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Pattern-recalling processes in quantum Hopfield networks far from saturation

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Abstract. As a mathematical model of associative memories, the Hopfield model was now well-established and a lot of studies to reveal the pattern-recalling process have been done from various different approaches. As well-known, a single neuron is itself an uncertain, noisy unit with a finite unnegligible error in the input-output relation. To model the situation artificially, a kind of 'heat bath' that surrounds neurons is introduced. The heat bath, which is a source of noise, is specified by the 'temperature'. Several studies concerning the pattern-recalling processes of the Hopfield model governed by the Glauber-dynamics at finite temperature were already reported. However, we might extend the 'thermal noise' to the quantum-mechanical variant. In this paper, in terms of the stochastic process of quantum-mechanical Markov chain Monte Carlo method (the quantum MCMC), we analytically derive macroscopically deterministic equations of order parameters such as 'overlap' in a quantum-mechanical variant of the Hopfield neural networks (let us call *quantum Hopfield model* or *quantum Hopfield networks*). For the case in which non-extensive number p of patterns are embedded via asymmetric Hebbian connections, namely, $p/N \to 0$ for the number of neuron $N \to \infty$ ('far from saturation'), we evaluate the recalling processes for one of the built-in patterns under the influence of quantum-mechanical noise.

1. Introduction

Basic concept of associative memories in artificial neural networks was already proposed in early 70' s by a Japanese engineer Kaoru Nakano [1]. Unfortunately, in that time, nobody interested in his model, however in 80's, J.J. Hopfield [2, 3] pointed out that there exists an energy function (Lyapunov function) in the so-called *associatron* (the Nakano model) and the system can be treated as a kind of spin glasses. After his study, a lot of researchers who were working in the research field of condensed matter physics picked the so-called *Hopfield model* up for their brand-new 'target materials'.

Among these studies, a remarkable progress has been done by three theoretical physicists Amit, Gutfreund and Sompolinsky [4] who clearly (mathematically) defined the concept of *storage capacity* in the Hopfield model as a critical point at which system undergoes phase transitions from ferromagnetic retrieval to spin glass phases by utilizing the replica method. They also introduced a noise to prevent the network from retrieving one of the built-in patterns as a 'heat bath' which surrounds the neurons. They draw the phase diagram which is composed of three distinct phases, namely, ferromagnetic-retrieval, paramagnetic and spin glass phases. These phase boundaries are specified by two control parameters, namely, storage capacity and temperature of the heat bath.

To evaluate the storage capacity of the Hopfield model without energy function (for instance, Hopfield model having a non-monotonic input-output function [5]), Shiino and Fukai [6] proposed the so-called Self-Consistent Signal-to-Noise Analysis (SCSNA) which enables us to derive a couple of self-consistent macroscopic equations of state by making use of the concept of the TAP equations [3].

As we mentioned, these theoretical arguments are constructed for the case in which each neuron is surrounded by a heat bath at finite temperature. In this sense, the above studies revealed the robustness of the associative memories against *thermal noises* in the artificial brain.

However, we might consider a different kind of such noises, that is, *quantum-mechanical noise*. As such successful attempts, Ma and Gong [7], Nishimori and Nonomura [8] independently introduced the quantum-mechanical noise to the conventional Hopfield model by adding the transverse field to the classical Hamiltonian (from now on, we shall refer the model as *quantum Hopfield model*). Especially, Nishimori and Nonomura [8] investigated the structure of retrieval phase diagrams by using of the standard replica method with the assistance of the static approximation. Therefore, we might say that the equilibrium properties of the Hopfield model were now well-understood and the methodology to investigate the model was already established.

On the other hand, theory of the dynamics to evaluate the pattern-recalling processes is not yet well-established. However, up to now, several powerful approaches were proposed. For instance, Amari and Maginu [9] pointed out that the relevant macroscopic quantities in the synchronous neuro-dynamics are the overlap (the direction cosine) and the noise variance. They derived the update equations with respect to these quantities. After their study, the so-called Amari-Maginu theory was improved by taking into account the correlations in the noise variances by Okada [10].

Whereas for the asynchronous dynamics, Coolen and his co-authors established a general approach, so-called *dynamical replica theory* [11, 12]. They utilized two assumptions, namely, equipartitioning in the sub-shells and self-averaging of the intrinsic noise distribution to derive the deterministic flow equations for relevant macroscopic quantities. However, there is no such theoretical framework so far to investigate the pattern-recalling process of the quantum Hopfield model systematically.

In this paper, we propose such a candidate of dynamical theory to deal with the pattern-recalling processes in the quantum Hopfield model. We shall consider the stochastic process of quantum Monte Carlo method which is applied for the quantum Hopfield model and investigate the quantum neuro-dynamics through the differential equations with respect to the macroscopic quantities such as the overlap.

This paper is organized as follows. In the next section 2, we explain the basics of the conventional Hopfield model and its generic properties. Then, we categorize the model into two distinct classes according to the origin of noises in artificial brain. The quantum Hopfield model is clearly defined. In section 3, we explain the quantum Monte Carlo method based on the Suzuki-Trotter decomposition [13] and consider the stochastic process in order to investigate the pattern-recalling dynamics of the quantum Hopfield model. In section 4, we derive the macroscopic deterministic flow of the overlap between the neuronal state and one of the built-in patterns from the microscopic master equation which describes the stochastic processes in the quantum Monte Calro method for the quantum Hopfield model [14]. The general solution of the dynamics is obtained under the so-called static approximation. In the next section 5, we apply our general solution to a special case, namely, sequential recalling the built-in two patterns via asymmetric Hebb connections. The effect of quantum-mechanical noise is compared with that of the conventional thermal noise. The last section is summary.

2. The Hopfield model

In this section, we briefly explain the basics of the conventional Hopfield model. Then, we shall divide the model into two classes, namely, the Hopfield model put in thermal noises (the model is referred to as *classical systems*) and the same model in the quantum-mechanical noise (the model is referred to as *quantum systems*). In following, we define each of the models explicitly.

2.1. The classical system

Let us consider the network having N-neurons. Each neuron S_i takes two states, namely, $S_i = +1$ (fire) and $S_i = -1$ (stationary). Neuronal states are given by the set of variables S_i, that is, $\boldsymbol{S} = (S_1, \cdots, S_N)$, $S_i \in \{+1, -1\}$. Each neuron is located on a complete graph, namely, graph topology of the network is 'fully-connected'. The synaptic connection between arbitrary two neurons, say, S_i and S_j is defined by the following Hebb rule:

$$J_{ij} = \frac{1}{N} \sum_{\mu,\nu} \xi_i^\mu A_{\mu\nu} \xi_j^\nu \tag{1}$$

where $\boldsymbol{\xi}^\mu = (\xi_1, \cdots, \xi_N), \xi_i^\mu \in \{+1, -1\}$ denote the embedded patterns and each of them is specified by a label $\mu = 1, \cdots, P$. $A_{\mu\nu}$ denotes $(P \times P)$-size matrix and P stands for the number of built-in patterns. We should keep in mind that there exists an energy function (a Lyapunov function) in the system if the matrix $A_{\mu\nu}$ is symmetric.

Then, the output of the neuron i, that is, S_i is determined by the sign of the local field h_i as

$$h_i = \sum_{\mu,\nu=1}^{p} \xi_i^\mu A_{\mu\nu} m^\nu + \frac{1}{N} \sum_{\mu',\nu'=p+1}^{P} \xi_i^{\mu'} A_{\mu'\nu'} \sum_j \xi_j^{\nu'} S_j \tag{2}$$

where $A_{\mu\nu}$ and $A_{\mu'\nu'}$ are elements of $p \times p, (P-p) \times (P-p)$-size matrices, respectively. We also defined the overlap (the direction cosine) between the state of neurons $\boldsymbol{S}$ and one of the built-in patterns $\boldsymbol{\xi}^\nu$ by

$$m^\nu \equiv \frac{1}{N} (\boldsymbol{S} \cdot \boldsymbol{\xi}^\nu) = \frac{1}{N} \sum_i \xi_i^\nu S_i. \tag{3}$$

Here we should notice that the Hamiltonian of the system is given by $-\sum_i h_i S_i$. The first term appearing in the left hand side of equation (2) is a contribution from $p \sim \mathcal{O}(1)$ what we call 'condensed patterns', whereas the second term stands for the so-called 'cross-talk noise'. In this paper, we shall concentrate ourselves to the case in which the second term is negligibly small in comparison with the first term, namely, the case of $P = p \sim \mathcal{O}(1)$. In this sense, we can say that the network is 'far from its saturation'.

2.2. The quantum system

To extend the classical system to the quantum-mechanical variant, we rewrite the local field h_i as follows.

$$\boldsymbol{\phi}_i = \sum_{\mu,\nu=1}^{p} \xi_i^\mu A_{\mu\nu} \left(\frac{1}{N} \sum_i \xi_i^\nu \boldsymbol{\sigma}_i^z \right) \tag{4}$$

where $\boldsymbol{\sigma}_i^z$ $(i = 1, \cdots, N)$ stands for the z-component of the Pauli matrix. Thus, the Hamiltonian $\boldsymbol{H}_0 \equiv -\sum_i \boldsymbol{\phi}_i \boldsymbol{\sigma}_i^z$ is a diagonalized $(2^N \times 2^N)$-size matrix and the lowest eigenvalue is identical to the ground state of the classical Hamiltonian $-\sum_i \phi_i S_i$ (S_i is an eigenvalue of the matrix $\boldsymbol{\sigma}_i^z$).

Then, we introduce quantum-mechanical noise into the Hopfield neural network by adding the transverse field to the Hamiltonian as follows.

$$\boldsymbol{H} = \boldsymbol{H}_0 - \Gamma \sum_{i=1}^{N} \boldsymbol{\sigma}_i^x \tag{5}$$

where $\boldsymbol{\sigma}_i^x$ is the x-component of the Pauli matrix and transitions between eigenvectors of the classical Hamiltonian $\boldsymbol{H}_0$ are induced due to the off-diagonal elements of the matrix $\boldsymbol{H}$ for $\Gamma \neq 0$. In this paper, we mainly consider the system described by (5).

3. Quantum Monte Carlo method

The dynamics of the quantum model (5) follows Schrödinger equation. Thus, we should solve it or investigate the time dependence of the state $|\psi(t)\rangle$ by using the time-evolutionary operator $\mathrm{e}^{-i\boldsymbol{H}\Delta t/\hbar}$ defined for infinitesimal time Δt as

$$|\psi(t+\Delta t)\rangle \quad = \quad \mathrm{e}^{-i\boldsymbol{H}\Delta t/\hbar}|\psi(t)\rangle. \tag{6}$$

However, even if we carry it out numerically, it is very hard for us to do it with reliable precision because $(2^N \times 2^N)$-size Hamilton matrix becomes huge for the number of neurons $N \gg 1$ as in a realistic brain. Hence, here we use the quantum Monte Carlo method to simulate the quantum system in our personal computer and consider the stochastic processes of Glauber-type to discuss the pattern-recalling dynamics of the quantum Hopfield model.

3.1. The Suzuki-Trotter decomposition

The difficulty to carry out algebraic calculations in the model system is due to the non-commutation operators appearing in the Hamiltonian (5), namely, $\boldsymbol{H}_0, \boldsymbol{H}_1 \equiv -\Gamma \sum_i \boldsymbol{\sigma}_i^x$. Thus, we use the following Suzuki-Trotter decomposition [13] in order to deal with the system as a classical spin system.

$$\mathrm{tr}\, \mathrm{e}^{\beta(\boldsymbol{H}_0+\boldsymbol{H}_1)} = \lim_{M\to\infty} \mathrm{tr} \left(\exp\left(\frac{\beta \boldsymbol{H}_0}{M}\right) \exp\left(\frac{\beta \boldsymbol{H}_1}{M}\right)\right)^M \tag{7}$$

where β denotes the 'inverse temperature' and M is the number of the Trotter slices, for which the limit $M \to \infty$ should be taken. Thus, one can deal with d-dimensional quantum system as the corresponding $(d+1)$-dimensional classical system.

4. Derivation of the deterministic flows

In the previous section, we mentioned that we should simulate the quantum Hopfield model by means of the quantum Monte Carlo method to reveal the quantum neuro-dynamics through the time-dependence of the macroscopic quantities such as the overlap. However, in general, it is also very difficult to simulate the quantum-mechanical properties at the ground state by a personal computer even for finite size systems $(N, M < \infty)$.

With this fact in mind, in this section, we attempt to derive the macroscopic flow equations from the microscopic master equation for the classical system regarded as the quantum system in terms of the Suzuki-Trotter decomposition. This approach is efficiently possible because the Hopfield model is a fully-connected mean-field model such as the Sherrington-Kirkpatrick model [15] for spin glasses and its equilibrium properties are completely determined by several order parameters.

4.1. The master equation

After the Suzuki-Trotter decomposition (7), we obtain the local field for the neuron i located on the k-th Trotter slice as follows.

$$\beta\phi_i(\boldsymbol{\sigma}_k : \sigma_i(k \pm 1)) = \frac{\beta}{M}\sum_{\mu,\nu}\xi_i^\nu A_{\mu\nu}\left\{\frac{1}{N}\sum_j \xi_j^\nu \sigma_j(k)\right\} + \frac{B}{2}\{\sigma_i(k-1)+\sigma_i(k+1)\} \quad (8)$$

where parameter B is related to the amplitude of the transverse field (the strength of the quantum-mechanical noise) Γ by

$$B = \frac{1}{2}\log\coth\left(\frac{\beta\Gamma}{M}\right). \quad (9)$$

In the classical limit $\Gamma \to 0$, the parameter B goes to infinity. For the symmetric matrix $A_{\mu\nu}$, the Hamiltonian (scaled by β) of the system is given by $-\sum_i \beta\phi_i(\boldsymbol{\sigma}_k : \sigma(k \pm 1))\sigma_i(k)$.

Then, the transition probability which specifies the Glauber dynamics of the system is given by $w_i(\boldsymbol{\sigma}_k) = (1/2)[1 - \sigma_i(k)\tanh(\beta\phi_i(\boldsymbol{\sigma}_k : \sigma(k \pm 1)))]$. More explicitly, $w_i(\boldsymbol{\sigma}_k)$ denotes the probability that an arbitrary neuron $\sigma_i(k)$ changes its state as $\sigma_i(k) \to -\sigma_i(k)$ within the time unit. Therefore, the probability that the neuron $\sigma_i(k)$ takes $+1$ is obtained by setting $\sigma_i(k) = -1$ in the above $w_i(\boldsymbol{\sigma}_k)$ and we immediately find $\sigma_i(k) = \sigma_i(k-1) = \sigma_i(k+1)$ with probability 1 in the limit of $B \to \infty$ which implies the classical limit $\Gamma \to 0$.

Hence, the probability that a microscopic state including the M-Trotter slices $\{\boldsymbol{\sigma}_k\} \equiv (\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_M), \boldsymbol{\sigma}_k \equiv (\sigma_1(k), \cdots, \sigma_N(k))$ obeys the following master equation:

$$\frac{dp_t(\{\boldsymbol{\sigma}_k\})}{dt} = \sum_{k=1}^{M}\sum_{i=1}^{N}[p_t(F_i^{(k)}(\boldsymbol{\sigma}_k))w_i(F_i^{(k)}(\boldsymbol{\sigma}_k)) - p_t(\boldsymbol{\sigma}_k)w_i(\boldsymbol{\sigma}_k)] \quad (10)$$

where $F_i^{(k)}(\cdot)$ denotes a single spin flip operator for neuron i on the Trotter slice k as $\sigma_i(k) \to -\sigma_i(k)$. When we pick up the overlap between neuronal state $\boldsymbol{\sigma}_k$ and one of the built-in patterns $\boldsymbol{\xi}^\nu$, namely,

$$m_k \equiv \frac{1}{N}(\boldsymbol{\sigma}_k \cdot \boldsymbol{\xi}^\nu) = \frac{1}{N}\sum_i \xi_i^\nu \sigma_i(k) \quad (11)$$

as a relevant macroscopic quantity, the joint distribution of the set of the overlaps $\{m_1, \cdots, m_M\}$ at time t is written in terms of the probability for realizations of microscopic states $p_t(\{\boldsymbol{\sigma}_k\})$ at the same time t as

$$P_t(m_1^\nu, \cdots, m_M^\nu) = \sum_{\{\boldsymbol{\sigma}_k\}} p_t(\{\boldsymbol{\sigma}_k\})\prod_{k=1}^{M}\delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k)) \quad (12)$$

where we defined the sums by

$$\sum_{\{\boldsymbol{\sigma}_k\}}(\cdots) \equiv \sum_{\boldsymbol{\sigma}_1}\cdots\sum_{\boldsymbol{\sigma}_M}(\cdots), \quad \sum_{\boldsymbol{\sigma}_k}(\cdots) \equiv \sum_{\sigma_1(k)=\pm 1}\cdots\sum_{\sigma_N(k)=\pm 1}(\cdots). \quad (13)$$

Taking the derivative of equation (12) with respect to t and substituting (10) into the result, we

have the following differential equations for the joint distribution

$$\begin{aligned}\frac{dP_t(m_1^\nu,\cdots,m_M^\nu)}{dt} &= \sum_k \frac{\partial}{\partial m_k^\nu}\{m_k^\nu P_t(m_1^\nu,\cdots,m_k^\nu,\cdots,m_M^\nu)\} \\ &- \sum_k \frac{\partial}{\partial m_k^\nu}\Bigg\{ P_t(m_1^\nu,\cdots,m_k^\nu,\cdots,m_M^\nu)\int_{-\infty}^{\infty} D[\xi^\nu]d\xi^\nu \\ &\times \frac{\sum_{\{\boldsymbol{\sigma}_k\}} p_t(\{\boldsymbol{\sigma}_k\})\xi^\nu \tanh[\beta\phi(k)]\prod_{k,i}\delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k))}{\sum_{\{\boldsymbol{\sigma}_k\}} p_t(\{\boldsymbol{\sigma}_k\})\prod_k \delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k))}\Bigg\} \\ &\times \delta(\sigma(k+1)-\sigma_i(k+1))\delta(\sigma(k-1)-\sigma_i(k-1)) \end{aligned} \tag{14}$$

where we introduced several notations

$$D[\xi^\nu] \equiv \frac{1}{N}\sum_i \delta(\xi^\nu - \xi_i^\nu) \tag{15}$$

$$\beta\phi(k) \equiv \frac{\beta\sum_{\mu\nu}\xi^\mu A_{\mu\nu}}{M} m_k^\nu + \frac{B}{2}\sigma(k-1) + \frac{B}{2}\sigma(k+1) \tag{16}$$

for simplicity.

Here we should notice that if the local field $\beta\phi(k)$ is independent of the microscopic variable $\{\boldsymbol{\sigma}_k\}$, one can get around the complicated expectation of the quantity $\tanh[\beta\phi(k)]$ over the time-dependent Gibbs measurement which is defined in the sub-shell: $\prod_k \delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k))$. As the result, only procedure we should carry out to get the deterministic flow is to calculate the data average (the average over the built-in patterns). However, unfortunately, we clearly find from equation (16) that the local field depends on the $\{\boldsymbol{\sigma}_k\}$. To overcome the difficulty and to carry out the calculation, we assume that the probability $p_t(\{\boldsymbol{\sigma}_k\})$ of realizations for microscopic states during the dynamics is independent of t, namely,

$$p_t(\{\boldsymbol{\sigma}_k\}) = p(\{\boldsymbol{\sigma}_k\}). \tag{17}$$

Then, our average over the time-dependent Gibbs measurement in the sub-shell is rewritten as

$$\begin{aligned}&\frac{\sum_{\{\boldsymbol{\sigma}_k\}} p_t(\{\boldsymbol{\sigma}_k\})\xi^\nu \tanh[\beta\phi(k)]\prod_{k,i}\delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k))}{\sum_{\{\boldsymbol{\sigma}_k\}} p_t(\{\boldsymbol{\sigma}_k\})\prod_k \delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k))} \\ &\times \delta(\sigma(k+1)-\sigma_i(k+1))\delta(\sigma(k-1)-\sigma_i(k-1)) \\ &\equiv \langle \xi^\nu \tanh[\beta\phi(k)]\prod_i \delta(\sigma(k+1)-\sigma_i(k+1))\delta(\sigma(k-1)-\sigma_i(k-1))\rangle_* \end{aligned} \tag{18}$$

where $\langle\cdots\rangle_*$ stands for the average in the sub-shell defined by $m_k^\nu = m_k^\nu(\boldsymbol{\sigma}_k)\,(\forall_k)$:

$$\langle\cdots\rangle_* \equiv \frac{\sum_{\{\boldsymbol{\sigma}_k\}} p(\{\boldsymbol{\sigma}_k\})(\cdots)\prod_k \delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k))}{\sum_{\{\boldsymbol{\sigma}_k\}} p(\{\boldsymbol{\sigma}_k\})\prod_k \delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k))} \tag{19}$$

If we notice that the Gibbs measurement in the sub-shell is rewritten as

$$\sum_{\{\boldsymbol{\sigma}_k\}} p(\{\boldsymbol{\sigma}_k\})\prod_k \delta(m_k^\nu - m_k^\nu(\boldsymbol{\sigma}_k)) = \mathrm{tr}_{\{\sigma\}}\exp\left[\beta\sum_{l=1}^{M}\phi(l)\sigma(l)\right] \tag{20}$$

$(\mathrm{tr}_{\{\sigma\}}(\cdots) \equiv \prod_k \sum_{\boldsymbol{\sigma}_k}(\cdots))$, and the quantity

$$\tanh\left[\beta\phi(k)\right] = \frac{\sum_{\sigma(k)=\pm 1}\sigma(k)\exp[\beta\phi(k)\sigma(k)]}{\sum_{\sigma(k)=\pm 1}\exp[\beta\phi(k)\sigma(k)]} \tag{21}$$

is independent of $\sigma(k)$, the average appearing in (18) leads to

$$\begin{aligned}\langle\xi^\nu \tanh[\beta\phi(k)]\prod_i \delta(\sigma(k\pm 1)-\sigma_i(k\pm 1))\rangle_* &= \frac{\mathrm{tr}_{\{\sigma\}}\xi^\nu\{\frac{1}{M}\sum_{l=1}^M\sigma(l)\}\exp[\beta\phi(k)\sigma(k)]}{\mathrm{tr}_{\{\sigma\}}\exp[\beta\phi(k)\sigma(k)]} \\ &\equiv \xi^\nu\langle\sigma\rangle_{path}^{(\xi^\nu)} \end{aligned} \tag{22}$$

in the limit of $M\to\infty$. This is nothing but a path integral for the effective single neuron problem in which the neuron updates its state along the imaginary time axis: $\mathrm{tr}_{\{\sigma\}}(\cdots) \equiv \sum_{\sigma(1)=\pm 1}\cdots\sum_{\sigma(M)=\pm 1}(\cdots)$ with weights $\exp[\beta\phi(k)\sigma(k)]$, $(k=1,\cdots,M)$.

Then, the differential equation (14) leads to

$$\begin{aligned}\frac{dP_t(m_1^\nu,\cdots,m_M^\nu)}{dt} &= \sum_k \frac{\partial}{\partial m_k^\nu}\{m_k^\nu P_t(m_1^\nu,\cdots,m_k^\nu,\cdots,m_M^\nu)\} \\ &- \sum_k \frac{\partial}{\partial m_k^\nu}\left\{P_t(m_1^\nu,\cdots,m_k^\nu,\cdots,m_M^\nu)\int_{-\infty}^{\infty}D[\xi^\nu]d\xi^\nu\xi^\nu\langle\sigma\rangle_{path}^{(\xi^\nu)}\right\}. \end{aligned} \tag{23}$$

In order to derive the compact form of the differential equations with respect to the overlaps, we substitute $P_t(m_1^\nu,\cdots,m_M^\nu)=\prod_{k=1}^M\delta(m_k^\nu-m_k^\nu(t))$ into the above (23) and multiplying m_l^ν by both sides of the equation and carrying out the integral with respect to $dm_1^\nu\cdots dm_M^\nu$ by part, we have for $l=1,\cdots,M$ as

$$\frac{dm_l^\nu}{dt} = -m_l^\nu + \int_{-\infty}^{\infty}D[\xi^\nu]d\xi^\nu\xi^\nu\langle\sigma\rangle_{path}^{(\xi^\nu)}. \tag{24}$$

Here we should notice that the path integral $\xi^\nu\langle\sigma\rangle_{path}^{(\xi^\nu)}$ depends on the embedded patterns $\boldsymbol{\xi}^\nu$. In the next subsection, we carry out the quenched average explicitly under the so-called static approximation.

4.2. Static approximation

In order to obtain the final form of the deterministic flow, we assume that macroscopic quantities such as the overlap are independent of the Trotter slices k during the dynamics. Namely, we must use the so-called static approximation:

$$m_k^\nu = m^\nu\,(\forall_k). \tag{25}$$

Under the static approximation, let us use the following inverse process of the Suzuki-Trotter decomposition (7):

$$\lim_{M\to\infty} Z_M = \mathrm{tr}\exp\left[\beta\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu\sigma_z + \beta\Gamma\sigma_x\right] \tag{26}$$

$$Z_M \equiv \mathrm{tr}_{\{\sigma\}}\exp\left[\frac{\beta\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu}{M}\sum_k\sigma(k) + B\sum_k\sigma(k)\sigma(k+1)\right] \tag{27}$$

In our previous study [14], we numerically checked the validity of static approximation by computer simulations and found that the approximation is successfully valid for the pure-ferromagnetic system but it is deviated from the good approximation for the disordered systems. The validity of the static approximation was recently argued by Takahashi and Matsuda [16] from the different perspective.

Then, one can calculate the path integral immediately as

$$\langle\sigma\rangle_{path}^{(\xi^\nu)} = \frac{\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu}{\sqrt{(\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu)^2+\Gamma^2}}\tanh\beta\sqrt{\left(\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu\right)^2+\Gamma^2}. \tag{28}$$

Inserting this result into (24), we obtain

$$\frac{dm^\nu}{dt} = -m^\nu + \mathbb{E}_{\boldsymbol{\xi}}\left[\frac{\xi^\nu\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu}{\sqrt{(\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu)^2+\Gamma^2}}\tanh\beta\sqrt{\left(\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu\right)^2+\Gamma^2}\right] \tag{29}$$

where we should bear in mind that the empirical distribution $D[\xi^\nu]$ in (24) was replaced by the built-in pattern distribution $\mathcal{P}(\xi^\nu)$ as

$$\lim_{N\to\infty}\frac{1}{N}\sum_i\delta(\xi_i^\nu-\xi^\nu)=\mathcal{P}(\xi^\nu) \tag{30}$$

in the limit of $N\to\infty$ and the average is now carried out explicitly as

$$\int D[\xi^\nu]d\xi^\nu(\cdots)=\int\mathcal{P}(\xi^\nu)d\xi^\nu(\cdots)\equiv\mathbb{E}_{\boldsymbol{\xi}}[\cdots]. \tag{31}$$

Equation (29) is a general solution for the problem in this paper.

4.3. The classical and zero-temperature limits

It is easy for us to take the classical limit $\Gamma\to 0$ in the result (29). Actually, we have immediately

$$\frac{dm^\nu}{dt}=-m^\nu+\mathbb{E}_{\boldsymbol{\xi}}\left[\xi^\nu\tanh\left(\beta\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu\right)\right]. \tag{32}$$

The above equation is identical to the result by Coolen and Ruijgrok [17] who considered the retrieval process of the conventional Hopfield model under thermal noise.

We can also take the zero-temperature limit $\beta\to\infty$ in (29) as

$$\frac{dm^\nu}{dt}=-m^\nu+\mathbb{E}_{\boldsymbol{\xi}}\left[\frac{\xi^\nu\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu}{\sqrt{(\sum_{\mu\nu}\xi^\mu A_{\mu\nu}m^\nu)^2+\Gamma^2}}\right]. \tag{33}$$

Thus, the equation (29) including the above two limiting cases is our general solution for the neuro-dynamics of the quantum Hopfield model in which $\mathcal{O}(1)$ patterns are embedded. Thus, we can discuss any kind of situations for such pattern-recalling processes and the solution is always derived from (29) explicitly.

5. Limit cycle solution for asymmetric connections

In this section, we discuss a special case of the general solution (29). Namely, we investigate the pattern-recalling processes of the quantum Hopfield model with asymmetric connections $\boldsymbol{A}\equiv\{A_{\mu\nu}\}$.

5.1. Result for two-patterns

Let us consider the case in which just only two patterns are embedded via the following matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{34}$$

Then, from the general solution (29), the differential equations with respect to the two overlaps m_1 and m_2 are written as

$$\begin{aligned} \frac{dm_1}{dt} &= -m_1 + \frac{m_1}{\sqrt{(2m_1)^2 + \Gamma^2}} - \frac{m_2}{\sqrt{(2m_2)^2 + \Gamma^2}} \\ \frac{dm_2}{dt} &= -m_2 + \frac{m_1}{\sqrt{(2m_1)^2 + \Gamma^2}} + \frac{m_2}{\sqrt{(2m_2)^2 + \Gamma^2}}. \end{aligned}$$

In Figure 1, we show the time evolutions of the overlaps m_1 and m_2 for the case of the amplitude $\Gamma = 0.01$. From this figure, we clearly find that the neuronal state evolves as

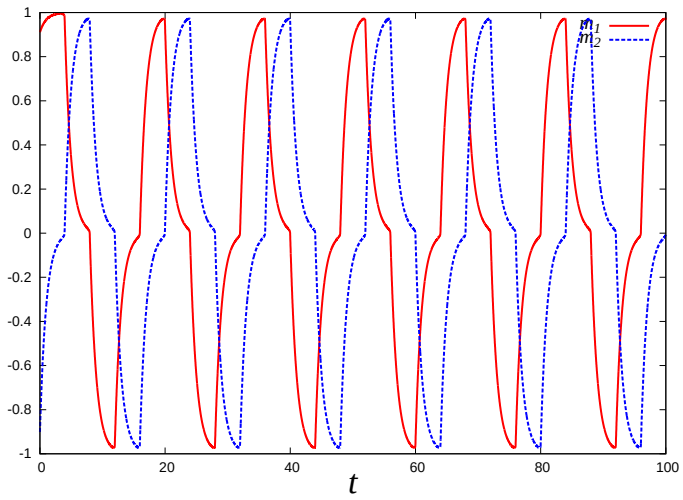

Figure 1. Time evolutions of m_1 and m_2 for the case of $\Gamma = 0.01$.

$A \to B \to \overline{A} \to \overline{B} \to A \to B \to \cdots$ ($\overline{A}, \overline{B}$ denote the 'mirror images' of A and B, respectively), namely, the network behaves as a limit cycle.

To compare the effects of thermal and quantum noises on the pattern-recalling processes, we plot the trajectories m_1-m_2 for $(T \equiv \beta^{-1}, \Gamma) = (0, 0.01), (0.01, 0)$ (left panel), $(T, \Gamma) = (0, 0.8), (0.8, 0)$ (right panel) in Figure 2. From these panels, we find that the limit cycles are getting collapsed as the strength of the noise level is increasing for both thermal and quantum-mechanical noises, and eventually the trajectories shrink to the origin $(m_1, m_2) = (0, 0)$ in the limit of $T, \Gamma \to \infty$.

6. Summary

In this paper, we considered the stochastic process of quantum Monte Carlo method applied for the quantum Hopfield model and investigated the quantum neuro-dynamics through the differential equations with respect to the macroscopic quantities such as the overlap. By using the present approach, one can evaluate the 'inhomogeneous' Markovian stochastic process of quantum Monte Carlo method (in which amplitude Γ is time-dependent [18, 19]) such as

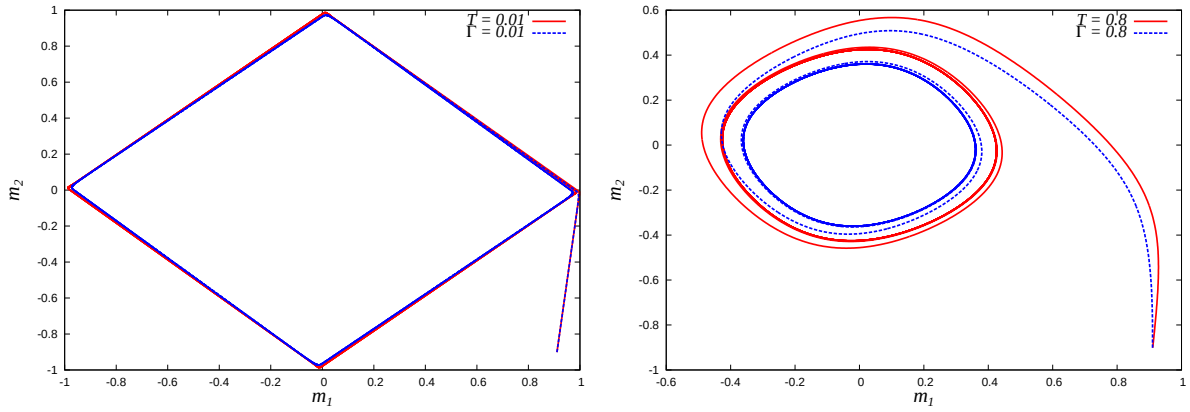

Figure 2. Trajectories m_1-m_2 for $(T, \Gamma) = (0, 0.01), (0.01, 0)$ (left panel), $(T, \Gamma) = (0, 0.8), (0.8, 0)$ (right panel).

quantum annealing [20, 21, 22, 23, 24, 25]. In the next step of the present study, we are planning to extend this formulation to the probabilistic information processing described by spin glasses including a peculiar type of antiferromagnet [26].

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