HOMOGENIZATION OF COMPOSITE BEAMS WITH PERIODIC MICROSTRUCTURES

I. Saiki[1*†], K. YARI[1], M. YAMADA[1], A. SETOGAWA[1] and T. IWAKUMA[1]

[1]*Department of Civil Engineering, School of Engineering, Tohoku University, Japan*

ABSTRACT

Some kinds of element test are generally conducted in order to evaluate mechanical characteristics of structural members that have complex microstructures such as steel and concrete composite beams. However, the element test with a part of the structural member can not faithfully reconstruct deformation state in actual structures. To that end, we formulate a beam with averaged mechanical properties in order to evaluate mechanical properties of Timoshenko beams with microstructures.

Keywords: homogenization method, periodic boundary condition, Timoshenko beam

1. INTRODUCTION

Nowadays, various types of steel and concrete hybrid structures are used. Hybrid structures enables us to get great performance that is never obtained by a single material. In this context, it is most important to ensure the integration of different materials for hybrid structures to provide high performance. For the steel and concrete hybrid structures, the mechanical integration by some kinds of shear connectors are usually employed. Since the shear connector is very small compared to the structure, it is hard to simulate the microstructure such as the shear connectors in the numerical analysis of the whole structure, no matter how computers are improved. Therefore, the averaged mechanical properties of the microstructure is generally evaluated prior to the analysis of the whole structure.

For the headed studs, the averaged mechanical properties such as shear strength and shear force-slip relationship are studied through the element test called push out test[1]. However, unlike the case of the element test of materials, the deformation and the stress distribution of the microstructure in the whole structure are not trivial. Moreover, the element test of the microstructure has limitations of the reproductivity of the complex deformation state. For instance, in the push out test of headed studs, the shear force is applied as single shear between steel and concrete interface whereas the shear stress in actual steel and concrete composite beam is distributed in the beam.

On the other hand, for the average property evaluation of the periodic microstructure, e.g., those of composite materials, the mathematical homogenization method is well suited. Hence, the mathematical homogenization method gets a lot of attention for an average technique of composite materials

[*] Corresponding author: Email: saiki@civil.tohoku.ac.jp

[†] Presenter: Email: saiki@civil.tohoku.ac.jp

with periodic microstructures. Although the shear connectors in the composite beam can be also regarded as the periodic microstructure, application of the homogenization method to the structural members such as beams and plates is rarely reported.

Approaches of the application of the homogenization to the structural members are twofold. One includes application of the homogenization to the second order governing differential equation of the continuum followed by introduction of the kinematic field of beams or plates to the microscale problem of the continuum. The other approach includes application of the homogenization method to the fourth order governing differential equation of beams or plates. The analysis of sandwich panels by Takano et. al[2] and the analysis of beam and plate structure by Okada et. al[3] can be classified into the former approach. The formulation of the homogenization of the plate bending based on the asymptotic expansion by Kohn and Vogelius[4] should be classified into the latter. On the other hand, authors have been applied the homogenization method on the basis of the generalized convergence theory to the linear[5] and nonlinear[6] bending problems of plates.

Although many reports on the homogenization for the structural members are published, those for the structural members considering the shear deformation has not been available so far. Therefore, in this paper, a method to evaluate the mechanical properties of the heterogeneous Timoshenko beam is developed. For this purpose, a periodic boundary condition that is applicable to the representative volume element discritized by the solid element is also introduced.

2. Formulation of two-scale boundary value problem of Timoshenko beam by generalized convergence method

A set of spatially fixed orthonormal base vectors $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ is defined as shown in Figure 1. Then the position vector $\boldsymbol{x}$ is expressed by its components x_i as $\boldsymbol{x} = x_i\boldsymbol{e}_i$ in terms of the base vectors.

The equivalent variational problem to the boundary value problem of the Timoshenko beam is the problem finding the stationary point of the functional

$$\begin{aligned}\Pi := \int W^{\mathrm{b}}\left(\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}x_1}\right)\mathrm{d}x_1 + \int W^{\mathrm{s}}\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}x_1} + \boldsymbol{e}_1 \times \boldsymbol{\theta}\right)\mathrm{d}x_1 \\ + \int W^{\mathrm{t}}\left(\frac{\mathrm{d}\psi}{\mathrm{d}x_1}\right)\mathrm{d}x_1 + \int W^{\mathrm{a}}\left(\frac{\mathrm{d}u}{\mathrm{d}x_1}\right)\mathrm{d}x_1 - \int \boldsymbol{q}\cdot\boldsymbol{v}\,\mathrm{d}x_1 - \int f u\,\mathrm{d}x_1\end{aligned} \tag{1}$$

The independent variables of the functional are $\boldsymbol{v}$, $\boldsymbol{\theta}$, u and ψ. Here, W^{b}, W^{s}, W^{t} and W^{a} are strain energy density functions of the bending strain, the shear strain, the torsional shear strain and the axial strain, respectively.

The generalized (Γ) convergence theory[7] states that the convex functional $\pi(w)$ of a function w with a parameter ϵ converges to

$$\pi^{\mathrm{H}}(w) := \int W^{\mathrm{H}}(\nabla_x w^0)\,\mathrm{d}x \tag{2}$$

as $\epsilon \to 0$. Here W^{H} is the homogenized strain energy density function defined as

$$W^{\mathrm{H}}(\nabla_x w^0) := \inf_{w^1}\left\langle W(\nabla_x w^0 + \nabla_y w^1)\right\rangle. \tag{3}$$

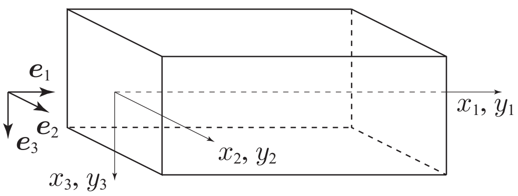

Figure 1: beam and basis vectors

Furthermore, it is known that ∇w converges to $\nabla_x w^0 + \nabla_y w^1$ as $\epsilon \to 0$, where $\langle \bullet \rangle$ denotes the volume average over Y defined as

$$\langle \bullet \rangle := \frac{1}{|Y|} \int_Y \bullet \, \mathrm{d}Y, \tag{4}$$

where Y does the representative volume element consists of unit periodic structure (unit cell). If the function w represents the displacement, w^0 and w^1 respectively correspond to the macroscale displacement and the microscale displacement which has the periodicity over the representative volume element (Y-periodicity). ∇ denotes the gradiant operator, which coincides with the derivative with respect to x_1 along the beam axis. $\boldsymbol{y}$ is the microscale coordinate defined as $\boldsymbol{y} = \frac{\boldsymbol{x}}{\epsilon}$.

The homogenized functional Π^{H} correspond to the original functional Π is given by

$$\begin{aligned} \Pi^{\mathrm{H}} = & \int \left\langle W^{\mathrm{b}} \left(\frac{\mathrm{d}\boldsymbol{\theta}^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\boldsymbol{\theta}^1}{\mathrm{d}y_1} \right) \right\rangle \mathrm{d}x_1 + \int \left\langle W^{\mathrm{s}} \left(\frac{\mathrm{d}\boldsymbol{v}^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\boldsymbol{v}^1}{\mathrm{d}y_1} + \boldsymbol{e}_1 \times \boldsymbol{\theta}^0 \right) \right\rangle \mathrm{d}x_1 \\ & + \int \left\langle W^{\mathrm{t}} \left(\frac{\mathrm{d}\psi^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\psi^1}{\mathrm{d}y_1} \right) \right\rangle \mathrm{d}x_1 + \int \left\langle W^{\mathrm{a}} \left(\frac{\mathrm{d}u^0}{\mathrm{d}x_1} + \frac{\mathrm{d}u^1}{\mathrm{d}y_1} \right) \right\rangle \mathrm{d}x_1 - \int \boldsymbol{q} \cdot \boldsymbol{v}^0 \mathrm{d}x_1 - \int f u^0 \mathrm{d}x_1, \end{aligned} \tag{5}$$

where $\boldsymbol{\theta}^0$ and $\boldsymbol{\theta}^1$ are respectively the macroscale rotational angle of the cross section and the microscale one, $\boldsymbol{v}^0$ and $\boldsymbol{v}^1$ are respectively the macroscale deflection (the displacement transverse to the beam axis) and the microscale one, ψ^0 and ψ^1 are respectively the macroscale torsional angle and the microscale one, u^0 and u^1 are respectively the macroscale axial displacement and the microscale one. It should be emphasized that all of $\boldsymbol{\theta}^1$, $\boldsymbol{v}^1$, ψ^1 and u^1 have Y-periodicity. The stationary condition of the homogenized functional is given by

$$\begin{aligned} 0 = \delta\Pi^{\mathrm{H}} = & \int \left\langle \boldsymbol{M}^0 \cdot \left(\frac{\mathrm{d}\delta\boldsymbol{\theta}^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\delta\boldsymbol{\theta}^1}{\mathrm{d}y_1} \right) \right\rangle \mathrm{d}x_1 - \int \left\langle \boldsymbol{Q}^0 \cdot \left(\frac{\mathrm{d}\delta\boldsymbol{v}^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\delta\boldsymbol{v}^1}{\mathrm{d}y_1} + \boldsymbol{e}_1 \times \delta\boldsymbol{\theta}^0 \right) \right\rangle \mathrm{d}x_1 \\ & - \int \left\langle T^0 \cdot \left(\frac{\mathrm{d}\delta\psi^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\delta\psi^1}{\mathrm{d}y_1} \right) \right\rangle \mathrm{d}x_1 - \int \left\langle N^0 \cdot \left(\frac{\mathrm{d}\delta u^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\delta u^1}{\mathrm{d}y_1} \right) \right\rangle \mathrm{d}x_1 \\ & - \int \boldsymbol{q} \cdot \delta\boldsymbol{v}^0 \mathrm{d}x_1 - \int f \delta u^0 \mathrm{d}x_1, \end{aligned} \tag{6}$$

where $\boldsymbol{M}^0$ is, for example, the bending moment defined as

$$\boldsymbol{M}^0 = \frac{\partial W}{\partial \left(\frac{\mathrm{d}\boldsymbol{\theta}^0}{\mathrm{d}x_1} + \frac{\mathrm{d}\boldsymbol{\theta}^1}{\mathrm{d}y_1} \right)} \tag{7}$$

arises from the total curvature $\dfrac{\mathrm{d}\boldsymbol{\theta}^0}{\mathrm{d}x_1} + \dfrac{\mathrm{d}\boldsymbol{\theta}^1}{\mathrm{d}y_1}$.

Equation (6) can be rearranged in terms of the variation of the independent variables as

$$\begin{aligned}\int\left\langle \boldsymbol{M}^0\cdot\frac{\mathrm{d}\delta\boldsymbol{\theta}^0}{\mathrm{d}x_1}-\boldsymbol{Q}^0\cdot\left(\boldsymbol{e}_1\times\delta\boldsymbol{\theta}^0\right)\right\rangle\mathrm{d}x_1 &= \int\widetilde{\boldsymbol{M}}\cdot\frac{\mathrm{d}\delta\boldsymbol{\theta}^0}{\mathrm{d}x_1}-\widetilde{\boldsymbol{Q}}\cdot(\boldsymbol{e}_1\times\delta\boldsymbol{\theta}^0)\,\mathrm{d}x_1\\ &= \left[\widetilde{\boldsymbol{M}}\cdot\delta\boldsymbol{\theta}^0\right]_0^{\ell}-\int\left(\frac{\mathrm{d}\widetilde{\boldsymbol{M}}}{\mathrm{d}x_1}+\boldsymbol{e}_1\times\boldsymbol{Q}^0\right)\cdot\delta\boldsymbol{\theta}^0\mathrm{d}x_1=0\end{aligned} \tag{8}$$

$$\int\left\langle \boldsymbol{q}\cdot\delta\boldsymbol{v}^0+\boldsymbol{Q}^0\cdot\frac{\mathrm{d}\delta\boldsymbol{v}^0}{\mathrm{d}x_1}\right\rangle\mathrm{d}x_1=\left[\widetilde{\boldsymbol{Q}}\cdot\delta\boldsymbol{v}^0\right]_0^{\ell}+\int\left(\boldsymbol{q}-\frac{\mathrm{d}\widetilde{\boldsymbol{Q}}}{\mathrm{d}x_1}\right)\cdot\delta\boldsymbol{v}^0\mathrm{d}x_1=0 \tag{9}$$

$$\int\left\langle T^0\cdot\frac{\mathrm{d}\delta\psi^0}{\mathrm{d}x_1}\right\rangle\mathrm{d}x_1=\left[\widetilde{T}\cdot\delta\psi^0\right]_0^{\ell}-\int\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}x_1}\,\delta\psi^0\mathrm{d}x_1=0 \tag{10}$$

$$\int\left\langle f\delta u^0+N^0\cdot\frac{\mathrm{d}\delta u^0}{\mathrm{d}x_1}\right\rangle\mathrm{d}x_1=\left[\widetilde{N}\cdot\delta u^0\right]_0^{\ell}+\int\left(f-\frac{\mathrm{d}\widetilde{N}}{\mathrm{d}x_1}\right)\delta u^0\mathrm{d}x_1=0 \tag{11}$$

where $\widetilde{\boldsymbol{M}}$, $\widetilde{\boldsymbol{Q}}$, $\widetilde{T}$ and $\widetilde{N}$ are the averaged bending moment, the averaged shear force, the averaged tortional moment and the averaged axial force, respectively. These internal forces are defined as

$$\widetilde{\boldsymbol{M}}:=\left\langle\boldsymbol{M}^0\right\rangle,\quad \widetilde{\boldsymbol{Q}}:=\left\langle\boldsymbol{Q}^0\right\rangle,\quad \widetilde{T}:=\left\langle T^0\right\rangle,\quad \widetilde{N}:=\left\langle N^0\right\rangle. \tag{12}$$

Considering periodicity of the microscale quantities, we also define, e.g., the averaged curvature

$$\nabla\boldsymbol{\theta}:=\left\langle\frac{\mathrm{d}\boldsymbol{\theta}^0}{\mathrm{d}x_1}+\frac{\mathrm{d}\boldsymbol{\theta}^1}{\mathrm{d}y_1}\right\rangle=\frac{\mathrm{d}\boldsymbol{\theta}^0}{\mathrm{d}x_1} \tag{13}$$

as the deformation counterparts of the averaged internal forces.

Finally, the macroscale problem is described by the original boundary conditions and the equilibrium equations Finally, the macroscale equillibrium is obtained as

$$\frac{\mathrm{d}\widetilde{\boldsymbol{M}}}{\mathrm{d}x_1}=-\boldsymbol{e}_1\times\boldsymbol{Q}^0\quad\text{that is}\quad\nabla\times\widetilde{\boldsymbol{M}}=\boldsymbol{Q}^0,\quad\frac{\mathrm{d}\boldsymbol{Q}^0}{\mathrm{d}x_1}=\boldsymbol{q},\quad\frac{\mathrm{d}T^0}{\mathrm{d}x_1}=0,\quad\frac{\mathrm{d}N^0}{\mathrm{d}x_1}=f. \tag{14}$$

Similarly, equation (6) also yield the microscale equilibrium equations

$$0=-\left\langle\boldsymbol{M}^0\cdot\frac{\mathrm{d}\delta\boldsymbol{\theta}^1}{\mathrm{d}y_1}\right\rangle=\left\langle\frac{\mathrm{d}\boldsymbol{M}^0}{\mathrm{d}y_1}\cdot\delta\boldsymbol{\theta}^1\right\rangle,\quad 0=-\left\langle\boldsymbol{Q}^0\cdot\frac{\mathrm{d}\delta\boldsymbol{v}^1}{\mathrm{d}y_1}\right\rangle=\left\langle\frac{\mathrm{d}\boldsymbol{Q}^0}{\mathrm{d}y_1}\cdot\delta\boldsymbol{v}^1\right\rangle \tag{15}$$

$$0=-\left\langle T^0\cdot\frac{\mathrm{d}\delta\psi^1}{\mathrm{d}y_1}\right\rangle=\left\langle\frac{\mathrm{d}T^0}{\mathrm{d}y_1}\cdot\delta\psi^1\right\rangle,\quad 0=-\left\langle N^0\cdot\frac{\mathrm{d}\delta u^1}{\mathrm{d}y_1}\right\rangle=\left\langle\frac{\mathrm{d}N^0}{\mathrm{d}y_1}\cdot\delta u^1\right\rangle. \tag{16}$$

3. Microscale problem of the homogenized Timoshenko beam

In this section, exploiting the homogenization method for the microstructures which are modeled as the frame structures proposed by the reference [8], we formulate the periodic boundary conditions for the three-dimensional beam with periodic microstructures and realize the multiscale finite element analysis of the beam.

We characterize the representative volume element by the unit structure of the beam which has the one dimensional periodicity with the periodic vector $\boldsymbol{r}$, the norm of which is ϵ, and focus on the domain Ω of the representative volume element as the domain of interest. Since the periodicity of the beam is one dimension, we can assume $\boldsymbol{r} \cdot \boldsymbol{e}_\alpha = 0$ ($\alpha = 2, 3$) and hence $\boldsymbol{r} = r\boldsymbol{e}_1$ without loss of generality. Then, the boundary $\partial\Omega$ of the representative volume element is expressed as the union of boundaries as

$$\partial\Omega = {}^{\mathrm{i}}\Gamma \cup {}^{\mathrm{d}}\Gamma \cup {}^{\mathrm{f}}\Gamma. \tag{17}$$

Although the cross sections at $y_1 = 0$ and $y_1 = r$ undergo same deformation because of the periodicity, the former cross section at $y_1 = 0$ is referred to as the independent cross section ${}^{\mathrm{i}}\Gamma$, which has independent degree of freedom, and the latter cross section at $y_1 = r$ is referred to as the dependent cross section ${}^{\mathrm{d}}\Gamma$, which is dependent on the independent cross section ${}^{\mathrm{i}}\Gamma$ for the sake of convenience.

3.1 Constraint of the relative displacement and the rigid body rotation

Integration of the total curvature and the total torsional angle per unit length yields the total rotation

$$\boldsymbol{\theta}_{\mathrm{r}}(y_1) = \nabla\boldsymbol{\theta}\, y_1 + \boldsymbol{\theta}^1 \tag{18}$$

and the total torsional angle

$$\psi_{\mathrm{r}}(y_1) = \nabla\psi\, y_1 + \psi^1, \tag{19}$$

respectively. Due to the periodicity of the rotational angle $\boldsymbol{\theta}^1$ and the torsional angle ψ^1, the relative rotation and the relative torsional angle can be written as

$$\boldsymbol{\theta}_{\mathrm{r}}(y_1 + r) - \boldsymbol{\theta}_{\mathrm{r}}(y_1) = \nabla\boldsymbol{\theta}\, r \quad \text{and} \quad \psi_{\mathrm{r}}(y_1 + r) - \psi_{\mathrm{r}}(y_1) = \nabla\psi\, r, \tag{20}$$

respectively. Under the assumption of infinitesimal displacement, both of the relative rotation and the relative torsional angle are also infinitesimal. Thus, the relative axial displacement arising from the relative rotation and the relative transversal displacement cased by the relative torsional angle can be written as

$$u_{\mathrm{r}}(y_1 + r, y_2, y_3) - u_{\mathrm{r}}(y_1, y_2, y_3) = \{(\nabla\boldsymbol{\theta}\, r) \times \boldsymbol{y}^{\mathrm{c}}\} \cdot \boldsymbol{e}_1 \tag{21}$$

and

$$\boldsymbol{v}_{\mathrm{r}}(y_1 + r, y_2, y_3) - \boldsymbol{v}_{\mathrm{r}}(y_1, y_2, y_3) = (\nabla\psi\, r\, \boldsymbol{e}_1) \times \boldsymbol{y}^{\mathrm{c}}, \tag{22}$$

respectively, where $\boldsymbol{y}^{\mathrm{c}}$ denotes the position vector of an arbitrary point which initial point is the centroid of the cross section and hence $\boldsymbol{y}^{\mathrm{c}} \cdot \boldsymbol{e}_1 = 0$.

By integrating the total shear strain and the total axial strain with respect to the microscale, the real lateral displacement and the real axial displacement are obtained, respectively, as

$$\boldsymbol{v}_{\mathrm{r}}(y_1) = \boldsymbol{\gamma}\, y_1 + \boldsymbol{v}^1, \quad u_{\mathrm{r}}(y_1) = \nabla u\, y_1 + u^1. \tag{23}$$

By virtue of the periodicity of the real lateral displacement $\boldsymbol{v}^1$, the torsional angle ψ^1 and the axial displacement u^1 in the microscale, the relative real lateral displacement and the relative real axial displacement are expressed as

$$\boldsymbol{v}_{\mathrm{r}}(y_1 + r) - \boldsymbol{v}_{\mathrm{r}}(y_1) = \boldsymbol{\gamma}\, r, \quad u_{\mathrm{r}}(y_1 + r) - u_{\mathrm{r}}(y_1) = \nabla u\, r. \tag{24}$$

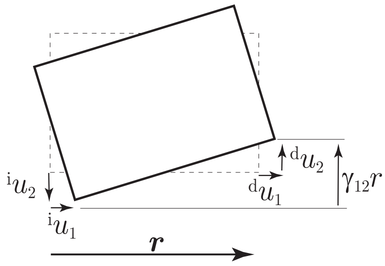

Figure 2: Rigid body rotation due to relative displacement

Since the real aixial displacement is induced by both the bending deformation and the axial one, the relative real axial displacement is obtained by summing both contributions as

$$u_{\mathrm{r}}(y_1 + r, y_2, y_3) - u_{\mathrm{r}}(y_1, y_2, y_3) = \{(\nabla\boldsymbol{\theta}\, r) \times \boldsymbol{y}^{\mathrm{c}}\} \cdot \boldsymbol{e}_1 + \nabla u\, r. \tag{25}$$

At the same time, since the real lateral displacement is induced by both the shear deformation and the torsional one, the relative real lateral displacement is obtained by summing both contributions as

$$\boldsymbol{v}_{\mathrm{r}}(y_1 + r, y_2, y_3) - \boldsymbol{v}_{\mathrm{r}}(y_1, y_2, y_3) = (\nabla\psi\, r\, \boldsymbol{e}_1) \times \boldsymbol{y}^{\mathrm{c}} + \boldsymbol{\gamma}\, r. \tag{26}$$

Then, prescription of the relative displacement determined by equations (25) and (26) is equivalent to the periodic boundary condtion in the microscale problems.

However, subscription of the relative real lateral displacement (26), which includes the periodic boundary condition related to shear deformation, induces only the rigid body rotation and does not realize shear deformation of the representative volume element as illustrated in Figure 2. Therefore, it is necessary to constrain the rigid body rotation. In this paper, we introduce constraint of the rigid body rotation of the representative volume element. Regarding the rigid body rotation of the representative volume element as the averaged rigid body rotation of the all cross sections in the representative volume element, the constraint of the rigid body rotation can be expressed as

$$g_\alpha := \frac{1}{r}\int_0^r \theta^{\mathrm{r}}_\alpha(y_1)\mathrm{d}y_1 = 0 \quad (\alpha = 2, 3), \tag{27}$$

where $\theta^{\mathrm{r}}_\alpha(y_1)$ is the real rotation of the cross section at y_1 about x_α-axis, i.e., $\theta^{\mathrm{r}}_\alpha := \boldsymbol{\theta}^{\mathrm{r}} \cdot \boldsymbol{e}_\alpha$. Here, the rotation of the cross section $\theta^{\mathrm{r}}_\alpha(y_1)$ is defined by the slope of the least-square regression line of the positions on a cross section. Hence, $\theta^{\mathrm{r}}_\alpha(y_1)$ which minimizes

$$R_3(y_1) := \int (u_1 + \theta^{\mathrm{r}}_3 y_2 - b_2)^2\ \mathrm{d}A \quad \text{and} \quad R_2(y_1) := \int (u_1 - \theta^{\mathrm{r}}_2 y_3 - b_3)^2\ \mathrm{d}A \tag{28}$$

is the rotation of the cross section. The surface of the integral in the above equation is the cross section at y_1 and b_α is the intersect with x_α-axis of the cross section. Owing to the condition that R_α becomes minimum

$$\frac{\partial R_\alpha}{\partial \theta^{\mathrm{r}}_\alpha} = 0 \quad \text{(summation convention is not adopted)}, \tag{29}$$

the cross sectional rotation $\theta^{\mathrm{r}}_{\alpha}$ can be expressed in terms of u_1 as

$$\theta^{\mathrm{r}}_3(y_1) = \frac{-\int y_2 u_1 \,\mathrm{d}A + b_2 G_3}{I_{33}} \quad \text{and} \quad \theta^{\mathrm{r}}_2(y_1) = \frac{\int y_3 u_1 \,\mathrm{d}A - b_3 G_2}{I_{22}}, \tag{30}$$

where G_α is the first moment of cross section about x_α-axis, $I_{\alpha\alpha}$ is the second moment of cross section about x_α-axis. Assuming that the cross section is uniform in y_1 direction and that the origins of y_2 and y_3 is coincide with the centroid of the cross section, the first moments of the cross section are

$$G_3 := \int y_2 \,\mathrm{d}A = 0, \quad G_2 := \int y_3 \,\mathrm{d}A = 0. \tag{31}$$

Then, the rotations of the cross section $\theta^{\mathrm{r}}_{\alpha}$ are obtained by

$$\theta^{\mathrm{r}}_3(y_1) = -\frac{\int y_2 u_1 \,\mathrm{d}A}{I_{33}}, \quad \theta^{\mathrm{r}}_2(y_1) = \frac{\int y_3 u_1 \,\mathrm{d}A}{I_{22}}. \tag{32}$$

Therefore, substituting the equation (32) into the equation (27), we obtain the constraint equations for the rigid body rotation as

$$\frac{1}{r}\int_0^r \theta^{\mathrm{r}}_3 \,\mathrm{d}y_1 = -\frac{\int y_2 u_1 \,\mathrm{d}V}{r\int I_{33}\,\mathrm{d}y_1} = 0 \quad \text{and} \quad \frac{1}{r}\int_0^r \theta^{\mathrm{r}}_2 \,\mathrm{d}y_1 = \frac{\int y_3 u_1 \,\mathrm{d}V}{r\int I_{22}\,\mathrm{d}y_1} = 0. \tag{33}$$

Since neither of the denominators in the right hand sides of the above equations are zero, the constraint equation for the rigid body rotation can be expressed in another form as

$$g_3 := \int y_2 u_1 \,\mathrm{d}V = 0, \quad g_2 := \int y_3 u_1 \,\mathrm{d}V = 0. \tag{34}$$

Then, the constraint equation (34) can be discretized and is expressed in terms of the nodal displacement u^n_1 as

$$\bar{g_\alpha} = \sum w^n_\alpha u^n_1 = 0. \tag{35}$$

In the above equation, a coefficient w^n_α of u^n_1 is defined by

$$w^n_\alpha := \mathcal{A}\sum_m \left\{ y^m_\alpha \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} N^m N^n \det\boldsymbol{J}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \right\}, \tag{36}$$

where $\mathcal{A}$ denotes the ensemble of the finite elements, $\sum$ is the sum of all nodes belonging to the element, $\boldsymbol{J}$ is the Jacobian of the natural coordinate of the isoparametric shape function.

4. Numerical example

In order to examine the accuracy of the proposed method, the averaged properties of a homogeneous beam with zero Poisson's ratio which analytical solution is available are computed.

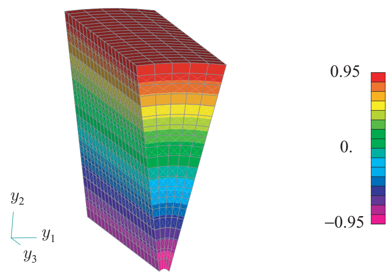

Figure 3: bending deformation (σ_{11})

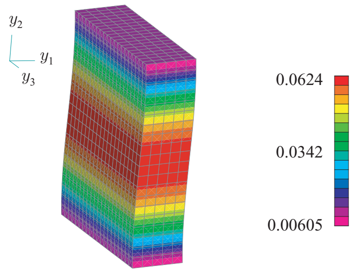

Figure 4: shear deformation (σ_{12})

First, the curvature $\phi_3^0 = 0.1/r$ about y_3-axis is applied to the homogeneous beam model. The bending stiffness obtained by beam theory is $EI = \frac{a^4 E}{12}$. On the other hand, the ratio of the averaged bending stiffness $(EI)^{\mathrm{H}}$ computed by the proposed method to EI is 1.00125. The deformation and the σ_{11} distribution of this case is illustrated in Figure 3.

Next, the macroscopic shear deformation $\gamma_{12}^0 = 0.1$ is applied. The shear stiffness by the Timoshenko beam theory is $kGA = \frac{5GA}{6}$. On the other hand, the ratio of the averaged shear stiffness $(kGA)^{\mathrm{H}}$ to kGA is 1.00208. The deformation and the σ_{12} distribution of this case is illustrated in Figure 4.

5. Conclusion

In this paper, the periodic boundary condition of the beam with the microstructures has been formulated. Moreover, the problem arising from the rigid body rotation has been solved by introducing the constraint of the rotation of the cross section of the beam. Finally, the method of the evaluation of the averaged mechanical properties of the beam with the microstructures has been proposed. The accuracy of the proposed method is examined through the numerical example of the homogeneous beam.

REFERENCES

Japan society of steel construction (1996). JSSC standard on push-out test for headed stud (draft). Technical Report 35 (in Japanese).

Takano N, ZAko M and Kikuchi N (1995). Analysis of Single-Layered Corrugated Core Sandwich Plate by Homogenization Method : On the Application of Homogenization Method to Composite Structures with Only Planar Periodicity. J. Soc. Mat. Sci. Japan 44(502). pp.905–910 (in Japanese).

Okada H, Fujitani S, Fukui Y and Kumazawa N (2000). Homogenization methods for beam/bar and plate structures. Proc. Comp. Eng. Conf. JSCES 5. pp.741-744 (in Japanese).

Kohn RV and Vogelius M (1984), A new model for thin plates with rapidly varying thickness. Int. J. Solids Struct. 20(4). pp.333–350.

Saiki I, Sano M and Nakajima A (2004). A consideration on application of homogenization theory to beam and plate structures. J. Applied Mechanics. JSCE. 7. pp.407-413 (in Japanese).

Saiki I, Honda H, Iwakuma T and Nakajima A (2006). An application of nonlinear homogenization theory to plate structures by the generalizaed convergence theory, J. Applied Mechanics. JSCE. 9. pp.203-210 (in Japanese).

Allaire G (1996). Mathematical approaches and methods. in: Hornung, U. ed., Homogenization and Porous Media. Springer. pp.225–250.

Saiki I, Ooue K, Nakajima A and Terada K (2002). Multi-scale Modeling for Microstructures with Structural Elements and its Application to Cellular Solids. Transactions of JSCES. 4. pp.139–144 (in Japanese).