Primitive filtrations of the modules of invariant logarithmic forms of Coxeter arrangements

Takuro Abe* and Hiroaki Terao†

December 10, 2010

Abstract

We define **primitive derivations** for Coxeter arrangements which may not be irreducible. Using those derivations, we introduce the **primitive filtrations** of the module of invariant logarithmic differential forms for an arbitrary Coxeter arrangement with an arbitrary multiplicity. In particular, when the Coxeter arrangement is irreducible with a constant multiplicity, the primitive filtration was studied in [2], which generalizes the Hodge filtration introduced by K. Saito (e.g., [6]).

1 Introduction

Let V be an ℓ-dimensional Euclidean space and $\mathcal{A}$ be an arrangement of hyperplanes in V. We use [4] as a general reference for arrangements. For each $H \in \mathcal{A}$, choose a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. Their product $Q := \prod_{H \in \mathcal{A}} \alpha_H$ lies in the symmetric algebra $S := \mathrm{Sym}^*(V^*)$. The quotient field of S is denoted by F. Let Ω_S and Ω_F denote the S-module of regular differential 1-forms on V and the F-vector space of rational differential 1-forms on V respectively. Define the S-module $\Omega(\mathcal{A}, \infty)$ of **logarithmic differential 1-forms** by

$$\Omega(\mathcal{A}, \infty) := \{\omega \in \Omega_F \mid Q^N \omega \text{ and } (Q/\alpha_H)^N d\alpha_H \wedge \omega \ \text{ are both regular for all } H \in \mathcal{A} \text{ for } N \gg 0\}.$$

*Department of Mechanical Engineering and Science, Kyoto University, Yoshida Honmachi, Sakyo-Ku, Kyoto 606-8501, Japan. email:abe.takuro.4c@kyoto-u.ac.jp.

†Department of Mathematics, Hokkaido University, Kita-10, Nishi-8, Kita-Ku, Sapporo, Hokkaido 060-0810, Japan. email:terao@math.sci.hokudai.ac.jp.

In other words, $\Omega(\mathcal{A},\infty)$ consists of all logarithmic differential 1-forms in the sense of Ziegler [9].

Suppose that $\mathcal{A}$ is a **Coxeter arrangement**. Then the corresponding **Coxeter group** $W = W(\mathcal{A})$ naturally acts on V, V^*, S and $\Omega(\mathcal{A},\infty)$. Note that we do not assume that $\mathcal{A}$ is irreducible. When $\mathcal{A}$ is irreducible, the primitive derivations play the central role to define the Hodge filtration introduced by K. Saito. (See [6] for example.) In this paper we develop a theory of primitive derivations and the Hodge filtration in the case of non-irreducible Coxeter arrangements. More precisely, in Section 2, we introduce primitive derivations even when $\mathcal{A}$ is not irreducible. Fix a primitive derivation D. Let $R := S^W$ be the W-invariant subring of S and

$$T := \{f \in R \mid D(f) = 0\}.$$

Consider the T-linear connection (covariant derivative)

$$\nabla_D : \Omega_F \to \Omega_F$$

characterized by (1) $\nabla_D(f\omega) = D(f)\,\omega + f\,(\nabla_D\,\omega)$ for $f \in F$ and $\omega \in \Omega_F$ and (2) $\nabla_D(d\alpha) = 0$ for all $\alpha \in V^*$. Our first main result is

Theorem 1.1
Let $\Omega(\mathcal{A},\infty)^W$ be the W-invariant part of $\Omega(\mathcal{A},\infty)$. Then the ∇_D induces a T-linear automorphism

$$\nabla_D : \Omega(\mathcal{A},\infty)^W \xrightarrow{\sim} \Omega(\mathcal{A},\infty)^W.$$

Note that the inverse map ∇_D^{-1} and ∇_D^k $(k \in \mathbb{Z})$ are also T-automorphisms. Under the assumption that $\mathcal{A}$ is irreducible, Theorem 1.1 was proved in [2, Theorem 1.2 (1)].

Definition 1.2
Let $I^* : \Omega_F \times \Omega_F \to F$ be the F-bilinear map induced from the inner product I of the Euclidean space V. Let $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ be an arbitrary multiplicity. Define

$$\Omega(\mathcal{A},\mathbf{m}) := \{\omega \in \Omega(\mathcal{A},\infty) \mid (Q/\alpha_H)^N \alpha_H^{\mathbf{m}(H)} I^*(d\alpha_H,\omega) \in S \quad \text{for all } H \in \mathcal{A} \text{ for } N \gg 0\}$$

and

$$\Omega(\mathcal{A},\mathbf{m})^W := \Omega(\mathcal{A},\mathbf{m}) \cap \Omega(\mathcal{A},\infty)^W.$$

The **primitive filtration of $\Omega(\mathcal{A},\infty)^W$ induced from $\mathbf{m}$** *is given by*

$$P_k^{(\mathbf{m})} := \nabla_D^k \Omega(\mathcal{A},\mathbf{m})^W \quad (k \in \mathbb{Z}).$$

Note that

$$\Omega(\mathcal{A},\mathbf{m}) = \{\omega \in \Omega_F \mid (\prod_{H\in\mathcal{A}} \alpha_H^{\mathbf{m}(H)})\omega \text{ and } (\prod_{H\neq H_0} \alpha_H^{\mathbf{m}(H)})(d\alpha_{H_0}\wedge\omega) \text{ are both regular for all } H_0 \in \mathcal{A}\}$$

if $\mathbf{m}(H) \geq 0$ for all $H \in \mathcal{A}$. In this case, $\Omega(\mathcal{A},\mathbf{m})$ was introduced by Ziegler [9].

Our second main result is an explicit description of the primitive filtration:

Theorem 1.3
The primitive filtration is an increasing filtration

$$\cdots \subset P_{-1}^{(\mathbf{m})} \subset P_0^{(\mathbf{m})} \subset P_1^{(\mathbf{m})} \subset \ldots$$

such that

$$P_k^{(\mathbf{m})} = P_0^{(\mathbf{m}+2k)} = \Omega(\mathcal{A},\mathbf{m}+2k)^W$$

where $(\mathbf{m}+2k)(H) = \mathbf{m}(H) + 2k$ $(k \in \mathbb{Z}, H \in \mathcal{A})$.

When $\mathcal{A}$ is irreducible and $\mathbf{m}$ is equal to the constant function $\mathbf{1}$ with $\mathbf{1}(H) = 1$ $(H \in \mathcal{A})$, the primitive filtration coincides with the filtration introduced in [2]. Its dual version in Theorem 4.4 generalizes the Hodge filtration introduced by K. Saito (e.g., [6]).

We construct bases for the primitive filtration induced from $\mathbf{1}$ in Theorem 2.6. The bases are used when we prove Theorems 1.1 and 1.3 in Section 3.

In Section 4, we translate our main results Theorems 1.1, 1.3 and 2.6 into the dual language in terms of the logarithmic derivations.

2 Primitive derivations

We first state a multiple version of Saito's criterion due to Abe [1].

Proposition 2.1
Let $\mathcal{A}$ be a central arrangement in V with an arbitrary multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$. Let $x_1, x_2, \ldots, x_\ell$ be a basis for V^*. Define

$$Q^{\mathbf{m}} := \prod_{H\in\mathcal{A}} \alpha_H^{\mathbf{m}(H)} \in F.$$

Let $\omega_1, \omega_2, \ldots, \omega_\ell \in \Omega(\mathcal{A},\mathbf{m})$. Then

(1) $Q^{\mathbf{m}}(\omega_1 \wedge \cdots \wedge \omega_\ell)$ is regular,

(2) $\omega_1, \omega_2, \ldots, \omega_\ell$ form an S-basis for $\Omega(\mathcal{A},\mathbf{m})$ if and only if

$$Q^{\mathbf{m}}(\omega_1 \wedge \cdots \wedge \omega_\ell) \in \mathbb{R}^\times(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell).$$

Remark. When $\mathbf{m} = \mathbf{1}$, this is due to K. Saito [5]. When $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, this is due to Ziegler [9].

The original proof in [1, Theorem 1.4] is written in a slightly different language from this paper, so we include our proof here.

Proof of Proposition 2.1. Pick $H \in \mathcal{A}$ arbitrarily and fix it. Let $m = \mathbf{m}(H)$. Choose an orthonormal basis $x_1, x_2, \ldots, x_\ell$ such that $H = \{x_1 = 0\}$.

(1) It is enough to show that $x_1^m(\omega_1 \wedge \cdots \wedge \omega_\ell)$ has no pole along H. Write

$$\omega_j = \sum_{i=1}^{\ell} f_{ij} dx_i \quad (j = 1, \ldots, \ell).$$

Since $\omega_j \in \Omega(\mathcal{A}, \mathbf{m})$, $x_1^m f_{1j} = x_1^m I^*(dx_1, \omega_j)$ has no pole along H for $j = 1, \ldots, \ell$ by Definition 1.2. Moreover,

$$\sum_{i \geq 2} f_{ij} dx_1 \wedge dx_i = dx_1 \wedge \omega_j$$

has no pole along H because $\omega_j \in \Omega(\mathcal{A}, \infty)$ for $j = 1, \ldots, \ell$. This implies that f_{ij} has no pole along H if $i \geq 2$. Therefore

$$x_1^m(\omega_1 \wedge \cdots \wedge \omega_\ell) = \begin{vmatrix} x_1^m f_{11} & x_1^m f_{12} & \ldots & x_1^m f_{1\ell} \\ f_{21} & f_{22} & \ldots & f_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\ell 1} & f_{\ell 2} & \ldots & f_{\ell\ell} \end{vmatrix} (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell)$$

has no pole along H.

(2) Suppose that $\omega_1, \omega_2, \ldots, \omega_\ell$ form an S-basis for $\Omega(\mathcal{A}, \mathbf{m})$. By (1) we may write

$$Q^{\mathbf{m}}(\omega_1 \wedge \cdots \wedge \omega_\ell) = f(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell)$$

with $f \in S$. In order to prove that f is a nonzero constant, it is enough to show that f is not divisible by x_1. Define a multiplicity $\mathbf{m}' : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ by

$$\mathbf{m}'(K) := \begin{cases} |\mathbf{m}(K)| & \text{if } K \neq H \\ 0 & \text{if } K = H. \end{cases}$$

Then it is not hard to see that

$$\eta_1 := Q^{\mathbf{m}'}(dx_1/x_1^m), \quad \eta_2 := Q^{\mathbf{m}'} dx_2, \quad \ldots, \quad \eta_\ell := Q^{\mathbf{m}'} dx_\ell$$

lie in $\Omega(\mathcal{A}, \mathbf{m})$. Thus

$$
\begin{aligned}
&(Q^{\mathbf{m}}/x_1^m)(Q^{\mathbf{m}'})^\ell (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell) \\
= &Q^{\mathbf{m}}(\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_\ell) \in S(\omega_1 \wedge \cdots \wedge \omega_\ell) = Sf(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell).
\end{aligned}
$$

This implies that $g := (Q^{\mathbf{m}}/x_1^m)(Q^{\mathbf{m}'})^\ell$ is divisible by f. Since g is not divisible by x_1, neither is f.

Suppose that $Q^{\mathbf{m}}(\omega_1 \wedge \cdots \wedge \omega_\ell) = dx_1 \wedge \cdots \wedge dx_\ell$. In order to prove that $\omega_1, \ldots, \omega_\ell$ form a basis it is enough to show that $\omega_1, \ldots, \omega_\ell$ span $\Omega(\mathcal{A}, \mathbf{m})$ over S. Fix $\omega \in \Omega(\mathcal{A}, \mathbf{m})$. By (1) we may write

$$Q^{\mathbf{m}}(\omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega \wedge \omega_{i+1} \cdots \wedge \omega_\ell) = f_i(dx_1 \wedge \cdots \wedge dx_\ell),$$

with $f_i \in S$ for $i = 1, \ldots, \ell$. Define $\eta := \omega - \sum_{i=1}^{\ell} f_i \omega_i$. Then we obtain

$$Q^{\mathbf{m}}(\omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \eta \wedge \omega_{i+1} \cdots \wedge \omega_\ell) = 0 \ \ (i = 1, \ldots, \ell).$$

Since $\omega_1, \ldots, \omega_\ell$ span the cotangent space of V at each point outside the hyperplanes, we have $\eta = 0$ and thus $\omega = \sum_{i=1}^{\ell} f_i \omega_i$. □

Next let $\mathcal{A}$ be an irreducible Coxeter arrangement. Then we may put

$$R = S^W = \mathbb{R}[P_1, \ldots , P_\ell]$$

with

$$\deg P_1 < \deg P_2 \leq \cdots \leq \deg P_{\ell-1} < \deg P_\ell$$

by [3]. The derivation

$$D := \frac{\partial}{\partial P_\ell}$$

is called a primitive derivation which was extensively studied by K. Saito. Although D depends upon the choice of P_ℓ, its ambiguity is only up to a constant multiple. Recall the T-linear connection

$$\nabla_D : \Omega_F \to \Omega_F.$$

Then the ∇_D induces a T-linear automorphism

$$\nabla_D : \Omega(\mathcal{A}, \infty)^W \xrightarrow{\sim} \Omega(\mathcal{A}, \infty)^W$$

by [2, Theorem 1.2 (1)]. Recall

Proposition 2.2
[2, Theorems 1.1 and 2.12] Suppose that $\mathcal{A}$ is an irreducible Coxeter arrangement. For any $k \in \mathbb{Z}$ and $1 \le j \le \ell$, define

$$\theta_j^{(k)} := \nabla_D^k \left(dP_j \right), \quad \Theta^{(k)} := \{\theta_j^{(k)}\}_{1 \le j \le \ell}, \quad \text{and} \quad \Theta := \bigcup_{k \in \mathbb{Z}} \Theta^{(k)}.$$

Then

(1) the S-module $\Omega(\mathcal{A}, 2k-1)$ is free with a basis $\Theta^{(k)}$,

(2) the R-module $\Omega(\mathcal{A}, 2k-1)^W$ is free with a basis $\Theta^{(k)}$,

(3) the T-module $\Omega(\mathcal{A}, 2k-1)^W$ is free with a basis $\bigcup_{p \le k} \Theta^{(p)}$, and

(4) the T-module $\Omega(\mathcal{A}, \infty)^W$ is free with a basis Θ.

Proposition 2.3
[2, Lemma 2.3 and Proposition 2.6 (4)] Let $G := \left[I^*(dP_i, dP_j) \right]_{1 \le i,j \le \ell}$. For each $k \in \mathbb{Z}$, there exists an $\ell \times \ell$-matrix G_k with entries in R such that

$$\left[\theta_1^{(k)}, \dots, \theta_\ell^{(k)} \right] = \left[\theta_1^{(k+1)}, \dots, \theta_\ell^{(k+1)} \right] G_k,$$

where G_k can be expressed as $G_k = B_k G B'_k$ with $B_k, B'_k \in GL_\ell(T)$.

From now on assume that $\mathcal{A}$ is an arbitrary Coxeter arrangement which may not be irreducible. Then one has the following decompositions:

$$V = V[1] \oplus \cdots \oplus V[t], \quad \mathcal{A} = \mathcal{A}[1] \times \cdots \times \mathcal{A}[t],$$
$$W = W[1] \times \cdots \times W[t], \quad S \simeq S[1] \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} S[t],$$

where each $\mathcal{A}[i]$ is an irreducible Coxeter arrangement in $V[i]$, $W[i] := W(\mathcal{A}[i])$, and

$$S[i] := S(V[i]^*) = \mathbb{R}[x_1[i], \dots, x_{\ell[i]}[i]]$$

for $i = 1 \dots, t$. We naturally regard $\mathcal{A}[i]$ as a subarrangement of $\mathcal{A}$, $S[i]$ as a subring of S, and $W[i]$ as a subgroup of W.

Let $1 \le i \le t$. Let $R[i]$ denote the $W[i]$-invariant subring of $S[i]$. Let $\ell[i] = \dim V[i]$. Then we may put

$$R[i] = \mathbb{R}[P_1[i], \dots, P_{\ell[i]}[i]]$$

with

$$\deg P_1[i] < \deg P_2[i] \le \cdots < \deg P_{\ell[i]}[i].$$

Then

$$R = S^W = \mathbb{R}[\{P_j[i]\}_{1\le i\le t, 1\le j\le \ell[i]}] \simeq R[1] \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} R[t].$$

Thus we may naturally regard $R[i]$ as a subring of R. Let $D[i] : R[i] \to R[i]$ denote a primitive derivation corresponding to the irreducible Coxeter arrangement $\mathcal{A}[i]$. We may naturally extend the derivation $D[i]$ to a derivation $\hat{D}[i] : R \to R$ by $\hat{D}[i](f) = 0$ for any $f \in R[j]$ $(i \neq j)$.

Definition 2.4
Let $\mathcal{A}$ be a Coxeter arrangement which may not be irreducible. Then the derivation

$$D := \sum_{i=1}^{t} \hat{D}[i] : R \to R$$

is called a **primitive derivation** *of W. Let $T := \ker(D : R \to R)$.*

Remark. The primitive derivations defined in Definition 2.4 are not necessarily homogeneous or unique up to a constant multiple unlike the irreducible case. However, those derivations play a similar role to irreducible primitive derivations as we show in this note.

We often write $P[i]$ instead of $P_{\ell[i]}[i]$ for simplicity. Then we have

Lemma 2.5
For $i = 1, \dots, t$, $R = T[P[i]]$.

Proof. It is obvious that $\{P_j[i]\}_{1\le i\le t, 1\le j\le \ell[i]-1} \subset T$. Note $P[j] - P[i] \in T$ because

$$D(P[j] - P[i]) = D(P[j]) - D(P[i]) = 1 - 1 = 0.$$

Thus $P[j] = (P[j] - P[i]) + P[i] \in T[P[i]]$. □

Theorem 2.6
For any $k \in \mathbb{Z}$, $1 \le i \le t$ and $1 \le j \le \ell[i]$, define

$$\theta_j^{(k)}[i] := \nabla_{D[i]}^{k} \left(dP_j[i]\right).$$

Let

$$\Theta^{(k)}[i] := \{\theta_j^{(k)}[i]\}_{1\le j\le \ell[i]},\ \Theta^{(k)} := \bigcup_{i=1}^{t} \Theta^{(k)}[i], \quad \textit{and} \ \ \Theta := \bigcup_{k\in\mathbb{Z}} \Theta^{(k)}.$$

Then

(1) *the S-module $\Omega(\mathcal{A}, 2k-1)$ is free with a basis $\Theta^{(k)}$,*

(2) *the R-module $\Omega(\mathcal{A}, 2k-1)^W$ is free with a basis $\Theta^{(k)}$,*

(3) *the T-module $\Omega(\mathcal{A}, 2k-1)^W$ is free with a basis $\bigcup_{p\leq k}\Theta^{(p)}$, and*

(4) *the T-module $\Omega(\mathcal{A}, \infty)^W$ is free with a basis Θ.*

Proof. (1) Let $t = 2$ for simplicity. By Proposition 2.2 (1), $\Theta^k[i]$ is an $S[i]$-basis for $\Omega(\mathcal{A}[i], 2k-1)$ for each i. Thus, by Proposition 2.1, we have

$$\begin{aligned} & Q^{2k-1}\left(\bigwedge_{j=1}^{\ell[1]}\theta_j^{(k)}[1]\right)\bigwedge\left(\bigwedge_{j=1}^{\ell[2]}\theta_j^{(k)}[2]\right) \\ = & \left(Q[1]^{2k-1}\bigwedge_{j=1}^{\ell[1]}\theta_j^{(k)}[1]\right)\bigwedge\left(Q[2]^{2k-1}\bigwedge_{j=1}^{\ell[2]}\theta_j^{(k)}[2]\right) \\ \in & \mathbb{R}^{\times}(dx_1[1]\wedge\cdots\wedge dx_{\ell[1]}[1]\wedge dx_1[2]\wedge\cdots\wedge dx_{\ell[2]}[2]), \end{aligned}$$

where $Q[i] = \prod_{H\in\mathcal{A}[i]}\alpha_H$ $(i = 1, 2)$. This implies (1) because of Proposition 2.1 again.

(2) Note that each $\theta_j^{(k)}[i]$ is W-invariant by definition. Thus

$$\Theta^{(k)} \subset \Omega(\mathcal{A}, 2k-1)^W.$$

Since $\Theta^{(k)}$ is linearly independent over S by (1), so is over R. An arbitrary element of $\Omega(\mathcal{A}, 2k-1)^W$ can be expressed as a linear combination of $\Theta^{(k)}$ with coefficients in S. Then it is obvious that each of the coefficients lies in R. This shows that $\Theta^{(k)}$ spans $\Omega(\mathcal{A}, 2k-1)^W$ over R.

(3) Let

$$\mathcal{T}[i] := \bigcup_{p\leq k}\Theta^{(p)}[i] \text{ and } \mathcal{T} := \bigcup_{i=1}^{t}\mathcal{T}[i] = \bigcup_{p\leq k}\Theta^{(p)}.$$

Step 1. $\mathcal{T}$ spans $\Omega(\mathcal{A}, 2p-1)^W$ over T.

Since

$$\Theta^{(p)} \subset \Omega(\mathcal{A}, 2p-1)^W \subseteq \Omega(\mathcal{A}, 2k-1)^W$$

for $p \leq k$, we have $\mathcal{T} \subset \Omega(\mathcal{A}, 2k-1)^W$. Let $\langle\mathcal{T}\rangle_T$ be the submodule of $\Omega(\mathcal{A}, 2k-1)^W$ generated by $\mathcal{T}$ over T. Let

$$T[i] := \ker(D[i] : R[i] \to R[i])$$

for each i. Then $T[i] \subseteq T$. By Proposition 2.2 (3) we know that $\langle \mathcal{T}[i] \rangle_{T[i]}$ is closed under the multiplication of $R[i]$ for each i. In particular,

$$P[i] \cdot \mathcal{T}[i] \subset \langle \mathcal{T}[i] \rangle_{T[i]} \subseteq \langle \mathcal{T}[i] \rangle_{T}$$

because $P[i] = P_{\ell[i]}[i] \in R[i]$. Therefore $\langle \mathcal{T}[i] \rangle_T$ is closed under the multiplication of R because $R = T[P[i]]$ by Lemma 2.5. Thus we obtain $\langle \mathcal{T}[i] \rangle_R = \langle \mathcal{T}[i] \rangle_T$ for each i. Therefore $\langle \mathcal{T} \rangle_R = \langle \mathcal{T} \rangle_T$. By (2) we have

$$\Omega(\mathcal{A}, 2k-1)^W = \left\langle \Theta^{(k)} \right\rangle_R \subseteq \langle \mathcal{T} \rangle_R = \langle \mathcal{T} \rangle_T \subseteq \Omega(\mathcal{A}, 2k-1)^W.$$

Therefore $\langle \mathcal{T} \rangle_T = \Omega(\mathcal{A}, 2k-1)^W$: $\mathcal{T}$ spans $\Omega(\mathcal{A}, 2k-1)^W$ over T.

Step 2. $\mathcal{T}$ is linearly independent over T.

It is enough to show that $\mathcal{T}[i]$ is linearly independent over T for each i. Let $1 \le i \le t$. Assume

$$\sum_{k \in \mathbb{Z}} \left[\theta_1^{(k)}[i], \ldots, \theta_{\ell[i]}^{(k)}[i]\right] \mathbf{g}_k = \mathbf{0}$$

with $\mathbf{g}_k = \left[g_{k,1}, \ldots g_{k,\ell[i]}\right]^T \in T^{\ell[i]}$, $k \in \mathbb{Z}$ such that there exist integers p and q such that $p \le q$, $\mathbf{g}_p \neq 0, \mathbf{g}_q \neq 0$ and $\mathbf{g}_k = 0$ for all $k < p$ and $k > q$. Then, by Proposition 2.3

$$\mathbf{0} = \sum_{k=p}^{q} \left[\theta_1^{(k)}[i], \ldots, \theta_{\ell[i]}^{(k)}[i]\right] \mathbf{g}_k = \left[\theta_1^{(q)}[i], \ldots, \theta_{\ell[i]}^{(q)}[i]\right] \sum_{k=p}^{q} H_k \mathbf{g}_k,$$

where

$$H_q := I_{\ell[i]}, \quad H_k := G_k G_{k+1} \ldots G_{q-1} \quad (p \le k < q).$$

This implies that

$$\mathbf{0} = \sum_{k=p}^{q} H_k \mathbf{g}_k.$$

Note that H_k can be expressed as a product of $(q-k)$ copies of

$$G[i] := [I^*(dP_a[i], dP_b[i])]_{1 \le a,b \le \ell[i]}$$

and matrices belonging to $\mathrm{GL}_{\ell[i]}(T[i])$. It is well-known that $D[G[i]] = D[i][G[i]] \in GL_{\ell[i]}(T[i])$ [2, Proposition 2.1]. Thus $D^{q-p}[H_k] = 0$ $(k > p)$. Applying D^{q-p} to the above, we obtain

$$D^{q-p}[H_p]\mathbf{g}_p = 0.$$

Note Since the matrix $D^{q-p}[H_p]$, which is a product of $(q-p)$ copies of $D[G[i]]$ and matrices in $\mathrm{GL}_{\ell[i]}(T[i])$, is nondegenerate, we get $\mathbf{g}_p = 0$, which is a contradiction. This implies that $\mathcal{T}$ is linearly independent over T.

(4) It follows from (3) and the fact that

$$\Omega(\mathcal{A}, \infty)^W = \bigcup_{k\in\mathbb{Z}} \Omega(\mathcal{A}, 2k-1)^W.$$

□

3 Proof of main theorems.

Proof of Theorem 1.1. Since

$$\nabla_D \theta_j^{(k)}[i] = \nabla_D \nabla_{D[i]}^k (dP_j[i]) = \nabla_{D[i]}^{k+1}(dP_j[i]) = \theta_j^{(k+1)}[i],$$

the connection ∇_D induces a bijection of Θ to itself. Thus ∇_D induces a T-automorphism of $\Omega(\mathcal{A}, \infty)^W$ because of Theorem 2.6 (4). □

For $f \in F$ with $f \neq 0$ and $\alpha \in V^* \setminus \{\mathbf{0}\}$ define

$$\mathrm{ord}_\alpha(f) := \min\{k \in \mathbb{Z} \mid \alpha^k f \in S_{(\alpha)}\},$$

where $S_{(\alpha)}$ is the localization of S at the prime ideal $(\alpha) = \alpha S$. In other words $\mathrm{ord}_\alpha(f)$ is the order of poles of f along the hyperplane $\ker(\alpha)$.

Lemma 3.1
Assume that $\mathcal{A}$ is a Coxeter arrangement which may not be irreducible. Let D be a primitive derivation of $\mathcal{A}$. Choose $\alpha \in V^*$ such that $\ker(\alpha) \in \mathcal{A}$. Then
(1) $\mathrm{ord}_\alpha D(\alpha) = 1$.
(2) For $f \in F \setminus \{0\}$ with $\mathrm{ord}_\alpha(f) \neq 0$, $\mathrm{ord}_\alpha(D(f)) = \mathrm{ord}_\alpha(f) + 2$.

Proof. (1) Assume that

$$\mathcal{A} = \mathcal{A}[1] \times \cdots \times \mathcal{A}[t]$$

such that each $\mathcal{A}[i]$ is irreducible. Suppose $\ker(\alpha) \in \mathcal{A}[k]$. Then $D[i](\alpha) = 0$ if $i \neq k$. This implies that we may assume that $\mathcal{A}$ is irreducible from the beginning. Choose an orthonormal basis $\alpha = x_1, x_2, \ldots, x_\ell$ and let $h_j := D(x_j)$ for $1 \leq j \leq \ell$. It is well-known (e.g., [7, pp. 249-250]) that h_j $(j > 1)$ has no poles along $x_1 = 0$. On the other hand, it is also known (e.g., [7, Corollary 3.32]) that

$$\det[\partial h_j / \partial x_i] = c\, Q^{-2}$$

for some nonzero constant c. Thus h_1 should have poles along $x_1 = 0$. Since $Qh_1 = (QD)(x_1)$ is regular, we have $\mathrm{ord}_\alpha D(\alpha) = \mathrm{ord}_\alpha h_1 = 1$.

(2) Suppose that $k := \mathrm{ord}_\alpha(f) \neq 0$. Put $f = g/\alpha^k$. Then $g \in S_{(\alpha)}$ and $g \notin \alpha S_{(\alpha)}$. Compute

$$D(f) = D(g/\alpha^k) = D(g)/\alpha^k - kD(\alpha)g/\alpha^{k+1}.$$

From (1) we have $\mathrm{ord}_\alpha(D(\alpha)) = 1$. Since

$$\mathrm{ord}_\alpha(D(\alpha)g/\alpha^{k+1}) = k+2, \quad \mathrm{ord}_\alpha(D(g)/\alpha^k) \leq k+1,$$

we obtain $\mathrm{ord}_\alpha(D(f)) = k+2$. □

Proof of Theorem 1.3. It is enough to prove $\nabla_D \Omega(\mathcal{A}, \mathbf{m})^W = \Omega(\mathcal{A}, \mathbf{m}+2)^W$. Let $\omega \in \Omega(\mathcal{A}, \infty)^W$ and $\alpha \in V^*$ with $\ker(\alpha) \in \mathcal{A}$.

We first verify:

$$\mathrm{ord}_\alpha I^*(\omega, d\alpha) \neq 0. \tag{3.1}$$

Let s_α be the orthogonal reflection through the hyperplane $\ker(\alpha)$. Since ω is W-invariant, we have $s_\alpha(I^*(\omega, d\alpha)) = -I^*(\omega, d\alpha)$. Suppose that $\mathrm{ord}_\alpha I^*(\omega, d\alpha) = 0$. Then, for a sufficiently large integer N,

$$g := (Q/\alpha)^N I^*(\omega, d\alpha) \in S \setminus \alpha S.$$

On the other hand, we obtain

$$s_\alpha(g) = \left(s_\alpha(Q/\alpha)^N\right) s_\alpha\left(I^*(\omega, d\alpha)\right) = -(Q/\alpha)^N I^*(\omega, d\alpha) = -g.$$

This shows that g is an antiinvariant with respect to the reflection group $\{\mathbf{1}, s_\alpha\}$. Therefore $g \in \alpha S$, which is a cotradiction. Thus (3.1) was verified. By Lemma 3.1, we have

$$\begin{aligned} &\alpha^k I^*(\omega, d\alpha) \in S_{(\alpha)} \Leftrightarrow \mathrm{ord}_\alpha I^*(\omega, d\alpha) \leq k \\ \Leftrightarrow\ &\mathrm{ord}_\alpha I^*(\nabla_D(\omega), d\alpha) = \mathrm{ord}_\alpha D(I^*(\omega, d\alpha)) \leq k+2 \\ \Leftrightarrow\ &\alpha^{k+2} I^*(\nabla_D(\omega), d\alpha) \in S_{(\alpha)}, \end{aligned}$$

where $k := \mathbf{m}(\ker(\alpha))$. This implies

$$\omega \in \Omega(\mathcal{A}, \mathbf{m})^W \Leftrightarrow \nabla_D(\omega) \in \Omega(\mathcal{A}, \mathbf{m}+2)^W.$$

□

4 Logarithmic derivation modules

In this section, we translate our main results Theorems 1.1, 1.3 and 2.6 into the corresponding theorems in the language of logarithmic derivation modules. Let Der_S and Der_F denote the S-module of $\mathbb{R}$-linear derivations from S to itself and the F-vector space of $\mathbb{R}$-linear derivations from F to itself. Recall the S-linear isomorphism

$$I^* : \Omega_F \to \mathrm{Der}_F, \qquad I^*(\omega)(f) := I^*(\omega, df) \qquad (\omega \in \Omega_F, f \in F).$$

The traslation of the main results is done by the isomorphism I^*.

Definition 4.1
Define the S-module $D(\mathcal{A}, -\infty)$ of **logarithmic derivations** *by*

$$D(\mathcal{A}, -\infty) := \{\xi \in \mathrm{Der}_F \mid Q^N\xi \text{ and } (Q/\alpha_H)^N\xi(\beta) \text{ are both regular for } N \gg 0, H \in \mathcal{A} \text{ and } \beta \in V^* \text{ with } I^*(d\alpha_H, d\beta) = 0\}.$$

Then the map I^* induces an S-linear isomorphism

$$I^* : \Omega(\mathcal{A}, \infty) \xrightarrow{\sim} D(\mathcal{A}, -\infty).$$

Let $\mathcal{A}$ be a Coxeter arrangement which may not be irreducible and D be a primitive derivation. The T-linear connection

$$\nabla_D : \mathrm{Der}_F \to \mathrm{Der}_F$$

is characterized by (1) $\nabla_D(f\xi) = D(f)\xi + f(\nabla_D\xi)$ and (2) $\nabla_D(\partial_v) = 0$ for all $v \in V$. Here the derivation ∂_v satisfies $\partial_v(\alpha) = \alpha(v)$ for any $\alpha \in V^*$. Then it is not hard to see $(\nabla_D\xi)(\alpha) = D(\xi(\alpha))$ for all $\alpha \in V^*$.

Lemma 4.2
For $\omega \in \Omega_F$ we have

$$I^*(\nabla_D(\omega)) = \nabla_D(I^*(\omega)).$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} \Omega(\mathcal{A}, \infty) & \xrightarrow{\nabla_D} & \Omega(\mathcal{A}, \infty) \\ {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow \\ D(\mathcal{A}, -\infty) & \xrightarrow{\nabla_D} & D(\mathcal{A}, -\infty). \end{array}$$

Proof. It is enough to prove $I^*(\nabla_D(\omega))(\alpha) = \nabla_D(I^*(\omega))(\alpha)$ for all $\alpha \in V^*$. Compute

$$(I^*(\nabla_D\omega))(\alpha) = I^*(\nabla_D\omega, d\alpha) = D(I^*(\omega, d\alpha)) = D(I^*(\omega)(\alpha)) = (\nabla_D I^*(\omega))(\alpha).$$

□

Definition 4.3
Let $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ *be an arbitrary multiplicity. Define*

$$D(\mathcal{A}, \mathbf{m}) := I^*(\Omega(\mathcal{A}, -\mathbf{m})) = \{\xi \in D(\mathcal{A}, -\infty) \mid (Q/\alpha_H)^N \xi(\alpha_H) \in \alpha_H^{\mathbf{m}(H)} S \text{ for all } H \in \mathcal{A} \text{ for } N \gg 0\}$$

and

$$D(\mathcal{A}, \mathbf{m})^W := D(\mathcal{A}, \mathbf{m}) \cap D(\mathcal{A}, -\infty)^W.$$

The **primitive filtration of** $D(\mathcal{A}, -\infty)^W$ **induced from m** *is given by*

$$R_k^{(\mathbf{m})} := \nabla_D^k D(\mathcal{A}, \mathbf{m})^W \quad (k \in \mathbb{Z}).$$

Note that

$$D(\mathcal{A}, \mathbf{m}) = \{\xi \in \mathrm{Der}_S \mid \xi(\alpha_H) \in \alpha_H^{\mathbf{m}(H)} S \text{ for all } H \in \mathcal{A}\}$$

if $\mathbf{m}(H) \geq 0$ for all $H \in \mathcal{A}$. In this case, $D(\mathcal{A}, \mathbf{m})$ was introduced by Ziegler [9].

Theorem 1.3 is translated into:

Theorem 4.4
The primitive filtration is an increasing filtration

$$\cdots \subset R_{-1}^{(\mathbf{m})} \subset R_0^{(\mathbf{m})} \subset R_1^{(\mathbf{m})} \subset \dots$$

such that

$$R_k^{(\mathbf{m})} = R_0^{(\mathbf{m}-2k)} = D(\mathcal{A}, \mathbf{m} - 2k)^W.$$

We construct bases for the primitive filtration of $D(\mathcal{A}, -\infty)^W$ induced from **1** by traslating Theorem 2.6 as follows:

Theorem 4.5
For any $k \in \mathbb{Z}$, $1 \leq i \leq t$ *and* $1 \leq j \leq \ell[i]$ *, define*

$$\xi_j^{(k)}[i] := \nabla_{D[i]}^k \left(I^*(dP_j[i])\right).$$

Let

$$\Xi^{(k)}[i] := \{\xi_j^{(k)}[i]\}_{1 \leq j \leq \ell[i]}, \ \Xi^{(k)} := \bigcup_{i=1}^{t} \Xi^{(k)}[i], \quad \text{and} \quad \Xi := \bigcup_{k \in \mathbb{Z}} \Xi^{(k)}.$$

Then

(1) the S-module $D(\mathcal{A}, -2k+1)$ is free with a basis $\Xi^{(k)}$,

(2) the R-module $D(\mathcal{A}, -2k+1)^W$ is free with a basis $\Xi^{(k)}$,

(3) the T-module $D(\mathcal{A}, -2k+1)^W$ is free with a basis $\bigcup_{p \le k} \Xi^{(p)}$, and

(4) the T-module $D(\mathcal{A}, \infty)^W$ is free with a basis Ξ.

References

[1] T. Abe, A generalized logarithmic module and duality of Coxeter multiarrangements. arXiv.0807.2552v1.

[2] T. Abe and H. Terao, A primitive derivation and logarithmic differential forms of Coxeter arrangements. *Math. Z.* **264** (2010), 813–828.

[3] C. Chevalley, Invariants of finite groups generated by reflections. *Amer. J. Math.* **77** (1955), 778–782.

[4] P. Orlik and H. Terao, *Arrangements of hyperplanes*. Grundlehren der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.

[5] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291.

[6] K. Saito, Uniformization of the orbifold of a finite reflection group. *RIMS preprint* **1414** (2003).

[7] L. Solomon and H. Terao, The double Coxeter arrangements. *Comment. Math. Helv.* **73** (1998), 237–258.

[8] H. Terao, The Hodge filtration and the contact-order filtration of derivations of Coxeter arrangements. *Manuscripta Math.* **118** (2005), 1–9.

[9] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness. in *Singularities* (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.