t is an injection it follows that $(i,j) \neq (i',j')$, and since $v \neq 0$ it follows that $T \neq T'$. It follows that α_T defines an injection between the sets $\Gamma_k(b)$ and $\mathcal{T}_{\pi^k,\lambda_{k,b}}$, and hence a bijection. □

Lemma 2.9 *Let* $0 \leq \ell \leq b$ *and* $\pi \vdash \ell$. *For any* $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$

$$\omega\,\alpha_T = \alpha_{\omega\,T} \qquad \textit{for all } \omega \in Sym_k.$$

Proof: For every $v \in V_b$ we have

$$\begin{aligned}
\alpha_{\omega\,T}(v) &= t(i,j) && \text{where} \quad (\omega,T)(i,j) = v \\
&= \omega\, t(i',j') && \text{where} \quad T(i',j') = (\omega,T)(i,j) = v \\
&= \omega\, \alpha_T(v).
\end{aligned}$$

□

Corollary 2.10 *Let* $0 \leq \ell \leq b$ *and* $\pi \vdash \ell$. *Then* $T \mapsto [\alpha_T]$ *defines an isomorphism between the* $\mathbb{C}Sym_k$*-modules* $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ *and* $\mathcal{V}_k(b)$. □

Following [20] Section 2.9, we define for every given $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$ the homomorphism

$$\theta_T : M^{\pi^k} \to \mathcal{V}_k(b) \qquad \text{by} \qquad \theta_T(\{t\}) = \sum_{S \in \{T\}} [\alpha_S]$$

and cyclic extension using cyclicity of M^{π^k}. That is, for every π^k-tableau t' there exists a $\omega \in \mathrm{Sym}_k$ such that $t' = \omega t$. Then

$$\theta_T(\{t'\}) = \theta_T(\{\omega t\}) = \omega \theta_T(\{t'\}) = \omega \sum_{S \in \{T\}} [\alpha_S].$$

The π^k-tabloid $\{T\}$ is defined in the obvious way. Note that in [20] Section 2.9 the homomorphism θ_T is into $\mathcal{T}_{\pi^k,\lambda_{k,b}}$, but with Corollary 2.10 we can extend it into $\mathcal{V}_k(b)$. From the cyclic extension it follows that θ_T respects the action of Sym_k. In particular

$$\theta_T(e_t) = \theta_T(\kappa_t\{t\}) = \kappa_t\theta_T\{t\} = \kappa_t \sum_{S\in\{T\}} [\alpha_S].$$

Denote by $\overline{\theta}_T : \mathcal{S}^{\pi^k} \to \mathcal{V}_k(b)$ the restriction of θ_T to $\mathcal{S}^{\pi^k}$.

We say that a tableau $T \in \mathcal{T}_{\lambda,\lambda_{k,b}}$ is *almost semistandard* if none of its columns has a repeated entry. In particular every semistandard tableau is also almost semistandard. From [20] Proposition 2.9.4 it follows that $\overline{\theta}_T$ is non-zero, i.e. is not the zero map, if and only if T is almost semistandard. Thus, the image $\mathrm{Im}(\overline{\theta}_T)$ is an irreducible submodule of $\mathcal{V}_k(b)$ isomorphic to $\mathcal{S}^{\pi^k}$. Denote this irreducible submodule by $\mathcal{U}_k(\pi, T, b)$.

Let $\mathcal{T}^0_{\pi^k,\lambda_{k,b}}$ be the set of semistandard π^k-tableaux of type $\lambda_{k,b}$. In [20] Theorem 2.10.1 it has been shown that

$$\left\{\overline{\theta}_T \mid T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}\right\}$$

is a basis of $\mathrm{Hom}(\mathcal{S}^{\pi^k}, \mathcal{V}_k(b))$. It follows that the $\mathcal{U}_k(\pi, T, b)$ are non-identical for different $T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}$.

Lemma 2.11 *Let* $0 \le \ell \le b$, $\pi \vdash \ell$ *and* $T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}$. *Then* $\mathcal{U}_k(\pi, T, b)$ *is an irreducible submodule of* $\mathcal{V}_k(b)$ *isomorphic to* $\mathcal{S}^{\pi^k}$ *with basis*

$$\left\{\omega\kappa_t \sum_{S\in\{T\}} [\alpha_S] \mid \omega \in \mathit{Sym}_k \text{ such that } \omega t \text{ is a standard } \pi^k\text{-tableau} \right\}.$$

Moreover, the $\mathcal{U}_k(\pi, T, b)$ *are non-identical for different* $T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}$ *and*

$$\left\{\mathcal{U}_k(\pi, T, b) \mid T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}\right\}$$

is the complete set of submodules of $\mathcal{V}_k(b)$ *isomorphic to* $\mathcal{S}^{\pi^k}$.

Proof: Recall that the set $\{e_t \mid \omega t \text{ is standard }\}$ is the standard basis for $\mathcal{S}^{\pi^k}$. From $e_{\omega t} = \omega e_t$ ([20] Lemma 2.3.3) using $\overline{\theta}_T$ it follows that the

$$\omega\kappa_t \sum_{S\in\{T\}} [\alpha_S]$$

with ωt being standard form a basis of $\mathcal{U}_k(\pi, T, b)$.

Further, since $\{\overline{\theta}_T \mid T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}\}$ is a basis of $\mathrm{Hom}(\mathcal{S}^{\pi^k}, \mathcal{V}_k(b))$ it follows that:

- $\mathcal{U}_k(\pi, T, b) \neq \mathcal{U}_k(\pi, T', b)$ for T, $T' \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}$ with $T \neq T'$.
- $\{\mathcal{U}_k(\pi, T, b) \mid T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}\}$ is the complete set of submodules of $\mathcal{V}_k(b)$ isomorphic to $\mathcal{S}^{\pi^k}$.

□

For every $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$ let $E_{T,t} = \kappa_t \sum_{S\in\{T\}} [\alpha_S]$. Then, by [20] Lemma 2.3.3:

$$\omega E_{T,t} = \omega \kappa_t \sum_{S\in\{T\}} [\alpha_S] = \kappa_{\omega t} \sum_{S\in\{T\}} [\omega\alpha_S] = E_{T,\omega t}$$

for all $\omega \in \mathrm{Sym}_k$, as required, and

$$\mathcal{U}_k(\pi, T, b) = \left\{\omega E_{T,t} \mid \omega t \text{ is a standard } \pi^k\text{-tableau for some } \omega \in \mathrm{Sym}_k\right\}.$$

Lemma 2.12 *Let $b \in \mathbb{N}$. Then*

$$\mathcal{V}_k(b) = \bigoplus_{\substack{0\leq \ell\leq b\\ \pi\vdash\ell}} \bigoplus_{T\in\mathcal{T}^0_{\pi^k,\lambda_{k,b}}} \mathcal{U}_k(\pi, T, b).$$

□

2.4.2 A change of basis

Let $X \subseteq V_b$ and let $g : X \to K$ be an injection. We define the function $[X \mid g] \in \mathcal{V}_k(b)$ by

$$[X \mid g](\alpha) = \begin{cases} 1 & \text{if } \alpha_X = g \\ 0 & \text{otherwise} \end{cases}$$

for every $\alpha \in \Gamma_k(b)$ where α_X is the restriction of α to X. Equivalently

$$[X \mid g] = \sum_{\delta_X = g} [\delta].$$

Let us proceed by expressing the irreducible submodules $\mathcal{U}_k(\pi, T, b)$ of $\mathcal{V}_k(b)$ in terms of linear combinations of $[X \mid g]$. Let $0 \leq \ell \leq b$ and $\pi \vdash \ell$ be given. **For the rest of this section let t be a fixed π^k-tableau.** For every tableau $T \in \mathcal{T}_{\pi^k, \lambda_{k,b}}$ denote by $T_\pi : [\pi] \to V_b \cup \{0\}$ the restriction of T to $[\pi]$. That is, $T_\pi(i,j) = T(i+1,j)$ for all $(i,j) \in [\pi]$. Denote by X_T the image of T_π. If T is semistandard of type $\lambda_{k,b}$ then $X_T \subseteq V_b$ and T_π is a standard tableau. Denote by $g_T : X_T \to K$ the restriction of α_T to X_T. That is, $g_T(x) = t(i,j)$ where $T(i,j) = x$ for all $x \in X_T$. Similarly, define t_π to be the restriction of t to $[\pi]$. Observe that the image of g_T as a set is independent of T and depends only on the choice of t.

Lemma 2.13 *Let $T \in \mathcal{T}^0_{\pi^k, \lambda_{k,b}}$. Then*

$$\sum_{S \in \{T\}} [\alpha_S] = \sum_{\omega \in R_{t_\pi}} [X_T \mid \omega g_T],$$

where R_{t_π} is the row-stabilizer corresponding to the tableau t_π.

Proof: First observe that $X_S = X_T$ for all $S \in \{T\}$. Partition $\{T\}$ into $r = \pi_1!\pi_2!\dots\pi_\ell!$ parts $\mathcal{B}_1$, $\mathcal{B}_2 \dots, \mathcal{B}_r$ each of size $(k-\ell)(k-\ell-1)\dots(k-b+1)$ by letting

$$S,\ S' \in \mathcal{B} \quad \text{if and only if} \quad S_\pi = S'_\pi.$$

Let $\mathcal{B}$ be any of these parts. Then $\alpha_S = \alpha_{S'}$ on X_T for all S', $S \in \mathcal{B}$. Denote by $g_{\mathcal{B}} : X_T \to K$ the restriction of α_S to X_T.

For every α with $\alpha = \alpha_S$ on X_T for some $S \in \mathcal{B}$ there exists a $S' \in \mathcal{B}$ such that $\alpha = \alpha_{S'}$ on V_b. Thus

$$\sum_{S \in \mathcal{B}} [\alpha_S] = [X_T \mid g_{\mathcal{B}}].$$

Let S', $S \in \{T\}$. Let x be any element of X_T. Then $\alpha_S(x) = t(i,j)$ and $\alpha_{S'}(x) = t(i',j')$ where $S(i,j) = x$ and $S'(i',j') = x$. Since the rows of S and S' are equal as sets it follows that $i = i'$. Hence $\alpha_S(x)$ and $\alpha_{S'}(x)$ are in the same row i of t_π and thus $\omega\alpha_S(x) = \alpha_{S'}(x)$ for some $\omega \in R_{t_\pi}$. This holds for all $x \in X_T$ and the result follows. □

For every $T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}$ it follows that $E_{T,t} = \kappa_t \sum_{\omega \in R_{t_\pi}} [X_T \mid \omega g_T]$ and by [20] Lemma 2.3.3 it follows that, as required,

$$\begin{aligned}\gamma E_{T,t} =& \kappa_{\gamma t} \sum_{\omega \in R_{t_\pi}} [X_T \mid \gamma\omega g_T] = \kappa_{\gamma t} \sum_{\omega \in \gamma R_{t_\pi}} [X_T \mid \omega g_T] \\ =& \kappa_{\gamma t} \sum_{\omega \in R_{(\gamma t)_\pi}} [X_T \mid \omega\gamma g_T] = E_{T,\gamma t}.\end{aligned}$$

2.4.3 The general case

Let us now return to the general case, i.e. B is not necessarily a complete graph with b vertices.

Denote by $\mathcal{U}_k(\pi, T, \mathcal{R})$ the irreducible submodule of $\langle \mathcal{R} \rangle$ isomorphic to $\mathcal{U}_k(\pi, T, |\mathcal{R}|)$ obtained via the isomorphism in Lemma 2.7.

Since $\mathcal{V}_k(B)$ is the direct sum the $\langle \mathcal{R} \rangle$ with $\mathcal{R} \in \Pi(B)$ it follows that:

Theorem 2.14 *Let B be any graph. Then*

$$\mathcal{V}_k(B) = \bigoplus_{\substack{0 \leq \ell \leq b \\ \pi \vdash \ell}} \mathcal{W}_k(\pi, B)$$

where

$$\mathcal{W}_k(\pi, B) = \bigoplus_{\substack{\mathcal{R} \in \Pi(B) \\ |\mathcal{R}| \geq \ell}} \bigoplus_{T \in \mathcal{T}^0_{\pi^k,\lambda_{k,|\mathcal{R}|}}} \mathcal{U}_k(\pi, T, \mathcal{R}).$$

Each submodule $\mathcal{U}_k(\pi, T, \mathcal{R})$ is isomorphic to $\mathcal{S}^{\pi^k}$, and $\mathcal{W}_k(\pi, B)$ is the direct sum of all irreducible submodules of $\mathcal{V}_k(B)$ isomorphic to $\mathcal{S}^{\pi^k}$. □

We say that $\mathcal{W}_k(\pi, B)$ is the submodule of $\mathcal{V}_k(B)$ at *level ℓ* and partition $\pi \vdash \ell$. If B is the complete graph K_b then we write $\mathcal{W}_k(\pi, b)$.

2.5 Examples

Example 2.10: Let B be the complete graph with 3 vertices K_3. Then $\mathcal{V}_k(3)$ splits up into three levels $\ell = 0, 1, 2, 3$.

At level $\ell = 0$ there is only the empty-partition $\pi = ()$. We assume that

$$t = \begin{matrix} 1 & 2 & \dots & k \end{matrix}\,.$$

There is only one semistandard tableau $T = \begin{matrix} 0 & 0 & \dots & 0 & 1 & 2 & 3 \end{matrix}$ in $\mathcal{T}^0_{(k),\lambda_{k,3}}$. The tabloid $\{T\}$ contains all (k)-tableaux of type $\lambda_{k,3}$. Then $\kappa_t = \epsilon$ and $\mathcal{U}_k((),T,3)$ consists of one element

$$E_{T,t} = \sum_{S\in\{T\}} [\alpha_S] = \sum_{\alpha\in\Gamma_k(3)} [\alpha],$$

that is the all-one function. The submodule $\mathcal{W}_k((),3)$ is equal to $\mathcal{U}_k((),T,3)$.

At level $\ell = 1$, again there is only one partition $\pi = (1)$. We assume that

$$t = \begin{matrix} 2 & 3 & \dots & k \\ 1 & & & \end{matrix}\,.$$

There are three semistandard tableaux in $\mathcal{T}^0_{(k-1,1),\lambda_{k,3}}$:

$$T_1 = \begin{matrix} 0 & 0 & \dots & 0 & 2 & 3 \\ 1 & & & & & \end{matrix}\,, \qquad T_2 = \begin{matrix} 0 & 0 & \dots & 0 & 1 & 3 \\ 2 & & & & & \end{matrix}$$

and

$$T_3 = \begin{matrix} 0 & 0 & \dots & 0 & 1 & 2 \\ 3 & & & & & \end{matrix}\,.$$

Then $\kappa_t = \epsilon - (12)$, and

$$E_{T_1,t} = \sum_{\alpha(1)=1} [\alpha] - \sum_{\alpha(1)=2} [\alpha], \qquad E_{T_2,t} = \sum_{\alpha(2)=1} [\alpha] - \sum_{\alpha(2)=2} [\alpha]$$

and

$$E_{T_3,t} = \sum_{\alpha(3)=1} [\alpha] - \sum_{\alpha(3)=2} [\alpha]$$

where $\alpha(i) = j$ means that α assigns the colour j to vertex i. The submodule $\mathcal{U}_k((1),T_1,3)$ is generated by the polytabloids $\omega E_{T_1,t} = E_{T_1,\omega t}$ with ωt being

$$\begin{matrix} 1 & 3 & 4 & \dots & k \\ 2 & & & & \end{matrix}\,, \quad \begin{matrix} 1 & 2 & 4 & \dots & k \\ 3 & & & & \end{matrix}\,, \quad \dots, \quad \begin{matrix} 1 & 2 & 3 & \dots & k-1 \\ k & & & & \end{matrix}\,.$$

Similarly $\mathcal{U}_k((1), T_2, 3)$ and $\mathcal{U}_k((1), T_3, 3)$. From Theorem 2.6 it follows that each of them is of dimension $n_{(k-1,1)} = k-1$ respectively. The submodule $\mathcal{W}_k((1), 3)$ is the direct sum of these three irreducible submodules.

At level $\ell = 2$ there are two partitions, $\pi = (2)$ and $\pi = (1^2)$. For $\pi = (2)$ we assume that $t = \begin{array}{cccccc} 3 & 4 & 5 & \dots & k \\ 1 & 2 \end{array}$. There are $\binom{3}{2}1 = 3$ semistandard tableaux in $\mathcal{T}^0_{(k-2,2),\lambda_{k,3}}$:

$$T_1 = \begin{array}{cccccc} 0 & 0 & 0 & \dots & 0 & 3 \\ 1 & 2 \end{array}, \qquad T_2 = \begin{array}{cccccc} 0 & 0 & 0 & \dots & 0 & 2 \\ 1 & 3 \end{array}$$

and

$$T_3 = \begin{array}{cccccc} 0 & 0 & 0 & \dots & 0 & 1 \\ 2 & 3 \end{array}.$$

Then $\kappa_t = (\epsilon - (13))(\epsilon - (24)) = \epsilon - (13) - (24) + (13)(24)$, and

$$\begin{aligned} E_{T_1,t} &= \Big(\sum_{\substack{\alpha(1)=1\\ \alpha(2)=2}} [\alpha] + \sum_{\substack{\alpha(2)=1\\ \alpha(1)=2}} [\alpha] \Big) - \Big(\sum_{\substack{\alpha(1)=3\\ \alpha(2)=2}} [\alpha] + \sum_{\substack{\alpha(2)=3\\ \alpha(1)=2}} [\alpha] \Big) \\ &\quad - \Big(\sum_{\substack{\alpha(1)=1\\ \alpha(2)=4}} [\alpha] + \sum_{\substack{\alpha(2)=1\\ \alpha(1)=4}} [\alpha] \Big) + \Big(\sum_{\substack{\alpha(1)=3\\ \alpha(2)=4}} [\alpha] + \sum_{\substack{\alpha(2)=3\\ \alpha(1)=4}} [\alpha] \Big), \\ E_{T_2,t} &= \Big(\sum_{\substack{\alpha(1)=1\\ \alpha(3)=2}} [\alpha] + \sum_{\substack{\alpha(3)=1\\ \alpha(1)=2}} [\alpha] \Big) - \Big(\sum_{\substack{\alpha(1)=3\\ \alpha(3)=2}} [\alpha] + \sum_{\substack{\alpha(3)=3\\ \alpha(1)=2}} [\alpha] \Big) \\ &\quad - \Big(\sum_{\substack{\alpha(1)=1\\ \alpha(3)=4}} [\alpha] + \sum_{\substack{\alpha(3)=1\\ \alpha(1)=4}} [\alpha] \Big) + \Big(\sum_{\substack{\alpha(1)=3\\ \alpha(3)=4}} [\alpha] + \sum_{\substack{\alpha(3)=3\\ \alpha(1)=4}} [\alpha] \Big) \end{aligned}$$

and similarly for $E_{T_3,t}$. The irreducible submodule $\mathcal{U}_k((2), T_1, 3)$ is generated by the polytabloids $\omega E_{T_1,t}$ with ωt being a standard tableau. Similarly $\mathcal{U}_k((2), T_2, 3)$ and $\mathcal{U}_k((2), T_3, 3)$. From Theorem 2.6 it follows that each of them is of dimension $n_{(k-2,2)} = \frac{1}{2}(k-3)k$ respectively. The submodule $\mathcal{W}_k((2), 3)$ is the direct sum of these three irreducible submodules.

For $\pi = (1^2)$ we assume that $t = \begin{array}{llll} 3 & 4 & \ldots & k \\ 1 & & & \\ 2 & & & \end{array}$. There are $\binom{3}{2}1 = 3$ semistandard tableaux in $\mathcal{T}^0_{(k-2,1^2),\lambda_{k,3}}$:

$$T_1 = \begin{array}{llllll} 0 & 0 & 0 & \ldots & 0 & 3 \\ 1 & & & & & \\ 2 & & & & & \end{array}, \qquad T_2 = \begin{array}{llllll} 0 & 0 & 0 & \ldots & 0 & 2 \\ 1 & & & & & \\ 3 & & & & & \end{array}$$

and

$$T_3 = \begin{array}{llllll} 0 & 0 & 0 & \ldots & 0 & 1 \\ 2 & & & & & \\ 3 & & & & & \end{array}.$$

Then $\kappa_t = \epsilon - (12) - (13) - (23) + (123) + (132)$ and for example

$$E_{T_1,t} = \sum_{\substack{\alpha(1)=1\\ \alpha(2)=2}} [\alpha] - \sum_{\substack{\alpha(1)=2\\ \alpha(2)=1}} [\alpha] - \sum_{\substack{\alpha(1)=3\\ \alpha(2)=2}} [\alpha] - \sum_{\substack{\alpha(1)=1\\ \alpha(2)=3}} [\alpha] + \sum_{\substack{\alpha(1)=2\\ \alpha(2)=3}} [\alpha] + \sum_{\substack{\alpha(1)=3\\ \alpha(2)=1}} [\alpha].$$

Similarly $E_{T_2,t}$ and $E_{T_3,t}$. Each of the irreducible submodules $\mathcal{U}_k((1^2), T_1, 3)$, $\mathcal{U}_k((1^2), T_2, 3)$ and $\mathcal{U}_k((1^2), T_3, 3)$ is of dimension $n_{(k-2,1^2)} = \frac{1}{2}(k-2)(k-1)$. The submodule $\mathcal{W}_k((1^2), 3)$ is the direct sum of these three irreducible submodules.

At level $\ell = 3$ there are three partitions $\pi = (3)$, $\pi = (2,1)$ and $\pi = (1^3)$. For $\pi = (3)$ we assume that $t = \begin{array}{llllll} 4 & 5 & 6 & 7 & \ldots & k \\ 1 & 2 & 3 & & & \end{array}$. There is only one semistandrd tableau $T = \begin{array}{llllll} 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 2 & 3 & & & \end{array}$ in $\mathcal{T}^0_{(k-3,3),\lambda_{k,3}}$. The tabloid $\{T\}$ contains all tableaux with 1, 2 and 3 in the second row in any order, and $\sum\limits_{S\in\{T\}} [\alpha_S]$ is the sum over all clourings using the colours 1, 2 and 3. It follows that

$$E_{T,t} = \kappa_t \sum_{\alpha(V)=\{1,2,3\}} [\alpha]$$

with $\kappa_t = \epsilon - (14) - (25) - (36) + (14)(25) + (14)(36) + (25)(36) - (14)(25)(36)$. The irreducible submodule $\mathcal{U}_k((3), T, 3)$ is of dimension $n_{(k-3,3)} = \frac{1}{6}(k-5)(k-1)k$, and the submodule $\mathcal{W}_k((3), 3)$ is equal to $\mathcal{U}_k((3), T, 3)$.

For $\pi = (2,1)$ we assume that $t = \begin{array}{llll} 4 & 5 & 6 & \dots & k \\ 1 & 2 \\ 3 \end{array}$. There are two semistandrd tableaux in $\mathcal{T}^0_{(k-3,2,1),\lambda_{k,3}}$:

$$T_1 = \begin{array}{lllll} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 2 \\ 3 \end{array} \quad \text{and} \quad T_2 = \begin{array}{lllll} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 3 \\ 2 \end{array}.$$

Writing colourings of K_3 as three-tuples, that is (h,i,j) is the colouring that assigns colour h to vertex 1, colour i to vertex 2 and colour j to vertex 3 it follows that:

$$\sum_{S\in\{T_1\}} [\alpha_S] = (1,2,3) + (2,1,3) \quad \text{and} \quad \sum_{S\in\{T_2\}} [\alpha_S] = (1,3,2) + (2,3,1)$$

and $E_{T_1,t} = \kappa_t\big((1,2,3) + (2,1,3)\big)$ and $E_{T_2,t} = \kappa_t\big((1,3,2) + (2,3,1)\big)$ where

$$\kappa_t = \big(\epsilon - (13) - (14) - (34) + (134) + (143)\big)\big(\epsilon - (25)\big).$$

The irreducible submodules $\mathcal{U}_k((2,1),T_1,3)$ and $\mathcal{U}_k((2,1),T_2,3)$ both are of dimension $n_{(k-3,2,1)} = \frac{2}{6}(k-4)(k-2)k$. The submodule $\mathcal{W}_k((2,1),3)$ is the direct sum of these two irreducible submodules.

For $\pi = (1^3)$ we assume that $t = \begin{array}{llll} 4 & 5 & \dots & k \\ 1 \\ 2 \\ 3 \end{array}$. There is one semistandard tableau

$T = \begin{array}{lllll} 0 & 0 & 0 & \dots & 0 \\ 1 \\ 2 \\ 3 \end{array}$ in $\mathcal{T}^0_{(k-3,1^3),\lambda_{k,3}}$. Then $\{T\} = T$ and $E_{T,t} = \kappa_t(1,2,3)$ where κ_t is the alternating sum of the elements of the group of permutations of the set $\{1,2,3,4\}$. The dimension of $\mathcal{U}_k((1^3),T,3)$ is $n_{(k-3,1^3)} = \frac{1}{6}(k-3)(k-2)(k-1)$. The submodule $\mathcal{W}_k((1^3),3)$ is equal to $\mathcal{U}_k((1^3),T,3)$.

The Table 2.1 summarizes this example. The module $\mathcal{V}_k(3)$ is the direct sum of 14 irreducible submodules $\mathcal{U}_k(\pi,T,3)$. Adding their dimensions up gives $k(k-1)(k-2) = \dim(\mathcal{V}_k(3))$.

ℓ	π	$\lvert\mathcal{T}^0_{\pi^k,\lambda_{k,3}}\rvert$	n_{π^k}
0	()	1	1
1	(1)	3	$k-1$
2	(2)	3	$\frac{1}{2}k(k-3)$
2	(1^2)	3	$\frac{1}{2}(k-1)(k-2)$
3	(3)	1	$\frac{1}{6}k(k-1)(k-5)$
3	(2,1)	2	$\frac{2}{6}k(k-2)(k-4)$
3	(1^3)	1	$\frac{1}{6}(k-1)(k-2)(k-3)$

Table 2.1: Summary of Example 2.10

Example 2.11: Let B be the path of length three, i.e. with three vertices and two edges. There are two colour-partitions $\mathcal{R} = \{1|2|3\}$ and $\mathcal{P} = \{1,3|2\}$. The submodule $\langle\mathcal{R}\rangle$ is equal to $\mathcal{V}_k(3)$ and $\langle\mathcal{P}\rangle$ is isomorphic to $\mathcal{V}_k(2)$. The decomposition of $\langle\mathcal{R}\rangle$ in terms of irreducible submodules has been obtained in Example 2.10. A decomposition of $\langle\mathcal{P}\rangle$ can be obtained similarly and thus:

At level $\ell = 0$ there is only the empty partition $\pi = ()$. For $\mathcal{P}$ there is only one semistandard tableau $T \in \mathcal{T}^0_{(k),\lambda_{k,2}}$ and $\mathcal{U}_k((),T,\mathcal{P})$ contains only one element $u_{\mathcal{P}} = \sum_{\alpha\models\mathcal{P}}[\alpha]$, that is the all-one function in $\langle\mathcal{P}\rangle$ (Similarly $u_{\mathcal{R}} = \sum_{\alpha\models\mathcal{R}}[\alpha]$). Then the submodule $\mathcal{W}_k((),B)$ is the direct sum of $\mathcal{U}_k((),T,\mathcal{R})$ and $\mathcal{U}_k((),T',\mathcal{P})$ where the second has been obtained in the previous example. They are spanned by the two functions $u_{\mathcal{R}}$ and $u_{\mathcal{P}}$ respectively.

At level $\ell = 1$ again there is only one partition $\pi = (1)$. For $\mathcal{P}$ there are two semistandard $(k-1,1)$-tableaux of type $\lambda_{k,2}$:

$$T_1 = \begin{matrix} 0 & 0 & \ldots & 0 & 2 \\ 1 & & & & \end{matrix} \qquad \text{and} \qquad T_2 = \begin{matrix} 0 & 0 & \ldots & 0 & 1 \\ 2 & & & & \end{matrix}\,.$$

Then $\mathcal{W}_k((1),B)$ is the direct sum of five irreducible submodules:

$$\mathcal{U}_k((1),T_1,\mathcal{P}),\ \mathcal{U}_k((1),T_2,\mathcal{P}),\ \mathcal{U}_k((1),T_1',\mathcal{R}),\ \mathcal{U}_k((1),T_2',\mathcal{R}) \text{ and } \mathcal{U}_k((1),T_3',\mathcal{R}),$$

where the last three have been obtained in the previous example. Each of them is of dimension $n_{(k-1,1)} = (k-1)$.

At level $\ell = 2$ there are two partitions $\pi = (2)$ and $\pi = (1^2)$. For $\mathcal{P}$ there is only one semistandard $(k-2, 2)$-tableau and one semistandard $(k-2, 1^2)$-tableau both of type $\lambda_{k,2}$:

$$\begin{matrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 2 & & & \end{matrix} \qquad \text{and} \qquad \begin{matrix} 0 & 0 & \dots & 0 \\ 1 & & & \\ 2 & & & \end{matrix} .$$

Then $\mathcal{W}_k((2), B)$ is the direct sum of four irreducible submodules:

$$\mathcal{U}_k((2), T, \mathcal{P}), \quad \mathcal{U}_k((2), T_1', \mathcal{R}), \quad \mathcal{U}_k((2), T_2', \mathcal{R}) \quad \text{and} \quad \mathcal{U}_k((2), T_3', \mathcal{R}),$$

where the last three have been obtained in the previous example. Each of them is of dimension $n_{(k-2,2)} = \frac{1}{2}k(k-3)$. Similarly, $\mathcal{W}_k((1^2), B)$ is the direct sum of four irreducible submodules:

$$\mathcal{U}_k((1^2), T, \mathcal{P}), \quad \mathcal{U}_k((1^2), T_1', \mathcal{R}), \quad \mathcal{U}_k((1^2), T_2', \mathcal{R}) \quad \text{and} \quad \mathcal{U}_k((1^2), T_3', \mathcal{R}),$$

where the last three have been obtained in the previous example. Each of them is of dimension $n_{(k-2,1^2)} = \frac{1}{2}(k-1)(k-2)$.

At level $\ell = 3$ there are three partitions $\pi = (3)$, $\pi = (2, 1)$ and $\pi = (1^3)$. For $\mathcal{P}$ all submodules $\mathcal{U}_k(\pi, T, \mathcal{P})$ are zero-modules. Thus the $\mathcal{W}_k(\pi, B)$ are the same as in Example 2.10.

Adding up the dimensions of all the $\mathcal{W}_k(\pi, B)$ gives $k(k-1)(k-2) + k(k-1) = \dim(\mathcal{V}_k(B))$

2.6 A new module

In the previous sections we considered the $\mathbb{C}\mathrm{Sym}_k$-modules $\mathcal{U}_k(\pi, T, |\mathcal{R}|)$ which are generated by the set

$$\{E_{T,t} \mid t \text{ is a standard } \pi^k\text{-tableau}\}$$

and T is a fixed almost semistandard π^k-tableau of type $\lambda_{k,|\mathcal{R}|}$. In this section we shall consider the modules generated by the set of $E_{T,t}$ where we keep t fixed and vary T (with some restrictions).

Let $X = \{x_1, x_2, \ldots, x_\ell\}$ be such that $x_1 < x_2 < \ldots < x_\ell$. We let Sym_ℓ act on X by

$$(\gamma, x_i) = x_{\gamma i} \quad \text{for all } x_i \in X \quad \text{and every} \quad \gamma \in \mathrm{Sym}_\ell.$$

We write γx_i instead of (γ, x_i). This induces an action of Sym_ℓ on the set

$$\left\{T \in \mathcal{T}_{\pi^k,\lambda_{k,b}} \mid T[\pi] = X\right\}.$$

That is, for every $\gamma \in \mathrm{Sym}_\ell$

$$(\gamma, T)(p,q) = \gamma x_i \qquad \text{where} \quad x_i = T(p,q) \quad \text{for all} \quad (p,q) \in [\pi].$$

Thus Sym_ℓ acts on the indices of T_π. We write γT instead of (γ, T). Let $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$ with $T[\pi] = X$. Then T induces a π tableaux t by replacing the entry x_i in T_π by i. For example

$$t = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{array} \qquad \text{corresponds to} \qquad T = \begin{array}{cccccc} 0 & x_7 & \cdots & 0 & x_8 & 0 \\ x_1 & x_3 & x_5 & & & \\ x_2 & x_6 & & & & \\ x_4 & & & & & \end{array}$$

For fixed X, this incuces a bijection between the set of tabloids $\{T\}$ with $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$ and $T[\pi] = X$, and the set of π tabloids $\{t\}$. This Bijection commutes with the action of Sym_ℓ. It follows that the $\mathbb{C}\mathrm{Sym}_\ell$-module generated by the set

$$\left\{ \sum_{S \in \{T\}} S \quad \mid \quad T \in \mathcal{T}_{\pi^k,\lambda_{k,b}} \quad \text{with} \quad T[\pi] = X \right\}$$

together with the action

$$\left(\gamma, \sum_{S \in \{T\}} S\right) = \sum_{S \in \{\gamma T\}} S$$

is isomorphic to M^π.

Now, let t be any π^k tableau. Recall the action of Sym_k on $\mathcal{T}_{\pi^k,\lambda_{k,b}}$ defined on Page 29, we get that the column stabilizer C_t permutes the positions rather than the entries of the elements of $\mathcal{T}_{\pi^k,\lambda_{k,b}}$. It follows that C_t is the column stabilizer for every elements in $\mathcal{T}_{\pi^k,\lambda_{k,b}}$. Thus

$$\left\{ \kappa_{t_\pi} \sum_{S \in \{T\}} S \quad \mid \quad T \in \mathcal{T}_{\pi^k,\lambda_{k,b}} \quad \text{with} \quad T[\pi] = X \right\}$$

generates a $\mathbb{C}\mathrm{Sym}_\ell$-module isomorphic to S^π. With the $\mathbb{C}\mathrm{Sym}_k$ isomorphism from Corollary 2.10 it follows that

$$\left\{\kappa_{t_\pi} \sum_{S\in\{T\}} [\alpha_S] \quad | \quad T \in \mathcal{T}_{\pi^k,\lambda_{k,b}} \quad \text{with} \quad T[\pi] = X\right\}$$

generates a $\mathbb{C}\mathrm{Sym}_\ell$-module isomorphic to S^π.

Lemma 2.15 *There exists some $Q_t \in \mathbb{C}Sym_k$ such that $\kappa_t = Q_t\kappa_{t_\pi}$.* □

Proof: The column stabilizer C_{t_π} is a subgroup of C_t Let D_t be a (left) transversal of C_{t_π} in C_t (i.e a complete set of (left) coset representatives). Then

$$Q_t = \sum_{\delta\in D_t} \mathrm{sign}(\delta)\delta$$

□

Corollary 2.16 *For every $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$ with $X_T \subseteq V_b$ the set*

$$\left\{E_{\gamma T,t} \mid \gamma \in Sym_\ell\right\}$$

together with the action $(\gamma, E_{T,t}) \mapsto E_{\gamma T,t}$ generates a $\mathbb{C}Sym_\ell$-module isomorphic to S^π. □

Example 2.12: Let $b = 3$. As in Example 2.10, for $\pi = (2,1)$ we assume that

$$t = \begin{matrix} 4 & 5 & 6 & \dots & k \\ 1 & 2 & & & \\ 3 & & & & \end{matrix}$$

. There are two semistandrd tableaux in $\mathcal{T}^0_{(k-3,2,1),\lambda_{k,3}}$:

$$T_1 = \begin{matrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 2 & & & & \\ 3 & & & & & \end{matrix} \quad \text{and} \quad T_2 = \begin{matrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 3 & & & & \\ 2 & & & & & \end{matrix} .$$

Writing colourings of K_3 as three-tuples, that is (h, i, j) is the colouring that assigns colour h to vertex 1, colour i to vertex 2 and colour j to vertex 3 it follows that:

$$\sum_{S\in\{T_1\}} [\alpha_S] = (1,2,3) + (2,1,3) \quad \text{and} \quad \sum_{S\in\{T_2\}} [\alpha_S] = (1,3,2) + (2,3,1)$$

and $E_{T_1,t} = \kappa_t\big((1,2,3) + (2,1,3)\big)$ and $E_{T_2,t} = \kappa_t\big((1,3,2) + (2,3,1)\big)$ where

$$\kappa_t = \big(\epsilon - (13)\big)\big(\epsilon - (14) - (34)\big)\big(\epsilon - (25)\big).$$

Let $Q_t = \big(\epsilon - (14) - (34)\big)\big(\epsilon - (25)\big)$ and $\kappa_{t_\pi} = \big(\epsilon - (13)\big)$. Then:

$$\big((12), E_{T_1,t}\big) = Q_t\big((2,1,3) - (2,3,1) + (1,2,3) - (3,2,1)\big) = E_{T_1,t},$$

$$\big((12), E_{T_2,t}\big) = Q_t\big((3,1,2) - (1,3,2) + (3,2,1) - (1,2,3)\big) = -E_{T_1,t} - E_{T_2,t},$$

$$\big((123), E_{T_1,t}\big) = Q_t\big((3,1,2) - (1,3,2) + (3,2,1) - (1,2,3)\big) = -E_{T_1,t} - E_{T_2,t}$$

$$\text{and } \big((123), E_{T_2,t}\big) = Q_t\big((2,1,3) - (2,3,1) + (1,2,3) - (3,2,1)\big) = E_{T_1,t}.$$

The corresponding matrices are $R(12) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and $R(123) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. It can easily be checked that they indeed generate a matrix representation for Sym_3 corresponding to $\pi = (2,1)$. Hence $E_{T_1,t}$ and $E_{T_1,t}$ together with the action $(\gamma, E_{T,t}) \mapsto E_{\gamma T,t}$ generate an irreducible submodule isomorphic to $\mathcal{S}^{(2,1)}$.

Chapter 3

The compatibility matrix method

In this chapter we shall describe the compatibility matrix method, introduced by N.L. Biggs in [2], and recently used and developed in [5], [8], [7], and [9]. We show how it can be used to obtain the chromatic polynomials for certain families of graphs.

The compatibility matrix commutes with the action of the symmetric group. Using the results from Representation Theory, introduced in the previous chapter, we show that the matrix is equivalent to a block-diagonal matrix, and the multiplicities and the sizes of the blocks are obtained. Using a repeated inclusion-exclusion argument the entries of the blocks can be calculated.

This method has previously been used by Biggs and co-workers in [7] and [9] in the case where the "base graph" is the complete graph K_b. Here this approach will be extended for general "base graphs".

3.1 Bracelets

Given a graph B, a set $L \subseteq V \times V$ and an integer $n \geq 3$ the *bracelet* $L_n(B)$ is the graph constructed as follows. Take n disjoint copies of B and link them by extra edges according to the rule: For every $i = 1, 2, \ldots, n$ and each pair $(v, w) \in L$ join the vertex v in the i^{th} copy of B, to the vertex w in the $(i+1)^{\text{th}}$ copy of B, with the convention that $n+1=1$. We obtain a "ring" of n copies of B linked by edges in the manner prescribed. The graph B is called *base graph* and the set L is called a *linking set.* The edges corresponding to L, that is the edges not part of a base graph, are also called *linking edges.*

Example 3.1: Let B be the complete graph K_3, $n=5$ and

$$L = \{(1,1),(2,2),(3,3)\}$$

be the "identity" linking set. The graph $L_5(3)$ is shown in Figure 3.1. The edges of the five copies of the complete graph are drawn as thick lines, the edges corresponding to the linking set are drawn as thin lines. In general, let B be the

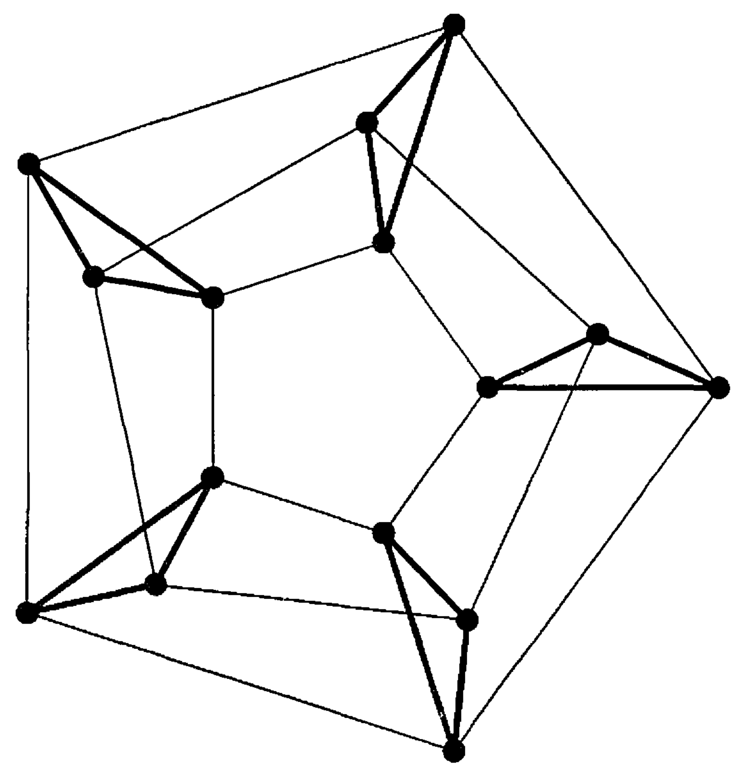


Figure 3.1: $L_5(3)$

complete graph K_b and

$$L = \{(1,1),(2,2),\ldots,(b,b)\}$$

be the "identity" linking set. The resulting graph is denoted by $B_n(b)$. For $b = 2$ the resulting bracelet is also called the ladder graph [5]. The case $b = 3$ has been covered in [10]. The chromatic polynomial in the case $b = 4$ has been obtained in [7] and [11], and the cases $b = 5, 6$ have been treated by this method in [12].

Example 3.2: Let B be the complete graph K_3 and

$$L = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}.$$

The resulting graph is a *cyclic octahedron* denoted by H_n. Its chromatic polynomial has been obtained in [9]. Figure 3.2 shows two adjacent copies of K_3 (thick lines) with the corresponding linking edges (thin lines).

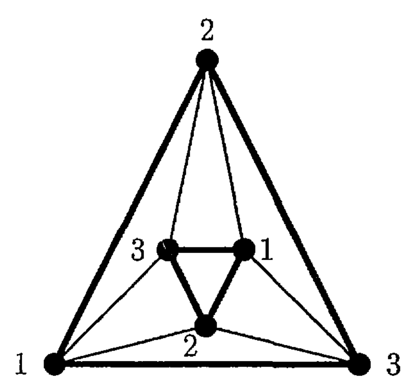


Figure 3.2: Two copies of K_3 and the linking set of the cyclic octahedron

Example 3.3: Let B be the cyclic graph C_b on b vertices and

$$L = \{(1,1), (2,2), \ldots, (b,b)\}$$

be the "identity" linking set. The resulting graph is denoted by $C_n(b)$.

Example 3.4: Let B be a path with vertex set $V = \{1,2,3,4\}$ (1 and 4 being the end-vertices). For $L = \{(1,1),\ (3,2),\ (4,4)\}$ the resulting cubic graph with $4n$ vertices is a *generalised dodecahedron* and is denoted by D_n. In particular D_5 is the graph of the regular dodecahedron. Two adjacent copies of B (thick lines) and

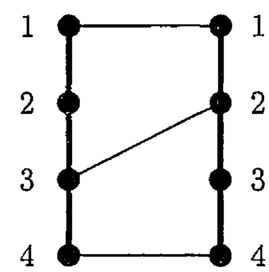


Figure 3.3: Two copies of path of length four and the linking set of the generalised dodecahedron

the linking edges (thin lines) are shown in Figure 3.3. To calculate the chromatic polynomial of D_n was a longstanding problem motivated by the question of whether chromatic roots can have a negative real part. D.A. Sands (in an unpublished thesis, 1972), Haggard (1976) obtained the chromatic polynomial of D_5. In 2001 S.C. Chang [11] calculated the chromatic polynomial for the general D_n and showed the existence of roots with negative real part for D_n for $n \geq 6$.

The n-fold symmetry of the bracelets allows us to use the compatibility matrix method, described in the next section, to calculate their chromatic polynomials.

3.2 The compatibility matrix method

Recall that $\Gamma_k(B)$ is the set of proper k-colourings of a graph B. $\mathcal{V}_k(B)$ is the vector space of complex-valued functions defined on $\Gamma_k(B)$. We say that a pair (α, β) of members of $\Gamma_k(B)$ is *compatible with* L if:

$$(v, w) \in L \implies \alpha(v) \neq \beta(w).$$

This means that if one copy of B is coloured according to α, a second copy of B according to β, and they are linked according to L, the resulting graph is properly k-coloured by α and β.

The *compatibility operator* $T_L = T_L(k)$ is defined by the matrix whose entries are

$$(T_L)_{\alpha\beta} = \begin{cases} 1 & \text{if } (\alpha,\beta) \text{ is compatible with } L; \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient to use the same symbol T_L for the linear operator represented by the matrix T_L, with respect to the standard basis of $\mathcal{V}_k(B)$. The connection between T_L and the chromatic polynomial $P(L_n(B);k)$ arises from the following theorem [5].

Theorem 3.1 *The number of k-colourings of $L_n(B)$ is equal to the trace of $T_L(k)^n$.*

Proof: Let $\alpha, \beta, \gamma, \ldots, \tau$ be n colourings in $\Gamma_k(B)$. Colour the first copy of B with α, the second copy with β and so on up to the n^{th} copy coloured with τ. The resulting colouring of $L_n(B)$ is a proper k-colouring if and only if

$$(T_L)_{\alpha\beta}(T_L)_{\beta\gamma}\ldots(T_L)_{\tau\alpha} = 1.$$

The number of proper k-colourings of $L_n(B)$ is equal to the sum of this product over all possible combinations of $\alpha, \beta, \gamma, \ldots, \tau$ in $\Gamma_k(B)$:

$$\sum_{\alpha,\beta,\gamma,\ldots,\tau} (T_L)_{\alpha\beta}(T_L)_{\beta\gamma}\ldots(T_L)_{\tau\alpha} = \sum_{\alpha}(T_L^n)_{\alpha\alpha} = \mathrm{tr}(T_L^n).$$

□

Observe, that for the moment k is still a fixed integer. Only later we will be able to show that the trace of $T_L(k)^n$ is indeed of the form of a polynomial in k and hence we can replace k with the complex variable z.

Since the trace of a matrix is equal to the sum of the eigenvalues multiplied by the corresponding algebraic multiplicities it follows [5]:

Corollary 3.2 *Suppose that $\lambda_1(k), \lambda_2(k), \ldots, \lambda_s(k)$ are the eigenvalues of $T_L(k)$ and*

$m_1(k), m_2(k), \ldots, m_s(k)$ are the corresponding algebraic multiplicities. Then the number of proper k-colourings of $L_n(B)$ is equal to

$$\sum_{i=1}^{s} m_i(k)\ \lambda_i^n(k).$$

□

3.3 Decomposition of the compatibility matrix

Recall from the previous chapter that we defined the action of the symmetric group Sym_k on $\Gamma_k(B)$ by

$$(\omega, \alpha)(v) = \omega(\alpha(v)) \qquad \text{for every } v \in V$$

for all $\omega \in \mathrm{Sym}_k$ and $\alpha \in \Gamma_k$. Clearly, for every $\omega \in \mathrm{Sym}_k$, if (α, β) is compatible with L then so is $(\omega\alpha, \omega\beta)$. Let $A(\omega)$ be the matrix representation corresponding to the $\mathbb{C}\mathrm{Sym}_k$-module $\mathcal{V}_k(B)$ with respect to the canonical basis. That is

$$(A(\omega))_{\alpha\beta} = \begin{cases} 1 & \text{if} \quad \omega\beta = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

It can easily be checked that $T_L(k)\, A(\omega) = A(\omega)\, T_L(k)$ for all $\omega \in \mathrm{Sym}_k$. This means that $T_L(k)$ belongs to the commutant algebra $\mathcal{C}(A)$ of $A(\omega)$. Moreover, this holds for any linking set L. Let $b = |V|$. From Lemma 2.1 and Theorem 2.6 it follows that $T_L(k)$ is equivalent to a matrix of the form

$$\bigoplus_{\substack{0 \le \ell \le b \\ \pi \vdash \ell}} (I_\pi \otimes N_L^\pi),$$

where I_π is the identity matrix of size n_{π^k} and N_L^π is a $m_{\pi^k} \times m_{\pi^k}$ matrix with entries depending on k.

Theorem 3.3 *For any given base graph B and any linking set L the number of k-colourings of $L_n(B)$ is equal to*

$$\sum_{\ell=0}^{b} \sum_{\pi \vdash \ell} \eta_\pi(k) \;\; tr(N_L^\pi)^n,$$

where $\quad \eta_\pi(k) = 1 \quad$ *if* $\quad \ell = 0 \quad$ *and*

$$\eta_\pi(k) = \tfrac{n_\pi}{\ell!} \prod_{i=1}^{\ell} (k - h_i(\pi)) \quad \textit{with} \quad h_i(\pi) = \pi_i + \ell - i \quad \textit{if} \quad \ell > 0,$$

N_L^π *is a matrix of size* $\sum_{\mathcal{R} \in \Pi(B)} \binom{|\mathcal{R}|}{\ell} n_\pi$ *with entries depending on k, and*

n_π *is the dimension of the Specht module S^π given in Lemma 2.3.*

Proof: From the argument preceding this theorem and from Theorem 3.1 it follows that for any given $k \in \mathbb{N}$ the number of k-colourings of $L_n(B)$ is equal to

$$\operatorname{tr} \bigoplus_{\substack{0 \le \ell \le b \\ \pi \vdash \ell}} (I_\pi \otimes (N_L^\pi)^n) = \sum_{\ell=0}^{b} \sum_{\pi \vdash \ell} \eta_\pi(k) \ \operatorname{tr}(N_L^\pi)^n$$

where $\eta_\pi(k) = n_{\pi^k}$ is the size of I_π, independent of B, given in Theorem 2.6. Also from Theorem 2.6 follows the size of N_L^π. □

Recall from Theorem 2.14 that $\mathcal{V}_k(B)$ is the direct sum of the submodules $\mathcal{W}_k(\pi, B)$ where

$$\mathcal{W}_k(\pi, B) = \bigoplus_{\substack{\mathcal{R} \in \Pi(B) \\ |\mathcal{R}| \ge \ell}} \quad \bigoplus_{T \in \mathcal{T}^0_{\pi^k, \lambda_{k,|\mathcal{R}|}}} \quad \mathcal{U}_k(\pi, T, \mathcal{R})$$

for all partitions $\pi \vdash \ell$ with $0 \le \ell \le b$.

It follows that each N_L^π corresponds to the submodule $\mathcal{W}_k(\pi, B)$. The rows and columns of N_L^π correspond to the $\mathcal{U}_k(\pi, T, |\mathcal{R}|)$.

Observe that the $\eta_\pi(k)$ are independent of L and they are given by an explicit formula. The matrix N_L^π is dependent on L and our main task is to explain how to calculate it.

Example 3.5: Let B be the complete graph K_3 and L any linking set. In Example 2.10 we expressed the module $\mathcal{V}_k(3)$ as a direct sum of seven submodules $\mathcal{W}_k(\pi, 3)$. Each of them is the direct sum of irreducible submodules. The number and the dimension of these irreducible submodules for each of the $\mathcal{W}_k(\pi, 3)$ has been given in Table 2.1. It follows that the sizes of the matrices N_L^π are as shown in the following table:

π	$()$	(1)	(2)	(1^2)	(3)	$(2,1)$	(1^3)
size of N_L^π	1×1	3×3	3×3	3×3	1×1	2×2	1×1

For any $n \in \mathbb{N}$ and every $k \in \mathbb{N}$ the number of proper k-colourings of $L_n(K_3)$ is equal to

$$\begin{aligned}
&\operatorname{tr}(N_L^{()})^n \;+\; (k-1)\operatorname{tr}(N_L^{(1)})^n \\
+&\frac{1}{2}k(k-3)\operatorname{tr}(N_L^{(2)})^n \;+\; \frac{1}{2}(k-1)(k-2)\operatorname{tr}(N_L^{(1^2)})^n \\
+&\frac{1}{6}k(k-1)(k-5)\operatorname{tr}(N_L^{(3)})^n \\
+&\frac{2}{6}k(k-2)(k-4)\operatorname{tr}(N_L^{(2,1)})^n \\
+&\frac{1}{6}(k-1)(k-2)(k-3)\operatorname{tr}(N_L^{(1^3)})^n
\end{aligned}$$

Later (Theorem 3.13), we will show that the entries of the matrices N_L^π are polynomials in k over $\mathbb{C}$. It follows that $\operatorname{tr}(N_L^\pi)^n$ is a polynomial in k and hence

$$P(L_n(B);k) = \sum_{\ell=0}^{b} \sum_{\pi\vdash\ell} \eta_\pi(k)\ \operatorname{tr}(N_L^\pi)^n \in \mathbb{C}[k]$$

Replacing k with the complex variable z we can make $P(L_n(B);z) \in \mathbb{C}[z]$ into a polynomial with complex variable z such that that $P(L_n(B);z)$ is the number of proper z-colourings for all $z \in \mathbb{N}$. Hence $P(L_n(B);z)$ is the chromatic polynomial of $L_n(B)$. In order to find $\operatorname{tr}(N_L^\pi)^n$ it is convenient if we can find the eigenvalues

$$\lambda_1(L,\pi;k), \lambda_2(L,\pi;k), \ldots, \lambda_s(L,\pi;k)$$

and the corresponding algebraic multiplicities $m_1(L,\pi), m_2(L,\pi), \ldots, m_s(L,\pi)$ of N_L^π. Then

$$\operatorname{tr}(N_L^\pi)^n = \sum_{i=1}^{s} m_i(L,\pi)\ \lambda_i^n(L,\pi;k).$$

However, the eigenvalues of N_L^π might not always be polynomials (but the sum of the their n^{th} powers is).

We refer to the $\eta_\pi(k)$ as the *global multiplicities* and to the $m_i(L,\pi)$ as the *local multiplicities*. As mentioned earlier, the global multiplicities do not depend on L whereas the local multiplicities do.

3.4 Reduction to the complete base graph

Let b and d be two positive integers and let L be a subset of $V_b \times V_d$, where V_b is the vertex set of K_b and V_d the vertex set of K_d. We consider the graph consisting of K_b and K_d with extra edges according to L. As before, we say that a pair of colourings $(\alpha, \beta) \in \mathcal{V}_k(b) \times \mathcal{V}_k(d)$ is compatible with L if $(v, w) \in L$ implies $\alpha(v) \neq \beta(w)$. We define the compatibility operator (and use the same symbol) $T_L(k)$, as before, as the matrix whose entry in position (α, β) is one if (α, β) is compatible with L and zero otherwise.

Let the graph B and the linking set L be given. Suppose that $\mathcal{P}$ and $\mathcal{R}$ are two colour-partitions of the vertex set of B consisting of b and d independent sets respectively. That is $\mathcal{R} = \{R_i\}_{i=1}^{b}$ and $\mathcal{P} = \{P_i\}_{i=1}^{d}$ where we assume that $\min(R_i) < \min(R_j)$ if $i < j$, and $\min(P_i) < \min(P_j)$ if $i < j$. We define $L_{\mathcal{RP}} \subseteq V_b \times V_d$ by

$$(i, j) \in L_{\mathcal{RP}} \quad \text{implies that there exists } (v, w) \in L \text{ such that } v \in R_i \text{ and } w \in P_j.$$

Recall that $\langle \mathcal{R} \rangle$ is the submodule of $\mathcal{V}_k(B)$ generated by the set $\{[\alpha] \mid \alpha \models \mathcal{R}\}$. By Lemma 2.7 each of the $\langle \mathcal{R} \rangle$ is isomorphic to $\mathcal{V}_k(b)$ if $b = |\mathcal{R}|$.

Lemma 3.4 *Let B be a base graph and L be a linking set. For any two colour-partitions $\mathcal{R}$ and $\mathcal{P}$ of B with $|\mathcal{R}| = b$ and $|\mathcal{P}| = d$ the diagram*

$$\begin{tikzcd} \langle \mathcal{P} \rangle \arrow[r, "T_L"] \arrow[d, "\cong"'] & \langle \mathcal{R} \rangle \arrow[d, "\cong"] \\ \mathcal{V}_k(d) \arrow[r, "T_{L_{\mathcal{RP}}}"'] & \mathcal{V}_k(b) \end{tikzcd}$$

is commutative.

Proof: Recall from Lemma 2.7 the $\mathbb{C}\mathrm{Sym}_k$-module isomorphism $\langle \mathcal{R} \rangle \to \mathcal{V}_k(|\mathcal{R}|)$ given by $[\alpha] \mapsto [\overline{\alpha}]$. Let $[\alpha] \in \langle \mathcal{R} \rangle$ and $[\beta] \in \langle \mathcal{P} \rangle$.

If $(T_L)_{\alpha\beta} = 1$ then $\alpha(v) \neq \beta(w)$ for all $(v,w) \in L$. From the definition of $L_{\mathcal{RP}}$ follows that $\overline{\alpha}(i) \neq \overline{\beta}(j)$ for all $(i,j) \in L_{\mathcal{RP}}$. Hence $(T_{L_{\mathcal{RP}}})_{\overline{\alpha}\overline{\beta}} = 1$.

If $(T_L)_{\alpha\beta} = 0$ then $\alpha(v) = \beta(w)$ for some $(v,w) \in L$. Let (i,j) be such that $v \in R_i$ and $w \in P_j$. Then $(i,j) \in L_{\mathcal{RP}}$ and it follows that $\overline{\alpha}(i) = \overline{\beta}(j)$. Hence $(T_{L_{\mathcal{RP}}})_{\overline{\alpha}\overline{\beta}} = 0$. $\square$

As in the previous section $T_{L_{\mathcal{RP}}} : \mathcal{V}_k(d) \to \mathcal{V}_k(b)$ commutes with the action of Sym_k and hence is equivalent to

$$\bigoplus_{\substack{0 \leq \ell \leq \min(b,d) \\ \pi \vdash \ell}} (I_\pi \otimes N^\pi_{L_{\mathcal{RP}}}),$$

where I_π is the identity matrix of size n_{π^k} and $N^\pi_{L_{\mathcal{RP}}}$ is a $\binom{b}{\ell} n_\pi \times \binom{d}{\ell} n_\pi$ matrix. Since $\mathcal{V}_k(B)$ is equal to the direct sum of the $\langle\mathcal{R}\rangle$ it follows from Theorem 2.14 that:

Lemma 3.5 *Let B be a base graph and L be a linking set. For any $0 \leq \ell \leq |V|$ and any $\pi \vdash \ell$ the matrix N^π_L consists of submatrices equivalent to $N^\pi_{L_{\mathcal{RP}}}$ with $\mathcal{R}$, $\mathcal{P} \in \Pi(B)$. Its rows correspond to the $\mathcal{U}_k(\pi, T, \mathcal{R})$ with $T \in \mathcal{T}^0_{\pi^k, \lambda_{k,|\mathcal{R}|}}$ and the columns correspond to the $\mathcal{U}_k(\pi, T', \mathcal{P})$ with $T' \in \mathcal{T}^0_{\pi^k, \lambda_{k,|\mathcal{P}|}}$.* $\square$

From this lemma it follows that in order to obtain the entries of N^π_L we may find the entries of each of the matrices $N^\pi_{L_{\mathcal{RP}}}$ individually and then use them to obtain the original matrix N^π_L. Hence we are interested in finding N^π_L for the case where we have two complete base graphs of not necessarily the same size and a linking set L. We write ${}_b\langle L\rangle_d$ for the graph consisting of one copy of K_b and one of K_d with extra edges according to L. Then ${}_{|\mathcal{R}|}\langle L_{\mathcal{RP}}\rangle_{|\mathcal{P}|}$ gives rise to $T_{L_{\mathcal{RP}}}$ and to $N^\pi_{L_{\mathcal{RP}}}$.

Each of the vertices in $K_{|\mathcal{R}|}$ corresponds to an independent set in $\mathcal{R}$. That is $i \in V_{|\mathcal{R}|}$ corresponds to R_i with respect to the labelling of the independent sets satisfying that $\min(R_i) < \min(R_j)$ if $i < j$.

Before further investigating N^π_L for general ${}_b\langle L\rangle_d$ in the next section we give an example.

Example 3.6: Let B be the path on three vertices. Let

$$L = \{(1,1),(2,2),(3,3)\}$$

be the "identity" linking set. There are two colour-partitions $\mathcal{R} = \{1|2|3\}$ and $\mathcal{P} = \{1,3|2\}$ and thus there are four induced graphs

$${}_3\langle 11,22,33\rangle_3, \qquad {}_3\langle 11,22,31\rangle_2, \qquad {}_2\langle 11,22,13\rangle_3 \quad \text{and} \quad {}_2\langle 11,22\rangle_2$$

where we write for example ${}_2\langle 11,22\rangle_2$ rather than ${}_2\langle\{(1,1),(2,2)\}\rangle_2$. These four graphs are shown in Figure 3.4 on Page 55. The edges of the base graphs are drawn as thick lines, the linking edges are drawn as thin lines.

For any $\ell = 0,1,2,3$ and any $\pi \vdash \ell$ the matrix N_L^π consists of four blocks:

$$N_L^\pi = \left(\begin{array}{c|c} N^\pi_{L_{\mathcal{R}\mathcal{R}}} & N^\pi_{L_{\mathcal{R}\mathcal{P}}} \\ \hline N^\pi_{L_{\mathcal{P}\mathcal{R}}} & N^\pi_{L_{\mathcal{P}\mathcal{P}}} \end{array}\right).$$

The sizes of these blocks and of N_L^π have been obtained in Example 2.11, and are as shown in the following table:

π	$()$	(1)	(2)	(1^2)	(3)	$(2,1)$	(1^3)
size of $N^\pi_{L_{\mathcal{R}\mathcal{R}}}$	1×1	3×3	3×3	3×3	1×1	2×2	1×1
size of $N^\pi_{L_{\mathcal{R}\mathcal{P}}}$	1×1	3×2	3×2	3×2	1×0	2×0	1×0
size of $N^\pi_{L_{\mathcal{P}\mathcal{R}}}$	1×1	2×3	2×3	2×3	0×1	0×2	0×1
size of $N^\pi_{L_{\mathcal{P}\mathcal{P}}}$	1×1	2×2	2×2	2×2	0×0	0×0	0×0
size of N^π_L	2×2	5×5	5×5	5×5	1×1	2×2	1×1

Observe that the "structure" of N_L^π, that is the sizes of the $N^\pi_{L_{\mathcal{R},\mathcal{R}'}}$, is independent of the linking set L.

3.5 The S_M operators

Let b and d be two positive integers, and as before let V_b be the vertex set of K_b and V_d the vertex set of K_d. A *matching* M is a triple (M_1, M_2, μ) with $M_1 \subseteq V_b$

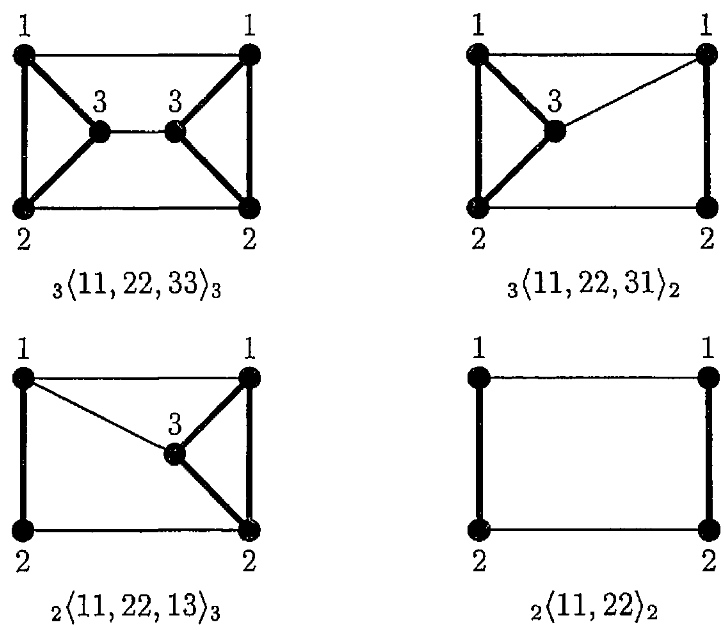


Figure 3.4: The four induced graphs

and $M_2 \subseteq V_d$ and $\mu : M_1 \to M_2$ being a bijection. Equivalently, the matching M is the subset of $V_b \times V_d$ consisting of the pairs $(v, \mu(v))$ for all $v \in M_1$.

Let $L \subseteq V_b \times V_d$ be a linking set. Denote by $\mathcal{M}(b, d, L)$ the set of matchings M that are subsets of L. For a given $M \in \mathcal{M}(b, d, L)$ we define the operator $S_M(k) : \mathcal{V}_k(d) \to \mathcal{V}_k(b)$ by the matrix (with respect to the canonical basis)

$$(S_M(k))_{\alpha\beta} = \begin{cases} 1 & \text{if} \quad \alpha_{M_1} = \beta\mu, \\ 0 & \text{otherwise} \end{cases}$$

where α_{M_1} is the restriction of α to M_1. Alternatively we can write $S_M(k)$ a linear operator:

$$S_M(k)[\beta] = \sum_{\delta_{M_1} = \beta\mu} [\delta].$$

The following theorem is a generalization of the result proved in [9].

Theorem 3.6 *Let the integers b and d, and the linking set L be given. Then*

$$T_L(k) = \sum_{M \in \mathcal{M}(b,d,L)} (-1)^{|M|} \, S_M(k).$$

Proof: For any $\alpha \in \Gamma_k(b)$ and $\beta \in \Gamma_k(d)$ we shall show that

$$(T_L)_{\alpha\beta} = \sum_{M \in \mathcal{M}(b,d,L)} (-1)^{|M|} \ (S_M)_{\alpha\beta}.$$

Let $M_{\alpha\beta}$ be the subset of L such that $\alpha(v) = \beta(w)$ for every $(v,w) \in M_{\alpha\beta}$. Since α and β are injections it follows that $M_{\alpha\beta} \in \mathcal{M}(b,d,L)$. Then, $(S_M)_{\alpha\beta} = 1$ if and only if $M \subseteq M_{\alpha\beta}$.

$$\sum_{M \in \mathcal{M}(b,d,L)} (-1)^{|M|} \ (S_M)_{\alpha\beta} = \sum_{M \subseteq M_{\alpha\beta}} (-1)^{|M|}.$$

If (α, β) is compatible with L then $M_{\alpha\beta}$ is the empty matching and the sum is equal to one. If (α,β) is not compatible with L then $M_{\alpha\beta}$ is not empty and

$$\sum_{M \subseteq M_{\alpha\beta}} (-1)^{|M|} = (1 + (-1))^{|M_{\alpha\beta}|} = 0.$$

□

It is easily verified that each of the $S_M(k)$ commutes with the action of Sym_k on the colourings. By a similar argument as in Section 3.3 it follows that:

Corollary 3.7 *Let the integers b and d, and the linking set L be given. Then there exist matrices U_M^π each of size $\binom{b}{\ell} n_\pi \times \binom{d}{\ell} n_\pi$ such that*

$$N_L^\pi = \sum_{M \in \mathcal{M}(b,d,L)} (-1)^{|M|} \quad U_M^\pi.$$

□

The matrix $(I_\pi \otimes U_M^\pi)$ represents the induced linear operator

$$S_M(k) : \mathcal{W}_k(\pi, d) \to \mathcal{W}_k(\pi, b)$$

where I_π is the identity matrix of size n_{π^k}. The columns of U_M^π corresponding to the irreducible submodules $\mathcal{U}_k(\pi, T, d)$ with $T \in \mathcal{T}^0_{\pi^k, \lambda_{k,d}}$ and rows corresponding to the irreducible submodules $\mathcal{U}_k(\pi, S, b)$ with $S \in \mathcal{T}^0_{\pi^k, \lambda_{k,b}}$. Later it will be shown that U_M^π is the all-zero matrix if $\ell > |M|$. The next aim is to find the entries of U_M^π.

3.6 Change of basis

Recall from Section 2.4.2 the following: Let $X \subseteq V_b$ and let $g : X \to K$ be an injection. We define the function $[X \mid g] \in \mathcal{V}_k(b)$ by

$$[X \mid g](\alpha) = \begin{cases} 1 & \text{if } \alpha_X = g \\ 0 & \text{otherwise.} \end{cases}$$

for every $\alpha \in \Gamma_k(b)$ where α_X is the restriction of α to X. Equivalently

$$[X \mid g] = \sum_{\delta_X = g} [\delta].$$

For every matching $M = (M_1, M_2, \mu)$ we can write

$$S_M(k)[\alpha] = \sum_{\delta_{M_1} = \alpha\mu} [\delta] = [M_1 \mid \alpha\mu].$$

Lemma 3.8 *Let $[X \mid g] \in \mathcal{V}_k(d)$ and $M \in \mathcal{M}(b, d, L)$ be given. Then*

$$S_M(k)[X \mid g] = \quad c \sum_{\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1} \quad (-1)^{|Y| - |X \cap M_2|} \sum_{\phi \in G_M(Y,X)} [Y \mid g\phi]$$

where $G_M(Y, X)$ is the set of injections $\phi : Y \to X$ such that $\phi\mu^{-1}$ is the identity map on $X \cap M_2$, and c is a non-zero constant.

Proof: From definitions follows on the left hand side that

$$\sum\nolimits_L = S_M[X \mid g] = S_M \sum_{\alpha_X = g} [\alpha] = \sum_{\alpha_X = g} \quad \sum_{\beta_{M_1} = \alpha\mu} [\beta] = \sum_{\beta \in \Gamma_k(b)} [\beta] \Big(\sum_{\substack{\alpha_X = g \\ \alpha\mu = \beta_{M_1}}} 1 \Big).$$

Denote by $\sum_R$ the map

$$\sum_{\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1} \quad c\,(-1)^{|Y| - |X \cap M_2|} \sum_{\phi \in G_M(Y,X)} [Y \mid g\phi]$$

Let $\gamma \in \Gamma_k(b)$. We are going to compare $\sum_L(\gamma)$ to $\sum_R(\gamma)$. We may assume that $\gamma\mu^{-1} = g$ on $X \cap M_2$, because otherwise both sides are zero. Indeed, for $\sum_L$ it follows immediately from $\alpha = g$ and $\alpha = \gamma\mu^{-1}$ on $X \cap M_2$. For $\sum_R$ by definition

$[Y \mid g\phi](\gamma) = 1$ only if $g\phi = \gamma$ on Y. Since $\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1$ it follows from the definition of $G_M(Y, X)$ that $g\phi\mu^{-1} = g = \gamma\mu^{-1}$ on $X \cap M_2$.

If $\gamma\mu^{-1}(v) \notin g(X)$ for all $v \in M_2 \setminus (X \cap M_2)$ then there exists a α such that $\alpha\mu = \gamma_{M_1}$ and $\alpha_X = g$. It follows that $\sum_L(\gamma)$ is non-zero. On the right hand side, since $Y \subseteq M_1$ and $g\phi(Y) \subseteq g(X)$ it follows that $[Y \mid g\phi](\gamma) \neq 0$ only if $Y = \mu^{-1}(X \cap M_2)$. And thus $\sum_R(\gamma) = c$.

If $\gamma\mu^{-1}(v) \in g(X)$ for some $v \in M_2 \setminus (X \cap M_2)$. Then, since $\alpha_X = g$ it follows that there exists $x \in X$ such that $\alpha(x) = \gamma(v)$. On the other hand $\gamma_{M_1} = \alpha\mu$ implies that $\alpha(v) = \alpha(x)$. Since $v \neq x$ it follows that $\sum_L(\gamma) = 0$. For the right hand side let

$$Q = \Big\{ v \in M_2 \setminus (X \cap M_2) \mid \gamma\mu^{-1}(v) \in g(X) \Big\}.$$

Then $[Y \mid g\phi](\gamma) \neq 0$ only if $\mu^{-1}(X \cap M_2) \subseteq Y \subseteq \mu^{-1}\big((X \cap M_2) \cup Q\big)$ and the injection ϕ is such that $g\phi = \gamma_Y$. Since we assumed that $g = \gamma\mu^{-1}$ on $X \cap M_2$ it follows that such a ϕ exists in $G_M(Y, X)$. And thus

$$\begin{aligned}
\sum_R(\gamma) &= \sum_{\mu^{-1}(X\cap M_2)\subseteq Y \subseteq \mu^{-1}(X\cap M_2)\cup Q} c\,(-1)^{|Y|-|X\cap M_2|}[Y \mid \gamma_Y](\gamma) \\
&= \sum_{\mu^{-1}(X\cap M_2)\subseteq Y \subseteq \mu^{-1}(X\cap M_2)\cup Q} c\,(-1)^{|Y|-|X\cap M_2|} \\
&= c \sum_{r=0}^{|Q|} (-1)^r \binom{|Q|}{r} = c\,(1-1)^{|Q|}.
\end{aligned}$$

□

Lemma 3.9 *Let $Y \subseteq V_b$ and $X \subseteq V_d$. Let $g : X \to K$ be an injection. Then the coefficient of $[Y \mid g\phi]$ in $S_M(k)[X \mid g]$ with $(M_1, M_2, \mu) \in \mathcal{M}(b, d, L)$ and injection $\phi : Y \to X$ is non-zero if and only if*

(i) $\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1$, *and*

(ii) $\phi\mu^{-1}$ *is the identity map on* $X \cap M_2$.

Let $f_s(d,k) = (k-s)_{d-s} = (k-s)(k-s-1)\dots(k-d+1)$ *be the falling factorial. If the conditions* (i) *and* (ii) *are satisfied the coefficient is*

$$(-1)^{|Y|-|X\cap M_2|}\, f_{|X\cup M_2|}(d,k)$$

Proof: The first part of the lemma follows directly from Lemma 3.8. Moreover it follows that when the conditions (i) and (ii) are satisfied the coefficient is $c\,(-1)^{|Y|-|X\cap M_2|}$. From the proof of Lemma 3.8 it follows that c is equal to the number of $\alpha \in \Gamma_k(d)$ satisfying $\alpha\mu = g'$ and $\alpha_X = g$. That is, α is fixed on X and on M_2, and there are $k - |X \cup M_2|$ colours left to be assigned to $d - |X \cup M_2|$ vertices to complete α. □

3.7 Action of $S_M(k)$ on the irreducible submodules of $\mathcal{V}_k(b)$

Let $0 \le \ell \le b$ and $\pi \vdash \ell$. **For the rest of this section let t be a fixed π^k-tableau**. Recall from Section 2.4.2 the following. For every tableau $T \in \mathcal{T}_{\pi^k,\lambda_{k,b}}$ we denote by $T_\pi : [\pi] \to V_b \cup \{0\}$ the restriction of T to $[\pi]$. The image of T_π is denoted by X_T. If T is semistandard of type $\lambda_{k,b}$ then $X_T \subseteq V_b$ and T_π is a standard tableau. Denote by $g_T : X_T \to K$ the restriction of α_T to X_T. That is, $g_T(x) = t(i,j)$ where $T(i,j) = x$ for all $x \in X_T$. Similarly, define t_π to be the restriction of t to $[\pi]$.

In Section 2.4.1 it has been shown that for every semistandard tableau $T \in \mathcal{T}^0_{\pi^k,\lambda_{k,b}}$ the set

$$\left\{E_{T,\gamma t} \mid \gamma \in \mathrm{Sym}_k \text{ such that } \gamma t \text{ is a standard } \pi^k\text{-tableau}\right\}$$

where

$$E_{T,t} = \kappa_t \sum_{S\in\{T\}} [\alpha_S] = \kappa_t \sum_{\omega\in R_{t_\pi}} [X_T \mid \omega g_T]$$

is the standard basis of the submodule $\mathcal{U}_k(\pi, T, b)$.

Since the $S_M(k)$ commute with the action of Sym_k it follows that we only have to consider the effect of $S_M(k)$ on $E_{T,t}$.

Lemma 3.10 *Let $M \in \mathcal{M}(b,d,L)$ and $T \in \mathcal{T}_{\pi^k,\lambda_{k,d}}$. Then $S_M(k)\, E_{T,t}$ is a linear combination of*

$$\kappa_t \sum_{\omega \in R_{t_\pi}} [Y \mid \omega g_T \phi]$$

where $\mu^{-1}(X_T \cap M_2) \subseteq Y \subseteq M_1$ with $|Y| = \ell$ and $\phi \in G_M^{Y,X_T}$.

Proof: From Lemma 3.8 and since S_M commutes with Sym_k it follows that

$$S_M(k)\Big(\kappa_t \sum_{\omega \in R_{t_\pi}} [X_T \mid \omega g_T]\Big)$$

$$= \sum_{\mu^{-1}(X_T \cap M_2) \subseteq Y \subseteq M_1} c\,(-1)^{|Y|-|X \cap M_2|} \sum_{\phi \in G_M(Y,X_T)} \kappa_t \sum_{\omega \in R_{t_\pi}} [Y \mid \omega g_T \phi].$$

Choose any $\mu^{-1}(X_T \cap M_2) \subseteq Y \subseteq M_1$ with $|Y| < \ell$ and any $\phi \in G_M^{Y,X_T}$. We can write

$$\kappa_t \sum_{\omega \in R_{t_\pi}} [Y \mid \omega g_T \phi] = \sum_{\omega \in R_{t_\pi}} \sum_{\delta \in C_t} \mathrm{sign}(\delta) \quad [Y \mid \delta \omega g_T \phi]$$

Choose any $\omega \in R_{t_\pi}$. Let $g : Y \to K$ be such that $g = \omega g_T \phi$. Partition C_t into parts $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r$ according to the rule that δ and δ' are in the same part if and only if $[Y \mid \delta g] = [Y \mid \delta' g]$. Since $|Y| < \ell$ it follows that each of the parts contains more than one element. For every $j = 1, 2, \ldots, \pi_1$ denote by D_j the set of colours that are in the j^{th} column of t but not in $g(Y)$. Let $H = \mathrm{Sym}_{D_1} \times \mathrm{Sym}_{D_2} \times \ldots \times \mathrm{Sym}_{D_{\pi_1}}$. Then for every $\mathcal{B}$ it holds that $\mathcal{B} = \delta H$ for some $\delta \in \mathcal{B}$. That is, $\mathcal{B}$ is a left coset of H. Thus

$$\begin{aligned}
\sum_{\delta \in \mathcal{B}} \mathrm{sign}(\delta) \quad [Y \mid \delta g] &= \mathrm{sign}(\delta) \sum_{\tau \in H} \mathrm{sign}(\tau) \quad [Y \mid \delta \tau g] \\
&= \mathrm{sign}(\delta) \quad [Y \mid \delta g] \sum_{\tau \in H} \mathrm{sign}(\tau) \\
&= \mathrm{sign}(\delta) \quad [Y \mid \delta g] \sum_{j=1}^{\pi_1} (1-1)^{D_j}.
\end{aligned}$$

This holds for all $\mathcal{B}$ and hence follows the result. $\square$

Let us recall Section 2.6 and study its implications. Let $X = \{x_1, x_2, \ldots, x_\ell\}$ be a subset of V_d such that $x_1 < x_2 < \ldots < x_\ell$. We let Sym_ℓ act on X by

$$(\gamma, x_i) = x_{\gamma i} \quad \text{for all } x_i \in X \quad \text{and every} \quad \gamma \in \mathrm{Sym}_\ell.$$

We write γx_i instead of (γ, x_i). This induces an action of Sym_ℓ on the set

$$\{T \in \mathcal{T}_{\pi^k, \lambda_{k,b}} \mid T[\pi] = X\}.$$

That is, for every $\gamma \in \mathrm{Sym}_\ell$

$$(\gamma, T)(p, q) = \gamma x_i \qquad \text{where} \quad x_i = T(p, q) \quad \text{for all} \quad (p, q) \in [\pi].$$

We write γT instead of (γ, T). We can assume that the π^k tableau t is such that $t[\pi] = \{1, 2, \ldots, \ell\}$. Let $Y = \{y_1, y_2, \ldots, y_\ell\}$ be a subset of V_b with $y_1 < y_2 < \ldots < y_\ell$. Choose $T_X \in \mathcal{T}_{\pi^k, \lambda_{k,d}}$ and $T_Y \in \mathcal{T}_{\pi^k, \lambda_{k,b}}$ such that $T_X[\pi] = X$ and $T_Y[\pi] = Y$, and

$$g_{T_X}(x_i) = g_{T_Y}(x_i) = i \qquad \text{for all} \quad i = 1, 2, \ldots, \ell.$$

For any matching $M = (M_1, M_2, \mu) \in \mathcal{M}(b, d, L)$ with $|M| \leq \ell$ denote by F_M^{YX} the subset of Sym_ℓ satisfying

$$F_M^{YX} = \{\rho \in \mathrm{Sym}_\ell \mid (y_i, x_j) \in (Y \times X) \cap M \Rightarrow i = \rho(j)\}$$

There is a one to one relationship between the elements of F_M^{YX} and G_M^{YX} such that

$$g_{T_Y} \rho^{-1} = g_{T_X} \phi \quad \text{on } Y.$$

Lemma 3.11 *Let $T \in \mathcal{T}_{\pi^k, \lambda_{k,b}}$ with $T[\pi] = X$. Then*

$$g_{\gamma T} = g_T \gamma^{-1} \qquad \textit{for all } \gamma \in \mathit{Sym}_\ell$$

Proof: By definition of g_T for every $x_i \in X$:

$$g_{\gamma T}(x_i) = t(p, q) \qquad \text{if} \quad \gamma T(p, q) = x_i.$$

It follows $T(p, q) = \gamma^{-1} x_i$ and thus $g_T \gamma^{-1}(x_i) = t(p, q)$. $\square$

It follows that

$$\kappa_t \sum_{\omega \in R_{t_\pi}} [X \mid \omega g_{T_X} \phi] = \kappa_t \sum_{\omega \in R_{t_\pi}} [Y \mid \omega g_{T_Y} \rho^{-1}] = E_{\rho T_Y, t}$$

and for every $\gamma \in \mathrm{Sym}_\ell$ it follows that

$$\kappa_t \sum_{\omega \in R_{t_\pi}} [X \mid \omega g_{\gamma T_X} \phi] = E_{\rho\gamma T_Y, t}.$$

Assume that γT_Y is semistandard then the restriction of γT_Y to $[\pi]$ is a standard π tableau. From Corollary 2.16 it follows that

$$E_{\rho\gamma T_Y} = \sum_{\substack{\sigma \in \mathrm{Sym}_\ell \\ \sigma T_Y \text{is semistandard}}} \Big(R^\pi(\rho)\Big)_{\sigma\gamma} \quad E_{\sigma T_Y, t}$$

where R^π is Young's natural representation corresponding to $\mathcal{S}^\pi$. Observe that the rows and columns of R^π correspond to the standard π tableaux, but we label them (for brevity) by the elements $\gamma \in \mathrm{Sym}_\ell$ such that the restriction of γT_Y to $[\pi]$ is a standard π tableau.

From Lemma 3.10 it follows that

$$S_M E_{\gamma T_X, t} = C_M(X) \sum_{\substack{\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1 \\ |Y| = \ell}} \sum_{\phi \in G_M^{Y,X}} \kappa_t \sum_{\omega \in R_{t_\pi}} [Y \mid \omega g_{\gamma T_X} \phi],$$

where

$$C_M(X) = (-1)^{\ell - |X \cap M_2|} f_{|X \cup M_2|}(b, k).$$

From the argument above it follows that

$$\begin{aligned} S_M E_{\gamma T_X, t} &= C_M(X) \sum_{\substack{\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1 \\ |Y| = \ell}} \sum_{\rho \in F_M^{Y,X}} \kappa_t \sum_{\omega \in R_{t_\pi}} [Y \mid \omega g_{\rho\gamma T_Y}] \\ &= C_M(X) \sum_{\substack{\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1 \\ |Y| = \ell}} \sum_{\rho \in F_M^{Y,X}} E_{\rho\gamma T_Y, t}. \end{aligned}$$

Corollary 3.12 *Let $M \in \mathcal{M}(b, d, L)$, $T \in \mathcal{T}_{\pi^k, \lambda_{k,d}}$ and $T' \in \mathcal{T}_{\pi^k, \lambda_{k,b}}$. If $\mu^{-1}(X_T \cap M_2) \subseteq X_{T'} \subseteq M_1$ and $\phi T'_\pi = T_\pi$ for some $\phi \in G_M^{X_{T'}, X_T}$, then $S_M(k) : \mathcal{U}_k(\pi, T, b) \to \mathcal{U}_k(\pi, T', b)$ is the isomorphism given by*

$$E_{T,t} \mapsto C_M(X_T)\, E_{T',t}, \quad \textit{where} \quad C_M(X) = (-1)^{\ell - |X \cap M_2|} f_{|X \cup M_2|}(d, k).$$

Otherwise $S_M(k) : \mathcal{U}_k(\pi, T, b) \to \mathcal{U}_k(\pi, T', b)$ *is the zero-map.* □

This result is no surprise since $\mathcal{U}_k(\pi, T', d)$ and $\mathcal{U}_k(\pi, T, b)$ are irreducible and from Schur's Lemma follows that $S_M(k) : \mathcal{U}_k(\pi, T, b) \to \mathcal{U}_k(\pi, T', b)$ is either the zero-map or a multiplication by a scalar.

Since γT_Y is semistandard if and only if γT_X is semistandard, it follows if γT_X is semistandard that:

$$S_M E_{\gamma T_X, t} = C_M(X) \sum_{\substack{\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1 \\ |Y| = \ell}} \sum_{\rho \in F_M^{Y,X}} \sum_{\substack{\sigma \in \mathrm{Sym}_\ell \\ \sigma T_Y \text{ is semistandard}}} \Big(R^\pi(\rho)\Big)_{\sigma\gamma} E_{\sigma T_Y, t}$$

It follows that:

Theorem 3.13 *Let* $0 \le \ell \le \min(b, d)$, $\pi \vdash \ell$ *and* $M \in \mathcal{M}(b, d, L)$. *Then the matrix* U_M^π *consists of square submatrices* $(U_M^\pi)^{YX}$ *where* $X \subseteq V_d$ *and* $Y \subseteq V_b$ *with* $|X| = |Y| = \ell$. *Each of the* $(U_M^\pi)^{YX}$ *is of the form*

$$(U_M^\pi)^{YX} = \begin{cases} C_M(X) \sum_{\rho \in F_M^{YX}} R^\pi(\rho) & \text{if } \mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1 \\ O & \text{otherwise.} \end{cases}$$

where

$$C_M(X) = (-1)^{\ell - |X \cap M_2|} f_{|X \cup M_2|}(d, k)$$

and $R^\pi(\rho)$ *is Young's natural representation, that is the* $n_\pi \times n_\pi$ *matrix representation corresponding to* S^π *and* O *is the all-zero matrix of size* $n_\pi \times n_\pi$. □

Observe that F_M^{YX} depends on ℓ but not on $\pi \vdash \ell$.

Theorem 3.14 *Let* B *be any base graph with vertex set* V *and* L *any linking set. Then*

$$P(L_n(B), k) = \sum_{\ell=0}^{|V|} \sum_{\pi \vdash \ell} \eta_\pi(k)\ tr(N_L^\pi)^n,$$

is the chromatic polynomial of $L_n(B)$ *in* k *where* $\eta_\pi(k) = 1$ *if* $\ell = 0$,

$$\eta_\pi(k) = \frac{n_\pi}{\ell!} \prod_{i=1}^{\ell} (k - h_i(\pi)) \quad \text{with} \quad h_i(\pi) = \pi_i + \ell - i \quad \text{if} \quad \ell > 0,$$

N_L^π is a square-matrix of size $\sum_{\mathcal{R}\in\Pi(B)} \binom{|\mathcal{R}|}{\ell} n_\pi$ with polynomials in k over $\mathbb{C}$ as entries, and n_π is the dimension of the Specht module S^π.

Proof: From Theorem 3.13 and Lemma 3.5 it follows that the entries of the matrices N_L^π in Theorem 3.3 are polynomials in k over $\mathbb{C}$. It follows that $\mathrm{tr}(N_L^\pi)^n$ is a polynomial in k and thus $P(L_n(B);k)$ is a polynomial in k. Extending k to a complex variable it follows that $P(L_n(B);k)$ is a polynomial such that its value at $k \in \mathbb{N}$ is equal to the number of proper k-colourings. Hence $P(L_n(B);k)$ is the chromatic polynomial of $L_n(B)$. □

Before concluding this chapter with a summary we give some examples.

3.8 Examples

Example 3.7: Let $b = d = 3$. In this example we shall determine all the matrices U_M^π for all levels $\ell = 0, 1, 2, 3$ and all $\pi \vdash \ell$, and all possible matchings $M \subset V_3 \times V_3$. This work is also published in [9] Section 6.

There are 1, 9, 18, 6 matchings M with $|M| = 0, 1, 2, 3$ respectively. We use Theorem 3.13 to evaluate the U_M^π.

At level $\boldsymbol{\ell = 0}$ and $\boldsymbol{\pi = ()}$ the matrices $U_M^{()}$ are of size 1×1 and for every matching $M \subset V_3 \times V_3$ it follows that $U_M^{()} = f_{|M|}(3, k)$. That is $U_M^{()}$ is

$$k(k-1)(k-2), \qquad (k-1)(k-2), \qquad (k-2), \qquad 1$$

for $|M| = 0, 1, 2, 3$ respectively.

At level $\boldsymbol{\ell = 1}$ and $\boldsymbol{\pi = (1)}$ all the matrices $U_M^{(1)}$ are of size 3×3. If M is the empty matching then $U_M^{(1)}$ is the all-zero matrix. Assume that M is not the empty matching. Let $X = \{x\}$ and $Y = \{y\}$ be two subsets of $\{1, 2, 3\}$ of size one. The set $F_M^{YX} = \mathrm{Sym}_1$ and

$$(U_M^\pi)^{YX} = \begin{cases} (k-|M|)_{3-|M|} & \text{if } y \in M_1 \text{ and } (y,x) \in M; \\ -(k-|M|-1)_{2-|M|} & \text{if } y \in M_1 \text{ and } x \notin M_2; \\ 0 & \text{if } y \notin M_1 \text{ or } x \in M_2 \text{ and } (y,x) \notin M. \end{cases}$$

For example

$$U_{11}^{(1)} = \begin{pmatrix} (k-1)(k-2) & -(k-2) & -(k-2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_{11,22}^{(1)} = \begin{pmatrix} k-2 & 0 & -1 \\ 0 & k-2 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_{12}^{(1)} = \begin{pmatrix} -(k-2) & (k-1)(k-2) & -(k-2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_{13,22}^{(1)} = \begin{pmatrix} -1 & 0 & k-2 \\ -1 & k-2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_{11,22,33}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{13,22,31}^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_{12,23,31}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

At level $\boldsymbol{\ell = 2}$ there are two partitions $\pi = \mathbf{(2)}$ and $\pi = \mathbf{(1^2)}$. In both cases the matrices U_M^π are of size 3×3. If $|M| \leq 1$ then U_M^π is the all-zero matrix. Assume that $|M| \geq 2$. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be two subsets of $\{1,2,3\}$ of size two with $x_1 < x_2$ and $y_1 < y_2$. Then $g_{T_Y}(y_1) = g_{T_X}(x_1)$ and $g_{T_Y}(y_2) = g_{T_X}(x_2)$ It follows that the subset F_M^{YX} of Sym_2 is of the form

$$F_M^{YX} = \begin{cases} \{\epsilon\} & \text{if } (y_1,x_1) \in M \text{ or } (y_2,x_2) \in M \\ \{(12)\} & \text{if } (y_1,x_2) \in M \text{ or } (y_2,x_1) \in M. \end{cases}$$

For $\boldsymbol{\pi} = \mathbf{(2)}$ since $R^{(2)}(\rho) = 1$ for all $\rho \in \mathrm{Sym}_2$ it follows that the submatrix $(U_M^{(2)})^{YX}$ is of the form

$$(U_M^{(2)})^{YX} = \begin{cases} (k-|M|)_{3-|M|} & \text{if } Y \subseteq M_1 \text{ and } X \subseteq M_2 \text{ and } \mu(Y) = X \\ -1 & \text{if } Y \subseteq M_1 \text{ and } X \nsubseteq M_2 \\ 0 & \text{if } Y \nsubseteq M_1 \text{ or } X \subseteq M_2 \text{ and } \mu(Y) \neq X. \end{cases}$$

For example

$$U^{(2)}_{11,22} = \begin{pmatrix} k-2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U^{(2)}_{11,23} = \begin{pmatrix} -1 & k-2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U^{(2)}_{11,22,33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{(2)}_{12,21,33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $\boldsymbol{\pi} = (\mathbf{1^2})$ since $R^{(1^2)}(\epsilon) = 1$ and $R^{(1^2)}(12) = -1$ it follows that the submatrix $(U^{(1^2)}_M)^{YX}$ is of the form

$$(U^{(1^2)}_M)^{YX} = \begin{cases} (k-|M|)_{3-|M|}\,\mathrm{sign}(\rho^{YX}_M) & \text{if } Y \subseteq M_1 \text{ and } X \subseteq M_2 \text{ and } \mu(Y) = X \\ -\mathrm{sign}(\rho^{YX}_M) & \text{if } Y \subseteq M_1 \text{ and } X \not\subseteq M_2 \\ 0 & \text{if } Y \not\subseteq M_1 \text{ or } X \subseteq M_2 \text{ and } \mu(Y) \neq X \end{cases}$$

where $F^{YX}_M = \{\rho^{YX}_M\}$. Then for example

$$U^{(1^2)}_{11,22} = \begin{pmatrix} k-2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U^{(1^2)}_{11,23} = \begin{pmatrix} -1 & k-2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U^{(1^2)}_{11,22,33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{(1^2)}_{12,21,33} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

At level $\boldsymbol{\ell} = \mathbf{3}$ there are three partitions $\boldsymbol{\pi} = (\mathbf{3})$, $\boldsymbol{\pi} = (\mathbf{2,1})$ and $\boldsymbol{\pi} = (\mathbf{1^3})$. The matrix U^{π}_M consists of a single submatrix with $X = Y = \{1,2,3\}$. Then, U^{π}_M is the all zero matrix unless $|M| = 3$. In this case $\mu \in \mathrm{Sym}_3$ and $F^{YX}_M = \{\mu^{-1}\}$. For $\pi = (3)$ it follows that $U^{(3)}_M = 1$. For $\pi = (1^3)$ it follows that $U^{(1^3)}_M = \mathrm{sign}(\mu^{-1})$. For $\pi = (2,1)$ it follows that $U^{(2,1)}_M = R^{(2,1)}(\mu^{-1})$ where the $R^{(2,1)}(\mu^{-1})$ are given in Example 2.7.

Example 3.8: Let $b = d = 3$ and $L = \{(1,1),(2,2),(3,3)\}$ be the identity linking set. We obtain the graphs $B_n(3)$ described in Example 3.1. From Corollary 3.7 it follows that:

$$N_L^\pi = U_\varnothing^\pi - (U_{11}^\pi + U_{22}^\pi + U_{33}^\pi) + U_{11,22}^\pi + U_{11,33}^\pi + U_{22,33}^\pi - U_{11,22,33}^\pi$$

where U_M^π is the all-zero matrix if $\ell > |M|$. Using Example 3.7 we obtain the U_M^π:

At level $\boldsymbol{\ell = 0}$ the 1×1 matrix

$$N_L^{()} = k(k-1)(k-2) - 3(k-1)(k-2) + 3(k-2) - 1$$

has the eigenvalue $k^3 - 6k^2 + 14k - 13$.

At level $\boldsymbol{\ell = 1}$ the 3×3 matrix

$$N_L^{(1)} = \begin{pmatrix} -k^2+5k-7 & k-3 & k-3 \\ k-3 & -k^2+5k-7 & k-3 \\ k-3 & k-3 & -k^2+5k-7 \end{pmatrix}$$

has eigenvalues $-k^2 + 7k - 13$ and $-k^2 + 4k - 4$ (twice).

At level $\boldsymbol{\ell = 2}$ the 3×3 matrices

$$N_L^{(2)} = \begin{pmatrix} k-3 & -1 & -1 \\ -1 & k-3 & -1 \\ -1 & -1 & k-3 \end{pmatrix} \quad \text{and} \quad N_L^{(1^2)} = \begin{pmatrix} k-3 & -1 & 1 \\ -1 & k-3 & -1 \\ 1 & -1 & k-3 \end{pmatrix}$$

have respective eigenvalues $k-5$ and $k-2$ (twice), and $k-1$ and $k-4$ (twice).

At level $\boldsymbol{\ell = 3}$ we have $F_M^{YX} = \{\epsilon\}$ and hence for each of the three partitions $\pi = (3)$, $\pi = (2,1)$ and $\pi = (1^3)$ the matrix N_L^π has the eigenvalue -1 with respective multiplicity 1, 2 and 1. From Theorem 3.14 follows that the chromatic polynomial of $B_n(3)$ is

$$\begin{aligned}
P(B_n(3),k) &= (k^3-6k^2+14k-13)^n \\
&+ (k-1)\left((-k^2+7k-13)^n+2(-k^2+4k-4)^n\right) \\
&+ \frac{1}{2}k(k-3)\left((k-5)^n+2(k-2)^n\right) \\
&+ \frac{1}{2}(k-1)(k-2)\left((k-1)^n+2(k-4)^n\right) \\
&+ \frac{1}{6}k(k-1)(k-5)\,(-1)^n \\
&+ \frac{2}{6}k(k-2)(k-4)\,2(-1)^n \\
&+ \frac{1}{6}(k-1)(k-2)(k-3)\,(-1)^n.
\end{aligned}$$

Compare this to the "structure" of $P(B_n(3),k)$ obtained in Example 3.5. In Figure 3.5 the roots of $B_{30}(3)$ are plotted.

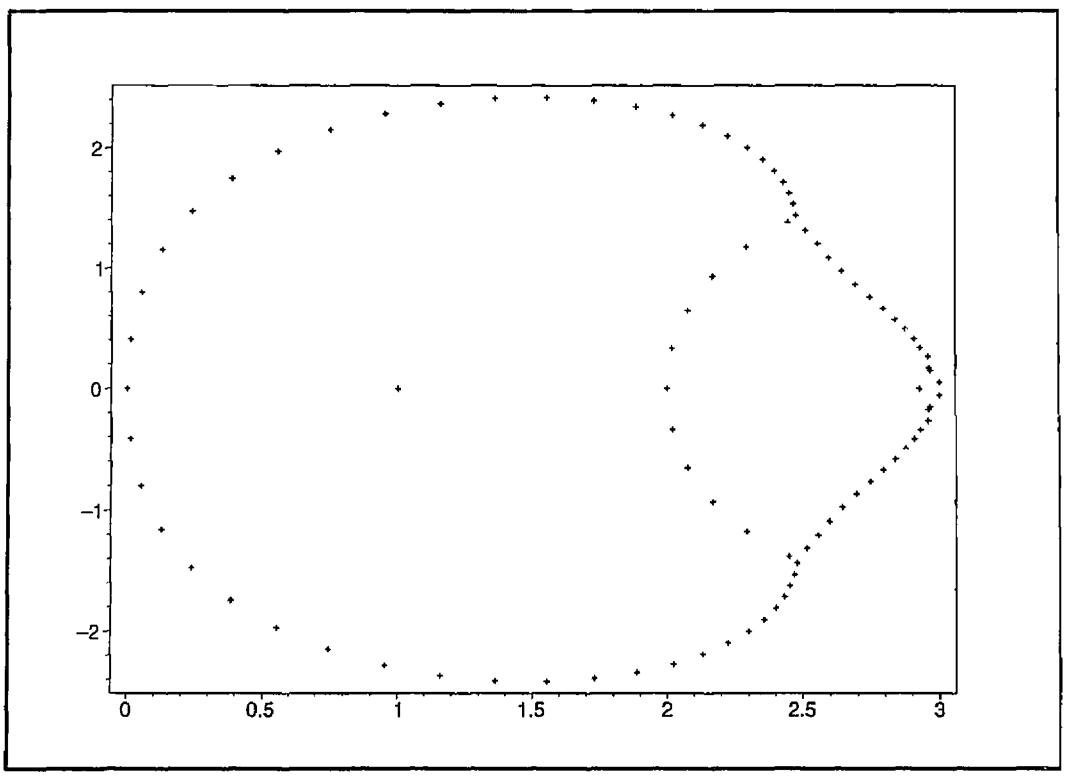


Figure 3.5: The roots of $B_{30}(3)$

Example 3.9: Let $b = d = 3$ and $H = \{12, 13, 21, 23, 31, 32\}$ be the linking set. The resulting graph $H_n(3)$ is a cyclic octahedron obtained in Example 3.2. From

Corollary 3.7 follows that:

$$\begin{aligned}N_H^\pi &= U_\varnothing^\pi - (U_{12}^\pi + U_{13}^\pi + U_{21}^\pi + U_{23}^\pi + U_{31}^\pi + U_{32}^\pi) \\ &+(U_{12,21}^\pi + U_{12,23}^\pi + U_{13,21}^\pi + U_{13,31}^\pi + U_{12,31}^\pi + U_{13,32}^\pi + U_{23,32}^\pi + U_{23,31}^\pi + U_{23,31}^\pi) \\ &-(U_{12,23,31}^\pi + U_{13,21,32}^\pi)\end{aligned}$$

At level $\boldsymbol{\ell} = \mathbf{0}$ the 1×1 matrix

$$N_H^{()} = k(k-1)(k-2) - 6(k-1)(k-2) + 9(k-2) - 2,$$

has the eigenvalue $k^3 - 9k^2 + 29k - 32$.

At level $\boldsymbol{\ell} = \mathbf{1}$ the 3×3 matrix

$$N_H^{(1)} = \begin{pmatrix} 2k-6 & -k^2+7k-13 & -k^2+7k-13 \\ -k^2+7k-13 & 2k-6 & -k^2+7k-13 \\ -k^2+7k-13 & -k^2+7k-13 & 2k-6 \end{pmatrix}$$

has eigenvalues $-2(k-4)^2$ and $k^2 - 5k + 7$ (twice).

At level $\boldsymbol{\ell} = \mathbf{2}$ the 3×3 matrices

$$N_H^{(2)} = \begin{pmatrix} k-4 & k-5 & k-5 \\ k-5 & k-4 & k-5 \\ k-5 & k-5 & k-4 \end{pmatrix} \text{ and } N_H^{(1^2)} = \begin{pmatrix} k-4 & -(k-3) & k-3 \\ -(k-3) & k-4 & -(k-3) \\ k-3 & -(k-3) & k-4 \end{pmatrix}$$

have respective eigenvalues $3k - 14$ and 1 (twice), and $k - 2$ and $-2k - 7$ (twice). At level $\boldsymbol{\ell} = \mathbf{3}$, the matrices $N_L^\pi = -(R^\pi(123) + R^\pi(132))$ with $\pi = (3)$, $\pi = (2,1)$ and $\pi = (1^3)$ are of size 1×1, 2×2 and 1×1 respectively. For $\pi = (3)$ and $\pi = (1^3)$ the eigenvalue is -2. For $\pi = (2,1)$ the eigenvalue is 1 (twice).

The global multiplicities do not depend on the linking set so they are the same as in Example 3.8. From Theorem 3.14 it follows that the chromatic polynomial of

$H_n(3)$ is

$$\begin{aligned} P(H_n(3);k) &= (k^3 - 9k^2 + 29k - 32)^n \\ &\quad + (k-1)\left((-2(k-4)^2)^n + 2(k^2 - 5k + 7)^n\right) \\ &\quad + \frac{1}{2}k(k-3)\left(3k - 14)^n + 2\right) \\ &\quad + \frac{1}{2}(k-1)(k-2)\left((k-2)^n + 2(-2k+7)^n\right) \\ &\quad + \frac{1}{6}k(k-1)(k-5)(-2)^n + \frac{1}{6}(k-1)(k-2)(k-3)(-2)^n \\ &\quad + \frac{1}{3}k(k-2)(k-4)(2). \end{aligned}$$

Example 3.10: Let $b = 3$ and $d = 2$. In this example we shall determine all the matrices U_M^π for all levels $\ell = 0, 1, 2$ and all $\pi \vdash \ell$, and all possible matchings $M \subset V_3 \times V_2$.

There are 1, 6, 6 matchings M with $|M| = 0, 1, 2$ respectively. We use Theorem 3.13 to evaluate the U_M^π. At level $\boldsymbol{\ell = 0}$ and $\boldsymbol{\pi = ()}$ the matrices $U_M^{()}$ are

$$k(k-1), \quad (k-1), \quad 1 \qquad \text{for } |M| = 0, 1, 2 \quad \text{respectively.}$$

At level $\boldsymbol{\ell = 1}$ and $\boldsymbol{\pi = (1)}$ the matrices $U_M^{(1)}$ are of size 3×2. Suppose that M is not the empty matching. Let $\{x\} \subset \{1, 2\}$ and $\{y\} \subset \{1, 2, 3\}$. The set $F_M^{YX} = \mathrm{Sym}_1$ and

$$(U_M^\pi)^{YX} = \begin{cases} (k - |M|)_{2-|M|} & \text{if } y \in M_1 \text{ and } (y, x) \in M; \\ -1 & \text{if } y \in M_1 \text{ and } x \notin M_2; \\ 0 & \text{if } y \notin M_1 \text{ or } x \in M_2 \text{ and } (y, x) \notin M. \end{cases}$$

Then for example

$$U_{11}^{(1)} = \begin{pmatrix} k-1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_{22}^{(1)} = \begin{pmatrix} 0 & 0 \\ -1 & k-1 \\ 0 & 0 \end{pmatrix}, \quad U_{31}^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ k-1 & -1 \end{pmatrix},$$

$$U_{11,22}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad U_{22,31}^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

At level $\ell = 2$ there are two partitions $\pi = (2)$ and $\pi = (1^2)$. In both cases the matrices U_M^π are of size 3×1. Suppose that $|M| \geq 2$. Let $X = \{1, 2\}$ and $Y = \{y_1, y_2\} \subset \{1, 2, 3\}$ with $y_1 < y_2$. The subset F_M^{YX} of Sym_2 is of the form

$$F_M^{YX} = \begin{cases} \{\epsilon\} & \text{if } (y_1, 1) \in M \text{ or } (y_2, 2) \in M \\ \{(12)\} & \text{if } (y_1, 2) \in M \text{ or } (y_2, 1) \in M. \end{cases}$$

Since $R^{(2)}(\rho) = 1$ and $R^{(1^2)}(\rho) = \mathrm{sign}(\rho)$ for all $\rho \in \mathrm{Sym}_2$ it follows that the submatrix $(U_M^{(2)})^{YX}$ is of the form

$$(U_M^\pi)^{YX} = \begin{cases} R^\pi(\rho_M^{YX}) & \text{if } Y \subseteq M_1 \\ 0 & \text{otherwise} \end{cases}$$

where ρ_M^{YX} is the element in F_M^{YX}. Then for example

$$U_{11,22}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U_{22,31}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad U_{11,22}^{(1^2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U_{22,31}^{(1^2)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Example 3.11: Let $b = 2$ and $d = 3$. In this example we shall determine all the matrices U_M^π for all levels $\ell = 0, 1, 2$ and all $\pi \vdash \ell$, and all possible matchings $M \subset V_2 \times V_3$. Although this case is very similar to the previous example we shall repeat all the calculations to avoid difficulties when referring to the results in later examples.

Again, there are 1, 6, 6 matchings M with $|M| = 0, 1, 2$ respectively. We use Theorem 3.13 to evaluate the U_M^π. At level $\ell = 0$ and $\pi = ()$ the matrices $U_M^{()}$ are

$$k(k-1)(k-2), \quad (k-1)(k-2), \quad k-2 \qquad \text{for } |M| = 0, 1, 2 \quad \text{respectively.}$$

At level $\ell = 1$ and $\pi = (1)$ the matrices $U_M^{(1)}$ are of size 2×3. Suppose that M is not the empty matching. Let $\{x\} \subset \{1, 2, 3\}$ and $\{y\} \subset \{1, 2\}$. The set $F_M^{YX} = \mathrm{Sym}_1$ and

$$(U_M^\pi)^{YX} = \begin{cases} (k - |M|)_{3-|M|} & \text{if } y \in M_1 \text{ and } (y, x) \in M; \\ -(k - |M| - 1)_{2-|M|} & \text{if } y \in M_1 \text{ and } x \notin M_2; \\ 0 & \text{if } y \notin M_1 \text{ or } x \in M_2 \text{ and } (y, x) \notin M. \end{cases}$$

Then for example

$$U_{11}^{(1)} = \begin{pmatrix} (k-1)(k-2) & -(k-2) & -(k-2) \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_{22}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ -(k-2) & (k-1)(k-2) & -(k-2) \end{pmatrix},$$

$$U_{31}^{(1)} = \begin{pmatrix} -(k-2) & -(k-2) & (k-1)(k-2) \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_{22,31}^{(1)} = \begin{pmatrix} -1 & 0 & k-2 \\ -1 & k-2 & 0 \end{pmatrix} \text{ and } U_{11,22}^{(1)} = \begin{pmatrix} k-2 & 0 & -1 \\ 0 & k-2 & -1 \end{pmatrix}.$$

At level $\boldsymbol{\ell = 2}$ there are two partitions $\boldsymbol{\pi = (2)}$ and $\boldsymbol{\pi = (1^2)}$. In both cases the matrices U_M^π are of size 1×3. Suppose that $|M| \geq 2$. Let $Y = \{1,2\}$ and $X = \{x_1, x_2\} \subset \{1,2,3\}$ with $x_1 < x_2$. The subset F_M^{YX} of Sym_2 is of the form

$$F_M^{YX} = \begin{cases} \{\epsilon\} & \text{if } (1,x_1) \in M \text{ or } (2,x_2) \in M \\ \{(12)\} & \text{if } (1,x_2) \in M \text{ or } (2,x_1) \in M. \end{cases}$$

Since $R^{(2)}(\rho) = 1$ and $R^{(1^2)}(\rho) = \mathrm{sign}(\rho)$ for all $\rho \in \mathrm{Sym}_2$ it follows that the submatrix $(U_M^{(2)})^{YX}$ is of the form

$$(U_M^\pi)^{YX} = \begin{cases} (k-2)R^\pi(\rho_M^{YX}) & \text{if } X \subseteq M_2 \\ -R^\pi(\rho_M^{YX}) & \text{otherwise} \end{cases}$$

where ρ_M^{YX} is the element in F_M^{YX}. Then for example

$$U_{11,22}^{(2)} = \begin{pmatrix} k-2 & -1 & -1 \end{pmatrix}, \qquad U_{22,31}^{(2)} = \begin{pmatrix} -1 & -1 & k-2 \end{pmatrix},$$

$$U_{11,22}^{(1^2)} = \begin{pmatrix} k-2 & -1 & 1 \end{pmatrix}, \qquad U_{22,31}^{(1^2)} = \begin{pmatrix} -1 & 1 & -(k-2) \end{pmatrix}.$$

Example 3.12: Let B be the path on three vertices and let

$$L = \{(1,1),(2,2),(3,3)\}$$

be the identity linking set. $\mathcal{R} = \{1|2|3\}$ and $\mathcal{P} = \{13|2\}$ are the two colour-partitions of the vertex set of B. As we saw in Example 3.6 the matrix T_L and every matrix N_L^π consist of four sub matrices

$$N_L^\pi = \left(\begin{array}{c|c} N_{L_{\mathcal{R}\mathcal{R}}}^\pi & N_{L_{\mathcal{R}\mathcal{P}}}^\pi \\ \hline N_{L_{\mathcal{P}\mathcal{R}}}^\pi & N_{L_{\mathcal{P}\mathcal{P}}}^\pi \end{array}\right)$$

one for each induced graph

$$_3\langle 11, 22, 33\rangle_3, \qquad _3\langle 11, 22, 31\rangle_2, \qquad _2\langle 11, 22, 13\rangle_3 \quad \text{and} \quad _2\langle 11, 22\rangle_2.$$

Hence, for every N_L^π we have to consider four cases:

(I) The case $(\mathcal{R}, \mathcal{R})$ corresponds to $b = d = 3$ and $L_{\mathcal{R}\mathcal{R}} = \{(1,1), (2,2), (3,3)\}$. This case has been dealt with in Examples 3.7 and 3.8, and all the matrices $N_{L_{\mathcal{R}\mathcal{R}}}^\pi$ have been obtained.

(II) The case $(\mathcal{R}, \mathcal{P})$ corresponds to $b = 3$, $d = 2$ and $L_{\mathcal{R},\mathcal{P}} = \{(1,1), (3,1), (2,2)\}$. From Corollary 3.7 follows that: $N_{L_{\mathcal{R},\mathcal{P}}}^\pi = U_\varnothing^\pi - (U_{11}^\pi + U_{31}^\pi + U_{22}^\pi) + U_{11,22}^\pi + U_{31,22}^\pi$. The U_M^π have been obtained in Example 3.10.

(III) The case $(\mathcal{P}, \mathcal{R})$ corresponds to $b = 2$, $d = 3$ and $L_{\mathcal{P},\mathcal{R}} = \{(1,1), (1,3), (2,2)\}$. From Corollary 3.7 follows that: $N_{L_{\mathcal{P},\mathcal{R}}}^\pi = U_\varnothing^\pi - (U_{11}^\pi + U_{13}^\pi + U_{22}^\pi) + U_{11,22}^\pi + U_{13,22}^\pi$. The U_M^π have been obtained in Example 3.11.

(IV) The case $(\mathcal{P}, \mathcal{P})$ corresponds to $b = 2$ and $d = 2$ with $L = \{(1,1), (2,2)\}$. This case is very similar to the one described in Example 3.7 and it can be easily checked that:

$$N_{L_{\mathcal{P}\mathcal{P}}}^{()} = k^2 - 3k + 3, \qquad N_{L_{\mathcal{P}\mathcal{P}}}^{(1)} = \begin{pmatrix} -(k-2) & 1 \\ 1 & -(k-2) \end{pmatrix},$$
$$N_{L_{\mathcal{P}\mathcal{P}}}^{(2)} = 1, \qquad \text{and} \qquad N_{L_{\mathcal{P}\mathcal{P}}}^{(1^2)} = 1.$$

It follows that:

$$N_L^{()} = \begin{pmatrix} k^3 - 6k^2 + 14k - 13 & k^2 - 4k + 5 \\ (k-2)(k^2 - 4k + 5) & k^2 - 3k + 3 \end{pmatrix},$$

with characteristic equation

$$\lambda^2 + (-k^3 + 5k^2 - 11k + 10)\lambda + k^4 - 7k^3 + 19k^2 - 24k + 11 = 0.$$

$$N_L^{(1)} = \begin{pmatrix} -k^2+5k-7 & k-3 & k-3 & -k+2 & 1 \\ k-3 & -k^2+5k-7 & k-3 & 1 & -k+3 \\ k-3 & k-3 & -k^2+5k-7 & -k+2 & 1 \\ -k^2+5k-7 & 2k-4 & -k^2+5k-7 & -k+2 & 1 \\ k-3 & -k^2+5k-6 & k-3 & 1 & -k+2 \end{pmatrix},$$

with characteristic equation

$$(1+\lambda)(\lambda + 4 - 4k + k^2)\big(\lambda^3 + a_2(k)\lambda^2 + a_1(k)\lambda + a_0(k)\big) = 0$$

where $a_2(k) = 2k^2 - 9k + 12$, $a_1(k) = k^4 - 10k^3 + 36k^2 - 56k + 31$ and

$$a_0(k) = -k^5 + 10k^4 - 38k^3 + 69k^2 - 62k + 22.$$

$$N_L^{(2)} = \begin{pmatrix} k-3 & -1 & -1 & 1 \\ -1 & k-3 & -1 & 0 \\ -1 & -1 & k-3 & 1 \\ k-3 & -2 & k-3 & 1 \end{pmatrix} \text{ and } N_L^{(1^2)} = \begin{pmatrix} k-3 & -1 & 1 & 1 \\ -1 & k-3 & -1 & 0 \\ 1 & -1 & k-3 & -1 \\ k-3 & 0 & -k+3 & 1 \end{pmatrix}.$$

both have the same characteristic equation

$$(1+\lambda)(k-1-\lambda)(k-2-\lambda)(k-4-\lambda) = 0.$$

At level $\ell = 3$ the matrix N_L^π is equal to $N_{L_{\mathcal{RR}}}^\pi$ for $\pi = (3)$, $\pi = (2,1)$ and $\pi = (1^3)$ respectively.

In principale, from here it easy to obtain the chromatic polynomial of $L_n(B)$, but only that some of the eigenvalues of $N_L^{()}$ are not polynomials in k. Although, in this case an explicit expression for this eigenvalues exists it is more convenient to use the so called "Newton's formula", described in Appendix A, to obtain the sum of their n^{th} powers recursively in n.

Then from Theorem 3.14 it follows that the chromatic polynomial of $L_n(B)$ can be written as

$$\begin{aligned} P(L_n(B);z) &= A_n(z) \\ &+ (z-1)\Big(B_n(z) + (-z^2+4z-4)^n + (-1)^n\Big) \\ &+ (z^2-3z+1)\Big((-1)^n + (z-1)^n + (z-2)^n + (z-4)^n\Big) \\ &+ (z^3-6z^2+8z-1)(-1)^n \end{aligned}$$

where $A_n(z)$ is the sum of the n^{th} power of the roots of

$$\lambda^2 + (-z^3+5z^2-11z+10)\lambda + z^4 - 7z^3 + 19z^2 - 24z + 11 = 0$$

and $B_n(z)$ is the sum of the n^{th} power of the roots of

$$\lambda^3 + a_2(z)\lambda^2 + a_1(z)\lambda - a_0(z) = 0$$

where

$$\begin{aligned} a_2(z) &= (2z^2 - 9z + 12), \\ a_1(z) &= (z^4 - 10z^3 + 36z^2 - 56z + 31) \quad \text{and} \\ a_0(z) &= z^5 + 10z^4 - 38z^3 + 69z^2 - 62z + 22 \end{aligned}$$

In Figure 3.6 the roots of $L_{30}(B)$ are plotted (the complex variable has been shifted to $z = c+2$). Clearly visible are two roots (and their conjugates) on the left of the line $c = -2$ (dotted). It follows that $L_n(B)$ for certain n has roots with negative real part.

3.9 Summary

Let b and d be integers, and let $L \subseteq V_b \times V_d$ be a linking set. The compatibility matrix T_L corresponding to the graph ${}_b\langle L\rangle_d$ is equivalent to

$$\bigoplus_{\substack{0\le \ell\le \min(b,d)\\ \pi\vdash \ell}} (I_\pi \otimes N_L^\pi),$$

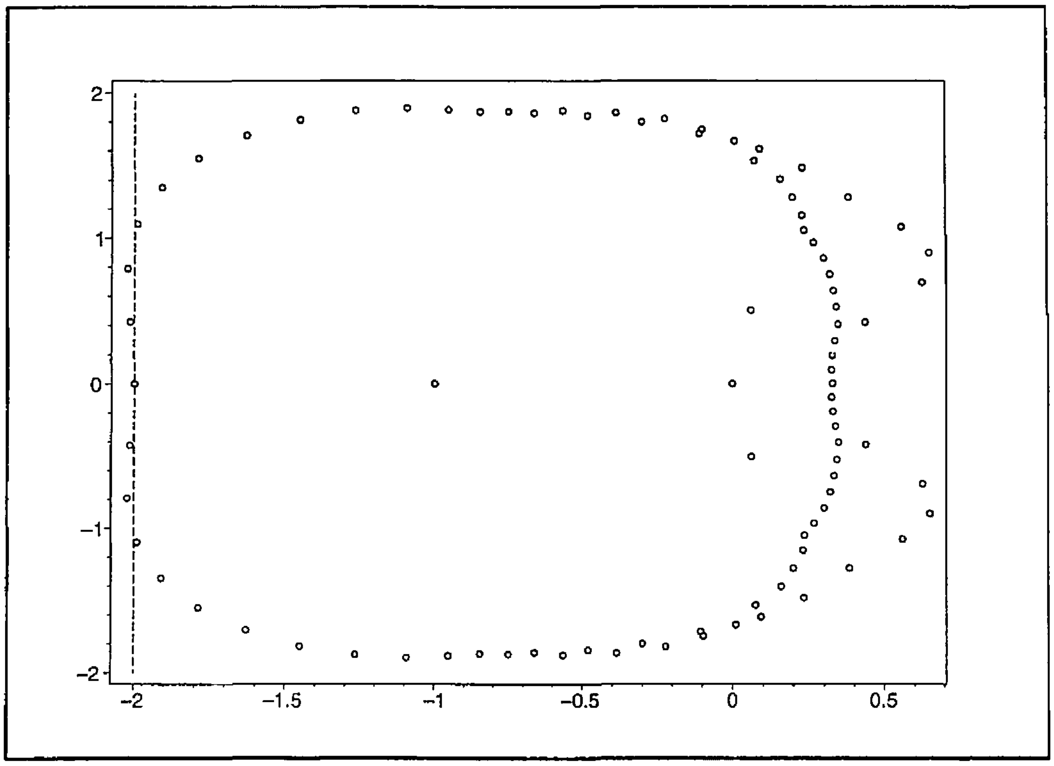


Figure 3.6: The roots of $L_{30}(B)$ where B is the path of length three and L is the identity linking set

where I_π is the identity matrix of size 1 if $\ell = 0$ and

$$\frac{n_\pi}{\ell!} \prod_{i=1}^{\ell} (k - h_i(\pi)) \quad \text{with} \quad h_i(\pi) = \pi_i + \ell - i \quad \text{if} \quad \ell > 0.$$

From Corollary 3.7 it follows that N_L^π can be written as an alternating sum of matrices

$$N_L^\pi = \sum_{M \in \mathcal{M}(b,d,L)} (-1)^{|M|} \; U_M^\pi.$$

If $|M| < \ell$ then U_M^π is the all zero matrix. For $|M| \geq \ell$ each matrix consists of $\binom{b}{\ell}\binom{d}{\ell}$ sub matrices $(U_M^\pi)^{YX}$ one for each pair (Y, X) where Y is a subset of V_b, X is a subset of V_d and both are of size ℓ. Each of the $(U_M^\pi)^{YX}$ is of the form (Theorem 3.13)

$$(U_M^\pi)^{YX} = \begin{cases} C_M(X) \sum_{\rho \in F_M^{YX}} R^\pi(\rho) & \text{if } \mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1 \\ O & \text{otherwise.} \end{cases}$$

where $\quad C_M(X) = (-1)^{\ell-|X\cap M_2|} f_{|X\cup M_2|}(d,k),$

$$F_M^{YX} = \{\rho \in \mathrm{Sym}_\ell \mid (y_i, x_j) \in (Y \times X) \cap M \Rightarrow i = \rho(j)\}$$

assuming that $x_1 < x_2 < \ldots < x_\ell$ and $y_1 < y_2 < \ldots < y_\ell$, and $R^\pi(\rho)$ is Young's natural representation corresponding to $\mathcal{S}^\pi$ and O is the all zero matrix of size $n_\pi \times n_\pi$.

Now, let B be any base graph with vertex set V and L any linking set. The corresponding compatibility matrix T_L is equivalent to

$$\bigoplus_{\substack{0\le \ell \le \min(b,d)\\ \pi\vdash \ell}} (I_\pi \otimes N_L^\pi),$$

where, as before, I_π is the identity matrix of size 1 if $\ell = 0$ and

$$\frac{n_\pi}{\ell!} \prod_{i=1}^{\ell} (k - h_i(\pi)) \quad \text{with} \quad h_i(\pi) = \pi_i + \ell - i \quad \text{if} \quad \ell > 0.$$

From Lemma 3.5 it follows that each of the matrices N_L^π consists of $\Pi(B)^2$ submatrices $N_{L_{\mathcal{RP}}}^\pi$ one for each pair $(\mathcal{R}, \mathcal{P}) \in \Pi(B) \times \Pi(B)$. The matrix $N_{L_{\mathcal{RP}}}^\pi$ is of size $\binom{|\mathcal{R}|}{\ell} n_\pi \times \binom{|\mathcal{P}|}{\ell} n_\pi$ and is equivalent to the matrix N_L^π, described above, corresponding to the graph ${}_{|\mathcal{R}|}\langle L_{\mathcal{RP}}\rangle_{|\mathcal{P}|}$, where $L_{\mathcal{RP}} \subseteq V_b \times V_d$ is defined by

$(i,j) \in L_{\mathcal{RP}} \quad$ implies that there exists $(v,w) \in L$ such that $v \in R_i$ and $w \in P_j$.

If $|\mathcal{R}| < \ell$ then $\binom{|\mathcal{R}|}{\ell} = 0$, and similarly $\binom{|\mathcal{P}|}{\ell} = 0$ if $|\mathcal{P}| < \ell$.

If

$$\lambda_1(L,\pi;k), \lambda_2(L,\pi;k), \ldots, \lambda_s(L,\pi;k)$$

are the eigenvalues of N_L^π and

$$m_1(L,\pi), m_2(L,\pi), \ldots, m_s(L,\pi)$$

the corresponding local multiplicities in N_L^π it follows that

$$\mathrm{tr}(N_L^\pi)^n = \sum_{i=1}^{s} \quad m_i(L,\pi) \ \ \lambda_i^n(L,\pi;k)$$

And (Theorem 3.14) the chromatic polynomial of $L_n(B)$ in k is

$$P(L_n(B), k) = \sum_{\ell=0}^{|V|} \sum_{\pi \vdash \ell} \eta_\pi(k) \ \mathrm{tr}(N_L^\pi)^n,$$

where $\eta_\pi(k) = 1$ if $\ell = 0$ and

$$\eta_\pi(k) = \frac{n_\pi}{\ell!} \prod_{i=1}^{\ell} (k - h_i(\pi)) \quad \text{with} \quad h_i(\pi) = \pi_i + \ell - i \quad \text{if} \quad \ell > 0.$$

Chapter 4

Explicit calculations of chromatic polynomials

In this chapter the theory developed in the previous chapters will be used to calculate the chromatic polynomials for various families of graphs. In particular we calculate the chromatic polynomials for the generalized dodecahedra described in Example 3.4, and four other families of cubic graphs.

In the following we assume that:

- We order the subsets X and X' of a vertex set $V \subset \mathbb{N}$ according to the dictionary ordering, that is according to their smallest non-common elements. For example we order the four subsets of size three of V_4 as follows:

$$\{1,2,3\} \qquad \{1,2,4\} \qquad \{1,3,4\} \qquad \{2,3,4\}.$$

  This fixes the order of the rows and columns in the submatrices N_L^π and U_M^π.

- We also use the dictionary odering for the independent sets of a colour-partition.

- In the following figures, the edges of the base graphs are represented by thick lines and the linking edges are represented by thin lines.

4.1 A catalogue of U_M^π

Recall, for given integers b and d, and any linking set $L \subseteq V_b \times V_d$ the graph ${}_b\langle L\rangle_d$ consists of K_b and K_d with extra edges according to L. The corresponding compatibility matrix T_L is equivalent to

$$\bigoplus_{\substack{0\leq \ell \leq \min(b,d)\\ \pi\vdash\ell}} (I_\pi \otimes N_L^\pi),$$

where I_π is the identity matrix of size 1 if $\ell = 0$ and

$$\frac{n_\pi}{\ell!}\prod_{i=1}^{\ell}(k - h_i(\pi)) \quad \text{with} \quad h_i(\pi) = \pi_i + \ell - i \quad \text{if} \quad \ell > 0.$$

From Corollary 3.7 it follows that N_L^π can be written as an alternating sum of matrices

$$N_L^\pi = \sum_{M\in\mathcal{M}(b,d,L)} (-1)^{|M|} \; U_M^\pi.$$

If $|M| < \ell$ then U_M^π is the all zero matrix. For $|M| \geq \ell$ each matrix consists of $\binom{b}{\ell}\binom{d}{\ell}$ sub matrices $(U_M^\pi)^{YX}$ one for each pair (Y, X) where Y is a subset of V_b, X is a subset of V_d and both are of size ℓ. Each of the $(U_M^\pi)^{YX}$ is of the form (Theorem 3.13)

$$(U_M^\pi)^{YX} = \begin{cases} C_M(X) \sum\limits_{\rho\in F_M^{YX}} R^\pi(\rho) & \text{if } \mu^{-1}(X\cap M_2) \subseteq Y \subseteq M_1 \\ O & \text{otherwise.}\end{cases}$$

where $\quad C_M(X) = (-1)^{\ell - |X\cap M_2|} f_{|X\cup M_2|}(d,k),$

$$F_M^{YX} = \{\rho \in \mathrm{Sym}_\ell \mid (y_i, x_j) \in (Y\times X)\cap M \Rightarrow i = \rho(j)\}$$

assuming that $x_1 < x_2 < \ldots < x_\ell$ and $y_1 < y_2 < \ldots < y_\ell$, and $R^\pi(\rho)$ is Young's natural representation corresponding to $\mathcal{S}^\pi$ and O is the all zero matrix of size $n_\pi \times n_\pi$.

In this section we describe all the U_M^π for the cases where $\min(b,d) = 3$. Recall that the case $b = d = 3$ has been done in Example 3.7 in the previous chapter. There are 1, $3m$, $6m(m-1)$, $m(m-1)(m-2)$ matchings $M \subset V_b \times V_d$ of size 0, 1, 2, 3 respectively where $m = \max(b,d)$. We consider two cases:

4.1.1 The case $b \geq 3$ and $d = 3$

At level $\ell = 0$ the 1×1 matrices $U_M^{()}$ are

$$k(k-1)(k-2), \qquad (k-1)(k-2), \qquad (k-2) \quad \text{and} \quad 1$$

for the matchings of size 0, 1, 2 and 3 respectively.

At level $\ell = 1$ the matrices $U_M^{(1)}$ are of size $b \times 3$. Assume that $|M| \geq 1$. Let $X \subset \{1,2,3\}$ and $Y \subset \{1,2,\ldots,b\}$ both be of size one. Then $F_M^{YX} = \mathrm{Sym}_1$ and the submatrix $(U_M^\pi)^{YX}$ is

$$(U_M)^{YX} = \begin{cases} (k-|M|)_{3-|M|} & \text{if } y \in M_1 \text{ and } (y,x) \in M \\ -(k-|M|-1))_{3-|M|-1} & \text{if } y \in M_1 \text{ and } x \notin M_2 \\ 0 & \text{if } y \notin M_1 \text{ or } x \in M_2 \text{ but } (y,x) \notin M. \end{cases}$$

For example for $b = 4$ and $d = 3$ the matrices $U_M^{(1)}$ are of size 4×3. Let $f_2 = (k-1)(k-2)$ and $f_1 = (k-2)$ then $U_{11}^{(1)}$, $U_{11,22}^{(1)}$, $U_{11,43}^{(1)}$ and $U_{11,22,33}^{(1)}$ are respectively

$$\begin{pmatrix} f_2 & -f_1 & -f_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} f_1 & 0 & -1 \\ 0 & f_1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} f_1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & f_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

At level $\ell = 2$ there are two partitions $\pi = (2)$ and $\pi = (1^2)$. In both cases the matrices U_M^π are of size $\binom{b}{2} \times 3$. Let $X = \{x_1, x_2\} \subset \{1,2,3\}$ with $x_1 < x_2$, and $Y = \{y_1, y_2\} \subset \{1,2,\ldots,b\}$ with $y_1 < y_2$. If $|M| \geq 2$ and $Y \subseteq M_1$ then $F_M^{YX} \subset \mathrm{Sym}_2$ contains one element:

$$\rho_M^{YX} = \begin{cases} \epsilon & \text{if } (y_1,x_1) \in M \text{ or } (y_2,x_2) \in M; \\ (12) & \text{if } (y_1,x_2) \in M \text{ or } (y_2,x_1) \in M. \end{cases}$$

The submatrix $(U_M^\pi)^{YX}$ is

$$(U_M^\pi)^{YX} = \begin{cases} (k-|M|)_{3-|M|} R^\pi(\rho_M^{YX}) & \text{if } Y \subseteq M_1 \text{ and } X \subseteq M_2 \text{ and } \mu(Y) = X; \\ -R^\pi(\rho_M^{YX}) & \text{if } Y \subseteq M_1 \text{ and } X \not\subseteq M_2; \\ 0 & \text{if } Y \not\subseteq M_1 \text{ or } X \subseteq M_2 \text{ but } \mu(Y) \neq X. \end{cases}$$

For example for $b = 4$ and $d = 3$ the matrices U^{π}_{M} are of size 6×3. Let $f_1 = (k-2)$; then $U^{\pi}_{11,22}$, $U^{\pi}_{11,43}$, $U^{\pi}_{11,22,33}$ and $U^{\pi}_{12,12,43}$ are respectively

$$\begin{pmatrix} f_1 R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(1\,2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -R^{\pi}(\epsilon) & f_1 R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} R^{\pi}(\epsilon) & 0 & 0 \\ 0 & R^{\pi}(\epsilon) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R^{\pi}(\epsilon) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R^{\pi}(1\,2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R^{\pi}(\epsilon) \\ 0 & 0 & 0 \\ 0 & R^{\pi}(\epsilon) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At level $\boldsymbol{\ell = 3}$ there are three partitions $\pi = \mathbf{(3)}$, $\pi = \mathbf{(2,1)}$ and $\pi = \mathbf{(1^3)}$. The matrices U^{π}_{M} are of size $\binom{b}{3} n_{\pi} \times n_{\pi}$ where n_{π} is equal to 1, 2 and 1 respectively. Let $X = \{1,2,3\}$ and $Y \subset \{1,2,\ldots,b\}$. The matrix is non zero only if $|M| = 3$ and $Y = M_1$. Let $\rho \in \mathrm{Sym}_3$ be such that $\mu(y_i) = x_{\rho^{-1}(i)}$ for $i = 1,2,3$. Then ρ is the only element ρ^{YX}_{M} in F^{YX}_{M}, so

$$(U^{\pi}_{M})^{YX} = \begin{cases} R^{\pi}(\rho^{YX}_{M}) & \text{if } Y = M_1 \\ O & \text{otherwise.} \end{cases}$$

If $b = 4$ and $d = 3$ the matrices U^{π}_{M} are of size $4n_{\pi} \times n_{\pi}$. For example

$$U^{\pi}_{12,21,43} = \begin{pmatrix} O \\ R^{\pi}(12) \\ O \\ O \end{pmatrix}, \qquad U^{\pi}_{12,23,41} = \begin{pmatrix} O \\ R^{\pi}(132) \\ O \\ O \end{pmatrix}.$$

4.1.2 The case $b = 3$ and $d \geq 3$

At level $\ell = 0$ the matrices $U_M^{()}$ are of size 1×1. For every matching $M \subset V_3 \times V_d$ it follows that $U_M^{()} = f_{|M|}(d, k)$. That is $U_M^{()}$ is equal to

$$(k)_d, \qquad (k-1)_{d-1}, \qquad (k-2)_{d-2} \quad \text{and} \quad (k-3)_{d-3}$$

for the matchings of size 0, 1, 2 and 3 respectively. For example for $b = 3$ and $d = 4$ the matrices $U_M^{()}$ are

$$k(k-1)(k-2)(k-3), \quad (k-1)(k-2)(k-3), \quad (k-2)(k-3), \quad (k-3)$$

for $|M| = 0,\ 1,\ 2,\ 3$ respectively.

At level $\ell = 1$ the matrices $U_M^{(1)}$ are of size $3 \times d$. Assume that $|M| \geq 1$. Let $Y \subset \{1,2,3\}$ and $X \subset \{1,2,\ldots,d\}$ both be of size one. Then $F_M^{YX} = \mathrm{Sym}_1$ and

$$(U_M)^{YX} = \begin{cases} (k-|M|)_{d-|M|} & \text{if } y \in M_1 \text{ and } (y,x) \in M \\ -(k-|M|-1))_{d-|M|-1} & \text{if } y \in M_1 \text{ and } x \notin M_2 \\ 0 & \text{if } y \notin M_1 \text{ or } x \in M_2 \text{ but } (y,x) \notin M. \end{cases}$$

For example for $b = 3$ and $d = 4$ the matrices $U_M^{(1)}$ are of size 3×4. Let

$$f_3 = (k-1)(k-2)(k-3), \qquad f_2 = (k-2)(k-3) \quad \text{and} \quad f_1 = (k-3);$$

then

$$U_{11}^{(1)} = \begin{pmatrix} f_3 & -f_2 & -f_2 & -f_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad U_{11,22}^{(1)} = \begin{pmatrix} f_2 & 0 & -f_1 & -f_1 \\ 0 & f_2 & -f_1 & -f_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$U_{11,34}^{(1)} = \begin{pmatrix} f_2 & -f_1 & -f_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -f_1 & -f_1 & f_2 \end{pmatrix} \quad \text{and} \quad U_{11,22,33}^{(1)} = \begin{pmatrix} f_1 & 0 & 0 & -1 \\ 0 & f_1 & 0 & -1 \\ 0 & 0 & f_1 & -1 \end{pmatrix}.$$

At level $\ell = 2$ there are two partitions $\pi = (2)$ and $\pi = (1^2)$. In both cases the matrices U_M^{π} are of size $3 \times \binom{d}{2}$. Assume that $|M| \geq 2$. Let $X \subset \{1,2,\ldots,d\}$ and

$Y \subset \{1,2,3\}$ both be of size two. The submatrix $(U_M^\pi)^{YX}$ is of the form

$$(U_M^\pi)^{YX} = \begin{cases} C_M(X) \sum_{\rho \in F_M^{YX}} R^\pi(\rho) & \text{if } \mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1; \\ 0 & \text{otherwise,} \end{cases}$$

where

$$C_M(X) = (-1)^{\ell - |X \cap M_2|} \; (k - |X \cup M_2|)_{d-|X \cup M_2|},$$

and $R^{(2)}(\rho) = 1$ and $R^{(1^2)}(\rho) = \operatorname{sign}(\rho)$ for all $\rho \in \mathrm{Sym}_2$. For example for $b = 3$ and $d = 4$ the matrices U_M^π are of size 3×6. Let $f_2 = (k-2)(k-3)$ and $f_1 = (k-3)$; then $U_{11,22}^\pi$, $U_{11,34}^\pi$ and $U_{11,22,33}^\pi$ are respectively

$$\begin{pmatrix} f_2R^\pi(\epsilon) & -f_1R^\pi(\epsilon) & -f_1R^\pi(\epsilon) & -f_1R^\pi(1\,2) & -f_1R^\pi(1\,2) & R^\pi(\epsilon)+R^\pi(1\,2) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -f_1R^\pi(\epsilon) & -f_1R^\pi(\epsilon) & f_2R^\pi(\epsilon) & R^\pi(\epsilon)+R^\pi(1\,2) & -f_1R^\pi(\epsilon) & -f_1R^\pi(\epsilon) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} f_1R^\pi(\epsilon) & 0 & -R^\pi(\epsilon) & 0 & -R^\pi(1\,2) & 0 \\ 0 & f_1R^\pi(\epsilon) & -R^\pi(\epsilon) & 0 & 0 & -R^\pi(1\,2) \\ 0 & 0 & 0 & f_1R^\pi(\epsilon) & -R^\pi(\epsilon) & -R^\pi(1\,2) \end{pmatrix}.$$

At level $\boldsymbol{\ell = 3}$ there are three partitions $\boldsymbol{\pi = (3)}$, $\boldsymbol{\pi = (2,1)}$ and $\boldsymbol{\pi = (1^3)}$. The matrices U_M^π are of size $n_\pi \times \binom{d}{3} n_\pi$ where n_π is equal to 1, 2 and 1 respectively. Let $X \subset \{1,2,\ldots,d\}$ and $Y = \{1,2,3\}$. The submatrices $(U_M^\pi)^{YX}$ is non zero only if $|M| = 3$ and then

$$(U_M^\pi)^{YX} = (-1)^{\ell - |X \cap M_2|} \; (k - |X \cup M_2|)_{d-|X \cup M_2|} \sum_{\rho \in F_M^{YX}} R^\pi(\rho).$$

If $b = 3$ and $d = 4$ the matrices U_M^π are of size $n_\pi \times 4n_\pi$. For example

$$U_{11,22,33} = \Big((k-3)R^\pi(\epsilon) \;\; -R^\pi(\epsilon) \;\; -R^\pi(23) \;\; -(R^\pi(123) \Big).$$

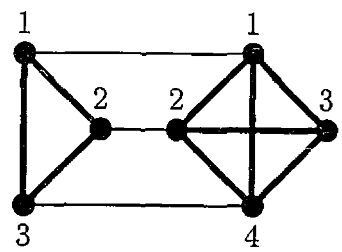


Figure 4.1: The graph ${}_3\langle 11, 22, 34\rangle_4$

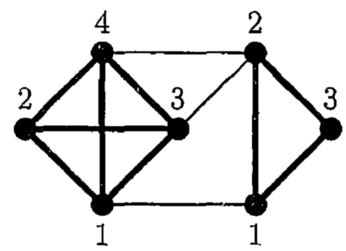


Figure 4.2: The graph ${}_4\langle 11, 32, 42\rangle_3$

4.2 Two Examples

Example 4.1: Let us find the matrices U_M^π for all levels and all matchings for the graph ${}_3\langle 11, 22, 34\rangle_4$ shown in Figure 4.1. There are 1, 3, 3 and 1 matchings of size 0, 1, 2 and 3 respectively:

$$\begin{array}{lll} \{\ \}, & & \\ \{(1,1)\}, & \{(2,2)\}, & \{(3,4)\}, \\ \{(1,1),(2,2)\}, & \{(1,1),(3,4)\}, & \{(2,2),(3,4)\}, \\ \{(1,1),(2,2),(3,4)\}. & & \end{array}$$

From Corollary 3.7 it follows that

$$N_L^\pi = U_\varnothing^\pi - (U_{11}^\pi + U_{22}^\pi + U_{34}^\pi) + (U_{11,22}^\pi + U_{11,34}^\pi + U_{22,34}^\pi) - U_{11,22,34}^\pi.$$

At level $\boldsymbol{\ell = 0}$ the matrices $U_M^{()}$ are

$$k(k-1)(k-2)(k-3), \quad (k-1)(k-2)(k-3), \quad (k-2)(k-3), \quad (k-3)$$

for the matchings of size 0, 1, 2 respectively. Thus

$$N_L^{()} = k(k-1)(k-2)(k-3) - 3(k-1)(k-2)(k-3) + 3(k-2)(k-3) - (k-3).$$

Let $\quad f_3 = (k-1)(k-2)(k-3), \quad f_2 = (k-2)(k-3) \quad$ and $\quad f_1 = (k-3).$

At level $\boldsymbol{\ell = 1}$ the matrices $U_M^{(1)}$ are

$$U_{11}^{(1)} = \begin{pmatrix} f_3 & -f_2 & -f_2 & -f_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad U_{22}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -f_2 & f_3 & -f_2 & -f_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$U_{34}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -f_2 & -f_2 & -f_2 & f_3 \end{pmatrix}, \qquad U_{11,22}^{(1)} = \begin{pmatrix} f_2 & 0 & -f_1 & -f_1 \\ 0 & f_2 & -f_1 & -f_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$U_{11,34}^{(1)} = \begin{pmatrix} f_2 & -f_1 & -f_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -f_1 & -f_1 & f_2 \end{pmatrix}, \qquad U_{22,34}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -f_1 & f_2 & -f_1 & 0 \\ -f_1 & 0 & -f_1 & f_2 \end{pmatrix},$$

and $$U_{11,22,34}^{(1)} = \begin{pmatrix} f_1 & 0 & -1 & 0 \\ 0 & f_1 & -1 & 0 \\ 0 & 0 & -1 & f_1 \end{pmatrix}.$$

Then

$$N_L^{(1)} = \begin{pmatrix} -f_3+2f_2-f_1 & f_2-f_1 & f_2-2f_1+1 & f_2-f_1 \\ f_2-f_1 & -f_3+2f_2-f_1 & f_2-2f_1+1 & f_2-f_1 \\ f_2-f_1 & f_2-f_1 & f_2-2f_1+1 & -f_3+2f_2-f_1 \end{pmatrix}.$$

At level $\ell = 2$ the matrices $U_{11,22}^{\pi}$, $U_{11,34}^{\pi}$, $U_{22,34}^{\pi}$ and $U_{11,22,34}^{\pi}$ are respectively

$$\begin{pmatrix} f_2R^{\pi}(\epsilon) & -f_1R^{\pi}(\epsilon) & -f_1R^{\pi}(\epsilon) & -f_1R^{\pi}(1\,2) & -f_1R^{\pi}(1\,2) & \left(R^{\pi}(1\,2)+R^{\pi}(\epsilon)\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -f_1R^{\pi}(\epsilon) & -f_1R^{\pi}(\epsilon) & f_2R^{\pi}(\epsilon) & \left(R^{\pi}(1\,2)+R^{\pi}(\epsilon)\right) & -f_1R^{\pi}(\epsilon) & -f_1R^{\pi}(\epsilon) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -f_1R^{\pi}(1\,2) & \left(R^{\pi}(1\,2)+R^{\pi}(\epsilon)\right) & -f_1R^{\pi}(\epsilon) & -f_1R^{\pi}(\epsilon) & f_2R^{\pi}(\epsilon) & -f_1R^{\pi}(\epsilon) \end{pmatrix},$$

and $$\begin{pmatrix} f_1R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & 0 & -R^{\pi}(1\,2) & 0 & 0 \\ 0 & -R^{\pi}(\epsilon) & f_1R^{\pi}(\epsilon) & 0 & 0 & -R^{\pi}(\epsilon) \\ 0 & 0 & 0 & -R^{\pi}(\epsilon) & f_1R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(2)}(1\,2) = R^{(1^2)}(\epsilon) = 1$ and $R^{(1^2)}(1\,2) = -1$. Then

$$N_L^{(2)} = \begin{pmatrix} f_2 - f_1 & -f_1+1 & -f_1 & -f_1+1 & -f_1 & 2 \\ -f_1 & -f_1+1 & f_2-f_1 & 2 & -f_1 & -f_1+1 \\ -f_1 & 2 & -f_1 & -f_1+1 & f_2-f_1 & -f_1+1 \end{pmatrix} \quad \text{and}$$

$$N_L^{(1^2)} = \begin{pmatrix} f_2 - f_1 & -f_1+1 & -f_1 & f_1-1 & f_1 & 0 \\ -f_1 & -f_1+1 & f_2-f_1 & 0 & -f_1 & -f_1+1 \\ f_1 & 0 & -f_1 & -f_1+1 & +f_2-f_1 & -f_1+1 \end{pmatrix}.$$

At level $\ell = 3$ the matrix U_M^π is

$$N_L^\pi = U_{11,22,34}^\pi = \begin{pmatrix} -R^\pi(\epsilon) & f_1 R^\pi(\epsilon) & -R^\pi(\epsilon) & -R^\pi(1\,2) \end{pmatrix},$$

where $R^{(3)}(\omega) = 1$, $R^{(1^3)}(\omega) = sign(\omega)$ and $R^{(2,1)}(\omega)$ is Young's natural representation corresponding to $\mathcal{S}^{(2,1)}$ (See Example 2.7). In particular:

$$R^{(2,1)}(1\,2) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Example 4.2: Let us find the matrices U_M^π for all levels and all matchings for the graph ${}_4\langle 11, 32, 42\rangle_3$ shown in Figure 4.2. There are 1, 3 and 2 matchings of size 0, 1 and 2 respectively:

$$\begin{array}{lll} \{\ \}, & & \\ \{(1,1)\}, & \{(3,2)\}, & \{(4,2)\}, \\ \{(1,1),(3,2)\}, & \{(1,1),(4,2)\}. & \end{array}$$

From Corollary 3.7 it follows that

$$N^\pi = U_\varnothing^\pi - (U_{11}^\pi + U_{32}^\pi + U_{42}^\pi) + U_{11,32}^\pi + U_{11,42}^\pi.$$

At level $\ell = 0$ the matrices $U_M^{()}$ are

$$k(k-1)(k-2), \qquad (k-1)(k-2) \quad \text{and} \quad (k-2)$$

for the matchings of size 0, 1 and 2 respectively. Thus

$$N_L^{()} = k(k-1)(k-2) - 3(k-1)(k-2) + 2(k-2).$$

Let $f_2 = (k-1)(k-2)$ and $f_1 = (k-2)$. At level $\boldsymbol{\ell} = \mathbf{1}$ the matrices $U_M^{(1)}$ are

$$U_{11}^{(1)} = \begin{pmatrix} f_2 & -f_1 & -f_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, U_{32}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_1 & f_2 & -f_1 \\ 0 & 0 & 0 \end{pmatrix}, U_{42}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f_1 & f_2 & -f_1 \end{pmatrix},$$

$$U_{11,32}^{(1)} = \begin{pmatrix} f_1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & f_1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_{11,42}^{(1)} = \begin{pmatrix} f_1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f_1 & -1 \end{pmatrix}.$$

Then

$$N_L^{(1)} = \begin{pmatrix} -f_2 + 2f_1 & f_1 & f_1 - 2 \\ 0 & 0 & 0 \\ f_1 & -f_2 + f_1 & f_1 - 1 \\ f_1 & -f_2 + f_1 & f_1 - 1 \end{pmatrix}$$

At level $\boldsymbol{\ell} = \mathbf{2}$ the matrices $U_{11,32}^{\pi}$ and $U_{11,42}^{\pi}$ are respectively

$$\begin{pmatrix} 0 & 0 & 0 \\ f_1 R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(1\,2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f_1 R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(1\,2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(2)}(1\,2) = R^{(1^2)}(\epsilon) = 1$ and $R^{(1^2)}(1\,2) = -1$. Then

$$N_L^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ f_1 & -1 & -1 \\ f_1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_L^{(1^2)} = \begin{pmatrix} 0 & 0 & 0 \\ f_1 & -1 & 1 \\ f_1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At level $\boldsymbol{\ell} = \mathbf{3}$ the matrices N_L^{π} are the 4×1 all-zero matrix for $\pi = (3)$ and $= (1^3)$ respectively, and the 8×2 all-zero matrix for $\pi = (2,1)$.

4.3 Permutations of the vertex sets

Let Sym_b and Sym_d act on V_b and V_d respectively in the obvious way. Let $M \subseteq V_b \times V_d$ be any matching. For every $\omega \in \mathrm{Sym}_b$ and $\tau \in \mathrm{Sym}_d$ we denote by ${}_{\omega}|\overset{\tau}{\overline{M}}$ the matching

$$\{(\omega(y), \tau(x)) \in V_b \times V_d \mid (y, x) \in M\}.$$

Similarly define ${}_{\omega}|\overset{\tau}{\overline{L}}$. For example, consider the graph ${}_4\langle 11, 32, 42\rangle_3$ shown in Figure 4.3. Then the graph ${}_4\langle L\rangle_3$ shown in Figure 4.4 satisfies

$$L = {}_{(13)}|\overset{(132)}{\overline{\{(1,1),(3,2),(4,2)\}}} = \{(1,1),(3,3),(4,1)\}.$$

We define ${}_{\omega}|\overset{\tau}{\overline{{}_b\langle L\rangle_d}} = {}_b\langle L'\rangle_d$ where $L' = {}_{\omega}|\overset{\tau}{\overline{L}}$. For example, denoting the graph

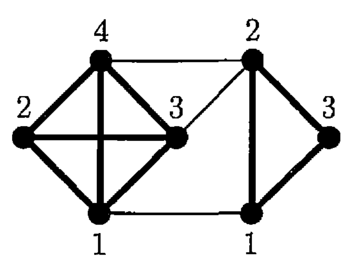


Figure 4.3: The graph ${}_4\langle 11, 32, 42\rangle_3$

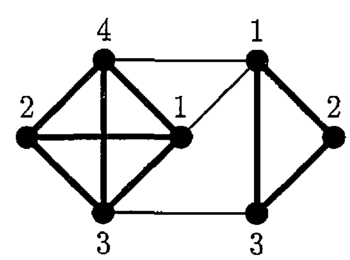


Figure 4.4: The graph ${}_3\langle 11, 33, 41\rangle_4$

${}_4\langle 11, 32, 42\rangle_3$ by H we may write ${}_3\langle 11, 33, 41\rangle_4 = {}_{(13)}|\overset{(132)}{\overline{H}}$.

Since $M \in \mathcal{M}(b, d, {}_{\omega}|\overset{\tau}{\overline{L}})$ if and only if ${}_{\omega}|\overset{\tau}{\overline{M}} \in \mathcal{M}(b, d, L)$, and $\left|{}_{\omega}|\overset{\tau}{\overline{M}}\right| = |M|$ for all matchings it follows that

$$N_L^{()} = N_{L'}^{()} \qquad \text{if } L' = {}_{\omega}|\overset{\tau}{\overline{L}} \text{ for some } \omega \in \mathrm{Sym}_b \text{ and } \tau \in \mathrm{Sym}_d.$$

Define ${}_{\omega}|\overset{\tau}{\overline{U_M^{(1)}}}$ to be the matrix obtained by replacing $(U_M^{(1)})^{yx}$ in $U_M^{(1)}$ by $(U_M^{(1)})^{\omega(y)\tau(x)}$. Recall

$$(U_M^{(1)})^{yx} = \begin{cases} (k - |M|)_{d-|M|} & \text{if } y \in M_1 \text{ and } (y,x) \in M \\ (k - |M| - 1))_{d-|M|-1} & \text{if } y \in M_1 \text{ and } x \notin M_2 \\ 0 & \text{if } y \notin M_1 \text{ or } x \in M_2 \text{ but } (y,x) \notin M. \end{cases}$$

It can easily be checked that ${}_{\omega}|\overset{\tau}{\overline{U^{(1)}_M}}$ is the equal to $U^{(1)}_{M'}$ where $M' = {}_{\omega}|\overset{\tau}{\overline{M}}$. The matrix ${}_{\omega}|\overset{\tau}{\overline{N^{\pi}_L}}$ is defined analogously to ${}_{\omega}|\overset{\tau}{\overline{U^{\pi}_M}}$.

Lemma 4.1 *Let the graph ${}_b\langle L\rangle_d$, $\omega \in Sym_b$ and $\tau \in Sym_d$ be given. Then $N^{()}_L = N^{()}_{L'}$ and ${}_{\omega}|\overset{\tau}{\overline{N^{(1)}_L}} = N^{(1)}_{L'}$, where $L' = {}_{\omega}|\overset{\tau}{\overline{L}}$.* □

The above results can be generalized for $\pi \vdash \ell$ with $\ell \geq 2$, but things are getting quite a bit more complicated since the F^{YX}_M are not trivial anymore. In the examples considered in the following sections it turns out to be more convenient to calculate the N^{π}_L "by hand" rather than using a generalization of the above.

4.4 Reduction of base graphs

We are now going to discuss another case where the matrices N^{π}_L corresponding to one graph can be obtained from the matrix $N^{*\pi}_L$ corresponding to another graph. Let us begin with an example.

Example 4.3: Let us consider the graphs ${}_3\langle 11, 22, 34\rangle_4$ and ${}_4\langle 11, 32, 44\rangle_4$ shown in Figures 4.5 and 4.6. Denote by $N^{\pi}_{11,22,34}$ and $N^{\pi}_{11,32,44}$ the respective matrices corresponding to these graphs. Let $M \in \mathcal{M}(b, d, L)$, $0 \leq \ell \leq 3$ and $\pi \vdash \ell$.

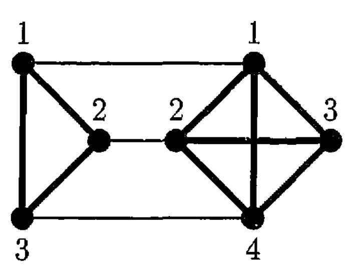


Figure 4.5: The graph ${}_3\langle 11, 22, 34\rangle_4$

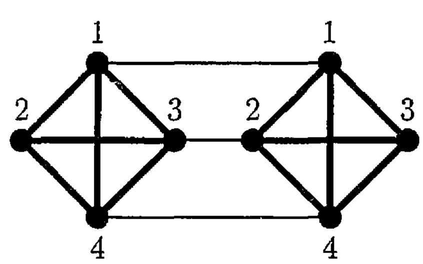


Figure 4.6: The graph ${}_4\langle 11, 32, 44\rangle_4$

From Theorem 3.13 it follows that $(U^{\pi}_M)^{YX}$ corresponding to graph ${}_4\langle 11, 32, 44\rangle_4$

is the all-zero submatrix if Y contains 2. Otherwise, if Y does not contain 2, the submatrix $(U_M^\pi)^{YX}$ depends only on Y being such that $\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1$ and the order of the elements in Y, but not on the size of V_b.

At level 0, since the matrices $U_M^{()}$ depend only on the size of the matching M, it follows that $N^{()}_{11,32,44} = N^{()}_{11,22,34}$.

At level 1, by removing the row indexed by $Y = \{2\}$, that is row two, in $N^{(1)}_{11,32,44}$ we obtain $N^{(1)}_{11,22,34}$.

At level 2 the matrix $N^\pi_{11,22,34}$ can be obtained by removing rows one, four and five in $N^\pi_{11,32,44}$.

At level 3, if $\pi = (3)$ or $\pi = (1^3)$ we obtain $N^\pi_{11,22,34}$ from $N^\pi_{11,32,44}$ by removing rows one, two and four. If $\pi = (2,1)$ we have to remove all rows except rows five and six.

The graph ${}_3\langle 11, 22, 34\rangle_4$ can be obtained from graph ${}_4\langle 11, 32, 44\rangle_4$ by removing the vertex 2 and all incident edges, and relabelling the vertices such that their order is preserved. That is, the vertices 3 and 4 in the obtained copy of K_3 become 2 and 3 respectively.

Let b and d be integers, and let $L \subseteq V_b \times V_d$ be any linking set. Let $Z \subset V_b$ be such that $(Z \times V_d) \cap L = \varnothing$. That is, none of the vertices in Z is incident with a linking edge. Delete all vertices in Z and all the adjacent edges, and relabel the vertices in $V_b \setminus Z$ such that their order is preserved. The resulting graph is of the form ${}_{b'}\langle L'\rangle_d$ where $b' = b - |Z|$ and L' is the induced linking set of the same size as L. We say that ${}_{b'}\langle L'\rangle_d$ has been obtained from ${}_b\langle L\rangle_d$ by deleting Z.

Lemma 4.2 *Let b and d be integers, and let $L \subseteq V_b \times V_d$ be any linking set. Let $Z \subset B_b$ be such that $(Z \times V_d) \cap L = \varnothing$. Assume that ${}_{b'}\langle L'\rangle_d$ has been obtained from ${}_b\langle L\rangle_d$ by deleting Z. Then for all ℓ and $\pi \vdash \ell$ the matrices $N^\pi_{L'}$ corresponding to ${}_{b'}\langle L'\rangle_d$ can be obtained from N^π_L corresponding to ${}_b\langle L\rangle_d$ by deleting all rows indexed by Y with $Y \cap Z \neq \varnothing$.*

Proof: From Theorem 3.13 it follows that all submatrices $(U_M^\pi)^{YX}$ corresponding to graph ${}_b\langle L\rangle_d$ is the all-zero submatrix if $Y \cap Z \neq \varnothing$. Otherwise, if $Y \cap Z = \varnothing$, the submatrix $(U_M^\pi)^{YX}$ depends only on Y being such that $\mu^{-1}(X \cap M_2) \subseteq Y \subseteq M_1$ and the order of the elements in Y, but not on the size of V_b. Removing all vertices in Z and all the adjacent edges does not change the order of the remaining vertices in V_b, hence does not change the submatrix $(U_M^\pi)^{YX}$. The result follows from Corollary 3.7. $\square$

4.5 Generalised dodecahedra

Let B be the path on four vertices and let $L = \{(1,1),(3,2),(4,4)\}$ be the linking set. The resulting graph D_n is the generalised dodecahedron introduced in Example 3.4. There are five colour-partitions of B:

$$\mathcal{R}_1 = \{1|2|3|4\}, \quad \mathcal{R}_2 = \{13|2|4\}, \quad \mathcal{R}_3 = \{1|24|3\},$$

$$\mathcal{R}_4 = \{14|2|3\} \quad \text{and} \quad \mathcal{R}_5 = \{13|24\}.$$

From Lemma 3.5 follows that the matrices N_L^π consist of 25 submatrices $N^\pi_{\mathcal{R}_i\mathcal{R}_j} = N^\pi_{L_{\mathcal{R}_i\mathcal{R}_j}}$:

$$N_L^\pi = \begin{pmatrix} N^\pi_{\mathcal{R}_1\mathcal{R}_1} & N^\pi_{\mathcal{R}_1\mathcal{R}_2} & N^\pi_{\mathcal{R}_1\mathcal{R}_3} & N^\pi_{\mathcal{R}_1\mathcal{R}_4} & N^\pi_{\mathcal{R}_1\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_2\mathcal{R}_1} & N^\pi_{\mathcal{R}_2\mathcal{R}_2} & N^\pi_{\mathcal{R}_2\mathcal{R}_3} & N^\pi_{\mathcal{R}_2\mathcal{R}_4} & N^\pi_{\mathcal{R}_2\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_3\mathcal{R}_1} & N^\pi_{\mathcal{R}_3\mathcal{R}_2} & N^\pi_{\mathcal{R}_3\mathcal{R}_3} & N^\pi_{\mathcal{R}_3\mathcal{R}_4} & N^\pi_{\mathcal{R}_3\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_4\mathcal{R}_1} & N^\pi_{\mathcal{R}_4\mathcal{R}_2} & N^\pi_{\mathcal{R}_4\mathcal{R}_3} & N^\pi_{\mathcal{R}_4\mathcal{R}_4} & N^\pi_{\mathcal{R}_4\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_5\mathcal{R}_1} & N^\pi_{\mathcal{R}_5\mathcal{R}_2} & N^\pi_{\mathcal{R}_5\mathcal{R}_3} & N^\pi_{\mathcal{R}_5\mathcal{R}_4} & N^\pi_{\mathcal{R}_5\mathcal{R}_5} \end{pmatrix}$$

Each $N^\pi_{\mathcal{R}_i\mathcal{R}_j}$ corresponds to a graph ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ which depends on the colour-partitions $\mathcal{R}_i$ and $\mathcal{R}_j$. The Table 4.1 shows the graphs corresponding to all pairs of colour-partitions.

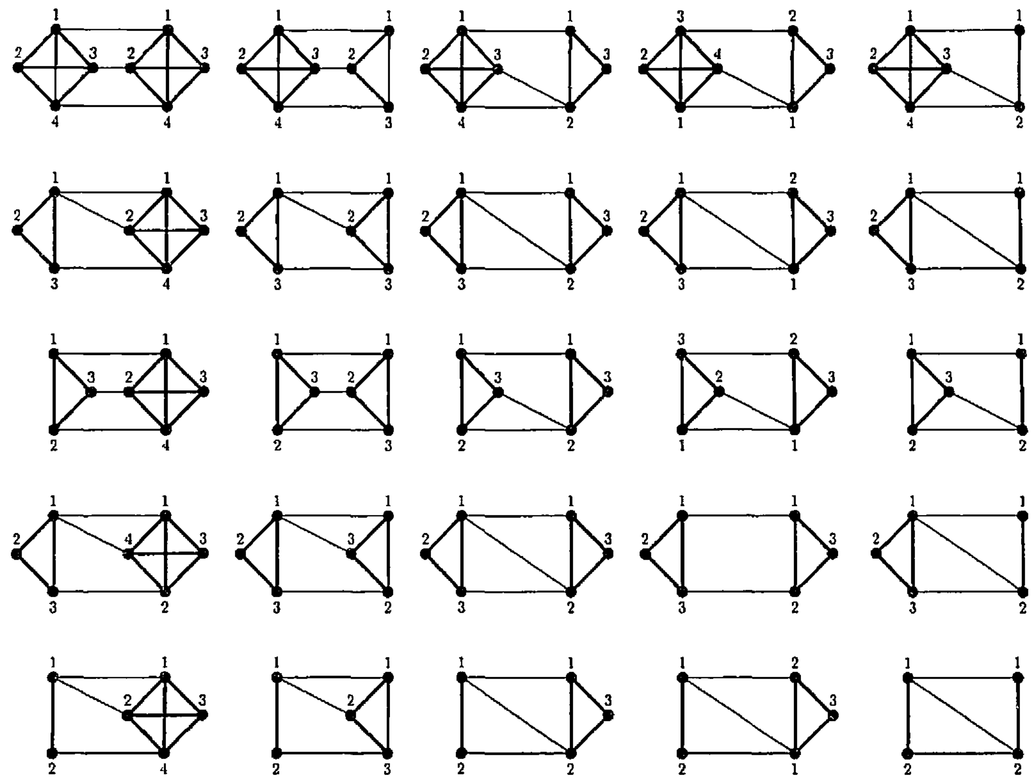


Table 4.1: The induced graphs $_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ in case of the family D_n

Let

$$H_{44} = {}_4\langle 11, 32, 44\rangle_4 \qquad H_{43a} = {}_4\langle 11, 32, 43\rangle_3 \qquad H_{43b} = {}_4\langle 11, 32, 42\rangle_3$$
$$H_{42} = {}_4\langle 11, 32, 42\rangle_2 \qquad H_{34} = {}_3\langle 11, 12, 34\rangle_4 \qquad H_{33a} = {}_3\langle 11, 12, 33\rangle_3$$
$$H_{33b} = {}_3\langle 11, 12, 32\rangle_3 \qquad H_{33c} = {}_3\langle 11, 22\rangle_3 \qquad H_{32a} = {}_3\langle 11, 12, 32\rangle_2$$

These nine graphs are shown in Table 4.2, and we call them the " H-series ". For all π we denote the matrices N_L^π corresponding to these graphs by N_{44}^π, N_{43a}^π , N_{43b}^π, N_{42}^π, N_{34}^π, N_{33a}^π, N_{33b}^π, N_{33c}^π and by N_{32a}^π respectively.

In Appendix B the non-trivial matrices N_L^π for the all levels and all the graphs in the H-series are given. Trivial means all-zero, like for example the level 4 of H_{44}.

Observe that in none of the graphs in the H-series is the linking set incident with vertex 2 on the left hand side. We indicate by a superscript " $*$ " the graph obtained by removing vertex 2 and all incident edges. The resulting graphs are shown in Table 4.3, and we call them the " H^*-series ".

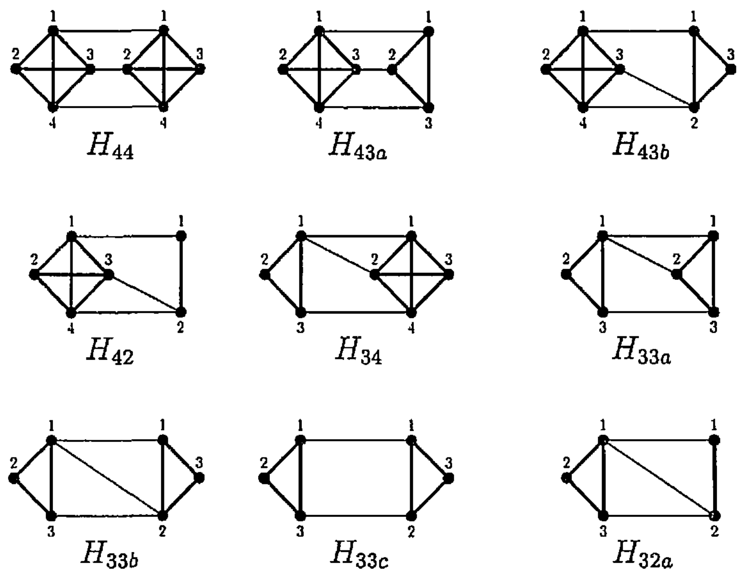


Table 4.2: The graphs of the H-series

Then, using the notation introduced in Section 4.3, we can rewrite Table 4.1 as:

$$
\begin{array}{ccccc}
H_{44} & H_{43a} & H_{43b} & (13)\,|\overline{H_{43b}}^{\,12} & H_{42} \\[1em]
H_{34} & H_{33a} & H_{33b} & \epsilon\,|\overline{H_{33b}}^{\,(12)} & H_{32a} \\[1em]
(23)\,|\overline{H^*_{44}}^{\,\epsilon} & (23)\,|\overline{H^*_{43a}}^{\,\epsilon} & H^*_{43b} & (13)\,|\overline{H^*_{43b}}^{\,(12)} & H^*_{42} \\[1em]
\epsilon\,|\overline{H_{34}}^{\,(24)} & \epsilon\,|\overline{H_{33a}}^{\,(23)} & H_{33b} & H_{33c} & H_{32a} \\[1em]
H^*_{34} & H^*_{33a} & H^*_{33b} & \epsilon\,|\overline{H^*_{33b}}^{\,(12)} & H^*_{32a}
\end{array}
$$

For all π we denote the matrices N^{π}_{L} corresponding to the graphs in the H^*-series by $N^{*\pi}_{44}$, $N^{*\pi}_{43a}$, $N^{*\pi}_{43b}$, $N^{*\pi}_{42}$, $N^{*\pi}_{34}$, $N^{*\pi}_{33a}$, $N^{*\pi}_{33b}$, $N^{*\pi}_{33c}$ and by $N^{*\pi}_{32a}$ respectively. From Lemma 4.2 it follows that for $\pi = ()$ and $\pi = (1)$ all these matrices can be obtained from the matrices corresponding to the H-series by removing the all-zero rows corresponding to Y containing 2. It follows that for levels zero and one we only need to obtain the matrices N^{π}_{L} for the graphs in the H-series. For levels two and three it is easier to calculate the matrices $N^{\pi}_{\mathcal{R}_i\mathcal{R}_j}$ directly. Level four is the all-zero matrix.

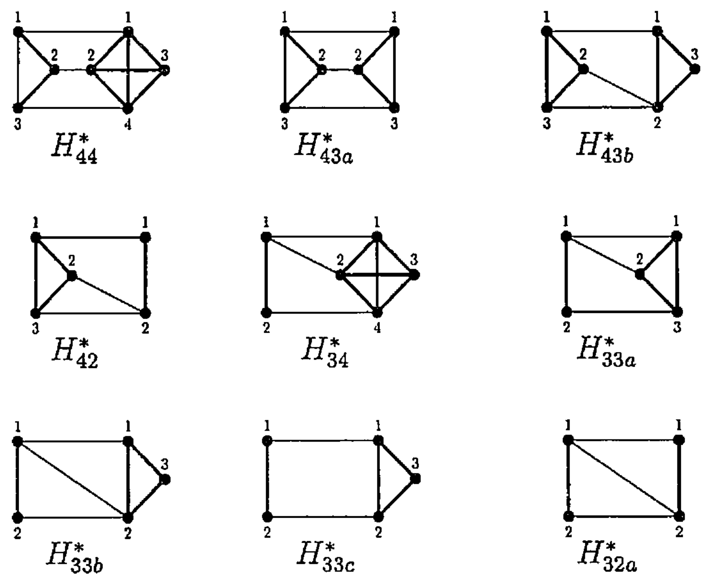


Table 4.3: The graphs of the H^*-series

Level 0: From Lemma 4.1 it follows that the matrix $N_L^{()}$ corresponding to the graph D_n can be written as

$$N_L^{()} = \begin{pmatrix} N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33b}^{()} & N_{32a}^{()} \\ N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33c}^{()} & N_{32a}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33b}^{()} & N_{32a}^{()} \end{pmatrix}$$

$$= \begin{pmatrix} (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c^3 & c^2 \\ (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c(c^2+c+1) & c^2 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c^3 & c^2 \end{pmatrix},$$

where $c = k - 2$. The characteristic equation is

$$\lambda^2\big(\lambda^3 + a_2^{()}(c)\lambda^2 + a_1^{()}(c)\lambda + a_0^{()}(c)\big) = 0$$

with

$$
\begin{aligned}
a_2^{()}(c) &= (-c^4 - 2c^3 - 4c^2 - 1) \\
a_1^{()}(c) &= (c^6 + 2c^5 + 3c^4 + 2c^3 + 2c^2 + 2c) \\
a_0^{()}(c) &= -c^4 - 2c^3 - c^2.
\end{aligned}
$$

Level 1: From Lemma 4.1 it follows that the 15×15 matrix $N_L^{()}$ corresponding to the graph D_n can be written as

$$
\begin{pmatrix}
N_{44}^{(1)} & N_{43a}^{(1)} & N_{43b}^{(1)} & {}_{(13)}|\overline{N_{43b}^{(1)}}^{(12)} & N_{42}^{(1)} \\
N_{34}^{(1)} & N_{33a}^{(1)} & N_{33b}^{(1)} & {}_{\epsilon}|\overline{N_{33b}^{(1)}}^{(12)} & N_{32a}^{(1)} \\
{}_{(34)}|\overline{N_{44}^{*(1)}}^{\epsilon} & {}_{(34)}|\overline{N_{43a}^{*(1)}}^{\epsilon} & N_{43b}^{*(1)} & {}_{(13)}|\overline{N_{43b}^{*(1)}}^{(12)} & N_{42}^{*(1)} \\
{}_{\epsilon}|\overline{N_{34}^{(1)}}^{(24)} & {}_{\epsilon}|\overline{N_{33a}^{(1)}}^{(23)} & N_{33b}^{(1)} & N_{33c}^{(1)} & N_{32a}^{(1)} \\
N_{34}^{*(1)} & N_{33a}^{*(1)} & N_{33b}^{*(1)} & {}_{\epsilon}|\overline{N_{33b}^{*(1)}}^{(12)} & N_{32a}^{*(1)}
\end{pmatrix}.
$$

Then the characteristic equation of $N_L^{(1)}$ is

$$
\lambda^8(\lambda-1)\Big(\lambda^6 + a_5^{(1)}(c)\lambda^5 + a_4^{(1)}(c)\lambda^4 + a_3^{(1)}(c)\lambda^3 + a_2^{(1)}(c)\lambda^2 + a_1^{(1)}(c)\lambda + a_0^{(1)}(c)\Big) = 0
$$

with

$$
\begin{aligned}
a_5^{(1)}(c) &= 2c^3 + 4c - 2 \\
a_4^{(1)}(c) &= c^6 - 2c^5 + 2c^4 - 4c^3 + 2c^2 - 2c - 1 \\
a_3^{(1)}(c) &= -2c^8 - 2c^7 - 6c^6 - 4c^5 - c^4 + 6c^3 - 4c + 1 \\
a_2^{(1)}(c) &= c^{10} + 2c^9 + 3c^8 + 4c^7 + 7c^6 + 6c^5 + 3c^4 - 2c^3 - 5c^2 + 1 \\
a_1^{(1)}(c) &= 2c - 6c^7 - 8c^4 - 2c^8 - 8c^5 - 4c^3 + 2c^2 - 8c^6 \\
a_0^{(1)}(c) &= 6c^4 + c^6 + 4c^5 + 4c^3 + c^2,
\end{aligned}
$$

where, as before, $c = k - 2$.

Level 2: There are $\pi = (2)$ and $\pi = (1^2)$. In both cases the matrix N_L^π corresponding to D_n is of size 16×16. We calculate all the submatrices $N^\pi_{\mathcal{R}_i\mathcal{R}_j}$ directly, as shown in the beginning of the chapter and then use them to obtain N_L^π. Omitting all the rows and columns corresponding to a sets Y and X containing 2 respectively we can reduce N_L^π to 9×9 matrices. This reduced matrices are shown in Appendix C. It turns out that the characteristic equation corresponding to $\pi = (2)$ and $\pi = (1^2)$ are both equal to

$$\lambda^{11}(\lambda - 1)\Big(\lambda^4 + a_3^\pi(c)\lambda^3 + a_2^\pi(c)\lambda^2 + a_1^\pi(c)\lambda + a_0^\pi(c)\Big) = 0$$

with

$$\begin{aligned} a_3^\pi(c) &= -c^2 + 2c - 2 & a_2^\pi(c) &= -2c^3 + c^2 - 2c - 1 \\ a_1^\pi(c) &= -c^4 + 1 & a_0^\pi(c) &= c^2 + 2c + 1, \end{aligned}$$

where $c = k - 2$.

Level 3: Here there are three partitions $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$. The matrix N_L^π corresponding to D_n is of size 7×7 for $\pi = (3)$ and $\pi = (1^3)$, and 14×14 for $\pi = (2,1)$. The set $\mathcal{M}(b, d, L)$ contains a matching of size three only in the case of the graphs H_{44} and H_{43a} (and the corresponding reduced graphs H_{44}^* and H_{43a}^*). Since all the rows in N_L^π corresponding to a Y containing 2 are zero it follows that N_{43a} and N_{43a}^* are the zero matrices. In N_{44} only the rows corresponding to $Y = \{1, 3, 4\}$ are non-zero. Since $F^{YX}_{11,32,44} = \{\epsilon\}$ where $Y = X = \{1, 3, 4\}$, it follows that N_L^π can be reduced to $R^\pi(\epsilon)$, Young's natural representation of ϵ. It follows that the characteristic equations for $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$ are

$$\lambda^6(\lambda - 1) = 0, \qquad \lambda^6(\lambda - 1) = 0 \quad \text{and} \quad \lambda^{12}(\lambda - 1)^2 = 0 \quad \text{respectively.}$$

Level 4: Here N_L^π corresponding to D_n is equal to N_{44}^π which is the all-zero matrix, and hence does not contribute to the chromatic polynomial.

The Chromatic Polynomial of $\mathbf{D_5}$: The global multiplicities are given in following table

ℓ	π	$\eta_\pi(k)$
0	()	1
1	(1)	$c+1$
2	(2)	$\frac{1}{2}(c+2)(c-1)$
2	(1^2)	$\frac{1}{2}(c+1)c$
3	(3)	$\frac{1}{6}(c+2)(c+1)(c-3)$
3	(2, 1)	$\frac{1}{3}(c+2)c(c-2)$
3	(1^3)	$\frac{1}{6}(c+1)c(c-1)$

As in Example 3.12 some of the eigenvalues are not polynomials. We use Newton's formula, given in Appendix A, to evaluate the sum $A_{\pi,n}$ of the n^{th} powers of the non-polynomial eigenvalues of the matrices N_L^π. For example for $n = 5$ we get, with $k = c + 2$:

$$\begin{aligned}
A_{(),5} =& c^{20} + 10c^{19} + 55c^{18} + 200c^{17} \\
&+ 535c^{16} + 1082c^{15} + 1705c^{14} + 2060c^{13} \\
&+ 1920c^{12} + 1230c^{11} + 529c^{10} - 110c^{9} \\
&- 80c^{8} - 290c^{7} + 110c^{6} - 180c^{5} + 125c^{4} - 80c^{3} + 35c^{2} - 10c + 1
\end{aligned}$$

$$\begin{aligned}
A_{(1),5} =& -2c^{15} - 10c^{14} - 40c^{13} - 80c^{12} - 170c^{11} - 115c^{10} \\
&- 350c^{9} + 260c^{8} - 870c^{7} + 1255c^{6} - 1674c^{5} \\
&+ 1825c^{4} - 1470c^{3} + 810c^{2} - 280c + 47
\end{aligned}$$

$$\begin{aligned}
A_{(2),5} =& c^{10} + 10c^{8} - 10c^{7} + 55c^{6} - 82c^{5} + 185c^{4} \\
&- 230c^{3} + 255c^{2} - 150c + 47
\end{aligned}$$

Hence the chromatic polynomial of D_5 is:

$$\begin{aligned}P(D_5; c) =& A_{(),5} + (c+1)(A_{(1),5} + 1) + (c^2 + c - 1)(A_{(2),5} + 1) + c^3 - 4c - 1 \\ &= c(c+1)(c+2)(c^{17} + 7c^{16} + 32c^{15} + 90c^{14} + 199c^{13} + 293c^{12} \\ &+ 378c^{11} + 220c^{10} + 255c^9 - 259c^8 + 340c^7 - 702c^6 \\ &+ 771c^5 - 831c^4 + 690c^3 - 400c^2 + 140c - 24)\end{aligned}$$

For general $n \in \mathbb{N}$, the chromatic polynomial of D_n is:

$$P(D_n; c) = A_{(),n} + (c+1)(A_{(1),n} + 1) + (c^2 + c - 1)(A_{(2),n} + 1) + c^3 - 4c - 1.$$

In Figure 4.7 the roots of $P(D_{30}; c)$ are plotted. It appears that they are clustering along curves. These curves will be the concern of the next chapter. Also clearly visible are the roots with negative real part.

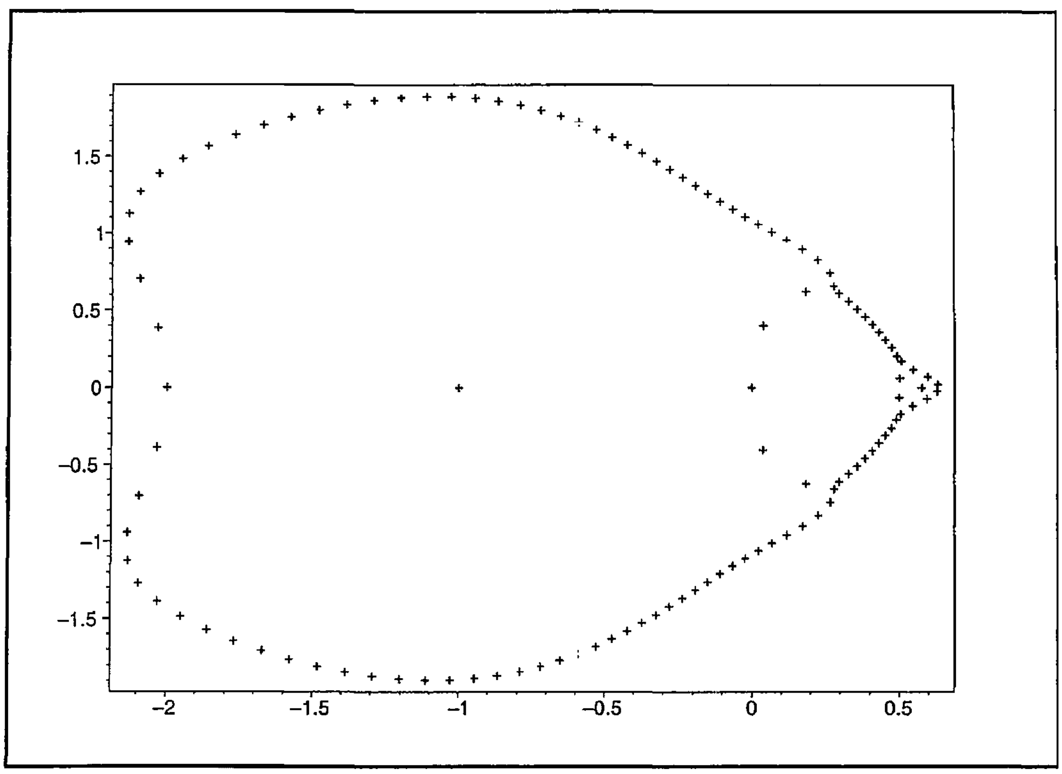


Figure 4.7: The roots of D_{30}

4.6 Four more families of cubic graphs

In the previous section we used the compatibility matrix method to calculate the chromatic polynomials of the generalized dodecahedra. There are four more families of cubic graphs all with the path on four vertices as base graph. In this section we obtain their chromatic polynomials.

Let B be the the path on four vertices. Then these families are $L_n(B)$ where L is

$$\{(1,4),(3,2),(4,1)\}, \qquad \{(1,4),(3,1),(4,2)\},$$

$$\{(1,2),(3,1),(4,4)\} \quad \text{and} \quad \{(1,2),(3,4),(4,1)\}$$

respectively. The graphs consisting of two adjacent copies of B linked by the respective L are shown in Figure 4.8. Each of them contains three cycles. The lengths of these cycles is characteristic for these graphs, and hence we denote them accordingly. That is, we denote them as 468, 477, 567 and 666. The graph consisting of two adjacent copies of B and the linking set corresponding to the generalized dodecahedron, shown in Figure 3.3, is 558.

We denote these five families of cubic graphs by

$$(558)_n, \qquad (468)_n, \qquad (477)_n, \qquad (567)_n \quad \text{and} \quad (666)_n$$

respectively, where $D_n = (558)_n$.

We denote the matrices T_L corresponding to these five families by

$$T_{558}, \qquad T_{468}, \qquad T_{477}, \qquad T_{567}, \quad \text{and} \quad T_{666}$$

respectively. Similarly we define the five matrices N^{π}_{558}, N^{π}_{468}, N^{π}_{477}, N^{π}_{567} and N^{π}_{666}. Note that all these five families have a trivial level 4. That is, for each of them the matrix N^{π}_{L} with $\pi \vdash 4$ is the all-zero matrix. Hence the level 4 does not contribute to the chromatic polynomial in any of the cases, and we are going to omit it in the following. For every π the matrices N^{π}_{558}, N^{π}_{468}, N^{π}_{477}, N^{π}_{567} and N^{π}_{666} are of equal size as shown in the following table:

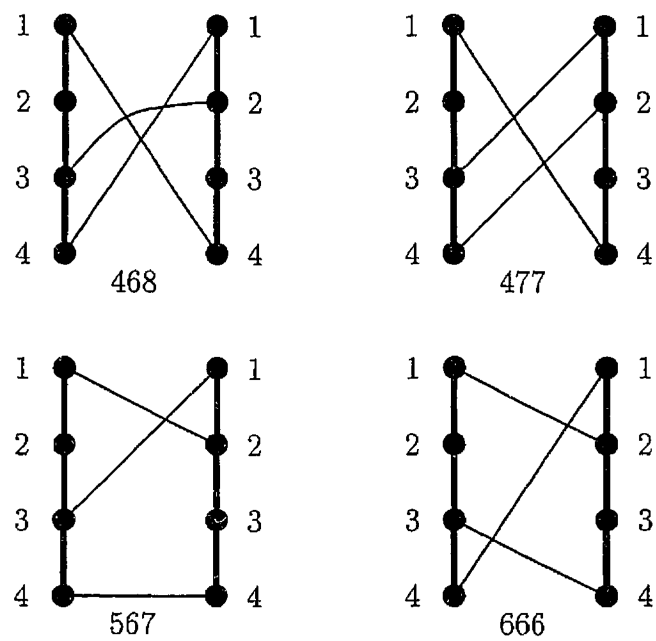


Figure 4.8: The graphs consisting of two adjacent copies of B linked by the respective linking set

π	Size of the matrices N^{π}_{558}, N^{π}_{468}, N^{π}_{477}, N^{π}_{567} and N^{π}_{666}
$()$	5×5
(1)	15×15
(2)	16×16
(1^2)	16×16
(3)	7×7
$(2,1)$	14×14
(1^3)	7×7

Since all these families have the path of length four as base graph it follows that we have for each of them the five colour-partitions:

$$\mathcal{R}_1 = \{1|2|3|4\}, \quad \mathcal{R}_2 = \{13|2|4\}, \quad \mathcal{R}_3 = \{1|24|3\},$$

$$\mathcal{R}_4 = \{14|2|3\} \quad \text{and} \quad \mathcal{R}_5 = \{13|24\}.$$

As before, from Lemma 3.5 follows that for all of them the matrices N_L^π consists of 25 submatrices $N^\pi_{\mathcal{R}_i\mathcal{R}_j} = N^\pi_{L_{\mathcal{R}_i\mathcal{R}_j}}$:

$$N_L^\pi = \begin{pmatrix} N^\pi_{\mathcal{R}_1\mathcal{R}_1} & N^\pi_{\mathcal{R}_1\mathcal{R}_2} & N^\pi_{\mathcal{R}_1\mathcal{R}_3} & N^\pi_{\mathcal{R}_1\mathcal{R}_4} & N^\pi_{\mathcal{R}_1\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_2\mathcal{R}_1} & N^\pi_{\mathcal{R}_2\mathcal{R}_2} & N^\pi_{\mathcal{R}_2\mathcal{R}_3} & N^\pi_{\mathcal{R}_2\mathcal{R}_4} & N^\pi_{\mathcal{R}_2\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_3\mathcal{R}_1} & N^\pi_{\mathcal{R}_3\mathcal{R}_2} & N^\pi_{\mathcal{R}_3\mathcal{R}_3} & N^\pi_{\mathcal{R}_3\mathcal{R}_4} & N^\pi_{\mathcal{R}_3\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_4\mathcal{R}_1} & N^\pi_{\mathcal{R}_4\mathcal{R}_2} & N^\pi_{\mathcal{R}_4\mathcal{R}_3} & N^\pi_{\mathcal{R}_4\mathcal{R}_4} & N^\pi_{\mathcal{R}_4\mathcal{R}_5} \\ N^\pi_{\mathcal{R}_5\mathcal{R}_1} & N^\pi_{\mathcal{R}_5\mathcal{R}_2} & N^\pi_{\mathcal{R}_5\mathcal{R}_3} & N^\pi_{\mathcal{R}_5\mathcal{R}_4} & N^\pi_{\mathcal{R}_5\mathcal{R}_5} \end{pmatrix}$$

Each $N^\pi_{\mathcal{R}_i\mathcal{R}_j}$ corresponds to a graph ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$.

For the rest of this chapter we let $k = c + 2$.

4.6.1 The family $(468)_n$

The Table 4.4 shows all the graphs ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ with $i = 1,2,3,4,5$ and $j = 1,2,3,4,5$.

Denote by H_{32b} the graph ${}_3\langle 12, 31\rangle_2$, and by H^*_{32b} the graph ${}_2\langle 12, 21\rangle_2$ (See Figure 4.9). The matrices N^π_{32b} can be found in Appendix B. Including these graphs into the H-series and the H^*-series respectively we can use the catalogue of graphs in Appendix B to evaluate levels zero and one. For levels two and three the matrices N^π_{468} are being calculated directly using the fact that many rows are zero.

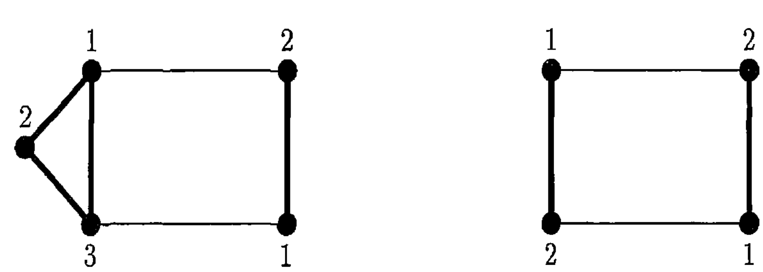


Figure 4.9: The graphs H_{32b} and H_{32b^*}

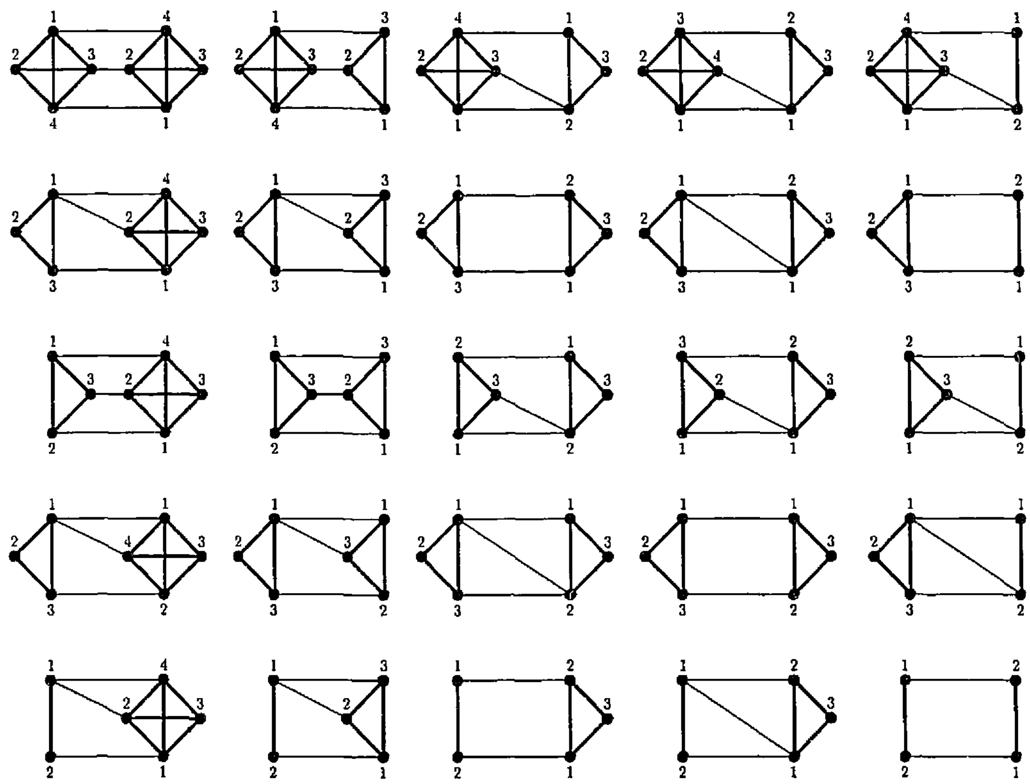


Table 4.4: The induced graphs ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ in case of the family $(468)_n$

Level 0: The matrix $N_L^{()}$ corresponding to the graph $(468)_n$ can be written as

$$N_{468}^{()} = \begin{pmatrix} N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33c}^{()} & N_{33b}^{()} & N_{32b}^{()} \\ N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33c}^{()} & N_{32a}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33c}^{()} & N_{33b}^{()} & N_{32b}^{()} \end{pmatrix}$$

$$= \begin{pmatrix} (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c(c^2+c+1) & c^3 & c^2+c+1 \\ (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c(c^2+c+1) & c^2 \\ c(c-1)(c^2+1) & c(c^2+1) & c(c^2+c+1) & c^3 & c^2+c+1 \end{pmatrix}$$

The characteristic equation is

$$\lambda^2\Big(\lambda^3 \,+\, a_2^{()}(c)\lambda^2 \,+\, a_1^{()}(c)\lambda \,+\, a_0^{()}(c)\Big) = 0$$

where

$$\begin{aligned}
a_2^{()}(c) &= -c^4 - 2c^3 - 4c^2 - c - 2\\
a_1^{()}(c) &= (c^6 + 2c^5 + 3c^4 + 3c^3 + 2c^2 + 2c + 1\\
a_0^{()}(c) &= c^6 + 2c^5 + c^4 - c^3 - 2c^2 - c.
\end{aligned}$$

Level 1: The matrix $N_{468}^{(1)}$ can be written as

$$\begin{pmatrix}
{}_{(14)}|\overline{N_{44}^{(1)}}^{\,\epsilon} & {}_{(14)}|\overline{N_{43a}^{(1)}}^{\,\epsilon} & {}_{(14)}|\overline{N_{43b}^{(1)}}^{\,\epsilon} & {}_{(13)}|\overline{N_{43b}^{(1)}}^{\,(12)} & {}_{(14)}|\overline{N_{42}^{(1)}}^{\,\epsilon}\\
{}_{\epsilon}|\overline{N_{34}^{(1)}}^{\,(14)} & {}_{\epsilon}|\overline{N_{33a}^{(1)}}^{\,(13)} & {}_{\epsilon}|\overline{N_{33c}^{(1)}}^{\,(12)} & {}_{\epsilon}|\overline{N_{33b}^{(1)}}^{\,(12)} & N_{32b}^{(1)}\\
{}_{(123)}|\overline{N_{44}^{*(1)}}^{\,\epsilon} & {}_{(123)}|\overline{N_{43a}^{*(1)}}^{\,\epsilon} & {}_{(123)}|\overline{N_{43b}^{*(1)}}^{\,\epsilon} & {}_{(13)}|\overline{N_{43b}^{*(1)}}^{\,(12)} & {}_{(12)}|\overline{N_{42}^{*(1)}}^{\,\epsilon}\\
{}_{\epsilon}|\overline{N_{34}^{(1)}}^{\,(24)} & {}_{\epsilon}|\overline{N_{33a}^{(1)}}^{\,(23)} & N_{33b}^{(1)} & N_{33c}^{(1)} & N_{32a}^{(1)}\\
{}_{\epsilon}|\overline{N_{34}^{*(1)}}^{\,(14)} & {}_{\epsilon}|\overline{N_{33a}^{*(1)}}^{\,(13)} & {}_{\epsilon}|\overline{N_{33c}^{*(1)}}^{\,(12)} & {}_{\epsilon}|\overline{N_{33b}^{*(1)}}^{\,(12)} & N_{32b}^{*(1)}
\end{pmatrix}.$$

Then the characteristic equation of $N_{468}^{(1)}$ is

$$\lambda^8(\lambda-1)\Big(\lambda^6 + a_5^{(1)}(c)\lambda^5 + a_4^{(1)}(c)\lambda^4 + a_3^{(1)}(c)\lambda^3 + a_2^{(1)}(c)\lambda^2 + a_1^{(1)}(c)\lambda + a_0^{(1)}(c)\Big) = 0$$

with

$$\begin{aligned}
a_5^{(1)}(c) &= -2c^2 - 3\\
a_4^{(1)}(c) &= -c^6 - 2c^5 - 4c^4 - 2c^3 - 3c^2 + 3c + 1\\
a_3^{(1)}(c) &= 2c^8 + 4c^7 + 8c^6 + 6c^5 + 7c^4 + 6c^3 + c^2 + 2\\
a_2^{(1)}(c) &= -c^{1}0 - 2c^9 - 3c^8 - 5c^7 - 5c^6 - 4c^5 - c^4 + 5c^3 + 2c^2 - 2c\\
a_1^{(1)}(c) &= c^9 + 3c^8 + 5c^7 + 3c^6 - 3c^5 - 6c^4 - 3c^3 + c^2 - 1\\
a_0^{(1)}(c) &= -c^9 - 3c^8 - 2c^7 + 3c^6 + 6c^5 + 3c^4 - 2c^3 - 3c^2 - c.
\end{aligned}$$

Level 2: There are $\pi = (2)$ and $\pi = (1^2)$. We calculate all the submatrices $N^{\pi}_{\mathcal{R}_i\mathcal{R}_j}$ directly, as shown in the beginning of the chapter and then use them to obtain N^{π}_{468}.

Omitting all the rows and columns corresponding to a sets Y and X containing 2 respectively we can reduce N^{π}_{468} to 9×9 matrices. These reduced matrices are shown in Appendix C. The characteristic equations of N^{π}_{468} with $\pi = (2)$ and $\pi = (1^2)$ are respectively

$$\lambda^{11}(\lambda-1)\Big(\lambda^4 + a_3(c)\lambda^3 + a_2(c)\lambda^2 + a_1(c)\lambda + a_0(c)\Big) = 0$$
$$\lambda^{11}(\lambda-1)\Big(\lambda^4 - a_3(c)\lambda^3 + a_2(c)\lambda^2 - a_1(c)\lambda + a_0(c)\Big) = 0$$

with $a_3(c) = -c^2 - 2$, $\quad a_2(c) = -c^2 + 2c - 1,$

$$a_1(c) = c^4 - c^2 + c + 1 \quad \text{and} \quad a_0(c) = -c^3 - c^2 + c + 1.$$

Level 3: Here there are three partitions $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$. Omitting all the zero rows and the corresponding columns, following a similar argument as in the case of the generalized dodecahedra, we can reduce these matrices to $R^{\pi}(13)$, Young's natural representation of (13). It follows that the characteristic equations for $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$ are

$$\lambda^6(\lambda-1) = 0, \qquad \lambda^6(\lambda+1) = 0 \qquad \text{and} \qquad \lambda^{12}(\lambda-1)(\lambda+1) = 0$$

respectively.

As in the case of the generalized dodecahedron, Newton's formula, given in Appendix A, can be used to evaluate the sum $A_{\pi,n}$ of the n^{th} powers of the non-polynomial eigenvalues of the matrices N^{π}_{L}. Then, for general $n \in \mathbb{N}$, the chromatic polynomial of $(468)_n$ is:

$$\begin{aligned} P((468)_n; c) = {} & A_{(),n} + (c+1)(A_{(1),n} + 1) \\ & + \frac{1}{2}(c^2 + c - 2)(A_{(2),n} + 1) + \frac{1}{2}(c^2 + c)(A_{(1^2),n} + 1) \\ & + \frac{1}{2}(c^3 - 3c)(-1)^n + \frac{1}{2}(c^3 - 5c - 2) \end{aligned}$$

In Figure 4.10 the roots of $(468)_{30}$ are plotted. Again, clearly visible are the roots with negative real part.

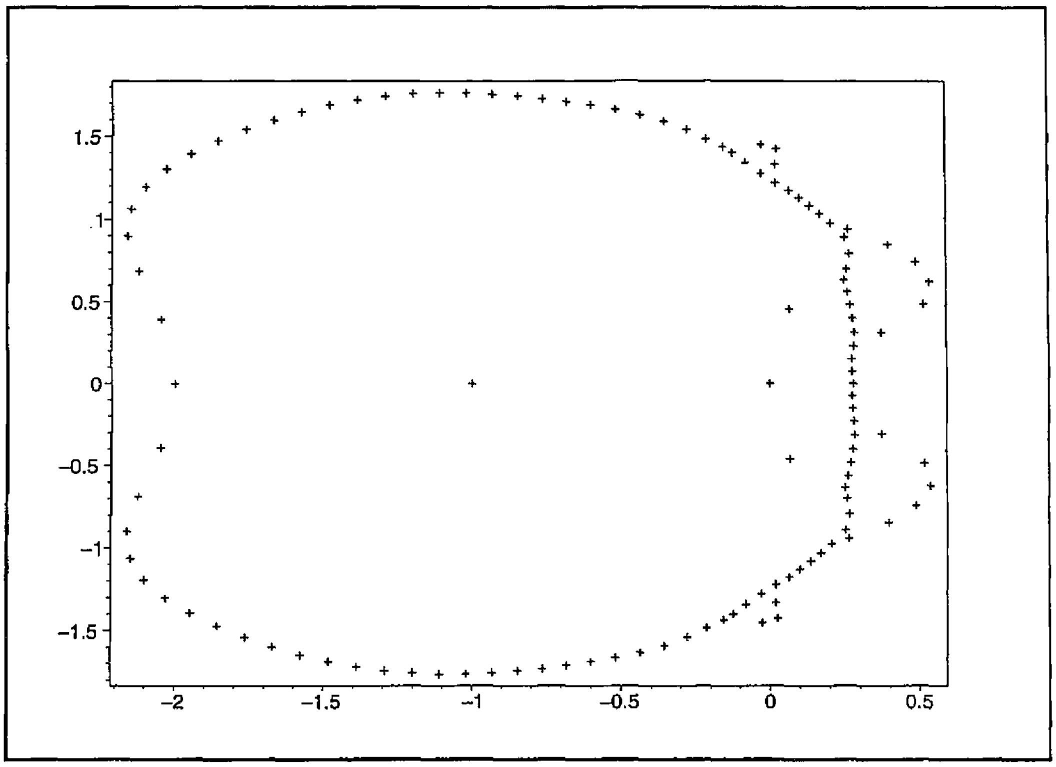


Figure 4.10: The roots of $(468)_{30}$

4.6.2 The family $(477)_n$

The Table 4.5 shows all this graphs ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ with $i = 1,2,3,4,5$ and $j = 1,2,3,4,5$.

All induced subgraphs are part of the catalogue of graphs in Appendix B which we will use to evaluate levels zero and one. For levels two and three we calculate the matrices directly.

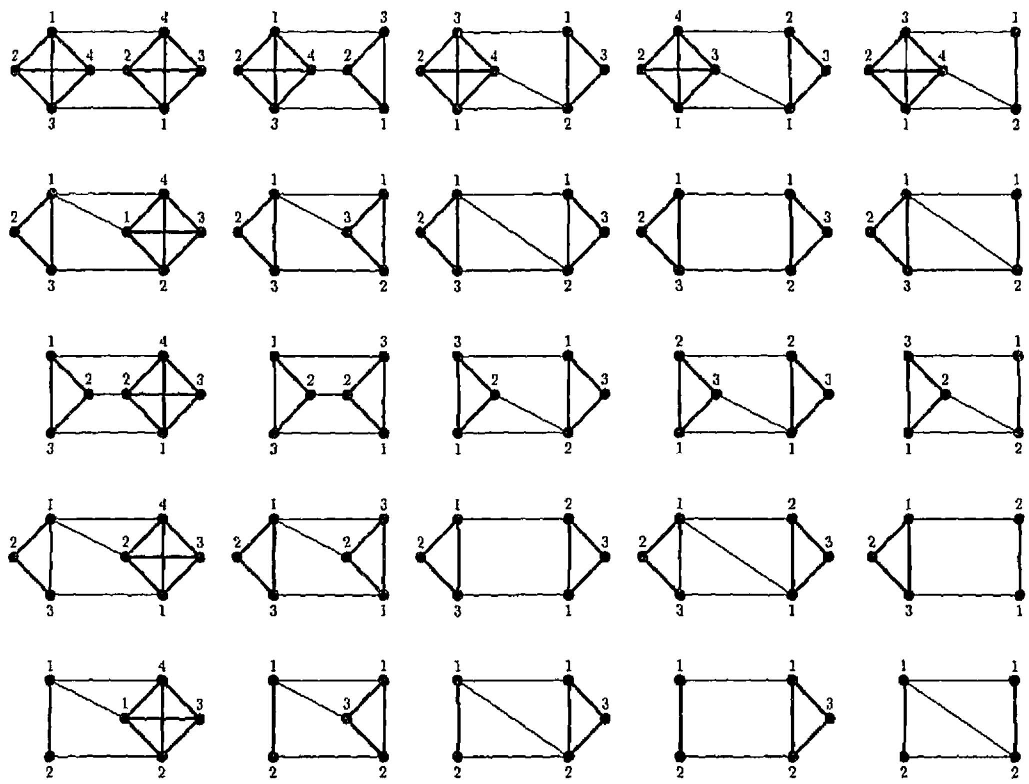


Table 4.5: The induced graphs ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ in case of the family $(477)_n$

Level 0: The matrix $N_L^{()}$ corresponding to the graph $(477)_n$ can be written as

$$N_{477}^{()} = \begin{pmatrix} N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33c}^{()} & N_{32a}^{()} \\ N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33c}^{()} & N_{33b}^{()} & N_{32b}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33c}^{()} & N_{32a}^{()} \end{pmatrix}$$

$$= \begin{pmatrix} (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c(c^2+c+1) & c^2 \\ (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c(c^2+c+1) & c^3 & c^2+c+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c(c^2+c+1) & c^2 \end{pmatrix}$$

The characteristic equation is

$$\lambda^2\Big(\lambda^3 + a_2^{()}(c)\lambda^2 + a_1^{()}(c)\lambda + a_0^{()}(c)\Big) = 0$$

where

$$\begin{aligned}
a_2^{()}(c) &= -c^4 - 2c^3 - 3c^2 + c - 1\\
a_1^{()}(c) &= -2c^5 - 5c^4 - 5c^3 - 2c^2\\
a_0^{()}(c) &= -c^6 - 2c^5 - c^4 + c^3 + 2c^2 + c.
\end{aligned}$$

Level 1: The matrix $N_{477}^{(1)}$ can be written as

$$\begin{pmatrix}
\overset{(142)}{\overline{\epsilon\,|N_{44}^{(1)}}} & \overset{132}{\overline{\epsilon\,|N_{43a}^{(1)}}} & \overset{\epsilon}{\overline{(13)\,|N_{43b}^{(1)}}} & \overset{(12)}{\overline{(14)\,|N_{43b}^{(1)}}} & \overset{\epsilon}{\overline{(13)\,|N_{42}^{(1)}}}\\
\overset{(24)}{\overline{\epsilon\,|N_{34}^{(1)}}} & \overset{(23)}{\overline{\epsilon\,|N_{33a}^{(1)}}} & N_{33b}^{(1)} & N_{33c}^{(1)} & N_{32a}^{(1)}\\
\overset{(14)}{\overline{\epsilon\,|N_{44}^{*(1)}}} & \overset{(13)}{\overline{\epsilon\,|N_{43a}^{*(1)}}} & \overset{\epsilon}{\overline{(13)\,|N_{43b}^{*(1)}}} & \overset{(12)}{\overline{(12)\,|N_{43b}^{*(1)}}} & \overset{\epsilon}{\overline{(13)\,|N_{42}^{*(1)}}}\\
\overset{(14)}{\overline{\epsilon\,|N_{34}^{(1)}}} & \overset{(13)}{\overline{\epsilon\,|N_{33a}^{(1)}}} & \overset{(12)}{\overline{\epsilon\,|N_{33c}^{(1)}}} & \overset{(12)}{\overline{\epsilon\,|N_{33b}^{(1)}}} & N_{32b}^{(1)}\\
\overset{(24)}{\overline{\epsilon\,|N_{34}^{*(1)}}} & \overset{(23)}{\overline{\epsilon\,|N_{33a}^{*(1)}}} & N_{33b}^{*(1)} & N_{33c}^{*(1)} & N_{32a}^{*(1)}
\end{pmatrix}.$$

Then the characteristic equation of $N_{477}^{(1)}$ is

$$\lambda^8\left(\lambda^7 + a_6^{(1)}(c)\lambda^6 + a_5^{(1)}(c)\lambda^5 + a_4^{(1)}(c)\lambda^4 + a_3^{(1)}(c)\lambda^3 + a_2^{(1)}(c)\lambda^2 + a_1^{(1)}(c)\lambda + a_0^{(1)}(c)\right) = 0$$

with

$$\begin{aligned}
a_6^{(1)}(c) &= 4c - 1\\
a_5^{(1)}(c) &= 2c^5 + 3c^4 + 7c^3 + 4c^2 + 2c - 2\\
a_4^{(1)}(c) &= -c^8 - 2c^7 + c^6 + 9c^5 + 10c^4 + 5c^3 + c - 3\\
a_3^{(1)}(c) &= -2c^9 - 5c^8 - 3c^7 + c^6 + c^5 - 2c^4 - 8c^3 - 2c^2 - 2c - 2\\
a_2^{(1)}(c) &= c^9 + 2c^8 + 3c^7 + 2c^6 - 5c^4 - 9c^3 - 6c^2 - 3c - 1\\
a_1^{(1)}(c) &= 4c^6 + 7c^5 + 2c^4 - 4c^3 - 6c^2 - 3c\\
a_0^{(1)}(c) &= -c^9 - 3c^8 - 2c^7 + 3c^6 + 6c^5 + 3c^4 - 2c^3 - 3c^2 - c.
\end{aligned}$$

Level 2: There are $\pi = (2)$ and $\pi = (1^2)$. We calculate all the submatrices $N^{\pi}_{\mathcal{R}_i\mathcal{R}_j}$ directly, as shown in the beginning of the chapter, and then use them to

obtain N^{π}_{477}. Omitting all the rows and columns corresponding to a sets Y and X containing 2 respectively we can reduce N^{π}_{477} to 9×9 matrices. These reduced matrices are shown in Appendix C. The characteristic equations of N^{π}_{477} for $\pi = (2)$ and $\pi = (1^2)$ are

$$\lambda^{11}\Big(\lambda^5 + a^{\pi}_4(c)\lambda^4 + a^{\pi}_3(c)\lambda^3 + a^{\pi}_2(c)\lambda^2 + a^{\pi}_1(c)\lambda + a^{\pi}_0(c)\Big) = 0$$

with $a_4^{(2)}(c) = 2c - 1, \qquad a_3^{(2)}(c) = c - 2, \qquad a_2^{(2)}(c) = -c^4 - 3,$

$$a_1^{(2)}(c) = c^3 + c^2 - 2c - 2, \qquad a_0^{(2)}(c) = c^3 + c^2 - c - 1,$$

or $a_4^{(1^2)}(c) = -2c + 1, \qquad a_3^{(1^2)}(c) = -3c, \qquad a_2^{(1^2)}(c) = -c^4 - 3,$

$$a_1^{(1^2)}(c) = -c^3 - c^2, \qquad a_0^{(1^2)}(c) = -c^3 - c^2 + c + 1.$$

Level 3: Here there are three partitions $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2, 1)$. Omitting all the zero rows and the corresponding columns, following a similar argument as in the case of the generalized dodecahedra, we can reduce these matrices to $R^{\pi}(123)$, Young's natural representation of (123) . It follows that the characteristic equations for $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2, 1)$ are

$$\lambda^6(\lambda - 1) = 0, \qquad \lambda^6(\lambda - 1) = 0 \qquad \text{and} \qquad \lambda^{12}(\lambda^2 + \lambda + 1) = 0$$

respectively.

As before, Newton's formula, given in Appendix A, can be used to evaluate the sum $A_{\pi,n}$ of the n^{th} powers of the non-polynomial eigenvalues of the matrices N^{π}_{L}. Then, for general $n \in \mathbb{N}$, the chromatic polynomial of $(477)_n$ is:

$$\begin{aligned} P((477)_n; c) = {} & A_{(),n} + (c+1)A_{(1),n} \\ & + \frac{1}{2}(c^2 + c - 2)A_{(2),n} + \frac{1}{2}(c^2 + c)A_{(1^2),n} \\ & + \frac{1}{3}(c^3 - 4c)A_{(2,1),n} + \frac{1}{3}(c^3 - 4c - 3) \end{aligned}$$

In Figure 4.11 the roots of $(477)_{30}$ are plotted. Again, clearly visible are the roots with negative real part.

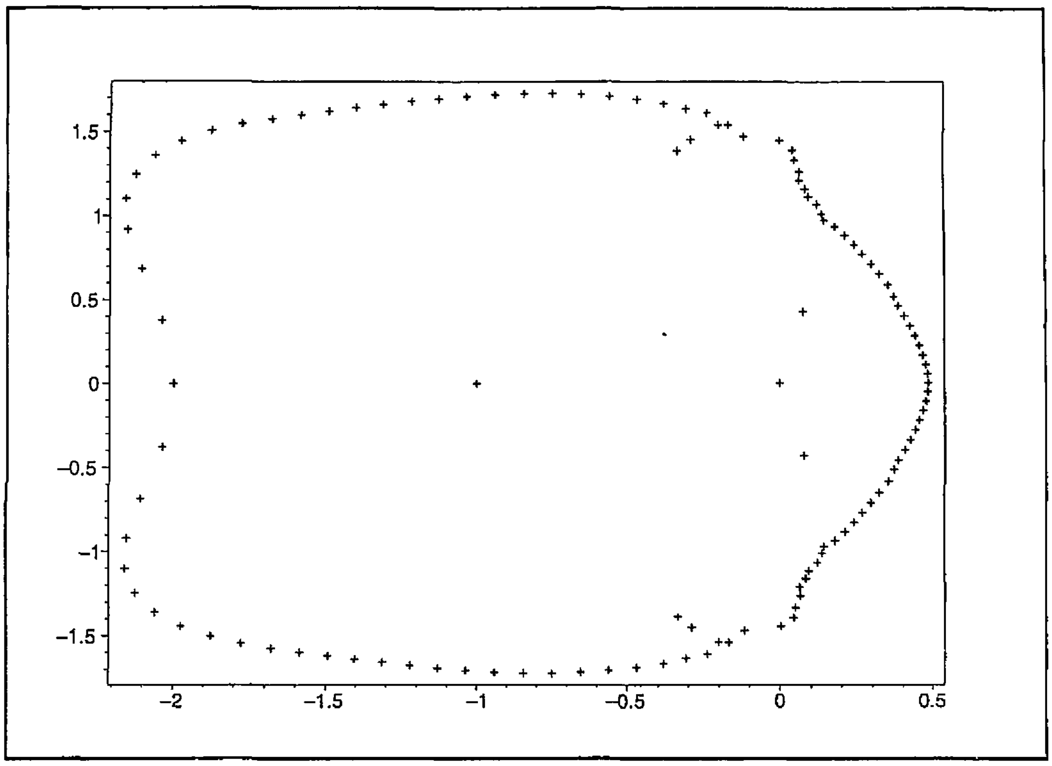


Figure 4.11: The roots of $(477)_{30}$

4.6.3 The family $(567)_n$

The Table 4.6 shows all this graphs ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ with $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4, 5$.

All induced subgraphs are part of the catalogue of graphs in Appendix B which we will use to evaluate levels zero and one. For levels two and three we calculate the matrices directly.

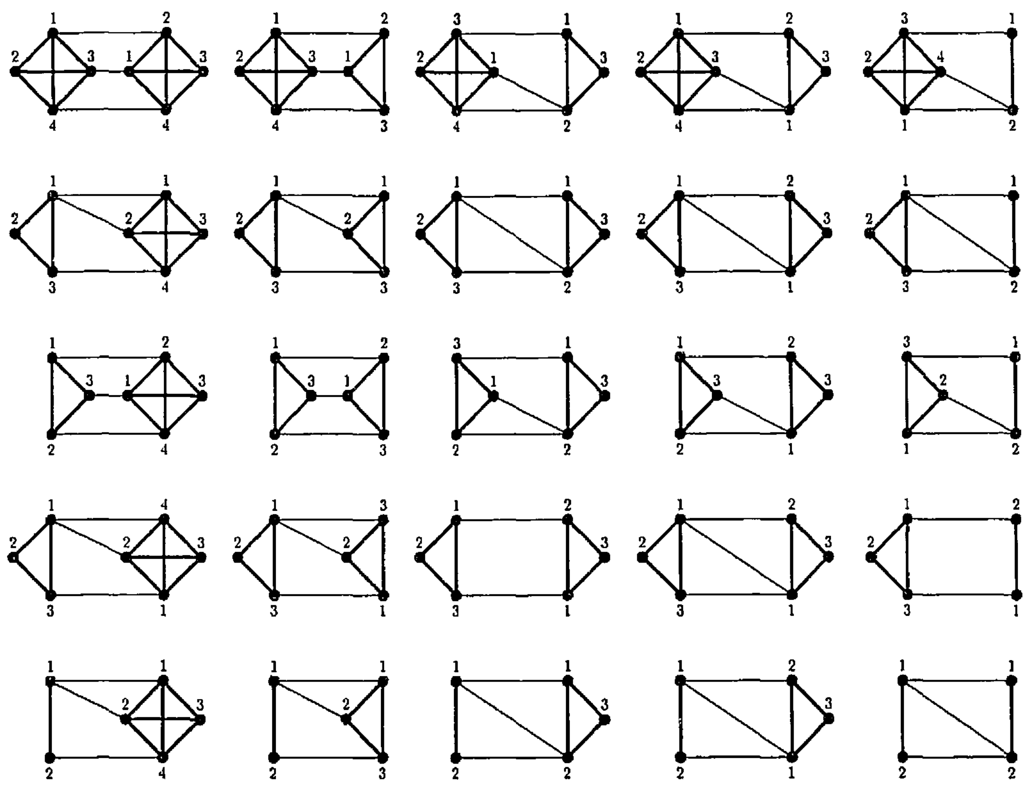


Table 4.6: The induced graphs $_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ in case of the family $(567)_n$

Level 0: The matrix $N_L^{()}$ corresponding to the graph $(567)_n$ can be written as

$$N_{567}^{()} = \begin{pmatrix} N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33b}^{()} & N_{32a}^{()} \\ N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33c}^{()} & N_{33b}^{()} & N_{32b}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33b}^{()} & N_{32a}^{()} \end{pmatrix}$$

$$= \begin{pmatrix} (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c^3 & c^2 \\ (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c(c^2+c+1) & c^3 & c^2+c+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c^3 & c^2 \end{pmatrix}$$

The characteristic equation is

$$\lambda^2\Big(\lambda^3 \;+\; a_2^{()}(c)\lambda^2 \;+\; a_1^{()}(c)\lambda \;+\; a_0^{()}(c)\Big) = 0$$

where

$$\begin{aligned}
a_2^{()}(c) &= -c^4 - 2c^3 - 3c^2 + c - 1 \\
a_1^{()}(c) &= -c^5 - 3c^4 - 2c^3 + c^2 + c \\
a_0^{()}(c) &= c^4 + 2c^3 + c^2.
\end{aligned}$$

Level 1: The matrix $N_{567}^{(1)}$ can be written as

$$\begin{pmatrix}
{}_{\epsilon}|\overline{N_{44}^{(1)}}^{(12)} & {}_{\epsilon}|\overline{N_{43a}^{(1)}}^{(12)} & {}_{(13)}|\overline{N_{43b}^{(1)}}^{\epsilon} & {}_{\epsilon}|\overline{N_{43b}^{(1)}}^{(12)} & {}_{(13)}|\overline{N_{42}^{(1)}}^{\epsilon} \\
N_{34}^{(1)} & N_{33a}^{(1)} & N_{33b}^{(1)} & {}_{\epsilon}|\overline{N_{33b}^{(1)}}^{(12)} & N_{32a}^{(1)} \\
{}_{\epsilon}|\overline{N_{44}^{*(1)}}^{(124)} & {}_{\epsilon}|\overline{N_{43a}^{*(1)}}^{(123)} & {}_{(13)}|\overline{N_{43b}^{*(1)}}^{\epsilon} & {}_{\epsilon}|\overline{N_{43b}^{*(1)}}^{(12)} & {}_{(13)}|\overline{N_{42}^{*(1)}}^{\epsilon} \\
{}_{\epsilon}|\overline{N_{34}^{(1)}}^{(14)} & {}_{\epsilon}|\overline{N_{33a}^{(1)}}^{(13)} & {}_{\epsilon}|\overline{N_{33c}^{(1)}}^{(12)} & {}_{\epsilon}|\overline{N_{33b}^{(1)}}^{(12)} & N_{32b}^{(1)} \\
N_{34}^{*(1)} & N_{33a}^{*(1)} & N_{33b}^{*(1)} & {}_{\epsilon}|\overline{N_{33b}^{*(1)}}^{(12)} & N_{32a}^{*(1)}
\end{pmatrix}.$$

Then the characteristic equation of $N_{567}^{(1)}$ is

$$\lambda^8\Big(\lambda^7+a_6^{(1)}(c)\lambda^6+a_5^{(1)}(c)\lambda^5+a_4^{(1)}(c)\lambda^4+a_3^{(1)}(c)\lambda^3+a_2^{(1)}(c)\lambda^2+a_1^{(1)}(c)\lambda+a_0^{(1)}(c)\Big)=0$$

with

$$\begin{aligned}
a_6^{(1)}(c) &= c^3 + 4c - 2 \\
a_5^{(1)}(c) &= 2c^4 + c^3 + 2c^2 - c - 2 \\
a_4^{(1)}(c) &= c^8 + 2c^7 + 4c^6 + 2c^5 + c^4 - 2c^3 - 2c^2 - 6c + 2 \\
a_3^{(1)}(c) &= c^9 + 3c^8 + 3c^7 - 2c^4 - 3c^3 - c^2 + 3c \\
a_2^{(1)}(c) &= -c^8 - 4c^7 - 8c^6 - 9c^5 - 8c^4 - 3c^3 + 4c^2 + 2c - 1 \\
a_1^{(1)}(c) &= -c^7 - 3c^6 - 3c^5 + 2c^4 + 5c^3 + c^2 - c \\
a_0^{(1)}(c) &= c^6 + 4c^5 + 6c^4 + 4c^3 + c^2.
\end{aligned}$$

Level 2: There are $\pi = (2)$ and $\pi = (1^2)$. We calculate all the submatrices $N^{\pi}_{\mathcal{R}_i\mathcal{R}_j}$ directly, as shown in the beginning of the chapter, and then use them to

obtain N_{567}^{π}. Omitting all the rows and columns corresponding to a sets Y and X containing 2 respectively we can reduce N_{567}^{π} to 9×9 matrices. These reduced matrices are shown in Appendix C. The characteristic equations of N_{567}^{π} for $\pi = (2)$ and $\pi = (1^2)$ are

$$\lambda^{11}\Big(\lambda^5 + a_4^{\pi}(c)\lambda^4 + a_3^{\pi}(c)\lambda^3 + a_2^{\pi}(c)\lambda^2 + a_1^{\pi}(c)\lambda + a_0^{\pi}(c)\Big) = 0$$

with $a_4^{(2)}(c) = 2c - 2, \quad a_3^{(2)}(c) = c^3 + c^2 - c - 2, \quad a_2^{(2)}(c) = c^4 - c + 2,$

$$a_1^{(2)}(c) = -c^3 - 2c^2 - c, \qquad a_0^{(2)}(c) = -c^2 - 2c - 1,$$

or $a_4^{(1^2)}(c) = -2, \quad a_3^{(1^2)}(c) = c^3 - c^2 + 3c, \quad a_2^{(1^2)}(c) = -c^4 - 3c + 2,$

$$a_1^{(1^2)}(c) = -c^3 - c - 2, \qquad a_0^{(1^2)}(c) = c^2 + 2c + 1.$$

Level 3: Here there are three partitions $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$. Omitting all the zero rows and the corresponding columns, following a similar argument as in the case of the generalized dodecahedra, we can reduce these matrices to $R^{\pi}(12)$, Young's natural representation of (12). It follows that the characteristic equations for $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$ are

$$\lambda^6(\lambda - 1) = 0, \qquad \lambda^6(\lambda + 1) = 0 \qquad \text{and} \qquad \lambda^{12}(\lambda - 1)(\lambda + 1) = 0$$

respectively.

As before, Newton's formula, given in Appendix A, can be used to evaluate the sum $A_{\pi,n}$ of the n^{th} powers of the non-polynomial eigenvalues of the matrices N_L^{π}. Then, for general $n \in \mathbb{N}$, the chromatic polynomial of $(567)_n$ is:

$$\begin{aligned} P((567)_n; c) = {} & A_{(),n} + (c+1)A_{(1),n} \\ & + \frac{1}{2}(c^2 + c - 2)A_{(2),n} + \frac{1}{2}(c^2 + c)A_{(1^2),n} \\ & + \frac{1}{2}(c^3 - 3c)(-1)^n + \frac{1}{2}(c^3 - 5c - 2) \end{aligned}$$

In Figure 4.12 the roots of $(567)_{30}$ are plotted. Again, clearly visible are the roots with negative real part.

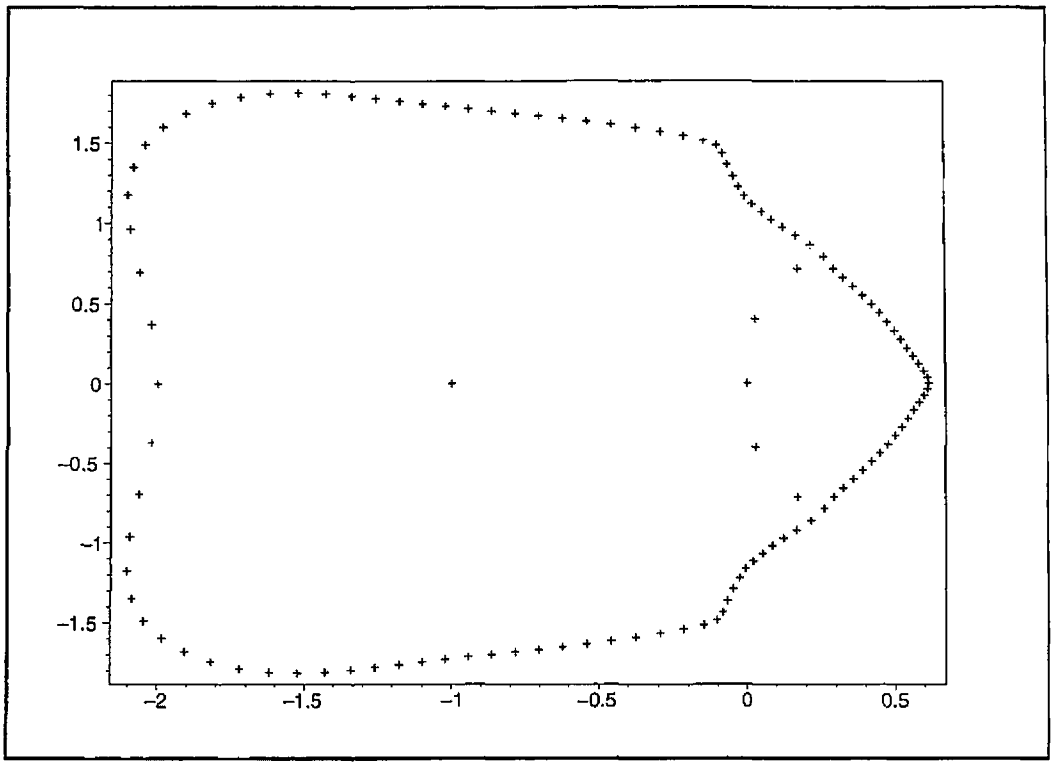


Figure 4.12: The roots of $(567)_{30}$

4.6.4 The family $(666)_n$

The Table 4.7 shows all this graphs ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ with $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4, 5$.

All induced subgraphs are part of the catalogue of graphs in Appendix B which we will use to evaluate levels zero and one. For levels two and three we calculate the matrices directly.

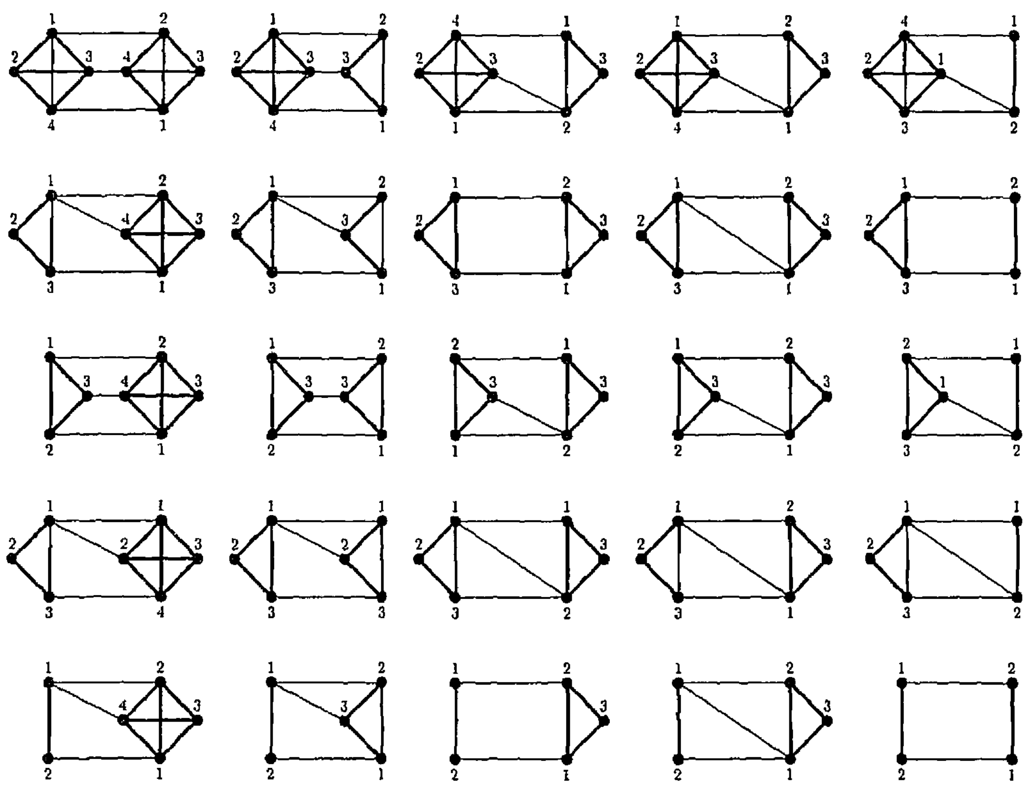


Table 4.7: The induced graphs ${}_{|\mathcal{R}_i|}\langle L_{\mathcal{R}_i,\mathcal{R}_j}\rangle_{|\mathcal{R}_j|}$ in case of the family $(666)_n$

Level 0: The matrix $N_L^{()}$ corresponding to the graph $(666)_n$ can be written as

$$N_{666}^{()} = \begin{pmatrix} N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33c}^{()} & N_{33b}^{()} & N_{32b}^{()} \\ N_{44}^{()} & N_{43a}^{()} & N_{43b}^{()} & N_{43b}^{()} & N_{42}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33b}^{()} & N_{33b}^{()} & N_{32a}^{()} \\ N_{34}^{()} & N_{33a}^{()} & N_{33c}^{()} & N_{33b}^{()} & N_{32b}^{()} \end{pmatrix}$$

$$= \begin{pmatrix} (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c(c^2+c+1) & c^3 & c^2+c+1 \\ (c-1)(c^3+2c-1) & c^3+2c-1 & c(c^2+1) & c(c^2+1) & c^2+1 \\ c(c-1)(c^2+1) & c(c^2+1) & c^3 & c^3 & c^2 \\ c(c-1)(c^2+1) & c(c^2+1) & c(c^2+c+1) & c^3 & c^2+c+1 \end{pmatrix}$$

The characteristic equation is

$$\lambda^2\Big(\lambda^3 + a_2^{()}(c)\lambda^2 + a_1^{()}(c)\lambda + a_0^{()}(c)\Big) = 0$$

where

$$
\begin{aligned}
a_2^{()}(c) &= -c^4 - 2c^3 - 3c^2 - 2\\
a_1^{()}(c) &= -c^4 + 1\\
a_0^{()}(c) &= -c^4 - 2c^3 - c^2.
\end{aligned}
$$

Level 1: The matrix $N_{666}^{(1)}$ can be written as

$$
\begin{pmatrix}
\epsilon|\overset{(124)}{\overline{N_{44}^{(1)}}} & \epsilon|\overset{(123)}{\overline{N_{43a}^{(1)}}} & (14)|\overset{\epsilon}{\overline{N_{43b}^{(1)}}} & \epsilon|\overset{(12)}{\overline{N_{43b}^{(1)}}} & (14)|\overset{\epsilon}{\overline{N_{42}^{(1)}}}\\
\epsilon|\overset{(14)}{\overline{N_{34}^{(1)}}} & \epsilon|\overset{(13)}{\overline{N_{33a}^{(1)}}} & \epsilon|\overset{(12)}{\overline{N_{33c}^{(1)}}} & \epsilon|\overset{(12)}{\overline{N_{33b}^{(1)}}} & N_{32b}^{(1)}\\
\epsilon|\overset{(12)}{\overline{N_{44}^{*(1)}}} & \epsilon|\overset{(12)}{\overline{N_{43a}^{*(1)}}} & (12)|\overset{\epsilon}{\overline{N_{43b}^{*(1)}}} & \epsilon|\overset{(12)}{\overline{N_{43b}^{*(1)}}} & (12)|\overset{\epsilon}{\overline{N_{42}^{*(1)}}}\\
N_{34}^{(1)} & N_{33a}^{(1)} & N_{33b}^{(1)} & \epsilon|\overset{(12)}{\overline{N_{33b}^{(1)}}} & N_{32a}^{(1)}\\
\epsilon|\overset{(14)}{\overline{N_{34}^{*(1)}}} & \epsilon|\overset{(13)}{\overline{N_{33a}^{*(1)}}} & \epsilon|\overset{(12)}{\overline{N_{33c}^{*(1)}}} & \epsilon|\overset{(12)}{\overline{N_{33b}^{*(1)}}} & N_{32b}^{*(1)}
\end{pmatrix}.
$$

Then the characteristic equation of $N_{666}^{(1)}$ is

$$\lambda^8\left(\lambda^7+a_6^{(1)}(c)\lambda^6+a_5^{(1)}(c)\lambda^5+a_4^{(1)}(c)\lambda^4+a_3^{(1)}(c)\lambda^3+a_2^{(1)}(c)\lambda^2+a_1^{(1)}(c)\lambda+a_0^{(1)}(c)\right)=0$$

with

$$
\begin{aligned}
a_6^{(1)}(c) &= -2c^2 - 4\\
a_5^{(1)}(c) &= -c^4 - 2c^2 + 4c + 4\\
a_4^{(1)}(c) &= -c^8 - 2c^7 - 5c^6 - 4c^5 - c^4 + 6c^3 + c^2 - 2\\
a_3^{(1)}(c) &= c^8 + 6c^7 + 10c^6 + 8c^5 + 3c^4 + 2c^3 - 2c^2 - 4c + 4\\
a_2^{(1)}(c) &= -2c^7 - 5c^6 - 6c^5 - 5c^4 - 2c^3 + 4c^2 - 4\\
a_1^{(1)}(c) &= 2c^6 + 6c^5 + 7c^4 + 2c^3 - 2c^2 + 1\\
a_0^{(1)}(c) &= -c^6 - 4c^5 - 6c^4 - 4c^3 - c^2.
\end{aligned}
$$

Level 2: There are $\pi = (2)$ and $\pi = (1^2)$. We calculate all the submatrices $N^{\pi}_{\mathcal{R}_i\mathcal{R}_j}$ directly, as shown in the beginning of the chapter, and then use them to

obtain N_{666}^{π}. Omitting all the rows and columns corresponding to a sets Y and X containing 2 respectively we can reduce N_{666}^{π} to 9×9 matrices. These reduced matrices are shown in Appendix C. The characteristic equations of N_{666}^{π} for $\pi = (2)$ and $\pi = (1^2)$ are

$$\lambda^{11}\Big(\lambda^5 + a_4^{\pi}(c)\lambda^4 + a_3^{\pi}(c)\lambda^3 + a_2^{\pi}(c)\lambda^2 + a_1^{\pi}(c)\lambda + a_0^{\pi}(c)\Big) = 0$$

with $a_4^{(2)}(c) = -3,\quad a_3^{(2)}(c) = -2c^2 + 2c + 1,\quad a_2^{(2)}(c) = -c^4 - 2c^2 - 1,$

$a_1^{(2)}(c) = 2c^3 + 3c^2 + 4c + 3,\quad a_0^{(2)}(c) = -c^2 - 2c - 1,$

or $a_4^{(1^2)}(c) = 1,\quad a_3^{(1^2)}(c) = 2c^2 - 2c + 1,\quad a_2^{(1^2)}(c) = -c^4 - 2c^2 - 1,$

$a_1^{(1^2)}(c) = -2c^3 - c^2 - 1,\quad a_0^{(1^2)}(c) = -c^2 - 2c - 1.$

Level 3: Here there are three partitions $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$. Omitting all the zero rows and the corresponding columns, following a similar argument as in the case of the generalized dodecahedra, we can reduce these matrices to $R^{\pi}(132)$, Young's natural representation of (132). It follows that the characteristic equations for $\pi = (3)$, $\pi = (1^3)$ and $\pi = (2,1)$ are

$$\lambda^6(\lambda - 1) = 0, \qquad \lambda^6(\lambda - 1) = 0 \qquad \text{and} \qquad \lambda^{12}(\lambda^2 + \lambda + 1) = 0$$

respectively.

As before, Newton's formula, given in Appendix A, can be used to evaluate the sum $A_{\pi,n}$ of the n^{th} powers of the non-polynomial eigenvalues of the matrices N_L^{π}. Then, for general $n \in \mathbb{N}$, the chromatic polynomial of $(666)_n$ is:

$$\begin{aligned} P((666)_n; c) = {} & A_{(),n} + (c+1)A_{(1),n} \\ & + \frac{1}{2}(c^2 + c - 2)A_{(2),n} + \frac{1}{2}(c^2 + c)A_{(1^2),n} \\ & + \frac{1}{3}(c^3 - 4c)A_{(2,1),n} + \frac{1}{3}(c^3 - 4c - 3) \end{aligned}$$

In Figure 4.13 the roots of $(666)_{30}$ are plotted. Again, clearly visible are the roots with negative real part.

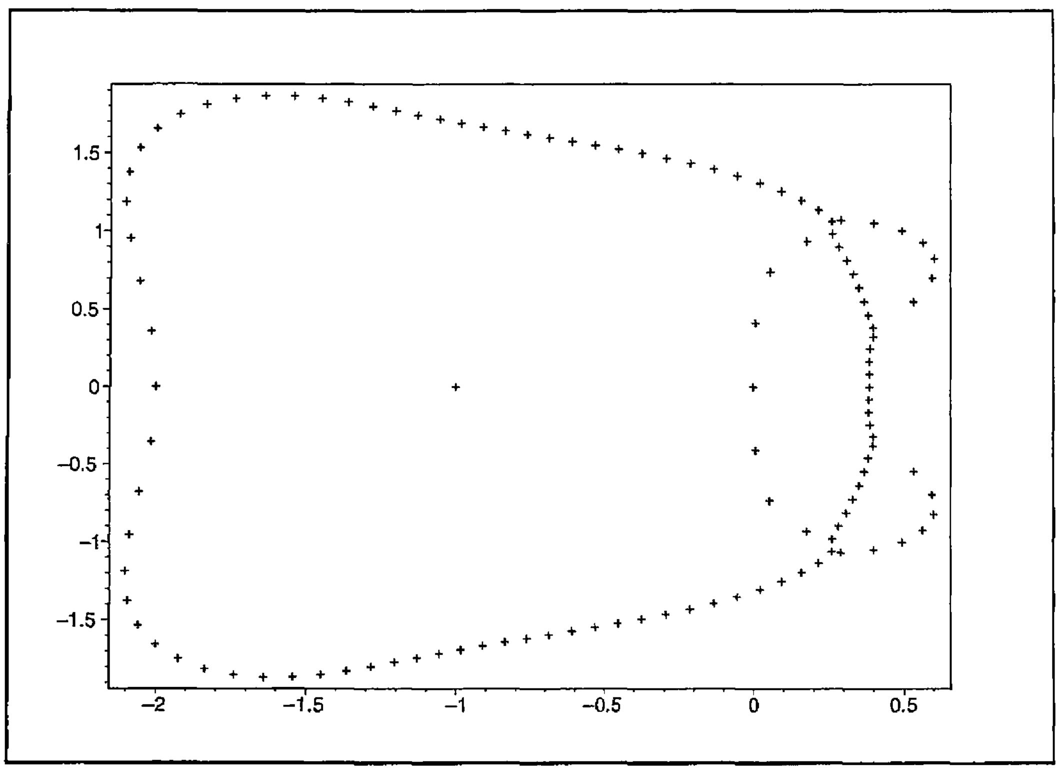


Figure 4.13: The roots of $(666)_{30}$

Chapter 5

Equimodular curves

In this chapter we discuss the behaviour of the roots of the chromatic polynomial as the number of copies of the base graph goes to infinity. We shall refer to them as *chromatic roots.* The framework for the following is taken from [4] and [6].

Recall that for any given base graph B, any linking set L and $k \in \mathbb{N}$ the compatibility matrix $T_L(k)$ corresponding to the family $L_n(B)$ is equivalent to a matrix of the form

$$\bigoplus_{\substack{0\leq \ell \leq b \\ \pi \vdash \ell}} (I_\pi \otimes N_L^\pi),$$

where I_π is the identity matrix of size $\eta_\pi(k)$ given in Theorem 3.3, and N_L^π is a matrix of size $\sum\limits_{\mathcal{R}\in\Pi(B)} \binom{|\mathcal{R}|}{\ell} n_\pi$ with entries depending on k (n_π is the dimension of the Specht module $\mathcal{S}^\pi$).

By Theorems 3.3 and 3.13 it follows that for every $n \in \mathbb{N}$ the chromatic polynomial of $L_n(B)$ is of the form

$$P(L_n(B), k) = \sum_{\ell=0}^{b} \sum_{\pi\vdash\ell} \eta_\pi(k) \;\; \mathrm{tr}(N_L^\pi)^n.$$

Further, recall that for every π, if $\lambda_1^\pi(k),\ \lambda_2^\pi(k), \ldots, \lambda_r^\pi(k)$ are the eigenvalues of N_L^π with respective algebraic multiplicities $m_1^\pi,\ m_2^\pi, \ldots, m_r^\pi$ in N_L^π then

$$\mathrm{tr}(N_L^\pi)^n = \sum_{i=1}^{r} m_i^\pi \left(\lambda_i^\pi(k)\right)^n$$

This particular structure of the chromatic polynomials allows us to use a theorem by Beraha, Kahane and Weiss to investigate the limiting behaviour of the chromatic roots as n goes to infinity.

5.1 A theorem of Beraha, Kahane and Weiss

The Figures 3.5, 4.7, 4.10, 4.11, 4.12 and 4.13 suggest that the roots approach some set of curves as n grows (plus some isolated points). This behaviour of the roots can be understood using a theorem of Beraha, Kahane and Weiss [18]: Suppose that we have a family of polynomials $\{P_n(z)\}$ of the form

$$P_n(z) = \sum_{i=1}^{s} m_i(z)\big(\lambda_i(z)\big)^n.$$

A complex number ζ is defined to be a limit point of roots of this family if there exists a sequence $\{z_j\}$ tending to ζ, such that z_j is a root of $P_j(z)$ for every j. We say that a root *dominates* the other roots if it has the largest modulus.

Theorem 5.1 [Beraha, Kahane, Weiss 1980] *Under the non-degeneracy conditions that $\{P_n(z)\}$ does not satisfy a lower order recurrence, and $\lambda_i(z)/\lambda_j(z)$ is not identically a constant of unit modulus for any $i \neq j$, the complex number ζ is a limit point of zeros of $\{P_n(z)\}$ if and only if, at $z = \zeta$, one of the following two conditions holds:*

- *One of the roots $\lambda_i(z)$ dominates all the other roots , and the corresponding $m_i(z) = 0$. Or,*
- *Two or more of the roots $\lambda_i(z)$ are of equal modulus and dominate the others.*

The chromatic polynomials obtained by the compatibility matrix method are exactly of the form required for the above theorem. The limit points of the first type are isolated points and easy to determine. We shall concentrate on finding limit points of the second type.

We first consider all points where two or more of the roots $\lambda_i(z)$ are of equal modulus, and then check their dominance. The $\lambda_i(z)$ in Theorem 5.1 are the eigenvalues of the matrices N_L^π.

Example 5.1: Recall the family $B(3)_n$ with complete base graph K_3 and identity linking set. Its chromatic polynomial has been obtained in Example 3.8. This example is particularly "nice" since all the characteristic equations of all the the N_L^π factorize into linear factors in λ. That is all the eigenvalues are all polynomials in k. There are eight distinct eigenvalues:

$$\begin{array}{lll} \lambda_1 = k^3 - 6k^2 + 14k - 13, & \lambda_2 = -k^2 + 7k - 13, & \lambda_3 = -k^2 + 4k - 4 \\ \lambda_4 = k - 2, & \lambda_5 = k - 5, & \lambda_6 = k - 1, \\ \lambda_7 = k - 4 \quad \text{and} & \lambda_8 = 1. & \end{array}$$

Then $|\lambda_i| = |\lambda_j|$ is equivalent to

$$\Re(\lambda_i)^2 + \Im(\lambda_i)^2 = \Re(\lambda_j)^2 + \Im(\lambda_j)^2$$

where $\Re(\lambda)$ denotes the real and $\Im(\lambda)$ the imaginary part of a complex function λ. Using the "`implicitplot`" function in `Maple 7` we obtain a a collection of "equimodular curves". There are 28 $|\lambda_i| = |\lambda_j|$ with $i \neq j$, but only a few of them contain points where the eigenvalues of equal modulus also dominate the other eigenvalues. We call them "dominant points". In Figure 5.1 the curves corresponding to $|\lambda_1| = |\lambda_2|$, $|\lambda_1| = |\lambda_5|$ and $|\lambda_2| = |\lambda_5|$ are shown. It turns out that these are the only curves containing dominant points. All three curves intersect in what will be called "triple points". In the following sections we will show that all points of a segment of an "equimodular curve" containing at most two "triple points" as endpoints are either all dominant or all are not. In Figure 5.1 the roots of B_{30} (circles) are also shown. Clearly, they cluster along segments of these curves, and the "dominance" property changes at the triple points.

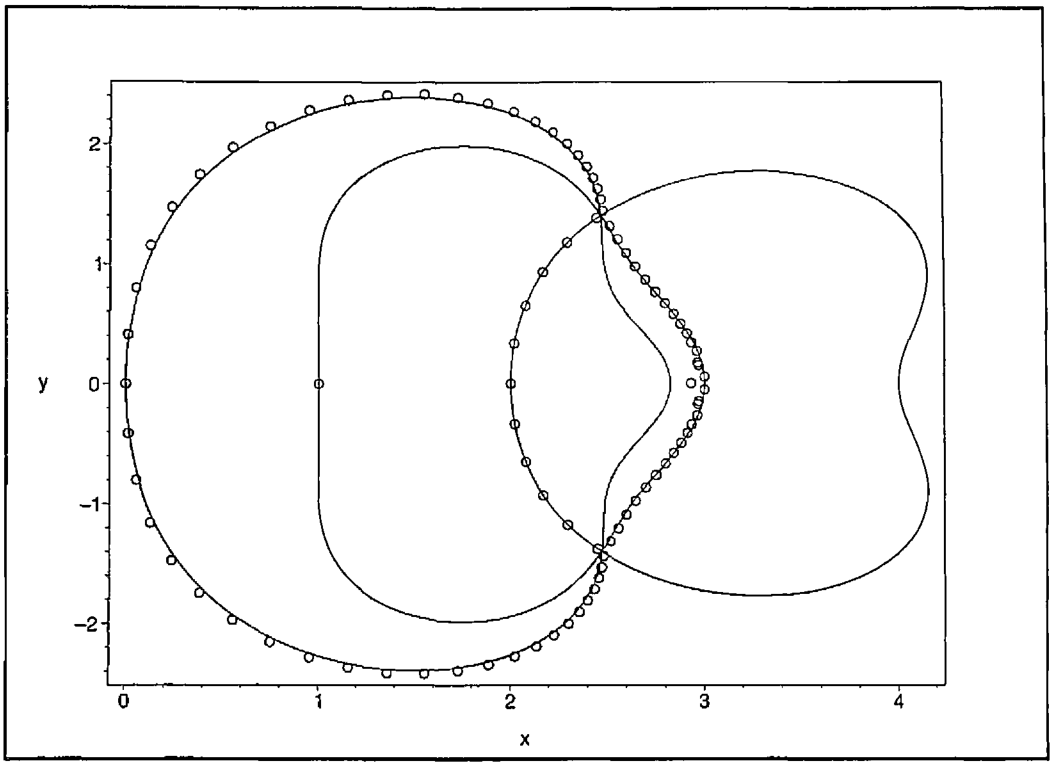


Figure 5.1: The curves $|\lambda_1| = |\lambda_2|$, $|\lambda_1| = |\lambda_5|$, $|\lambda_2| = |\lambda_5|$ and the roots of B_{30}

5.2 Equimodular points

Before exploring these curves further, let us first introduce some more general notation. Let $f(\lambda, z)$ be a polynomial of degree m in the complex variable λ of the form

$$f(\lambda, z) = \lambda^m + f_1(z)\lambda^{m-1} + f_2(z)\lambda^{m-2} + \ldots + f_{m-1}(z)\lambda + f_m(z)$$

where the coefficients $f_i(z)$ of $f(\lambda, z)$ are polynomials in the complex variable z with integer coefficients. We assume that $f(\lambda, z)$ does not contain repeated factors.

An *equimodular point* is a point z_0 in $\mathbb{C}$ where two roots of $f(\lambda, z_0) = 0$ are of equal modulus. Denote by $E(f)$ the subset of $\mathbb{C}$ consisting of all equimodular points of f. This includes roots of algebraic multiplicity two or more.

If $f(\lambda, z)$ and $g(\lambda, z)$ are two such polynomials, possibly of different degrees, we say that a point z_0 in $\mathbb{C}$ is an *equimodular point* of f and g if one of the roots of $f(\lambda, z_0) = 0$ and one of the roots of $g(\lambda, z_0) = 0$ are of equal modulus. Denote by

$E(f,g)$ the subset of $\mathbb{C}$ containing all the equimodular points of f and g.

Suppose that f factors as $f(\lambda, z) = u(\lambda, z)w(\lambda, z)$, where $u(\lambda, z)$ and $w(\lambda, z)$ are polynomials of the same form as $f(\lambda, z)$ with degrees l and d respectively. Then

$$E(f) = E(u) \cup E(w) \cup E(u, w).$$

Recall that for all π the matrices N_L^π corresponding to a graph $L_n(B)$ are such that every component is a polynomial in k with integer coefficients. The coefficients of its characteristic polynomial are sums of principal minors of N_L^π and thus polynomials in k with integer coefficients. Thus we can replace k by a complex variable z and it follows that the characteristic polynomial of N_L^π is of the form $f(\lambda, z)$.

5.3 The resultant

If $f(\lambda, z)$ factorizes into linear factors in λ then all the roots are polynomial functions in z, and we can equate their moduli one by one, as done in Example 5.1. From algebraic geometry it follows that the $E(f)$ are collections of continuous and almost everywhere differentiable curves on the Riemann Sphere. Unfortunately the polynomials $f(\lambda, z)$ do not always factorize completely as we saw in the previous chapter. That is the roots are not all polynomial functions. More powerful tools are needed. Assume that $m = l + d$ and let

$$u(\lambda) = \lambda^l + u_1\lambda^{l-1} + u_2\lambda^{l-2} + \ldots + u_{l-1}\lambda + u_l$$

and

$$w(\lambda) = \lambda^d + w_1\lambda^{d-1} + w_2\lambda^{d-2} + \ldots + w_{d-1}\lambda + w_d$$

be two polynomials in $\mathbb{C}[\lambda]$. The key idea is that if λ_u is a root of $u(\lambda)$ and λ_w is a root of $w(\lambda)$ with $|\lambda_u| = |\lambda_w|$ then there exists $s \in \mathbb{C}$ with $|s| = 1$ such that $\lambda_w = s\lambda_u$. It follows that λ_u is a common root of the polynomials $u(\lambda)$ and

$$w_s(\lambda) = w(s\lambda) = s^d\lambda^d + w_1 s^{d-1}\lambda^{d-1} + w_2 s^{d-2}\lambda^{d-2} + \ldots + w_{d-1}s\lambda + w_d$$

It is a standard result [14] that $u(\lambda)$ and $w_s(\lambda)$ have a common root if and only if the *resultant* $\det R_{u,w_s}$ vanishes for some $s \in \mathbb{C}$ with $|s| = 1$, where R_{u,w_s} is the $ld \times ld$ matrix:

$$\begin{pmatrix}
1 & u_1 & u_2 & \dots & u_{l-1} & u_l & & & & \\
 & 1 & u_1 & u_2 & \dots & u_{l-1} & u_l & & & \\
 & & \cdots & \cdots & \cdots & \cdots & & & & \\
 & & & & 1 & u_1 & u_2 & \dots & u_{l-1} & u_l \\
s^d & s^{d-1}w_1 & s^{d-2}w_2 & \dots & sw_{d-1} & w_d & & & & \\
 & s^d & s^{d-1}w_1 & s^{d-2}w_2 & \dots & sw_{d-1} & w_d & & & \\
 & & \cdots & \cdots & \cdots & \cdots & & & & \\
 & & & & s^d & s^{d-1}w_1 & w_2 & \dots & sw_{d-1} & w_d
\end{pmatrix}.$$

The blank spaces are supposed to be filled with zeros. Many properties of the resultant have been discussed in [4] and [6]. If $u(\lambda) = w(\lambda)$ then R_{u,u_s} is a $2l \times 2l$ matrix and $\det R_{u,u_s}$ vanishes if $u(\lambda)$ has two roots of equal modulus. Returning to our polynomials $f(\lambda, z)$ and $g(\lambda, z)$ it follows that $\det R_{f_s,g}$ is a polynomial in s and z with integer coefficients, and

$$E(f, g) = \{z \in \mathbb{C} \mid \det R_{f_s,g}(s, z) = 0 \text{ for some } s \in S^1\},$$

where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in $\mathbb{C}$.

In particular, if $f = g$ then from the explicit form of the determinant it can be seen that $\det R_{f,f_s}$ has a factor of $(s-1)^m$. Let $r_f(s, z)$ be the polynomial in s and z such that

$$\det R_{f,f_s} = (s-1)^m \, r_f(s, z).$$

Thus, provided there are no factors of r_f that are independent of z,

$$E(f) = \{z \in \mathbb{C} \mid r_f(s, z) = 0 \text{ for some } s \in S^1\}.$$

From the property [14]: $\det R_{uw,g} = \det R_{u,g} \det R_{w,g}$, for any polynomials $u(\lambda, z)$, $w(\lambda, z)$ and $g(\lambda, z)$, it follows that:

Lemma 5.2 *If $f(\lambda,z) = u(\lambda,z)w(\lambda,z)$, where $u(\lambda,z)$ and $w(\lambda,z)$ are polynomials of degrees l and d in λ respectively, then*

$$(s-1)^{l+d} r_f(s,z) = \det R_{f,f_s} = (s-1)^{l+d} r_u(s,z)\, r_w(s,z)\ \det R_{u_s,w}\ \det R_{u,w_s}.$$

□

Note that in [4] and [6] using the substitution $t = s + s^{-1} + 2$ the polynomial $v_f : [0,4] \times \mathbb{C} \to \mathbb{C}$ is obtained where $s \in S^1$ implies that $t \in [0,4]$. This has the advantage that t is a real variable. In this case

$$E(f) = \{z \in \mathbb{C} \mid v_f(t,z) = 0 \text{ for some } t \in [0,4]\}.$$

5.4 Equimodular curves

Let $\det R_{f_s,g} : S^1 \times \mathbb{C} \to \mathbb{C}$ and suppose that $(s_0, z_0) \in S^1 \times \mathbb{C}$ is such that $\det R_{f_s,g}(s_0,z_0) = 0$. From the Implicit Function Theorem it follows that if the Jacobian of the mapping $z \mapsto \det R_{f_s,g}(s,z)$ is not zero at (s_0,z_0) then there exists a unique continuous and differentiable map $\phi : \Omega \to \mathbb{C}$ defined on some open neighborhood $\Omega \subset S^1$ such that $\phi(s_0) = z_0$ and $\det R_{f_s,g}(s,\phi(s)) = 0$ for all $s \in \Omega$. Since there is only a finite number of points $(s_0,z_0) \in S^1 \times \mathbb{C}$ with $\det R_{f_s,g}(s_0,z_0) = 0$ and a vanishing Jacobian, it follows that $E(f,g)$ is the union of homeomorphic images of the open intervals Ω. We refer to $E(f,g)$ (or $E(f)$ if $f = g$) as the sets of the *equimodular curves* corresponding to $f(\lambda,z)$ and $g(\lambda,z)$.

Denote by $E(T_L)$ the set of equimodular curves corresponding to the characteristic polynomial of T_L for some given $L_n(B)$. Denote by $f^\pi(\lambda,z)$ the characteristic polynomial of N_L^π. From Lemma 5.2 it follows that:

Corollary 5.3 *The set of equimodular points $E(T_L)$ consists, except for some isolated points ("degenerate curves"), of the union of piecewise differentiable curves. Moreover, it is the union of all the $E(f^\pi)$ and $E(f^\pi, f^{\pi'})$ for distinct π and π'.*

□

Note that the "degenerate curves" in the previous corollary correspond to factors of the resultant that are independent of s.

Lemma 5.4 *Let the polynomial $f(\lambda, z)$ be given. Then $r_f(s, z) = r_f(\overline{s}, z)$, where $\overline{s}$ denotes the conjugate of s.*

Proof: Recall that $r_f(s_0, z_0) = 0$ for some $(s_0, z_0) \in S^1 \times \mathbb{C}$ if and only if

$$f(s_0\lambda, z_0) = 0 \quad \text{and } f(\lambda, z_0) = 0 \quad \text{for some } \lambda \in \mathbb{C}.$$

Let $\lambda' = s_0\lambda$ then

$$f(s_0^{-1}\lambda', z_0) = f(\lambda, z_0) = 0 \quad \text{and } f(\lambda', z_0) = f(s_0\lambda, z_0) = 0.$$

Hence $r_f(s_0^{-1}, z_0) = 0$. This holds for all $(s_0, z_0) \in S^1 \times \mathbb{C}$ with $r_f(s_0, z_0) = 0$. Since $s^{-1} = \overline{s}$, the result follows. □

Let $S^+ = \{z \in S^1 \mid \Im(z) \geq 0\}$ and $S^- = \{z \in S^1 \mid \Im(z) \leq 0\}$. From the previous lemmas we have the following:

Corollary 5.5 *Except for some isolated points ("degenerate curves"), $E(f)$ is the union of piecewise differentiable curves where the points corresponding to $r_f(1, z) = 0$ and to $r_f(-1, z) = 0$ are endpoints. Further:*

$$\begin{aligned} E(f) &= \{z \in \mathbb{C} \mid r_f(s, z) = 0 \textit{ for some } s \in S^+\} \\ &= \{z \in \mathbb{C} \mid r_f(s, z) = 0 \textit{ for some } s \in S^-\}. \end{aligned}$$

□

In [4] Section 5 it has been shown that every point corresponding to $r_f(-1, z) = 0$ is a double root. Hence, the above curves occur in pairs that coincide in the points corresponding to $r_f(-1, z) = 0$.

5.4.1 Examples

Let B be the path on three vertices and let $L = \{(1,1),(2,2),(3,3)\}$ be the identity linking set. The resulting graph $L_n(B)$ has been considered in Example 3.12 and the matrices N_L^π for all π have been obtained. The polynomials

$$f(\lambda, z) = \lambda^2 + (-z^3 - z^2 - 3z)\lambda + z^4 + z^3 + z^2 - 1$$

and $g(\lambda, z) = \lambda - c + 2$ are irreducible factors in the characteristic polynomials at levels zero and two respectively. In the following two examples we shall study the sets $E(f)$ and $E(f,g)$ in detail. In Section 5.6.1 all levels corresponding to this example are analyzed numerically.

Example 5.2: Let us analyze the set $E(f)$. Here

$$r_f(s,z) = (z+1)(z^3+z-1)q(s,z) \qquad \text{where}$$

$$q(s,z) = -sz^6 - 2sz^5 + (s^2-5s+1)z^4 + (s^2-4s+1)z^3 + (s^2-7s+1)z^2 - s^2 - 2s - 1.$$

In Figure 5.2 the equimodular curves $E(f)$ are shown. The points z satisfying $r_f(s,z) = 0$ for $s = 1$ and $s = -1$ are indicated by $\diamond$ and $\square$ respectively. One can see very nicely that $E(f)$ consists of a union of curves with endpoints $\diamond$ and $\square$. The points $\square$ ($-0.5 \pm 1.6583i$ and 0) are double roots where pairs of curves coincide. Further there are four points ("degenerate curves") at -1, $-0.3412 \pm 1.1615i$ and 0.6823 corresponding to the equation $(z+1)(z^3+z-1) = 0$. Here and in the following, points are represented by approximations to four decimal places.

Example 5.3: Let us analyze the set $E(f,g)$. Here

$$\det R_{f_s,g}(s,z) = (-s+1)z^4 + (s+1)z^3 + (s^2-s+1)z^2 + (-4s^2+6s)z + 4s^2 - 1.$$

The coefficient for z^4 vanishes for $s = 1$. This implies that one of the points in $E(f,g)$ corresponding to $s = 1$ is at infinity. For all other $s \in S^1$ there are four solutions of $\det R_{f_s,g}(s,z) = 0$. Hence there are four segments in $E(f,g)$. In Figure 5.3

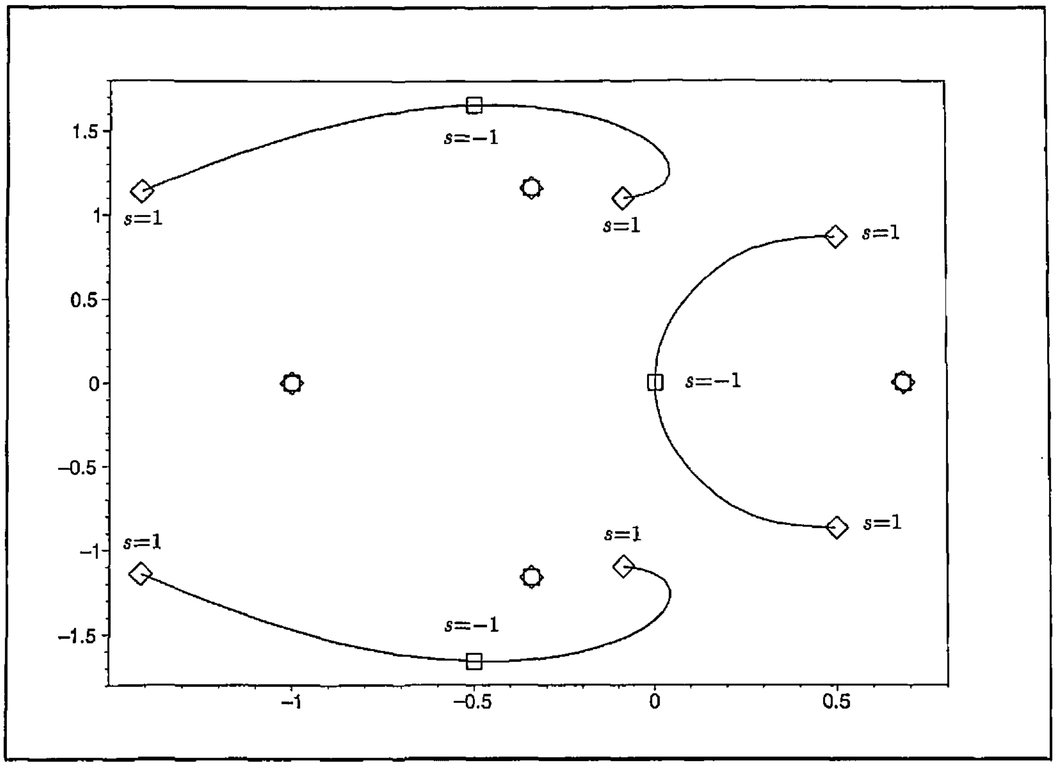


Figure 5.2: $E(f)$

the equimodular curves $E(f,g)$ are shown. The points z satisfying $\det R_{f_s,g} = 0$ for $s = 1$ and $s = -1$ are indicated by $\diamond$ and $\square$ respectively. The closed curve on the left hand side contains three points $0.2500 \pm 1.1990i$ and -1 corresponding to $s = 1$, and three points $-0.7906 \pm 10.7193i$ and 0.3365 corresponding to $s = -1$. In this case, as s runs over the values in S^1 in an anti-clockwise direction we move along the curve in an anti-clockwise direction (as indicated).

The curve has two cusps at ω and ω^5 where ω is a primitive sixth root of unity. They are represented by circles. These points can be obtained as follows:

$$D(z) = (z^2 - z + 1)(z^4 + 3z^3 + 5z^2 + 4z + 4)(z - 2)^2$$

is the discriminant of $\det R_{f_s,g}(s,z)$ with respect to s. For each z where $D(z)$ vanishes there exists a root s of $\det R_{f_s,g}(s,z)$ of multiplicity at least two. Only the two roots $(z^2 - z + 1)$ correspond to $s \in S^1$. In fact $z = \omega$ corresponds to $s = \omega^5$; and $z = \omega^5$ corresponds to $s = \omega$.

The curve on the right hand side has one point corresponding to $s = -1$ at 1.2446. Its point corresponding to $s = 1$ is at infinity. Its points in the half-plane with

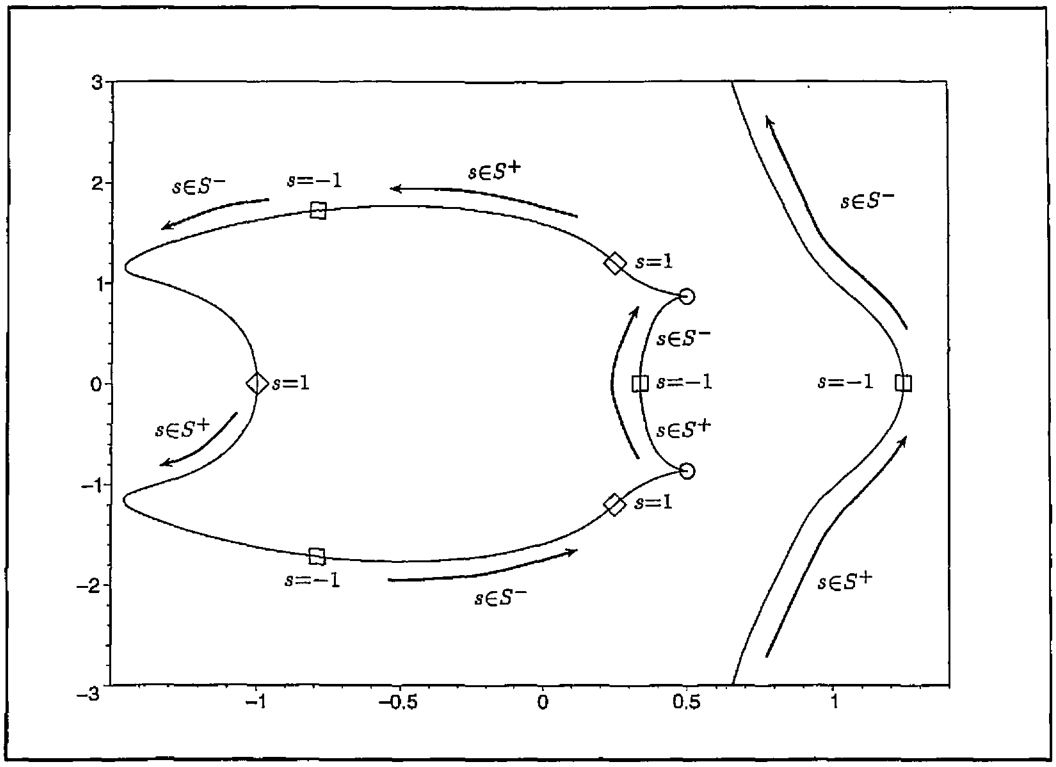


Figure 5.3: $E(f, g)$

negative imaginary values correspond to $s \in S^+$ and the points in the half-plane with positive imaginary values correspond to $s \in S^-$.

5.5 Dominance

Once we have obtained $E(T_L)$ for some T_L we can can concentrate on finding the subset $D(T_L)$ of $E(T_L)$ containing the *dominant points,* i.e. the points where the two (or more) eigenvalues of equal modulus also dominate the other eigenvalues in modulus. This is equivalent to saying that the dominant equimodular eigenvalues are equal to the spectral radius of T_L [6]. A point that is not dominant we refer to as a *sub-dominant point.*

Lemma 5.6 *Let Γ be a segment of an equimodular curve not intersecting with other equimodular curves. Then the points of Γ are either all dominant or all sub-dominant.*

Proof: Suppose that Γ is a segment of an equimodular curve containing dominant and sub-dominant points. The moduli of the eigenvalues are continuous functions in z. Going along Γ from dominant to sub-dominant points there has to be at least one point where three or more non-equal eigenvalues are of equal modulus. Such a point is a point of intersection of at least three equimodular curves. □

It follows that a equimodular curve can only change its dominance property at an intersection point of equimodular curves. We refer to those as *triple points* [6]. At a triple point three equimodular curves, one for each pair of eigenvalues, intersect as shown in Figure 5.4 for the three eigenvalues λ_1, λ_2 and λ_3. Then either there is a fourth eigenvalue bigger in modulus and there is no change in dominance, or each of the curves changes from being dominant to sub-dominant or vice versa, as shown in Figure 5.5. The sub-dominant parts are represented by thin lines, the dominant parts are represented by thick lines.

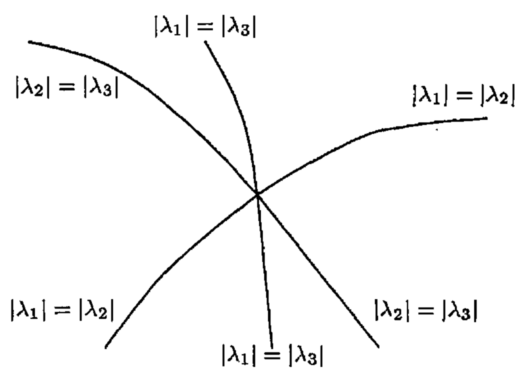


Figure 5.4: A triple point

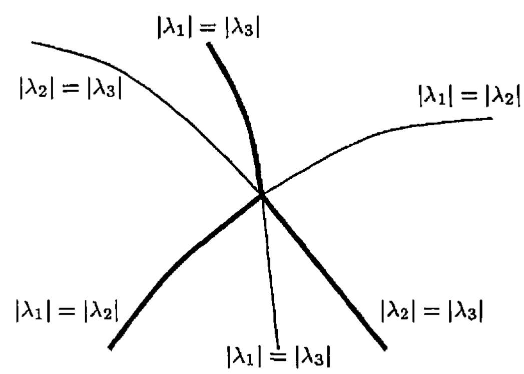


Figure 5.5: A triple point with change of dominance

Thus, in theory, once $E(T_L)$ has been obtained one just has to check the segments between triple points of the equimodular curves for dominance. In practice however there are too many equimodular curves.

We need some procedure to generate the equimodular curves and check for dominance. Assume that $u(\lambda, z)$ and $w(\lambda, z)$ are two distinct, irreducible factors of the characteristic polynomial of T_L. Consider $E(u)$, $E(w)$ and $E(u, w)$ separately and check their "equimodular segments" for dominance with respect to $u(\lambda, z)$, $w(\lambda, z)$

and $u(\lambda, z)w(\lambda, z)$ respectively. We obtain $D(u)$, $D(w)$ and $D(u, w)$. Then $D(T_L)$ is a subset of the union of $D(u)$ and $D(u, w)$ taken over all distinct factors $u(\lambda, z)$ and $w(\lambda, z)$ of the characteristic polynomial of T_L. By Lemma 5.6 it follows that we just have to check each segment Γ in this union for dominance with respect to the characteristic polynomial of T_L. In all the examples considered here this reduces the number of equimodular curves under consideration at each step to a manageable size.

5.6 Numerical computations

The numerical computations in the following examples are done in `Maple 7` using the program `EquiDominantPoints` (and some sub-programs) shown in Appendix D.1. For two given polynomials $u(\lambda, z)$ and $w(\lambda, z)$, with possibly $u = w$, the program returns a list R of equimodular points of u and w which are dominant with respect to uw. The points in R belong to $D(u, w)$ (or $D(u)$ if $u = w$).

The program `DomTest`, given in Appendix D.2, tests the points in a list R for dominance with respect to a given polynomial. Taking this polynomial to be the minimum characteristic polynomial of T_L, `DomTest` returns the points of R belonging to $D(T_L)$. That is, `DomTest` returns the intersection of $D(T_L)$ and $D(u, w)$ (or $D(u)$ if $u = w$).

In the program `EquiDominantPoints`, s is of course a discrete variable with a finite number of points, and there is no guarantee that the program does not miss out parts of the equimodular curves corresponding to values of s not checked for by the program. One way to avoid this would be to calculate the triple points in $E(T_L)$ and check them for dominance. Unfortunately the author is not aware of any feasible way of calculating these triple points. Programs similar to `Slices`, given in the Appendix D.3, can be used to check the results given by `EquiDominantPoints`: Let Z be a list of points on a line in $\mathbb{C}$. `Slices` evaluates the absolute values of the roots of a given polynomial at the points $z \in Z$, and plots them against Z.

In the following we are using these programs to obtain approximations of dominant points of the equimodular curves for various examples. This is not a full analysis of these curves, and in fact there appear to be many unexplained phenomena leaving space for future research. We assume that:

- The output of `EquiDominantPoints` with respect to $u(\lambda, z)$ and $w(\lambda, z)$ is denoted by $R(u, w)$ (or $R(u)$ if $u = w$).
- We use $100 \leq n \leq 200$ and $\varepsilon = 10^{-6}$ in `EquiDominantPoints`.
- In the following plots the intersection of the sets $R(u)$ or $R(u, w)$ with $D(T_L)$ is represented by solid lines.
- Since the sets $R(u)$, $R(u, w)$ and $D(T_L)$ are symmetric with respect to the real axis we only show the positive half plane in the following plots.
- Points are represented by approximations to four decimal places.
- The complex variable z has been shifted by -2.

5.6.1 The path of length three with the identity linking set

Let B be the path on three vertices and let $L = \{(1,1), (2,2), (3,3)\}$ be the identity linking set. The resulting graph $L_n(B)$ has been considered in Example 3.12 and the matrices N_L^π for all π have been obtained. The characteristic polynomials are:

$$f^{()}(\lambda, z) = \lambda^2 + (-z^3 - z^2 - 3z)\lambda + z^4 + z^3 + z^2 - 1,$$

$$f^{(1)}(\lambda, z) = (1 + \lambda)(\lambda + z^2)\big(\lambda^3 + a_2(z)\lambda^2 + a_1(z)\lambda + a_0(z)\big),$$

$$f^{(2)}(\lambda, z) = f^{(1^2)}(\lambda, z) = (1 + \lambda)(\lambda - z - 1)(z - \lambda)(z - 2 - \lambda),$$

$$f^{(3)}(\lambda, z) = f^{(1^3)}(\lambda, z) = \lambda + 1 \quad \text{and} \quad p^{(2,1)}(\lambda, z) = (\lambda + 1)^2,$$

where $a_2(z) = 2z^2 - z + 2, \quad a_1(z) = z^4 - 2z^3 - 1$ and

$$a_0(z) = -z^5 + 2z^3 + z^2 - 2z - 2.$$

There are seven different irreducible polynomials, and hence in total there are 49 sets $R(u)$ and $R(u, w)$ to be considered in this example. Most of them have an empty intersection with $D(T_L)$. We discuss only those contributing to $D(T_L)$. Let $u(\lambda, z)$ be the irreducible cubic factor of $f^{(1)}(\lambda, z)$. The set $R(f^{()}, u)$ is shown in Figure 5.6. Figure 5.7 is a detail of $R(f^{()}, u)$ showing that $E(f^{()}, u)$ crosses the line

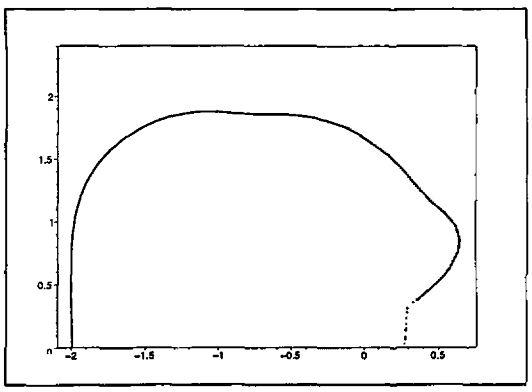


Figure 5.6: $R(f^{()}, u)$

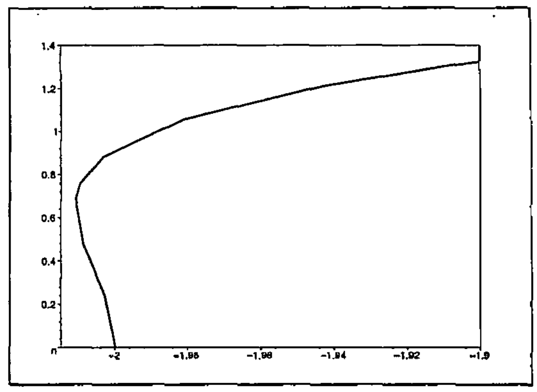


Figure 5.7: A detail of $R(f^{()}, u)$

$\Re(z) = -2$. It follows that $L_n(B)$ with n large enough has roots with negative real part.

Figure 5.8 shows parts (one of the "branches" goes off to infinity) of the set $R(u)$. Note that $E(u)$ has a singularity at $z = 0.2889$ corresponding to $s = 0.9878 \pm 0.1557i$, that is the partial derivative of $r_u(s, z)$ with respect to z is zero. There are two conjugate roots of $u(\lambda, z)$ for all points z along the real axis between 0.2256 and 0.3519. Figure 5.9 shows the triple point with change of dominance in $R(u)$. Let $v(\lambda, z) = z - 2 - \lambda$. Figure 5.10 shows (parts) of $R(f^{()}, v)$. The corresponding

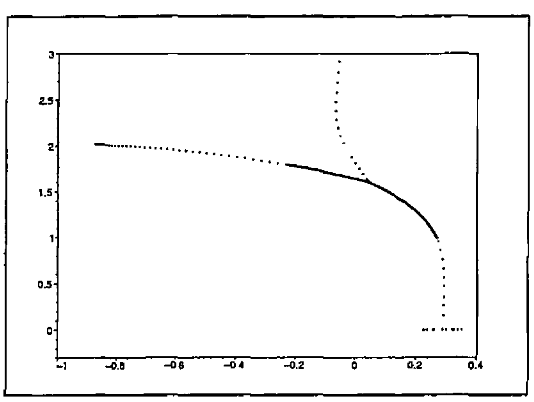


Figure 5.8: $R(u)$

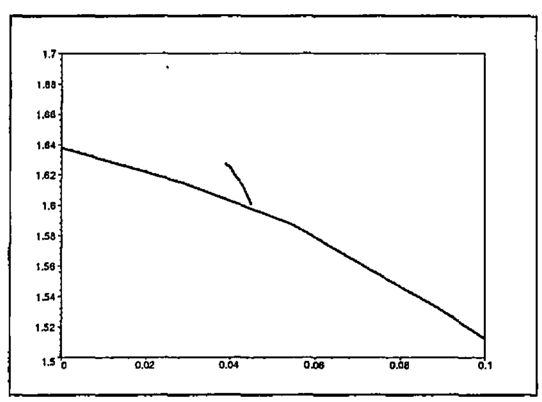


Figure 5.9: The triple point in $R(u)$

curve $E(f^{()}, v)$ has been analyzed in detail in Example 5.3. Figure 5.11 shows

$R(u, v)$. Figure 5.12 shows the union of of $R(f^{()}, u)$, $R(f^{()}, v)$, $R(u)$ and $R(u, v)$.

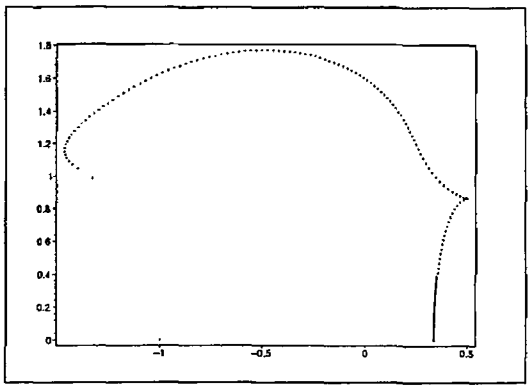


Figure 5.10: $R(f^{()}, v)$

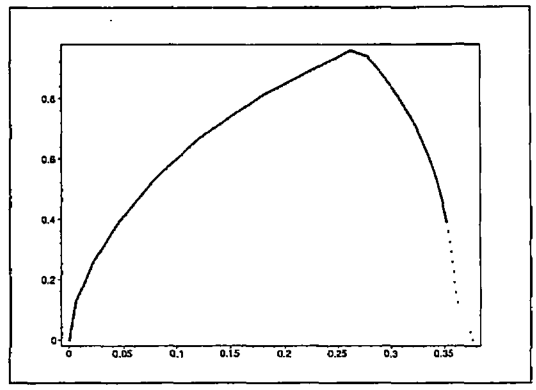


Figure 5.11: $R(u, v)$

Figure 5.13 shows a detail of the union of $R(f^{()}, u)$ and $R(u)$. And Figure 5.14

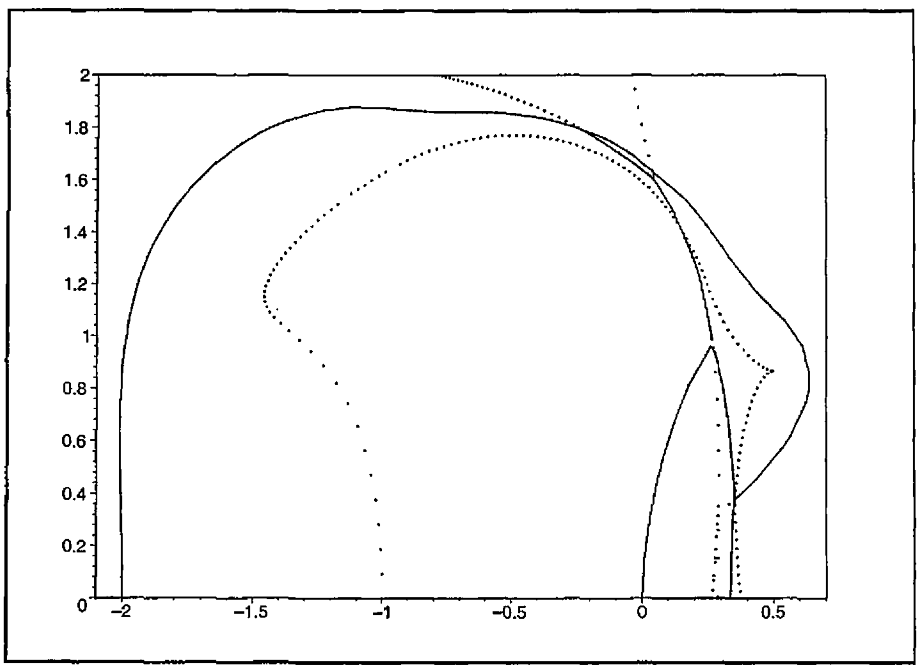


Figure 5.12: The union of $R(f^{()}, u)$, $R(f^{()}, v)$, $R(u)$ and $R(u, v)$

shows $D(T_L)$ and the roots of $L_{30}(B)$ represented by circles.

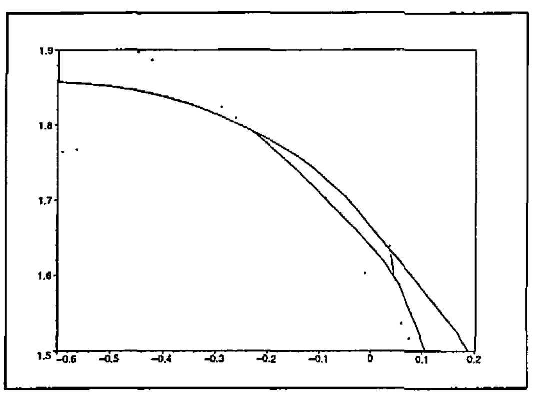


Figure 5.13: A detail of the union of $R(f^0, u)$ and $R(u)$

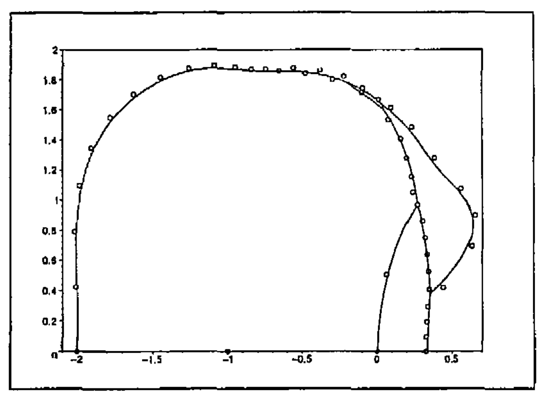


Figure 5.14: $D(T_L)$ and the roots of $L_{30}(B)$ (circles)

5.6.2 Generalised dodecahedra

Let us now consider the family of generalised dodecahedra D_n with the path on four vertices as base graph and the linking set $L = \{(1,1),(3,2),(4,4)\}$. Its chromatic polynomial has been obtained in Section 4.5. The characteristic polynomials of the N_L^π are given on pages following Page 95. Again, we will only consider the irreducible factors of the characteristic polynomials that contribute to $D(T_L)$. Let $u(\lambda, z)$, $w(\lambda, z)$ and $v(\lambda, z)$ be such that:

$$\lambda^2 u(\lambda, z) = 0, \qquad \lambda^8(\lambda - 1)w(\lambda, z) = 0 \quad \text{and} \quad \lambda^{11}(\lambda - 1)v(\lambda, z) = 0$$

are the characteristic equations corresponding to $\pi = ()$, $\pi = (1)$ and $\pi = (2)$ respectively.

The sets $R(u,w)$, $R(u,v)$, $R(w,v)$ and $R(v)$ shown in the Figures 5.15, 5.16, 5.17 and 5.18 respectively have been obtained using the program `EquiDominantPoints`. The respective intersections with $D(T_L)$ indicated by solid lines have been calculated using `DomTest`. Figure 5.19 shows the union of the sets $R(u,w)$, $R(u,v)$,

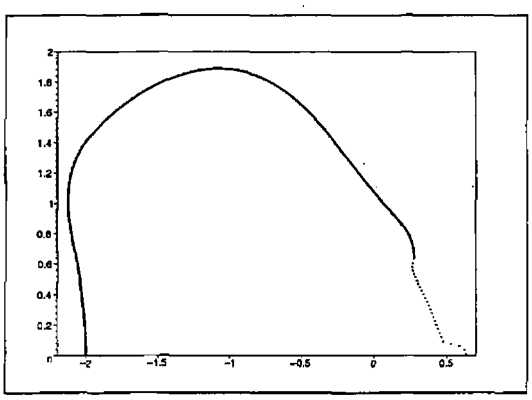


Figure 5.15: $R(u,w)$

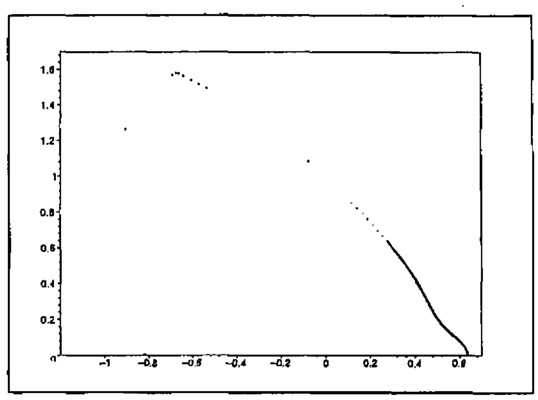


Figure 5.16: $R(u,v)$

$R(w,v)$ and $R(v)$. And the Figure 5.20 shows the set $D(T_L)$ and the roots of D_{30} represented by circles.

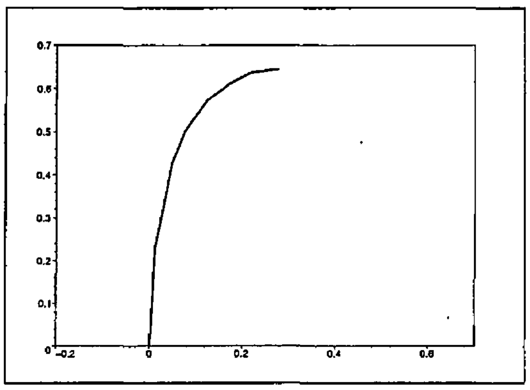


Figure 5.17: $R(w, v)$

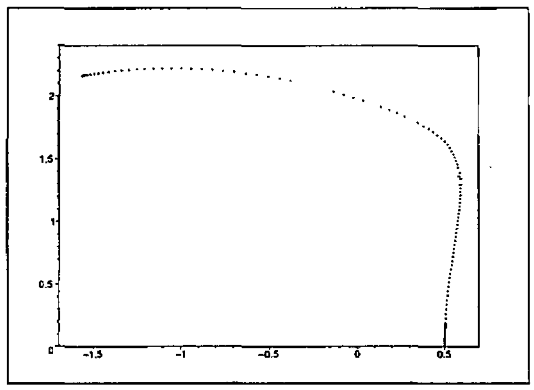


Figure 5.18: $R(v)$

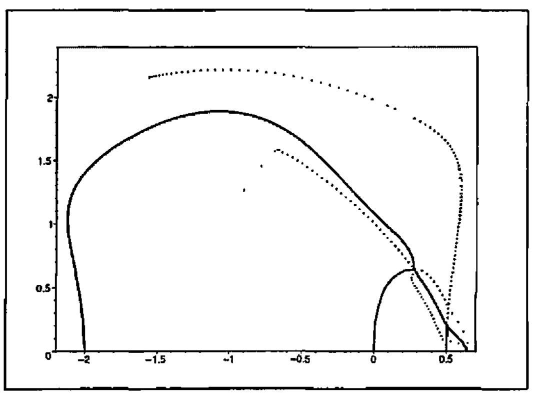


Figure 5.19: The Union of $R(u, w)$, $R(u, v)$, $R(w, v)$ and $R(v)$.

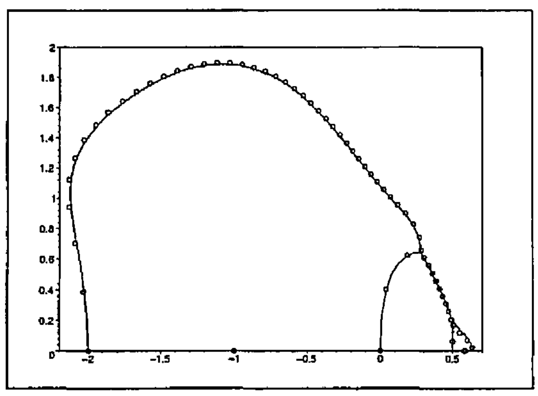


Figure 5.20: $D(T_L)$ and the roots of D_{30} (circles)

5.6.3 The family $(468)_n$

Recall the family of graphs $(468)_n$ with the path on four vertices as base graph and linking set $L = \{(1,4),(3,2),(4,1)\}$. Its chromatic polynomial has been obtained in Section 4.6.1. The characteristic polynomials of the matrices N^{π}_{468} for all π have been obtained on the pages following Page 103. Denote by $u(\lambda,z)$, $w(\lambda,z)$ and $v(\lambda,z)$ the three polynomials such that

$$\lambda^2 u(\lambda,z)=0, \quad \lambda^8(\lambda-1)w(\lambda,z)=0 \quad \text{and} \quad \lambda^{11}(\lambda-1)v(\lambda,z)=0$$

are the characteristic equations corresponding to $\pi=()$, $\pi=(1)$ and $\pi=(2)$ respectively.

Only the sets $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$ have non-empty intersections with $D(T_L)$. They are shown in Figures 5.21, 5.22, 5.23 and 5.24 respectively. The respective intersections with $D(T_L)$ are shown as solid lines. Figure 5.25

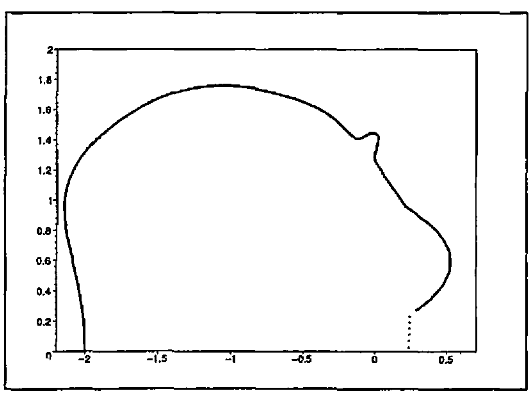


Figure 5.21: $R(u,w)$

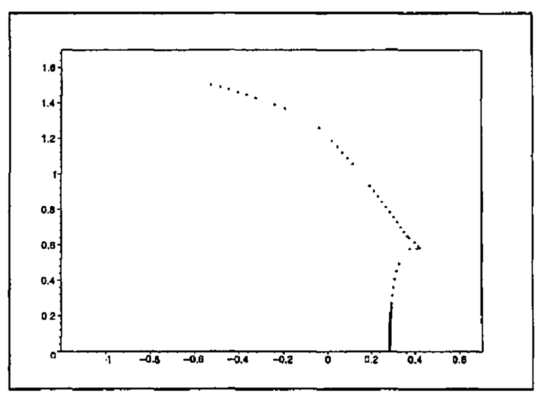


Figure 5.22: $R(u,v)$

show a detail of the union of the sets $R(u,w)$ and $R(w)$. The dotted line is the sub-dominant part of $R(w)$. According to our previous discussion of triple points, there should be two more sub-dominant parts at each of the triple points shown here. These sub-dominant parts belong to $E(u,w)$ but they are not in $R(u,w)$ since there is a level 0 eigenvalue of bigger modulus.

Figure 5.26 shows a detail of the union of $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$.

And the Figure 5.27 shows the set $D(T_L)$ and the roots of $(468)_{30}$ represented by circles.

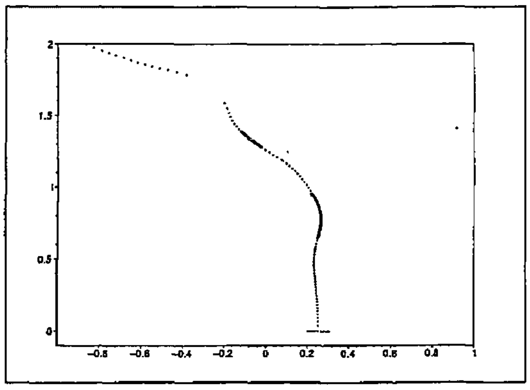


Figure 5.23: $R(w)$

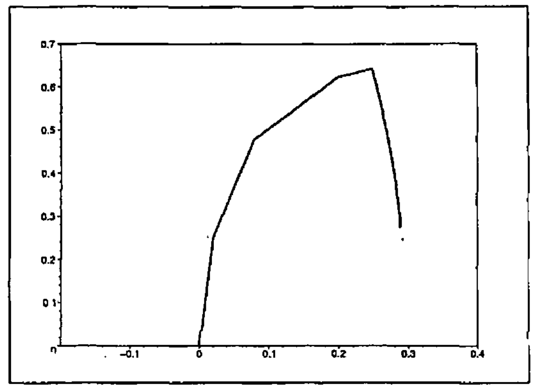


Figure 5.24: $R(w, v)$

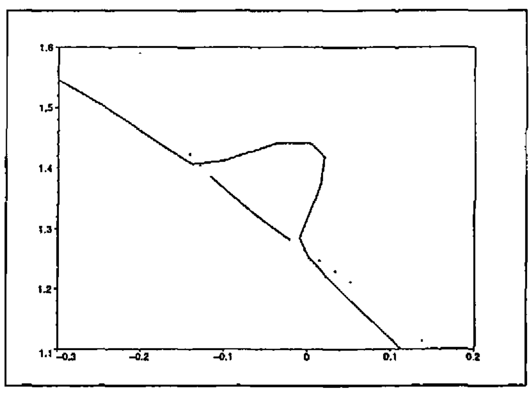


Figure 5.25: A detail of the union of $R(u, w)$ and $R(w)$

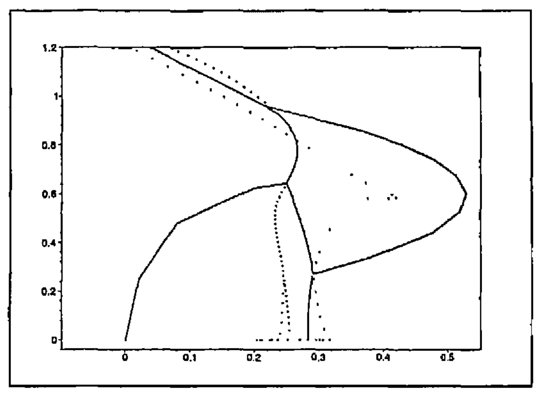


Figure 5.26: A detail of the union of $R(u, w)$, $R(u, v)$, $R(w)$ and $R(w, v)$.

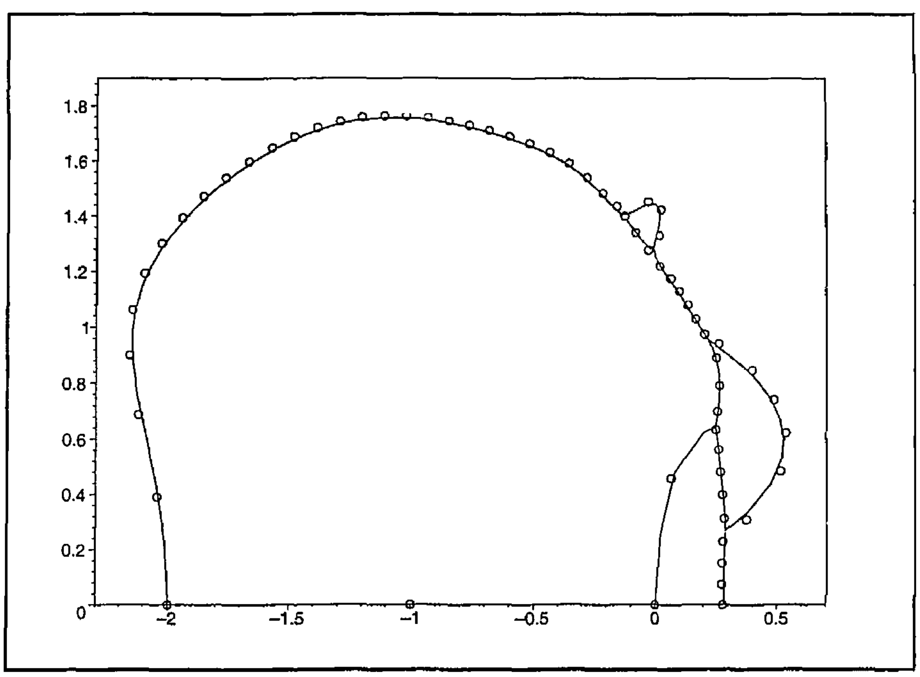


Figure 5.27: $D(T_L)$ and the roots of $(468)_{30}$ (circles)

5.6.4 The family $(477)_n$

Take the path on four vertices as base graph B, and let $L = \{(1,4),(3,1),(4,2)\}$ be the linking set. The family of graphs $L_n(B)$ obtained is $(477)_n$. Its chromatic polynomial has been obtained in Section 4.6.2. The characteristic polynomials of the matrices N_{477}^{π} for all π have been obtained on the pages following Page 107. Denote by $u(\lambda,z)$, $w(\lambda,z)$ and $v(\lambda,z)$ the three polynomials such that

$$\lambda^2 u(\lambda,z) = 0, \quad \lambda^8 w(\lambda,z) = 0 \quad \text{and} \quad \lambda^{11} v(\lambda,z) = 0$$

are the characteristic equations corresponding to $\pi = ()$, $\pi = (1)$ and $\pi = (2)$ respectively.

Only the sets $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$ have non empty intersections with $D(T_L)$. They are shown in Figures 5.28, 5.29, 5.30 and 5.31 respectively. The respective intersections with $D(T_L)$ are shown as solid lines.

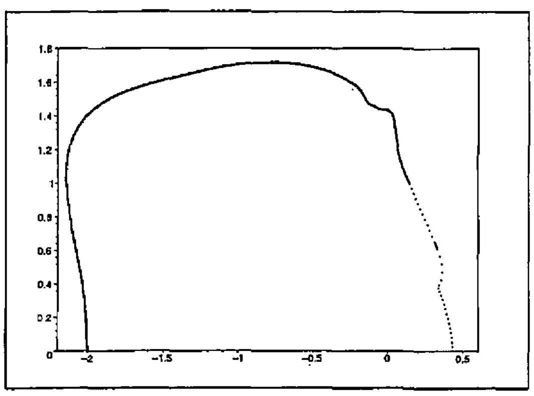


Figure 5.28: $R(u,w)$

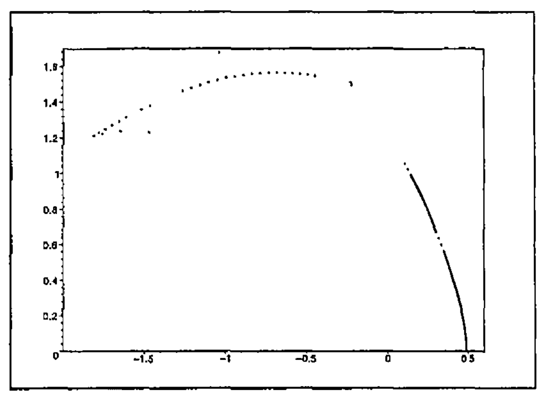


Figure 5.29: $R(u,v)$

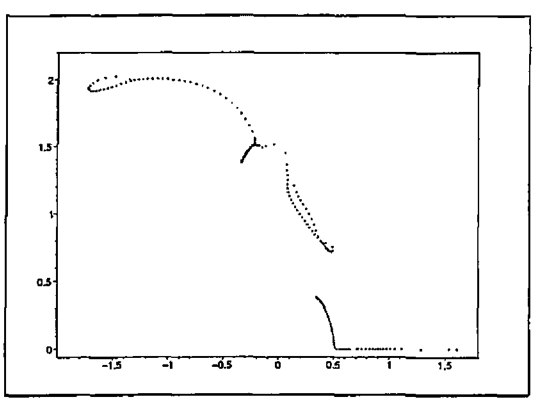


Figure 5.30: $R(w)$

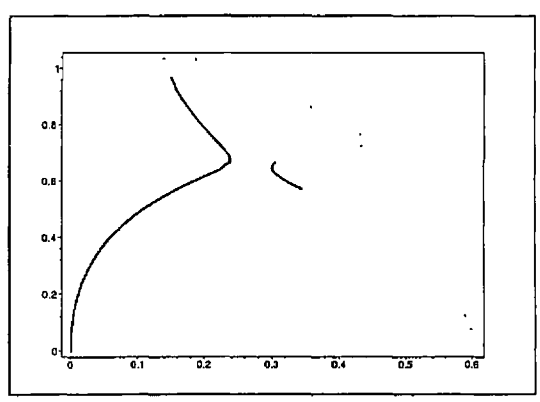


Figure 5.31: $R(w,v)$

Figure 5.32 show a detail of the union of the sets $R(u,w)$ and $R(w)$. Figure 5.33 shows a detail of the union of $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$.

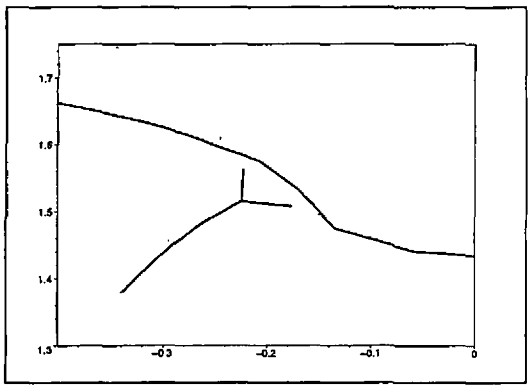


Figure 5.32: A detail of the union of $R(u,w)$ and $R(w)$

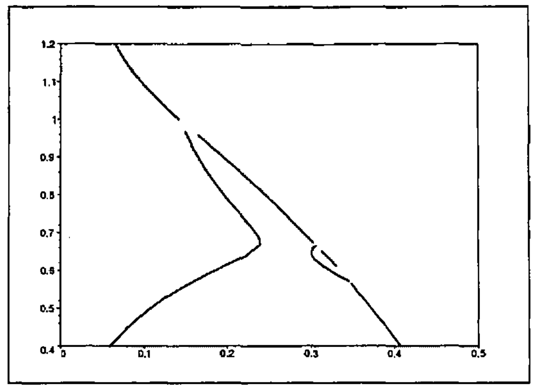


Figure 5.33: A detail of the union of $R(u,w)$, $R(u,v)$ and $R(w,v)$.

And the Figure 5.34 shows the set $D(T_L)$ and the roots of $(477)_{30}$ represented by circles.

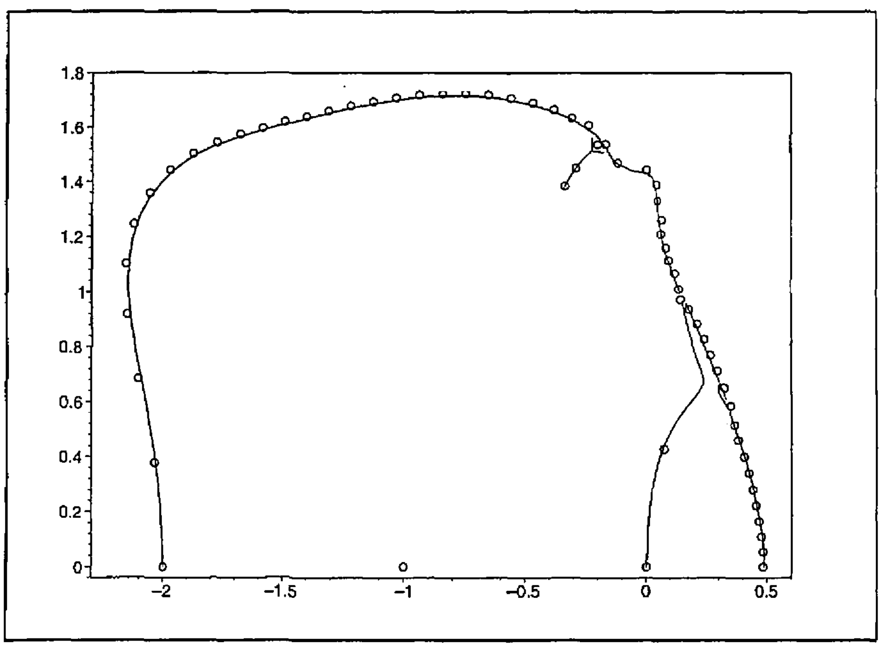


Figure 5.34: $D(T_L)$ and the roots of $(477)_{30}$ (circles)

5.6.5 The family $(567)_n$

Take the path on four vertices as base graph B, and let $L = \{(1,4),(3,1),(4,2)\}$ be the linking set. The family of graphs $L_n(B)$ obtained is $(567)_n$. Its chromatic polynomial has been obtained in Section 4.6.3. The characteristic polynomials of the matrices N_{567}^{π} for all π have been obtained on the pages following Page 111. Denote by $u(\lambda, z)$, $w(\lambda, z)$ and $v(\lambda, z)$ the three polynomials such that

$$\lambda^2 u(\lambda, z) = 0, \quad \lambda^8 w(\lambda, z) = 0 \quad \text{and} \quad \lambda^{11} v(\lambda, z) = 0$$

are the characteristic equations corresponding to $\pi = ()$, $\pi = (1)$ and $\pi = (2)$ respectively.

Only the sets $R(u, w)$, $R(u, v)$ and $R(w, v)$ have non empty intersections with $D(T_L)$. They are shown in Figures 5.35, 5.36 and 5.37 respectively. The respective intersections with $D(T_L)$ are shown as solid lines.

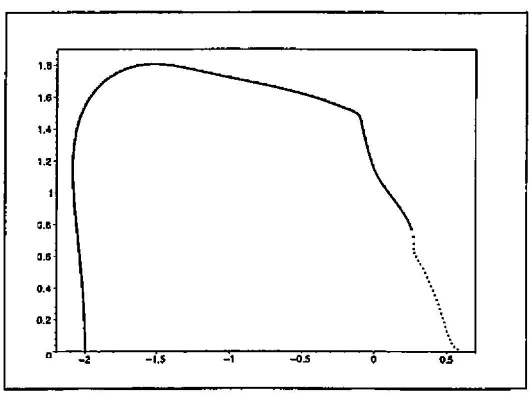


Figure 5.35: $R(u, w)$

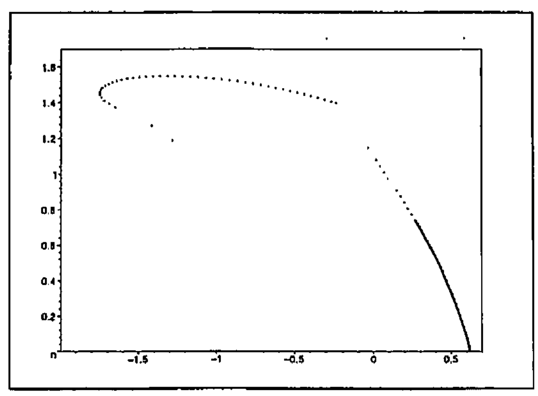


Figure 5.36: $R(u, v)$

Figure 5.38 shows a detail of the union of $R(u, w)$, $R(u, v)$ and $R(w, v)$. Figure 5.39 shows the union of $R(u, w)$, $R(u, v)$ and $R(w, v)$. And the Figure 5.40 shows the set $D(T_L)$ and the roots of $(567)_{30}$ represented by circles.

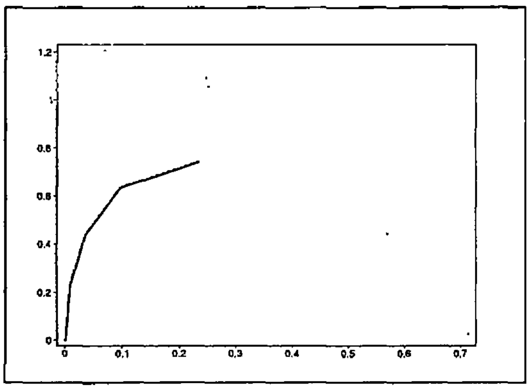


Figure 5.37: $R(w, v)$

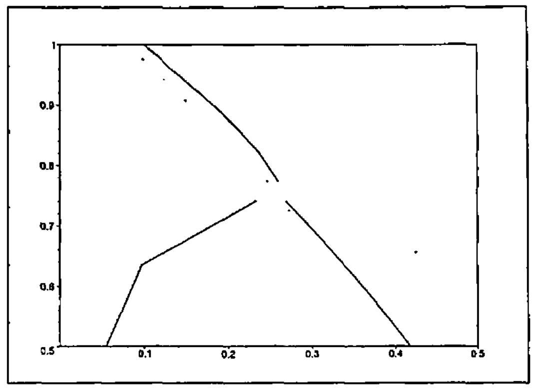


Figure 5.38: A detail of the union of $R(u, w)$, $R(u, v)$ and $R(w, v)$.

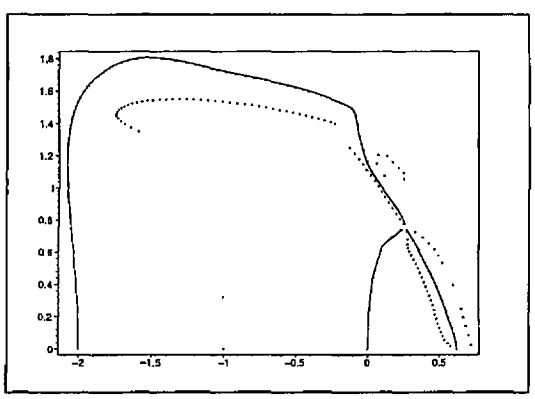


Figure 5.39: The union of $R(u, w)$, $R(u, v)$ and $R(w, v)$.

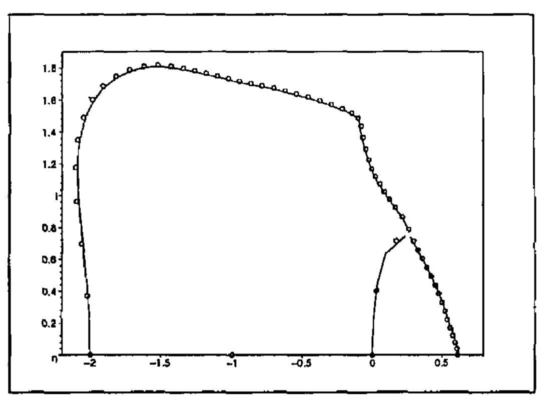


Figure 5.40: $D(T_L)$ and the roots of $(567)_{30}$ (circles)

5.6.6 The family $(666)_n$

Take the path on four vertices as base graph B, and let $L = \{(1,4),(3,1),(4,2)\}$ be the linking set. The family of graphs $L_n(B)$ obtained is $(666)_n$. Its chromatic polynomial has been obtained in Section 4.6.4. The characteristic polynomials of the matrices N^{π}_{666} for all π have been obtained on the pages following Page 115. Denote by $u(\lambda, z)$, $w(\lambda, z)$ and $v(\lambda, z)$ the three polynomials such that

$$\lambda^2 u(\lambda, z) = 0, \quad \lambda^8 w(\lambda, z) = 0 \quad \text{and} \quad \lambda^{11} v(\lambda, z) = 0$$

are the characteristic equations corresponding to $\pi = ()$, $\pi = (1)$ and $\pi = (2)$ respectively.

Only the sets $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$ have non empty intersections with $D(T_L)$. They are shown in Figures 5.41, 5.42, 5.43 and 5.44 respectively. The respective intersections with $D(T_L)$ are shown as solid lines.

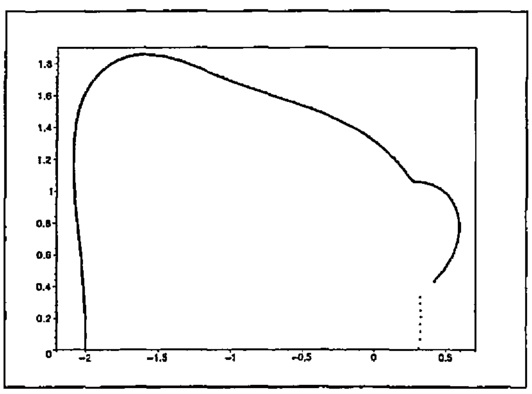


Figure 5.41: $R(u,w)$

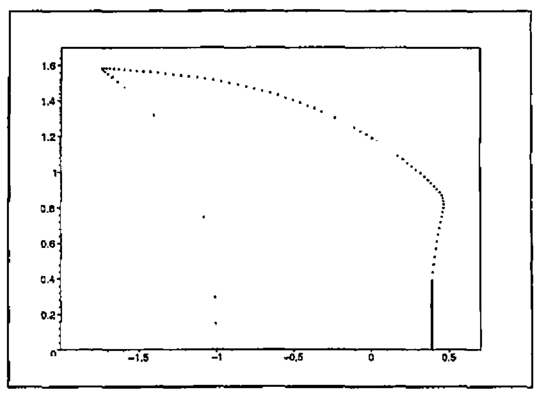


Figure 5.42: $R(u,v)$

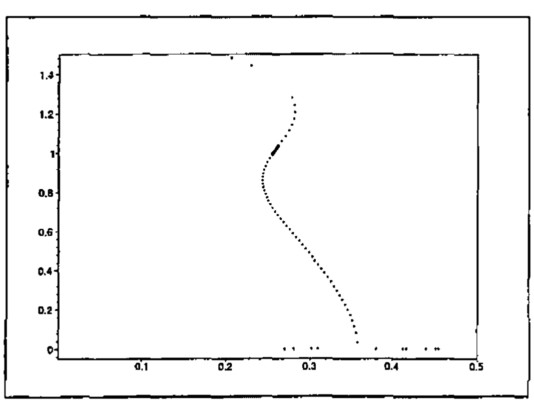


Figure 5.43: $R(w)$

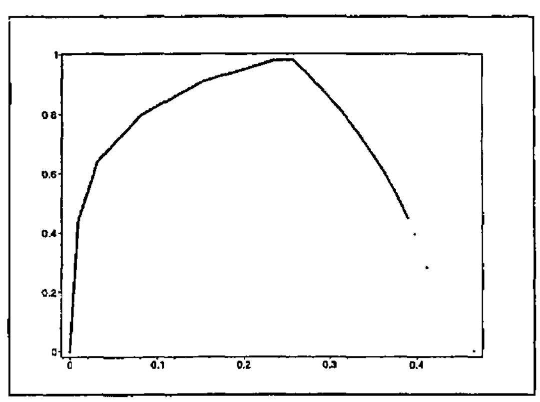


Figure 5.44: $R(w,v)$

Figure 5.45 shows a detail of the union of $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$. Figure 5.46 shows the union of $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$. And the Figure 5.47 shows the set $D(T_L)$ and the roots of $(666)_{30}$ represented by circles

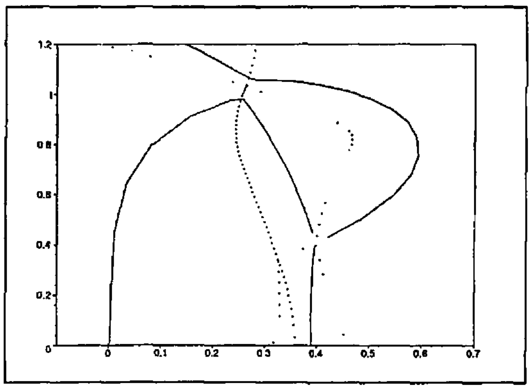


Figure 5.45: A detail of the union of $R(u,w)$, $R(u,v)$, $R(w)$ and $R(w,v)$.

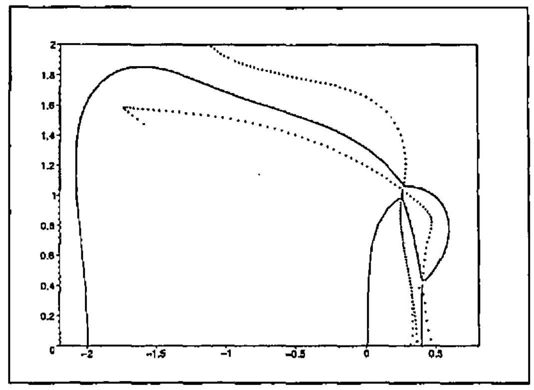


Figure 5.46: $D(T_L)$

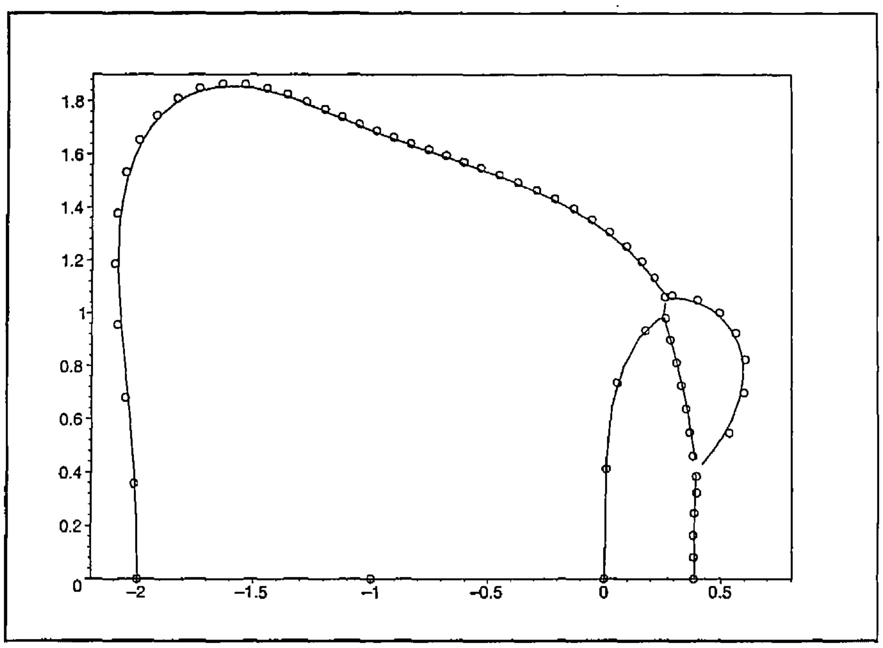


Figure 5.47: $D(T_L)$ and the roots of $(666)_{30}$ (circles)

Chapter 6

The operator algebras $\mathcal{A}_b(k)$ and $\mathcal{A}_\pi(k)$

Let k be an integer. Recall from Chapter 3 for a given base graph B and a linking set $L \subseteq V \times V$ the compatibility operator $T_L = T_L(k)$ is defined by the following matrix. The rows and columns correspond to the colourings of B and the entry $(T_L)_{\alpha\beta}$ is one if the pair (α, β) is compatible with L, and zero otherwise. For a pair (α, β) to be compatible with L means that:

$$(v, w) \in L \quad \Longrightarrow \quad \alpha(v) \neq \beta(w).$$

In this chapter we consider the case where B is the complete base graph K_b with vertex set V_b. Recall that for a given matching $M \subset V_b \times V_b$ the operator $S_M = S_M(k)$ is given by the matrix

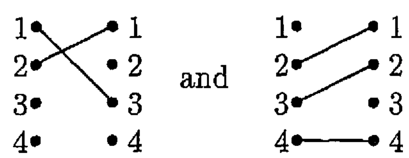
$$(S_M)_{\alpha\beta} = \begin{cases} 1 & \text{if} \quad \alpha(x) = \beta(y) \quad \forall (x, y) \in M \\ 0 & \text{otherwise} \end{cases}$$

In Theorem 3.6 it has been shown that the compatibility operator T_L can be written as a linear combination of operators S_M. In the following we show that that the operators S_M form an algebra. A minimal generating set is found. In case of the identity linking set some properties of the spectrum are proved and the full set of eigenvalues conjectured.

6.1 A binary operation for matchings

Recall, a matching M in $V_b \times V_b$ is a triple (M_1, M_2, μ) where M_1 and M_2 are subsets of V_b and $\mu : M_1 \to M_2$ is a bijection. Equivalently, M is the subset of $V_b \times V_b$ consisting of all $(x, \mu(x))$ with $x \in M_1$.

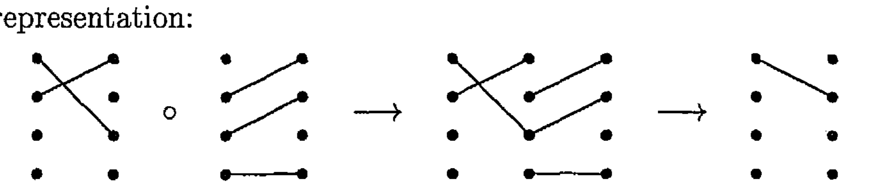
A matching can be represented by a diagram in the obvious way: Take two disjoint copies of the vertex set and arrange the vertices in each of them as columns. The vertex x in the first copy is linked to the vertex y in the second copy if $(x, y) \in M$. For example, for $b = 4$, the respective diagrams corresponding to the matchings $M = \{(1,3),(2,1)\}$ and $M' = \{(2,1),(3,2),(4,4)\}$ are:



We arrange the vertices in the diagrams in increasing order from top to bottom. We usually omit the numbering of the vertices. For given M and M' we define the binary operation $\circ$ on the set of matchings by:

$$M \circ M' = \big\{(x, y') \mid \text{ there exists } z \in V_b \text{ with } (x, z) \in M \text{ and } (z, y') \in M'\big\}.$$

In the case of our previous example this is $M \circ M' = \{(1,2)\}$. And in terms of the diagram representation:



6.2 The operator algebra $\mathcal{A}_b(k)$

We are going to investigate the structure obtained by multiplying and adding the matrices S_M. Let $M = (M_1, M_2, \mu)$ and $M' = (M_1', M_2', \mu')$ be two matchings. For given α, $\beta \in \Gamma_k(b)$, where $\Gamma_k(b)$ is the the set of colourings of K_b:

$$(S_M S_{M'})_{\alpha\beta} = \sum_{\gamma \in \Gamma_k(b)} (S_M)_{\alpha\gamma}(S_{M'})_{\gamma\beta} = \big|\{\gamma \in \Gamma_k \mid (S_M)_{\alpha\gamma} = (S_{M'})_{\gamma\beta} = 1\}\big|$$

Observe that: $(S_M)_{\alpha\gamma} = 1$ if $\gamma = \alpha\mu^{-1}$ on M_2

and $(S_{M'})_{\gamma\beta} = 1$ if $\gamma = \beta\mu'$ on M'_1.

It follows that necessary conditions for $(S_M S_{M'})_{\alpha\beta} \neq 0$ are:

- $\alpha\mu^{-1} = \beta\mu'$ on $M_2 \cap M'_1$

and since γ is an injection:

- $\alpha\mu^{-1}\big(M_2 \setminus (M_2 \cap M'_1)\big) \cap \beta\mu'\big(M'_1 \setminus (M_2 \cap M'_1)\big) = \varnothing$.

Suppose that α and β are such that above conditions are satisfied. Then γ is completely determined on $M_2 \cup M'_1$. For γ on $V_b \setminus (M_2 \cup M'_1)$ there are

$$(k - |M_2 \cup M'_1|)(k - |M_2 \cup M'_1| - 1) \dots (k - b + 1)$$

choices. Recall that we denoted the falling factorial by $f_s(d,k) = (k-s)_{d-s} = (k-s)(k-s-1)\dots(k-d+1)$. Hence the above conditions are also sufficient for $(S_M S_{M'})_{\alpha\beta} \neq 0$, and in this case $(S_M S_{M'})_{\alpha\beta} = f_{|M_2 \cup M'_1|}(b,k)$. Then: $(S_M S_{M'})_{\alpha\beta} = f_{|M_2 \cup M'_1|}(b,k)$ if for all $(x,y) \in M$ and $(x',y') \in M'$ it holds that

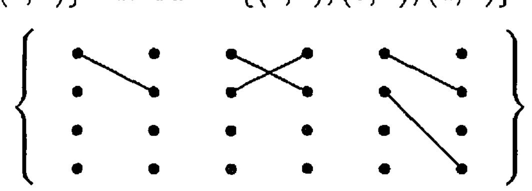
$$\alpha(x) = \beta(y') \quad \text{if and only if} \quad y = x',$$

and $(S_M S_{M'})_{\alpha\beta} = 0$ otherwise.

Using the binary operation $\circ$ on the matchings the first of the above necessary conditions can be formulated as:

$$(S_M S_{M'})_{\alpha\beta} \neq 0 \quad \text{only if} \quad (S_{M \circ M'})_{\alpha\beta} = 1$$

Define the set of matchings $\mathcal{N}_{MM'} = \{\hat{M} \mid M \circ M' \subseteq \hat{M} \subset M_1 \times M'_2\}$. For example for $M = \{(1,3),(2,1)\}$ and $M' = \{(2,1),(3,2),(4,4)\}$

$\mathcal{N}_{MM'} =$  .

Lemma 6.1 *Let $M = (M_1, M_2, \mu)$ and $M' = (M'_1, M'_2, \mu')$ be two matchings. Then*

$$S_M S_{M'} = (-1)^{|M \circ M'|} f_{|M_2 \cup M'_1|}(b,k) \sum_{\hat{M} \in \mathcal{N}_{MM'}} (-1)^{|\hat{M}|} \, S_{\hat{M}}$$

Proof: Let $\alpha,\ \beta \in \Gamma_k(b)$ be given. We are comparing $(S_M S_{M'})_{\alpha\beta}$ to

$$(-1)^{|M\circ M'|} f_{|M_2 \cup M_1'|}(b,k) \sum_{\hat{M}\in\mathcal{N}_{MM'}} (-1)^{|\hat{M}|}\ (S_{\hat{M}})_{\alpha\beta}$$

denoted by $\sum_{\alpha\beta}$. Let $W(\alpha,\beta) = \{(x,y') \in M_1 \times M_2' \mid \alpha(x) = \beta(y')\}$. Then $(S_{\hat{M}})_{\alpha\beta} = 1$ for any $\hat{M} \in \mathcal{N}_{MM'}$ only if $\hat{M} \subseteq W(\alpha,\beta)$, and $(S_{\hat{M}})_{\alpha\beta} = 0$ otherwise. We have to consider three cases:

(i) If $M \circ M' \not\subseteq W(\alpha,\beta)$ clearly both $(S_M S_{M'})_{\alpha\beta}$ and $\sum_{\alpha\beta}$ are equal to zero.

(ii) If $M \circ M' = W(\alpha,\beta)$ from the argument preceding the lemma it follows that $(S_M S_{M'})_{\alpha\beta} = f_{|M_2 \cup M_1'|}(b,k)$. In the sum $\sum_{\alpha\beta}$ we have $(S_{\hat{M}})_{\alpha\beta} = 1$ only if $\hat{M} = M \circ M'$. Hence follows equality.

(iii) If $M \circ M' \subset W(\alpha,\beta)$ then $(S_M S_{M'})_{\alpha\beta} = 0$. And in the sum $\sum_{\alpha\beta}$ it holds that $(S_{\hat{M}})_{\alpha\beta} = 1$ only if $\hat{M} \subseteq W(\alpha,\beta)$. It follows that:

$$\begin{aligned}\sum_{\hat{M}\in\mathcal{N}_{MM'}} (-1)^{|\hat{M}|}\ (S_{\hat{M}})_{\alpha\beta} &= \sum_{M\circ M' \subseteq \hat{M} \subseteq W(\alpha,\beta)} (-1)^{|\hat{M}|} \\ &= (-1)^{|M\circ M'|}(1-1)^{(|W(\alpha,\beta)|-|M\circ M'|)}.\end{aligned}$$

This completes the proof. □

Corollary 6.2 *For every integer k the operators S_M with $M \subset V_b \times V_b$ a matching form an algebra $\mathcal{A}_b(k)$ over $\mathbb{C}$.* □

6.3 A minimal generating set

In this section we find a minimal generating set for $\mathcal{A}_b(k)$. Clearly there is a one-to-one relationship between the matchings of size b and the elements of Sym_b. For any $\omega \in \mathrm{Sym}_b$ the corresponding matching is defined as $M_\omega = \{(\omega(y), y) \mid y = 1, 2, \ldots, b\}$. Observe that if we write M_ω as a triple (M_1, M_2, μ) then $\mu = \omega^{-1}$.

Lemma 6.3 *For any ω and τ in Sym_b the following holds:* $\quad M_\omega \circ M_\tau = M_{\omega\tau}$.

Proof: Recall

$$M_\omega \circ M_\tau = \Big\{(x,y) \mid \text{ there exists } z \in V_b \text{ with } (x,z) \in M_\omega \text{ and } (z,y) \in M_\tau\Big\}$$

where $(x,z) \in M_\omega$ if $x = \omega(z)$, and $(z,y) \in M_\tau$ if $z = \tau(y)$. Thus $(x,y) \in M_\omega \circ M_\tau$ if $x = \omega(\tau(y))$. □

Let $\sigma = (1\,2\,3\ldots b)$ and $\phi = (1\,2)$. The corresponding matchings are:

$$M_\sigma = \{(2,1),(3,2),\ldots,(b,b-1),(1,b)\}$$

and $$M_\phi = \{(1,2),(2,1),(3,3),(4,4),\ldots,(b,b)\}.$$

Then, for example $M_\phi \circ M_\sigma = M_{\phi\sigma} = \{(3,1),(2,2),(4,3),(5,4),\ldots,(1,b)\}$. From Lemma 6.1 it follows that $S_{M_\tau} S_{M_\omega} = S_{M_\omega \circ M_\tau} = S_{M_{\omega\tau}}$. Since σ and ϕ generate Sym_b it follows that M_σ and M_ϕ generate all matchings of size b. And thus, S_{M_σ} and S_{M_ϕ} generate every $S_{\hat{M}}$ with $|\hat{M}| = b$.

For any $X \subseteq V_b$ denote by M_X the following matching

$$M_X = \{(y,y) \mid y \in V_b \setminus X\}.$$

Then M_{V_b} is the empty matching and $M_\varnothing = M_\epsilon$ the "identity matching". If $X = \{x\}$ we write M_x.

Choose any two x and y in V_b, and any $\omega \in \mathrm{Sym}_b$ such that $\omega(y) = x$. Then

$$M_{\omega^{-1}} \circ M_x \circ M_\omega = M_y.$$

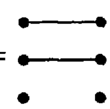
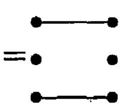
From Lemma 6.1 it follows that $S_{M_y} = S_{M_{\omega^{-1}}} S_{M_x} S_{M_\omega}$. Then, for some given $x \in V_b$ every matching M with $|M| = b-1$ can be written in the form $M_\gamma \circ M_y \circ M_\tau$ for some $\gamma, \tau \in \mathrm{Sym}_b$. And $S_M = S_{M_\gamma} S_{M_y} S_{M_\tau}$.

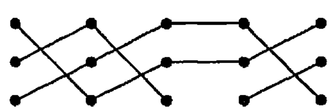
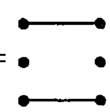
Lemma 6.4 *The set $\{S_{M_\sigma}, S_{M_\phi}, S_{M_x}\}$ for any given $x \in V_b$ is a generating set for the algebra $\mathcal{A}_b(k)$*

Proof: We will use a proof by induction on the size s of the matchings. The base step is given by the argument preceding the lemma for $s = b$. Suppose now that for some $0 < s < b$ all S_M with $|M| \geq s$ are generated a by finite product of the elements in $\{S_{M_\sigma},\, S_{M_\phi},\, S_{M_x}\}$. Let M' be any matching with $|M'| = s-1$. Let $\hat{M}$ be a matching of size s containing M'. That is, $\hat{M} = M' \cup \{(\hat{x}, \hat{y})\}$ for some $(\hat{x}, \hat{y}) \notin M'_1 \times M'_2$. Then $M_{\hat{x}} \circ \hat{M} = M'$ and by Lemma 6.1 it follows that $S_{M_{\hat{x}}} S_{\hat{M}} = S_{M'} + \sum$, where $\sum$ is a linear combination of S_M with $|M| \geq s$. This completes the proof. □

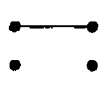
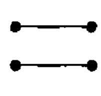
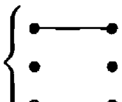
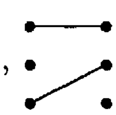
This implies that for every integer k and every matching M there exists a polynomial F_M over $\mathbb{C}$ in three *non commutative* variables ζ_σ, ζ_ϕ and ζ_x such that $F_M(S_{M_\sigma},\, S_{M_\phi},\, S_{M_x}) = S_M$. For example $F_{M_{\sigma\phi}} = \zeta_\sigma \zeta_\phi$. Then Theorem 3.6 can be written as

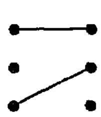
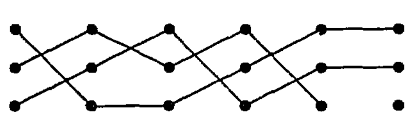
$$T_L = \sum_{M \in \mathcal{M}(L)} (-1)^{|M|} F_M(S_{M_\sigma},\, S_{M_\phi},\, S_{M_x}).$$

Example 6.1: Let $b = 3$. We write for example $F_{11,22}$ rather than $F_{\{(1,1),(2,2)\}}$ or F_{M_x}. Then $F_{11,22,33} = \zeta_\phi^2$ and $F_{11,22} = \zeta_3$. It is convenient to use the diagrams in order to obtain the F_M. For example if $M_3 =$  and $M_2 =$  then

$M_\sigma \circ M_\sigma \circ M_3 \circ M_\sigma =$  $=$ .

From Lemma 6.1 it follows that $F_{11,33} = \zeta_\sigma^2 F_{11,22} \zeta_\sigma = \zeta_\sigma^2 \zeta_3 \zeta_\sigma$ and similarly $F_{22,33} = \zeta_\sigma \zeta_3 \zeta_\sigma^2$.

For $M_2 =$  and $M_3 =$  the set $\mathcal{N}_{M_2 M_3} = \Big\{$  ,  $\Big\}$.

With  $=$  $= M_\sigma \circ M_\phi \circ M_\sigma \circ M_\sigma \circ M_3$

and from Lemma 6.1 it follows that:

$$F_{11} = F_{11,33}F_{11,22} + \zeta_\sigma\zeta_\phi\zeta_\sigma^2\zeta_3 = \zeta_\sigma^2\zeta_3\zeta_\sigma\zeta_3 + \zeta_\sigma\zeta_\phi\zeta_\sigma^2\zeta_3,$$

$$F_{22} = \zeta_\sigma F_{11}\zeta_\sigma^2, \qquad F_{33} = \zeta_\sigma^2 F_{11}\zeta_\sigma \quad \text{and}$$

$$F_\varnothing = F_{11}F_{22}/(k-2) + F_{12} = F_{11}F_{22}/(k-2) + F_{11}\zeta_\phi$$

For $L = \{(1,1),(2,2),(3,3)\}$, the identity linking set, we obtain the graphs $B_n(3)$ (see Example 3.1). The chromatic polynomial has been obtained in Example 3.8. With the above, the matrix T_L is equal to

$$F_\varnothing - (F_{11} + F_{22} + F_{33}) + F_{11,22} + F_{11,33} + F_{22,33} - F_{11,22,33}$$

evaluated at S_{M_σ}, S_{M_ϕ} and S_{M_3}.

Observe that if $cS_{\overline{M}} = S_M S_{M'}$ then $|\overline{M}| \leq \min(|M|,|M'|)$, where $c \in \mathbb{C}$. It follows that for every $0 \leq \ell \leq b$ the set of S_M with $|M| \leq \ell$ generates a subalgebra of $\mathcal{A}_b(k)$.

6.4 The operator algebra $\mathcal{A}_\pi(k)$

Recall that each S_M is equivalent to a matrix of the form

$$\bigoplus_{\substack{0\leq\ell\leq b\\ \pi\vdash\ell}} (I_\pi \otimes U_M^\pi),$$

where I_π is the identity matrix of size equal to the dimension of the Specht module $\mathcal{S}^{\pi^k}$, and U_M is a matrix of size $\binom{b}{\ell}n_\pi$ with entries depending on k. From Lemma 6.1 it follows that:

Lemma 6.5 *Let $\pi \vdash \ell$ for some $0 \leq \ell \leq b$. For any integer k and any two matchings M and M' in $V_b \times V_b$ we have*

$$U_M^\pi U_{M'}^\pi = (-1)^{|M \circ M'|} f_{|M_2 \cup M_1'|}(b,k) \sum_{\hat{M} \in \mathcal{N}_{MM'}} (-1)^{|\hat{M}|}\ U_{\hat{M}}^\pi.$$

The set of the $U_M^\pi(k)$ forms an algebra $\mathcal{A}_\pi(k)$ over $\mathbb{C}$. □

Lemma 6.6 *Let $\pi \vdash \ell$ for some $0 \leq \ell \leq b$. For any integer k the algebra $\mathcal{A}_\pi(k)$ is isomorphic to the quotient algebra $\mathcal{A}_b(k) \big/ \left\langle S_M \,\middle|\, |M| < \ell \right\rangle$, where $\left\langle S_M \,\middle|\, |M| < \ell \right\rangle$ is the subalgebra generated by all S_M with $|M| < \ell$.*

Proof: The result follows from the observation that U_M^π is the zero matrix if $|M| < \ell$. □

From Lemma 6.4 follows that:

Corollary 6.7 *Let $\pi \vdash \ell$ for some $0 \leq \ell \leq b$. The set $\{U_{M_\sigma}^\pi, U_{M_\phi}^\pi, U_{M_x}^\pi\}$ for some $x \in V_b$ is a generating set for the algebra $\mathcal{A}_\pi(k)$* □

Clearly the polynomials F_M over $\mathbb{C}$ in three non commutative variables ζ_σ, ζ_ϕ and ζ_x introduced in the previous section satisfy the condition that

$$F_M(S_{M_\sigma}, S_{M_\phi}, S_{M_x}) = S_M$$

then

$$F_M(U_{M_\sigma}^\pi, U_{M_\phi}^\pi, U_{M_x}^\pi) = U_M^\pi$$

with the extra property that $F_M(U_{M_\sigma}^\pi, U_{M_\phi}^\pi, U_{M_x}^\pi)$ is the zero matrix if $|M| < \ell$. Further it is the case that

$$N_L^\pi = \sum_{M \in \mathcal{M}(L)} (-1)^{|M|} F_M(U_{M_\sigma}^\pi, U_{M_\phi}^\pi, U_{M_x}^\pi).$$

The problem of calculating all the U_M^π for a given b and π is thus equivalent to finding the relevant F_M and the three matrices $U_{M_\sigma}^\pi$, $U_{M_\phi}^\pi$ and $U_{M_x}^\pi$.

Example 6.2: In Example 6.1 we obtained the polynomials for $b = 3$ and the identity linking set. The matrices $U_{M_\sigma}^\pi$, $U_{M_\phi}^\pi$ and $U_{M_3}^\pi$ for all π shown in Table 6.1 on Page 154 have been obtained in Example 3.7. Then N_L^π is equal to

$$F_\varnothing - (F_{11} + F_{22} + F_{33}) + F_{11,22} + F_{11,33} + F_{22,33} - F_{11,22,33}$$

evaluated at $U_{M_\sigma}^\pi$, $U_{M_\phi}^\pi$ and $U_{M_3}^\pi$. It can easily be checked that the matrices obtained here agree with the ones in Example 3.8.

	$U^\pi_{M_\sigma}$	$U^\pi_{M_\phi}$	$U^\pi_{M_3}$
$\pi = ()$	1	1	$k-2$
$\pi = (1)$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} k-2 & 0 & -1 \\ 0 & k-2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
$\pi = (2)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} k-2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\pi = (1^2)$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} k-2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\pi = (3)$	1	1	0
$\pi = (21)$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\pi = (1^3)$	1	-1	0

Table 6.1: The matrices $U^\pi_{M_\sigma}$, $U^\pi_{M_\phi}$ and $U^\pi_{M_3}$ in the case $b = 3$

6.5 The level $b-1$ for the identity linking set

Let $\pi \vdash b-1$. Then U_M^π is the all zero matrix if $|M| < b-1$. For $L = \{(1,1),(2,2),\ldots,(b,b)\}$ it follows that

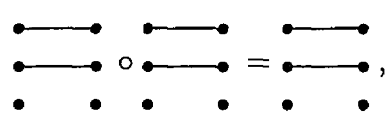
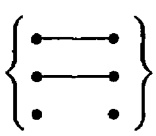
$$N_L^\pi = (-1)^{b-1}\Big(\sum_{x\in V_b} U_{M_x}^\pi - U_{M_\epsilon}^\pi\Big)$$

where M_ϵ is the identity matching of size b, and $U_{M_\epsilon}^\pi$ is the identity matrix.

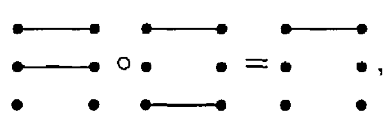
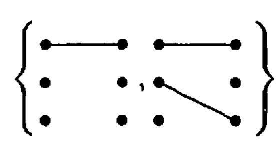
Every matching M of size $b-1$ can be written as $M = M_\omega M_x$ for some $\omega \in \mathrm{Sym}_b$ and some $x \in V_b$. Then for any $y \in V_b$ we have to consider two cases: If $y = x$ then $M \circ M_y = M$ and $\mathcal{N}_{MM_y} = \{M\}$. Otherwise $M \circ M_y = M_\omega \circ M_{\{x,y\}}$ which is of size $b-2$, and $\mathcal{N}_{MM_y} = \{M_\omega \circ M_{\{x,y\}},\ M_\omega \circ M_{(x,y)} \circ M_y\}$. From Lemma 6.5 it follows that:

$$U_M^\pi U_{M_y}^\pi = \begin{cases} (k-b+1)\,U_M^\pi & \text{if } x = y \\ -U_{M_{\omega(x,y)}\circ M_y} & \text{otherwise.} \end{cases}$$

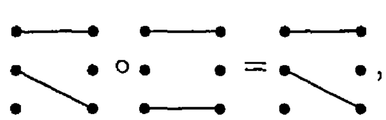
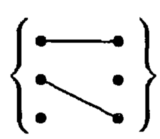
Example 6.3: Let $b = 3$ and $\pi \vdash 2$:

(*i*)  $\mathcal{N}_{M_3M_3} =$ 

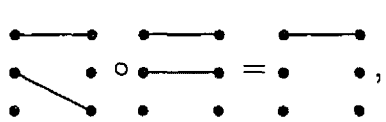
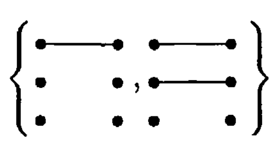
and $U_{M_3}^\pi U_{M_3}^\pi = (k-2)U_{M_3}^\pi$.

(*ii*)  $\mathcal{N}_{M_3M_2} =$ 

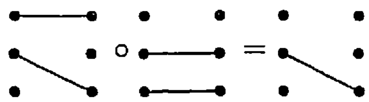
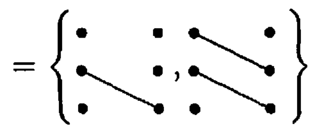
and $U_{M_3}^\pi U_{M_2}^\pi = -U_{M_{(23)}\circ M_2}^\pi$.

(*iii*)  $\mathcal{N}_{M_{(23)}\circ M_2M_2} =$ 

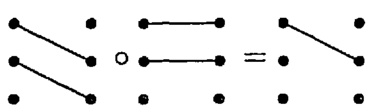
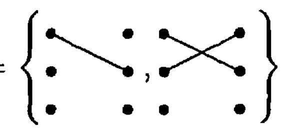
and $U_{M_{(23)}\circ M_2}^\pi U_{M_2}^\pi = (k-2)U_{M_{(23)}\circ M_2}^\pi$.

(*iv*)  $\mathcal{N}_{M_{(23)}\circ M_2M_2} =$ 

and $\quad U^{\pi}_{M_{(23)}\circ M_2} U^{\pi}_{M_3} = -U^{\pi}_{M_3}.$

(*v*)  $\quad \mathcal{N}_{M_{(23)}\circ M_2 M_1} =$ 

and $\quad U^{\pi}_{M_{(23)}\circ M_2} U^{\pi}_{M_1} = -U^{\pi}_{M_{(23)(12)}\circ M_1} = -U^{\pi}_{M_{(132)}\circ M_1}.$

(*vi*)  $\quad \mathcal{N}_{M_{(132)}\circ M_1 M_3} =$ 

and $\quad U^{\pi}_{M_{(132)}\circ M_1} U^{\pi}_{M_3} = -U^{\pi}_{M_{(132)(13)}\circ M_3} = -U^{\pi}_{M_{(12)}\circ M_3}.$

Example 6.4: (The case $b = 3$ continued.) In the previous example we saw that the product $U^{\pi}_{M} U^{\pi}_{M'}$ is independent of the choice of $\pi \vdash 2$ and hence we write U_M instead U^{π}_{M}. Let $H_1 = U_{M_1} + U_{M_2} + U_{M_3}$. Then:

$$H_1 H_1 = (k-2)H_1 - H_2 \quad \text{where}$$

$$H_2 = U_{M_{(12)}\circ M_1} + U_{M_{(12)}\circ M_2} + U_{M_{(13)}\circ M_1} + U_{M_{(13)}\circ M_3} + U_{M_{(23)}\circ M_2} + U_{M_{(23)}\circ M_3};$$

$$H_2 H_1 = (k-2)H_2 - H_3 - 2H_1 \quad \text{where}$$

$$H_3 = U_{M_{(123)}\circ M_1} + U_{M_{(123)}\circ M_2} + U_{M_{(123)}\circ M_3} + U_{M_{(132)}\circ M_1} + U_{M_{(132)}\circ M_2} + U_{M_{(132)}\circ M_3};$$

$$H_3 H_1 = (k-2)H_3 - H_2 - 2H_{2,1} \quad \text{where}$$

$$H_{2,1} = U_{M_{(12)}\circ M_3} + U_{M_{(13)}\circ M_2} + U_{M_{(23)}\circ M_1};$$

and $\quad H_{2,1} H_1 = (k-2)H_{2,1} - H_3.$

This can be written as

$$\begin{pmatrix} k-2 & -2 & 0 & 0 \\ -1 & k-2 & -1 & 0 \\ 0 & -1 & k-2 & -1 \\ 0 & 0 & -2 & k-2 \end{pmatrix} = (k-2)I - \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

where I is the 4×4 identity matrix. The second matrix on the right has eigenvalues 2, 1, -1 and -2. It follows that H_1 has eigenvalues k, $k-1$, $k-3$ and $k-4$ and thus the eigenvalues of $N_L^{(2)}$ and $N_L^{(1^2)}$ are $k-1$, $k-2$, $k-4$ and $k-5$.

Let us now generalize these results for general $b \in \mathbb{N}$. Define the following two operations on the set of partitions λ of b:

- For any two distinct nonzero parts λ_i and λ_j denote by $\lambda^{\lambda_i \vee \lambda_j}$ the partition obtained by joining the part λ_i and λ_j. For example $(5,3,2^2,1)^{3\vee 2} = (5^2,2,1)$.

- For any λ_i and $1 \leq q < \lambda_i$ denote by $\lambda^{q \wedge \lambda_i - q}$ the partition obtained by splitting the part λ_i into a parts of size $\lambda_i - q$ and q. For example $(5,3,2^2,1)^{2\wedge 1} = (5,2^3,1^2)$.

For the rest of this section we assume that all permutations are written as the product of disjoint cycles. Let $\omega \in \mathrm{Sym}_b$ and let x, $y \in V_b$. We consider the product $\omega(xy)$. There are two cases:

- If x and y are in the same orbit under ω we can write $\omega = \overline{\omega}(xa_1a_2\dots a_{q-1}ya_{q+1}\dots a_i)$ for some $\overline{\omega} \in \mathrm{Sym}_b$. Then

$$\omega(xy) = \overline{\omega}(xa_1a_2\dots a_{q-1}ya_{q+1}\dots a_i)(xy) = \overline{\omega}(xa_{q+1}a_{q+2}\dots a_i)(ya_1a_2\dots a_{q-1}).$$

- If x and y are in different orbits under ω we can write $\omega = \overline{\omega}(a_1a_2\dots a_{q-1}x)$ $(b_1b_2\dots b_{p-i}y)$ for some $\overline{\omega} \in \mathrm{Sym}_b$. Then

$$\omega(xy) = \overline{\omega}(a_1a_2\dots a_{q-1}x)(b_1b_2\dots b_{p-i}y)(xy) = \overline{\omega}(xb_1b_2\dots b_{p-1}ya_1a_2\dots a_{q-1}).$$

A conjugacy class of Sym_b is a subset of Sym_b containing all permutations with a certain cycle type. Hence there is a natural bijection between the conjugacy classes of Sym_b and the partitions λ of b. Denote by $\mathcal{C}_\lambda$ the conjugacy class corresponding to λ. For any $\omega \in \mathcal{C}_\lambda$ and any $x, y \in V_b$ it follows by the above arguments that:

- $\omega(xy) \in \mathcal{C}_{\lambda^{q\wedge\lambda_i - q}}$ if x and y are in the same orbit of size λ_i and $\omega^q(x) = y$ or $\omega^{\lambda_i - q}(x) = y$.

- $\omega(xy) \in \mathcal{C}_{\lambda^{\lambda_i\vee\lambda_j}}$ if x and y are in different orbits of respective size λ_i and λ_j.

Since $U_M^\pi U_{M'}^\pi$ is independent of the choice of $\pi \vdash b-1$ we write U_M instead of U_M^π. For any $\omega \in \mathrm{Sym}_b$ and $1 \leq i \leq b$ let

$$X(\omega, i) = \{x \in V_b \mid x \text{ is in an orbit of length } i \text{ under } \omega\}.$$

Define the operators

$$H(\lambda, i) = \sum_{\omega \in C_\lambda} \sum_{x \in X(\omega, i)} U_{M_\omega \circ M_x}.$$

For example in Example 6.4 we can write: $H_1 = H((1^3), 1)$, $H_2 = H((2,1), 2)$, $H_3 = H((3), 3)$ and $H_{2,1} = H((2,1), 1)$.

Lemma 6.8 *Let $\omega \in C_\lambda$ for some partition λ of b. Let $x \in V_b$ be in an ω-orbit under ω of size λ_i, and let y be any vertex in V_b. Then $U_{M_\omega \circ M_x} U_{M_y}$ is either:*

- *equal to $(k-b+1)U_{M_\omega \circ M_x}$ if $x = y$, or*
- *a term in $H(\lambda^{\lambda_i \vee \lambda_j}, \lambda_i + \lambda_j)$ if y is in an orbit under ω of size λ_j not containing x, or*
- *a term in $H(\lambda^{q \wedge \lambda_i - q}, q)$ where $\omega^q(x) = y$.*

Proof: Recall from the beginning of this section:

$$U_M^\pi U_{M_y}^\pi = \begin{cases} (k-b+1)\, U_M^\pi & \text{if } x = y \\ -U_{M_{\omega(x,y)} \circ M_y} & \text{otherwise.} \end{cases}$$

The first two cases follow directly from the argument preceding the lemma. For the case when x and y are in the same orbit of ω, recall that

$$\omega(xy) = \overline{\omega}(x a_1 a_2 \ldots a_{q-1} y a_{q+1} \ldots a_i)(xy) = \overline{\omega}(x a_{q+1} a_{p+2} \ldots a_i)(y a_1 a_2 \ldots a_{q-1}).$$

Hence, y is in a orbit of size q if $\omega^q(x) = y$. □

Lemma 6.9 *Let λ be a partition of b. Let λ_i and λ_j be two non-zero parts with $i \neq j$, but possibly $\lambda_i = \lambda_j$. Then every term in $H(\lambda^{\lambda_i \vee \lambda_j}, \lambda_i + \lambda_j)$ can be written as $U_{M_\omega \circ M_x} U_{M_y}$ where $\omega \in C_\lambda$, $x \in V_b$ is in an orbit under ω of size λ_i and $y \in V_b$ is in an orbit under ω of size λ_j not containing x.*

Proof: By definition every term in $H(\lambda^{\lambda_i \vee \lambda_j}, \lambda_i + \lambda_j)$ is of the form $U_{M_\tau \circ M_y}$ for some $\tau \in \mathcal{C}_{\lambda^{\lambda_i \vee \lambda_j}}$ and y in an orbit of τ of size $\lambda_i + \lambda_j$. Let x be the element in the orbit containing y such that $\tau^{\lambda_j}(x) = y$. Then

$$\begin{aligned}\tau &= \overline{\tau}(a_1 a_2 \dots a_{\lambda_i - 1} x a_{\lambda_i + 1} \dots a_{\lambda_i + \lambda_j - 1} y)\\ &= \overline{\tau}(a_1 a_2 \dots a_{\lambda_i - 1} x)(a_{\lambda_i + 1} \dots a_{\lambda_i + \lambda_j - 1} y)(xy)\end{aligned}$$

for some $\overline{\tau} \in \mathrm{Sym}_b$. Let $\omega = \overline{\tau}(a_1 a_2 \dots a_{\lambda_i - 1} x)(a_{\lambda_i + 1} \dots a_{\lambda_i + \lambda_j - 1} y)$. Then $\omega \in \mathcal{C}_\lambda$ and the result follows. □

Lemma 6.10 *Let λ be a partition of b, and λ_i be a non-zero part. For any $1 \leq q \leq \lambda_i - 1$ every term in $H(\lambda^{q \wedge \lambda_i - q}, q)$ can be written as $U_{M_\omega \circ M_x} U_{M_y}$ where $\omega \in \mathcal{C}_\lambda$, $x, y \in V_b$ are in an orbit under ω of size λ_i and $\omega^q(x) = y$.*

Proof: By definition every term in $H(\lambda^{q \wedge \lambda_i - q}, q)$ is of the form $U_{M_\tau \circ M_y}$ for some $\tau \in \mathcal{C}_{\lambda^{q \wedge \lambda_i - q}}$ and y is in an orbit under τ of size q. Choose any of the orbits under τ of size $\lambda_i - q$ not containing y, and denote one of the vertices in this orbit by x. Then

$$\begin{aligned}\tau &= \overline{\tau}(a_1 a_2 \dots a_{\lambda_i - q - 1} x)(a_{\lambda_i - q + 1} \dots a_{\lambda_i - 1} y)\\ &= \overline{\tau}(a_1 a_2 \dots a_{\lambda_i - q - 1} x a_{\lambda_i - q + 1} \dots a_{\lambda_i - 1} y)(xy)\end{aligned}$$

for some $\overline{\tau} \in \mathrm{Sym}_b$. Let $\omega = \overline{\tau}(a_1 a_2 \dots a_{\lambda_i - q - 1} x a_{\lambda_i - q + 1} \dots a_{\lambda_i - 1} y)$. Then $\omega \in \mathcal{C}_\lambda$ and the result follows. □

Theorem 6.11 *Let λ be any partition of b, and λ_i be any non-zero part. Then*

$$\begin{aligned}H(\lambda, \lambda_i)\, H((1^b), 1) = (k - b + 1)\, H((1^b), 1) - \sum_{\substack{\lambda_j \text{ distinct}\\ \text{in size}}} H(\lambda^{\lambda_i \vee \lambda_j}, \lambda_i + \lambda_j)\\ - \sum_{q=1}^{\lambda_i - 1} r(\lambda^{q \wedge \lambda_i - q}, \lambda_i - q)\, (\lambda_i - q)\, H(\lambda^{q \wedge \lambda_i - q}, q)\end{aligned}$$

where $r(\lambda^{q \wedge \lambda_i - q}, \lambda_i - q)$ is equal to the number of parts of size $\lambda_i - q$ in $\lambda^{q \wedge \lambda_i - q}$ if $\lambda_i - q \neq q$, and is equal to the number of parts of size $\lambda_i - q$ in $\lambda^{q \wedge \lambda_i - q}$ minus one if $\lambda_i - q = q$. The first sum in the above equation is over all parts λ_j in λ of distinct size. This includes the possibility that $\lambda_j = \lambda_i$ with $j \neq i$.

Proof: This result follows from Lemma 6.8, Lemma 6.9 and Lemma 6.10. The factors in the second sum follow from counting in the proof of Lemma 6.10 the number of ways to choose an orbit under τ of size $\lambda_i - q$, and the number of ways to choose x in this orbit. □

For any given integer b denote by $M(b)$ the operator on the space spanned by the operators $H(\lambda, \lambda_i)$ such that $(k-b+1)I - M(b)$ is the operator corresponding to the multiplication on the right by $H((1^b), 1)$ as given in the previous theorem. Here I is the identity operator.

Example 6.5: Let $b = 4$. There are five partitions of 4: (1^4), $(2, 1^2)$, (2^2), $(3, 1)$ and (4), and thus there are seven operators:

$$\begin{array}{llll} H((1^4), 1), & H((2^2), 2), & H((3,1), 3), & H((3,1), 1), \\ H((2,1^2), 2), & H((2,1^2), 1) & \text{and} & H((4), 4). \end{array}$$

Then

$$H((1^4),1)\,H((1^4),1) = (k-3)H((1^4),1) - H((2,1^2),2);$$

$$H((2^2),2)\,H((1^4),1) = (k-3)H((2^2),2) - H((4),4) - H((2,1^2),1);$$

$$\begin{aligned} H((3,1),3)\,H((1^4),1) = (k-3)H((3,1),3) - H((4),4) - 2H((2,1^2),2) \\ - 2H((2,1^2),1); \end{aligned}$$

$$H((3,1),1)\,H((1^4),1) = (k-3)H((3,1),1) - H((4),4);$$

$$H((2,1^2),2)\,H((1^4),1) = (k-3)H((2,1^2),2) - H((3,1),3) - 3H((1^4),1);$$

$$H((2,1^2),1)\,H((1^4),1) = (k-3)H((2,1^2),1) - H((3,1),3) - H((2^2),2);$$

$$\begin{aligned} H((4),4)\,H((1^4),1) = (k-3)H((4),4) - H((3,1),3) - 3H((3,1),1) \\ - 2H((2^2),2). \end{aligned}$$

Then $M(4)$ is equivalent to the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Its eigenvalues are ± 3, ± 2, ± 1 and 0. Thus $B_n(4)$ has eigenvalues k, $k-1$, $k-2$, $k-3$, $k-4$, $k-5$ and $k-6$ at level 3.

In general, N.L. Biggs conjectures the following.

Conjecture 1 *Let b be an integer. Then the level $b-1$ eigenvalues of $B_n(b)$ are*

$$k, \quad k-1, \quad k-2, \ldots, k-(b-1).$$

The rest of this section is joint work with Jan van den Heuvel. Observe that $M(4)$ is equivalent to a matrix of the form:

$$\left(\begin{array}{c|c} O & * \\ \hline * & O \end{array}\right)$$

where O are a all-zero submatrices and $*$ are submatrices with integer entries. In general, if λ is a partition of b with an odd number of non-zero parts then $H(\lambda, \lambda_i)H((1^b), 1)$ is a linear combination of $H(\mu, \mu_j)$ where μ has an even number of non-zero parts, and vice versa. It follows that:

Lemma 6.12 *Let b be an integer. Then $M(b)$ is equivalent to a matrix of the form:*

$$\left(\begin{array}{c|c} O & A \\ \hline B & O \end{array}\right)$$

where A and B are submatrices with integer entries of sizes $m \times n$ and $n \times m$ respectively, and O are all-zero submatrices. $\square$

Corollary 6.13 *Let b be an integer. If λ is an eigenvalue of $M(b)$ then $-\lambda$ is also an eigenvalue of $M(b)$.*

Proof: With Lemma 6.12 we can assume that $M(b)$ is of the form:

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix}$$

where O are all-zero submatrices and A and B are submatrices with integer entries of sizes $m \times n$ and $n \times m$ respectively. Suppose that λ is a non-zero eigenvalue of $M(b)$ and $\begin{pmatrix} v \\ u \end{pmatrix}$ is the corresponding eigenvector where v and u are the subblocks in sizes m and n respectively. From

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} Au \\ Bv \end{pmatrix} = \lambda \begin{pmatrix} v \\ u \end{pmatrix}$$

it follows that $Au = \lambda v$ and $Bv = \lambda u$. Since λ is non-zero it follows that v and u are non-zero. And thus

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} v \\ -u \end{pmatrix} = \begin{pmatrix} -Au \\ Bv \end{pmatrix} = \begin{pmatrix} -\lambda v \\ \lambda u \end{pmatrix} = -\lambda \begin{pmatrix} v \\ -u \end{pmatrix}$$

That is $\begin{pmatrix} v \\ -u \end{pmatrix}$ is an eigenvector of $M(b)$ with eigenalue $-\lambda$. □

Lemma 6.14 *Let b be an integer. Then a matrix corresponding to $M(b)$ has constant row sums equal to $b-1$.*

Proof: Let λ be any partition of b and λ_p be any part in λ. Let μ be an other partition of b. Then

$$H(\mu^{\mu_i \vee \mu_j}, \mu_i + \mu_j) = H(\lambda, \lambda_p)$$

if and only if λ is $\mu^{\mu_i \vee \mu_j}$ for some parts μ_i and μ_j with $\mu_i + \mu_j = \lambda_p$. The number of pairs (μ_i, μ_j) such that $\mu_i + \mu_j = \lambda_p$ is equal to $\lambda_p - 1$.

Further:

$$H(\mu^{q \wedge \mu_i - q}, q) = H(\lambda, \lambda_p)$$

if and only if $q = |\lambda_p|$ and μ is equal to $\lambda^{\lambda_j \vee \lambda_p}$ for some part λ_j. For each such μ there is an operator $H(\lambda, \lambda_p)$ with coefficient $r(\mu^{\lambda_p \wedge \lambda_j}, \lambda_j)\,(\lambda_j)$. The sum of all this coefficients over all such μ is equal to

$$\sum_{\substack{\lambda_j \\ j \neq i}} \lambda_j = b - |\lambda_p|$$

It follows that the row sum of the row corresponding to $H(\lambda, \lambda_p)$ is equal to

$$b - |\lambda_p| + |\lambda_p| - 1 = b - 1.$$

Since this holds for all choices of λ and λ_p the result follows. □

Corollary 6.15 *Let b be an integer. Then $M(b)$ has eigenvalues $\pm(b-1)$.*

Proof: From Lemma 6.14 follows that the all-one vector v is an eigenvector with eigenvalue $b-1$. From Corollary 6.13 follows that $1-b$ is also an eigenvalue. □

Appendix A

Newton's formula

Suppose that $p(x)$ is a polynomial of degree d with coefficients $a_i \in \mathbb{R}$:

$$p(x) = \sum_{i=0}^{d} a_{d-i}\, x^i$$

The sum of the n^{th} powers of the roots of $p(x)$ is A_n where A_n satisfies

$$A_n = -n\, a_n - \sum_{i=1}^{n-1} a_i\, A_{n-i} \qquad (1 \leq n \leq d)$$

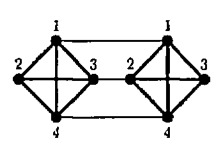
and $$A_n = -\sum_{i=1}^{d} a_i A_{n-i} \qquad (n > d).$$

Appendix B

The H-series catalogue

In this section all the matrices N^π for all levels corresponding to the graphs H^π_{44}, H^π_{43a} , H^π_{43b}, H^π_{42}, H^π_{34}, H^π_{33a}, H^π_{33b}, H^π_{33c}, H^π_{32a} and H^π_{32b} are given. In all the graphs rows corresponding to sets Y containing 2 are all-zero.

1) The graph H_{44}



Let

$$f_4 = k(k-1)(k-2)(k-3), \quad f_3 = (k-1)(k-2)(k-3),$$

$$f_2 = (k-2)(k-3), \quad f_1 = (k-3).$$

Level 0: $N^{()}_{44} = f_4 - 3f_3 + 3f_2 - f_1.$

Level 1:

$$N^{(1)}_{44} = \begin{pmatrix} -f_3+2f_2-f_1 & f_2-f_1 & f_2-2f_1+1 & f_2-f_1 \\ 0 & 0 & 0 & 0 \\ f_2-f_1 & -f_3+2f_2-f_1 & f_2-2f_1+1 & f_2-f_1 \\ f_2-f_1 & f_2-f_1 & f_2-2f_1+1 & -f_3+2f_2-f_1 \end{pmatrix}.$$

Level 2: The transpose of N_{44}^{π} is

$$\begin{pmatrix} 0 & (f_2-f_1)R^{\pi}(\epsilon) & -f_1R(\epsilon)^{\pi} & 0 & 0 & -f_1R^{\pi}(12) \\ 0 & (-f_1+1)R^{\pi}(\epsilon) & (-f_1+1)R^{\pi}(\epsilon) & 0 & 0 & R^{\pi}(\epsilon)+R^{\pi}(1\,2) \\ 0 & -f_1R^{\pi}(\epsilon) & (f_2-f_1)R^{\pi}(\epsilon) & 0 & 0 & -f_1R^{\pi}(\epsilon) \\ 0 & (-f_1+1)R^{\pi}(12) & R^{\pi}(\epsilon)+R^{\pi}(12) & 0 & 0 & (-f_1+1)R^{\pi}(\epsilon) \\ 0 & -f_1R^{\pi}(12) & -f_1R^{\pi}(\epsilon) & 0 & 0 & (f_2-f_1)R^{\pi}(\epsilon) \\ 0 & R^{\pi}(\epsilon)+R^{\pi}(1\,2) & (-f_1+1)R^{\pi}(\epsilon) & 0 & 0 & (-f_1+1)R^{\pi}(\epsilon) \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = R^{(2)}(1\,2) = 1$ and $R^{(1^2)}(1\,2) = -1$.

Level 3:

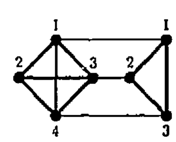
$$N_{44}^{\pi} = \begin{pmatrix} O & O & O & O \\ O & O & O & O \\ R^{\pi}(\epsilon) & -f_1R^{\pi}(\epsilon) & R^{\pi}(\epsilon) & R^{\pi}(12) \\ O & O & O & O \end{pmatrix},$$

where $R^{(3)}(\epsilon) = R^{(3)}(12) = R^{(1^3)}(\epsilon) = 1$, $R^{(1^3)}(12) = -1$ and O is equal to 0. Or, if $\pi = (2,1)$ then

$$R^{(2,1)}(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R^{(2,1)}(\epsilon) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and O is the 2×2 all-zero matrix.

2) The graph H_{43a}



Let

$$f_3 = k(k-1)(k-2), \quad f_2 = (k-1)(k-2), \quad f_1 = (k-2).$$

Level 0: $N_{43a}^{()} = f_3 - 3f_2 + 3f_1 - 1.$

Level 1:

$$N_{43a}^{(1)} = \begin{pmatrix} -f_2+2f_1-1 & f_1-1 & f_1-1 \\ 0 & 0 & 0 \\ f_1-1 & -f_2+2f_1-1 & f_1-1 \\ f_1-1 & f_1-1 & -f_2+2f_1-1 \end{pmatrix}$$

Level 2:

$$N_{43a}^{\pi} = \begin{pmatrix} 0 & 0 & 0 \\ (f_1-1)R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(12) \\ -R^{\pi}(\epsilon) & (f_1-1)R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -R^{\pi}(12) & -R^{\pi}(\epsilon) & (f_1-1)R^{\pi}(\epsilon) \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = 1$.

Level 3:

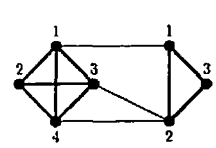
$$N_{43a}^{\pi} = \begin{pmatrix} O \\ O \\ R(\epsilon)^{\pi} \\ O \end{pmatrix}$$

where $R(\epsilon)^{(3)} = R(\epsilon)^{(1^3)} = 1$ with O equal to 0, or if $\pi = (2,1)$ then

$$R^{(2,1)}(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and O is the 2×2 all-zero matrix.

3) The graph H_{43b}



Let

$$f_3 = k(k-1)(k-2), \quad f_2 = (k-1)(k-2), \quad f_1 = (k-2).$$

Level 0: $N_{43b}^{()} = f_3 - 3f_2 + 2f_1.$

Level 1:

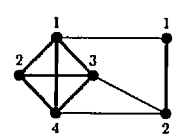
$$N_{43b}^{(1)} = \begin{pmatrix} -f_2+2f_1 & f_1 & f_1-2 \\ 0 & 0 & 0 \\ f_1 & -f_2+f_1 & f_1-1 \\ f_1 & -f_2+f_1 & f_1-1 \end{pmatrix}$$

Level 2:

$$N_{43b}^{\pi} = \begin{pmatrix} 0 & 0 & 0 \\ f_1R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(12) \\ f_1R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(12) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = 1$ and $R^{(1^2)}(12) = -1$.

4) The graph H_{42}



Let

$$f_2 = k(k-1), \quad f_1 = (k-1).$$

Level 0: $N_{42}^{()} = f_2 - 3f_1 + 2.$

Level 1:

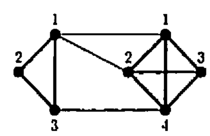
$$N_{42}^{(1)} = \begin{pmatrix} -f_1+2 & 1 \\ 0 & 0 \\ 1 & -f_1+1 \\ 1 & -f_1+1 \end{pmatrix}$$

Level 2:

$$N_{42}^{\pi} = \begin{pmatrix} 0 \\ R^{\pi}(\epsilon) \\ R^{\pi}(\epsilon) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = 1$.

5) The graph H_{34}



Let

$$f_4 = k(k-1)(k-2)(k-3), \quad f_3 = (k-1)(k-2)(k-3),$$
$$f_2 = (k-2)(k-3), \quad f_1 = (k-3).$$

Level 0: $N_{34}^{()} = f_4 - 3f_3 + 2f_2$.

Level 1:

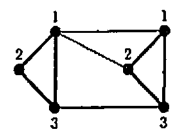
$$N_{34}^{(1)} = \begin{pmatrix} -f_3+2f_2-f_1 & -f_3+2f_2-f_1 & 2f_2-2f_1 & 2f_2 \\ 0 & 0 & 0 & 0 \\ f_2-f_1 & f_2-f_1 & f_2-2f_1 & -f_3+2f_2 \end{pmatrix}$$

Level 2: The transpose of N_{34}^{π} is

$$\begin{pmatrix} 0 & -f_1\big(R^{\pi}(\epsilon)+R^{\pi}(12)\big)\big) & 0 \\ 0 & (-f_1+1)R^{\pi}(\epsilon)+R^{\pi}(12) & 0 \\ 0 & (f_2-f_1)R^{\pi}(\epsilon) & 0 \\ 0 & (-f_1+1)R^{\pi}(\epsilon)+R^{\pi}(12) & 0 \\ 0 & (f_2-f_1)R^{\pi}(\epsilon) & 0 \\ 0 & -2f_1R^{\pi}(\epsilon) & 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = R^{(2)}(1\,2) = 1$ and $R^{(1^2)}(1\,2) = -1$.

6) The graph H_{33a}



Let

$$f_3 = k(k-1)(k-2), \quad f_2 = (k-1)(k-2), \quad f_1 = (k-2).$$

Level 0: $N_{33a}^{()} = f_3 - 3f_2 + 2f_1$.

Level 1:

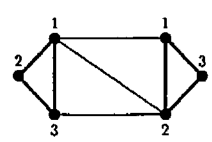
$$N_{33a}^{(1)} = \begin{pmatrix} -f_2 + 2f_1 - 1 & -f_2 + 2f_1 - 1 & 2f_1 \\ 0 & 0 & 0 \\ f_1 - 1 & f_1 - 1 & -f_2 + 2f_1 \end{pmatrix}$$

Level 2:

$$N_{33a}^{\pi} = \begin{pmatrix} 0 & 0 & 0 \\ -R^{\pi}(\epsilon) - R^{\pi}(12) & (f_1 - 1)R^{\pi}(\epsilon) & (f_1 - 1)R^{\pi}(\epsilon) \\ 0 & 0 & 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = R^{(2)}(1\,2) = 1$ and $R^{(1^2)}(1\,2) = -1$.

7) The graph H_{33b}



Let

$$f_3 = k(k-1)(k-2), \quad f_2 = (k-1)(k-2), \quad f_1 = (k-2).$$

Level 0: $N_{33b}^{()} = f_3 - 3f_2 + f_1$.

Level 1:

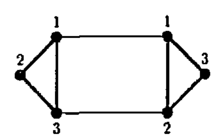
$$N_{33b}^{(1)} = \begin{pmatrix} -f_2 + 2f_1 & -f_2 + f_1 & 2f_1 - 1 \\ 0 & 0 & 0 \\ f_1 & -f_2 + f_1 & f_1 - 1 \end{pmatrix}$$

Level 2:

$$N^{\pi}_{33b} = \begin{pmatrix} 0 & 0 & 0 \\ f_1 R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(12) \\ 0 & 0 & 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = R^{(2)}(1\,2) = 1$ and $R^{(1^2)}(1\,2) = -1$.

8) The graph H_{33c}



Let

$$f_3 = k(k-1)(k-2), \quad f_2 = (k-1)(k-2), \quad f_1 = (k-2).$$

Level 0: $N^{()}_{33c} = f_3 - 2f_2 + f_1$.

Level 1:

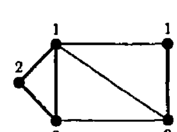
$$N^{(1)}_{33c} = \begin{pmatrix} -f_2 + f_1 & f_1 & f_1 - 1 \\ 0 & 0 & 0 \\ f_1 & -f_2 + f_1 & f_1 - 1 \end{pmatrix}$$

Level 2:

$$N^{\pi}_{33c} = \begin{pmatrix} 0 & 0 & 0 \\ f_1 R^{\pi}(\epsilon) & -R^{\pi}(\epsilon) & -R^{\pi}(12) \\ 0 & 0 & 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = R^{(2)}(1\,2) = 1$ and $R^{(1^2)}(1\,2) = -1$.

9) The graph H_{32a}



Let

$$f_2 = k(k-1), \quad f_1 = (k-1).$$

Level 0: $N^{()}_{32a} = f_2 - 3f_1 + 1.$

Level 1:

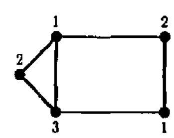
$$N^{(1)}_{32a} = \begin{pmatrix} -f_1+2 & -f_1+1 \\ 0 & 0 \\ 1 & -f_1+1 \end{pmatrix}$$

Level 2:

$$N^{\pi}_{32a} = \begin{pmatrix} 0 \\ R^{\pi}(\epsilon) \\ 0 \end{pmatrix},$$

where $R^{(2)}(\epsilon) = R^{(1^2)}(\epsilon) = 1.$

10) The graph H_{32b}



Let

$$f_2 = k(k-1), \quad f_1 = (k-1).$$

Level 0: $N^{()}_{32b} = f_2 - 2f_1 + 1.$

Level 1:

$$N^{(1)}_{32b} = \begin{pmatrix} -f_1+1 & 1 \\ 0 & 0 \\ 1 & -f_1+1 \end{pmatrix}$$

Level 2:

$$N^{\pi}_{32b} = \begin{pmatrix} 0 \\ R^{\pi}(12) \\ 0 \end{pmatrix},$$

where $R^{(2)}(12) = 1$ and $R^{(1^2)}(12) = -1.$

Appendix C

The reduced matrices for level 2

In this section the reduced matrices $\overline{N}_L^{\pi}$ for level two with $\pi = (2)$ and $\pi = (1^2)$ for the graphs

$$(558)_n, \qquad (468)_n, \qquad (477)_n, \qquad (567)_n \quad \text{and} \quad (666)_n$$

are given. The rows and columns correspond to the pairs of independent sets:

$$\{1,3\},\ \{1,4\},\ \{3,4\},\ \{13,4\},\ \{1,24\},\ \{1,3\},\ \{24,3\},\ \{14,3\} \text{ and } \{13,24\}.$$

As before $c = k-2$. **The graph** $(558)_n$:

$$\overline{N}_{558}^{(1^2)} = \begin{pmatrix}
-c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\
-c+2 & c(c-1)-c+1 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\
2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\
-c+3 & c(c-1)-c+1 & -2c+2 & c-1 & -1 & c & -1 & -1 & 1 \\
-c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\
-c+2 & c(c-1)-c+1 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\
2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\
-c+3 & -2c+2 & -c+3 & -2 & -1 & c & -1 & -1 & 1 \\
-c+3 & c(c-1)-c+1 & -2c+2 & c-1 & -1 & c & -1 & -1 & 1
\end{pmatrix}$$

$$\overline{N}_{558}^{(1^2)} = \begin{pmatrix} -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -c+2 & c(c-1)-c+1 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+3 & c(c-1)-c+1 & -2c+2 & c-1 & -1 & c & -1 & -1 & 1 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -c+2 & c(c-1)-c+1 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+3 & -2c+2 & -c+3 & -2 & -1 & c & -1 & -1 & 1 \\ -c+3 & c(c-1)-c+1 & -2c+2 & c-1 & -1 & c & -1 & -1 & 1 \end{pmatrix}$$

The graph $(468)_n$:

$$\overline{N}_{468}^{(2)} = \begin{pmatrix} 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -2c+2 & (c-1)^2 & -c+3 & c-1 & -1 & c & -1 & -1 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -c+3 & -2c+2 & -c+3 & -2 & -1 & c & -1 & -1 & 1 \\ -2c+2 & (c-1)^2 & -c+3 & c-1 & -1 & c & -1 & -1 & 1 \end{pmatrix}$$

$$\overline{N}_{468}^{(1^2)} = \begin{pmatrix} 0 & c-1 & c-2 & 1 & 0 & 0 & 0 & -1 & 0 \\ c-2 & -(c-1)^2 & c-2 & -c+1 & 1 & -c & 1 & 0 & -1 \\ c-2 & c-1 & 0 & 1 & 1 & -c & 1 & 1 & -1 \\ 2c-2 & -(c-1)^2 & c-1 & -c+1 & 1 & -c & 1 & 1 & -1 \\ 0 & c-1 & c-2 & 1 & 0 & 0 & 0 & -1 & 0 \\ c-2 & -(c-1)^2 & c-2 & -c+1 & 1 & -c & 1 & 0 & -1 \\ c-2 & c-1 & 0 & 1 & 1 & -c & 1 & 1 & -1 \\ -c+1 & 0 & c-1 & 0 & -1 & c & -1 & -1 & 1 \\ 2c-2 & -(c-1)^2 & c-1 & -c+1 & 1 & -c & 1 & 1 & -1 \end{pmatrix}$$

The graph $(477)_n$:

$$\overline{N}^{(2)}_{477} = \begin{pmatrix} -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -c+3 & -2c+2 & -c+3 & -2 & -1 & c & -1 & -1 & 1 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -2c+2 & (c-1)^2 & -c+3 & c-1 & -1 & c & -1 & -1 & 1 \\ -c+3 & -2c+2 & -c+3 & -2 & -1 & c & -1 & -1 & 1 \end{pmatrix}$$

$$\overline{N}^{(1^2)}_{477} = \begin{pmatrix} c-2 & -(c-1)^2 & c-2 & -c+1 & 1 & -c & 1 & 0 & -1 \\ 0 & c-1 & c-2 & 1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & -c+1 & 0 & -1 & -1 & c & -1 & -1 & 1 \\ -c+1 & 0 & c-1 & 0 & -1 & c & -1 & -1 & 1 \\ c-2 & -(c-1)^2 & c-2 & -c+1 & 1 & -c & 1 & 0 & -1 \\ 0 & c-1 & c-2 & 1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & -c+1 & 0 & -1 & -1 & c & -1 & -1 & 1 \\ 2c-2 & -(c-1)^2 & c-1 & -c+1 & 1 & -c & 1 & 1 & -1 \\ -c+1 & 0 & c-1 & 0 & -1 & c & -1 & -1 & 1 \end{pmatrix}$$

The graph $(567)_n$:

$$\overline{N}^{(2)}_{567} = \begin{pmatrix} -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ -c+3 & (c-1)^2 & -2c+2 & c-1 & -1 & c & -1 & -1 & 1 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ -2c+2 & (c-1)^2 & -c+3 & c-1 & -1 & c & -1 & -1 & 1 \\ -c+3 & (c-1)^2 & -2c+2 & c-1 & -1 & c & -1 & -1 & 1 \end{pmatrix}$$

$$\overline{N}^{(1^2)}_{567} = \begin{pmatrix} c-2 & c-1 & 0 & 1 & 1 & -c & 1 & 1 & -1 \\ 0 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & 1 & 0 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ -c+1 & (c-1)^2 & -2c+2 & c-1 & -1 & c & -1 & 1 & 1 \\ c-2 & c-1 & 0 & 1 & 1 & -c & 1 & 1 & -1 \\ 0 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & 1 & 0 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ 2c-2 & -(c-1)^2 & c-1 & -c+1 & 1 & -c & 1 & 1 & -1 \\ -c+1 & (c-1)^2 & -2c+2 & c-1 & -1 & c & -1 & 1 & 1 \end{pmatrix}$$

The graph $(666)_n$:

$$\overline{N}^{(2)}_{666} = \begin{pmatrix} 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ -2c+2 & (c-1)^2 & -c+3 & c-1 & -1 & c & -1 & -1 & 1 \\ 2 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -c+2 & -c+1 & 2 & -1 & -1 & c & -1 & -1 & 1 \\ -c+2 & (c-1)^2 & -c+2 & c-1 & -1 & c & -1 & 0 & 1 \\ -c+3 & (c-1)^2 & -2c+2 & c-1 & -1 & c & -1 & -1 & 1 \\ -2c+2 & (c-1)^2 & -c+3 & c-1 & -1 & c & -1 & -1 & 1 \end{pmatrix}$$

$$\overline{N}^{(1^2)}_{666} = \begin{pmatrix} 0 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & 1 & 0 \\ c-2 & c-1 & 0 & 1 & 1 & -c & 1 & 1 & -1 \\ c-2 & -(c-1)^2 & c-2 & -c+1 & 1 & -c & 1 & 0 & -1 \\ 2c-2 & -(c-1)^2 & c-1 & -c+1 & 1 & -c & 1 & 1 & -1 \\ 0 & -c+1 & -c+2 & -1 & 0 & 0 & 0 & 1 & 0 \\ c-2 & c-1 & 0 & 1 & 1 & -c & 1 & 1 & -1 \\ c-2 & -(c-1)^2 & c-2 & -c+1 & 1 & -c & 1 & 0 & -1 \\ -c+1 & (c-1)^2 & -2c+2 & c-1 & -1 & c & -1 & 1 & 1 \\ 2c-2 & -(c-1)^2 & c-1 & -c+1 & 1 & -c & 1 & 1 & -1 \end{pmatrix}$$

Appendix D

Maple programs

D.1 The program EquiDominantPoints

The program `EquiDominantPoints` has as input a natural number n, two polynomials $f(\lambda, z)$ and $g(\lambda, z)$ and a real number $\varepsilon > 0$.

It first calculates the resultant $\det R(f_s, g)$ with respect to λ. This is a polynomial in s and z with integer coefficients. For each point s_i in a given sequence $\{s_i\}_{i=1}^{n} \subset S^1$ the program evaluates the roots of the resultant and saves them as a list R.

For each of these roots R_i the sub-program `EquidomOne` if $f = g$ or `EquidomTwo` if $f \neq g$ is called. In the case of `EquidomOne` the roots of $f(\lambda, R_i)$ are calculated, the absolute values found and the biggest two compared. If their difference is less than ε the value R_i is saved as P_k and k increases by one.

In case of `EquidomTwo` the roots of $f(\lambda, R_i)$ and $f(\lambda, R_i)$ are calculated, the absolute values found and the respective biggest ones are compared. If their difference is less than ε the value R_i is saved as P_k and k increases by one. In the end `EquiDominantPoints` returns the list P_k lying on the dominant equimodular curves $D(f, g)$ (or $D(f)$ if $f = g$).

```
EquiDominantPoints:=proc(n,Poly1,Poly2,epsilon)
local Rab,R,S,P,k,i,s;
k:=1;
Rab:=(s,z)->factor(resultant(Poly1(lambda*s,z),
                              Poly2(lambda,z),lambda)):
if Poly1(lambda,z)=Poly2(lambda,z) then
          for S from 1 to n do
              R:=[fsolve(Rab(exp(I*S*Pi/n),z),z,complex)];
              for i from 1 to nops(R) do
                 P[k]:=R[i];
                 k:=k+EquidomOne(Poly1,R[i],epsilon);
              end do;
          end do;
else
          for S from 1 to n do
              R:=[fsolve(Rab(exp(I*S*2*Pi/n),z),z,complex)];
              for i from 1 to nops(R) do
                 P[k]:=R[i];
                 k:=k+EquidomTwo(Poly1,Poly2,R[i],epsilon);
              end do;
          end do;
end if;
return([seq(P[i],i=1..k-1)]);
end:

EquidomOne:=proc(Poly,z_0,epsilon)
local r,R,i;
R:=[fsolve(Poly(lambda,z_0),lambda,complex)]; r:=nops(R);
R:=[seq(abs(R[i]),i=1..r)]; R:=sort(R);
if abs(R[r]-R[r-1])<epsilon then return(1) end if;
return(0);
```

```
end:

EquidomTwo:=proc(Poly1,Poly2,z_0,epsilon)
local r1,r2,R1,R2,i;
R1:=[fsolve(Poly1(lambda,z_0),lambda,complex)];
r1:=nops(R1);
R2:=[fsolve(Poly2(lambda,z_0),lambda,complex)];
r2:=nops(R2);
if abs(max(seq(abs(R1[i]),i=1..r1))
      -max(seq(abs(R2[i]),i=1..r2)))<epsilon
   then return(1) end if;
return(0);
end:
```

D.2 The program DomTest

The program DomTest has as input a polynomial $f(\lambda, z)$, a list of points R in $\mathbb{C}$ and a real number $\varepsilon > 0$. For each of the points R_i the sub-program EquidomOne is called and the "biggest" two eigenvalues (in modulus) are compared. If their difference is less than ε the point is saved. The program returns a sub-list of R of dominant points with respect to the polynomial.

```
DomTest:=proc(PolyAll,R,epsilon)
local i,k,P;
k:=1;
for i from 1 to nops(R) do
    P[k]:=R[i];
    k:=k+EquidomOne(PolyAll,R[2],epsilon);
end do;
return([seq(P[i],i=1..k-1)]);
end:
```

D.3 The program Slices

The program Slices has as input a polynomial $f(\lambda, z)$, two real numbers $x_{\min}$ and $x_{\max}$ with $x_{\min} < x_{\max}$, a natural number n and a real number y.

It evaluates the absolute values of the roots of $f(\lambda, z)$ at the points $z = [x_j + iy]$ where $x_j = x_{\min} + j(x_{\max} - x_{\min})/n$ for $j = 0, 1, \ldots, n$. It returns the plot of these absolute values against the values x_j. The program can easily adapted to "slice" along a different line than $\Im(z) = y$.

```
Slice:=proc(Poly,xmin,xmax,n,y)
local x,i,j,R1,R2,P;
for j from 0 to n do
   x:=xmin+j*(xmax-xmin)/n;
   R1:=[fsolve(Poly(lambda,x+y*I),lambda,complex)]:
   for i from 1 to nops(R1) do R2[i,j]:=x+I*abs(R1[i]);
   end do;
end do;
for j from 1 to nops(R1) do
   P[j]:=complexplot([seq(R2[j,i],j=0..n)],
                     axes=boxed,color=black):
end do;
print(display(seq(P[j],j=1..nops(R1))));
end:
```

Bibliography

[1] Biggs N.L., Meredith G.H.J.
Approximations for Chromatic Polynomials.
J. Combinatorial Theory (B), **20** (1976) 5-19.

[2] Biggs N.L.
Colouring of square lattice graphs.
Bulletin of the London Mathematical Society, **9** (1977) 54-56.

[3] Biggs N.L.
Algebraic Graph Theory. (1993)
Cambridge University Press, Cambridge.

[4] Biggs N.L.
Equimodular Curves.
to appear in *Discrete Mathematics.*

[5] Biggs N.L.
A matrix method for chromatic polynomials.
J. Combinatorial Theory (B), **82** (2001) 19-29.

[6] Biggs N.L.
Equimodular Curves for Reducible Matrices.
CDAM Research Report Series, LSE-CDAM-**2001-01**.

[7] Biggs N.L.
Chromatic polynomials and representations of the symmetric group.
to appear in *Linear Algebra and its Applications.*

[8] Biggs N.L.
Chromatic polynomials for twisted bracelets.
Bulletin of the London Mathematical Society, **34** (2002) 129-139.

[9] Biggs N.L., Klin M.H., Reinfeld P.
Algebraic Methods for Chromatic Polynomials.
to appear in the *European Journal of Combinatorics*.

[10] Biggs N.L. Shrock R.
$T = 0$ partition functions for Potts antiferromagnets on square lattice strips with (twisted) periodic boundary conditions. *J. Phys. A: Math. Gen.* **32** (1999) L1-L5.

[11] Chang S.C.
Chromatic Polynomials for Lattice Strips with Cyclic Boundary Conditions.
Physice A, **296** (2001) 495-522.

[12] Chang S.C.
Exact Chromatic Polynomials of Toroidal Chains of Complete Graphs.
arXiv: math-ph/0111028v1 16 Nov 2001.

[13] Chang S.C., Shrock R.
$T = 0$ partition functions for Potts antiferomagnets on lattice strips with fully periodic boundary conditions.
Physice A, **292** (2001) 307-345.

[14] Cox D., Little J., O'Shea D.
Using Algebraic Geometry. (1998)
Springer-Verlag New York, Inc., New York.

[15] Jacobsen J.L., Salas J., Sokal A.D.
Transfer matrices and partition-function zeros for antiferromagnetic Potts models III. Triangular-lattice chromatic polynomial.
e-Print Archive: cond-mat/0204587.

[16] James G.D.
The Representation Theory of the Symmetric Groups. (1978)
Lecture Notes in Mathematics 682, Springer Verlag, Berlin.

[17] Ledermann W.
Introduction to Group Characters. (1977)
Cambridge University Press, Cambridge.

[18] Read R., Royle G.F.
Chromatic roots of families of graphs.
In: Y. Alavi, G. Chartrand, O.R. Ollermann, A.J. Schwenk, editors,
Graph Theory, Combinatorics and Applications. (1991)
John Wiley and Sons, Inc., New York.

[19] Reinfeld P.
Chromatic Polynomials and the Spectrum of the Kneser Graph.
CDAM Research Report Series, LSE-CDAM-**2000-02**.

[20] Sagan B.E.
The Symmetric Group. (2001)
Springer-Verlag New York, Inc., New York.

[21] Salas J., Sokal A.D.
Transfer Matrices and Partition-Function Zeros for Antiferromagnetic Potts Models.
J. Statist. Phys. **104** (2001) 609-699.

[22] Sokal A.D.
Chromatic polynomials, Potts models and all that.
Physica A **279** (2000) 324-332.

[23] Sokal A.D.
Bounds on the complex zeros of (di)chromatic polynomials and Potts model partition functions.
J. Comb. Probab. Comput **10** (2001) 41-77.

[24] Shrock R., Tsai S.H.

Asymptotic limits and zeros of chromatic polynomials and ground state entropy of Potts antiferromagnets.

Phys. Rev., **E55** (1997) 5165-5179.

[25] Shrock R., Tsai S.H.

Exact partition functions for Potts antiferromagnets on cyclic lattice strips.

Physice A, **275** (2000) 429-449.