Preliminary Study of Uncertainty-Driven Plasma Diffusion II

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We have constructed a semiclassical collisional diffusion model. In this model, a field particle is treated as either a point charge or a spatially distributed charge. The test particle is treated as a distributed point charge with Gaussian distribution. It was shown that the collisional changes in velocity in our model is of the same order as the classical theory for a typical proton in a fusion plasma of $T = 10\,\mathrm{keV}$ and $n = 10^{20}\,\mathrm{m}^{-3}$. It was also shown that the spatial extent of the distribution, or the quantum-mechanical uncertainty in position, for the test particle obtained in our model grows in time, and becomes of the order of the average interparticle separation $\Delta\ell \equiv n^{-1/3}$ during a time interval $\tau_r \sim \times 10^7 \Delta\ell/g_{\mathrm{th}}$, where $g_{\mathrm{th}} = \sqrt{2T/m}$ is the thermal speed, with m being the mass of the particle under consideration. The time interval is 3-4 order of magnitudes smaller than the collision time. This suggests that particle transport cannot be understood in the framework of classical mechanics, and that the quantum-mechanical distribution of individual particles in plasmas may cause the anomalous diffusion.

Keywords: anomalous diffusion, uncertainty, distribution function, Coulomb potential, magnetic length, Schrödinger equation

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1. Introduction

We have pointed out in the first paper [1] that (i) for distant encounters in typical fusion plasmas of $T = 10\,\mathrm{keV}$ and $n = 10^{20}\,\mathrm{m}^{-3}$, the average potential energy $\langle U\rangle \sim 30\,\mathrm{meV}$ is as small as the uncertainty in energy $\Delta E \sim 40\,\mathrm{meV}$, and (ii) in the presence of a magnetic field $B = 3\,\mathrm{T}$, the spatial size of the wavefunction in the plane perpendicular to the magnetic field is as large as $\ell_{\mathrm{B}} \sim 1.4\times10^{-8}\,\mathrm{m}$ which is much larger than the typical electron wavelength $\lambda_{\mathrm{e}} \sim 10^{-11}\,\mathrm{m}$, as well as ion wavelengths $\lambda_{\mathrm{i}} \sim 10^{-13}\,\mathrm{m}$.

In considering diffusion of plasmas correctly, it was also pointed out more than half a century ago [2, 3] that one must consider the wave character of charged particles when the temperature T is high, i.e. the relative speeds of interacting particles are fast. The criterion of the classical theory to be valid in terms of relative speed g in a hydrogen plasma is given in Ref. [3], as

$$g \ll \frac{2e^2}{4\pi\epsilon_0\hbar} = 4.4\times10^6\ \mathrm{m/s}, \tag{1}$$

where $e = 1.60\times10^{-19}$ C and $\hbar \equiv h/2\pi = 1.05\times10^{-34}$ J·s stand for the elementary electric charge and the reduced Planck constant. In contemporary fusion plasmas with $T \sim 10\,\mathrm{keV}$ or higher, ions as well as electrons should be treated quantum-mechanically. In current plasma physics, however, the quantum-mechanical effects enters as a minor correction to the Coulomb logarithm, $\ln\Lambda$, in the case of close encounters [4]. Nonetheless, the neoclassical theory [5] is capable of predicting a lot of phenomena such as those related to the current conduction. Such phenomena linearly depend on the change in velocity $\Delta\boldsymbol{v}$ or in position $\Delta\boldsymbol{r}$. The quantum-mechanical changes, e.g. in the velocity ${}^{\mathrm{QM}}\Delta\boldsymbol{v}$, are stochastic. The average or expectation value of $\Delta\boldsymbol{v}$ conforms to the classical prediction ${}^{\mathrm{CL}}\Delta\boldsymbol{v}$ due to the Ehrenfest's theorem: for $\boldsymbol{\xi} = \boldsymbol{v}, \boldsymbol{r}$

$$\langle\Delta\boldsymbol{\xi}\rangle = \left\langle {}^{\mathrm{CL}}\Delta\boldsymbol{\xi} + {}^{\mathrm{QM}}\Delta\boldsymbol{\xi}\right\rangle = {}^{\mathrm{CL}}\Delta\boldsymbol{\xi}. \tag{2}$$

However, diffusion is quadratic in $\Delta\boldsymbol{g}$ or $\Delta\boldsymbol{r}$:

$$\left\langle(\Delta\boldsymbol{\xi})^2\right\rangle = \left({}^{\mathrm{CL}}\Delta\boldsymbol{\xi}\right)^2 + \left\langle\left({}^{\mathrm{QM}}\Delta\boldsymbol{\xi}\right)^2\right\rangle > \left({}^{\mathrm{CL}}\Delta\boldsymbol{\xi}\right)^2. \tag{3}$$

This might be the reason why we cannot understand the so-called anomalous diffusion using classical theories that only give correct $\langle\Delta\boldsymbol{\xi}\rangle$.

In quantum mechanics [1, 6], the *size* of a charge q in the presence of a magnetic induction $\boldsymbol{B}$, becomes the magnetic length $\ell_{\mathrm{B}} = \sqrt{\hbar/qB}$, where $\hbar = h/2\pi$ stands for Dirac constant. In typical fusion plasma with a temperature T and a density n, ℓ_{B} is as large as one tenth of the inter-particle separation $\Delta\ell \equiv n^{-1/3}$, which is considerably longer than the typical de Broglie wavelength, $\lambda \approx h/\sqrt{2mT}$:

$$\lambda \ll \ell_{\mathrm{B}} \sim \Delta\ell. \tag{4}$$

2. Interaction Potential

We will not solve the Schrödinger equation here, instead we will adopt an alternative method as described in what follows. Let us assume that the initial wave function of a *field* particle is Gaussian with the center at the origin:

$$f(\boldsymbol{r}) = \frac{1}{\pi^{3/2}\ell_{\mathrm{B}}^3}\exp\left(-\frac{r^2}{\ell_{\mathrm{B}}^2}\right). \tag{5}$$

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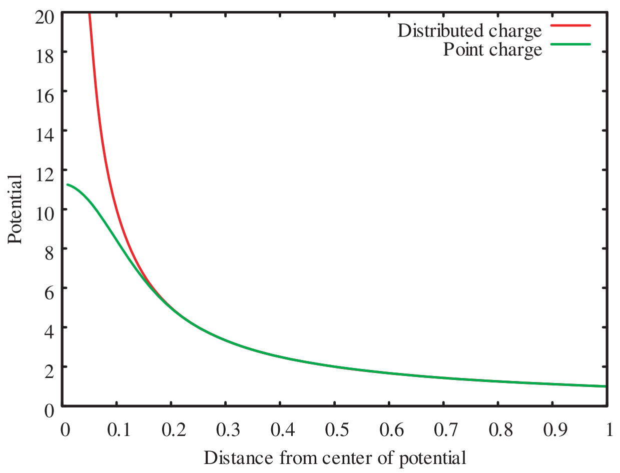

Fig. 1 Interaction potentials U; due to a point charge at the origin, and due to a distributed charge centered at the origin.

If the *test* particle with the same charge q as the field particle has the similar distribution as Eq. (5), the probability $\mathrm{d}P(\boldsymbol{r})$ of finding the test particle within an infinitesimal volume $\mathrm{d}^3\boldsymbol{r}$ around a position $\boldsymbol{r}$ is given as

$$\mathrm{d}P(\boldsymbol{r}) = \frac{1}{\pi^{3/2}\ell_{\mathrm{B}}^3} \exp\left(-\frac{(\boldsymbol{r}-\boldsymbol{r}_0)^2}{\ell_{\mathrm{B}}^2}\right) \mathrm{d}^3\boldsymbol{r}. \tag{6}$$

The Coulomb potential energy in this case is given by

$$U(\boldsymbol{r}) = \frac{q^2}{4\pi\epsilon_0 r} \mathrm{erf}\left(\frac{r}{\ell_{\mathrm{B}}}\right), \tag{7}$$

whereas the potential due to a point charge is

$$U(\boldsymbol{r}) = \frac{q^2}{4\pi\epsilon_0 r}, \tag{8}$$

as shown in Fig. 1.

3. Method

We solve a set of classical equations of motion, in which the test particle q for several initial positions at $\boldsymbol{r} = \boldsymbol{r}_0$ with a velocity $\boldsymbol{v} = \boldsymbol{v}(0)$ in the presence of the potential field given by Eq. (7). For each initial position, Eq. (6) is used to mimic the quantum-mechanical distribution of the test particle in order to calculate particle scattering in the plasma.

For simplicity, initial speed is fixed to be the thermal velocity v_{th} and initial positions are restricted within the sphere of a radius $3\ell_{\mathrm{B}}$ centered at the initial position $\boldsymbol{r} = \boldsymbol{r}_0$, as shown in Fig. 2. The test particle moves during $\Delta t = \Delta\ell/g_{\mathrm{th}}$, the time for the electron with its thermal speed to travel the average interparticle separation in classical mechanics.

4. Numerical Results

In the calculations, we have ignored the effect of magnetic field $\boldsymbol{B}$, because $\Delta t \approx 10^{-13}$ sec is much shorter than the cyclotron period of the order of 10^{-8} sec for protons in a plasma with $n = 10^{20}\,\mathrm{m}^{-3}$ and $T = 10\,\mathrm{keV}$.

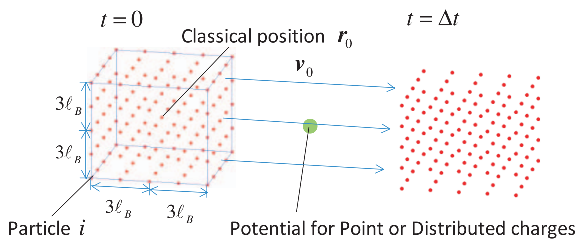

Fig. 2 Distributed system before ($t = 0$) and after ($t = \Delta t$) the interaction.

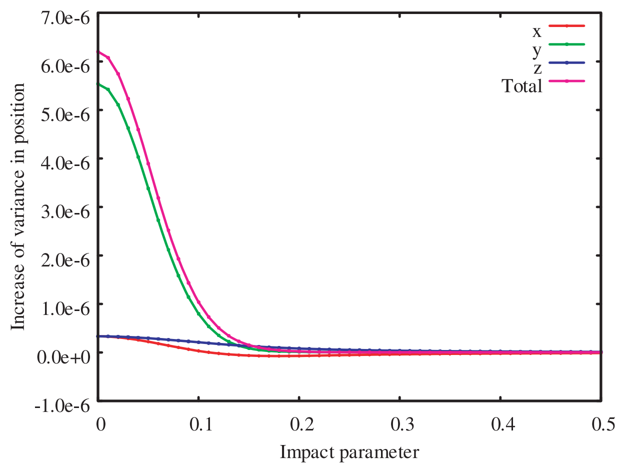

Fig. 3 Incremental variance of a particle in position as a function of classical impact parameter b, in the case of an interaction potential $U \propto 1/r$ due to a point charge at the origin.

4.1 Case 1: Potential due to a point charge

Let us define the time-dependent variance in position as

$$\sigma^2(\Delta t) \equiv \mathrm{VAR}[\boldsymbol{r}(\Delta t)] = \left\langle (\boldsymbol{r}(\Delta t) - \bar{\boldsymbol{r}}(\Delta t))^2 \right\rangle, \tag{9}$$

where $\bar{\boldsymbol{r}} \equiv \langle \boldsymbol{r} \rangle$ stands for the averaged position using Eq. (5), with $\langle\cdot\rangle$ being the ensemble average over the impact parameter b.

Figure 3 shows the increase in variance in position during the time interval Δt as a function of the classical impact parameter, defined as

$$\sigma^2(\Delta t) - \sigma^2(0), \tag{10}$$

the average over the impact parameter b of which is

$$\left\langle \sigma^2(\Delta t) - \sigma^2(0) \right\rangle_b \sim 1.44 \times 10^{-7} \Delta\ell^2, \tag{11}$$

from which

$$N \times 1.44 \times 10^{-7} \Delta\ell^2 = \Delta\ell^2, \tag{12}$$

$$\therefore\ N \sim 10^7. \tag{13}$$

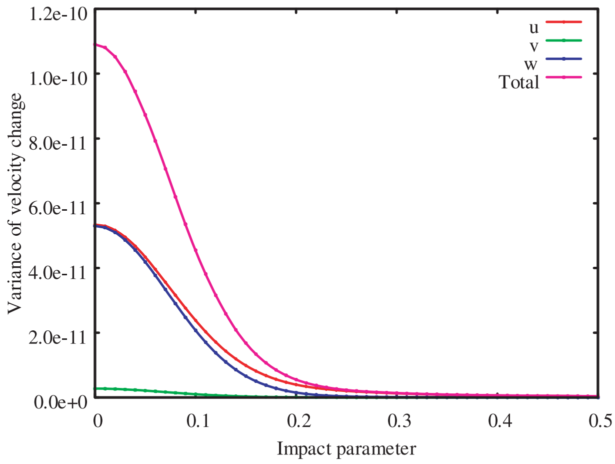

Fig. 4 Variance of a particle in velocity in the case of an interaction potential U due to a point charge at the origin.

Therefore, the variance in position, i.e., the spatial extent of the proton under consideration, becomes as large as the average interparticle separation $\Delta\ell$

$$\tau_r \sim 10^7 \Delta t = 10^{-6} \text{ sec}, \quad (14)$$

under the condition that the increase in the variance be constant. Quantum-mechanically, the wavefunction of each proton overlaps with each other at the time $t \sim \tau_r$, which is much smaller than the collision time for protons of the order of several milliseconds in the plasma.

Figure 4 shows the variance in velocity. The averaged variance, over the impact parameter, of velocity change is

$$\left\langle (\Delta v)^2 \right\rangle \approx 0.6 \times 10^{-11} v_{\text{th}}^2. \quad (15)$$

The corresponding variance in classical mechanics is given by

$$\left\langle (\Delta v)^2 \right\rangle = 4\pi \left(\frac{b_0}{\Delta t} \right)^2 \ln\Lambda \approx 2.3 \times 10^{-11} v_{\text{th}}^2, \quad (16)$$

where $b_0 = q^2/4\pi\epsilon_0 \mu v_{\text{th}}^2$ and $\ln\Lambda \approx 17$ are the impact parameter for $\pi/2$ scattering and the Coulomb logarithm.

4.2 Case 2: Potential due to a distributed charge

Figures 5 and 6 show the variances in position and in velocity, respectively, as a function of the classical impact parameter normalized by the average interparticle separation $\Delta\ell \equiv n^{-1/3}$ and the thermal speed $v_{\text{th}} \equiv \sqrt{2T/m}$.

The incremental variance in position during the time interval $\Delta t \equiv \Delta\ell / g_{\text{th}}$, averaged over the impact parameter b, in this case is

$$\left\langle \sigma^2(\Delta t) - \sigma^2(0) \right\rangle_b \sim 1.34 \times 10^{-7}. \quad (17)$$

Therefore, the variance in position, i.e., the spatial extent of a particle, becomes as large as the average interparticle separation $\Delta\ell$

$$\tau_r \sim 10^7 \Delta t, \quad (18)$$

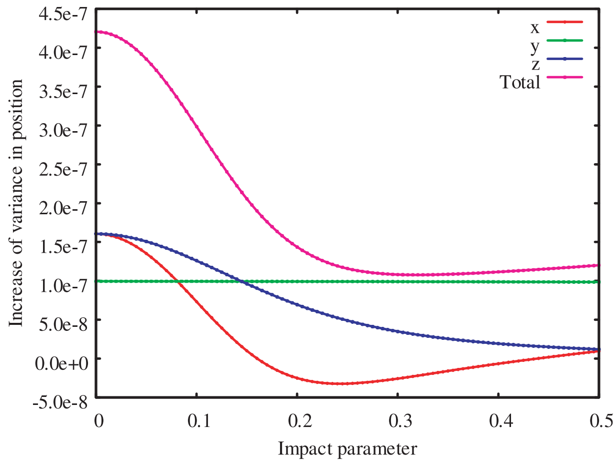

Fig. 5 Incremental variance of a particle in position as a function of classical impact parameter b. Interaction potential $U \propto r^{-1}\text{erf}(r/\ell_{\text{B}})$ is due to a distributed charge centered at the origin.

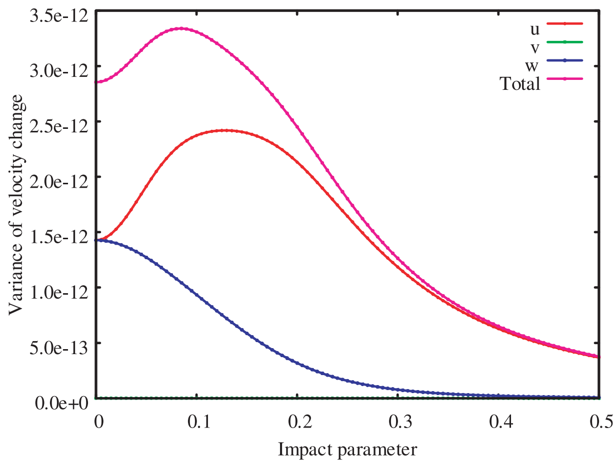

Fig. 6 Variance of a particle in velocity. Interaction potential U is due to a distributed charge centered at the origin.

which is approximately the same as Case 1 given in Eq. (14), i.e., the field particle being a point charge.

The variance of velocity change for a distributed potential is

$$\left\langle (\Delta v)^2 \right\rangle \approx 0.12 \times 10^{-11} v_{\text{th}}^2. \quad (19)$$

5. Summary

We have constructed a semiclassical collisional diffusion model. In this model, a field particle is treated as either a point charge or a spatially distributed charge. The test particle is treated as a distributed point charge with Gaussian distribution. The collisional changes in velocity in our model is of the same order as the classical theory for a typical proton in a fusion plasma of $T = 10\,\text{keV}$ and $n = 10^{20}\,\text{m}^{-3}$. The spatial extent of the distribution, or the quantum-mechanical uncertainty in position, for the test particle obtained in our model grows in time, and, irrespective of the interaction potential $U(r)$, becomes of the order of the average interparticle separation $\Delta\ell \equiv n^{-1/3}$ during a time interval $\tau_r \sim 10^7\Delta t \sim 10^{-6}$ sec, which is 3-4 orders of magnitudes smaller than the collision time. This suggests that particle transport cannot be understood in the framework of classical mechanics, and that the quantum-mechanical distribution of a charged particle in plasmas may cause the anomalous diffusion.

In magnetically confined fusion plasmas, diffusion is governed by the banana particle motion due to the toroidicity of the magnetic field $\boldsymbol{B}$ and the plasma current I_{p}, with which we have not dealt in this study. The diffusion model presented here is semiclassical, so we will need to solve Schrödinger's equation for exact analysis; this will be reported soon.

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[1] S. Oikawa, T. Oiwa and T. Shimazaki, Plasma Fusion Res. **5**, S1050 (2009).
[2] R. E. Marshak, Ann. N. Y. Acad. Sci. **410**, 49 (1941).
[3] R. S. Cohen, L. Spitzer, Jr. and P. McR. Routly, Phys. Rev. **80**, 230 (1950).
[4] S. I. Braginskii, *Reviews of Plasma Physics*, M. A. Leontovich (ed.), (Consultants Bureau, New York, 1965).
[5] R. J. Hawryluk, Rev. Mod. Phys. **70**, 537 (1998).
[6] L. D. Landau and E. M. Lifshitz, *Quantum mechanics: non-relativistic theory*, 3rd ed., translated from the Russian by J. B. Sykes and J. S. Bell (Pergamon Press, Oxford, 1977).
[7] V. Sokolov and A. K. Sen, Phys. Rev. Lett. **92**, 165002 (2004).
[8] P. Yushmanov *et al.*, Nucl. Fusion **30**, 1999 (1990).