Critical points for spread-out self-avoiding walk, percolation and the contact process above the upper critical dimensions

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Abstract

We consider self-avoiding walk and percolation in $\mathbb{Z}^d$, oriented percolation in $\mathbb{Z}^d \times \mathbb{Z}_+$, and the contact process in $\mathbb{Z}^d$, with $p\,D(\,\cdot\,)$ being the coupling function whose range is denoted by $L < \infty$. For percolation, for example, each bond $\{x, y\}$ is occupied with probability $p\,D(y-x)$. The above models are known to exhibit a phase transition when the parameter p varies around a model-dependent critical point p_c. We investigate the value of p_c when $d > 6$ for percolation and $d > 4$ for the other models, and $L \gg 1$. We prove in a unified way that $p_c = 1 + C(D) + O(L^{-2d})$, where the universal term 1 is the mean-field critical value, and the model-dependent term $C(D) = O(L^{-d})$ is written explicitly in terms of the random walk transition probability D. We also use this result to prove that $p_c = 1 + cL^{-d} + O(L^{-d-1})$, where c is a model-dependent constant. Our proof is based on the lace expansion for each of these models.

1 Introduction and main results

Self-avoiding walk, percolation, and the contact process are well-known models that exhibit critical phenomena. For percolation in two or higher dimensions, for example, there exists a percolation threshold p_c^{pe} such that there is almost surely no infinite cluster for $p < p_c^{pe}$, while for $p > p_c^{pe}$ there is almost surely a unique infinite cluster. As $p \uparrow p_c^{pe}$, the average cluster size and the correlation length diverge. The precise value of p_c^{pe} depends on the details of the model, and is only explicitly known in a few cases, such as for two-dimensional nearest-neighbor bond percolation [20].

In this paper, we will consider self-avoiding walk, percolation, oriented percolation and the contact process, where the interaction range L is taken to be large. When $L \gg 1$, the interaction in the considered models is relatively weak, and therefore the critical values can be expected to be close to the critical value 1 of the respective mean-field models, i.e., random walk and branching

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random walk. We study the difference of the critical values and 1 for the above four models as $L \to \infty$. It turns out that, above the respective upper critical dimensions, we can write this difference to leading order as simple functions of the underlying random walk.

1.1 Models

First, we define the models. A self-avoiding walk is a path ω in the d-dimensional integer lattice $\mathbb{Z}^d$ with $\omega(i) \neq \omega(j)$ for every distinct $i, j \in \{0, 1, \ldots, |\omega|\}$. We also take the zero-step walk into account. We define the weight of a non-zero path ω by

$$W_p(\omega) = p^{|\omega|} \prod_{i=1}^{|\omega|} D\big(\omega(i) - \omega(i-1)\big), \tag{1.1}$$

where D is a probability distribution on $\mathbb{Z}^d$, and let $W_p(\omega) = 1$ if $|\omega| = 0$. We suppose that D is symmetric with respect to the lattice symmetries and that $D(o) = 0$, where o is the origin in $\mathbb{Z}^d$. A more detailed definition will be given below. The self-avoiding walk two-point function is defined by

$$\tau_p^{\text{sa}}(x) = \sum_{\substack{\omega: o \longrightarrow x \\ \text{saw}}} W_p(\omega), \tag{1.2}$$

where the sum is over all self-avoiding paths from o to x. It is known (see, e.g., [23]) that there is a critical value p_{c}^{sa} such that

$$\chi_p^{\text{sa}} = \sum_{x \in \mathbb{Z}^d} \tau_p^{\text{sa}}(x) \tag{1.3}$$

is finite if and only if $p < p_{\text{c}}^{\text{sa}}$ and diverges as $p \uparrow p_{\text{c}}^{\text{sa}}$.

For percolation, each bond $\{x, y\}$ is occupied with probability $p\,D(y-x)$ and vacant with probability $1 - p\,D(y-x)$, independently of the other bonds, where $p \in [0, \|D\|_\infty^{-1}]$. Since $\sum_x D(x) = 1$, the percolation parameter p is the expected number of occupied bonds per site. We denote by $\mathbb{P}_p$ the probability distribution of the bond variables. We say that x is connected to y, and write $x \longleftrightarrow y$, if either $x = y$ or there is a path of occupied bonds between x and y. The percolation two-point function and its sum over $\mathbb{Z}^d$ are denoted by

$$\tau_p^{\text{pe}}(x) = \mathbb{P}_p(o \longleftrightarrow x), \qquad \chi_p^{\text{pe}} = \sum_{x \in \mathbb{Z}^d} \tau_p^{\text{pe}}(x). \tag{1.4}$$

Similarly to self-avoiding walk, there is a critical value p_{c}^{pe} such that χ_p^{pe} is finite if and only if $p < p_{\text{c}}^{\text{pe}}$ and diverges as $p \uparrow p_{\text{c}}^{\text{pe}}$ (see, e.g., [8]).

Oriented percolation is a time-directed version of percolation. Each bond $((x, t), (y, t+1))$ is an ordered pair of sites in $\mathbb{Z}^d \times \mathbb{Z}_+$, and is occupied with probability $p\,D(y-x)$ and vacant with probability $1 - p\,D(y-x)$, independently of the other bonds, where $p \in [0, \|D\|_\infty^{-1}]$. We say that (x, s) is connected to (y, t), and write $(x, s) \longrightarrow (y, t)$, if either $(x, s) = (y, t)$ or there is an oriented path of occupied bonds from (x,s) to (y,t). Let $\mathbb{P}_p$ be the probability distribution of the bond variables. The oriented percolation two-point function and its sum over $\mathbb{Z}^d \times \mathbb{Z}_+$ are denoted by

$$\tau_p^{\rm op}(x,t) = \mathbb{P}_p((o,0) \longrightarrow (x,t)), \qquad\qquad \chi_p^{\rm op} = \sum_{t\in\mathbb{Z}_+} \sum_{x\in\mathbb{Z}^d} \tau_p^{\rm op}(x,t). \tag{1.5}$$

Also oriented percolation exhibits a phase transition such that $\chi_p^{\rm op} < \infty$ if and only if p is less than the critical value $p_{\rm c}^{\rm op}$, and that $\chi_p^{\rm op} \uparrow \infty$ as $p \uparrow p_{\rm c}^{\rm op}$ (see, e.g., [9]).

The contact process is a model of the spread of an infection in $\mathbb{Z}^d$, and is a continuous-time version of oriented percolation in $\mathbb{Z}^d \times \mathbb{R}_+$. We now describe a graphical representation for the contact process. Along each time line $\{x\} \times \mathbb{R}_+$, where $x \in \mathbb{Z}^d$, we place points according to a Poisson process with intensity 1, independently of the other time lines. For each ordered pair of distinct time lines from $\{x\} \times \mathbb{R}_+$ to $\{y\} \times \mathbb{R}_+$, we place oriented bonds $((x,t),(y,t))$, $t \geq 0$, according to a Poisson process with intensity $p\,D(y-x)$, independently of the other Poisson processes, where the parameter $p \geq 0$ is the infection rate. We say that (x,s) is connected to (y,t), and write $(x,s) \longrightarrow (y,t)$, if either $(x,s) = (y,t)$ or there is an oriented path in $\mathbb{Z}^d \times \mathbb{R}_+$ from (x,s) to (y,t) using the Poisson bonds and time-line segments traversed in the increasing-time direction without traversing the Poisson points. Let $\mathbb{P}_p$ be the corresponding probability distribution. We denote the contact process two-point function and its integro-sum over $\mathbb{Z}^d \times \mathbb{R}_+$ by

$$\tau_p^{\rm cp}(x,t) = \mathbb{P}_p((o,0) \longrightarrow (x,t)), \qquad\qquad \chi_p^{\rm cp} = \int_0^\infty dt \sum_{x\in\mathbb{Z}^d} \tau_p^{\rm cp}(x,t). \tag{1.6}$$

Again there is a critical value $p_{\rm c}^{\rm cp}$ such that $\chi_p^{\rm cp}$ is finite if and only if $p < p_{\rm c}^{\rm cp}$ and diverges as $p \uparrow p_{\rm c}^{\rm cp}$ (see, e.g., [21]).

We will omit the superscript referring to the precise model, and write $p_{\rm c}$ when referring to the critical values in all models simultaneously. The goal in this paper is to study $p_{\rm c}$ when the range L of D is sufficiently large. In the proofs, we will have versions of D in mind which are such that $L^d D(Lx)$ is a discrete approximation of a function on $\mathbb{R}^d$. We will formalize this assumption on D in the following definition:

Definition 1.1. Let h be a probability distribution over $\mathbb{R}^d \backslash \{o\}$, which is invariant under rotations by $\pi/2$ and reflections in the coordination hyperplanes. We suppose that h is piecewise continuous, so that $\int_{\mathbb{R}^d} h(x)\, d^d x \equiv 1$ can be approximated by the Riemann sum $L^{-d}\sum_{x\in\mathbb{Z}^d} h(x/L)$. Then, we define

$$D(x) = \frac{h(x/L)}{\sum_{y\in\mathbb{Z}^d} h(y/L)}. \tag{1.7}$$

We will make heavy use of results proved elsewhere for the models under consideration. For these results, some further assumptions are made on D, of which we now list the most important ones. We require that there exist $c > 0$, $C < \infty$, $\eta \in (0,1)$ such that

$$\sup_{x\in\mathbb{Z}^d} D(x) \leq CL^{-d}, \qquad\qquad \eta \wedge (cL^2|k|^2) \leq 1 - \hat{D}(k) \leq 2 - \eta, \tag{1.8}$$

where $\hat{D}(k) = \sum_{x\in\mathbb{Z}^d} D(x)\, e^{ik\cdot x}$ and $|k|^2 = \sum_{j=1}^d k_j^2$. There are a few more minor requirements that depend on the precise model under investigation. For details, see [11] for percolation and [14, 15, 16, 17] for the other three models, for which the requirements are virtually identical. A simple example of D, where all the above assumptions are satisfied, is

$$D(x) = \frac{\mathbb{1}_{\{0<\|x\|_\infty\le L\}}}{(2L+1)^d-1}, \tag{1.9}$$

for which $h(x) = 2^{-d}$ if $0 < \|x\|_\infty \le 1$ and $h(x) = 0$ otherwise.

We denote by $D * G$ the convolution of D and a function G in $\mathbb{Z}^d$, and by D^{*n} the n-fold convolution of D in $\mathbb{Z}^d$, where we define $D^{*0}(x) = \delta_{o,x}$. We will frequently use

$$D^{*n}(x) \le \delta_{0,n}\delta_{o,x} + \frac{O(\beta)}{(1\vee n)^{d/2}}, \tag{1.10}$$

where

$$\beta = L^{-d}. \tag{1.11}$$

The inequality (1.10) is a consequence of (1.8), as we will show in Appendix A.

1.2 Main results

Let d_c denote the respective upper critical dimensions, i.e., $d_\mathrm{c} = 6$ for percolation and $d_\mathrm{c} = 4$ for the other three models. In this paper, we investigate the respective critical values when $d > d_\mathrm{c}$ and $L \gg 1$, in a unified fashion.

Theorem 1.1. *For each model with $d > d_\mathrm{c}$, as $L \to \infty$,*

$$p_\mathrm{c}^{\mathrm{sa}},\ p_\mathrm{c}^{\mathrm{cp}} = 1 + \sum_{n=2}^{\infty} D^{*n}(o) + O(\beta^2), \tag{1.12}$$

$$p_\mathrm{c}^{\mathrm{op}} = 1 + \frac{1}{2}\sum_{n=2}^{\infty} D^{*2n}(o) + O(\beta^2), \tag{1.13}$$

$$p_\mathrm{c}^{\mathrm{pe}} = 1 + D^{*2}(o) + \frac{1}{2}\sum_{n=3}^{\infty}(n+1)\, D^{*n}(o) + O(\beta^2). \tag{1.14}$$

The universal term 1 is the critical value for the mean-field models (random walk and branching random walk). Note that, by (1.10), the model-dependent terms in (1.12)–(1.14) are $O(\beta)$. In Section 1.3, we will intuitively explain why the model-dependent terms have the above respective forms.

We next compute the dependence on β more explicitly, and compute the coefficients of β in $p_\mathrm{c} - 1$ explicitly. For this, we let U be the uniform probability distribution over $[-1,1]^d \subset \mathbb{R}^d$, i.e., for $x \in \mathbb{R}^d$,

$$U(x) = 2^{-d}\mathbb{1}_{\{\|x\|_\infty\le 1\}}, \tag{1.15}$$

and denote by $U^{\star n}$ the n-fold convolution of U in $\mathbb{R}^d$. Then, the leading order coefficient in β for p_c is given in the following theorem:

Theorem 1.2. *Fix D as in* (1.9)*, and let $d > d_c$. As $L \to \infty$,*

$$p_c^{\rm sa},\ p_c^{\rm cp} = 1 + \beta \sum_{n=2}^{\infty} U^{\star n}(o) + O(\beta L^{-1}), \tag{1.16}$$

$$p_c^{\rm op} = 1 + \frac{\beta}{2} \sum_{n=2}^{\infty} U^{\star 2n}(o) + O(\beta L^{-1}), \tag{1.17}$$

$$p_c^{\rm pe} = 1 + \beta \bigg[U^{\star 2}(o) + \frac{1}{2} \sum_{n=3}^{\infty} (n+1)\, U^{\star n}(o) \bigg] + O(\beta L^{-1}). \tag{1.18}$$

We now comment on the relation between the asymptotics in Theorems 1.1–1.2. The advantage of Theorem 1.2 is that it is more concrete, and the continuum limit of the critical points appears explicitly. However, the error term in Theorem 1.1 is $O(\beta^2)$, while in Theorem 1.2 it is equal to $O(\beta L^{-1})$, which is much larger. In order to compute the critical value more precisely, Theorem 1.1 gives a much more powerful result, at the expense of having to compute the random walk terms appearing in its statement. In principle, it should be possible to compute the coefficients of $\beta L^{-1}, \beta L^{-2}, \dots, \beta L^{-d+1}$, but this requires a substantial amount of work. Finally, it should be possible to compute the random walk sums in Theorem 1.1 for other examples than the one in (1.9), but we refrain from doing so.

We now summarize previous results on the critical values. We start with self-avoiding walk. Penrose's result in [25] implies that the critical value for self-avoiding walk defined by (1.9) with $L \gg 1$ satisfies

$$1 + c\,\beta^{2/7} \log \beta^{-1} \geq p_c^{\rm sa} \geq \begin{cases} 1, & \text{if } d \geq 3, \\ 1 + c'\beta \log \beta^{-1}, & \text{if } d = 2, \\ 1 + c''\beta^{4/5}, & \text{if } d = 1, \end{cases} \tag{1.19}$$

for some β-independent constants c, c', c''. For *spread-out lattice trees*, a related result with a different leading term, namely $= e^{-1}$, was also obtained in [25]. For $d > 4$, Madras and Slade [23, Corollary 6.2.7] improved (1.19) to $p_c^{\rm sa} = 1 + O(\beta)$. In [15, 17], this result was extended to more general D as defined in Definition 1.1. We will rely on the results in [15, 17], whose proof is based on the *lace expansion* and a generalized inductive approach. We will also use the lace expansion to derive the expression of the $O(\beta)$ term in (1.12).

For percolation, the best previous result is $p_c^{\rm pe} = 1 + O(\beta^{2/d-\epsilon})$ for $d > 6$ and $L \gg 1$, where $\epsilon > 0$ is an arbitrarily small number [10]. However, if we combine Lemma 3.1 proved below and the estimates for the lace expansion in [11], then we obtain the better estimate $p_c^{\rm pe} = 1 + O(\beta)$. The result in (1.14), which is also obtained by an application of the lace expansion, identifies the expression of this $O(\beta)$ term.

When $d > 4$ and $L \gg 1$, both $p_c^{\rm op}$ and $p_c^{\rm cp}$ were proved to be $1 + O(\beta)$ [14, 15, 16]. Similarly to self-avoiding walk, the proofs of these results rely on the lace expansion and an adaptation of the inductive approach. The contact process defined in terms of D of (1.9) was first considered by

Bramson, Durrett and Swindle [4], and they proved that, as $L \to \infty$,

$$p_c^{cp} - 1 \asymp f(\beta) \equiv \begin{cases} \beta, & \text{if } d \geq 3, \\ \beta \log \beta^{-1}, & \text{if } d = 2, \\ \beta^{2/3}, & \text{if } d = 1, \end{cases} \tag{1.20}$$

where $p_c^{cp} - 1 \asymp f(\beta)$ means that the ratio $(p_c^{cp} - 1)/f(\beta)$ is bounded away from zero and infinity. Later, Durrett and Perkins [7] proved that

$$\lim_{L\to\infty} \frac{p_c^{cp} - 1}{f(\beta)} = \begin{cases} \sum_{n=2}^{\infty} U^{\star n}(o), & \text{if } d \geq 3, \\ 3/(2\pi), & \text{if } d = 2. \end{cases} \tag{1.21}$$

Our result (1.16) in Theorem 1.2 is stronger when $d > 4$ in the sense that not only the coefficient of β, but also the speed of convergence in (1.21) is identified. In [14], we also obtained certain lace expansion results for a local mean-field limit, where the range and time grow large simultaneously, for the contact process in $d \leq 4$, and we expect that these results could be used to prove a stronger version of (1.21) for $d = 3, 4$, as well as for oriented percolation. However, this will need serious work using block constructions as used in [7].

We expect that (1.12)–(1.14) remain valid for $d = d_c - 1$ and d_c when we change $O(\beta^2)$ to $o(\beta)$. As mentioned above, this is the case for the contact process [7]. When $d \leq d_c - 2$, the second terms in (1.12)–(1.14) diverge, so that Theorem 1.1 cannot hold. However, we expect that the asymptotics of the critical point will, as for the contact process, again be described by the divergence of the sums in (1.12)–(1.14).

When $d > d_c$, we expect that the $O(\beta^2)$ terms could be identified in terms of D as well, using a similar method as in this paper, but to do so will require a serious amount of work.

A related problem is to obtain the asymptotics of the critical points for the nearest-neighbor models, when $D(x) = (2d)^{-1}\mathbb{1}_{\{|x|=1\}}$ and $d \to \infty$. In [12], p_c^{sa} was proved to have an asymptotic expansion into powers of $(2d)^{-1}$, and the first six coefficients were obtained. For unoriented percolation, the first three coefficients were computed in [12] and [18], but the proof of the asymptotic expansion only appeared in [19]. The proofs of these results are again based on the lace expansion. For nearest-neighbour oriented percolation and the nearest-neighbour contact process, it is proved that $p_c^{op} = 1 + O(d^{-2})$ (see [6]) and $p_c^{cp} = 1 + O(d^{-1})$ (see, e.g., [22]), using different methods.

1.3 Overview of the proof

To prove Theorem 1.1, we will apply the lace expansion (see, e.g., [11, 14, 17, 23, 24]). For example, the lace expansion for self-avoiding walk gives the recurrence relation

$$\tau_p^{sa}(x) = \delta_{o,x} + \sum_v [p\, D(v) + \Pi_p^{sa}(v)]\, \tau_p^{sa}(x - v), \tag{1.22}$$

where $\Pi_p^{sa}(x)$ is a certain expansion coefficient. It was proved in [15, 17] that $\hat{\Pi}_p^{sa} \equiv \sum_x \Pi_p^{sa}(x) = O(\beta)$ for $p \leq p_c^{sa}$, if $d > 4$ and $L \gg 1$ (see Section 2). Summing both sides of (1.22) over $x \in \mathbb{Z}^d$

and solving the resulting equation in terms of χ_p^{sa}, we obtain

$$\chi_p^{\mathrm{sa}} = (1 - p - \hat{\Pi}_p^{\mathrm{sa}})^{-1}, \tag{1.23}$$

and thus

$$p_{\mathrm{c}}^{\mathrm{sa}} = 1 - \hat{\Pi}_{p_{\mathrm{c}}^{\mathrm{sa}}}^{\mathrm{sa}}. \tag{1.24}$$

To estimate $p_{\mathrm{c}}^{\mathrm{sa}}$, we thus need to investigate $\hat{\Pi}_{p_{\mathrm{c}}^{\mathrm{sa}}}^{\mathrm{sa}}$. We will prove that, since $p_{\mathrm{c}}^{\mathrm{sa}} = 1 + O(\beta)$, we can replace $\hat{\Pi}_{p_{\mathrm{c}}^{\mathrm{sa}}}^{\mathrm{sa}}$ by $\hat{\Pi}_1^{\mathrm{sa}}$ up to an error of order $O(\beta^2)$. When $p = 1$, the exponentially growing factor $p^{|\omega|}$ in (1.1) does not play any role, and $\hat{\Pi}_1^{\mathrm{sa}}$ can be investigated in terms of random walks. This is the key ingredient for the proof of Theorem 1.1.

The strategy for percolation models is the same as above. There is a similar recursion relation to (1.22), with some model-dependent expansion coefficient $\Pi_p(x)$. Therefore, to obtain the formulae in Theorem 1.1, we will have to investigate $\hat{\Pi}_1 = \sum_x \Pi_1(x)$, again in terms of random walks.

As we will explain in Sections 2–3, $\hat{\Pi}_1^{\mathrm{sa}}$ and $\hat{\Pi}_1$ can be described by an alternating sum of a model-dependent sequence $\hat{\pi}_1^{(N)}$ for $N \geq 0$, where $\hat{\pi}_1^{(N)}$ for $N \geq 1$ decays as β^N for all models. For self-avoiding walk, $\hat{\pi}_1^{(0)}$ equals zero, while $\hat{\pi}_1^{(0)}$ for percolation models is nearly a half of $\hat{\pi}_1^{(1)}$. (This is why we have the factor $\frac{1}{2}$ in (1.13)–(1.14).) Therefore, roughly speaking, we only need to investigate $\hat{\pi}_1^{(1)}$ to obtain (1.12)–(1.14). We will show later that the diagrammatic interpretation of $\hat{\pi}_1^{(1)}$ for self-avoiding walk is a single random walk taking more than one step and going back to the starting point (cf., (1.12)), while the diagrammatic interpretation of $\hat{\pi}_1^{(1)}$ for percolation models is that two random walks, at least one of which is non-vanishing, meet at some point. Therefore, the correction to the mean-field value 1 are related to *random walk loops*.

For loops in the time-oriented percolation models, the lengths in the time-increasing direction of these two walks have to be equal (which explains the sum over even convolution powers in (1.13)), while for unoriented percolation this is not the case (which explain the sum over all powers and the factor $n + 1$ in (1.14)).

For the contact process, the two paths are *continuous time* random walk paths, for which the number of convolution powers of D is equal to the number of spatial steps made by the random walk up to a given time, which has a Poisson distribution. Therefore, the sum over the convolution powers of D is not restricted to even powers, and we see that the correction to the mean-field value for the contact process and oriented percolation are different. For the contact process, it will turn out that also the factor $\frac{1}{2}$ in (1.13) disappears, which is due to the fact that the two walks are in fact *avoiding* each other, and which will be explained in more detail in Section 3.1. This is an intuitive explanation of the model-dependent terms in (1.12)–(1.14).

We organize the rest of this paper as follows. We begin with self-avoiding walk in Section 2, and explain the key steps to estimate $p_{\mathrm{c}}^{\mathrm{sa}}$. Following the same steps, we estimate $p_{\mathrm{c}}^{\mathrm{op}}$ and $p_{\mathrm{c}}^{\mathrm{cp}}$ in Section 3.1, and $p_{\mathrm{c}}^{\mathrm{pe}}$ in Section 3.2. Finally, we prove an extension of (1.10) in Appendix A, and Theorem 1.2 in Appendix B.

2 Critical point for self-avoiding walk

In this section, we prove (1.12), using (1.24). Throughout this section, we will omit the superscript "sa" and write, e.g., $p_c = p_c^{sa}$ and $\hat{\Pi}_p = \hat{\Pi}_p^{sa}$.

Before computing the asymptotics of $\hat{\Pi}_{p_c}$ in (1.24), we first note that $p_c \geq 1$. This is because the removal of the self-avoidance constraint in (1.2) results in $\sum_{\omega:o \longrightarrow x} W_p(\omega)$, whose sum over $x \in \mathbb{Z}^d$ equals $(1-p)^{-1}$ for any $p \leq 1$. For self-avoiding walk,

$$\Pi_p(x) = \sum_{N=1}^{\infty} (-1)^N \pi_p^{(N)}(x), \tag{2.1}$$

where, e.g., $\pi_p^{(1)}(x)$ is a "1-loop diagram" at the origin [23]:

$$\pi_p^{(1)}(x) = \delta_{o,x}\ (pD * \tau_p)(o) = \delta_{o,x} \sum_{\substack{\omega:o \longrightarrow o \\ |\omega| \geq 1}} W_p(\omega)\, I(\omega), \tag{2.2}$$

where $I(\omega) = 1$ if there are no self-intersection points except for $\omega(0) = \omega(|\omega|)$, otherwise $I(\omega) = 0$.

For $d > 4$ and $L \gg 1$, it was proved in [17] that, for $\hat{\pi}_p^{(N)} = \sum_x \pi_p^{(N)}(x)$, we have

$$\hat{\pi}_p^{(N)} \leq O(\beta)^N, \qquad\qquad p\, \partial_p \hat{\Pi}_p \leq O(\beta), \tag{2.3}$$

for all $p \leq p_c$ and $N \geq 1$. Together with (1.24) and (2.1), we immediately obtain that $p_c = 1 + O(\beta)$. Moreover, by the mean-value theorem, there is a $p \in (1, p_c)$ such that

$$p_c = 1 - \hat{\Pi}_1 - (\hat{\Pi}_{p_c} - \hat{\Pi}_1) = 1 - \hat{\Pi}_1 - (p_c - 1)\ \partial_p \hat{\Pi}_p = 1 + \hat{\pi}_1^{(1)} + O(\beta^2), \tag{2.4}$$

where

$$\hat{\pi}_1^{(1)} = \sum_{\substack{\omega:o \longrightarrow o \\ |\omega| \geq 1}} W_1(\omega)\, I(\omega) = \sum_{n=2}^{\infty} D^{*n}(o) - \sum_{\substack{\omega:o \longrightarrow o \\ |\omega| \geq 1}} W_1(\omega)\ [1 - I(\omega)]. \tag{2.5}$$

To complete the proof of (1.12), it thus suffices to prove that the second term in the right-hand side of (2.5) is $O(\beta^2)$ if $d > 4$. We first note that $I(\omega)$ is an indicator function. If $I(\omega) = 0$, so that $1 - I(\omega) = 1$, then there must be a pair $\{s, t\} \neq \{0, |\omega|\}$ with $0 \leq s < t \leq |\omega|$ such that $\omega(s) = \omega(t)$. Denoting the parts of ω corresponding to these three time intervals by ω_i, $i = 1, 2, 3$, respectively, we obtain

$$\sum_{\substack{\omega:o \longrightarrow o \\ |\omega| \geq 1}} W_1(\omega)\ [1 - I(\omega)] \leq \sum_{x \in \mathbb{Z}^d} \sum_{\substack{\omega_1, \omega_3 : o \longrightarrow x \\ |\omega_1| + |\omega_3| \geq 1}} \sum_{\substack{\omega_2 : x \longrightarrow x \\ |\omega_2| \geq 1}} \prod_{i=1}^{3} W_1(\omega_i) = (D * G^{*2})(o)\ (D^{*2} * G)(o), \tag{2.6}$$

where $G(x) = \sum_{n=0}^{\infty} D^{*n}(x)$, and $(D^{*2} * G)(o) = \sum_{n=2}^{\infty} D^{*n}(o) = O(\beta)$ if $d > 2$. Moreover, by (1.10),

$$(D * G^{*2})(o) = \sum_{n=1}^{\infty} n D^{*n}(o) = O(\beta) \tag{2.7}$$

if $d > 4$. This completes the proof of (1.12) for self-avoiding walk. □

3 Critical points for percolation models

In this section, we compute the asymptotics of the critical values for the other three models, and thus complete the proof of Theorem 1.1.

To discuss oriented percolation and the contact process simultaneously, it is convenient to introduce the following oriented percolation on $\mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$, which is the time-discretized contact process with a discretization parameter $\varepsilon \in (0, 1]$. A bond is a directed pair $((x, t), (y, t + \varepsilon))$ of sites in $\mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$. Each bond is either occupied or vacant, independently of the other bonds, and a bond $((x, t), (y, t + \varepsilon))$ is occupied with probability

$$q_p(y - x) = \begin{cases} 1 - \varepsilon, & \text{if } x = y, \\ p\varepsilon D(y - x), & \text{if } x \neq y, \end{cases} \tag{3.1}$$

provided that $\sup_x q_p(x) \leq 1$. In this notation, the model with $\varepsilon = 1$ is the usual oriented percolation model as defined in Section 1.1, and the weak limit as $\varepsilon \downarrow 0$ is the contact process [3]. Similarly to oriented percolation with $\varepsilon = 1$, for each $\varepsilon \in (0, 1]$, there is a critical value $p_c^{(\varepsilon)}$ for every $\varepsilon \in (0, 1]$, such that $p_c^{(1)} = p_c^{\mathrm{op}}$ and $\lim_{\varepsilon\downarrow 0} p_c^{(\varepsilon)} = p_c^{\mathrm{cp}}$ [26]. We will call the model with $\varepsilon \in (0, 1]$ the *time-discretized contact process*.

To summarise notation for percolation and the time-discretized contact process, we will write $\Lambda = \mathbb{Z}^d$ for percolation and $\Lambda = \mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$ for oriented percolation. For notational convenience, we will take $\varepsilon = 1$ for percolation. We will also use bold letters to represent elements of Λ. For example, $\boldsymbol{o} = o$, $\boldsymbol{x} = x$ for percolation, and $\boldsymbol{o} = (o, 0)$, $\boldsymbol{x} = (x, t)$ for the time-discretized contact process. For a bond $b = (\boldsymbol{u}, \boldsymbol{v})$, we write $\underline{b} = \boldsymbol{u}$ and $\overline{b} = \boldsymbol{v}$. We also omit the superscripts ε, pe, op and cp, if no confusion can arise.

As mentioned in Section 1, the lace expansion for percolation models takes a similar form as in (1.22), and reads (see, e.g., [11, 14])

$$\tau_p(\boldsymbol{x}) = [\delta_{\boldsymbol{o},\boldsymbol{x}} + \Pi_p(\boldsymbol{x})] + \sum_{\boldsymbol{u},\boldsymbol{v}\in\Lambda} [\delta_{\boldsymbol{o},\boldsymbol{u}} + \Pi_p(\boldsymbol{u})]\, q_p(\boldsymbol{v} - \boldsymbol{u})\, \tau_p(\boldsymbol{x} - \boldsymbol{v}). \tag{3.2}$$

In particular, $q_p(\boldsymbol{v} - \boldsymbol{u}) = p\, D(v - u)$ for percolation and oriented percolation for which $\varepsilon = 1$. (To unify notation, we recall that we regard unoriented percolation as a model with $\varepsilon = 1$.) The lace expansion coefficient $\Pi_p(\boldsymbol{x})$ equals

$$\Pi_p(\boldsymbol{x}) = \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(\boldsymbol{x}), \tag{3.3}$$

where $\pi_p^{(N)}(\boldsymbol{x})$, $N \geq 0$, are model-dependent diagram functions. The result of the lace expansion will be explained in Sections 3.1–3.2. For the time-discretized contact process with $\varepsilon \in (0, 1]$, $d > d_c$ and $L \gg 1$, it has been proved [14, 16] that $\hat{\Pi}_p \equiv \varepsilon \sum_{\boldsymbol{x}\in\Lambda} \Pi_p(\boldsymbol{x})$ is $O(\beta)\, \varepsilon^2$ for all $p \leq p_c$. The same estimate is proved to hold for unoriented percolation (with $\varepsilon = 1$), using the lace expansion in [11] and Lemma 3.1 proved below in Section 3.2.

As in the derivation of (1.23), solving (3.2) in terms of $\chi_p = \varepsilon \sum_{\boldsymbol{x}\in\Lambda} \tau_p(\boldsymbol{x})$ gives

$$\chi_p = \frac{1 + \frac{1}{\varepsilon}\hat{\Pi}_p}{1 - p - (1 - \varepsilon + p\varepsilon)\frac{1}{\varepsilon^2}\hat{\Pi}_p}, \tag{3.4}$$

and thus, equating the denominator to zero,

$$p_c = 1 - \frac{1}{\varepsilon^2}\hat{\Pi}_{p_c} - (p_c - 1)\frac{1}{\varepsilon}\hat{\Pi}_{p_c}. \tag{3.5}$$

This expression holds uniformly in ε. We will use it to compute $p_c^{\rm op}$ and $p_c^{\rm pe}$ by taking $\varepsilon = 1$ and $p_c^{\rm cp}$ by taking the limit when $\varepsilon \downarrow 0$ [26], respectively. In particular, the third term is $O(\beta^2)$ when $\varepsilon = 1$, and it has no contribution in the limit $\varepsilon \downarrow 0$. Therefore, we are left to prove that, apart from an error term of order $O(\beta^2)$, the second term in (3.5) equals the second term in (1.12) when $\varepsilon \downarrow 0$, and equals the second term in (1.13) for oriented percolation and that in (1.14) for (unoriented) percolation when $\varepsilon = 1$. We again note that $p_c^{(\varepsilon)} \geq 1$, since $\chi_p \leq \varepsilon \sum_{n=0}^{\infty} \sum_x q_p^{*n}(x) = (1-p)^{-1}$ for $p \leq 1$. In addition, similarly to (1.10), when $p = 1$ and $\varepsilon < 1$, we have

$$q^{*n}(x) \equiv q_1^{*n}(x) \leq (1-\varepsilon)^n\, \delta_{o,x} + \frac{O(\beta)}{[1 \vee (n\varepsilon)]^{d/2}}. \tag{3.6}$$

We will prove (3.6) in Appendix A. Note that when $\varepsilon = 1$, (3.6) reduces to (1.10).

To complete the proof of Theorem 1.1, we investigate $\hat{\Pi}_{p_c}$ for oriented percolation and the contact process in Section 3.1, and for unoriented percolation in Section 3.2.

3.1 Asymptotics of $p_c^{\rm op}$ and $p_c^{\rm cp}$

In this section, we investigate $\hat{\Pi}_{p_c}$ for the discretized contact process, and derive (1.13) for oriented percolation (i.e., $\varepsilon = 1$) and (1.12) for the contact process (i.e., $\varepsilon \downarrow 0$).

To describe the diagram functions $\pi_p^{(N)}(\boldsymbol{x})$, $N \geq 0$, we need some definitions. We say that $\boldsymbol{x}$ is *doubly connected to* $\boldsymbol{y}$, if either $\boldsymbol{x} = \boldsymbol{y}$ or there are at least two nonzero bond-disjoint occupied paths from $\boldsymbol{x}$ to $\boldsymbol{y}$. Following the notation in [16] as closely as possible, we denote this event by $\boldsymbol{x} \Longrightarrow \boldsymbol{y}$, and define

$$\hat{\pi}_p^{(0)} = \varepsilon \sum_{\boldsymbol{x}\in\Lambda} \pi_p^{(0)}(\boldsymbol{x}), \qquad \text{where} \qquad \pi_p^{(0)}(\boldsymbol{x}) = \mathbb{P}_p(\boldsymbol{o} \Longrightarrow \boldsymbol{x}) - \delta_{\boldsymbol{o},\boldsymbol{x}}. \tag{3.7}$$

If $\boldsymbol{o}$ is connected but not doubly connected to $\boldsymbol{x}$, there is a *pivotal bond* $b = (\underline{b}, \overline{b})$ for $\boldsymbol{o} \longrightarrow \boldsymbol{x}$ such that both $\boldsymbol{o} \longrightarrow \underline{b}$ and $\overline{b} \longrightarrow \boldsymbol{x}$ occur, and that $\boldsymbol{o} \longrightarrow \boldsymbol{x}$ occurs if and only if b is set occupied. For $A \subseteq \Lambda$, we say that $\boldsymbol{y}$ *is connected to* $\boldsymbol{x}$ *through* A when every occupied path from $\boldsymbol{y} \longrightarrow \boldsymbol{x}$ has at least one bond with an endpoint in A. We define $E(b, \boldsymbol{x}; A)$ to be the event that b is occupied, that $\overline{b} \longrightarrow \boldsymbol{x}$ through A, and that there are no pivotal bonds b' for $\overline{b} \longrightarrow \boldsymbol{x}$ such that $\overline{b} \longrightarrow \underline{b}'$ through A. Let $\tilde{C}^b(\boldsymbol{o})$ be the set of vertices in Λ connected from $\boldsymbol{o}$ without using b. Then,

$$\hat{\pi}_p^{(1)} = \varepsilon \sum_{\boldsymbol{x}\in\Lambda} \pi_p^{(1)}(\boldsymbol{x}), \qquad \text{where} \qquad \pi_p^{(1)}(\boldsymbol{x}) = \sum_b \mathbb{P}_p\big(\boldsymbol{o} \Longrightarrow \underline{b};\ E(b, \boldsymbol{x}; \tilde{C}^b(\boldsymbol{o}))\big). \tag{3.8}$$

The higher order diagram functions $\pi_p^{(N)}(\boldsymbol{x})$, $N \geq 2$, are defined in a similar way, but are irrelevant in this paper (see [14, Section 3] for a complete definition, with slightly different notation).

For $d > 4$ and $L \gg 1$, it was proved in [14] that, for $\hat{\pi}_p^{(N)} = \varepsilon \sum_{\boldsymbol{x}\in\Lambda} \pi_p^{(N)}(\boldsymbol{x})$, we have

$$\hat{\pi}_p^{(N)} \leq O(\beta)^{N\vee 1}\,\varepsilon^2, \qquad\qquad p\,\partial_p\hat{\Pi}_p \leq O(\beta)\,\varepsilon^2, \tag{3.9}$$

for all $p \leq p_{\rm c}$ and $N \geq 0$. Together with (3.3) and (3.5), we obtain $p_{\rm c} = 1 + O(\beta)$. Moreover, by the mean-value theorem, there is a $p \in (1, p_{\rm c})$ such that

$$\begin{aligned} p_{\rm c} = 1 - \frac{1}{\varepsilon^2}\hat{\Pi}_{p_{\rm c}} - (p_{\rm c}-1)\frac{1}{\varepsilon}\hat{\Pi}_{p_{\rm c}} &= 1 - \frac{1}{\varepsilon^2}\hat{\Pi}_1 - (p_{\rm c}-1)\frac{1}{\varepsilon^2}\partial_p\hat{\Pi}_p + O(\beta^2)\,\varepsilon \\ &= 1 - \frac{1}{\varepsilon^2}\hat{\pi}_1^{(0)} + \frac{1}{\varepsilon^2}\hat{\pi}_1^{(1)} + O(\beta^2). \end{aligned} \tag{3.10}$$

To prove (1.12)–(1.13), it thus suffices to investigate $\hat{\pi}_1^{(0)}$ and $\hat{\pi}_1^{(1)}$.

Analysis of $\hat{\pi}_1^{(0)}$. We prove

$$\frac{1}{\varepsilon^2}\hat{\pi}_1^{(0)} \begin{cases} = \frac{1}{2}\sum_{n=2}^{\infty} D^{*2n}(o) + O(\beta^2), & \text{for } \ \varepsilon = 1, \\ \to \sum_{n=2}^{\infty} D^{*n}(o) + O(\beta^2), & \text{when } \ \varepsilon \downarrow 0. \end{cases} \tag{3.11}$$

Recall (3.7). To describe a double connection by a pair of two random walk paths, we order the support of D in an arbitrary but fixed manner. For x, y in the support of D, we write $x \prec y$ if x is *lower* than y in that order. For a pair of paths consisting of bonds in Λ, $\omega = (b_1, \dots, b_N)$ and $\omega' = (b_1', \dots, b_N')$ with $\underline{b}_1 = \underline{b}_1'$ and $\overline{b}_N = \overline{b}_N'$, we say that ω is *lower* than ω', denoted by $\omega \prec \omega'$, if at the first time $n \in \{1, \dots, N\}$ when ω is incompatible with ω' (therefore $b_i = b_i'$ for all $i < n$) we have $\overline{b}_n \prec \overline{b}_n'$. We also say that ω_2 is *higher* than ω_1.

A path $\omega = (b_1, \dots, b_{|\omega|})$ is said to be *occupied* if all bonds along ω are occupied. We define $E_\prec(\omega)$ to be the event that ω is the lowest occupied path among all occupied paths from $\underline{b}_1$ to $\overline{b}_{|\omega|}$, and that there is another occupied path ω' from $\underline{b}_1$ to $\overline{b}_{|\omega|}$ which is bond-disjoint from ω (denoted by $\omega \cap \omega' = \varnothing$). Given a path ω, we also define $E_\succ(\omega';\omega)$ to be the event that ω' is the highest occupied path among all occupied paths from $\underline{b}_1$ to $\overline{b}_{|\omega|}$ that are bond-disjoint from ω. Such an occupied path ω' exists on $\{\underline{b}_1 \Longrightarrow \overline{b}_{|\omega|}\} \cap E_\prec(\omega)$ by definition.

Using the above notation, we have, for $\boldsymbol{x} \neq \boldsymbol{o}$,

$$\{\boldsymbol{o} \Longrightarrow \boldsymbol{x}\} = \dot{\bigcup_{\substack{\omega_1,\omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x} \\ \omega_1\cap\omega_2=\varnothing \\ \omega_1\prec\omega_2}}} \big\{\omega_1, \omega_2 \text{ occupied; } E_\prec(\omega_1) \cap E_\succ(\omega_2;\omega_1)\big\}. \tag{3.12}$$

We define the right-hand side to be empty if $\boldsymbol{x} = \boldsymbol{o}$. Then,

$$\hat{\pi}_1^{(0)} = \varepsilon \sum_{\boldsymbol{x}\in\Lambda} \sum_{\substack{\omega_1,\omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x} \\ \omega_1\cap\omega_2=\varnothing \\ \omega_1\prec\omega_2}} \mathbb{P}_1\big(\omega_1, \omega_2 \text{ occupied; } E_\prec(\omega_1) \cap E_\succ(\omega_2;\omega_1)\big). \tag{3.13}$$

Since $\mathbb{P}_1$ is a product measure, if we ignore $E_{\prec}(\omega_1) \cap E_{\succ}(\omega_2;\omega_1)$, then we obtain

$$\sum_{\substack{\omega_1,\omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x}\\ \omega_1\cap\omega_2=\varnothing\\ \omega_1\prec\omega_2}} \mathbb{P}_1(\omega_1,\omega_2 \text{ occupied})$$
$$= \sum_{\substack{u,v:u\prec v\\ y,z:y\neq z}} q(u)\,q(v)\,q(x-y)\,q(x-z) \sum_{\substack{\omega_1:\boldsymbol{u}\longrightarrow\boldsymbol{y}\\ \omega_2:\boldsymbol{v}\longrightarrow\boldsymbol{z}\\ \omega_1\cap\omega_2=\varnothing}} \mathbb{P}_1(\omega_1 \text{ occupied})\ \mathbb{P}_1(\omega_2 \text{ occupied}), \tag{3.14}$$

where $\boldsymbol{u} = (u,\varepsilon)$, $\boldsymbol{v} = (v,\varepsilon)$, $\boldsymbol{y} = (y, t-\varepsilon)$, $\boldsymbol{z} = (z, t-\varepsilon)$, and $q(x) = q_1(x)$ (cf., (3.6)). By an inclusion-exclusion relation, the correction is bounded by

$$\sum_{\substack{\omega_1,\omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x}\\ \omega_1\cap\omega_2=\varnothing\\ \omega_1\prec\omega_2}} \Big[\mathbb{P}_1\big(\omega_1,\omega_2 \text{ occupied};\ E_{\prec}(\omega_1)^{\mathrm{c}}\big) + \mathbb{P}_1\big(\omega_1,\omega_2 \text{ occupied};\ E_{\succ}(\omega_2;\omega_1)^{\mathrm{c}}\big)\Big].$$

We will prove below that, for E equal to $E_{\prec}(\omega_1)$ or $E_{\succ}(\omega_2;\omega_1)$,

$$\varepsilon \sum_{\boldsymbol{x}\in\Lambda} \sum_{\substack{\omega_1,\omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x}\\ \omega_1\cap\omega_2=\varnothing}} \mathbb{P}_1(\omega_1,\omega_2 \text{ occupied};\ E^{\mathrm{c}}) = O(\beta^2)\,\varepsilon^2. \tag{3.15}$$

We investigate (3.14) to obtain the expression of $O(\beta)$ from (3.13). If we ignore the restriction $\omega_1 \cap \omega_2 = \varnothing$, then we obtain

$$\sum_{\substack{u,v:u\prec v\\ y,z:y\neq z}} q(u)\,q(v)\,q(x-y)\,q(x-z)\ q^{*(t/\varepsilon-2)}(y-u)\ q^{*(t/\varepsilon-2)}(z-v), \tag{3.16}$$

where $t/\varepsilon \in [2,\infty) \cap \mathbb{Z}_+$. We will prove below that the correction satisfies

$$\varepsilon \sum_{\boldsymbol{x}\in\Lambda} \sum_{\substack{u,v:u\prec v\\ y,z:y\neq z}} q(u)\,q(v)\,q(x-y)\,q(x-z) \sum_{\substack{\omega_1:\boldsymbol{u}\longrightarrow\boldsymbol{y}\\ \omega_2:\boldsymbol{v}\longrightarrow\boldsymbol{z}\\ \omega_1\cap\omega_2\neq\varnothing}} \mathbb{P}_1(\omega_1 \text{ occupied})\ \mathbb{P}_1(\omega_2 \text{ occupied}) = O(\beta^2)\,\varepsilon^2. \tag{3.17}$$

Therefore, we only need to consider the contribution to (3.13) from (3.16). By changing variables as $y' = x - y$ and $z' = x - z$ and using the symmetry between $u \prec v$ and $u \succ v$, the sum of (3.16) over $x \in \mathbb{Z}^d$ equals

$$\sum_{\substack{u,v:u\prec v\\ y',z':y'\neq z'}} q(u)\,q(v)\,q(y')\,q(z') \sum_{x} q^{*(t/\varepsilon-2)}(x-y'-u)\ q^{*(t/\varepsilon-2)}(x-z'-v)$$
$$= \frac{1}{2} \sum_{\substack{u,v:u\neq v\\ y,z:y\neq z}} q(u)\,q(v)\,q(y)\,q(z)\ q^{*(2t/\varepsilon-4)}(v+z-y-u). \tag{3.18}$$

Recall (3.1). Since there is at most one temporal (or vertical) bond growing out of every site in Λ, we must have $q(u) = \varepsilon D(u)$ or $q(v) = \varepsilon D(v)$, so that we obtain at least one factor of ε. By the same reason, we should have $q(y) = \varepsilon D(y)$ or $q(z) = \varepsilon D(z)$, so that we obtain a second factor of

ε. Therefore, the number of combinations for the product of four factors of q in (3.18) is nine: one combination is proportional to ε^4, four others are proportional to $(1-\varepsilon)\,\varepsilon^3$, and the remaining four are proportional to $(1-\varepsilon)^2\,\varepsilon^2$. Only the first case arises for oriented percolation for which $\varepsilon = 1$, while only the third case arises for the contact process for which $\varepsilon \downarrow 0$, respectively.

We first complete the proof of (3.11) for oriented percolation. When $\varepsilon = 1$, and using inclusion-exclusion on the restrictions $u \neq v$ and $y \neq z$, the sum of (3.18) over $t \geq 2$ equals

$$\frac{1}{2}\sum_{u,v,y,z} D(u)\,D(v)\,D(y)\,D(z)\sum_{t=2}^{\infty} D^{*(2t-4)}(v+z-y-u) + O(\beta^2) = \frac{1}{2}\sum_{t=2}^{\infty} D^{*2t}(o) + O(\beta^2), \tag{3.19}$$

where we use (1.8) to obtain an error of order $O(\beta^2)$ that comes from contributions where $u = v$ or $y = z$.

For the contact process, for which $\varepsilon \downarrow 0$, the leading contribution is due to the four combinations of order $(1-\varepsilon)^2\,\varepsilon^2$ mentioned above, where either u or v is o, and either y or z is o. Therefore, the coefficient of $(1-\varepsilon)^2\,\varepsilon^2$ in (3.18) is

$$\begin{aligned}\frac{1}{2}\bigg[&\sum_{u,y} D(u)\,D(y)\;q^{*(2t/\varepsilon-4)}(-y-u) + \sum_{u,z} D(u)\,D(z)\;q^{*(2t/\varepsilon-4)}(z-u)\\ &+\sum_{v,y} D(v)\,D(y)\;q^{*(2t/\varepsilon-4)}(v-y) + \sum_{v,z} D(v)\,D(z)\;q^{*(2t/\varepsilon-4)}(v+z)\bigg] = 2\big(D^{*2} * q^{*(2t/\varepsilon-4)}\big)(o).\end{aligned}$$

Summing this expression (multiplied by ε) over $t/\varepsilon \in [2,\infty)\cap\mathbb{Z}_+$ gives

$$\begin{aligned}2\int_{\square_\pi}\frac{d^dk}{(2\pi)^d}\;\hat{D}(k)^2\;\varepsilon\sum_{n=0}^{\infty}\big[1-\varepsilon+\varepsilon\hat{D}(k)\big]^{2n} &= \int_{\square_\pi}\frac{d^dk}{(2\pi)^d}\;\frac{2\hat{D}(k)^2}{[1-\hat{D}(k)][2-\varepsilon+\varepsilon\hat{D}(k)]}\\ &\xrightarrow{\varepsilon\downarrow 0}\int_{\square_\pi}\frac{d^dk}{(2\pi)^d}\;\frac{\hat{D}(k)^2}{1-\hat{D}(k)} = \sum_{n=2}^{\infty} D^{*n}(o),\end{aligned} \tag{3.20}$$

where $\square_\pi = [-\pi,\pi]^d$. This completes the proof of (3.11). $\square$

Analysis of $\hat{\pi}_1^{(1)}$. We prove that $\frac{1}{\varepsilon^2}\hat{\pi}_1^{(1)}$ is asymptotically twice as large as the right-hand side of (3.11):

$$\frac{1}{\varepsilon^2}\hat{\pi}_1^{(1)}\begin{cases} = \sum_{n=2}^{\infty} D^{*2n}(o) + O(\beta^2), & \text{for }\ \varepsilon = 1,\\ \to 2\sum_{n=2}^{\infty} D^{*n}(o) + O(\beta^2), & \text{when }\ \varepsilon\downarrow 0.\end{cases} \tag{3.21}$$

For a bond b, let $\{b \Longrightarrow \boldsymbol{x}\}$ be the event that b is occupied and $\overline{b} \Longrightarrow \boldsymbol{x}$. We define $\{\boldsymbol{u} \longrightarrow b\}$ and a joint event $\{\boldsymbol{u} \longrightarrow b \Longrightarrow \boldsymbol{x}\}$ similarly. For events E_1 and E_2, we denote by $E_1 \circ E_2$ the event that E_1 and E_2 occur *disjointly*, i.e., using disjoint bond sets of bonds (see e.g., [8, Section 2.3]). Recalling (3.8) and distinguishing between $\underline{b} = \boldsymbol{o}$ and $\underline{b} \neq \boldsymbol{o}$, we can rewrite $\hat{\pi}_1^{(1)}$ as

$$\begin{aligned}\hat{\pi}_1^{(1)} = \varepsilon\sum_{\boldsymbol{u},\boldsymbol{x}\in\Lambda}\mathbb{P}_1\big(\{(\boldsymbol{o},\boldsymbol{u}) \longrightarrow \boldsymbol{x}\}\circ\{\boldsymbol{o}\longrightarrow\boldsymbol{x}\}\big) + \varepsilon\sum_{\boldsymbol{x}\in\Lambda}\sum_{b:\underline{b}\neq\boldsymbol{o}}\mathbb{P}_1\big(\boldsymbol{o}\Longrightarrow\underline{b};\ E(b,\boldsymbol{x};\tilde{C}^b(\boldsymbol{o}))\big)\\ -\varepsilon\sum_{\boldsymbol{u},\boldsymbol{x}\in\Lambda}\mathbb{P}_1\Big(\big\{\{(\boldsymbol{o},\boldsymbol{u}) \longrightarrow \boldsymbol{x}\}\circ\{\boldsymbol{o}\longrightarrow\boldsymbol{x}\}\big\}\setminus E((\boldsymbol{o},\boldsymbol{u}),\boldsymbol{x};\tilde{C}^{(\boldsymbol{o},\boldsymbol{u})}(\boldsymbol{o}))\Big).\end{aligned} \tag{3.22}$$

We will extract the leading contribution from the first term. Note that $\{(\boldsymbol{o},\boldsymbol{u})\longrightarrow\boldsymbol{x}\}\circ\{\boldsymbol{o}\longrightarrow\boldsymbol{x}\}$ is almost identical to $\{\boldsymbol{o}\Longrightarrow\boldsymbol{x}\}=\{\boldsymbol{o}\longrightarrow\boldsymbol{x}\}\circ\{\boldsymbol{o}\longrightarrow\boldsymbol{x}\}$. However, the symmetry between the two connections from $\boldsymbol{o}$ to $\boldsymbol{x}$ is lost in the former event, due to the bond $(\boldsymbol{o},\boldsymbol{u})$. We will use this symmetry breaking in a convenient manner. Recall that below (3.11), the support of D was ordered in an arbitrary way. Now, instead, we choose the ordering such that, for $\boldsymbol{u}=(u,\varepsilon)$, the element u in the support of D is *minimal*. This will ensure that the lowest occupied path ω_1 from $\boldsymbol{o}$ to $\boldsymbol{x}$ will use the bond $(\boldsymbol{o},\boldsymbol{u})$. We also write $E^{\boldsymbol{u}}_{\prec}(\omega_1)$ and $E^{\boldsymbol{u}}_{\succ}(\omega_2;\omega_1)$ for $E_{\prec}(\omega_1)$ and $E_{\succ}(\omega_2;\omega_1)$ in this $\boldsymbol{u}$-dependent ordering. Therefore, (cf., (3.12)),

$$\{(\boldsymbol{o},\boldsymbol{u})\longrightarrow\boldsymbol{x}\}\circ\{\boldsymbol{o}\longrightarrow\boldsymbol{x}\}=\dot{\bigcup_{\substack{\omega_1:(\boldsymbol{o},\boldsymbol{u})\longrightarrow\boldsymbol{x}\\ \omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x}\\ \omega_1\cap\omega_2=\varnothing}}}\big\{\omega_1,\omega_2\text{ occupied; }E^{\boldsymbol{u}}_{\prec}(\omega_1)\cap E^{\boldsymbol{u}}_{\succ}(\omega_2;\omega_1)\big\},\tag{3.23}$$

and its contribution to (3.22) is

$$\varepsilon\sum_{\boldsymbol{x}\in\Lambda}\sum_{\substack{\omega_1:(\boldsymbol{o},\boldsymbol{u})\longrightarrow\boldsymbol{x}\\ \omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x}\\ \omega_1\cap\omega_2=\varnothing}}\mathbb{P}_1\big(\omega_1,\omega_2\text{ occupied; }E^{\boldsymbol{u}}_{\prec}(\omega_1)\cap E^{\boldsymbol{u}}_{\succ}(\omega_2;\omega_1)\big),\tag{3.24}$$

where $\omega_1:(\boldsymbol{o},\boldsymbol{u})\longrightarrow\boldsymbol{x}$ is a path from $\boldsymbol{o}$ to $\boldsymbol{x}$ starting by the bond $(\boldsymbol{o},\boldsymbol{u})$. Ignoring the condition $E^{\boldsymbol{u}}_{\prec}(\omega_1)\cap E^{\boldsymbol{u}}_{\succ}(\omega_2;\omega_1)$ as in (3.14) and following the same strategy as in estimating $\hat{\pi}_1^{(0)}$, we obtain the main contribution to (3.21). The leading term of $\frac{1}{\varepsilon^2}\hat{\pi}_1^{(1)}$ is twice as large as that of $\frac{1}{\varepsilon^2}\hat{\pi}_1^{(0)}$, because the symmetry is broken and we do not obtain the factor $\frac{1}{2}$ as in (3.18) (cf., (3.13) and (3.24)).

To complete the proof of (3.21), it suffices to show that the second and third terms in (3.22) are both $O(\beta^2)\,\varepsilon^2$. The event in the second term of (3.22) implies the existence of $\boldsymbol{y}\in\Lambda$ such that $\{\boldsymbol{o}\longrightarrow\boldsymbol{y}\longrightarrow\underline{b}\}\circ\{\boldsymbol{o}\longrightarrow\underline{b}\}$ and $\{\boldsymbol{y}\longrightarrow\boldsymbol{x}\}\circ\{\overline{b}\longrightarrow\boldsymbol{x}\}$ occur disjointly. Let ω_1 denote a path from $\boldsymbol{o}$ to $\boldsymbol{x}$ through $\boldsymbol{y}$, ω_2 denote another path from $\boldsymbol{o}$ to $\boldsymbol{x}$ via the the bond b with $\underline{b}=\boldsymbol{z}$, and ω_3 denote another path from $\boldsymbol{y}$ to $\boldsymbol{z}$. Then, the second term in (3.22) is bounded by

$$\varepsilon\sum_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\Lambda\\ \boldsymbol{z}\neq\boldsymbol{o},\boldsymbol{x}}}\sum_{\substack{\omega_1:\boldsymbol{o}\longrightarrow\boldsymbol{y}\longrightarrow\boldsymbol{x}\\ \omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{z}\longrightarrow\boldsymbol{x}\\ \omega_3:\boldsymbol{y}\longrightarrow\boldsymbol{z}\\ \omega_i\cap\omega_j=\varnothing,\ i\neq j}}\prod_{i=1}^{3}\mathbb{P}_1(\omega_i\text{ occupied}),\tag{3.25}$$

since $\mathbb{P}_1$ is a product measure. The third term in (3.22) is also bounded by the above expression. This is because the event in the third term in (3.22) implies existence of $\boldsymbol{y}\in\Lambda$ and a pivotal bond $b=(\boldsymbol{z},\cdot\,)$ for $\boldsymbol{u}\longrightarrow\boldsymbol{x}$ such that $\{\boldsymbol{o}\longrightarrow\boldsymbol{y}\longrightarrow\boldsymbol{x}\}$, $\{(\boldsymbol{o},\boldsymbol{u})\longrightarrow b\longrightarrow\boldsymbol{x}\}$ and $\{\boldsymbol{y}\longrightarrow\boldsymbol{z}\}$ occur disjointly. We thus obtain (3.25) by the same random walk representation.

Therefore, it is sufficient to prove that (3.25) is bounded by $O(\beta^2)\,\varepsilon^2$. When $\varepsilon=1$, we simply ignore the restriction $\omega_i\cap\omega_j=\varnothing$, $i\neq j$, and apply the Gaussian bound (1.10) to the part of ω_1 from $\boldsymbol{y}$ to $\boldsymbol{x}$ and to the part of ω_2 from $\boldsymbol{o}$ to $\boldsymbol{z}$. Since $\boldsymbol{y}\neq\boldsymbol{x}$ and $\boldsymbol{z}\neq\boldsymbol{o}$, the term $\delta_{\boldsymbol{o},\boldsymbol{x}}$ in (1.10) does not contribute, so that (3.25) is bounded by

$$\sum_{\substack{t,s,s'\in\mathbb{Z}_+\\0\le s\le s'\le t}} \frac{O(\beta)}{[1\vee(t-s)]^{d/2}}\,\frac{O(\beta)}{(1\vee s')^{d/2}} \le \sum_{t=0}^{\infty}\frac{O(\beta^2)}{(1\vee t)^{d/2}} \le O(\beta^2), \tag{3.26}$$

where s, s' are the time variables of $\boldsymbol{y}$ and $\boldsymbol{z}$, respectively. When $\varepsilon < 1$, we use the restriction $\omega_i \cap \omega_j = \varnothing$, $i \neq j$, to extract factors of q with pairwise different arguments, as in (3.14), out of the four intersection points $\boldsymbol{o}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{x}$. As explained above (3.19), each pair gives rise to a factor ε, and we obtain a total factor ε^4. With the help of (3.6), (3.25) with $\varepsilon < 1$ is bounded by ε^{1+4} times the left-hand side of (3.26) with the region of summation being replaced by $\varepsilon\mathbb{Z}_+$. This is further bounded by $O(\beta^2)\,\varepsilon^2$, since the sum over $t, s, s' \in \varepsilon\mathbb{Z}_+$ eats up a factor ε^3 for the Riemann sum approximation. This completes the proof of (3.21). ☐

Proof of (3.15). We only consider the case $E^{\rm c} = E^{\boldsymbol{u}}_{\preceq}(\omega_1)^{\rm c}$, which is the event that there is an $\eta \prec \omega_1$ from $\boldsymbol{o}$ to $\boldsymbol{x}$, which must share at least one step with ω_1, such that $E^{\boldsymbol{u}}_{\preceq}(\eta)$ occurs; the other case $E = E^{\boldsymbol{u}}_{\succ}(\omega_2;\omega_1)$ can be estimated in a similar way. Let ω_3 be the part of η from the point, say $\boldsymbol{y}$, where η starts disagreeing from ω_1 until it hits ω_1 or ω_2 at $\boldsymbol{z}$. Since $\mathbb{P}_1$ is a product measure, (3.15) is bounded by

$$\varepsilon \sum_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\Lambda\\ \boldsymbol{z}\neq\boldsymbol{o},\boldsymbol{x}}} \sum_{\substack{\omega_1:\boldsymbol{o}\longrightarrow\boldsymbol{y}\longrightarrow\boldsymbol{x}\\ \omega_2:\boldsymbol{o}\longrightarrow\boldsymbol{x}\\ \omega_3:\boldsymbol{y}\longrightarrow\boldsymbol{z}\\ \omega_i\cap\omega_j=\varnothing,\ i\neq j}} \left(\mathbb{1}_{\{\boldsymbol{z}\in\omega_1\setminus\{\boldsymbol{y}\}\}} + \mathbb{1}_{\{\boldsymbol{z}\in\omega_2\}}\right)\prod_{i=1}^{3}\mathbb{P}_1(\omega_i \text{ occupied}).$$

Since the contribution from $\mathbb{1}_{\{\boldsymbol{z}\in\omega_2\}}$ is equal to (3.25), we only need to investigate the contribution due to the other indicator $\mathbb{1}_{\{\boldsymbol{z}\in\omega_1\setminus\{\boldsymbol{y}\}\}}$. We again discuss the case $\varepsilon = 1$ first, and then adapt the argument to the case $\varepsilon < 1$, as done below (3.26). When $\varepsilon = 1$, we ignore the restriction $\omega_i \cap \omega_j = \varnothing$, $i \neq j$, and apply (1.10) to the probability of ω_2 and ω_3 being occupied. By denoting the time variables of $\boldsymbol{y}$ and $\boldsymbol{z}$ by s and s' respectively, the contribution from $\mathbb{1}_{\{\boldsymbol{z}\in\omega_1\setminus\{\boldsymbol{y}\}\}}$ is bounded by

$$\sum_{\substack{t,s,s'\in\mathbb{Z}_+\\0\le s< s'\le t}} \frac{O(\beta)}{(1\vee t)^{d/2}}\,\frac{O(\beta)}{[1\vee(s'-s)]^{d/2}} \le \sum_{t=0}^{\infty}\frac{O(\beta^2)}{(1\vee t)^{(d-2)/2}} \le O(\beta^2). \tag{3.27}$$

When $\varepsilon < 1$, we use the restriction $\omega_i \cap \omega_j = \varnothing$, $i \neq j$, along each of the four intersection points and obtain the eight factors of q with pairwise different arguments. Following the argument below (3.26), we obtain the desired bound $O(\beta^2)\,\varepsilon^2$. This completes the proof of (3.15). ☐

Proof of (3.17). Since $\omega_1 \cap \omega_2 \neq \varnothing$, there is a sequence of bonds $b_1, \dots, b_n$ such that ω_1 and ω_2 meet for the first time at $\underline{b}_1$, share $b_1, \dots, b_n$, and split at $\overline{b}_n$ (ω_1 and ω_2 may share a bond again after $\overline{b}_n$). This means that, together with $q(u)\,q(v)\,q(x-y)\,q(x-z)$ in (3.17), the left-hand side of

(3.17) is bounded by the convolution of two non-vanishing bubbles and $\prod_{i=1}^n q(w_i)^2$, where each w_i is the spatial component of $\overline{b}_i - \underline{b}_i$. Using (3.6) and $\sum_w q(w)^2 \leq \|q\|_\infty$, we can bound (3.17) by

$$\varepsilon \sum_{\substack{t,s,s' \in \varepsilon\mathbb{Z}_+ \\ \varepsilon < s < s' < t-\varepsilon}} \frac{O(\beta)\,\varepsilon^2}{(1 \vee s)^{d/2}} \big[(1-\varepsilon) \vee (\varepsilon \|D\|_\infty)\big]^{\frac{s'-s}{\varepsilon}} \frac{O(\beta)\,\varepsilon^2}{[1 \vee (t-s')]^{d/2}} \leq O(\beta^2)\,\varepsilon^2, \tag{3.28}$$

where, as before, ε^3 is used up for the Riemann sum approximation. The above estimate can be improved to $O(\beta^3)$ for oriented percolation, using (1.8). This completes the proof of (3.17). ☐

3.2 Asymptotics of $p_{\rm c}^{\rm pe}$

In this section, we compute the asymptotics of the critical point for (unoriented) percolation. We follow the strategy in Section 3.1 as closely as possible. However, there are a number of changes due to the fact that we have less control of the lace expansion coefficients. For example, the bounds on the derivative of $\hat{\Pi}_p$ with respect to p are not available in the literature, even though in the unpublished manuscript [13], this derivative is computed. To make this paper self-contained, we avoid the use of the derivative, which causes changes in the proof.

We start with some notation. Let

$$T_p = \sup_{x \in \mathbb{Z}^d} (pD * \tau_p^{*3})(x), \qquad T'_p = \sup_{x \in \mathbb{Z}^d} \tau_p^{*3}(x). \tag{3.29}$$

We will use the following bounds:

Lemma 3.1. *Fix $d > 6$. For L sufficiently large, and all $p \leq p_{\rm c}$,*

$$T_p \leq C\beta, \qquad T'_p \leq 1 + C\beta. \tag{3.30}$$

We will defer the proof of Lemma 3.1 to the end of this section.

To compute the asymptotics of $\hat{\Pi}_p$, we use (3.3) and the bound (cf., [5, Proposition 4.1])

$$\hat{\pi}_p^{(N)} \leq T'_p (2T_p T'_p)^{N \vee 1}. \tag{3.31}$$

Note that Lemma 3.1 together with (3.5) and (3.31) immediately imply

$$p_{\rm c} = 1 + O(\beta). \tag{3.32}$$

We now start the proof to improve (3.32) one term further. Together with Lemma 3.1, (3.31) proves that the contribution to $\sum_{N=2}^\infty \hat{\pi}_{p_{\rm c}}^{(N)}$ is $O(\beta^2)$. Thus, we are left to compute $\hat{\pi}_{p_{\rm c}}^{(0)}$ and $\hat{\pi}_{p_{\rm c}}^{(1)}$. The goal of this section is to prove

$$\hat{\pi}_{p_{\rm c}}^{(0)} = \frac{1}{2} \sum_{n=3}^\infty (n-1) D^{*n}(o) + O(\beta^2), \qquad \hat{\pi}_{p_{\rm c}}^{(1)} = D^{*2}(o) + \sum_{n=3}^\infty n D^{*n}(o) + O(\beta^2). \tag{3.33}$$

Using (3.5) and (3.33), we arrive at (1.14). Thus, we are left to prove (3.33).

We again investigate $\hat{\pi}^{(0)}_{p_c}$ and $\hat{\pi}^{(1)}_{p_c}$ separately. First, we compute $\hat{\pi}^{(0)}_{p_c}$. For percolation, we denote by $\{w \Longleftrightarrow x\}$ the event that w is doubly connected to x. By definition [11], $\hat{\pi}^{(0)}_p = \sum_{x\in\mathbb{Z}^d} \pi^{(0)}_p(x)$, where

$$\pi^{(0)}_p(x) = \mathbb{P}_p(o \Longleftrightarrow x) - \delta_{o,x}. \tag{3.34}$$

We wish to use *Russo's formula* (see, e.g., [8]) to prove that $\hat{\pi}^{(0)}_{p_c} = \hat{\pi}^{(0)}_1 + O(\beta^2)$. However, Russo's formula is restricted to events that only depend on a *finite* number of bonds, so that we will first show that Russo's formula may be applied to $\pi^{(0)}_p(x)$.

Let $B_\ell = \{x \in \mathbb{Z}^d : |x| \le \ell\}$. We note that, since $\hat{\pi}^{(0)}_p$ is finite for any $p \le p_c$, there is an $r < \infty$ such that $\sum_{x\notin B_r} \pi^{(0)}_p(x) = O(\beta^2)$ for any $p \le p_c$. In fact, using the *BK inequality* (see, e.g., [8]) and the bound $\tau_{p_c}(x) \le K|x|^{2-d}$ for $x \ne o$ [10, Proposition 2.2][1], we have

$$\sum_{x\notin B_r} \pi^{(0)}_p(x) \le \sum_{x\notin B_r} \tau_p(x)^2 \le \sum_{x\notin B_r} \tau_{p_c}(x)^2 \le c \sum_{\ell>r} \ell^{(d-1)+2(2-d)} = O(r^{4-d}) = O(\beta^2), \tag{3.35}$$

where we assume $r = O(L^{2d/(d-4)})$. Let $\{E \text{ in } B_R\}$ be the set of bond configurations whose restriction on bonds $\{u,v\}$ with $u,v \in B_R$ are in E. Similarly to (3.35), if $R = O(L^{2d/(d-6)})$, then for any $p \le p_c$ we have[2]

$$\sum_x \mathbb{P}_p\big(\{o \Longleftrightarrow x\} \setminus \{o \Longleftrightarrow x \text{ in } B_R\}\big) \le O(\beta^2). \tag{3.36}$$

By (3.35)–(3.36) and the mean-value theorem, there is a $p \in (1, p_c)$ such that

$$\begin{aligned}
\hat{\pi}^{(0)}_{p_c} &= \sum_{x\in B_r} \pi^{(0)}_{p_c}(x) + O(\beta^2) = \sum_{x\in B_r\setminus\{o\}} \mathbb{P}_{p_c}(o \Longleftrightarrow x \text{ in } B_R) + O(\beta^2) \\
&= \sum_{x\in B_r\setminus\{o\}} \mathbb{P}_1(o \Longleftrightarrow x \text{ in } B_R) + (p_c - 1) \sum_{x\in B_r\setminus\{o\}} \partial_p \mathbb{P}_p(o \Longleftrightarrow x \text{ in } B_R) + O(\beta^2) \\
&= \hat{\pi}^{(0)}_1 + (p_c - 1) \sum_{x\in B_r\setminus\{o\}} \partial_p \mathbb{P}_p(o \Longleftrightarrow x \text{ in } B_R) + O(\beta^2).
\end{aligned} \tag{3.37}$$

We will later identify $\hat{\pi}^{(0)}_1$, and first show that the second term is $O(\beta^2)$. Since the event $\{o \Longleftrightarrow x$ in $B_R\}$ depends only on finitely many bonds, we are now allowed to apply Russo's formula to obtain

$$\sum_{x\in B_r\setminus\{o\}} \partial_p \mathbb{P}_p(o \Longleftrightarrow x \text{ in } B_R) = \sum_{x\in B_r\setminus\{o\}} \sum_{(u,v)} D(v-u)\, \mathbb{P}_p\big((u,v) \text{ pivotal for } \{o \Longleftrightarrow x \text{ in } B_R\}\big), \tag{3.38}$$

[1] In [10, Proposition 2.2], K is of order $O(L^{-2+\epsilon})$ with an arbitrarily small number $\epsilon > 0$, and thus is small when L is large. Here, we do not care about the dependence of K on L, and will take $K = O(1)$.

[2] The event $\{o \Longleftrightarrow x\} \setminus \{o \Longleftrightarrow x \text{ in } B_R\}$ implies the existence of $y \notin B_R$ such that $o \longleftrightarrow x$, $x \longleftrightarrow y$ and $y \longleftrightarrow o$ occur disjointly. Therefore, by the BK inequality and Propositions 1.7(i) and 2.2 of [10], we obtain

$$\sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\big(\{o \Longleftrightarrow x\} \setminus \{o \Longleftrightarrow x \text{ in } B_R\}\big) \le \sum_{\substack{x\in\mathbb{Z}^d \\ y\notin B_R}} \tau_p(x)\, \tau_p(y-x)\, \tau_p(y) \le c \sum_{y\notin B_R} |y|^{(4-d)+(2-d)} = O(R^{6-d}).$$

where the factor $D(v-u)$ arises from the derivative of the bond occupation probability of $\{u,v\}$ with respect to p, and where a bond is pivotal for $o \Longleftrightarrow x$ when $o \Longleftrightarrow x$ in the (possibly modified) configuration where the bond is made occupied, while $o \Longleftrightarrow x$ does not occur in the (possibly modified) configuration where the bond is made occupied.

Since $p_{\rm c} = 1 + O(\beta)$, and since, by the BK inequality, (3.38) is bounded by

$$\sum_{x,(u,v)} D(v-u)\ \mathbb{P}_p\big(\{o \longleftrightarrow u\} \circ \{v \longleftrightarrow x\} \circ \{o \longleftrightarrow x\}\big)$$
$$\leq \sum_{x,(u,v)} D(v-u)\ \tau_p(u)\, \tau_p(x-v)\, \tau_p(x) \leq p^{-1} T_p \leq T_p, \tag{3.39}$$

so that the second term in (3.37) is $O(\beta^2)$. We are left to analyse the first term $\hat{\pi}_1^{(0)}$. We follow the strategy around (3.12), but the details change somewhat.

Let $\mathcal{S}_x$ denote all self-avoiding paths from o to x, and order the elements in $\mathcal{S}_x$ in an arbitrary way. Then we can write

$$\hat{\pi}_1^{(0)} = \sum_{x \neq o} \sum_{\substack{\omega_1,\omega_2 \in \mathcal{S}_x \\ \omega_1 \cap \omega_2 = \varnothing \\ \omega_1 \prec \omega_2}} \mathbb{P}_1\big(\omega_1, \omega_2 \text{ occupied; } E_{\prec}(\omega_1) \cap E_{\succ}(\omega_2; \omega_1)\big), \tag{3.40}$$

where $E_{\prec}(\omega_1)$ and $E_{\succ}(\omega_2;\omega_1)$ were defined between (3.11) and (3.12). In words, the event $E_{\prec}(\omega_1)$ holds when ω_1 is the lowest occupied self-avoiding walk path from o to x such that there is an occupied bond disjoint path from o to x. The event $E_{\succ}(\omega_2;\omega_1)$ holds when ω_2 is the highest occupied self-avoiding walk path from o to x that is bond disjoint from ω_1. Since $\mathbb{P}_1$ is a product measure, if we ignore $E_{\prec}(\omega_1) \cap E_{\succ}(\omega_2;\omega_1)$, we obtain

$$\sum_{\substack{\omega_1,\omega_2 \in \mathcal{S}_x \\ \omega_1 \cap \omega_2 = \varnothing \\ \omega_1 \prec \omega_2}} \mathbb{P}_1(\omega_1, \omega_2 \text{ occupied}) = \sum_{\substack{\omega_1,\omega_2 \in \mathcal{S}_x \\ \omega_1 \cap \omega_2 = \varnothing \\ \omega_1 \prec \omega_2}} \mathbb{P}_1(\omega_1 \text{ occupied})\ \mathbb{P}_1(\omega_2 \text{ occupied}). \tag{3.41}$$

We can then follow the rest of the argument between (3.14) and (3.19) to arrive at

$$\hat{\pi}_1^{(0)} = \frac{1}{2} \sum_{x \neq o} \sum_{\substack{\omega_1,\omega_2 \in \mathcal{S}_x \\ |\omega_1| + |\omega_2| \geq 3}} W_1(\omega_1)\ W_1(\omega_2) + O(\beta^2), \tag{3.42}$$

where we recall the definition of $W_p(\omega)$ in (1.1). Here the factor $1/2$ has the same origin as the one in (3.18), and the restriction that $|\omega_1| + |\omega_2| \geq 3$ is due to the fact that the smallest cycle in percolation has length 3. In (3.42), each ω_j is a self-avoiding path from o to x. However, as estimated in Section 2, the contribution in which ω_1 or ω_2 has a self-intersection is $O(\beta^2)$. Therefore, we can remove the self-avoidance constraint in (3.42). Performing the sum over $x \neq o$ and writing $\omega = (\omega_1, \omega_2)$, which is a random walk path starting and ending at o of length at least 3, we obtain

$$\hat{\pi}_1^{(0)} = \frac{1}{2} \sum_{\substack{\omega : o \longrightarrow o \\ |\omega| \geq 3}} (|\omega| - 1)\, W_1(\omega) + O(\beta^2) = \frac{1}{2} \sum_{n=3}^{\infty} (n-1)\, D^{*n}(o) + O(\beta^2), \tag{3.43}$$

where $|\omega|-1=n-1$ is the number of vertices along ω, excluding the starting and ending point of ω. This completes the computation of the leading asymptotics of $\hat\pi^{(0)}_{p_c}$.

We next derive the asymptotics of $\hat\pi^{(1)}_{p_c}$, following the strategy in [18, 19], where the first three coefficients of the asymptotic expansion into powers of $(2d)^{-1}$ of the critical value p_c for nearest-neighbour percolation were computed. The details of the argument are changed considerably compared to [18, 19]. Indeed, since we are only interested in the leading order term, while in [18] the first three coefficients are computed, many terms that need explicit computation in [18, 19] will be error terms for us. On the other hand, since in [18, 19] the asymptotics in nearest-neighbour models for large dimensions are considered, long loops lead to error term in [18, 19], whereas they contribute to the leading asymptotics here. We follow the proof in [18, Section 4.2] as closely and as long as possible, and indicate where the argument diverges.

To define $\hat\pi^{(1)}_p$, we need the following definitions. Given a bond configuration and $A\subseteq\mathbb{Z}^d$, we recall that x and y are *connected through* A, and write $x\xleftrightarrow{A}y$, if every occupied path connecting x to y has at least one bond with an endpoint in A. As defined below (3.7), the directed bond (u,v) is said to be *pivotal* for $x\longleftrightarrow y$, if $x\longleftrightarrow u$ and $v\longleftrightarrow y$ occur, and if $x\longleftrightarrow y$ occurs only when $\{u,v\}$ is set occupied. (Note that there is a distinction between the events $\{(u,v)$ is pivotal for $x\longleftrightarrow y\}$ and $\{(v,u)$ is pivotal for $x\longleftrightarrow y\}=\{(u,v)$ is pivotal for $y\longleftrightarrow x\}$.) Let

$$E'(v,x;A)=\{v\xleftrightarrow{A}x\}\cap\big\{\nexists(u',v')\text{ occupied \& pivotal for }v\longleftrightarrow x\text{ s.t. }v\xleftrightarrow{A}u'\big\}. \tag{3.44}$$

Then, by definition [11],

$$\hat\pi^{(1)}_p=\sum_x\sum_{(u,v)}p\,D(v-u)\;\mathbb{E}_0\Big[\mathbb{1}_{\{o\Longleftrightarrow u\}}\;\mathbb{P}_1\big(E'(v,x;\tilde C^{(u,v)}_0(o))\big)\Big], \tag{3.45}$$

where the sum over (u,v) is a sum over directed bonds. On the right-hand side, we use subscripts to identify the different expectations. Thus, the subscripts do *not* refer to the percolation parameter p. The cluster $\tilde C^{(u,v)}_0(o)$ appearing on the right hand side of (3.45) is random with respect to the expectation $\mathbb{E}_0$, but $\tilde C^{(u,v)}_0(o)$ should be regarded as a *fixed* set inside the probability $\mathbb{P}_1$. The latter introduces a second percolation model which depends on the original percolation model via the set $\tilde C^{(u,v)}_0(o)$. We refer to the bond configuration corresponding to the j^{th}-expectation as the "level-j" configuration.

By (3.31),

$$0\le\hat\pi^{(1)}_p\le 2T'_pT_pT'_p. \tag{3.46}$$

We will use refinements of this bound in the following.

We first claim that the contribution to (3.45) due to $u\ne o$ is an error term of order $O(\beta^2)$. Indeed, if $u\ne o$ then at level-0 the origin is in a cycle of length at least 3. Standard diagrammatic estimates then allow for the replacement in (3.46) of a factor T'_p by a constant multiple of T_p. This improves the bound (3.46) from $O(\beta)$ to $O(\beta^2)$, by (3.30).

We are left with the contribution to (3.45) due to $u=o$, namely

$$\sum_{x,v}p\,D(v)\;\mathbb{E}_0\big[\mathbb{P}_1\big(E'(v,x;\tilde C^{(o,v)}_0(o))\big)\big]. \tag{3.47}$$

If $x \notin \tilde{C}_0^{(o,v)}(o)$, then to obtain a non-zero contribution to $\mathbb{P}_1(E'(v,x;\tilde{C}_0^{(o,v)}(o)))$, x must be in an occupied cycle of length at least 3, in level-1 (in the language of [5, Section 3], the sausage containing x must consist of a cycle containing both x and an endpoint of the last pivotal bond for the connection from o to x). In this case, in (3.46), we may again replace a factor T_p' by a constant multiple of T_p, and again this contribution is $O(\beta^2)$. We are left to consider

$$\sum_{x,v} p\,D(v)\;\mathbb{E}_0\Big[\mathbb{1}_{\{x\in\tilde{C}_0^{(o,v)}(o)\}}\;\mathbb{P}_1\big(E'(v,x;\tilde{C}_0^{(o,v)}(o))\big)\Big]. \tag{3.48}$$

This is as far as the analogy with the argument in [18, Section 4.2] goes. We now need to adapt the proof there to compute the asymptotics of $\hat{\pi}_{p_c}^{(1)}$ when $L\to\infty$.

If $x\in\tilde{C}_0^{(o,v)}(o)$, and if $v\longleftrightarrow x$, then $v \xleftrightarrow{\tilde{C}_0^{(o,v)}(o)} x$. We next claim that the intersection with the second event in (3.44) leads to an error term. We write

$$\begin{aligned}
&I_0[x\in\tilde{C}_0^{(o,v)}(o)]\;I_1[E'(v,x;\tilde{C}_0^{(o,v)}(o))]\\
&\quad= I_0[x\in\tilde{C}_0^{(o,v)}(o)]\;I_1[v\longleftrightarrow x]\\
&\qquad\times\Big(1-I_1\big[\exists(u',v')\text{ occupied \& pivotal for }v\longleftrightarrow x\text{ s.t. }v\xleftrightarrow{\tilde{C}_0^{(o,v)}(o)}u'\big]\Big),
\end{aligned}$$

where we write I_0 and I_1 for the indicator functions on levels 0 and 1, respectively. The latter term can be bounded by

$$\sum_{(u',v')}\sum_z I_0[z\in\tilde{C}_0^{(o,v)}(o)]\;I_1\big[\{v\longleftrightarrow z\}\circ\{z\longleftrightarrow u'\}\circ\{(u',v')\text{ occupied}\}\circ\{v'\longleftrightarrow x\}\big], \tag{3.49}$$

which, using the BK inequality, yields a bound of the form

$$\sum_{x,v,z}\sum_{(u',v')} p\,D(v)\;\mathbb{P}_0(o\longleftrightarrow x,\;o\longleftrightarrow z)\;\tau_p(z-v)\;\tau_p(u'-z)\;p\,D(v'-u')\;\tau_p(x-v'). \tag{3.50}$$

By the tree-graph inequality [1]

$$\mathbb{P}_0(o\longleftrightarrow x,\;o\longleftrightarrow z)\le\sum_y\tau_p(y)\;\tau_p(x-y)\;\tau_p(z-y), \tag{3.51}$$

so that we end up with

$$\sum_{x,z,y}(p\,D*\tau_p)(y)\;\tau_p(x-y)\;\tau_p(z-y)\;\tau_p(z-v)\;(\tau_p*p\,D*\tau_p)(x-z)\le T_p^2=O(\beta^2), \tag{3.52}$$

which indeed is an error term. Thus, using the identity

$$\{x\in\tilde{C}_0^{(o,v)}(o)\}=\{o\longleftrightarrow x\text{ without using }(o,v)\}, \tag{3.53}$$

we end up with

$$\hat{\pi}_p^{(1)}=\sum_{v,x}p\,D(v)\;\tau_p^{(o,v)}(x)\;\tau_p(x-v)+O(\beta^2), \tag{3.54}$$

where

$$\tau_p^{(o,v)}(x) = \mathbb{P}(o \longleftrightarrow x \text{ without using } (o,v)). \tag{3.55}$$

Note that we can think of $\tau_p^{(o,v)}(x)$ as the two-point function on $\mathbb{Z}^d$, where the bond (o,v) is removed. We will denote the resulting graph with vertex set $\mathbb{Z}^d$ and edge set $\big\{\{x,y\} : x,y \in \mathbb{Z}^d, \{x,y\} \neq \{o,v\}\big\}$ by $\mathbb{Z}^d_{(o,v)}$, so that $\tau_p^{(o,v)}(x)$ is the two-point function on $\mathbb{Z}^d_{(o,v)}$. We will use this observation to compute $\tau_p^{(o,v)}(x)$.

We investigate the main term in the right-hand side of (3.54) further. Russo's formula, together with the BK inequality, yields that

$$\partial_p \tau_p(x) = \sum_{(y,z)} D(z-y)\ \mathbb{P}((y,z) \text{ pivotal for } o \longleftrightarrow x) \leq (\tau_p * D * \tau_p)(x), \tag{3.56}$$

$$\partial_p \tau_p^{(o,v)}(x) = \sum_{(y,z)} D(z-y)\ \mathbb{P}((y,z) \text{ pivotal for } o \longleftrightarrow x \text{ in } \mathbb{Z}^d_{(o,v)}) \leq (\tau_p * D * \tau_p)(x). \tag{3.57}$$

Therefore, we obtain that for $p = p_c$,

$$\begin{aligned}
\hat{\pi}^{(1)}_{p_c} &= \sum_{x,v} p_c D(v) \tau_{p_c}^{(o,v)}(x) \tau_{p_c}(x-v) + O(\beta^2) \\
&= \sum_{x,v} D(v) \tau_1^{(o,v)}(x) \tau_1(x-v) + O((p_c - 1)T_{p_c}) + O(\beta^2) \\
&= \sum_{x,v} D(v) \tau_1^{(o,v)}(x) \tau_1(x-v) + O(\beta^2),
\end{aligned} \tag{3.58}$$

since $p_c = 1 + O(\beta)$. Furthermore, an argument similar to the one for $\hat{\pi}^{(0)}_p$ shows that

$$\tau_1(x) = G(x) + O\big((G * g * G)(x)\big), \tag{3.59}$$

$$\tau_1^{(o,v)}(x) = G(x)\,(1 - \delta_{v,x}) + \delta_{v,x}(D^{*2} * G)(x) + O\big((G * g * G)(x)\big) + O\big(D * (G - \delta_o)(x)\big). \tag{3.60}$$

where we recall $G(x) = \sum_{n=0}^{\infty} D^{*n}(x)$ and define

$$g(x) = G(x)(D * G)(x). \tag{3.61}$$

We will prove (3.59)–(3.60) in full detail below, and first complete the proof subject to (3.59)–(3.60). Using (3.59)–(3.60), together with the fact that for $u \neq o$, we have $G(u) = (D * G)(u)$, we end up with

$$\begin{aligned}
\hat{\pi}^{(1)}_{p_c} &= \sum_{x \neq v} G(x)\, D(v)\, G(x-v) + \sum_{x} (D^{*2} * G)(x)\, D(x) + O(\beta^2) + O((D * G^{*3} * g)(o)) \\
&= \sum_{n=2}^{\infty} (n-1) D^{*n}(o) + \sum_{n=3}^{\infty} D^{*n}(o) + O(\beta^2) = D^{*2}(o) + \sum_{n=3}^{\infty} n D^{*n}(o) + O(\beta^2),
\end{aligned} \tag{3.62}$$

where we use

$$(D * G^{*3} * g)(o) \leq \|D * G^{*3}\|_\infty \, \|g\|_1 \leq O(\beta^2), \tag{3.63}$$

for $d > 6$, by (1.10).

This completes the proof subject to (3.59)–(3.60) and Lemma 3.1. □

Proof of (3.59)–(3.60). We start by proving (3.59), and then adapt the argument to prove (3.60). To see (3.59), we recall the arbitrary ordering of the elements in $\mathcal{S}_x$ introduced above (3.40). Then we have that

$$\tau_1(x) = \sum_{\omega\in\mathcal{S}_x} \mathbb{P}_p(\omega \text{ occupied}; F_{\succ}(\omega)), \tag{3.64}$$

where $F_{\succ}(\omega)$ is the event that ω is the lowest occupied path in $\mathcal{S}_x$. Thus, we can write

$$\tau_1(x) = \sum_{\omega\in\mathcal{S}_x} \mathbb{P}_p(\omega \text{ occupied}) - \sum_{\omega\in\mathcal{S}_x} \mathbb{P}_p(\omega \text{ occupied}; F_{\succ}(\omega)^c). \tag{3.65}$$

The former term equals

$$\delta_{o,x} + (1-\delta_{o,x}) \sum_{\omega\in\mathcal{S}_x} \prod_{i=0}^{|\omega|-1} D(\omega(i+1)-\omega(i)). \tag{3.66}$$

Clearly, by using inclusion-exclusion on the fact that ω is self-avoiding, as in (2.5), (3.66) equals

$$G(x) + O((G*G)(x)(G(o)-1)), \tag{3.67}$$

which is a contribution to the error in (3.59) when we note that $G(o)-1=(D*G)(o)$. Similarly, the second term in (3.65) is bounded by $O\big((G*g*G)(x)\big)$ using the fact that there must exist a $u\in\mathbb{Z}^d$ such that there exist bond disjoint occupied paths from o to u, two occupied paths from u to v (of which at least one is non-vanishing) and one from v to x. Thus, by the BK inequality, this term is bounded by

$$\sum_{u,v} G(u)\, G(v-u)\, (D*G)(v-u)\, G(x-v) = (G*g*G)(x).$$

The proof of (3.60) follows the same ideas. In (3.65) and (3.66), we only need to sum over self-avoiding walk paths that do not use the bond (o,v). When $x=v$, this means that $|\omega|\geq 2$, so that we obtain

$$\tau_1^{(o,v)}(v) = (D^{*2}*G)(v) + O((G*g*G)(v)) + O(D(v)(G(o)-1)). \tag{3.68}$$

When $x\neq v$, we can use inclusion-exclusion on the fact that the bond (o,v) is not used, and obtain

$$\tau_1^{(o,v)}(x) = \tau_1(x) + O(D(v)G(x-v)), \tag{3.69}$$

and then use (3.59). □

Proof of Lemma 3.1. We use [11, (5.20)], which states that uniformly in $p\in[1,p_c)$ and for L large enough

$$\hat{\tau}_p(k) \leq \frac{1+o(1)}{1-\hat{D}(k)}, \tag{3.70}$$

where $o(1)$ tends to 0 when $L \to \infty$. We also use the standard bound (see e.g. [5]) that for $x \neq 0$,

$$\tau_p(x) \le (pD * \tau_p)(x). \tag{3.71}$$

We then follow the proof as in [5]. For T_p, we fix x and extract the term in (3.29) due to the case where every argument of τ_p is o, which is $pD(x) \le p\,C\beta$ (see (1.8)). This gives

$$T_p(x) \le p\,C\beta + p \sum_{(u,y,z)\neq(x,o,o)} \tau_p(y)\,\tau_p(z-y)\,D(u)\,\tau_p(x+z-u). \tag{3.72}$$

Therefore, by (3.71),

$$T_p \le p\,C\beta + 3p^2 \sup_x (D^{*2} * \tau_p^{*3})(x), \tag{3.73}$$

where the factor 3 comes from the 3 factors τ_p whose argument can differ from o. In terms of the Fourier transform, this gives

$$T_p \le p\,C\beta + 3p^2 \sup_x \int_{\square_\pi} \frac{d^dk}{(2\pi)^d}\,\hat{D}(k)^2\,\hat{\tau}_p(k)^3\,e^{-ik\cdot x} = p\,C\beta + 3p^2 \int_{\square_\pi} \frac{d^dk}{(2\pi)^d}\,\hat{D}(k)^2\,\hat{\tau}_p(k)^3, \tag{3.74}$$

where $\square_\pi = [-\pi,\pi]^d$ and we use $\hat{\tau}_p(k) \ge 0$ [1]. For $L \gg 1$, by (3.70) we obtain

$$T_p \le p\,C\beta + 4p^2 \int_{\square_\pi} \frac{d^dk}{(2\pi)^d}\,\frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^3}. \tag{3.75}$$

Let $\hat{B}_{1/L} = \{k \in \square_\pi : cL^2|k|^2 \le \eta\}$. Using (1.8), we estimate the contribution to the integral in (3.75) from $k \in \square_\pi \setminus \hat{B}_{1/L}$ by

$$\int_{\square_\pi \setminus \hat{B}_{1/L}} \frac{d^dk}{(2\pi)^d}\,\frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^3} \le \eta^{-3} \int_{\square_\pi} \frac{d^dk}{(2\pi)^d}\,\hat{D}(k)^2 = O(\beta). \tag{3.76}$$

On the other hand, the contribution from $k \in \hat{B}_{1/L}$ is, again using (1.8), bounded by

$$\int_{\hat{B}_{1/L}} \frac{d^dk}{(2\pi)^d}\,\frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^3} \le \int_{\hat{B}_{1/L}} \frac{d^dk}{(2\pi)^d}\,(cL^2|k|^2)^{-3} = O(\beta). \tag{3.77}$$

This proves the bound on T_p.

The bound on T'_p is a consequence of $T'_p \le 1+3T_p$. Here the term 1 is due to the contribution to (3.29) where the arguments of the three factors of τ_p in T'_p in (3.29) are equal to o. If at least one of these arguments is nonzero, then we can use (3.71) for the corresponding two-point function. $\square$

A Bounds on $D^{*n}(x)$ and $q^{*n}(x)$

In this appendix, we prove (3.6) for any $\varepsilon \in (0,1]$ assuming (1.8). The inequality (1.10) follows by taking $\varepsilon = 1$.

First, we note that

$$q^{*n}(x)=(1-\varepsilon)^n\delta_{o,x}+\sum_{j=0}^{n-1}(1-\varepsilon)^{n-1-j}\varepsilon(D*q^{*j})(x), \tag{A.1}$$

where we suppose that the empty sum equals zero. When $n\le N\equiv\varepsilon^{-1}$, we use (A.1) to obtain

$$q^{*n}(x)\le(1-\varepsilon)^n\delta_{o,x}+\sum_{l=0}^{n-1}(1-\varepsilon)^l\varepsilon\,\|D\|_\infty\le(1-\varepsilon)^n\delta_{o,x}+O(\beta)\le(1-\varepsilon)^n\delta_{o,x}+\frac{O(\beta)}{[1\vee(n\varepsilon)]^{d/2}}, \tag{A.2}$$

as required. On the other hand, when $n>N$, we use

$$q^{*n}(x)=(1-\varepsilon)^n\delta_{o,x}+(1-\varepsilon)^{n-1}n\varepsilon D(x)+\sum_{j=0}^{n-2}(n-1-j)(1-\varepsilon)^{n-2-j}\varepsilon^2(D^{*2}*q^{*j})(x), \tag{A.3}$$

which is obtained by substituting (A.1) into q^{*j} in the right-hand side of (A.1). Since the second term in the right-hand side is bounded by $O(\beta)n\varepsilon e^{-n\varepsilon}\le O(\beta)[1\vee(n\varepsilon)]^{-d/2}$, it suffices to investigate the third term. Let S_1 be the sum over $j<N$, and let S_2 be the remaining sum, i.e.,

$$S_1=\sum_{0\le j<N}(n-1-j)(1-\varepsilon)^{n-2-j}\varepsilon^2(D^{*2}*q^{*j})(x), \tag{A.4}$$

$$S_2=\sum_{N\le j\le n-2}(n-1-j)(1-\varepsilon)^{n-2-j}\varepsilon^2(D^{*2}*q^{*j})(x). \tag{A.5}$$

For S_1, we use (A.2) to obtain

$$S_1\le(1-\varepsilon)^{n-2}\varepsilon^2D^{*2}(o)\sum_{l=n-N}^{n-1}l+O(\beta)\,\varepsilon^2\sum_{l=n-N}^{n-1}l\,(1-\varepsilon)^{l-1}\le O(\beta)\,(n\varepsilon)^2e^{-n\varepsilon}. \tag{A.6}$$

For S_2, we recall the definition $\hat{B}_{1/L}=\{k\in\square_\pi:cL^2|k|^2\le\eta\}$ below (3.75), and let $\hat{B}_{1/L}^+=\{k\in\hat{B}_{1/L}:\hat{q}(k)\ge0\}$. Note that

$$(D^{*2}*q^{*j})(x)\le\int_{\hat{B}_{1/L}^+}\frac{d^dk}{(2\pi)^d}\,\hat{q}(k)^j+\int_{\square_\pi\setminus\hat{B}_{1/L}^+}\frac{d^dk}{(2\pi)^d}\,\hat{D}(k)^2|\hat{q}(k)|^j. \tag{A.7}$$

Recall (1.8). For the first integral, we use

$$\hat{q}(k)=1-\varepsilon[1-\hat{D}(k)]\le e^{-\varepsilon[1-\hat{D}(k)]}\le e^{-c\varepsilon L^2|k|^2}, \tag{A.8}$$

while for the second integral in (A.7), we use, noting that without loss of generality, we may assume that $\eta\le1$,

$$|\hat{q}(k)|\le|1-\varepsilon(2-\eta)|\vee(1-\varepsilon\eta)\le1-\varepsilon\eta. \tag{A.9}$$

Therefore, we have

$$
\begin{aligned}
(D^{*2} * q^{*j})(x) &\le \int_{\hat{B}^+_{1/L}} \frac{d^dk}{(2\pi)^d}\, e^{-cj\varepsilon L^2|k|^2} + (1-\varepsilon\eta)^j \int_{\Box_\pi\setminus\hat{B}^+_{1/L}} \frac{d^dk}{(2\pi)^d}\, \hat{D}(k)^2 \\
&= O(\beta)\,(j\varepsilon)^{-d/2} + O(\beta)\,(1-\varepsilon\eta)^j. \qquad\qquad (A.10)
\end{aligned}
$$

Substituting (A.10) into (A.5) and separating the sum of the first term in (A.10) into the sum over $N \le j < \frac{n}{2} - 1$ and the sum over $\frac{n}{2} - 1 \le j \le n-2$, we obtain

$$
\begin{aligned}
S_2 &\le O(\beta)\,(n\varepsilon - \varepsilon - 1)\,(1-\varepsilon)^{\frac{n}{2}-1}\,\varepsilon \sum_{N\le j<\frac{n}{2}-1} (j\varepsilon)^{-d/2} \\
&\quad + O(\beta)\,(\tfrac{n\varepsilon}{2} - \varepsilon)^{-d/2}\,\varepsilon^2 \sum_{\frac{n}{2}-1\le j\le n-2} (n-1-j)\,(1-\varepsilon)^{n-2-j} \\
&\quad + O(\beta)\,(1-\varepsilon\eta)^{n-2}\,\varepsilon^2 \sum_{N\le j\le n-2} (n-1-j) \\
&\le O(\beta)\,(n\varepsilon)e^{-n\varepsilon/2} + O(\beta)\,(n\varepsilon)^{-d/2} + O(\beta)\,(n\varepsilon)^2\,e^{-\eta n\varepsilon}. \qquad\qquad (A.11)
\end{aligned}
$$

The proof of (3.6) is completed by combining (A.2)–(A.3), (A.6) and (A.11), and $(n\varepsilon)^2 e^{-n\varepsilon/2} \le C[1 \vee (n\varepsilon)]^{-d/2}$ for all $n \ge 1/\varepsilon$. ☐

B Computation for the spread-out uniform model

In this appendix, we compute the model-dependent terms of $p_{\rm c} - 1$ in (1.12)–(1.14) when the probability distribution D is defined as in (1.9). Recalling (1.15), we have

$$
D^{*2}(o) = \frac{1}{(2L+1)^d - 1} = \frac{\beta}{2^d} + O(\beta L^{-1}) = \beta\, U^{\star 2}(o) + O(\beta L^{-1}). \qquad (B.1)
$$

This relation can be extended as follows:

Proposition B.1. *Let D be the function defined in* (1.9)*. For $\alpha = 0, 1$, as $L \to \infty$,*

$$
\sum_{n=3}^{\infty} (n+1)^\alpha\, D^{*n}(o) = \beta \sum_{n=3}^{\infty} (n+1)^\alpha\, U^{\star n}(o) + O(\beta L^{-1}), \qquad (B.2)
$$

$$
\sum_{n=2}^{\infty} D^{*2n}(o) = \beta \sum_{n=2}^{\infty} U^{\star 2n}(o) + O(\beta L^{-1}), \qquad (B.3)
$$

where $d > 4 + 2\alpha$ in (B.2) *and $d > 4$ in* (B.3).

Theorem 1.2 is an immediate consequence of (B.1) and Proposition B.1. Note further that the coefficients of β in (B.2)–(B.3) are bounded if $d > 2 + 2\alpha$ and $d > 2$, respectively, which suggests that also for $d = 3 + 2\alpha$ and $d = 4 + 2\alpha$ the leading order contributions should be given by the first terms in (B.2)–(B.3).

Proof. For $x \in \mathbb{Z}^d$, define

$$D_o(x) = \frac{\mathbb{1}_{\{\|x\|_\infty \le L\}}}{(2L+1)^d}, \tag{B.4}$$

to be a regularized version of D in (1.9). It is obvious that $D_o^{*2}(o)$ satisfies the same estimate as in (B.1). We note that

$$\begin{aligned} D^{*m}(o) - D_o^{*m}(o) &= \sum_{j=1}^m \Big((D - D_o) * D^{*(j-1)} * D_o^{*(m-j)}\Big)(o) \\ &= \sum_{j=1}^m \left[\sum_{x:0<\|x\|_\infty \le L} \frac{\big(D^{*(j-1)} * D_o^{*(m-j)}\big)(x)}{(2L+1)^d[(2L+1)^d - 1]} - \frac{\big(D^{*(j-1)} * D_o^{*(m-j)}\big)(o)}{(2L+1)^d} \right] \\ &= \frac{1}{(2L+1)^d} \sum_{j=1}^m \Big[\big(D^{*j} * D_o^{*(m-j)}\big)(o) - \big(D^{*(j-1)} * D_o^{*(m-j)}\big)(o) \Big]. \end{aligned} \tag{B.5}$$

By this identity and the fact that $D_o^{*n}(x)$ also satisfies (1.10), we can approximate the expressions in the left-hand side of (B.2)–(B.3) by the corresponding expressions defined in terms of D_o instead of D, up to $O(\beta^2)$ when $d > 4 + 2\alpha$ and $d > 4$, respectively. For example, for (B.2) with $\alpha = 0$, we use (1.10) to obtain

$$\begin{aligned} &\bigg| \sum_{n=3}^\infty D^{*n}(o) - \sum_{n=3}^\infty D_o^{*n}(o) \bigg| \\ &\qquad \le \frac{1}{(2L+1)^d} \sum_{j=1}^\infty \sum_{n=j\vee 3}^\infty \Big[\big(D^{*j} * D_o^{*(n-j)}\big)(o) + \big(D^{*(j-1)} * D_o^{*(n-j)}\big)(o) \Big] \\ &\qquad = \frac{1}{(2L+1)^d} \Big((D + \delta_o) * \big(D_o^{*2} + D * D_o + D^{*2} * G\big) * G_o \Big)(o) = O(\beta^2), \end{aligned} \tag{B.6}$$

where $\delta_o(x) = \delta_{o,x}$ and $G_o(x) = \sum_{n=0}^\infty D_o^{*n}(x)$. Therefore, to prove Proposition B.1, it suffices to show that, for $\alpha = 0, 1$,

$$\sum_{n=3}^\infty (n+1)^\alpha D_o^{*n}(o) = \beta \sum_{n=3}^\infty (n+1)^\alpha U^{\star n}(o) + O(\beta L^{-1}), \tag{B.7}$$

$$\sum_{n=2}^\infty D_o^{*2n}(o) = \beta \sum_{n=2}^\infty U^{\star 2n}(o) + O(\beta L^{-1}). \tag{B.8}$$

We prove (B.7) for $\alpha = 0$ by comparing the Fourier transform of the first term in the right-hand side of (B.7), i.e.,

$$\beta \sum_{n=3}^\infty U^{\star n}(o) = \beta \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{\hat{U}(k)^3}{1 - \hat{U}(k)}, \tag{B.9}$$

with the Fourier transform of the left-hand side of (B.7), i.e.,

$$\sum_{n=3}^{\infty} D_o^{*n}(o) = \int_{\square_\pi} \frac{d^dk}{(2\pi)^d}\, \frac{\hat{D}_o(k)^3}{1-\hat{D}_o(k)} = \beta_o \int_{\square_{(L+\frac{1}{2})\pi}} \frac{d^dk}{(2\pi)^d}\, \frac{\hat{D}_o\big(\frac{k}{L+\frac{1}{2}}\big)^3}{1-\hat{D}_o\big(\frac{k}{L+\frac{1}{2}}\big)}, \tag{B.10}$$

where $\square_\ell = [-\ell, \ell]^d$ and

$$\beta_o = (L+\tfrac{1}{2})^{-d} = \beta + O(\beta L^{-1}), \tag{B.11}$$

and also

$$\hat{U}(k) = \int_{\mathbb{R}^d} d^dx\; U(x)\; e^{ik\cdot x} = \prod_{j=1}^d \frac{\sin k_j}{k_j}, \tag{B.12}$$

$$\hat{D}_o(k) = \sum_{x\in\mathbb{Z}^d} D_o(x)\; e^{ik\cdot x} = \prod_{j=1}^d \frac{\sin[(2L+1)\frac{k_j}{2}]}{(2L+1)\sin\frac{k_j}{2}} = \frac{\hat{U}((L+\frac{1}{2})k)}{\hat{U}(\frac{k}{2})}. \tag{B.13}$$

The simple product formula (B.13) is the main advantage of using D_o instead of D. It follows from (B.13) that $\hat{D}_o\big(\frac{k}{L+\frac{1}{2}}\big) = \hat{U}(k)/\hat{U}(\frac{k}{2L+1})$, which approximates $\hat{U}(k)$ for large L. We write

$$\beta\sum_{n=3}^{\infty} U^{\star n}(o) - \sum_{n=3}^{\infty} D_o^{*n}(o) = (\beta-\beta_o)\sum_{n=3}^{\infty} U^{\star n}(o) + \beta_o(I_1+I_2), \tag{B.14}$$

where, by (B.11), the first term is $O(\beta L^{-1})$, and

$$I_1 = \int_{\mathbb{R}^d\setminus\square_{(L+\frac{1}{2})\pi}} \frac{d^dk}{(2\pi)^d}\, \frac{\hat{U}(k)^3}{1-\hat{U}(k)}, \qquad I_2 = \int_{\square_{(L+\frac{1}{2})\pi}} \frac{d^dk}{(2\pi)^d} \left[\frac{\hat{U}(k)^3}{1-\hat{U}(k)} - \frac{\hat{D}_o\big(\frac{k}{L+\frac{1}{2}}\big)^3}{1-\hat{D}_o\big(\frac{k}{L+\frac{1}{2}}\big)}\right]. \tag{B.15}$$

We prove below that each I_j is $O(L^{-1})$ if $d>2$. This suffices to prove (B.7) for $\alpha=0$. If fact, we will prove that each I_j is $O(L^{-2}\log L)$ if $d>2$, which also identifies the coefficient of βL^{-1}.

To estimate each I_j, we use the following properties of $\hat{U}(k)$ and $\hat{D}_o\big(\frac{k}{L+\frac{1}{2}}\big)$ that follow from the standard estimates for the trigonometric functions: for any k,

$$|\hat{U}(k)| \le \prod_{j=1}^d (1\vee|k_j|)^{-1}, \qquad\qquad 1-\hat{U}(k) \ge c_1(1\wedge|k|^2), \tag{B.16}$$

and for $k\in\square_{(L+\frac{1}{2})\pi}$,

$$\big|\hat{D}_o\big(\tfrac{k}{L+\frac{1}{2}}\big)\big| \le c_2\prod_{j=1}^d (1\vee|k_j|)^{-1}, \qquad\qquad 1-\hat{D}_o\big(\tfrac{k}{L+\frac{1}{2}}\big) \ge c_3(1\wedge|k|^2), \tag{B.17}$$

$$\big|\hat{U}(k)-\hat{D}_o\big(\tfrac{k}{L+\frac{1}{2}}\big)\big| \le c_4L^{-2}|\hat{U}(k)|\,|k|^2, \tag{B.18}$$

where each $c_i \in (0,\infty)$ is independent of L and k. To see the first inequality in (B.16), we only need to use $|\frac{\sin r}{r}| \le (1\vee r)^{-1}$ for any r. For the first inequality in (B.17), we recall $\hat{D}_o\big(\frac{k}{L+\frac12}\big) = \hat{U}(k)/\hat{U}(\frac{k}{2L+1})$ and use $\frac{\sin r}{r} \ge \frac{2}{\pi}$ for any $r \in [-\frac{\pi}{2},\frac{\pi}{2}]$, so that $|\hat{U}(\frac{k}{2L+1})^{-1}| \le (\frac{\pi}{2})^d$, and then use the bound on $|\hat{U}(k)|$ in (B.16). The second inequalities in (B.16)–(B.17) follow from (B.12) and (1.8), respectively. Finally, for (B.18), we again use $\frac{2}{\pi} \le \frac{\sin r}{r} \le 1$ for any $r \in [-\frac{\pi}{2},\frac{\pi}{2}]$ to obtain

$$\big|\hat{U}(k) - \hat{D}_o\big(\tfrac{k}{L+\frac12}\big)\big| \le \Big(\frac{\pi}{2}\Big)^d |\hat{U}(k)| \left|\prod_{j=1}^d \frac{\sin\frac{k_j}{2L+1}}{\frac{k_j}{2L+1}} - 1\right| \le \Big(\frac{\pi}{2}\Big)^d |\hat{U}(k)| \sum_{i=1}^d \left|\frac{\sin\frac{k_i}{2L+1}}{\frac{k_i}{2L+1}} - 1\right|, \tag{B.19}$$

which is bounded by $|\hat{U}(k)|\, O(|k|^2L^{-2})$, using $0 \le \frac{\sin r}{r} - 1 + \frac{r^2}{3!} \le \frac{r^2}{5!}(\frac{\pi}{2})^2$ for any $r \in [-\frac{\pi}{2},\frac{\pi}{2}]$. This completes the proof of (B.16)–(B.18).

First, we consider I_1. Using (B.16), we obtain

$$|I_1| \le c\int_{\mathbb{R}^d\setminus\square_{(L+\frac12)\pi}} d^dk\ |\hat{U}(k)|^3 \le c'\sum_{i=1}^d\bigg[\int_{(L+\frac12)\pi}^\infty \frac{dk_i}{k_i^3}\bigg]\bigg[\prod_{j\neq i}\int_{-\infty}^\infty \frac{dk_j}{(1\vee|k_j|)^3}\bigg] = O(L^{-2}). \tag{B.20}$$

For I_2, we use

$$\left|\frac{u^3}{1-u} - \frac{d^3}{1-d}\right| = \frac{|u-d|\,|u^2+ud+d^2-ud(u+d)|}{(1-u)(1-d)} \le \frac{|u-d|\,(u^2+3|ud|+d^2)}{(1-u)(1-d)}, \tag{B.21}$$

with $u = \hat{U}(k)$ and $d = \hat{D}_o\big(\frac{k}{L+\frac12}\big)$. Using (B.16)–(B.18), we obtain

$$|I_2| \le cL^{-2}\int_{\square_{(L+\frac12)\pi}} d^dk\ \frac{|k|^2\prod_{j=1}^d(1\vee|k_j|)^{-3}}{(1\wedge|k|^2)^2} = O(L^{-2}\log L), \tag{B.22}$$

for $d>2$. The proof of (B.7) for $\alpha=0$ is completed by (B.14), (B.20) and (B.22).

The same strategy explained above also applies to the proof of (B.2) for $\alpha=1$ and (B.3), using the following expressions:

$$\sum_{n=3}^\infty (n+1)\,D_o^{*n}(o) = \beta_o\int_{\square_{(L+\frac12)\pi}} \frac{d^dk}{(2\pi)^d}\ \frac{\hat{D}_o\big(\frac{k}{L+\frac12}\big)^3\big[4-3\hat{D}_o\big(\frac{k}{L+\frac12}\big)\big]}{\big[1-\hat{D}_o\big(\frac{k}{L+\frac12}\big)\big]^2}, \tag{B.23}$$

$$\sum_{n=2}^\infty D_o^{*2n}(o) = \beta_o\int_{\square_{(L+\frac12)\pi}} \frac{d^dk}{(2\pi)^d}\ \frac{\hat{D}_o\big(\frac{k}{L+\frac12}\big)^4}{1-\hat{D}_o\big(\frac{k}{L+\frac12}\big)^2}. \tag{B.24}$$

This completes the proof of Proposition B.1. $\square$

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